Problem 1

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases} \Rightarrow \begin{cases} 0 = x_2 \\ 0 = -x_1^3, x_1 = 0 \end{cases}$$

equilibrium point $(0,0)$

$$V(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

$$\dot{V}(x_1, x_2) = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$$
$$= x_1^3 x_2 + (-x_2 x_1^3 - x_2^4)$$
$$= -x_2^4 \le 0$$

So $\dot{V}(x_1, x_2) = 0$ when $(x_1, x_2) = (0,0)$, which is asymptotic stable. More over, $\|x\| \to \infty$ lead to $V(x_1, x_2) \to \infty$. Hence, system is globally asymptotically stable

Problem 2

1. $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^3 - \delta x_2 \end{cases} \Rightarrow \begin{cases} 0 = x_2 \\ 0 = x_1 - x_1^3 \Rightarrow x_1 = 0, \pm 1 \end{cases}$$

$$J = \begin{bmatrix} 0 & 1 \\ 1-3x_1^3 & -\delta \end{bmatrix} (\delta > 0) \qquad J = \begin{bmatrix} 0 & 1 \\ 1-3x_1^3 & 0 \end{bmatrix} (\delta = 0)$$

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \qquad J_{(\pm 1, 0)} = \begin{bmatrix} 0 & 1 \\ -2 & -\delta \end{bmatrix}$$

$$\Rightarrow \lambda^2 + \delta\lambda - 1 = 0 \qquad \lambda^2 + \delta\lambda + 2 = 0$$

$$\lambda = \frac{-\delta \pm \sqrt{\delta^2 + 4}}{2} \qquad \lambda = \frac{-\delta \pm \sqrt{\delta^2 - 8}}{2}$$

$\lambda_1 < 0 < \lambda_2$

Hence, saddle $\Rightarrow$ unstable.

$$\begin{cases} \delta > 2\sqrt{2}, \ \lambda_1 < \lambda_2 < 0. \ (\text{stable}) \\ 0 < \delta < 2\sqrt{2}, \ \text{Re}\{\lambda_1, \lambda_2\} \ (\text{stable focus}) \end{cases}$$

$\delta = 0$

$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow$ saddle $\Rightarrow$ unstable

$J_{(\pm1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \Rightarrow \lambda^2 + 2 = 0 \quad \lambda = \pm\sqrt{2}i \Rightarrow$ can not be determined

2. $V(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

$\dot{V}(x_1, x_2) = -x_1\dot{x}_1 + x_1^3\dot{x}_1 + x_2\dot{x}_2$
$= -\delta x_2^2 \leq 0$

Hence, $V(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

3. Find set of all points for $x_2 = 0$.

$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = x_1 - x_1^3 = 0 \end{cases}$, where $x_2 = 0$

Find region of attraction. $(\delta > 0) \Rightarrow$ Restrict x_1 by $x_1 = \pm1, 0$. So $|x_1| > 1$.

$V(x)|_{(\pm1,0)} = \frac{1}{4}$. Hence, region of attraction $V(x) > \frac{1}{4}$

Problem 3

$\dot{x}_1 = -x_1 + x_2 + (x_1^2 + x_2^2)\sin(t)$
$\dot{x}_2 = -x_1 - x_2 + (x_1^2 + x_2^2)\cos(t)$

Set $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(x_1, x_2) = \dot{x}_1 x_1 + \dot{x}_2 x_2$

$= -x_1^2 + \cancel{x_1 x_2} + x_1(x_1^2 + x_2^2)\sin t - x_2^2 - \cancel{x_1 x_2} + x_2(x_1^2 + x_2^2)\cos(t)$
$= -(x_1^2 + x_2^2) + \underline{(x_1^2 + x_2^2)(x_1\sin(t) + x_2\cos(t))}$
$\leq -\|x\|^2 + \|x\|^3\sqrt{\sin^2(t) + \cos^2(t)} = -\|x\|^2 + \|x\|^3$

let $\|x\| \leq r$. for $r < 1$

$\dot{V}(x_1, x_2) = -(1-r)\|x\|^2$

Thus, $(0,0)$ is exponentially stable, with region of attraction $\|x\| \leq r$ for $r<1$.

Problem 4.

1. $V(x_1,x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

$\dot{V}(x_1,x_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(h(t)x_2 - g(t)x_1^5) + x_2(-h(t)x_1 - g(t)x_2^3)$

$= -g(t)x_1^6 - g(t)x_2^4 \leq -k(x_1^6 + x_2^4)$

Knowing $\dot{V}=0$ when $(0,0)$, $\dot{V}<0$ for $(x_1,x_2) \neq (0,0)$

Hence, $x=0$ is uniformly asymptotically stable

2. $J_{(0,0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$ linear system at $(0,0)$

Hence, $\begin{cases} \dot{x}_1 = h(t)x_2 \\ \dot{x}_2 = -h(t)x_1 \end{cases}$

$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2h(t) - x_1x_2h(t) = 0$

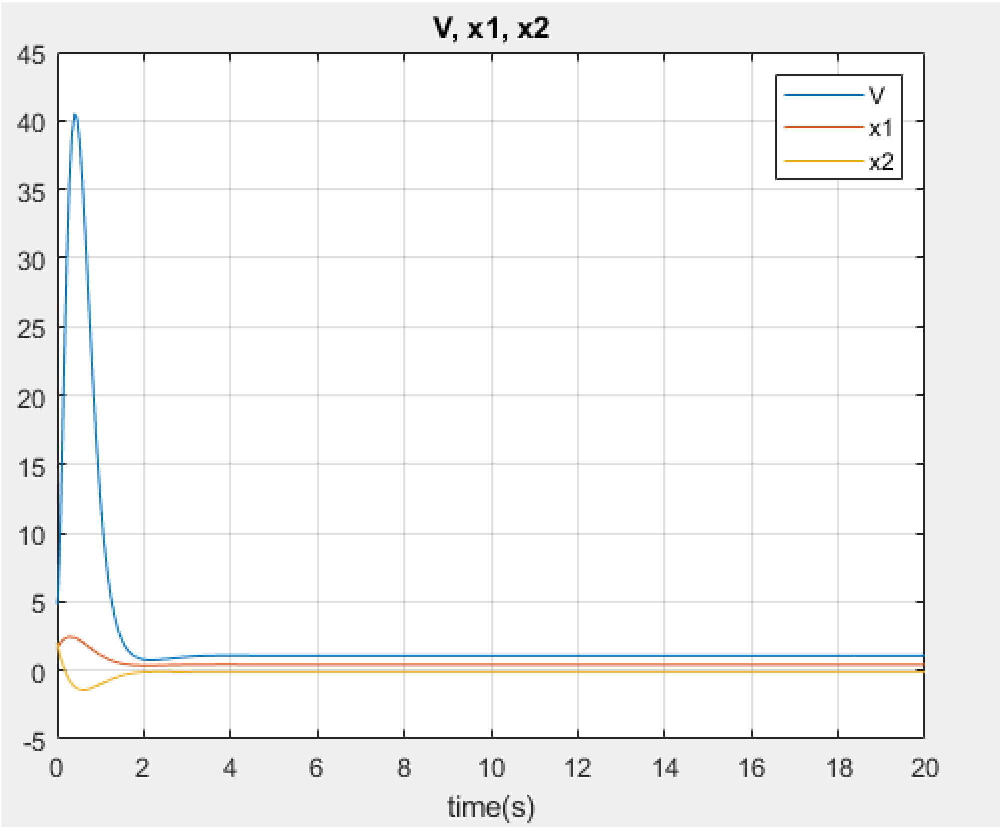
Hence, $(0,0)$ of linear system isn't exponentially stable, which implies to nonlinear syst

3. $V(x) = \frac{1}{2}(x_1^2+x_2^2) > 0$ and $\dot{V}(x) \leq -k(x_1^4+x_2^4) < 0$ and $\dot{V}(x)=0$ only when $(x_1,x_2)=(0,0)$

Hence, $(0,0)$ is globally uniformly asymptotically stable

4. System is not globally exponentially stable since it's not exponentially stable

Problem 5:



Control feedback gain:
3.0439
6.9763

Function defined:

```
function sys_states_dot = f(t, sys_states)
disp(sys_states);
r = 1;

A = [1, 3; -1, 2];
B = [0; 1];
Q = [4.5, 2.8; 2.8, 2.5];


[X, L, G] = care(A, B, Q);

Am = A - B*G;

temp = Am*sys_states;
disp(temp)
```

```
x1_dot = temp(1,:);
x2_dot = temp(2,:)+r;

sys_states_dot = [x1_dot; x2_dot];
end
```

Main function:

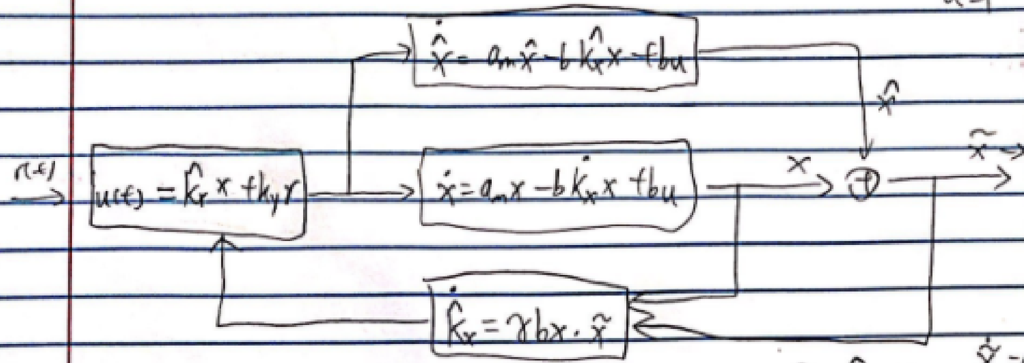
```
A = [1, 3; -1, 2];
B = [0; 1];
Q = [4.5, 2.8; 2.8, 2.5];

x0 = [1.5, 2];

[X, L, G] = care(A, B, Q);

Am = A - B*G;

P = lyap(Am, Q);

tspan = [0,20];
[t,states] = ode45(@q5_func, tspan, x0);
x = states;
x_trans = x.';

V = zeros(size(x,1),1);
Vdot = zeros(size(x,1),1);

for i = 1:(size(V,1))
   V(i) = x(i,:)*P*x_trans(:,i);
end

figure(1);
plot(t, V, t, x);
xlabel('time(s)');
grid on;
title('V, x1, x2');
legend('V', 'x1', 'x2');
```

Problem 6:

$bk_y = b_m$

$a_m = -2 \quad b = 2 \quad k_x = \frac{-5}{2}$

$a = 1 \quad a_p = 3$



$\tilde{x} = \hat{x} - x$

$\dot{\tilde{x}} = a_m \tilde{x} - b \cdot k_x x$

$\begin{cases} \tilde{x} = \hat{x} - x \\ \Delta k_x = \hat{k}_x - k_x^* \end{cases}$

passive identifier: $\dot{\hat{x}} = \underbrace{a_m \hat{x} - b\hat{k}_x x}_{} + bu \qquad \hat{x}(0) = 0$

$= -2\hat{x} - 2\hat{k}_x x + 2u$

$= -2\hat{x} + 2k_y r$

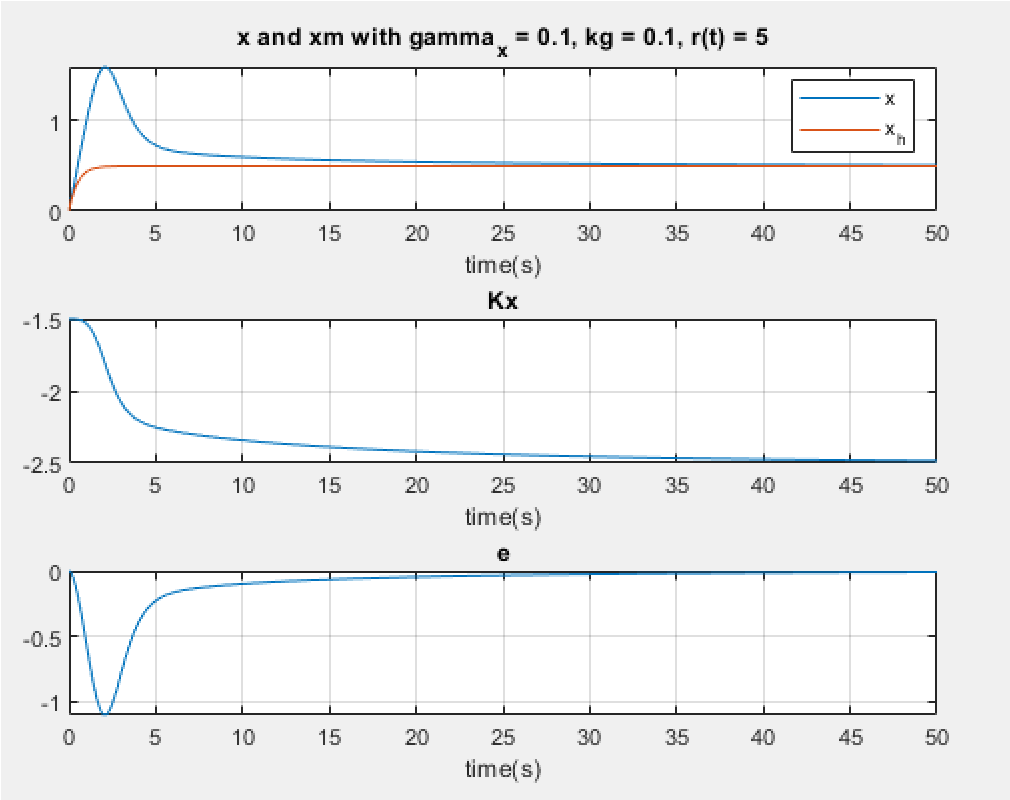
plant: $\dot{x} = (a_m - b\dot{k}_x)x + bu = (a_m - b\dot{k}_x + b\hat{k}_x)x + bk_y \cdot r = 3x + 2u$

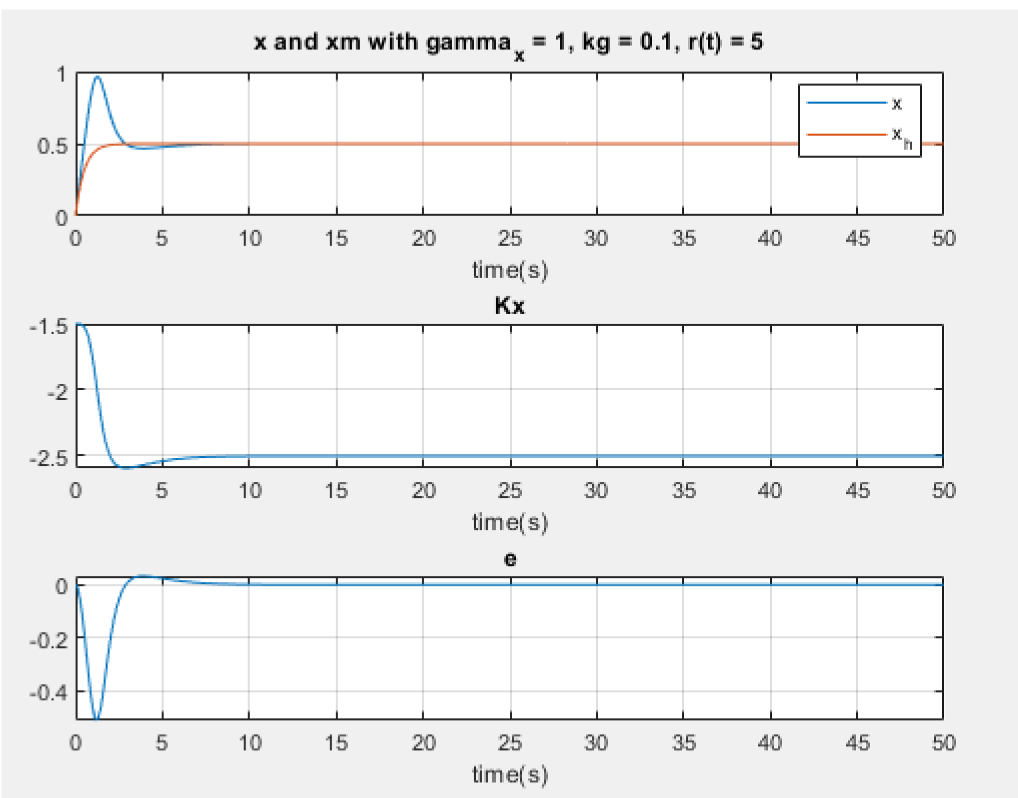
$= (3 + 2\hat{k}_x)x + 2k_y r$

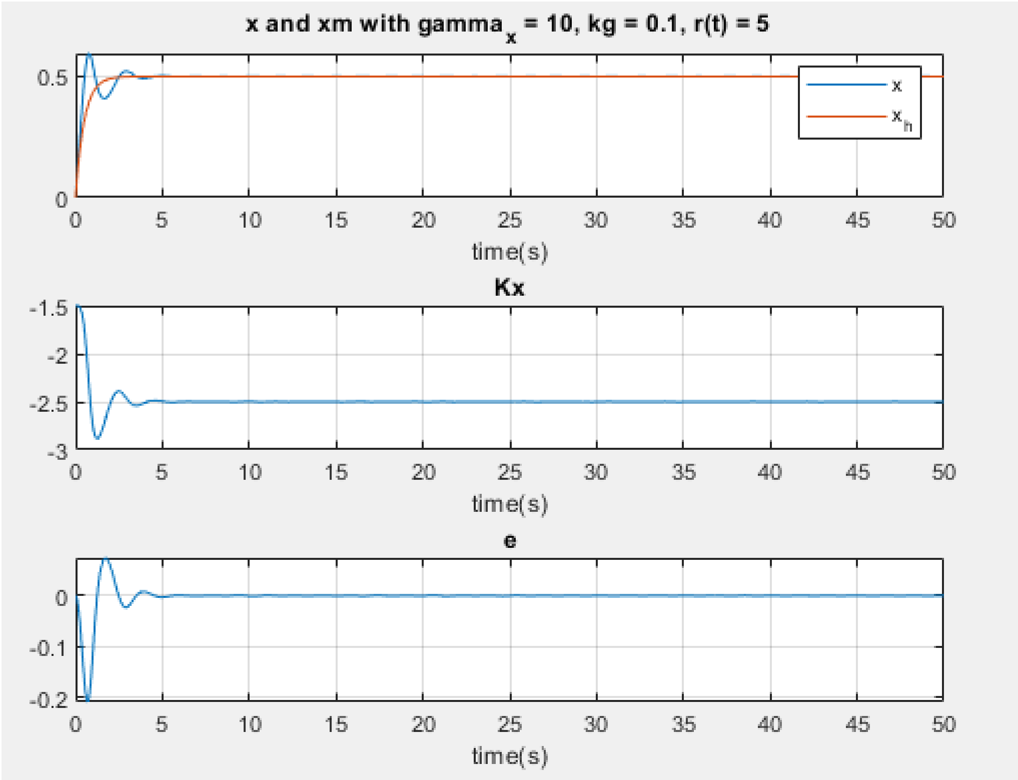
$x(0) = 0$

estimate plant:

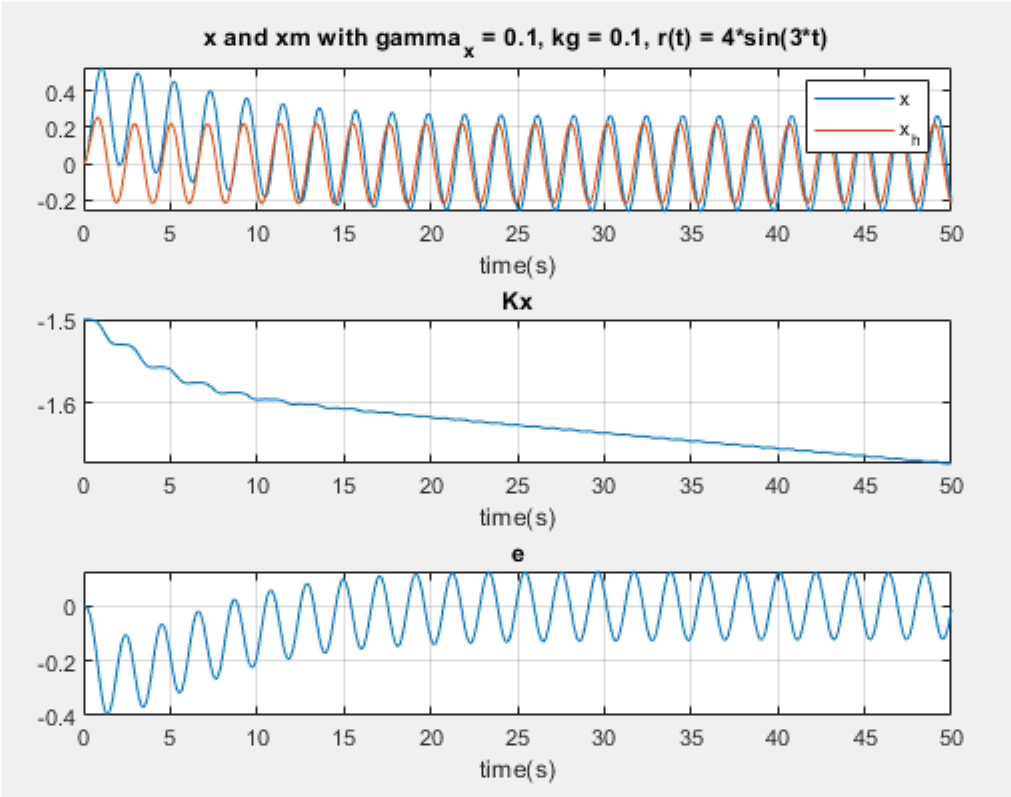
$\dot{x} = ax + bu = (a + b\hat{k}_x)x + bk_y r = 1x + 2u = (1 + 2\hat{k}_x)x + 2k_y r$

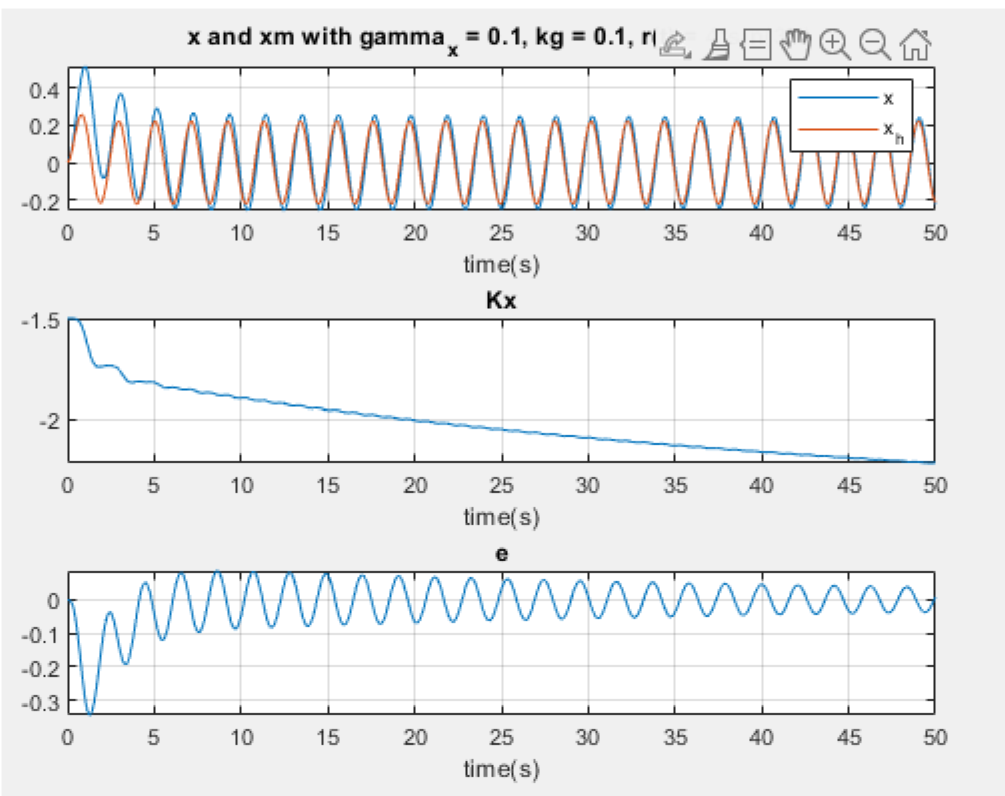

x and xm with gamma_x = 0.1, kg = 0.1, r(t) = 5
x
x_h
time(s)
Kx
time(s)
e
time(s)

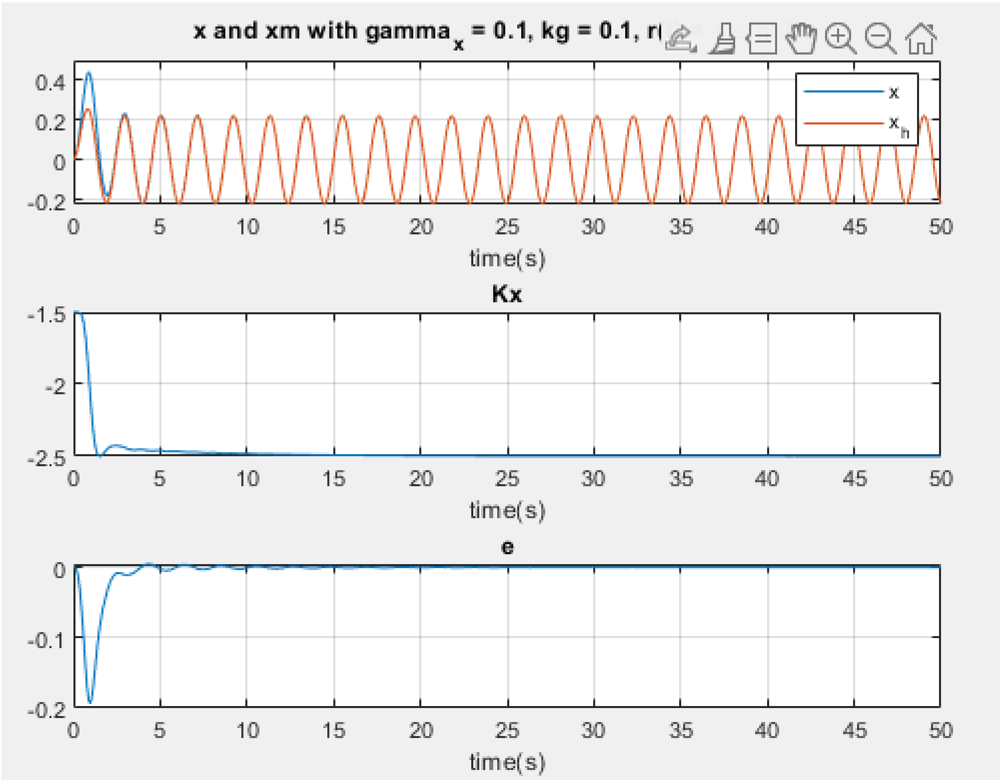

x and xm with gamma_x = 1, kg = 0.1, r(t) = 5
x
x_h
time(s)
Kx
time(s)
e
time(s)



When the reference signal is r(t) = 5, as the adaptation gain, gamma x increases, we find the plant model will better track the model system. Comparing the plant system x's behavior using different gamma x values, we find the initial transient behavior of the plant system will have a longer overshoot and settling time. When gamma x increases, the plant system's overshoot is decreasing, and settling time is shortened. The error plot will also converge to 0 with a faster rate when gamma x is large enough.


x and xm with gamma$_x$ = 0.1, kg = 0.1, r(t) = 4*sin(3*t)
x
x_h
time(s)
Kx
time(s)
e
time(s)


x and xm with gamma$_x$ = 0.1, kg = 0.1, r(
x
x_h
time(s)
Kx
time(s)
e
time(s)



When the reference signal is r(t) = 4sin(3t), as the adaptation gain, gamma x increases, we find the plant model will better track the model system. Comparing the plant system x's behavior using different gamma x values, we find the initial transient behavior of the plant system will have a longer overshoot and settling time. When gamma x increases, the plant system's overshoot is decreasing, and settling time is shortened. The error plot will also converge to 0 with a faster rate when gamma x is large enough.

Function defined:

```
function sys_dot = f(t, sys_states)
% Passive Identifier
am = -2; b = 2;
% Plant
ap = 3;
kx_star = (ap - am)/(-b);
% Estimate Plant
a = 1;

% Adaptive control gains
gamma_x = 10;
kg = 0.1;
% Reference i/p:
```

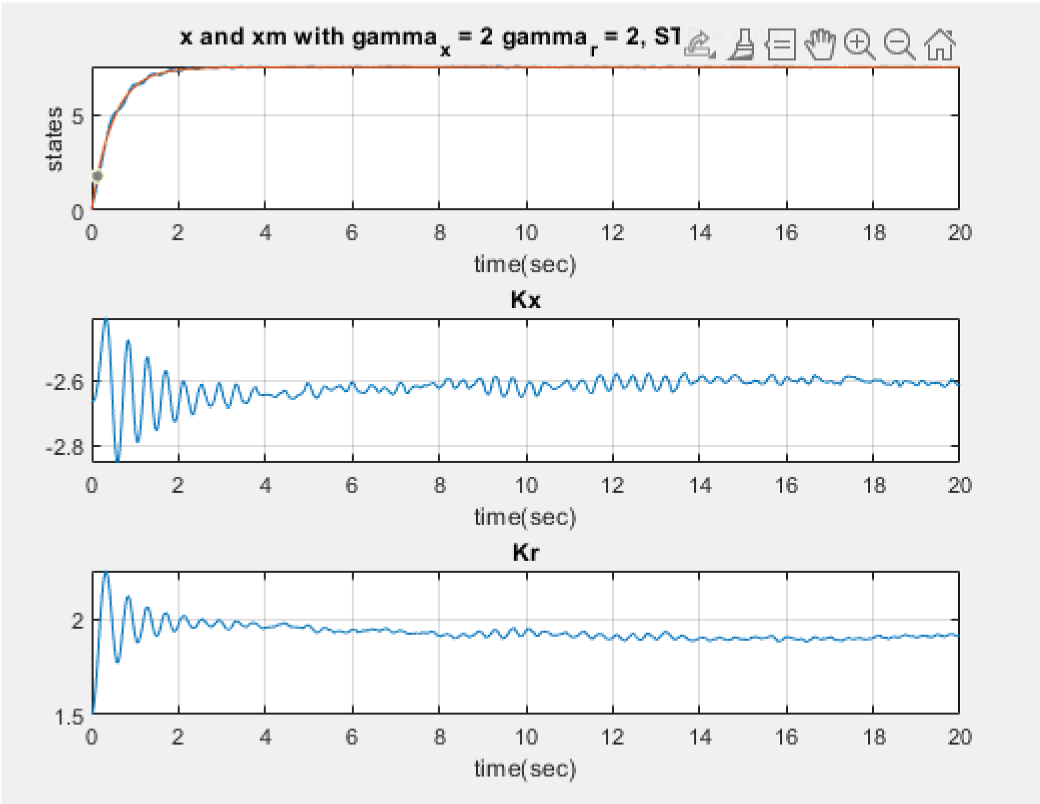
```
% r = 5;
r = 4*sin(3*t);


% Taking out the states accordingly
x = sys_states(1);
x_h = sys_states(2);
kx_h = sys_states(3);

e = x_h - x;

% Updating the state variables
x_dot = (am-b*kx_star + b*kx_h)*x + b*kg*r;
x_h_dot = am*x_h + b*kg*r;
kx_h_dot = gamma_x*b*x*e;

% Putting the states together and return
sys_dot = [x_dot; x_h_dot; kx_h_dot];

end
```

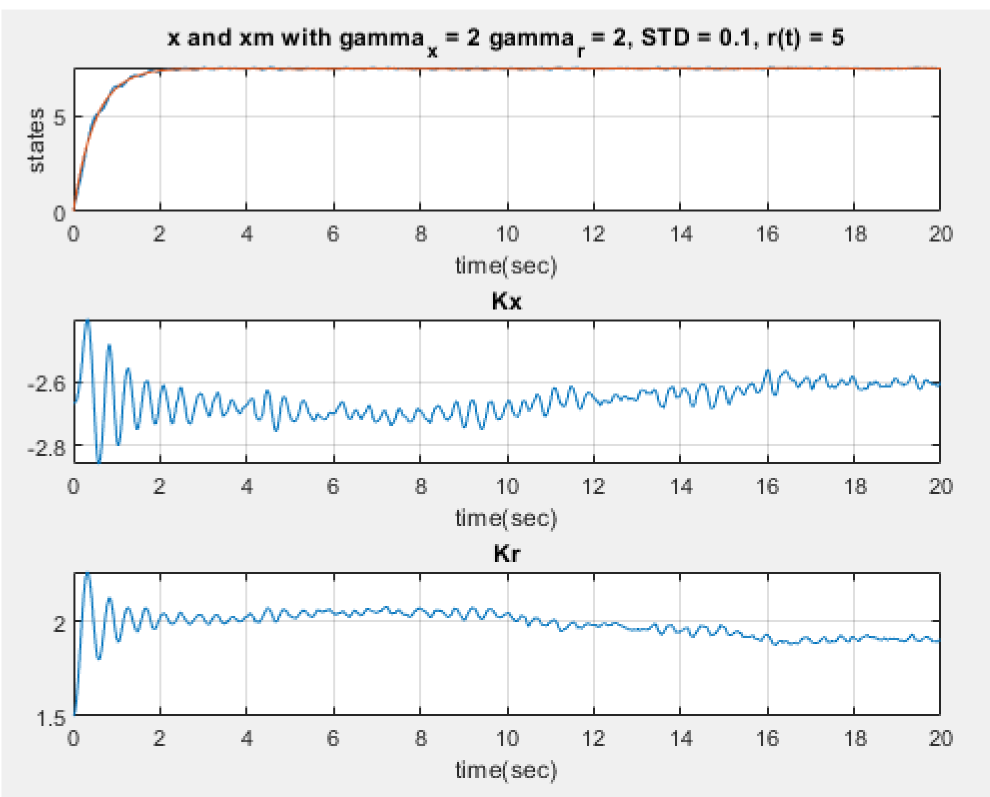
Main function:

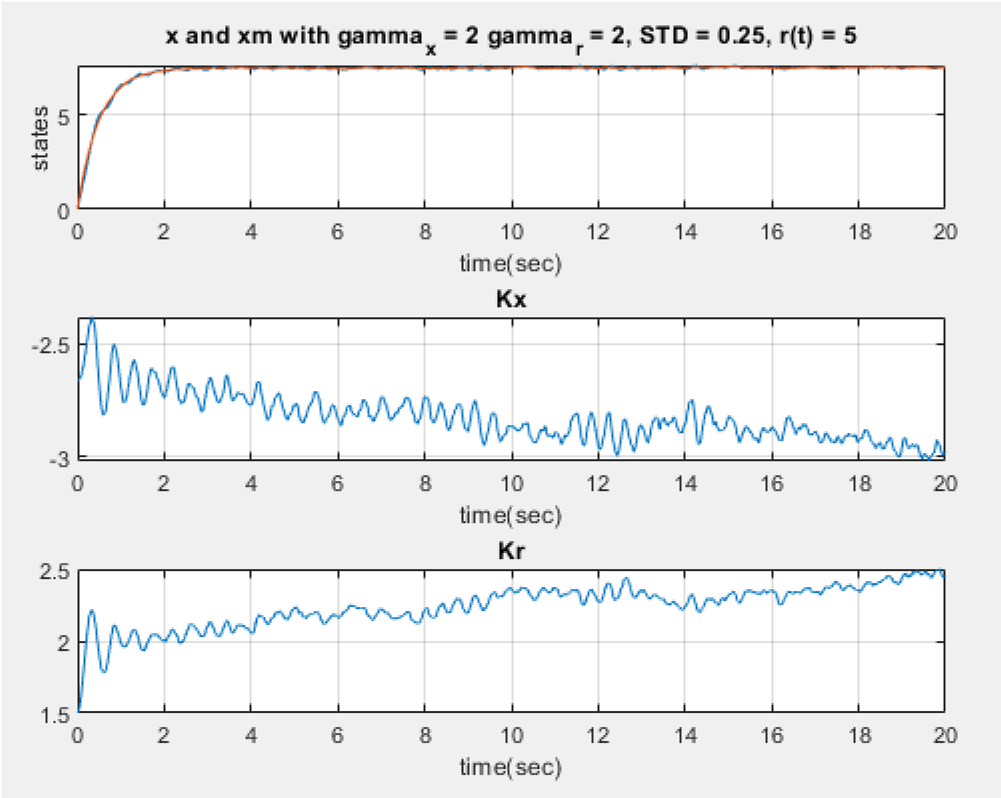
```
% Passive Identifier
am = -2; b = 2;
% Plant
ap = 3;
% Estimate Plant
a = 1;

% Initialization
xh_0 = 0;
x_0 = 0;
kx_0 = (am-a)/b;

sys_states_0 = [x_0, xh_0, kx_0];

% Time
tmax = 50;
tspan = [0, tmax];



% Simulation
[t, sys_states] = ode45(@q6_func, tspan, sys_states_0);
```

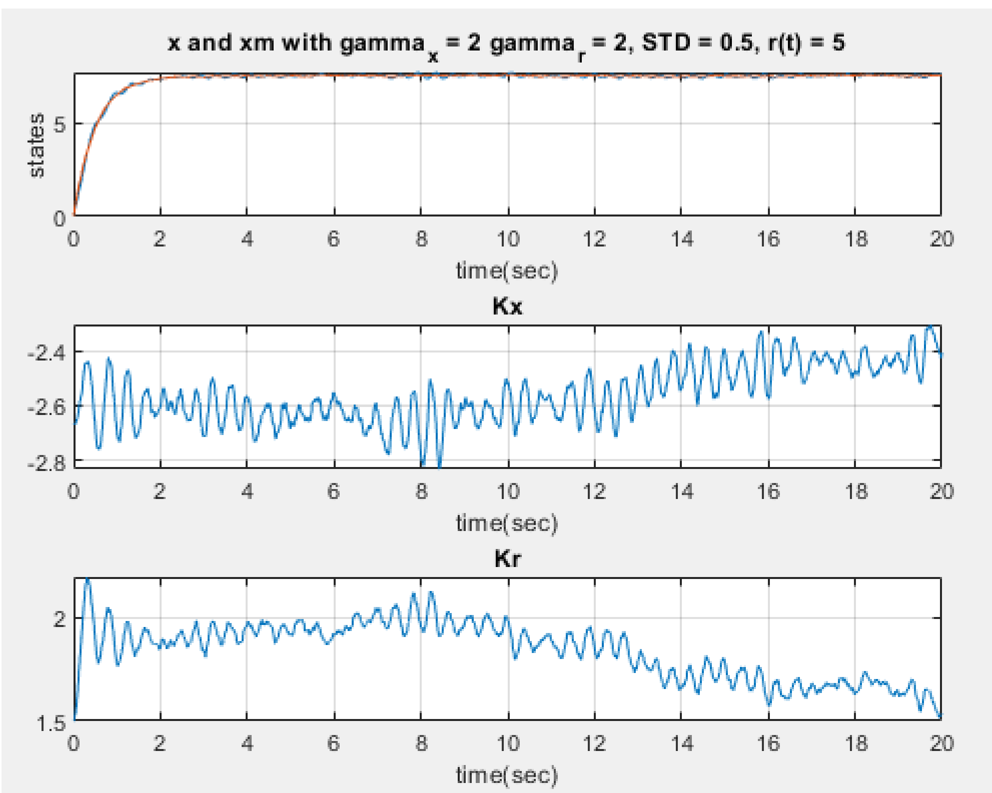
```
% Taking out the sys_states
x = sys_states(:,1);
x_h = sys_states(:,2);
kx_h = sys_states(:,3);

e = x_h-x;

figure(1);
    subplot(3,1,1);
    plot(t, x, t, x_h);
    xlabel('time(s)');

    title('x and xm with gamma_x = 0.1, kg = 0.1, r(t) = 4*sin(3*t)')

    legend('x','x_h');
    grid on;

    subplot(3,1,2);
    plot(t, kx_h);
    xlabel('time(s)');
    title('Kx')
    grid on;

    subplot(3,1,3);
    plot(t, e);
    xlabel('time(s)');
    title('e')
    grid on;
```

Problem 7

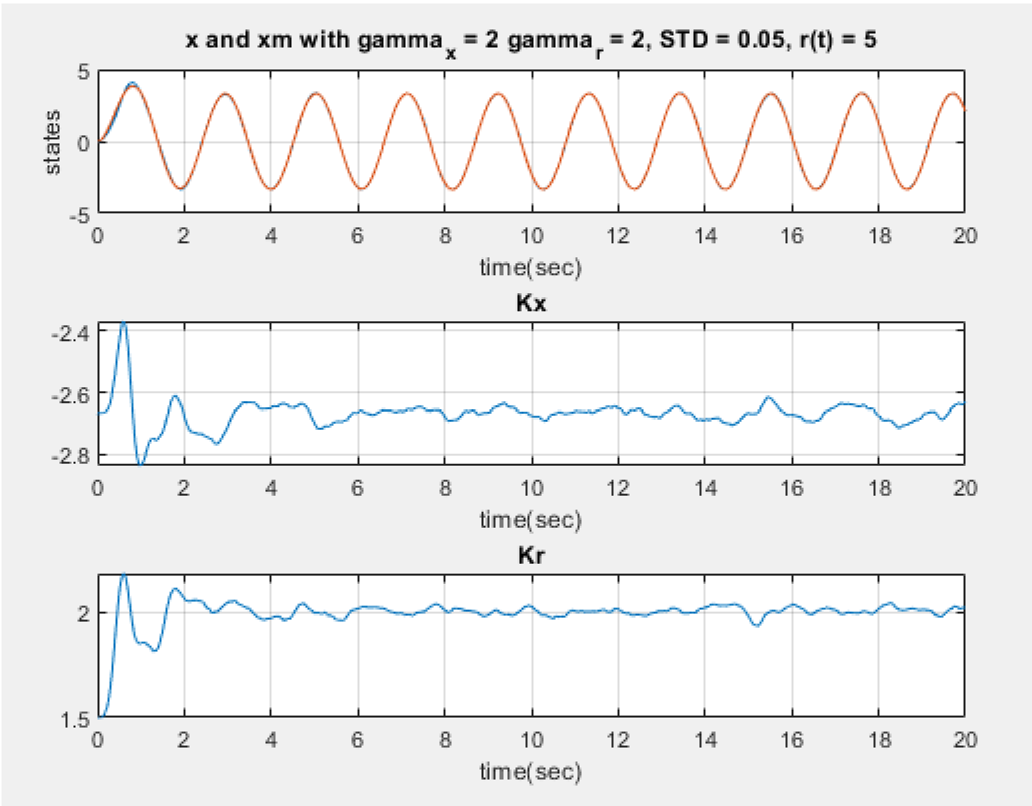

x and xm with gamma$_x$ = 2 gamma$_r$ = 2, ST
states
time(sec)
Kx
time(sec)
Kr
time(sec)

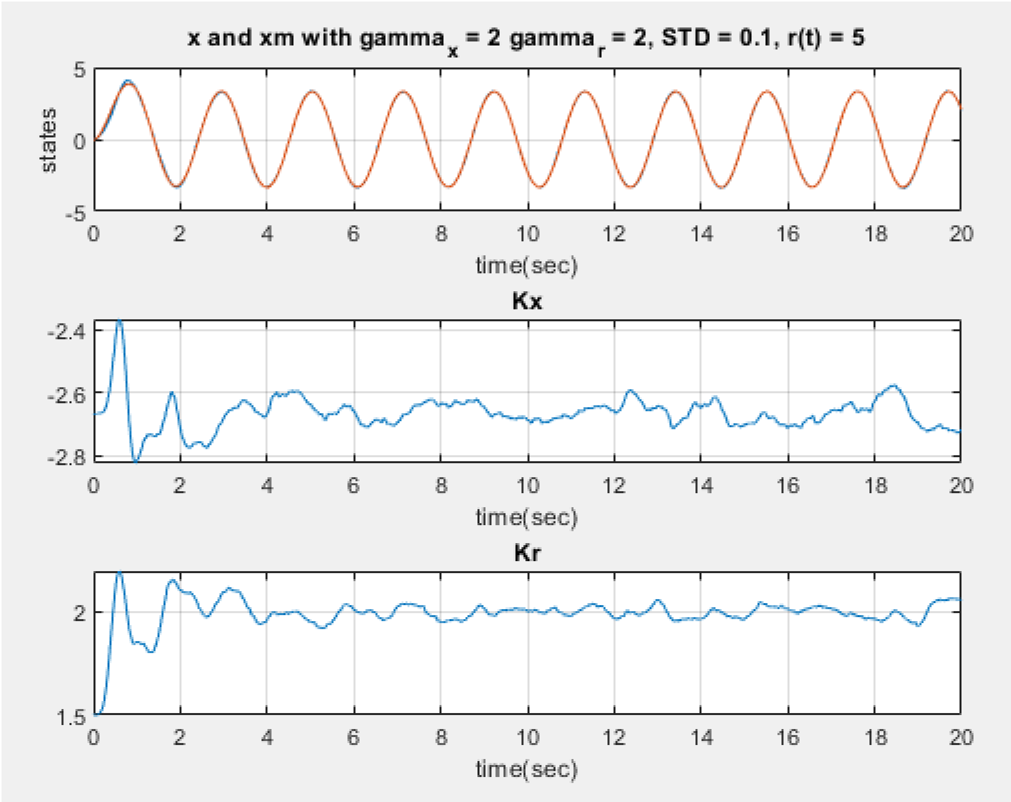

x and xm with gamma$_x$ = 2 gamma$_r$ = 2, STD = 0.1, r(t) = 5
states
time(sec)
Kx
time(sec)
Kr
time(sec)

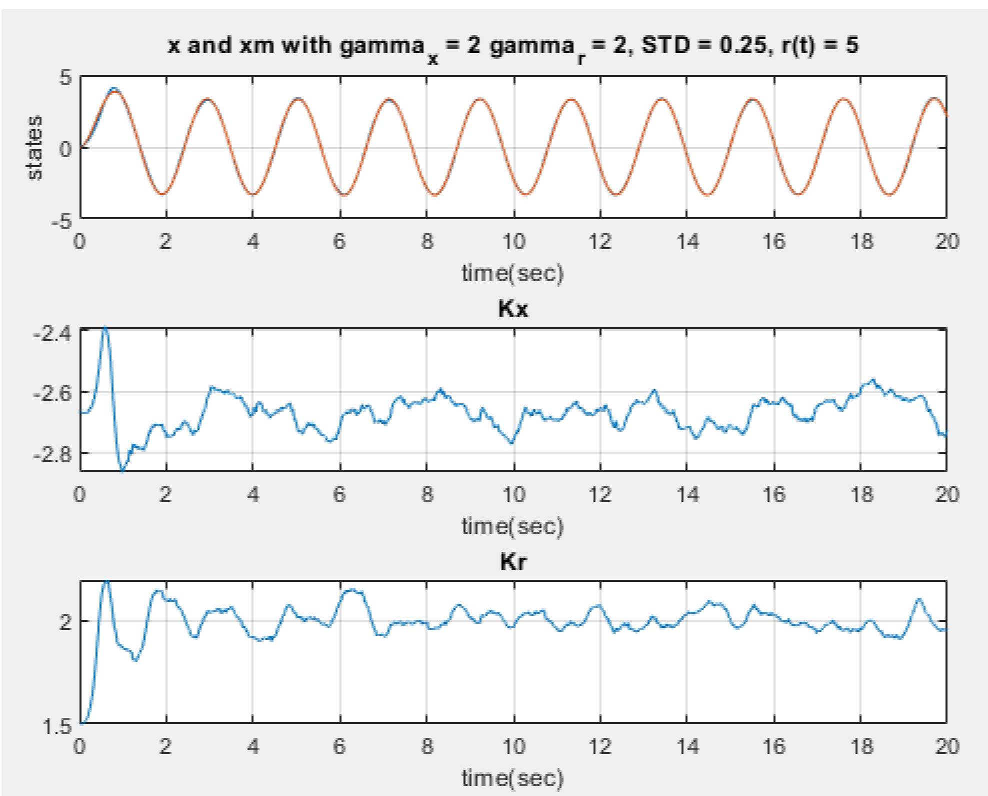

x and xm with gamma_x = 2 gamma_r = 2, STD = 0.25, r(t) = 5
states
time(sec)
Kx
time(sec)
Kr
time(sec)

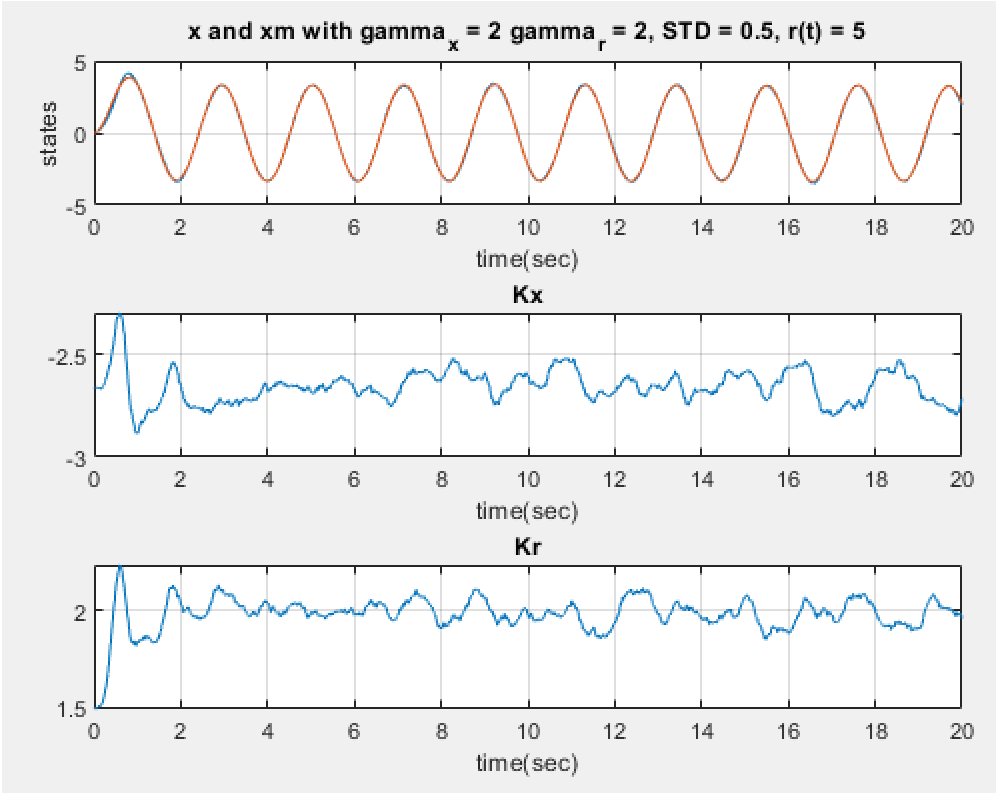

x and xm with gamma_x = 2 gamma_r = 2, STD = 0.5, r(t) = 5
states
time(sec)
Kx
time(sec)
Kr
time(sec)

For reference signal of r(t) = 5, as the value of the additive Gaussian noise standard deviation increases, the plant system will have a worse tracking performance. When the noise standard deviation increases, the system's adaptive control gains will also increase rather than converge to a value.




x and xm with gamma_x = 2 gamma_r = 2, STD = 0.1, r(t) = 5
states
time(sec)
Kx
time(sec)
Kr
time(sec)


x and xm with gamma_x = 2 gamma_r = 2, STD = 0.25, r(t) = 5
states
time(sec)
Kx
time(sec)
Kr
time(sec)



For reference signal of r(t) = 5, as the value of the additive Gaussian noise standard deviation increases, the plant system will have a worse tracking performance. When the noise standard deviation increases, the system's adaptive control gains will also increase but with a larger oscillation.

Main function:

```
clear; clc;

% Adaptive control gains
gamma_x = 2;
gamma_r = 2;

% reference model
am = -2;
bm = 3;

% estimated plant
a_hat = 1;
b_hat = 2;

% initialization
```

```
xm0 = 0;
x0  = 0;
kx0 = -8/3;%(am-a_hat)/b_hat;
kr0 = 1.5;%bm/b_hat;

system_states_init = [x0, xm0, kx0, kr0];

% setting time
tmax = 20;
T=0:0.001:tmax;
tspan = [0, tmax];
STD = 0.5;
global Noise_1;
Noise_1 = STD*randn(1,length(T));
global Noise_2;
Noise_2 = STD*randn(1,length(T));
global Noise_3;
Noise_3 = STD*randn(1,length(T));
global Noise_4;
Noise_4 = STD*randn(1,length(T));

% simulation
[t, system_states] = ode45(@Copy_of_q5ode, tspan, system_states_init);

% system_states
x  = system_states(:,1);
xm = system_states(:,2);
kx = system_states(:,3);
kr = system_states(:,4);

figure(1);
subplot(3,1,1);
plot(t, x, t, xm);
xlabel('time(sec)');
ylabel('states');
title(['x and xm with gamma_x = ',num2str(gamma_x),' gamma_r = ',num2str(gamma_x),', STD =
',num2str(STD),', r(t) = 5'])
grid on;

subplot(3,1,2);
plot(t, kx);
xlabel('time(sec)');
title('Kx')
grid on;
```

```
subplot(3,1,3);
plot(t, kr);
xlabel('time(sec)');
title('Kr')
grid on;
```

Function defined:

```
function system_states_dot = f(t, system_states)
global Noise_1;
global Noise_2;
global Noise_3;
global Noise_4;
% Plant
a = 2;
b = 3/2;
% Reference
am = -2;
bm = 3;

% Adaptive control gains
gamma_x = 2;
gamma_r = 2;

% Reference i/p:
% r = 5;
r = 4*sin(3*t);

% Taking out the states accordingly
x  = system_states(1);
xm = system_states(2);
kx = system_states(3);
kr = system_states(4);

% error
e = x - xm;

%control input
u = kx*x + kr*r;

tmax = 20;
T=0:0.001:tmax;
noise_1 = interp1(T,Noise_1,t,'nearest');
```

```
noise_2 = interp1(T,Noise_2,t,'nearest');
noise_3 = interp1(T,Noise_3,t,'nearest');
noise_4 = interp1(T,Noise_4,t,'nearest');

% Updating the state variables
% x_dot = (a+b*kx)*x + b*kr*r + noise;
% xm_dot = am*xm + bm*r + noise;
% kx_dot = -gamma_x*x*e*sign(b) + noise;
% kr_dot = -gamma_r*r*e*sign(b) + noise;

x_dot = a*x+b*u +noise_1;%x
xm_dot = am*xm+bm*r+noise_2;%xm
kx_dot = -gamma_x*x*e+noise_3;%kx
kr_dot = -gamma_r*r*e+noise_4;%kr

% Putting the states together and return
system_states_dot = [x_dot; xm_dot; kx_dot; kr_dot];
end
```

Problem 8

Problem 8.

$\dot{x} = -(1+x^2)x$

$V(t) = x^2(t)$

$\dot{V}(t) = 2x\cdot\dot{x}$

$= 2x(-(1+x^2)x) \leq -2x^2$

$V(0) = a^2$

$\dot{u} = -2u \quad u(0) = a^2$

$\Rightarrow u(t) = a^2e^{-2t} \Rightarrow |x(t)| = \sqrt{V(t)} = |a|e^{-t}$

When $a>0$, $x(t) \leq ae^{-t}$, $\Rightarrow x(t)$ never goes negative since $\sqrt{V}$ is differentiable $\Rightarrow x(t)$ is differentiable

$x(t)$ can only be greater or equal than 0 or less than or equal to 0.

$x(0) = a$ and $a>0$. $\Rightarrow$ $x(0)$ never goes to negative