Problem 1

Problem 1

let $V(x_1, x_2) = x_1^2 + \frac{1}{2} x_2^2$

$$\dot{V}(x_1, x_2) = 2x_1 \dot{x}_1 + x_2 \dot{x}_2 = 2x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3)$$

$$= -2x_1^2 - 2x_1x_2 + 2x_1x_2 - x_2^4$$

$$= -2x_1^2 - x_2^4$$

$$\dot{V}(x_1 = 0, x_2 = 0) = 0$$

$$\dot{V}(x_1, x_2) < 0, \ \forall x \neq 0$$

Hence, it's globally asymptotically stable

Problem 2

Problem 2.

1. given $u_i = 0$, system becomes

$$J_1 \dot{w}_1 = (J_2 - J_3) w_2 w_3$$

$$J_2 \dot{w}_2 = (J_3 - J_1) w_3 w_1$$

$$J_3 \dot{w}_3 = (J_1 - J_2) w_1 w_2$$

first let's let $V(w) = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2)$

$\dot{V}(w) = w_1 \cdot \dot{w}_1 + w_2 \cdot \dot{w}_2 + w_3 \cdot \dot{w}_3$ (not ideal enough)

Let's try another one .... let $V(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2)$

$$\dot{V}(w) = J_1 w_1 \dot{w}_1 + J_2 w_2 \dot{w}_2 + J_3 w_3 \dot{w}_3$$

$$= w_1 (\cancel{J_2} - \cancel{J_3}) w_2 w_3 + w_2 (\cancel{J_3} - \cancel{J_1}) w_3 w_1 + w_3 (\cancel{J_1} - \cancel{J_2}) w_1 w_2$$

[redacted]

$$= 0, \ \forall w_i$$

Hence, origin is stable but not asymptotically stable

2. $u_i = -k_i w_i \qquad (k_i > 0)$

$$\begin{cases} J_1 \dot{w}_1 = (J_2 - J_3) w_2 w_3 - k_1 w_1 \\ J_2 \dot{w}_2 = (J_3 - J_1) w_3 w_1 - k_2 w_2 \\ J_3 \dot{w}_3 = (J_1 - J_2) w_1 w_2 - k_3 w_3 \end{cases}$$

Let $V(w) = \frac{1}{2}\left(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2\right)$

$$\dot{V}(w) = J_1 w_1 \dot{w}_1 + J_2 w_2 \dot{w}_2 + J_3 w_3 \dot{w}_3$$
$$= w_1 \cdot \left((J_2 - J_3) w_2 w_3 - k_1 w_1\right) + w_2 \cdot \left((J_3 - J_1) w_3 w_1 - k_2 w_2\right)$$
$$+ w_3 \cdot \left((J_1 - J_2) w_1 w_2 - k_3 w_3\right)$$

$$= w_1 \cdot (J_2 w_2 w_3 - J_3 w_2 w_3 - k_1 w_1) + w_2 \cdot (J_3 w_3 w_1 - J_1 w_3 w_1 - k_2 w_2)$$
$$+ w_3 (J_1 w_1 w_2 - J_2 w_1 w_2 - k_3 w_3)$$
$$= \cancel{w_1 J_2 w_2 w_3} - \cancel{w_1 J_3 w_2 w_3} - w_1 k_1 w_1 + \cancel{w_2 J_3 w_3 w_1} - \cancel{w_2 J_1 w_3 w_1} - w_2 k_2 w_2$$
$$+ \cancel{w_3 J_1 w_1 w_2} - \cancel{w_3 J_2 w_1 w_2} - w_3 k_3 w_3$$
$$= -w_1 k_1 w_1 - w_2 k_2 w_2 - w_3 k_3 w_3$$
$$= -w_1^2 k_1 - w_2^2 k_2 - w_3^2 k_3 \quad (k_i > 0)$$
$$\leq 0$$

Since $J_1 > J_2 > J_3 > 0$ as $V(w) = \frac{1}{2}\left(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2\right)$

as $w \to \infty$. $V(w) \to \infty$ (global)

Hence, global asymptotic stabe

Problem 3

let $V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$

$\dot{V}(x_1, x_2) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 + x_1 x_2) + x_2(-x_2)$

$= -x_1^2 + x_1^2 x_2 - x_2^2$

$\leq -x_1^2 - x_2^2 + |x| |x_1| |x_2|$

$\dot{V}(x_1 = 0, x_2 = 0) = 0$

In set $\{\|x\| \leq r^2\}$, we have $|x_1| \leq r$. ($\lim_{r \to \infty} r = 0$, $r > 0$)

Hence, $\dot{V}(x) \leq -x_1^2 - x_2^2 + r|x_1||x_2| \leq -2\|x\|^2 + r\|x\| < 0$, which is negative definite

$$\begin{cases} \dot{x}_2 = -x_2 \Rightarrow x_2 = e^{-t} x_2(0) \\ \dot{x}_1 = -x_1 + x_1 x_2 \Rightarrow (-1 + e^{-t} x_2(0)) x_1 \Rightarrow x_1 = e^{-t(1 + e^{-t} x_2(0)) t} x_1(0) \end{cases}$$

Hence, $\lim_{t \to \infty} x_2(t) = 0$, $\lim_{t \to \infty} x_1(t) = 0$. origin is globally asymptotically stable

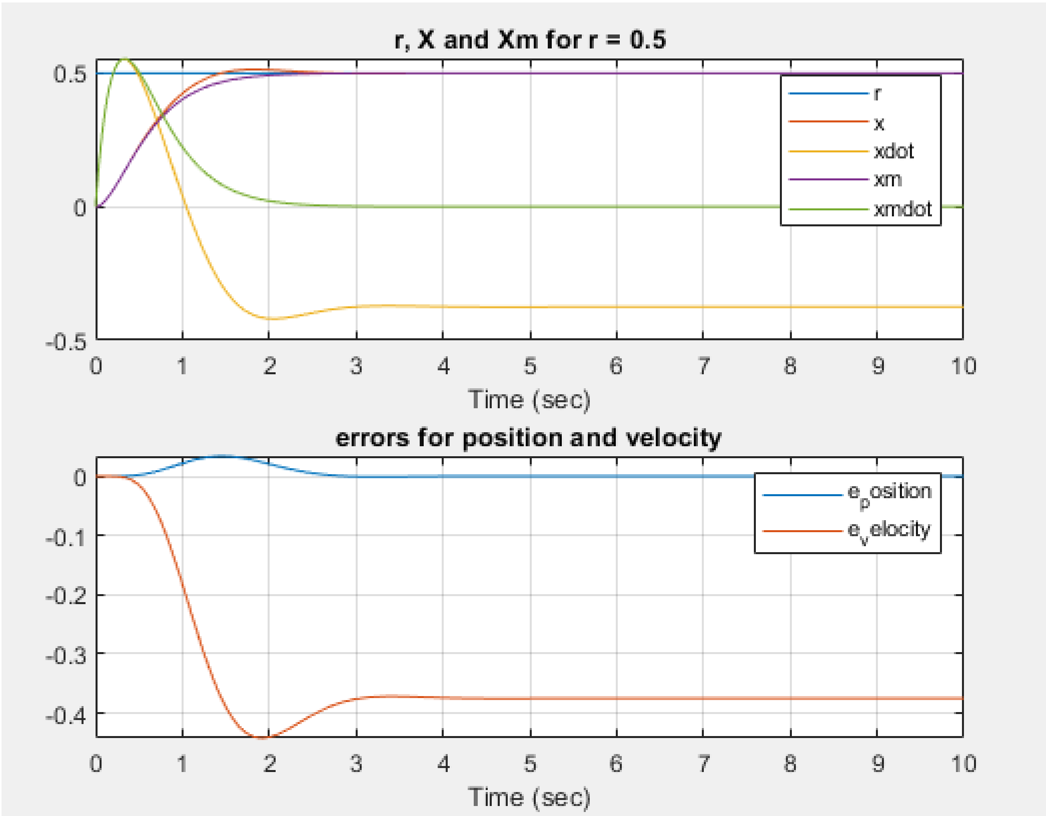
Problem 4

Linear Part:

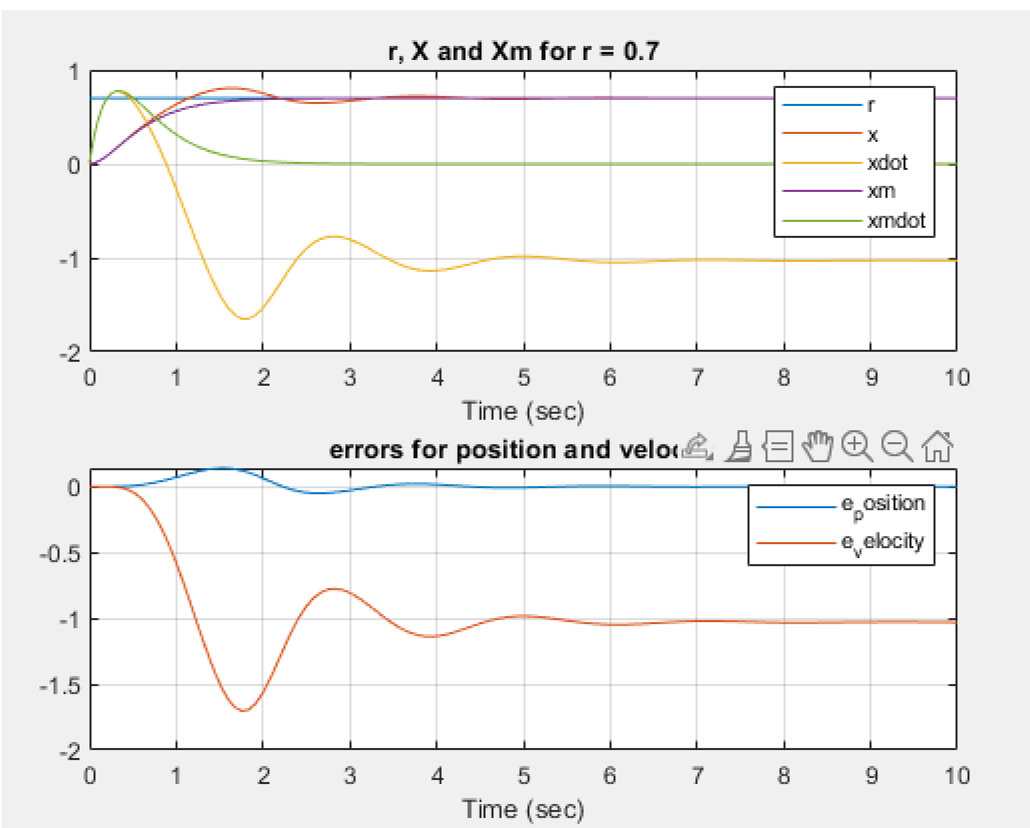
Case 1:

r(t) = A

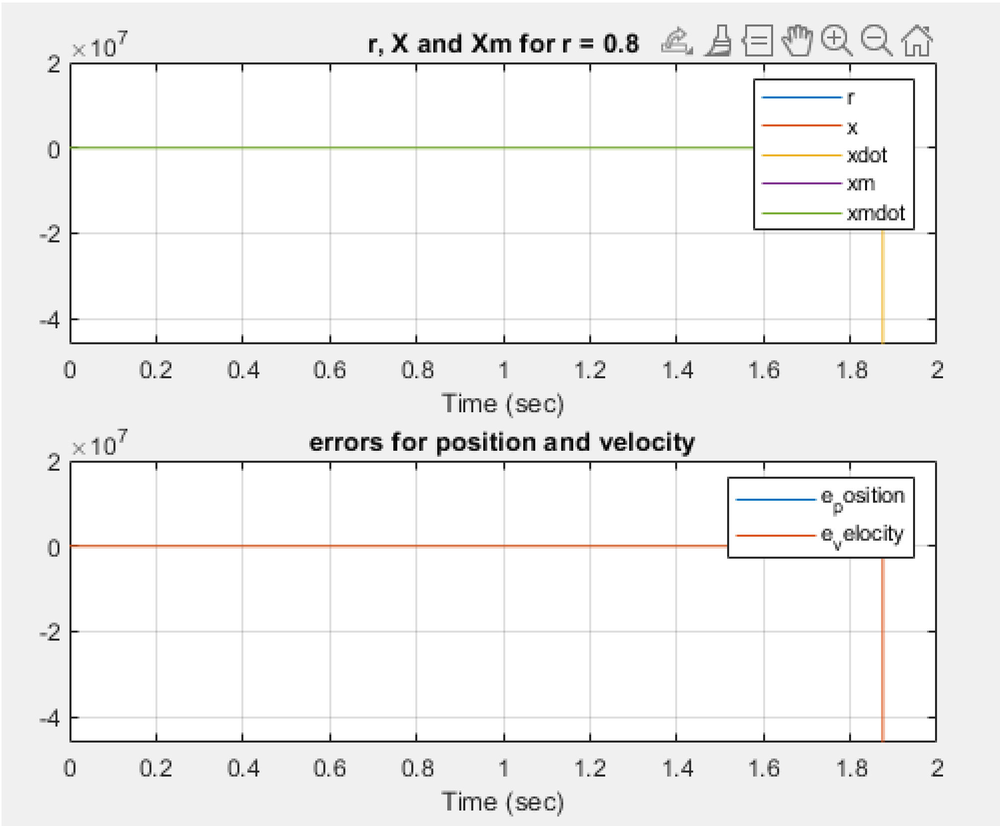
Subcase 1: A = 0.5



Subcase 2: A = 0.7



Subcase 3: A = 0.8

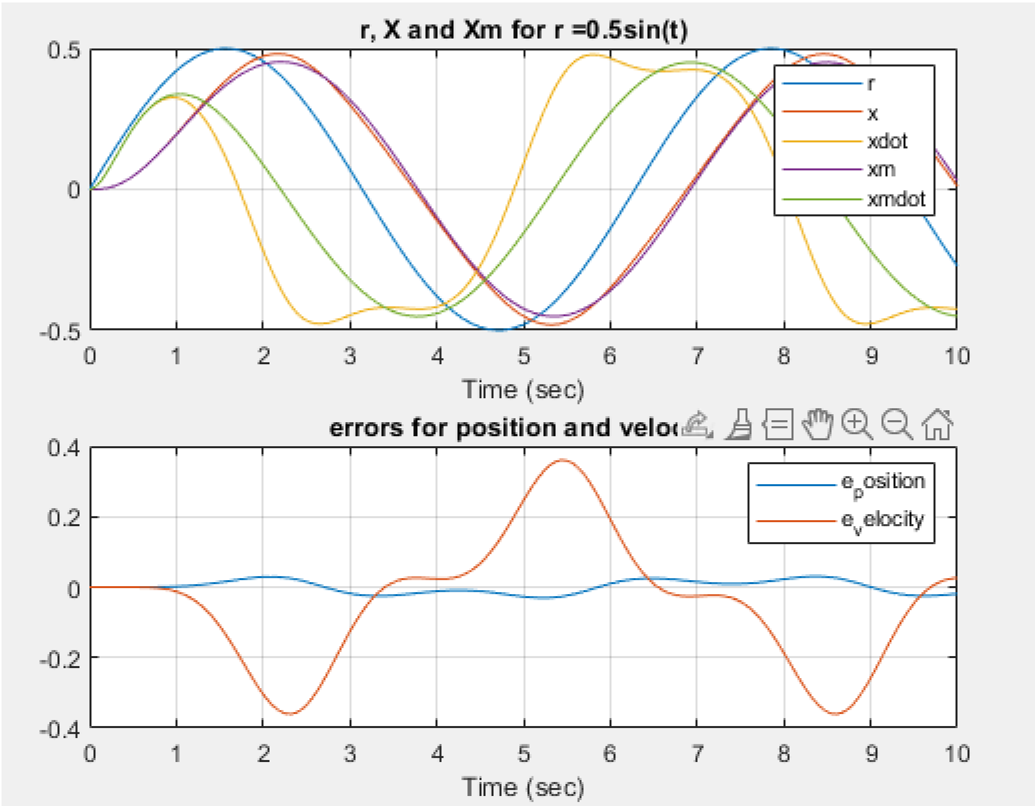


At this value, we saw that the system exploded

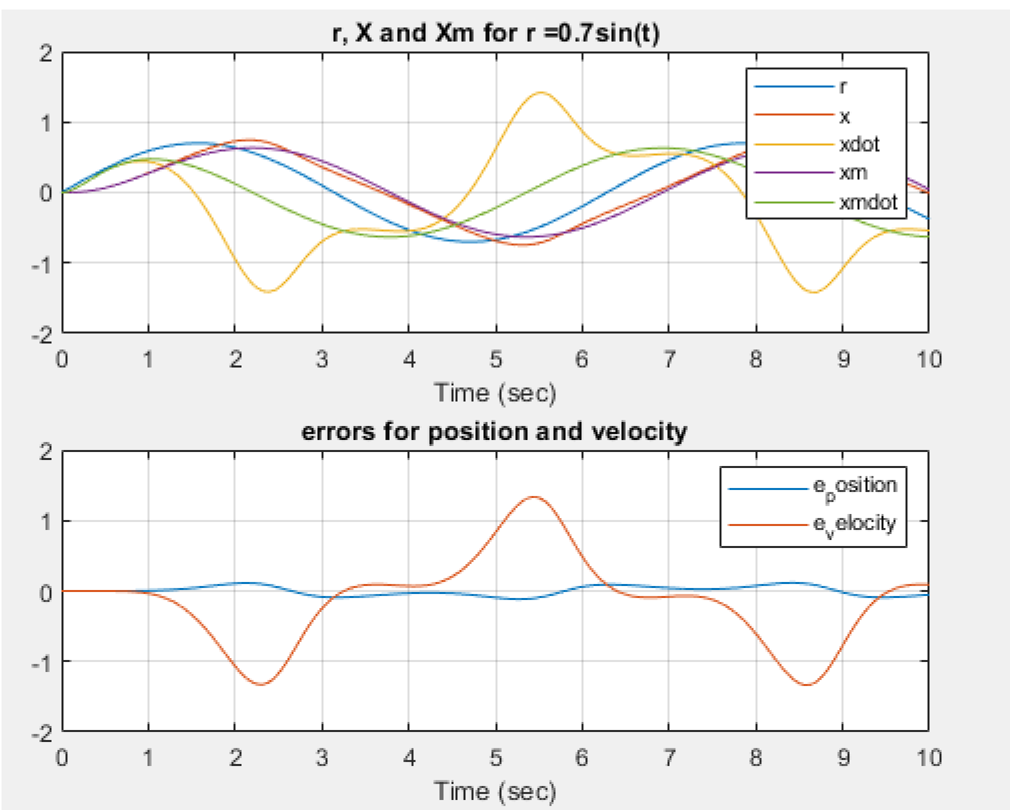
Case2:

r(t) = Asin(t)

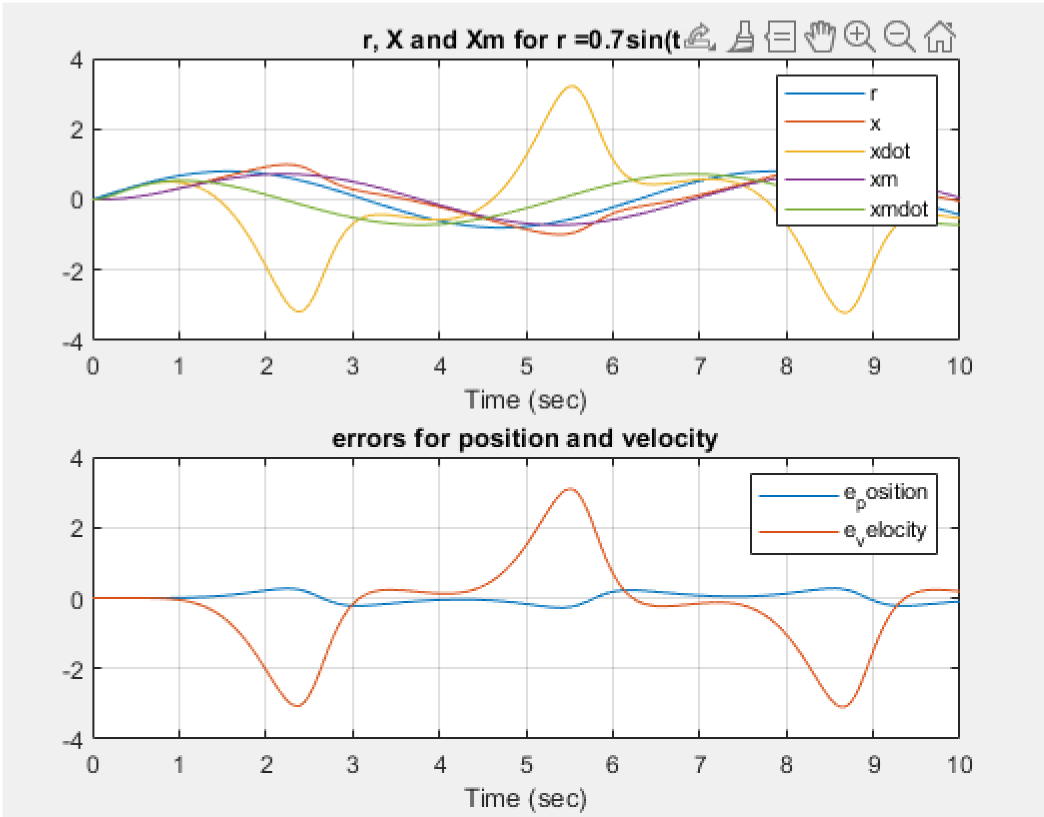
Subcase 1: A = 0.5



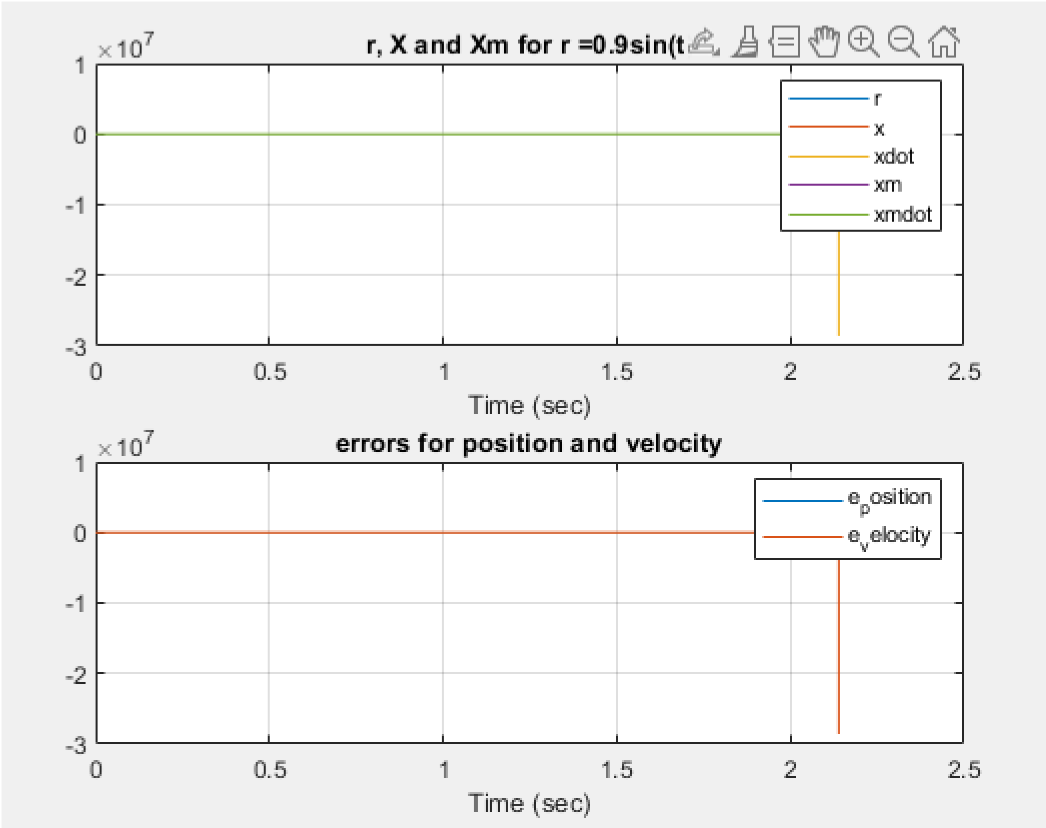
Subcase 2: A = 0.7



Subcase 3: A = 0.8



Subcase 4: A = 0.9



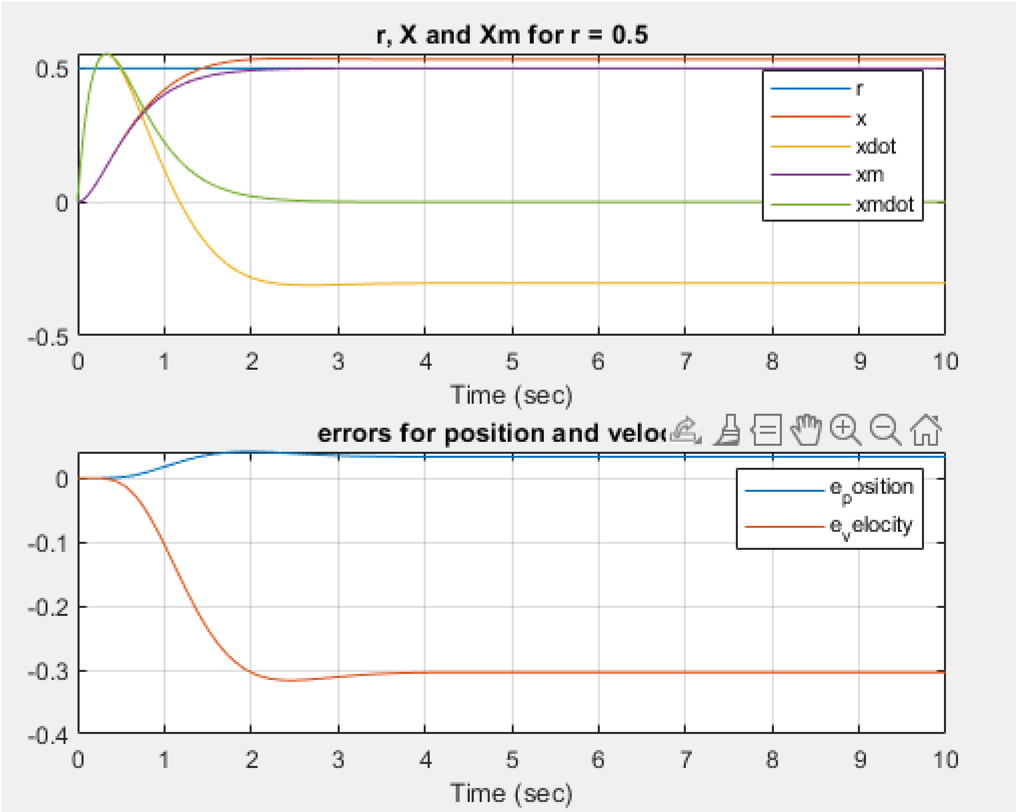
The system explodes at this value

Non-Linear Part:

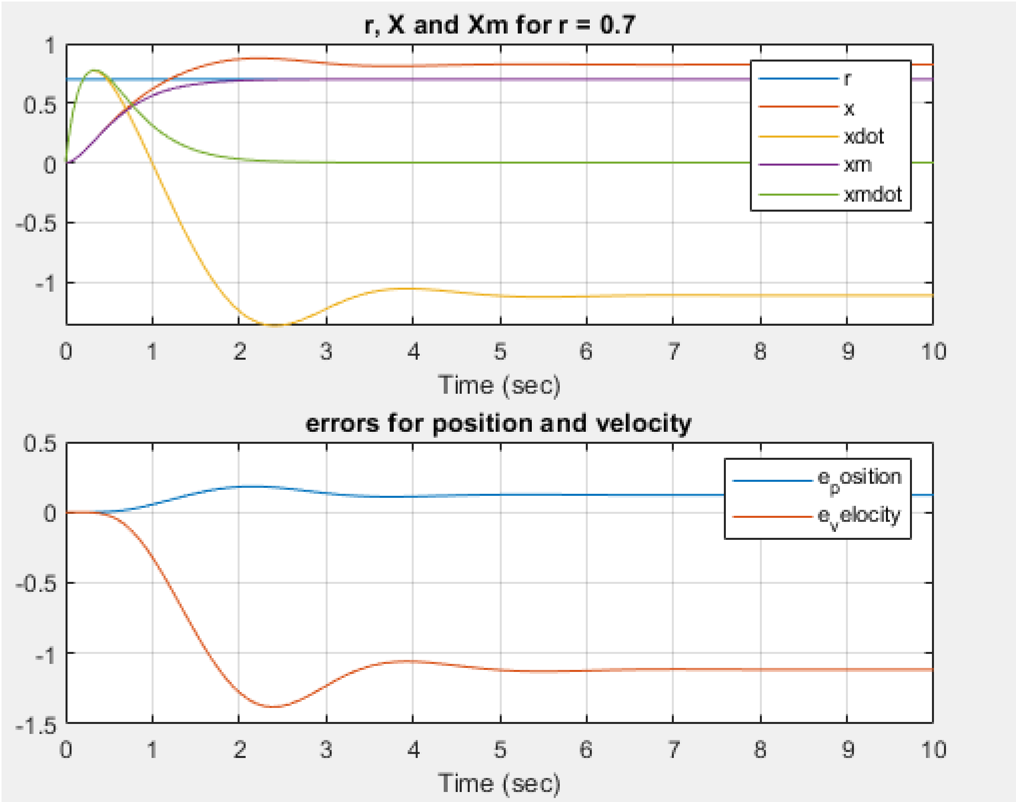
Case 1:

r(t) = A

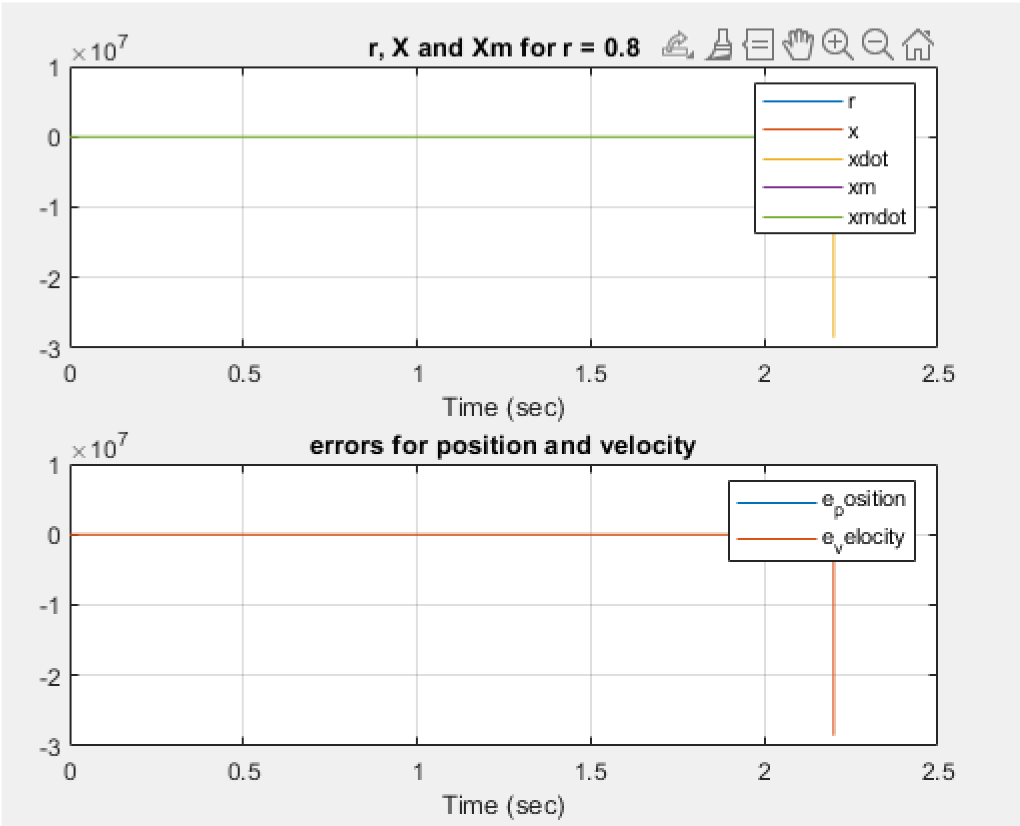
Subcase 1: A = 0.5



Subcase 2: A = 0.7

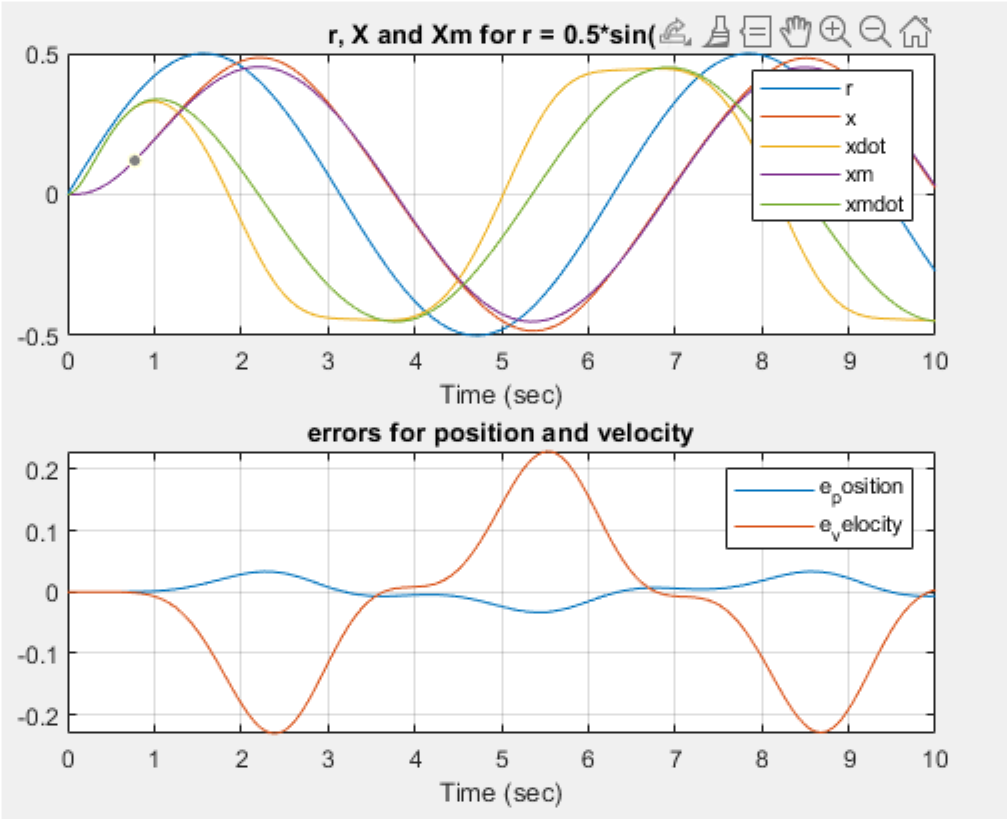


Subcase 2: A = 0.8

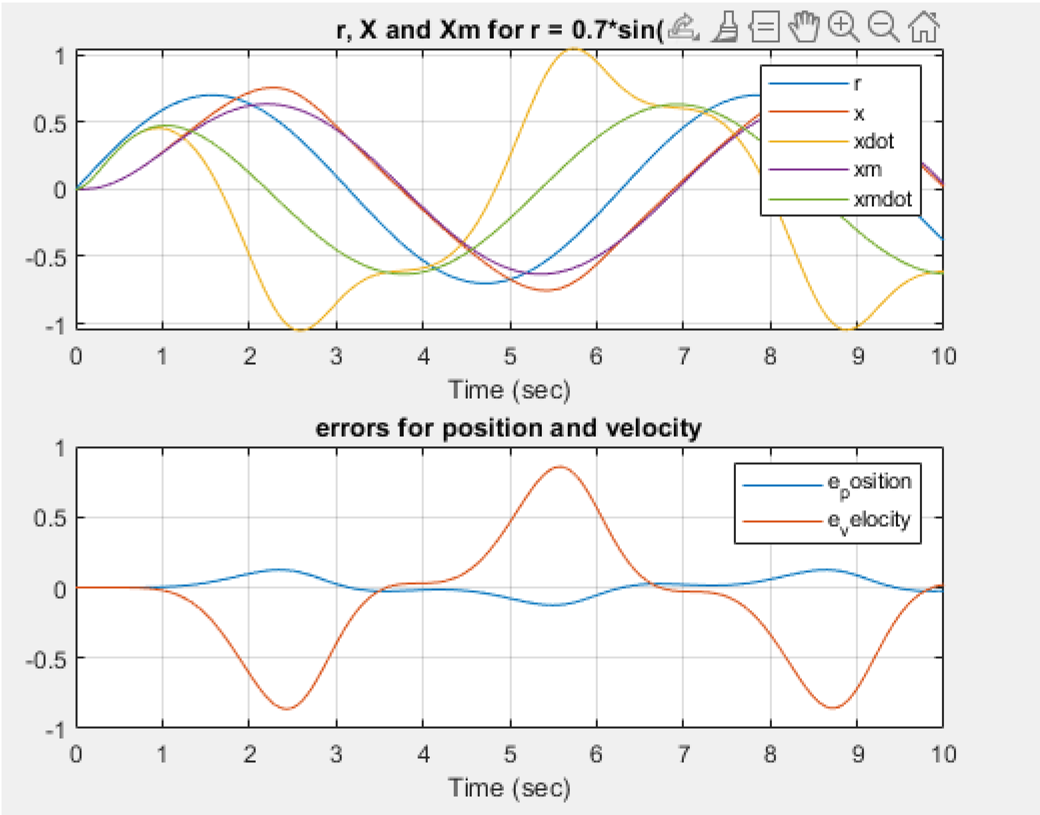


The system explode at this value

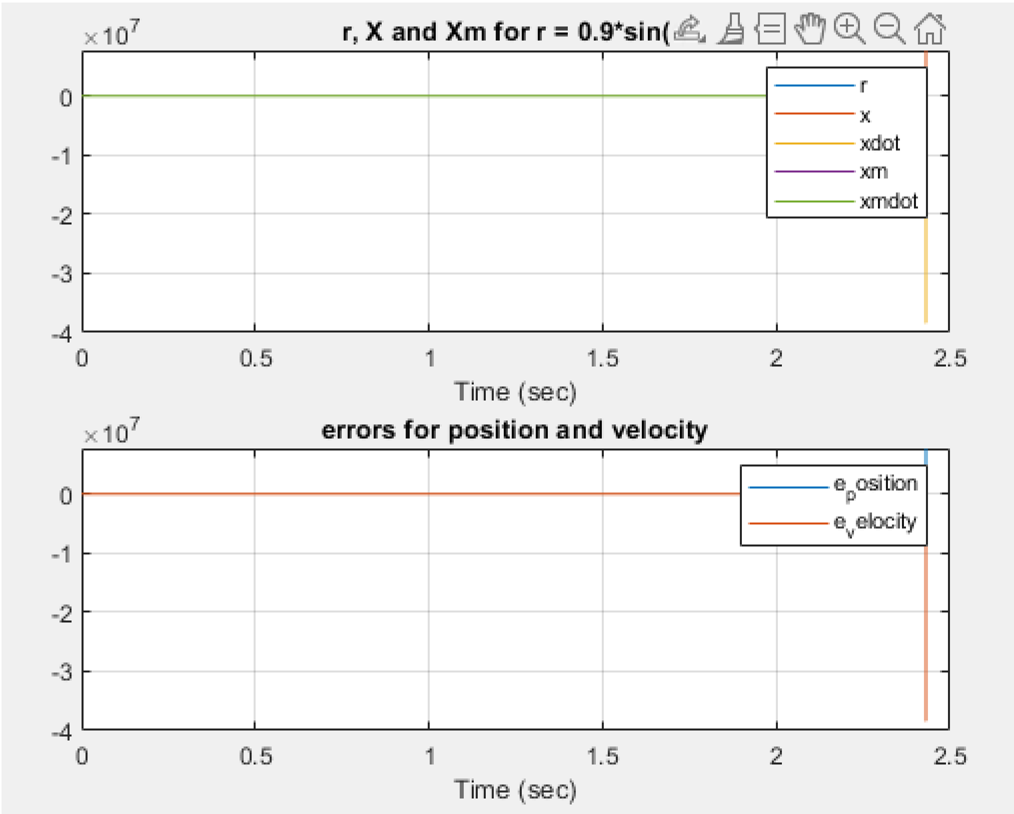
Case 2:
r(t) = Asin(t)
Subcase 1: A = 0.5



Subcase 2: A = 0.7sin(t)



Subcase 3: A = 0.9sin(t)

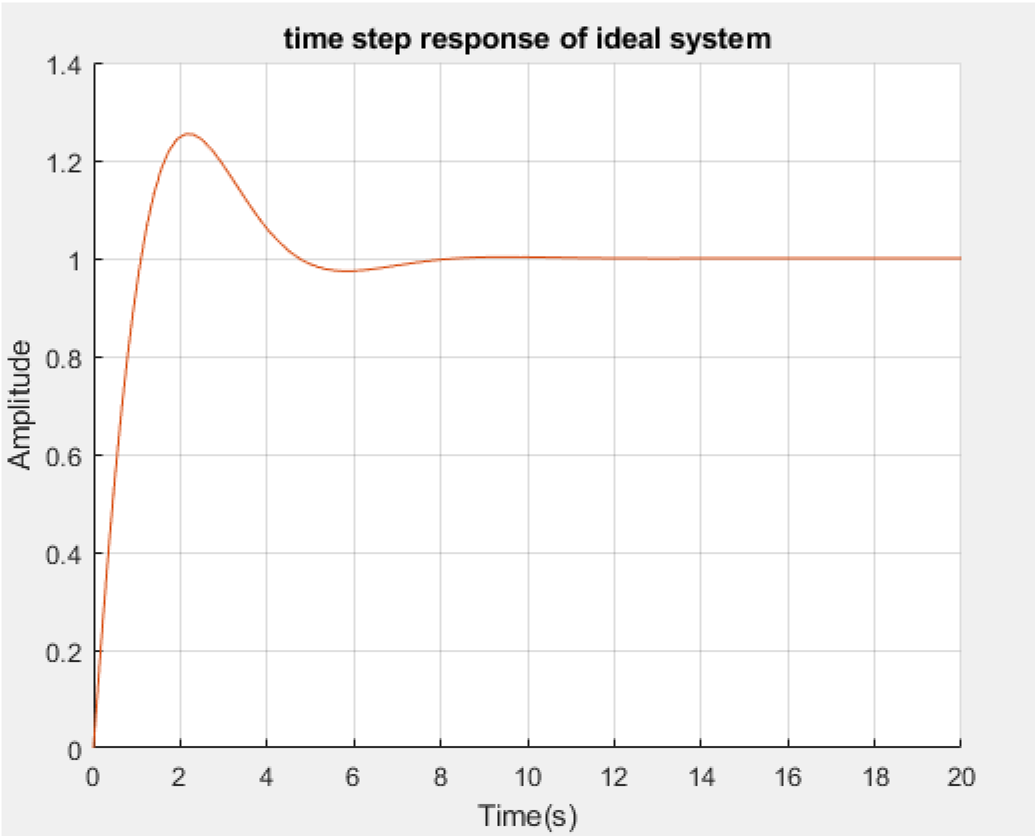


The system explodes at this value

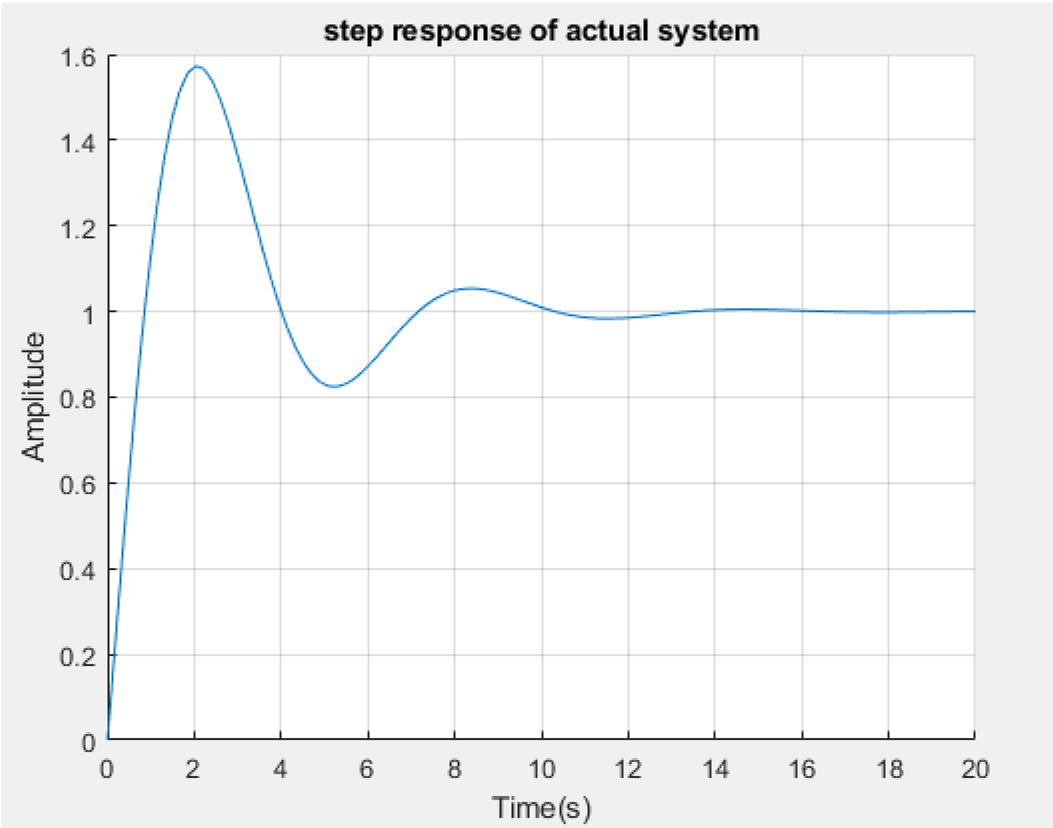
Explanation: For both r(t) = A and r(t) = Asin(t), we find the system will converge when A<1. However, when A = 1 or larger, the system explodes (output diverges).
We also saw for r(t) = A, it diverges within a shorter period of time compare to r(t) =Asin(t). The reason could be the input signals tolerance. There's a certain threshold if the tolerance is reached and makes the output of the system unstable.

From the simulation above, we also found for the system introduced with a non-linearity term, we find the value of A when the system explodes are A = 0.76 and A = 0.85, where the first is a linear system and the latter one is the nonlinear system. The time for the system to go unstable is longer than the original system introduced with the non-linearity term.

Problem 5
a)

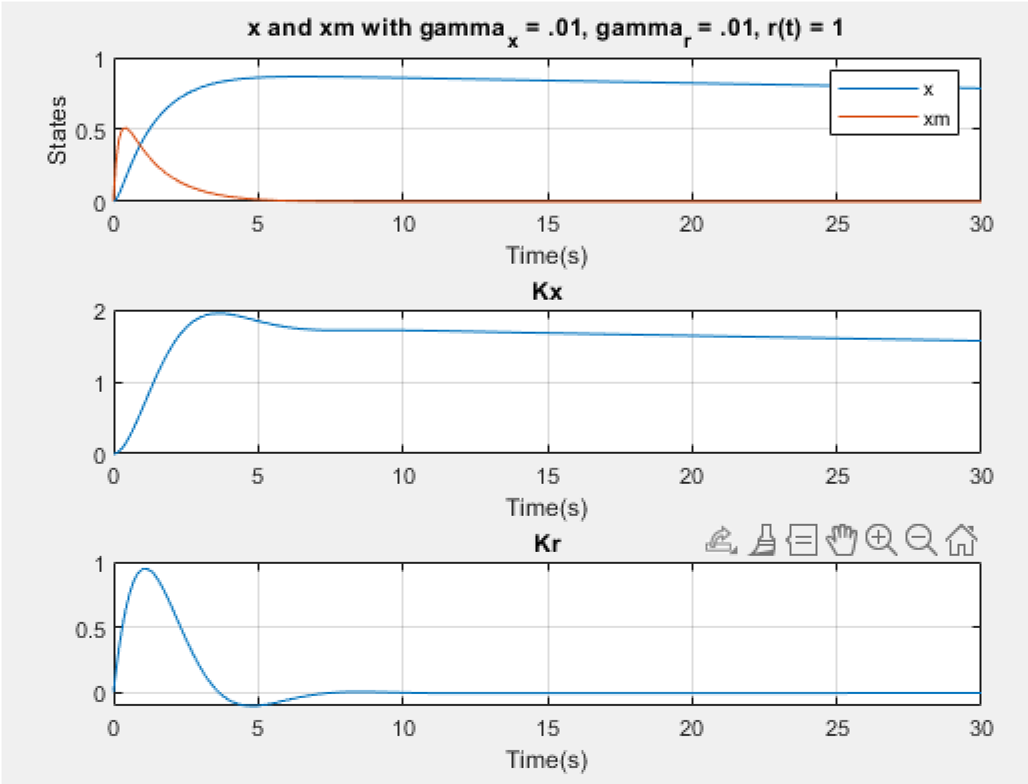

time step response of ideal system
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
2
4
6
8
10
12
14
16
18
20
Time(s)

b)

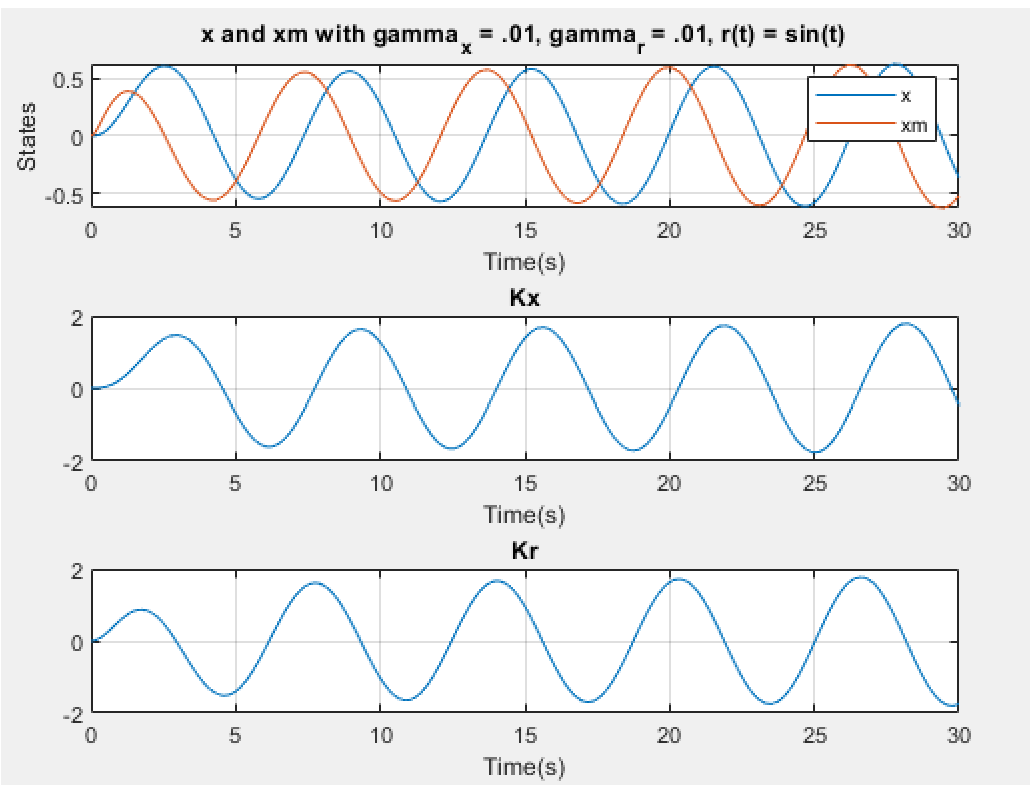


We find the overshoot for the actual system is larger than the ideal system, so does the settling time compared and the damping effect.

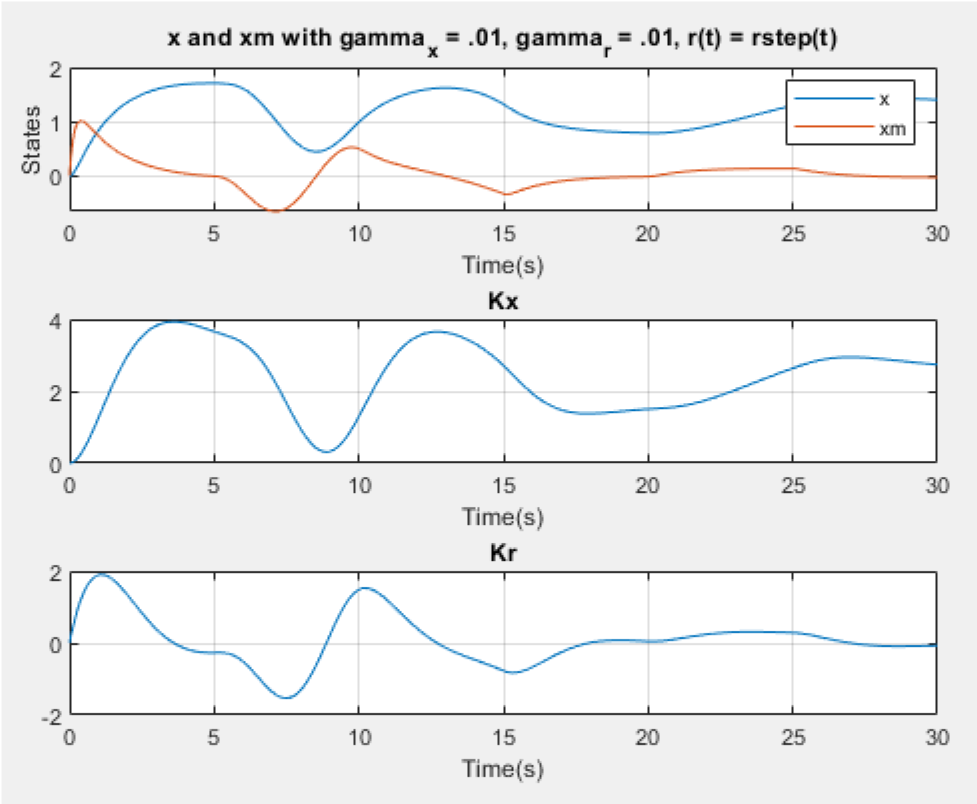
c)


x and xm with gamma$_x$ = .01, gamma$_r$ = .01, r(t) = 1
States
x
xm
Time(s)
Kx
Time(s)
Kr
Time(s)

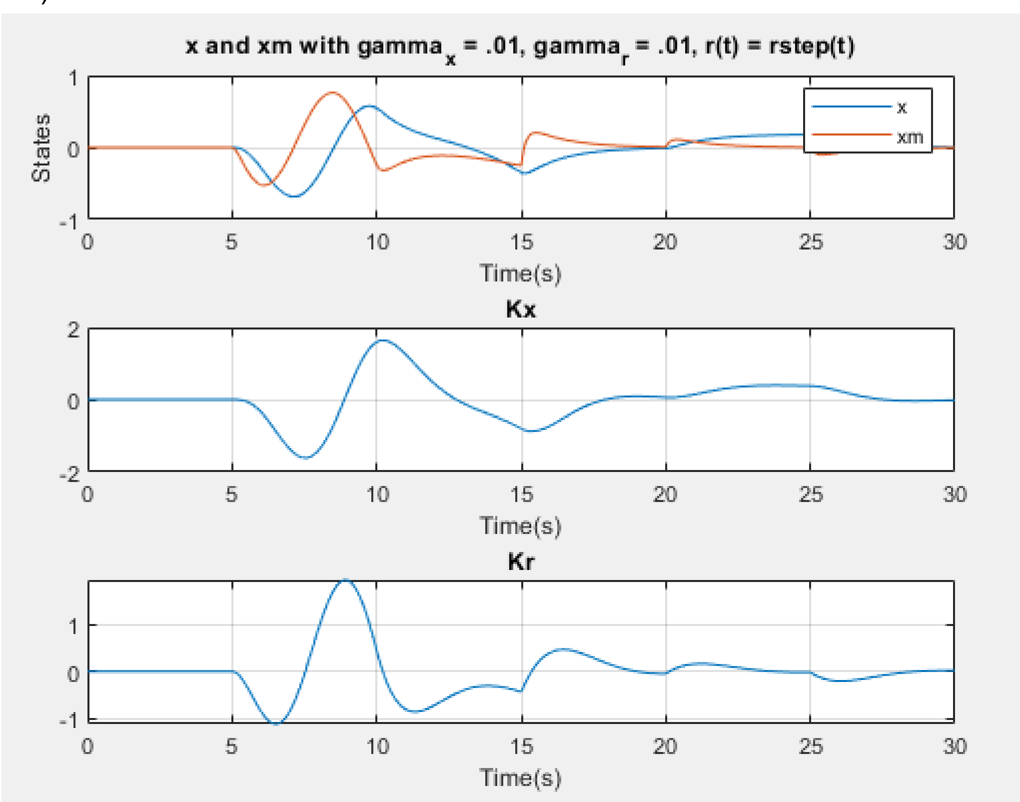
d)


x and xm with gamma$_x$ = .01, gamma$_r$ = .01, r(t) = sin(t)
States
x
xm
Time(s)
Kx
Time(s)
Kr
Time(s)

e-1)

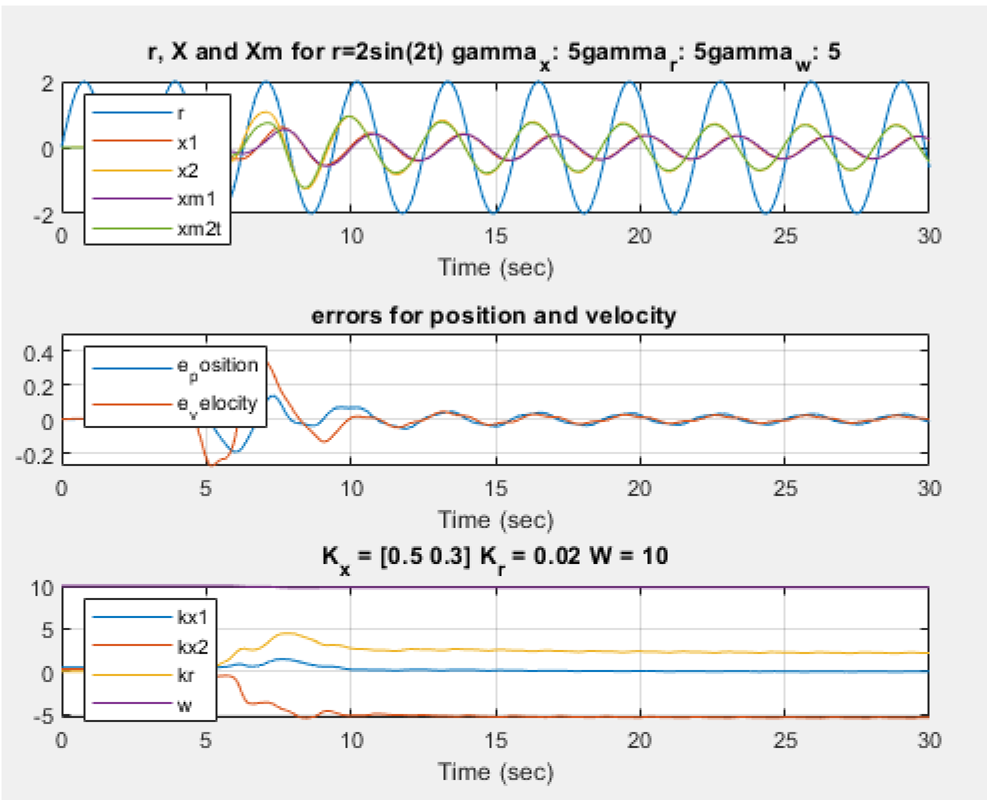

x and xm with gamma$_x$ = .01, gamma$_r$ = .01, r(t) = rstep(t)
States
x
xm
Time(s)
Kx
Time(s)
Kr
Time(s)

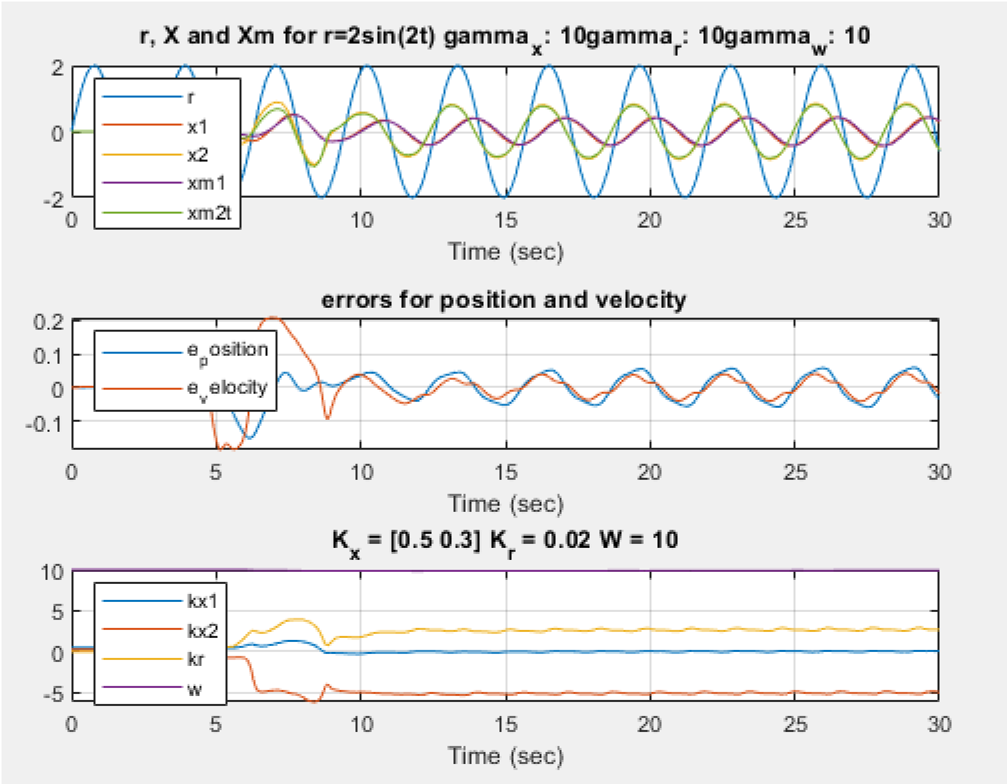
e-2)

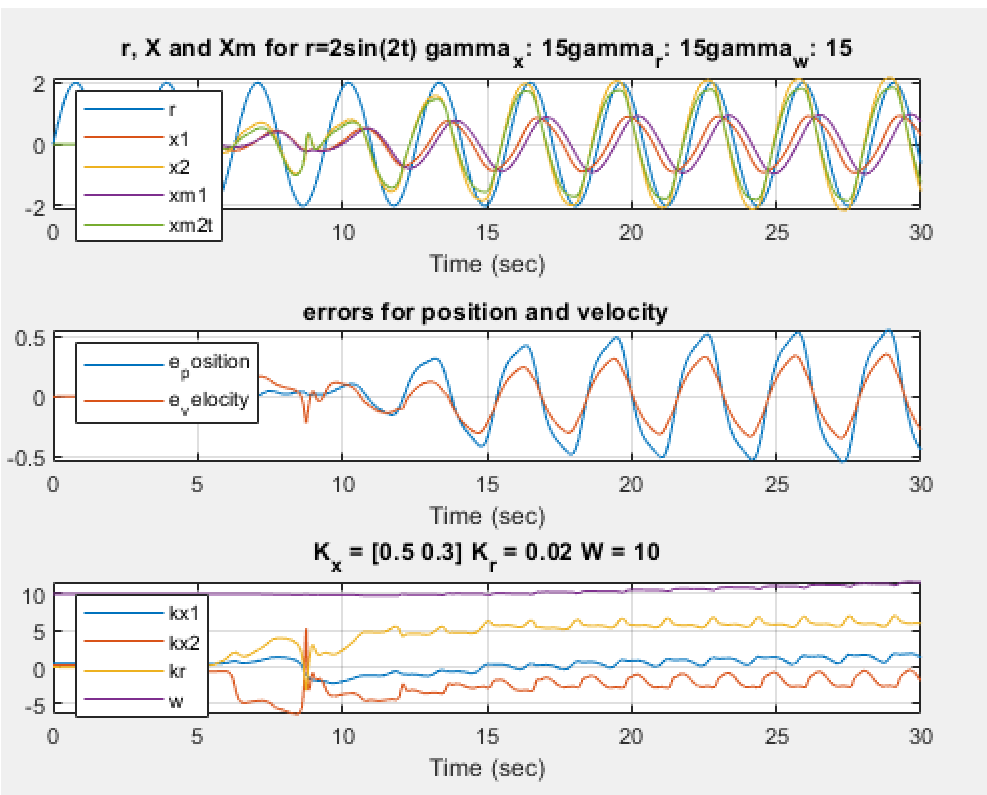

x and xm with gamma$_x$ = .01, gamma$_r$ = .01, r(t) = rstep(t)
States
x
xm
Time(s)
Kx
Time(s)
Kr
Time(s)

We find that the introduction of the adaptive controller into the system will make the unstable plant match with the ideal system's response. We saw that with the input signal differentiates, the output signal also changes accordingly. And the output of the adaptive system will reach a steady-state that the output of the plant is tracking the characteristics of our ideal closed-loop system. Kx and Kr will also reach a steady-state value so the system adaptive controller will not change much. We saw when the system does not have an adaptive controller, the output signal will have a larger overshoot and longer settling time. However, with the introduction of an adaptive controller, the signal no longer has an overshoot and the adaptive system reached a steady state faster than the closed-loop system.

Problem 6:



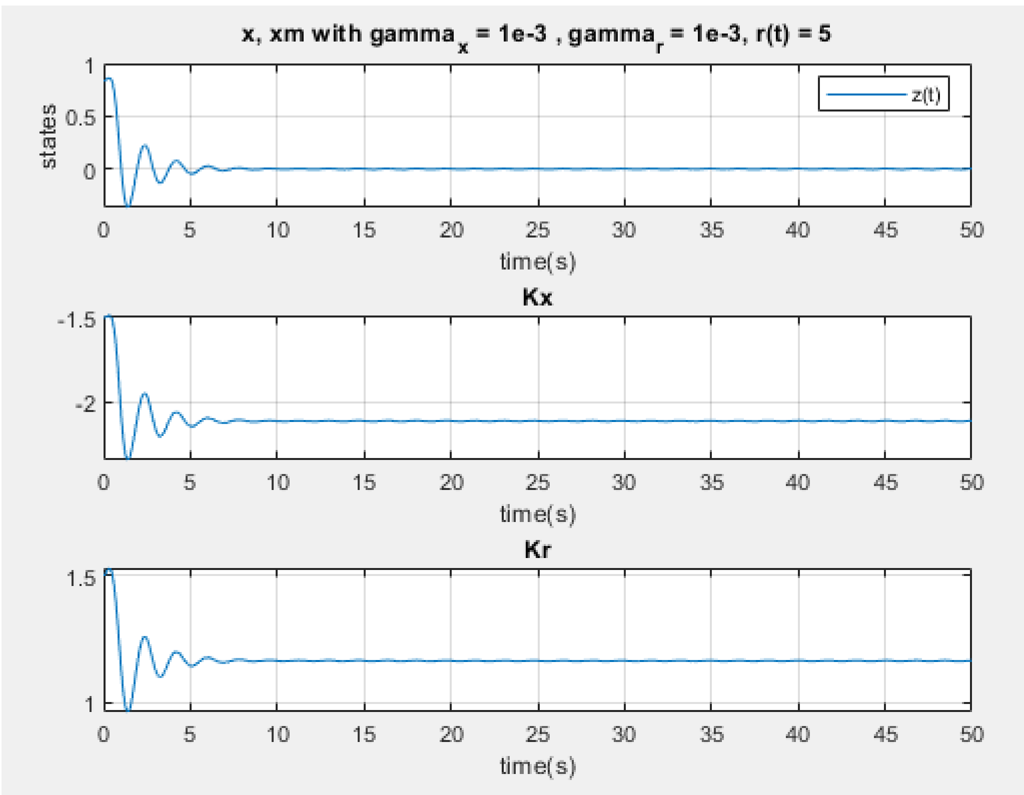




As the adaptive gains gamma_x,gamma_r,gamma_w increase from 5 to 15, we saw the output error of the system increase.

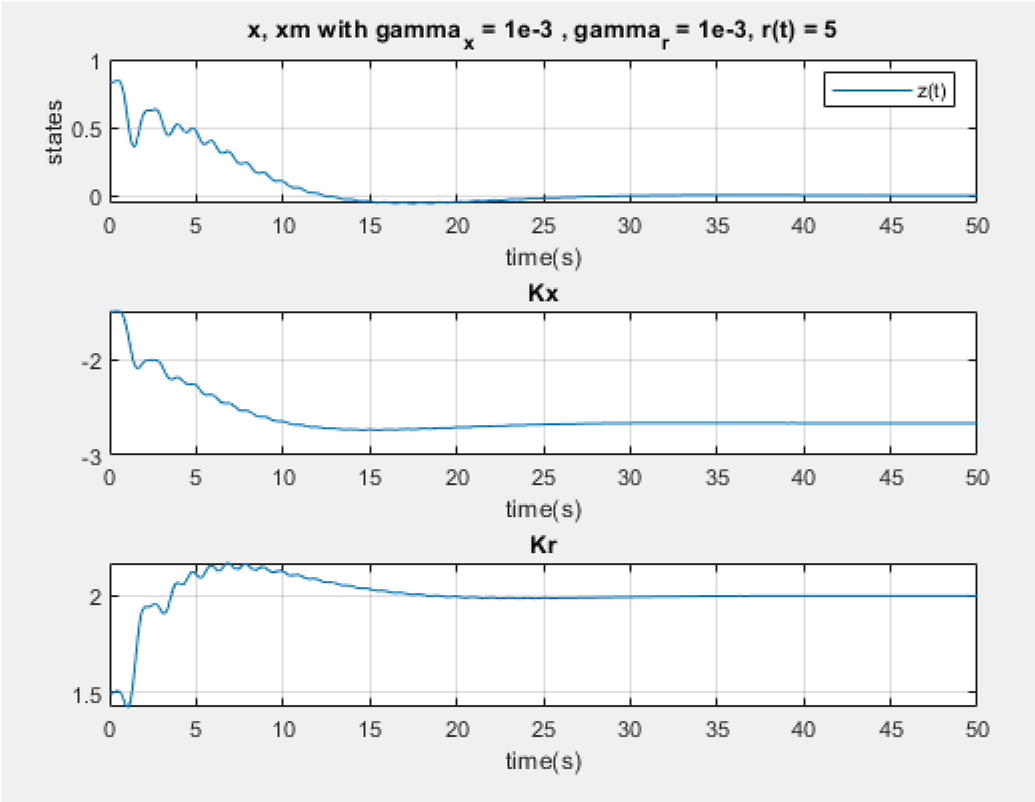
Compare the performance of direct SISO MRAC with scalar case, direct SISO MRAC has a better tolerance to different system parameters and input reference signal. This tolerance is also adapted by changing values of the adaptive controller gains, the direct SISO MRAC system is still able to get plant track the modeled output.
Therefore, we concluded that the SISO MRAC has a better performance than scalar case.

Problem 7
Scalar reference:



Convergence is seen.

Sinusoidal reference:



Convergence behavior is seen

Problem 8

Problem 8. $\nabla V(x) = \left(\frac{\partial V}{\partial x}\right)^T$

1. $\dot V(x) = \frac{\partial V}{\partial x}\dot x = \frac{\partial V}{\partial x}\left(-(\nabla V(x))^T\right) = -(\nabla V)^T \nabla V \le 0$

we also see that $\dot V(x) = 0 \Rightarrow \nabla V(x) = 0 \Rightarrow \dot x = 0$

Hence, $\dot V(x) \le 0$ and $\dot V(x) = 0$ if and only if x is an equilibrium point.

2. Solution starts with $c \geq V(x_0)$ in set Ω_c. Since $\dot V \le 0$ in Ω_c. the solution remains in Ω_c for all $t \geq 0$. Since Ω_c is compact, we say that solution is defined for $t \geq 0$.

Problem 9

1.
$$\begin{cases} 0 = y^3 - 4x & \cdots ① \\ 0 = y^3 - y - 3x & \cdots ② \end{cases}$$

①: $x = \frac{1}{4}y^3$

②: $0 = y^3 - y - 3 \cdot \frac{1}{4}y^3$

$y = \pm 2, 0$

fixed points: $(0,0)$ $(2,2)$ $(-2,-2)$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{bmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix} \quad \lambda = -1, 4 \Rightarrow \text{stable}$$

$$J_{(2,2)} = J_{(-2,-2)} = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix} \quad \lambda = -1, 8 \quad \text{saddle}$$

Therefore, $(0,0)$ is stable and $(2,2)$ $(-2,-2)$ are saddle points.

2. let $x-y=k$, $\dot{k}=\dot{x}-\dot{y}=(y^2-4x)-(y^2-y-3x)=-x+y=-(x-y)=-S$

$S=e^{-t}(x_0-y_0)$

if $x_0=y_0$. $S\Rightarrow 0$. $(\forall t\in\mathbb{R})$ where $k=x-y$.

Hence, $x(t)=y(t)$ for $\forall t\geq 0$. for $x_0=y_0$. $x=y$ is invariant

3. $S(t)=e^{-t}(x_0-y_0)$ is known.

$\lim_{t\to\infty}|S(t)|=\lim_{t\to\infty}|x(t)-y(t)|=\lim_{t\to\infty}|e^{-t}(x_0-y_0)|=0$

Hence, $\lim_{t\to\infty}|x(t)-y(t)|=0$