MA 381 Exam 3
Summer 2013, Dr. Evans
100 points possible
Time Limit: 3 hours

Name:_________________**Xiangbo Meng**________________

Signature:_____________Xiangbo Meng_______________________
I agree that I have not consulted anyone about this exam besides Dr. Evans.

♣ There are 12 problems with parts on this exam, and each problem is worth 10 points. **Choose** your favorite **10 problems** and put a big **X** through the two problem numbers that you **don't want me to grade***:

1 2 3 4 5 6 7 8 **9** 10 11 **12**

* If you don't exclude two problems, then I'll just exclude the last two problems.

♣ You will have 3 hours once you download this exam to submit it to the Exam 3 Drop Box. When you download it, make sure you have 3 hours to work on it. I am setting ANGEL to report to me the time you download and upload the exam so that I can make sure your submission is within 3 hours of downloading it. You will be penalized for time over 3 hours – there will be a 5 point reduction for each additional ½ hour.

♣ You must submit this exam as a **single PDF document**. Please do not submit a zipped folder with multiple PDF's in it.

♣ You may use a calculator, Maple, prepared notes, textbooks, the internet ... basically **anything except other people!** I am thoroughly checking exams for collaboration. If it appears that there could be collaboration on your part, **I will require you to take the final exam on campus when you return from summer break.**

♣ By Rose's Honor Code, I am requiring that the only living people who are allowed to look at your exam are you and me. If you do not follow this rule, you may receive a 0 on this portion of the exam and disciplinary action within and perhaps beyond the math department.

♣ **Please include the supports** with all univariate and joint distributions.

♣ Please write scratch work on this exam. It's much easier for me to give you partial credit when I can see your work and thought process for each problem.

	Your Points		**Your Points**
Problem 1		**Problem 8**	
Problem 2		**Problem 9**	
Problem 3		**Problem 10**	
Problem 4		**Problem 11**	
Problem 5		**Problem 12**	
Problem 6		***Subtotal 2***	
Problem 7		***Subtotal 1***	
Subtotal 1		**Bonus**	
		TOTAL	

1. Did you complete the SurveyMonkey survey? If your answer is "yes", then you get 10 points for this problem.

If your answer is "no", then the only way that you can get points for this problem is to go to the survey site now and do it – the address is in an email that I sent you. It will take you less than 5 minutes to complete the survey. Otherwise, you will get 0 points if you choose this problem without having completed the survey.

Yes

2.(a) The time it takes Dr. Grimaldi to walk from the ISU Stadium to Rose-Hulman on Heritage Trail is a random variable *X* that is normally distributed with mean 25 minutes and standard deviation 3 minutes. If he leaves the stadium at 8 a.m. and walks the trail to Rose, what's the probability that he'll make it to Rose in less than 20 minutes? Please write down the appropriate expression and evaluate it.

Solution: normal(25,3) = P(x<20) = P(z<(20-25)/3) = P(z<-1.67) = **0.0475**

(b) At the end of the day, Dr. Grimaldi can be a little tired from a full day of classes and beating new faculty (often 30 or more years younger than him) at tennis matches. The time is takes him to walk from Rose to the stadium on Heritage Trail is normally distributed with mean 27 minutes and standard deviation 4 minutes. What's the probability that his TOTAL walking time to and from school on a given day is longer than 55 minutes? (Note: His walking time from Rose to the stadium is independent of his walking time from the stadium to Rose.) Please write down the appropriate expression and evaluate it.

Solution: Suppose x = the time he walk from the stadium to rose. And y = the time he walk from rose to the stadium.

X~ normal(25,3)
Y~ normal(27,4)
K = x+y
L~ normal(25+27, sqrt(3^2+4^2)) = normal(47, 5)
P(k>55) =1- P(z<(55-47)/5) = 1- 0.9452 = **0.0548**

3.(a) The time is takes Dr. Goulet to bike from the ISU Stadium to Rose-Hulman on Heritage Trail is a random variable Y that has an exponential distribution with mean time 10 minutes. If he leaves the stadium at 8 a.m. and rides his bike on the trail to Rose, what's the probability that he'll make it to Rose in more than 12 minutes? Please write down the appropriate expression and evaluate it.

Solution:
Lambda = 1/E(T) = 1/ 10
PDF = Lambda * exp(-lambda*x) = 1/10*exp(-1/10 *x)
P(X> 12) = int(1/10*exp(-1/10 *x), x = 12..infinity) = **0.3012**

(b) On a certain morning when he is biking to school on the trail from the stadium, if we know it's already taken him 10 minutes and he hasn't arrived, what's the probability that it will take him a total of more than 12 minutes? Please write down the appropriate expression and evaluate it.

Solution:
As exponential random variable is memoryless P(x>12|x>10) = P(x>2) = int(1/10*exp(-1/10 *x), x = 2..infinity) = **0.8187**

4. On a given MA 381 exam, the number of problems that I can grade in an hour has a Poisson distribution with mean 2 problems per hour.

(a) Suppose I'm in a hurry to get problems graded on this exam to meet the registrar's grade submission deadline. What's the probability that I can grade 3 or more problems in a half hour? Please write down the appropriate expression and evaluate it.

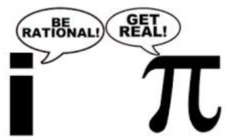
Solution:
As we are talking about half hour and 2 problems per hour, you can finish 1 problem per half hour
So the lambda = 1
P(x) = lambda^x*exp(lambda)/x! = (1^x * exp(1)/x!)
As poisson distribution is descrete random variable so we need to sum up all the condition
P(x>=3) = $\sum_{x=3}^{infinity} \frac{1^x * e^1}{x!}$ = **0.0803**

(b) Suppose that I have one problem left to grade. What's the probability that I can grade it in less than 45 minutes? Please write down the appropriate expression and evaluate it.

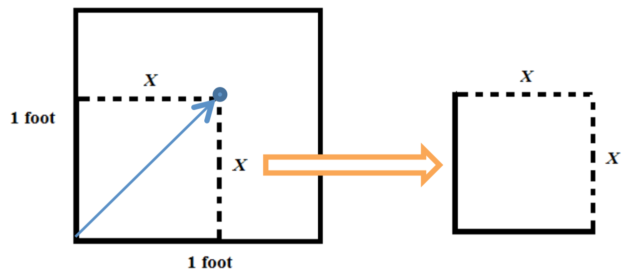
Solution: Let x be the questions that you graded per hour. Then x~poisson(lambda = 2). Let Y be the number of questions you graded in the next 45.
Grade 1 question less than 45 min is the same as in 45 min grade more than 1 question so

P(Y>1) = $\sum_{x=1}^{infinity} \frac{\frac{3}{2}^x * e^{\frac{3}{2}}}{x!}$ = **0.7769**

5. I want to make greeting cards for all of my math colleagues coming back to Rose after a long summer. Here's the image that I want to put on the cards:



Anyway, I have a square sheet of paper that is 1 foot by 1 foot. I'm going to start in the left bottom corner of the sheet of paper and move diagonally across the square to a point. Then I'll cut out an x feet by x feet square. I'm going to cut out that square to place the card image on.



Let X be the random variable associated with the length of the cut in each direction (with the lengths being identical). The probability density function of X is given by:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[You can check – it's valid!] I think it's important to note that the length X is not chosen uniformly on the interval from $x = 0$ to $x = 1$; it is chosen according to the distribution given above.

(a) What's the probability that the length of the cut in one direction is more than 0.25 foot? Write down the appropriate expression and evaluate it.

Solution: P(x>0.25) = int(2x, x = 0.25..1) = **0.9375**.

(b) What's the probability that the area of the square that I cut out (x feet by x feet) will be more than 0.5 foot^2?
Solution: in order to make the area be more than 0.5 square foot the length of the edge is more than sqrt(0.5)

P(x> sqrt(0.5)) = int(2x, x = sqrt(0.5)..1) = 0.5

(c) What's the expected area of the square that I cut out?

Solution: as the area is x^2, so we are calculating E(x^2) = int(2x * x^2, x = 0..1) = 0.5

6. Let X be a random variable that is uniformly distributed on the interval $x = 1$ to $x = 5$.

(a) What are the mean and variance of X? Set up the appropriate expressions and evaluate them.
Solution: so the f(x) = 1/(5-1) = ¼
E(x) = int(1/4*x, x = 1..5) = 3
E(x^2) = int(1/4*x^2, x = 1..5)
Var(x) = E(x^2) – E(x)^2 = 1.333

(b) Let $Y = \frac{1}{X^2}$. Determine the probability density function of Y. Include the support of Y.

Solution: f(x) = ¼
F(Y) = P(Y<= y) = P(1/x^2 <= y) = P(x>=sqrt(1/y)) = int(1/4, x = sqrt(1/y)..5) = 5/4 – ¼*sqrt(1/y)
f(y) = F'(Y) = 1/(8*sqrt(1/y)*y^2) 1/25<=y <= 1

7. Suppose that we have an electronics device that has two batteries that operate independently of each other and have identical performance characteristics. Let X and Y denote the batteries' life spans, where the lifetimes are both exponentially distributed with mean lifetimes of 100 hours.

(a) What is the joint density function $f(x, y)$? Please include its support.
Solution: lambda = 1/100
f(x) = 1/100*(-(1/100)*x)
f(y) = 1/100*(-(1/100)*y)
as x and y are independent, f(x,y) = f(x)*f(y) = 1/100*(-(1/100)*x) * 1/100*(-(1/100)*y)

(b) Suppose there is a money-back guarantee (refund) on the batteries if either one or both of the batteries fails within 10 hours. What's the probability of getting a refund on a pair of new batteries that you just put in the electronics device? [Hint: Draw a picture of the "refund" region.] Please set up the appropriate expression and evaluate it.
Solution: int(int(f(x,y), x=0..10),y= 10..infinity)+ int(int(f(x,y), y=0..10),x= 10..infinity)+ int(int(f(x,y), x=0..10),y= 0..10) = **0.1813**

8. Let X be a random variable with moment generating function (MGF) given by:

$$M_X(t) = \frac{\frac{1}{4}e^t}{1 - \frac{3}{4}e^t} = \frac{e^t}{4 - 3e^t} \text{ for } t < -\log\left(\frac{3}{4}\right).$$

(a) Determine the mean of X. There are several different ways you can determine this. Clearly show your work.

Solution: E(x) = M(t)'|t=0 = $subs(t = 0, \frac{e^t}{4-3*e^t} + 3 * \frac{(e^t)^2}{(4-3e^t)})$ = 4

(b) Let Y be the sum of 5 independent X's whose moment generating function is given above. That is, $Y = X_1 + X_2 + \ldots + X_5$. Determine the moment generating function of Y. That is, determine $M_Y(t)$.

Solution: MY(t) = MX1+X2+X3+x4+X5(t) = Mx1(t)* Mx2(t)* Mx3(t)* Mx4(t)* Mx5(t) $= \left(\frac{e^t}{4-3e^t}\right)^5$

BONUS (+1): What is Y's distribution (e.g., Binomial, Poisson, Normal, Exponential, etc)?

Geometric distribution.

9. I did a little data collecting on my plane trip to Montreal. The amount of time that a customer (who arrives one hour early for his/her domestic flight) spends waiting at an airport check-in counter is a random variable with mean 10.5 minutes and standard deviation 1.5 minutes. Suppose that we observe 49 customers waiting at an airport check in (who are one hour early for a domestic flight). Determine the probability that their **average** wait time in the check-in line is:

(a) between 10 and 10.75 minutes?

Solution:

(b) With the information that we are given, why can't we determine the probability that the wait time for only one customer is between 10 and 10.75 minutes?

10. (Another airlines scenario). Calls are made to check the airline schedule at your departure city. You monitor the number of bars of signal strength on your cell phone and the number of times that you need to state the name of your departure city before the voice system recognizes the name.

Let X denote the number of bars of signal strength on your cell phone, and let Y denote the number of times that you need to state your departure city.

The joint probability mass function of X and Y is given in the following table:

		X = number of bars of signal strength		
		1	**2**	**3**
Y = number of times that the city name is stated	**1**	0.01	0.02	0.25
	2	0.02	0.03	0.2
	3	0.02	0.1	0.05
	4	0.15	0.1	0.05

(a) Write down (as a piecewise function) the marginal mass function of Y, where Y is the number of times that the city name is stated.

Solution: f(y) = 0.28 y= 1
= 0.25 y=2
= 0.17 y=3
=0.3 y=4

(b) Determine the probability that the number of times that the city name is stated is 1 given that the signal strength is 3. That is, determine $P(Y = 1 \mid X = 3)$.
P(Y=1|x=3) = 0.25

(c) Determine $E(Y \mid X = 3)$.
E(y|x=3) = 0.25 * 1+0.2*2+0.05*3+0.05*4 = 1

11. Let the joint probability density function of random variables X and Y be $f(x, y) = c \cdot (x + y)$ over the region $0 < x < 3$ and $x < y < x + 2$.

(a) Determine the value of c that makes the joint probability density function $f(x, y) = c \cdot (x + y)$ valid over the region $0 < x < 3$ and $x < y < x + 2$.

Solution: int(int(f(x,y), y = x..x+2),x = 0..3) = 1
24c = 1 c = 1/24

(b) Determine $P(X < 1, Y > 2)$.

int(int(f(x,y), y = 2..x+2),x = 1..3) = 7/8

(c) Determine $P(1 < X < 2)$.

Solution: int(int(f(x,y),y= x..x+2),x = 1..2) = 1/3

(d) Determine $P(Y > 2 \mid X = 1)$.

Solution: f(x) = int(f(x,y), y = x..x+2) = $\left(\frac{1}{12}\right) * x + \frac{1}{48}(x+2)^2 - \frac{1}{48}x^2$

f(y|x) = f(x,y)/f(x) = $\dfrac{\frac{1}{24}*(x+y)}{\left(\frac{1}{12}\right)*x+\frac{1}{48}(x+2)^2-\frac{1}{48}x^2}$

f(y|x, x = 1) = $\dfrac{\frac{1}{24}*(1+y)}{\left(\frac{1}{12}\right)+\frac{1}{48}(3)^2-\frac{1}{48}}$

int($\dfrac{\frac{1}{24}*(1+y)}{\left(\frac{1}{12}\right)+\frac{1}{48}(3)^2-\frac{1}{48}}$, y= 2..1+2) = 7/12

12. The lifetime X of a certain electronic device is exponentially distributed with mean Y, where Y is the lifetime of the device's main battery. Assume that the lifetime of this battery is uniformly distributed between 5 and 10 years. Compute the probability that a randomly chosen device of this type will last more than 3 years. I'll walk you through determining this probability below.

(a) First, determine an expression for $f_{X|Y}(x \mid y)$, the conditional probability density function for the lifetime of the device X given the lifetime of the device's main battery Y. Include the support of the conditional density function.

(b) Determine an expression for the joint probability density function $f(x, y)$. You will need to first determine the probability density function $f_Y(y)$ in order to construct $f(x, y)$. Include the support of the joint density function.

(c) Last, determine $P(X > 3)$. Although the expression for $f_X(x)$ will look "nasty" or "unfamiliar," you can still integrate it to determine this probability.