Singular Value Decomposition

Given a matrix A we can decompose it into 3 matrices,

- U which contains left singular vectors, $AA^T - \lambda I$.
- Σ has diagonal singular values which are the square roots the eigenvalues.
- V are the right singular vectors, $A^TA - \lambda I$.

The dimension of the matrices are as follows:

$$\underset{m\times n}{A} = \underbrace{\underbrace{\underset{m\times k}{U}\ \underset{k\times k}{\Sigma}}_{m\times k}\ \underset{n\times k}{V}}_{m\times k}$$

Step 1 - Find Symmetric Matrix of $A \to S$:

Suppose $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, convert to a square matrix by A^TA

$$\underset{3\times 2}{\begin{bmatrix} 2 & -1 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 2\cdot 2 + (-1)(-1) & 2\cdot 2 + -1\cdot 1 & 0 \\ 2\cdot 2 + 1(-1) & 2\cdot 2 + 1\cdot 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underset{\text{symmetric matrix}}{S}$$

3×3

Step 2 - Find Eigenvalues of S:

Now with the symmetric matrix S, compute the eigenvalues, $A^TA - \lambda I \Rightarrow S - \lambda I$:

$$\det(S - \lambda I) \Rightarrow \begin{vmatrix} 5-\lambda & 3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \Rightarrow ((-1)^{3+3} - \lambda)((5-\lambda)(5-\lambda) - (3\cdot 3))$$

$$= -\lambda \cdot ((25 + \lambda^2 - 5\lambda - 5\lambda) - 9)$$

$$= -\lambda(\lambda^2 - 10\lambda + 16) = -\lambda^3 + 10\lambda^2 - 16\lambda = 0$$

Since we want to find the roots of the characteristic equation $\det(S - \lambda I) = 0$, find the factors of λ: $\underline{-\lambda}((\underline{\lambda - 8})(\underline{\lambda - 2})) = 0$

$$\lambda = \{0, 2, 8\}$$

This also gives us the singular values of S, which are the square roots of λ.

$\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{2}$, $\sigma_3 = 0$. σ_3 is not a positive singular value so it's discarded.

Step 3 - Find eigenvectors for each eigenvalues.

Given that $\lceil \lambda = \{0, 2, 8\} \rfloor$ we start by:

Let $\lambda = 8$, $S - 8I = \begin{bmatrix} 5-8 & 3 & 0 \\ 3 & 5-8 & 0 \\ 0 & 0 & 0-8 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{bmatrix}$

Now we find the Nullspace of the matrix: $u = N(S - \lambda I)$

$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \xrightarrow{\text{Reduce}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{0} = \begin{matrix} u_1 - u_2 = 0 \Rightarrow u_1 = u_2 \\ -u_3 = 0 \Rightarrow u_3 = 0 \end{matrix}$$

Now we've $u_1 = u_2$, let $u_1 = u_2 = t$, then $u\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

Now the unit vector is $\frac{1}{\|u\|}\vec{u}$, norm of $u = \|u\| \rightarrow \sqrt{1^2+1^2+0^2} = \sqrt{2}$, nice!

Now $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$. We do the same w/ eigenvalues $\lambda = \{2, 0\}$.

$$\text{Now } S - 2I = 0 \Rightarrow \begin{bmatrix} 5-2 & 3 & 0 \\ 3 & 5-2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{\text{RRE}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now, $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{matrix} u_1 + u_2 = 0 \Rightarrow u_1 = -u_2 \\ -u_3 = 0 \Rightarrow u_3 = 0 \end{matrix}$. $u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\|u\| = \sqrt{2}$, $u_2 = \frac{1}{\|u\|}\vec{u} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$.

Lastly for $\lambda = 0$: $N(S - 0 \cdot I) = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RRE}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{0}$

$u_1 = 0$, $u_2 = 0$, u_3 is free $\Rightarrow u\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\|u\| = \sqrt{1} = 1$. $u_3 = \frac{1}{\|u\|}u = \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Now we've all three eigenvectors for S, our symmetric matrix of A.

$u_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Nice :)

Now, with the eigenvectors:

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \lambda = 8, \quad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \lambda = 2, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \lambda = 0$$

we assemble $V = \{v_1, v_2, v_3\} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We also have the eigenvalues $\lambda = \{0, 2, 8\}$

Assemble Σ as diagonalized singular values in its pivots.

Singular values are $\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{2}$

$$\Sigma = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}.$$

← Only non-zero (pivots) columns are diagonalize, discard $\lambda = 0 = \sigma_3 = 0$ ← Not applied

Awesome pic!

Step 4 · Finding U

U is the left singular eigenvectors $AA^T - \lambda I$, following the same process for V, accounting for its singular values.

$$u_1 = A\frac{v_1}{\sigma_1} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{8}} \Rightarrow u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Similarly for u_2,

$$u_2 = A\frac{v_2}{\sigma_2} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \Rightarrow u_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

} u_1 & u_2 are orthonormal

Now $U = \{u_1, u_2\} \Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Finally, we have U, Σ and V,

thus we can decompose A as well as compose it

as $A = U \Sigma V^T$. $\Rightarrow$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}}_{2\times 3} \cdot \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T}_{3\times 3}$$

$A = 2\times 3$

$$\therefore \quad A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$ ← Absolute Wizardry!