Singular Value Decomposition for Images Processing

Description

Singular Value Decomposition (SVD) is a matrix factorization technique that provides a representation of any matrix by decomposing it into three matrices, it is a factorization A = $U\Sigma V^T$.

U: An $m \times r$ orthogonal matrix which columns are the left singular vectors of A, columns of U are orthonormal eigenvectors of AA^T.
Σ: An $r \times r$ diagonal matrix with singular values $\sigma_1, \ldots, \sigma_n$. The number of non-zero singular values is equal to the rank of A, σ are the square roots of the eigenvalues AA^T and A^TA.
V^T: An $r \times n$ orthogonal matrix which rows are the right singular vectors of A, columns of V are orthonormal eigenvectors of A^TA.

The columns of U and rows of V^T are orthogonal eigenvectors of AA^T and A^TA respectively. The matrices AA^T and A^TA have the same nonzero eigenvalues. The entries in the diagonal matrix Σ are the square roots of the eigenvalues, called singular values.

Fundamental Subspaces

The columns of U and V provide orthogonal bases for the four fundamental subspaces of A:

- Column Space
- Row Space
- Null Space
- Left Null Space

Orthogonality and Orthonormal Values

Two vectors are orthogonal if their dot product is zero,
$u \cdot v = 0$, which implies they lie at a right angle to each other.

For matrices, they are orthogonal if its columns are orthogonal and satisfies $A^TA = I$.

Normalization

Orthonormality defines the norm or magnitude of a vector, $||u|| = 1$ implies u is orthonormal because it is of norm 1. For matrices the columns can be normalized referred to as unit vectors. For SVD normalizing a vector scales it to have a unit norm

Computing the Singular Value Decomposition of a Matrix

Given a matrix A, we'll decompose it into three matrices,

- U which contains left singular vectors, $AA^T - \lambda I$.
- Σ has diagonal singular values which are the square roots of the eigenvalues.
- V has the right singular vectors, $A^T A - \lambda I$.

Significance of SVD Components

Singular Values:

The diagonal entries of Σ are the square roots of the non-negative eigenvalues of $A^T A$. These values determine the 'importance' of each component in the decomposition.

Orthonormality:

The rows of V^T are eigenvectors of $A^T A$, and the columns of U are eigenvectors of AA^T. Their orthonormality ensures that the decomposition captures the structure of A without redundancy.

Step 1 - Find Symmetric Matrix of A → S

Suppose , $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ we convert into symmetric matrix S by $S = A^T A$

$$S = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2 - Find Eigenvalues of S

Now with the symmetric matrix S, compute the eigenvalues, that is $A^T A - \lambda I \rightarrow S - \lambda I$:
We can compute the determinant as

$$det(S - \lambda I) \rightarrow -\lambda((\lambda - 8)(\lambda - 2)) = 0$$

we will find the eigenvalues $\lambda_1 = 8$, $\lambda_2 = 2$, $\lambda_3 = 0$. This also gives us the singular values that composes Σ, $\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{2}$, because the third singular value is not a positive value, it is discarded.

Step 3 - Find eigenvectors for each eigenvalues of S

Given that $\lambda_1 = 8, \lambda_2 = 2, \lambda_3 = 0$, we start by finding the null spaces of the matrices with the corresponding eigenvalues as

$$N(S - \lambda_i I)$$

First $\lambda_1 = 8 \rightarrow N(S - 8I)$ will give the eigenvector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Likewise for $\lambda_2 = 2, \lambda_3 = 0$.

We will find $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ v_2, v_3 eigenvectors respectively.

Now we 'normalize' the vectors to obtain the 'unit vector', $\frac{1}{||v||}v$, with $||v||$ being the norm of the vector, defined as $||v|| = \sum_{i=1}^{n} \sqrt{v_i^2}$.

We find that for the eigenvectors, their norms corresponds to

$$\left\|v_1\right\| = \sqrt{2}\,,\ \left\|v_2\right\| = \sqrt{2}\,,\ \left\|v_3\right\| = 1$$

Then the unit vectors $\frac{1}{\|v_i\|}v_i$ are $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\ v_3 = \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We then have the unit vectors $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Now with our unit vectors which are given by the varying eigenvalues we assemble our matrix V.

$$V = \{v_1, v_2, v_3\} \rightarrow V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the obtained singular values $\sigma_1 = \sqrt{8}, \sigma_2 = \sqrt{2}$, we assemble Σ as diagonalized singular values in its pivots.

$\Sigma = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \leftarrow$ Only non-zero pivot columns are diagonalized, as $\lambda_3 = 0 \Rightarrow \sigma_3 = 0$.

Step 4 - Finding U

U is the left singular eigenvectors $AA^T - \lambda I$, following the same process as V, accounting for its singular values

We may obtain the vectors $u_i = Av_i/\sigma_i$ as they are orthonormal for $i = 1,\dots, r$. They are a basis for the column space of A. And the $u's$ are eigenvectors of the symmetric matrix AA^T, which is usually different from $S = A^TA$ (but the eigenvalues σ_r^2 are the same).

$$u_1 = A\frac{v_1}{\sigma_1} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{8}} \rightarrow u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = A\frac{v_2}{\sigma_2} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \rightarrow u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we assemble $U = \{u_1, u_2\} \rightarrow$ $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Step 5 - Composing A

After finally obtaining the three factorized matrices we can assemble A = $U\Sigma V^T$.

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

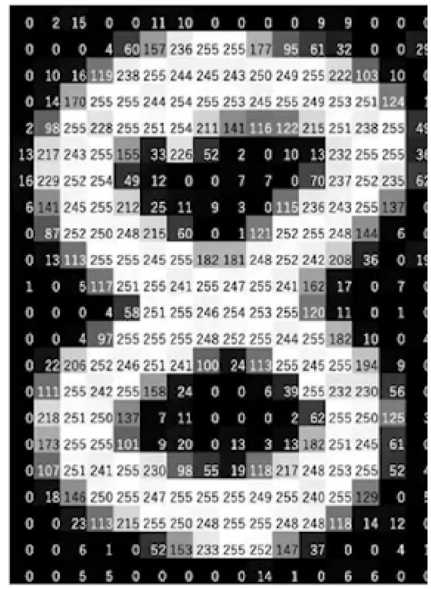
Notice their dimensions, 2×2 , 2×3 , 3×3 . The final resulting matrix has dimension 2×3, just as our original matrix A.

$$A = U\Sigma V^T \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{8} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$.

[*Proof*](#) 🐧

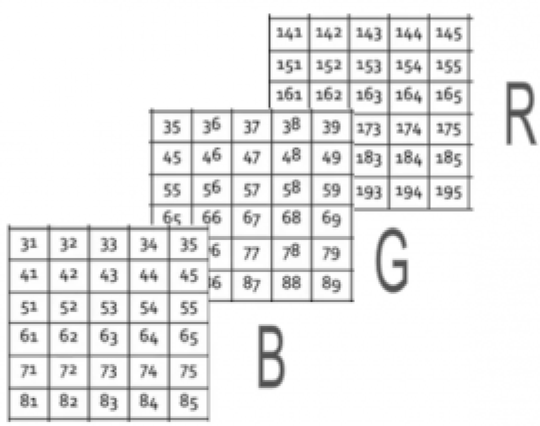
Image Representation:

Grayscale Images: These are stored as a 2D matrix where each element represents the pixel's intensity value that ranges from 0 to 255 <black> to <white>.



Color Images: These are stored using three 2D matrices which correspond to Red Green Blue (RGB) values.

Colored images will have 1 matrix for each color referred to as channel



Applying SVD for Image Processing

Given an image A, represented as a data matrix, we can use SVD to decompose into the three components A = $U\Sigma V^T$.

- U $(left\ singular\ vectors)$: Captures the patterns in the rows, which represent individual observations e.g., pixels across different columns, they can be horizontal patterns like a stripe or texture along rows as well as changes in brightness along rows.
- Σ: Diagonal matrix containing singular values, these quantifies the strength of each pattern, the larger singular values indicate more important patterns.
- V^T $(right\ singular\ vectors)$: Captures the patterns in the columns, which represents features or variables e.g., pixel intensity in different rows, these are vertical patterns like an edge or gradient along columns.

Applications

Why use SVD for Image Processing? It helps by analyzing and processing images by reducing data redundancy and compressing information whilst retaining important information.

Compression: By retaining only the largest singular values, we're able to reduce the matrix's size whilst preserving its most important features. We can do this by approximating A as $A_k = U_k \Sigma_k V_k^T$, retaining top k singular values in Σ_k.

Dimensionality Reduction: The subspaces defined by U and V are useful for reducing the dimensionality of the matrix's data.

Noise Filtering: Smaller singular values often implies noise, which can be identified and then removed by truncation.

<Short Example>:
Suppose we've a 100×100 grayscale image, we can compress it by retaining only the top 10 singular values, this will result in a smaller storage without noticeable quality degradation.

Resources:

What's SVD
https://www.geeksforgeeks.org/singular-value-decomposition-svd/
Image Processing with SVD example
https://medium.com/@maydos/image-processing-with-singular-value-decomposition-ce8db3f78ce0
SVD Image Processing paper <just in case>
https://sites.math.washington.edu/~morrow/498_13/svdphoto.pdf
Video with hands-on example
https://youtu.be/cOUTpqlX-Xs?si=t3EnThW7twYgBzMu
Hands-on example with a 2x2 matrix
https://medium.com/intuition/singular-value-decomposition-svd-working-example-c2b6135673b5
How images are stored Grayscale & Color format
https://www.analyticsvidhya.com/blog/2021/03/grayscale-and-rgb-format-for-storing-images/

Differential Equations and Linear Algebra (2014) - Gilbert Strang