10 Composite Adaptation

[Read [19], pp.382-391, for an alternative viewpoint on this.]

Summarizing the lecture notes until today, we can say that adaptive control is all about how to do adaptation. You tried different adaptive laws, like Projection-based, e-modification based, σ-modification based, etc. You know that all these adaptive laws implied robustification of the basic adaptive laws in (60) coming out from a straightforward Lyapunov proof. Composite adaptation is another form of writing adaptation laws. It is not about robustification, but about using a different error signal in (60), in addition to the tracking error. All the modifications for robustification, like Projection, e-modification, σ-modification, can be similarly considered. As observed in simulations, composite adaptation leads to less oscillatory behavior in output tracking and control signal.

10.1 State Predictors

Composite adaptation implicitly combines features of direct and indirect MRAC in one adaptive law, by making use of so-called state predictors. State predictor is a structure that enables you to predict the state of your dynamical system one-step ahead in time. It is important to distinguish between state predictor and state observer. State observers estimate the state of your system at every time instant from available measurements, while state predictors predict your system output at the next time instant using your state measurement at the given time instant. For example, the most simple structure of state predictor for the system dynamics

$$\dot{x}(t) = ax(t) + bu(t)\,, \quad x(0) = x_0 \tag{202}$$

with unknown a and b, would be

$$\dot{\hat{x}}(t) = a_p(\hat{x}(t) - x(t)) + \hat{a}(t)x(t) + \hat{b}(t)u(t), \quad a_p < 0, \quad \hat{x}(0) = x_0\,. \tag{203}$$

If we denote the prediction error by $\tilde{x}(t) = \hat{x}(t) - x(t)$, then it leads to the following error dynamics

$$\dot{\tilde{x}}(t) = a_p\tilde{x}(t) + \Delta a(t)x(t) + \Delta b(t)u(t), \quad \tilde{x}(0) = 0\,, \tag{204}$$

where $\Delta a(t) = \hat{a}(t) - a$ and $\Delta b(t) = \hat{b}(t) - b$ denote the parametric errors. Notice that we did not need to define a control law, but we immediately came up with the same structure of error dynamics that we had for the indirect adaptive control scheme (69). We can immediately write the same adaptation laws what we had for indirect MRAC:

$$\begin{aligned} \dot{\hat{a}}(t) &= -\gamma_a x(t)\tilde{x}(t)\,, \quad \hat{a}(0) = \hat{a}_0 \\ \dot{\hat{b}}(t) &= -\gamma_b u(t)\tilde{x}(t), \quad \hat{b}(0) = \hat{b}_0\,. \end{aligned} \tag{205}$$

A straightforward Lyapunov proof will ensure that $\tilde{x}(t)$, $\Delta a(t)$, $\Delta b(t)$ are bounded. While all the errors can be bounded, both signals $x(t)$ and $\hat{x}(t)$ can drift to infinity. However, this does not imply asymptotic stability for the following reasons:

- We have not specified any control signal. In the above analysis, $u(t)$ is treated just as time-varying signal. As a matter of fact, since we did not specify any control signal we did not run into any need of the knowledge of the $\mathrm{sgn}(b)$.
- Even if we assume that a bounded (open-loop or closed-loop) control signal is applied, we have no guarantee if the predictor state or system state is bounded. In application of Barbalat's lemma, it was crucial that the reference model was bounded. With the $\dot{V}(t) \leq 0$, implying that $e(t)$ is bounded, boundedness of the reference model state $x_m(t)$ was helping to conclude that $x(t)$ was bounded, which eventually was leading to $e(t) \to 0$ as $t \to \infty$. In this case, we do not know if the system state is bounded or not (this depends upon the sign of a: even if $u(t)$ is a bounded signal, if $a > 0$, then the system state $x(t) \to \infty$ as $t \to \infty$).

Let's consider our indirect adaptive control structure:

$$u(t) = \frac{1}{\hat{b}(t)} \left(-\hat{a}(t)x(t) + a_m x(t) + b_m r(t) \right) . \tag{206}$$

Remark 10.1. It is important to keep in mind that once we are using a feedback of this structure, the adaptive laws need to be modified to avoid $\hat{b}(t) = 0$ for all $t \geq 0$. So, we need to use the projection-type modification of adaptive laws from (70) instead of simply (205), replacing the tracking error $e(t) = x(t) - x_m(t)$ by the prediction error $\tilde{x}(t) = \hat{x}(t) - x(t)$ to ensure boundedness of $\tilde{x}(t), \Delta a(t), \Delta b(t)$.

If we substitute (206) into (203), we will get a closed-loop system predictor:

$$\dot{\hat{x}}(t) = a_p(\hat{x}(t) - x(t)) + a_m x(t) + b_m r(t), \quad \hat{x}(0) = x_0 . \tag{207}$$

If we substitute (206) into (202), we will get the following closed-loop system:

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) + \hat{a}(t)x(t) + \hat{b}(t)u(t) - \hat{a}(t)x(t) - \hat{b}(t)u(t) \\ &= \hat{a}(t)x(t) + \hat{b}(t)u(t) - \Delta a(t)x(t) - \Delta b(t)u(t) \\ &= a_m x(t) + b_m r(t) - \Delta a(t)x(t) - \Delta b(t)u(t) \end{aligned} \tag{208}$$

If we subtract (208) from (207), we will get the closed-loop prediction error dynamics in the same form as its open-loop one was in (204):

$$\dot{\tilde{x}}(t) = a_p \tilde{x}(t) + \Delta a(t)x(t) + \Delta b(t)u(t), \quad \tilde{x}(0) = 0 . \tag{209}$$

All we know until this point is that $\tilde{x}(t)$, $\Delta a(t)$ and $\Delta b(t)$ are bounded. We have not so far solved a tracking problem for our system. We did not prove that the state of the reference model $x_m(t)$ can be tracked. If we had used the adaptive laws from (70) that operate over $e(t) = x(t) - x_m(t)$, then, by applying Barbalat's lemma, we would have had tracking of the reference model state $x_m(t)$ by the system state $x(t)$. But our adaptive laws in (205) operate over the prediction error $\tilde{x}(t)$ and not the tracking error $e(t)$.

10.2 Composite Adaptation

So, now we are interested in defining a control signal that can achieve tracking of the desired reference model making use of the state predictor. The idea is that making use of a state predictor can help to achieve better performance. So, consider our desired reference system:

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t)\,, \quad a_m < 0, \quad x_m(0) = x_0\,. \tag{210}$$

All that we are going to borrow from the previous analysis is the closed-loop predictor structure in (207):

$$\dot{\hat{x}}(t) = a_p(\hat{x}(t) - x(t)) + a_m x(t) + b_m r(t), \quad \hat{x}(0) = x_0\,. \tag{211}$$

Notice that the structure in (211) can be defined using **only** the state $x(t)$ of the system from (202) and the bounded reference input $r(t)$. The state of the predictor $\hat{x}(t)$ can be defined as the solution of (211). They key point here is to use this structure (211), **independently of its closed-loop origin**, and to write a direct MRAC scheme for our system (202) to track the reference system in (210). The above derivation in Section 10.1 was done to show you how the structure (207) can be derived or justified. But there is nothing wrong with treating (211) as an independent differential equation on its own, forgetting about the feedback that was embedded into it.

We know that if we use the direct feedback from (56)

$$u(t) = k_x(t)x(t) + k_r(t)r(t) \tag{212}$$

along with adaptive laws from (60)

$$\begin{aligned} \dot{k}_x(t) &= -\gamma_x x(t)e(t)\mathrm{sgn}(b)\,, \quad k_x(0) = k_{x0} \\ \dot{k}_r(t) &= -\gamma_r r(t)e(t)\mathrm{sgn}(b)\,, \quad k_r(0) = k_{r0}, \end{aligned} \tag{213}$$

subject to the matching assumptions in (58) and knowledge of the sgn(b), where $e(t) = x(t) - x_m(t)$ is the conventional tracking error, we can achieve $e(t) \to 0$ as $t \to \infty$ with the help of Barbalat's lemma. The closed-loop system dynamics with the feedback from (212) has the form in (57):

$$\dot{x}(t) = (a + bk_x(t))x(t) + bk_r(t)r(t)\,, \tag{214}$$

while tracking error dynamics has the structure in (59):

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)\,, \quad e(0) = 0\,. \tag{215}$$

If we define the prediction error to be $\tilde{x}(t) = \hat{x}(t) - x(t)$, then using (214) and (211), we obtain:

$$\begin{aligned}\dot{\tilde{x}}(t) &= a_p(\hat{x}(t) - x(t)) + a_m x(t) + b_m r(t) - (a + bk_x(t))x(t) - bk_r(t)r(t) \\ &= a_p\tilde{x}(t) + (a_m - a - bk_x(t))x(t) + (b_m - bk_r(t))r(t)\end{aligned}$$

With our matching assumptions we are back to conventional error dynamics structure

$$\dot{\tilde{x}}(t) = a_p\tilde{x}(t) - b\Delta k_x(t)x(t) - b\Delta k_r(t)r(t)\,, \quad \tilde{x}(0) = 0\,.$$

Now we can write a candidate Lyapunov function in the form:

$$V(e(t), \tilde{x}(t), \Delta k_x(t), \Delta k_r(t)) = e^2(t) + \gamma\tilde{x}^2(t) + \left(\frac{1}{\gamma_x}\Delta k_x^2(t) + +\frac{1}{\gamma_r}\Delta k_r^2(t)\right)|b|\,, \quad \gamma > 0.$$

Its derivative is

$$\begin{aligned}\dot{V}(t) &= 2e(t)\left(a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)\right) \\ &\quad +2\gamma\tilde{x}(t)\left(a_p\tilde{x}(t) - b\Delta k_x(t)x(t) - b\Delta k_r(t)r(t)\right) \\ &\quad +2\gamma_x^{-1}|b|\Delta k_x(t)\Delta\dot{k}_x(t) + 2\gamma_r^{-1}|b|\Delta k_r(t)\Delta\dot{k}_r(t) \\ &= -2|a_m|e^2(t) - 2\gamma|a_p|\tilde{x}^2(t) \qquad (216) \\ &\quad +2|b|\Delta k_x(t)\left(x(t)e(t)\mathrm{sgn}(b) - \gamma x(t)\tilde{x}(t)\mathrm{sgn}(b) + \gamma_x^{-1}\Delta\dot{k}_x(t)\right) \\ &\quad +2|b|\Delta k_r(t)\left(r(t)e(t)\mathrm{sgn}(b) - \gamma r(t)\tilde{x}(t)\mathrm{sgn}(b) + \gamma_r^{-1}\Delta\dot{k}_r(t)\right) \qquad (217)\end{aligned}$$

So, if we choose our adaptive laws like

$$\begin{aligned}\dot{k}_x(t) &= -\gamma_x x(t)(e(t) - \gamma\tilde{x}(t))\mathrm{sgn}(b)\,, \quad k_x(0) = k_{x0} \\ \dot{k}_r(t) &= -\gamma_r r(t)(e(t) - \gamma\tilde{x}(t))\mathrm{sgn}(b)\,, \quad k_r(0) = k_{r0},\end{aligned} \tag{218}$$

then the derivative of the candidate Lyapunov function will be

$$\dot{V}(t) = -2|a_m|e^2(t) - 2\gamma|a_p|\tilde{x}^2(t) \le 0,$$

which is obviously more negative than in the case of simple adaptive laws in (60), where we ended up having $\dot{V}(t) = -2|a_m|e^2(t)$. Thus, we get <u>**a better rate of decrease**</u> for the candidate Lyapunov function as compared to the conventional direct MRAC case. Whether this can be defined as a criteria for better performance in adaptive system, is indeed hard to answer. But the simulations show less oscillations with composite MRAC as compared to direct MRAC. Finally, since $e(t), \tilde{x}(t), \Delta k_x(t), \Delta k_r(t)$ are all bounded, application of Barbalat's lemma implies that both $e(t), \tilde{x}(t) \to 0$ as $t \to \infty$.

Remark 10.2. Notice that setting $\gamma = 0$, you're back to conventional direct MRAC structure and direct MRAC type adaptive laws. The parameter γ was inserted into the candidate Lyapunov function to give you the opportunity to set it to 0 to recover the conventional direct MRAC. If you're building a general model, it would be good to have this parameter and predictor as a part of your overall simulation, so that externally you can always set $\gamma = 0$ and recover direct MRAC if needed.

Remark 10.3. You'll see a clear advantage of composite adaptation over the conventional direct MRAC, if in your simulations you choose $|a_p| > |a_m|$. In this case, the prediction error dynamics converges much faster than your tracking error dynamics. Intuitively this implies that by predicting the system output one-step-ahead in time you are able to have better adaptation against the uncertainties in your system.

Remark 10.4. Honestly, one should have started directly from Section 10.2, omitting Section 10.1. Section 10.2 is completely self-contained, independent of any preliminaries stated in Section 10.1. Section 10.1 was stated to give you an idea as how some of the structures have appeared or how they can be justified. There is no random walk in deterministic systems.

Homework Problems 10.1. Go back to your scalar system, for which you have done direct and indirect MRAC, and try out the composite MRAC. Convince yourself that choosing $|a_p| > |a_m|$ indeed leads to less oscillations in your system performance.