Game Theory Individual Coursework: An Investigation of Truels

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1 Abstract

This will be an investigation into the game theory governing three player duels, known as truels. We will examine two cases, simultaneous and sequential firing, and aim to analyse the situation in which the third player's ability to shoot is secretly removed by player one.

2 Introduction

There are many different ways to model a truel mathematically, for example we could consider a probabilistic approach and determine the expected number of shots a player would have to make in order to hit his target and the probability, contingent on the other duelist's accuracy, of surviving to that point, as done in [1]. Instead we will model a truel as a game theory normal form game, making the following assumptions:

- The players are perfect shots, i.e. the probability of hitting their chosen target is one.
- Players shoot as fast as they can, they cannot strategically wait for others to fire.
- Intentionally missing is not allowed and players cannot shoot themselves or choose not to shoot.
- The players have only one bullet/they will play only one "round".
- Imperfect information/players have no way of knowing the other players' choices before shooting.

We will first consider the game in which all players fire at exactly the same time and then move on to the game in which players fire sequentially. Finally we will see how the latter game changes when the third (and fastest) player's strategies are removed.

3 The Game

Our normal form game is defined[2] by:

- The player set {Player 1, Player 2, Player 3} $\equiv$ {Row player, Column player, Matrix player}
- Strategy spaces $S_1 =$ {Shoot Player 2, Shoot Player 3}, $S_2 =$ {Shoot Player 1, Shoot Player 3}, $S_3 =$ {Shoot Player 1, Shoot Player 2}.
- Payoff functions mapping strategy profiles to the utilities for each player. We let the utilities for each player be 0 if they die, 1 if they live, and 2 if they are the only surviving player.

Each player then has two strategies, the row/col player chooses the row/col of the matrices as usual and the third 'matrix' player chooses which half of the matrices we are in. Let the row, column, and matrix player have payoff matrices A, B, and C, respectively.

3.1 Simultaneous firing

Consider the simple form of the truel in which all players fire simultaneously. Clearly there will always be three shots fired and hence, by our assumptions, either all players will be killed, or one player will be shot twice and there will be exactly one player left alive. This game can be represented as follows:

$$A = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] \qquad B = \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{cc:cc} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In some sense this game is symmetric as each player has the same "utility space", i.e. the game is the same for each player. It is clear to see that a player's utility is not affected by his own strategy and is only dependent on the strategies of the other players. Therefore no player ever has a reason to change his strategy, that is, any strategy is a best response to any other two strategies. So we see that any combination of strategies, either pure or mixed, is a Nash equilibrium for this game.

3.2 Sequential firing

Now suppose that the players do not fire simultaneously, instead let t_i be the time it takes player i to fire and assume $t_3 < t_1 < t_2$. Now it is possible that two players will survive as a player may be shot before shooting. The payoff matrices become:

$$A = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right] \qquad B = \left[\begin{array}{cc:cc} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{cc:cc} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

The game loses its symmetry and now we wish to use rationalisation[2] to predict the outcome of the game by removing players' dominated strategies. We can use python's nashpy library to compute the dominated strategies of a player from their utility matrix.

Input:

```python
import numpy as np
A = np.array([[0, 0, 1, 1], [0, 0, 2, 2]])  # Create utility matrices as numpy arrays
B = np.array([[1, 2, 0, 0], [1, 2, 0, 0]])
C = np.array([[1, 0, 1, 1], [1, 0, 0, 0]])
A_domd_strats, B_domd_strats, C_domd_strats = set([]), set([]), set([])  # Create empty sets.

for i in range(len(A)):
    for j in range(len(A)):
        if i != j:  # Prevents checking if a strategy dominates itself.
            if all(A[i,:] <= A[j,:]) and any(A[i,:] < A[j,:]):  # Finds weakly dominated strategies.
                A_domd_strats.add(i)
```

We can use a custom, more general, function to obtain the dominated strategies of a given player:

```python
Get_dominated_strats(B, B_domd_strats, 'col')
Get_dominated_strats(C, C_domd_strats, 'matrix')
print(A_domd_strats, B_domd_strats, C_domd_strats)
```

Output: $\{0\}$ $\{0\}$ $set()$

So it seems that the row and column players' first strategies (strategy 0 in the code) are dominated and the matrix player has no dominated strategies. Using this we can also find that the order in which we remove dominated strategies changes the predicted strategy profile and in some orders we cannot predict a complete strategy profile. For instance, if we remove first the row then the column players' strategies we cannot predict the matrix player's strategy. But if we remove the matrix player's dominated strategies second then we can predict a complete profile (which will be different depending on which way round we remove the row and column strategies).

Instead we can look at the best responses for each player, underlining all best responses:

$$A = \left[\begin{array}{cc:cc} \underline{0} & \underline{0} & 1 & 1 \\ \underline{0} & \underline{0} & \underline{2} & \underline{2} \end{array}\right] \qquad B = \left[\begin{array}{cc:cc} 1 & \underline{2} & \underline{0} & \underline{0} \\ 1 & \underline{2} & \underline{0} & \underline{0} \end{array}\right] \qquad C = \left[\begin{array}{cc:cc} \underline{1} & 0 & \underline{1} & \underline{1} \\ \underline{1} & \underline{0} & 0 & \underline{0} \end{array}\right]$$

The row and column player will certainly play their second strategies (both shooting the matrix player) as this is always a best response, but we cannot predict the matrix player's strategy as there is not one strategy which is a best response to every other combination of strategies. The Matrices will reduce to:

$$A = \left[\begin{array}{c:c} \underline{0} & \underline{2} \end{array}\right] \qquad B = \left[\begin{array}{c:c} \underline{2} & \underline{0} \end{array}\right] \qquad C = \left[\begin{array}{c:c} \underline{0} & \underline{0} \end{array}\right]$$

And we see that in this version of the game player 3 is indifferent to his move, so we have found two pure equilibria and an infinite number of mixed equilibria. Note that player 3 does however choose who lives and dies between the other two.

3.3 Reduction to two player game

Now we investigate the situation in which player 1 secretly removes the bullets from the gun of player 3. So, as player 3 now has no moves, the payoff matrices become:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Clearly player 1's first strategy weakly dominates his second and he will shoot the column player. Player 1 is also now the fastest shooter with bullets in his gun, so the column player will die before firing and hence players 1 and 3 will survive. Note that the column player will still try to shoot the matrix player as he still views the payoff matrices as a three player game, and player 3 will try to shoot either player 1 or 2 but will find his gun empty.

4 Summary

We have seen that in the case of simultaneous firing the players are indifferent to their own strategies and thus any strategy profile is a Nash equilibrium. In the case of sequential firing we can reduce the game by identifying best responses and we see that the fastest shooter inevitably dies, and, depending on his strategy, one of the other players will live and one will die. We also found that in the reduction of a sequential truel to a two player game we can fully predict the outcome and payoffs of each player concluding that players 1 and 3 will both survive.

We have investigated only a few versions of the truel as a game theory game, it would be interesting to relax some of our assumptions and examine the games that arise.

References

[1] Ariadna Alpizar, Cameron McKenzie, Justin Schuetz. [A Mathematical Analysis of the Truel] Available at: `https://nanopdf.com/download/a-mathematical-analysis-of-the-truel_pdf` Access date: 27/02/2018.

[2] Knight, V. (2017). [Game Theory] Available at: `https://vknight.org/gt/` Access date: 27/02/2018.