Testing an ansatz for the quark mass matrices

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Abstract

The quark mass matrices are defined by two matrices whose eigenvalues are the quark masses (one matrix for each of the up-like and down-like quarks):

$$M_i^{'} = \begin{bmatrix} \frac{(r_i^{'})^2}{s_i^{'}} & p_i^{'} & 0 \\ p_i^{'} & r_i^{'} & q_i^{'} \\ 0 & q_i^{'} & s_i^{'} \end{bmatrix} = U_i^T D_i^{'} U_i, \tag{1}$$

where

$$D_i^{'} = \begin{bmatrix} \lambda_1^{'} & 0 & 0 \\ 0 & \lambda_2^{'} & 0 \\ 0 & 0 & \lambda_3^{'} \end{bmatrix}_i = \begin{bmatrix} m_{u,d} & 0 & 0 \\ 0 & m_{c,s} & 0 \\ 0 & 0 & m_{t,b} \end{bmatrix}, \tag{2}$$

and U_i is the unitary matrix that diagonalises $M_i^{'}$ to $D_i^{'}$ whose rows are the eigenvectors of $M_i^{'}$. The eigenvectors make up the columns of $U^{\dagger}$ and therefore make up the rows of $U^{*}=U$ in the real case.

We investigate the normalised mass matrices, $M_i=\frac{M_i^{'}}{s_i^{'}}$ (with eigenvalues approximately equal to the ratio of quark masses):

$$M_i=\begin{bmatrix} r_i^2 & p_i & 0 \\ p_i & r_i & q_i \\ 0 & q_i & 1 \end{bmatrix}=U_i^T D_i U_i, \tag{3}$$

Since M_i is only changed by a scale factor from $M_i^{'}$ it is diagonalised by the same U_i.

Hence, setting $i=u$:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_u = D_u = U_u M_u U_u^T = U_u \frac{M_u^{'}}{s_u^{'}} U_u^T = D_u^{'}/s_u^{'} = \begin{bmatrix} m_u/s_u^{'} & 0 & 0 \\ 0 & m_c/s_u^{'} & 0 \\ 0 & 0 & m_t/s_u^{'} \end{bmatrix}, \tag{4}$$

Thus:

$$\frac{1}{\lambda_3^u} D_u = \begin{bmatrix} \frac{\lambda_1}{\lambda_3} & 0 & 0 \\ 0 & \frac{\lambda_2}{\lambda_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}_u = \begin{bmatrix} m_u/m_t & 0 & 0 \\ 0 & m_c/m_t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

and since $\lambda_3 \approx 1$:

$$D_u \approx \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}_u \approx \begin{bmatrix} m_u/m_t & 0 & 0 \\ 0 & m_c/m_t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

Hence,

$$D_u = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_u \approx \begin{bmatrix} m_u/m_t & 0 & 0 \\ 0 & m_c/m_t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Similarly for M_d

The purpose of this project is to start with some "known" values of the eigenvalues of the M_i (i.e. the approximations $\lambda_1=m_u/m_t$ etc.) and to construct reasonable estimates for the M_i ($i=u,d$) by comparing values of the constructed CKM matrix $V=U_u U_d^T$ to previously measured values. And then to do the same allowing the eigenvalues to vary, and for the case that the p_i and q_i are complex.

Fixed eigenvalues

Set up

We first want to find equations for the p_i and q_i as functions of the λ_i and r_i. Start by noting that, by the unitarity of U_i (dropping the i for simplicity):

$$Det(M)=Det(D) \quad \Rightarrow \quad r^3-q^2r^2-p^2 \quad = \quad \prod_{n=1}^{3}\lambda_n \approx \lambda_1\lambda_2, \tag{8}$$

$$Tr(M)=Tr(D) \quad \Rightarrow \quad r^2+r+1 \quad = \quad \sum_{n=1}^{3}\lambda_n \approx \lambda_1+\lambda_2+1, \tag{9}$$

$$\sum_{ij=11,22,33}[M]_{ij} = \sum_{ij=11,22,33}[D]_{ij} \quad \Rightarrow \quad r^3+r^2+r-q^2-p^2 \quad = \quad \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3 \approx \lambda_1\lambda_2+\lambda_2+\lambda_1. \tag{10}$$

Hence,

$$\sum_{ij=11,22,33}[M]_{ij}-Det(M) \quad = \quad q^2(r^2-1)+r^2+r\approx\lambda_1+\lambda_2, \tag{11}$$

$$q^2\approx\frac{\lambda_1+\lambda_2-r^2-r}{r^2-1}. \tag{12}$$

And substituting q^2 into $Det(M)$:

$$r^3-(\frac{\lambda_1+\lambda_2-r^2-r}{r^2-1})r^2-p^2\approx\lambda_1\lambda_2, \tag{13}$$

$$p^2\approx\frac{\lambda_1\lambda_2-(\lambda_1+\lambda_2+\lambda_1\lambda_2)r^2+r^4+r^5}{r^2-1}. \tag{14}$$

Next we want to find the U_i, we start by finding the ratios of the elements of the eigenvectors. Again dropping the i suffix and using $(M-\lambda_k I)\vec{v}_k=\vec{0}$ (k=1,2,3):

$$\begin{bmatrix} r^2-\lambda_k & p & 0 \\ p & r-\lambda_k & q \\ 0 & q & 1-\lambda_k \end{bmatrix}\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}=\vec{0}. \tag{15}$$

Thus,

$$(r^2-\lambda_k)x_k+py_k \;=\; 0, \tag{16}$$

$$\Rightarrow \quad \frac{x_k}{y_k} \;=\; \frac{p}{\lambda_k-r^2}, \tag{17}$$

$$px_k+(r-\lambda_k)y_k+qz_k \;=\; 0, \tag{18}$$

$$qy_k+(1-\lambda_k)z_k \;=\; 0 \quad , \tag{19}$$

$$\Rightarrow \quad \frac{y_k}{z_k} \;=\; \frac{\lambda_k-1}{q}. \tag{20}$$

And multiplying (11) and (14) gives:

$$\frac{x_k}{z_k}=\frac{p(\lambda_k-1)}{q(\lambda_k-r^2)}. \tag{21}$$

Recall that the $\vec{v}_k$ make up the rows of U so:

$$U=\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}. \tag{22}$$

And as U is a unitary matrix we can write (since we have assumed real parameters i.e. the phase $\delta=0$ and therefore $e^{i\delta}=1$):

$$U=\begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13} & c_{12}c_{23}-s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13} & -c_{12}s_{23}-s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix}, \tag{23}$$

where $s_{ij}=\sin\theta_{ij}$, $c_{ij}=\cos\theta_{ij}$ and the angles θ_{ij} can lie in any of the four quadrants, so $-1\leq s_{ij}\leq 1$, $-1\leq c_{ij}\leq 1$.

Now we have:

From (11):

$$\frac{\lambda_1-r^2}{p}=\frac{y_1}{x_1}=\frac{s_{12}c_{13}}{c_{12}c_{13}}=\tan\theta_{12}, \tag{24}$$

$$\Rightarrow \quad \theta_{12}=\arctan\Big(\frac{\lambda_1-r^2}{p}\Big), \tag{25}$$

From (15):

$$\frac{q(\lambda_1-r^2)}{p(\lambda_1-1)}=\frac{z_1}{x_1}=\frac{s_{13}}{c_{12}c_{13}}, \tag{26}$$

$$\Rightarrow \quad \theta_{13}=\arctan\Big(\frac{c_{12}q(\lambda_1-r^2)}{p(\lambda_1-1)}\Big), \tag{27}$$

From (14):

$$\frac{q}{\lambda_2-1}=\frac{z_2}{y_2}=\frac{s_{23}c_{13}}{c_{12}c_{23}-s_{12}s_{23}s_{13}}\approx\frac{s_{23}c_{13}}{c_{12}c_{23}}=\frac{c_{13}\tan\theta_{23}}{c_{12}}, \tag{28}$$

$$\Rightarrow \quad \theta_{23}=\arctan\Big(\frac{c_{12}q}{c_{13}(\lambda_2-1)}\Big). \tag{29}$$

So now we can substitute the θ_{ij} into the U_i to get the U_i as functions of λ_1^i, λ_2^i, and r_i and we can easily construct the CKM matrix:

$$V(ru,rd,\lambda_1^u,\lambda_2^u,\lambda_1^d,\lambda_2^d) = U_u(r_u,\lambda_1^u,\lambda_2^u)U_d(r_d,\lambda_1^d,\lambda_2^d)^T. \tag{30}$$

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Tests

To test the accuracy of our constructed CKM matrix, V, we evaluate the χ^2 of the elements of the matrix against their values given by the PDG [1]. Due to the orthogonality of V we only test V_{12}, V_{13}, and V_{23}. We take the observed values of the CKM matrix to be:

$$|V_{ij}^{measured}| = \begin{bmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{bmatrix}. \tag{31}$$

We also define:

$$\sigma_{12} = 0.0025,\ \sigma_{13} = 0.0006, \text{ and } \sigma_{23} = 0.004. \tag{32}$$

And now

$$\chi^2 = \sum_{ij} \frac{(|V_{ij}| - |V_{ij}^{measured}|)^2}{\sigma_{ij}^2} \quad (ij = 12,\ 13,\ 23). \tag{33}$$

We take the quark masses as known values from the PDG [1]:

$$m_u = 0.0023,\ m_c = 1.275,\ m_t = 173.21,\ m_d = 0.0048,\ m_s = 0.095,\ m_b = 4.66. \tag{34}$$

Now complications arise as we do not know the signs of the eigenvalues or the p_i and q_i. So there are 256 possible combinations of signs. But due to the invariance of the D_i and V under unitary transformations on the M_i of the form $M_i' = U^\dagger M_i U$ (see Appendix: A), we have that V is unchanged by two transformations: i) flipping the signs of both p_u and p_d, and ii) flipping the signs of both q_u and q_d. Therefore every solution that we obtain will be in a set of four equivalent solutions (the three extra solutions gained by the transformations i), ii), i) and ii)).

Since we have assumed $p_i,\ q_i \in$ IR we have $p_i^2 > 0$, $\quad q_i^2 > 0$. We can use this to obtain allowed ranges for the r_i (we also assume $|r_i| \leq 0.5$). We use Wolfram Mathematica to gain solutions of $Min(\chi^2) < 20$ within these bounds, keeping p_u and q_u positive and varying the signs of λ_1^u, λ_2^u, λ_1^d, λ_2^d, p_d, q_d, so that each returned solution represents a set of 4 solutions gained by the above described transformations.

Recall, we have the assumed values: $|\lambda_1^u| = \frac{m_u}{m_t} = 0.0000132787$, $|\lambda_2^u| = \frac{m_c}{m_t} = 0.00736101$, $|\lambda_1^d| = \frac{m_d}{m_b} = 0.00103004$, $|\lambda_2^d| = \frac{m_s}{m_b} = 0.0203863$.

The headings (e.g. (+,-,+,-,+,+)) refer to the signs of (λ_1^u -ve,λ_2^u +ve,λ_1^d +ve,λ_2^d +ve, p_d -ve, q_d +ve) respectively.

Table 1: Results

	(-,+,+,+,-,+)	(+,-,-,+,-,+)	(+,-,+,+,+,+)	(+,+,+,+,-,+)
p_u	1.23×10^{-3}	2.67×10^{-4}	2.51×10^{-4}	1.16×10^{-3}
q_u	0.0832	0.0731	0.0976	0.0835
p_d	-1.07×10^{-3}	-5.33×10^{-3}	3.57×10^{-3}	-1.26×10^{-3}
q_d	0.113	0.0289	0.148	0.114
$Min(\chi^2)$	17.9	10.3	12.4	15.3
Min point (r_u,r_d)	(0.0141, 0.0330)	(-2.01×10^{-3}, 0.0198)	(2.20×10^{-3}, 0.0414)	(0.0141, 0.0334)
rubounds	$7.30\times10^{-3} \leq r_u \leq 0.082$	$-3.64\times10^{-3} \leq r_u \leq 3.64\times10^{-3}$	$-3.64\times10^{-3} \leq r_u \leq 3.64\times10^{-3}$	$7.32\times10^{-3} \leq r_u \leq 0.0825$
rdbounds	$0.0321 \leq r_d \leq 0.134$	$0.0190 \leq r_d \leq 0.134$	$0.0321 \leq r_d \leq 0.134$	$0.0321 \leq r_d \leq 0.134$
V_{12}	0.225	0.223	-0.224	0.225
V_{13}	-5.50×10^{-3}	-1.70×10^{-3}	2.30×10^{-3}	-5.40×10^{-3}
V_{23}	-0.0304	0.0429	-0.0530	-0.0320
V_{31}	-1.50×10^{-3}	0.0112	9.60×10^{-3}	-1.90×10^{-3}

Treating the eigenvalues as variables

Set up

Now, instead of holding the eigenvalues as fixed values, we wish to find ansatza for

$$M_i = \begin{bmatrix} r_i^2 & p_i & 0 \\ p_i & r_i & q_i \\ 0 & q_i & 1 \end{bmatrix}, \quad i = u,d \tag{35}$$

by allowing the eigenvalues to vary. The approximation $\lambda_3^i = 1$ is automatically no longer used.

We take the λ_k as functions of p, q, and r, λ_1 being the smallest eigenvalue and λ_3 being the biggest. Now we want to find the U_i. Recall

$$U = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix}, \tag{36}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$ and the angles θ_{ij} can lie in any of the four quadrants, so $-1 \le s_{ij} \le 1$, $-1 \le c_{ij} \le 1$.

Again we have:

$$\theta_{12} = \arctan\left(\frac{\lambda_1 - r^2}{p}\right), \tag{37}$$

$$\theta_{13} = \arctan\left(\frac{c_{12}q(\lambda_1 - r^2)}{p(\lambda_1 - 1)}\right), \tag{38}$$

$$\theta_{23} = \arctan\left(\frac{c_{12}q}{c_{13}(\lambda_2 - 1)}\right). \tag{39}$$

So now, remembering that the λ_k^i are function of p_i, q_i, and r_i we can substitute the θ_{ij} into the U_i to get the U_i as functions of p_i, q_i, and r_i and we can easily construct the CKM matrix:

$$V(r_u,p_u,q_u r_d,p_d,q_d) \;=\; U_u(r_u,p_u,q_u)U_d(r_d,p_d,q_d)^T. \tag{40}$$

.

Tests

To test the accuracy of our constructed CKM matrix, V, we again evaluate the χ^2 of the elements of the matrix against their values given by the PDG. Due to the orthogonality of V we only test V_{12}, V_{13}, and V_{23}. We take the observed values of the CKM matrix to be:

$$|V_{ij}^{measured}| = \begin{bmatrix} 0.97427 & 0.22536 & 0.00355 \\ 0.22522 & 0.97343 & 0.0414 \\ 0.00886 & 0.0405 & 0.99914 \end{bmatrix}. \tag{41}$$

We also define:

$$\sigma_{12} = 0.0025,\ \sigma_{13} = 0.0006, \text{ and } \sigma_{23} = 0.004. \tag{42}$$

And now

$$\chi_v^2 = \sum_{ij} \frac{(|V_{ij}| - |V_{ij}^{measured}|)^2}{\sigma_{ij}^2} \quad (ij = 12,\ 13,\ 23). \tag{43}$$

We also test the accuracy of our values for the eigenvalues:

$$\chi^2_{\lambda_k} = \sum_k \frac{(|\lambda_k| - |\lambda_k^{measured}|)^2}{\sigma^2_{\lambda_k}} \quad (k = 1,\ 2,\ 3,\ 4), \tag{44}$$

where $\lambda_1 = \lambda_1^u$, $\lambda_2 = \lambda_2^u$, $\lambda_3 = \lambda_1^d$, $\lambda_4 = \lambda_2^d$, and:

$\sigma_{\lambda_k} = (0.000002, 0.0003, 0.00005, 0.0008)$.

We now use two methods to minimise $\chi^2 = \chi^2_v + \chi^2_{\lambda_k}$ with Wolfram Mathematica.

We can recover the solutions from the previous case:

Table 2: Recovered solutions

	1	2	3	4
λ_1^u	-1.31×10^{-5}	1.39×10^{-5}	1.39×10^{-5}	1.35×10^{-5}
λ_2^u	7.36×10^{-3}	-7.36×10^{-3}	-7.45×10^{-3}	7.37×10^{-3}
λ_3^u	1.01	1.01	1.01	1.01
r_u	0.0136	-1.91×10^{-3}	2.19×10^{-3}	0.0141
p_u	1.19×10^{-3}	2.75×10^{-4}	2.60×10^{-4}	1.16×10^{-3}
q_u	0.0798	0.0740	0.0985	0.0827
λ_1^d	1.01×10^{-3}	-1.06×10^{-3}	9.98×10^{-4}	1.01×10^{-3}
λ_2^d	0.0207	0.0201	0.0205	0.0204
λ_3^d	1.01	1.00	1.02	1.01
r_d	0.0329	0.0195	0.0413	0.0332
p_d	-1.24×10^{-3}	-5.33×10^{-3}	3.67×10^{-3}	-1.32×10^{-3}
q_d	0.110	0.0290	0.145	0.113
$Min(\chi^2)$	16.4	9.36	9.87	14.7
V_{12}	0.225	0.225	-0.225	0.225
V_{13}	-5.33×10^{-3}	-1.80×10^{-3}	2.24×10^{-3}	-5.19×10^{-3}
V_{23}	-0.0307	0.0438	-0.0497	-0.0308
V_{31}	-1.72×10^{-3}	0.0116	8.95×10^{-3}	-1.87×10^{-3}

And obtain new solutions:

Table 3: New solutions

	5	6	7	8
λ_1^u	1.40×10^{-5}	1.44×10^{-5}	1.40×10^{-5}	1.41×10^{-5}
λ_2^u	7.20×10^{-3}	-7.32×10^{-3}	7.22×10^{-3}	-7.36×10^{-3}
λ_3^u	1.00	1.01	1.00	1.01
r_u	8.31×10^{-3}	3.56×10^{-4}	7.21×10^{-3}	1.35×10^{-3}
p_u	-6.26×10^{-4}	3.24×10^{-4}	5.22×10^{-4}	-3.00×10^{-4}
q_u	0.0340	0.0878	5.09×10^{-3}	-0.0936
λ_1^d	9.26×10^{-4}	1.03×10^{-3}	9.42×10^{-4}	-1.04×10^{-3}
λ_2^d	-0.0221	-0.0204	-0.0218	0.0203
λ_3^d	1.00	1.00	1.00	1.00
r_d	-0.0216	-0.0178	-0.0198	0.0212
p_d	-3.22×10^{-3}	-3.84×10^{-3}	3.50×10^{-3}	5.45×10^{-3}
q_d	-4.85×10^{-3}	0.0437	-0.0390	-0.0487
$Min(\chi^2)$	10.4	6.32	7.96	6.64
V_{12}	-0.227	-0.225	0.227	-0.225
V_{13}	-3.37×10^{-3}	-2.13×10^{-3}	2.99×10^{-3}	-2.06×10^{-3}
V_{23}	0.0388	0.0440	0.0430	-0.0430
V_{31}	-5.52×10^{-3}	-7.84×10^{-3}	6.83×10^{-3}	0.0120

Complex case

Now we test

$$M_i = \begin{bmatrix} r_i^2 & p_i & 0 \\ p_i^* & r_i & q_i \\ 0 & q_i^* & 1 \end{bmatrix} = U_i^\dagger D_i U_i, \tag{45}$$

with

$$D_i = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_i, \tag{46}$$

where the r_i, λ_k^i are real and the p_i, q_i are complex.

We have $D_i = U_i M_i U_i^\dagger$. Let

$$\Phi_i = \begin{bmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{bmatrix}_i, \tag{47}$$

then, since the Φ_i are diagonal

$$D_i = \Phi_i D_i \Phi_i^\dagger = \Phi_i U_i M_i U_i^\dagger \Phi_i^\dagger = U_i^{'} M_i (U_i^{'})^\dagger. \tag{48}$$

I.E. the D_i are invariant under transformations of the form $U_i \to \Phi_i U_i$. So we can choose the Φ_i such that the diagonal elements of the U_i are real.

Note that $V = U_u U_d^\dagger$ is not invariant under such transformations, instead we have: $V^{'} = \Phi_u U_u U_d^\dagger \Phi_d^\dagger$. But the physics are unchanged (observable quantities such as $|V|$)...

We parameterise the complex (transformed) U_i with the general parameterisation (using the orthogonality of U_i)

$$U_i \approx \begin{bmatrix} 1 & U_{12} & U_{13} \\ -U_{12}^* & 1 & U_{23} \\ U_{31} & -U_{23}^* & 1 \end{bmatrix}_i. \tag{49}$$

Set up

We want to find the U_i, we start by finding the ratios of the elements of the eigenvectors. Dropping the i suffix and using $(M - \lambda_k I)\vec{v}_k = \vec{0}$ $(k = 1,2,3)$:

$$\begin{bmatrix} r^2 - \lambda_k & p & 0 \\ p^* & r - \lambda_k & q \\ 0 & q^* & 1 - \lambda_k \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \vec{0}. \tag{50}$$

Thus,

$$\frac{x_k}{y_k} = \frac{p}{\lambda_k - r^2}, \tag{51}$$

$$\frac{y_k}{z_k} = \frac{\lambda_k - 1}{q^*}, \tag{52}$$

$$\frac{x_k}{z_k} = \frac{p(\lambda_k - 1)}{q^*(\lambda_k - r^2)}. \tag{53}$$

The eigenvectors make up the columns of $U^\dagger$ and therefore make up the rows of U^*:

$$U^* = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \Rightarrow U = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^*. \tag{54}$$

Hence,

$$U \approx \begin{bmatrix} 1 & U_{12} & U_{13} \\ -U_{12}^* & 1 & U_{23} \\ U_{31} & -U_{23}^* & 1 \end{bmatrix} \approx \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^*, \tag{55}$$

where $U_{12} = |U_{12}|e^{i\theta_{12}}$ etc.

So, From (51):

$$-U_{12}^* = (\frac{x_2}{y_2})^* = \frac{p^*}{\lambda_2 - r^2} = \frac{|p|e^{-i\theta_p}}{\lambda_2 - r^2}, \tag{56}$$

$$\Rightarrow |U_{12}|e^{-i\theta_{12}} = \frac{|p|e^{i(\pi - \theta_p)}}{\lambda_2 - r^2}, \tag{57}$$

and since $\lambda_2 \gg r^2$:

$$|U_{12}| = \frac{|p|}{\lambda_2 - r^2}, \text{ and } \quad \theta_{12} = \pi + \theta_p. \tag{58}$$

From (52):

$$U_{23} = (\frac{z_2}{y_2})^* = \frac{q}{\lambda_2 - 1} = \frac{|q|e^{i\theta_q}}{\lambda_2 - 1} = \frac{|q|e^{i(\pi + \theta_q)}}{1 - \lambda_2}, \tag{59}$$

$$\Rightarrow |U_{23}|e^{i\theta_{23}} = \frac{|q|e^{i(\pi + \theta_q)}}{1 - \lambda_2}, \tag{60}$$

and since $1 \gg \lambda_2$:

$$|U_{23}| = \frac{|q|}{1 - \lambda_2}, \text{ and } \quad \theta_{23} = \pi + \theta_q. \tag{61}$$

Now:

$$\frac{U_{13}}{U_{12}} = (\frac{z_1}{y_1})^* = \frac{q}{\lambda_1 - 1} = \frac{|q|e^{i(\pi + \theta_q)}}{1 - \lambda_1}, \tag{62}$$

hence:

$$|U_{13}|e^{i\theta_{13}} = \frac{|q|e^{i(\pi + \theta_q)}}{1 - \lambda_1} \frac{|p|}{\lambda_2 - r^2} e^{i(\pi + \theta_p)}, \tag{63}$$

and since $1 \gg \lambda_1$ and $\lambda_2 \gg r^2$:

$$|U_{13}| = \frac{|q||p|}{(1 - \lambda_1)(\lambda_2 - r^2)}, \text{ and } \quad \theta_{13} = \theta_q + \theta_p. \tag{64}$$

Furthermore:

$$\frac{U_{31}}{-U_{23}^*} = (\frac{x_3}{y_3})^* = \frac{p^*}{\lambda_3 - r^2} = \frac{|p|e^{-i\theta_p}}{\lambda_3 - r^2}, \tag{65}$$

hence:

$$|U_{31}|e^{i\theta_{31}}=\frac{|p|e^{-i\theta_p}}{\lambda_3-r^2}\left(-\frac{|q|e^{-i(\pi+\theta_q)}}{1-\lambda_2}\right)=\frac{|p||q|e^{-i(\theta_q+\theta_p)}}{(\lambda_3-r^2)(1-\lambda_2)}, \tag{66}$$

and since $1\gg\lambda_2$ and $\lambda_3\gg r^2$:

$$|U_{31}|=\frac{|p||q|}{(\lambda_3-r^2)(1-\lambda_2)}, \text{ and } \quad \theta_{31}=-(\theta_q+\theta_p). \tag{67}$$

Finally:

$$U\approx\begin{bmatrix}1 & -\frac{|p|}{\lambda_2-r^2}e^{\theta_p} & \frac{|q||p|}{(1-\lambda_1)(\lambda_2-r^2)}e^{i(\theta_q+\theta_p)}\\ -U_{12}^* & 1 & -\frac{|q|}{1-\lambda_2}e^{i\theta_q}\\ \frac{|p||q|}{(\lambda_3-r^2)(1-\lambda_2)}e^{-i(\theta_q+\theta_p)} & -U_{23}^* & 1\end{bmatrix}=\begin{bmatrix}1 & -\frac{p}{\lambda_2-r^2} & \frac{qp}{(1-\lambda_1)(\lambda_2-r^2)}\\ -U_{12}^* & 1 & -\frac{q}{1-\lambda_2}\\ \frac{p^*q^*}{(\lambda_3-r^2)(1-\lambda_2)} & -U_{23}^* & 1\end{bmatrix}, \tag{68}$$

thus, as $\lambda_3\approx 1$, $\lambda_2\gg r^2$, $\lambda_1\ll 1$:

$$U\approx\begin{bmatrix}1 & -\frac{p}{\lambda_2} & \frac{qp}{\lambda_2}\\ \frac{p^*}{\lambda_2} & 1 & -q\\ p^*q^* & q^* & 1\end{bmatrix}. \tag{69}$$

We can now construct the CKM matrix, $V=U_uU_d^\dagger$:

$$V\approx\begin{bmatrix}1 & -\frac{p}{\lambda_2} & \frac{qp}{\lambda_2}\\ \frac{p^*}{\lambda_2} & 1 & -q\\ p^*q^* & q^* & 1\end{bmatrix}_u\begin{bmatrix}1 & \frac{p}{\lambda_2} & qp\\ -\frac{p^*}{\lambda_2} & 1 & q\\ \frac{p^*q^*}{\lambda_2} & -q^* & 1\end{bmatrix}_d, \tag{70}$$

$$\approx\begin{bmatrix}1 & \frac{p_d}{\lambda_2^d}-\frac{p_u}{\lambda_2^u} & \frac{p_u(q_u-q_d)}{\lambda_2^u}\\ \frac{p_u^*}{\lambda_2^u}-\frac{p_d^*}{\lambda_2^d} & 1 & q_d-q_u\\ \frac{p_d^*(q_d^*-q_u^*)}{\lambda_2^d} & q_u^*-q_d^* & 1\end{bmatrix}, \tag{71}$$

hence,

$$V\approx\begin{bmatrix}1 & U_{12}^u-U_{12}^d & U_{12}^u(V_{23})\\ -V_{12}^* & 1 & U_{23}^u-U_{23}^d\\ (-U_{12}^d(V_{23}))^* & -V_{23}^* & 1\end{bmatrix}. \tag{72}$$

Tests

To test the accuracy of our constructed CKM matrix, V, we again evaluate the χ^2 of the elements of the matrix against their values given by the PDG. Due to the orthogonality of V we only test V_{12}, V_{13}, V_{31}, and V_{23}. We take the observed values of the CKM matrix to be:

$$|V_{ij}^{measured}|=\begin{bmatrix}0.97427 & 0.22536 & 0.00355\\ 0.22522 & 0.97343 & 0.0414\\ 0.00886 & 0.0405 & 0.99914\end{bmatrix}. \tag{73}$$

We calculate numerical exact values for our constructed V using mathematica.

We also define:

$$\sigma_{12}=0.0025,\ \sigma_{13}=0.0006,\ \sigma_{31}=0.001 \text{ and } \sigma_{23}=0.004. \tag{74}$$

And now

$$\chi_v^2=\sum_{ij}\frac{(|V_{ij}|-|V_{ij}^{measured}|)^2}{\sigma_{ij}^2} \quad (ij=12,\ 13,\ 23,\ 31). \tag{75}$$

We also test the accuracy of our values for the eigenvalues:

$$\chi^2_{\lambda_k}=\sum_k \frac{(|\lambda_k|-|\lambda_k^{measured}|)^2}{\sigma^2_{\lambda_k}} \quad (k=1,\,2,\,3,\,4), \tag{76}$$

where $\lambda_1=\lambda_1^u$, $\lambda_2=\lambda_2^u$, $\lambda_3=\lambda_1^d$, $\lambda_4=\lambda_2^d$, and:

$\sigma_{\lambda_k}=(0.000002,0.0003,0.00005,0.0008)$.

We now minimise $\chi^2=\chi^2_v+\chi^2_{\lambda_k}$. Note that, since we have two unphysical phase degrees of freedom, we can fix a phase convention such that $Arg(p_u)=-Arg(p_d)$ and $Arg(q_u)=-Arg(q_d)$ (see Appendix: A).

And obtain new solutions:

<u>Solution 9 - $\chi^2=5.68\times10^{-1}$</u>

$$|M_u|=\begin{bmatrix}7.8\times10^{-5} & 6.9\times10^{-4} & 0\\ 6.9\times10^{-4} & 8.8\times10^{-3} & 3.8\times10^{-2}\\ 0 & 3.8\times10^{-2} & 1\end{bmatrix},\quad |M_d|=\begin{bmatrix}1.7\times10^{-4} & 4.3\times10^{-3} & 0\\ 4.3\times10^{-3} & -1.3\times10^{-2} & 8.1\times10^{-2}\\ 0 & 8.1\times10^{-2} & 1\end{bmatrix} \tag{77}$$

<u>Solution 10 - $\chi^2=1.56$</u>

$$|M_u|=\begin{bmatrix}5.5\times10^{-5} & 7.1\times10^{-4} & 0\\ 7.1\times10^{-4} & 7.4\times10^{-3} & 1.1\times10^{-2}\\ 0 & 1.1\times10^{-2} & 1\end{bmatrix},\quad |M_d|=\begin{bmatrix}4.6\times10^{-4} & 5.4\times10^{-3} & 0\\ 5.4\times10^{-3} & 2.1\times10^{-2} & 4.8\times10^{-2}\\ 0 & 4.8\times10^{-2} & 1\end{bmatrix} \tag{78}$$

<u>Solution 11 - $\chi^2=2.97$</u>

$$|M_u|=\begin{bmatrix}7.7\times10^{-5} & 6.8\times10^{-4} & 0\\ 6.8\times10^{-4} & 8.8\times10^{-3} & 4.0\times10^{-2}\\ 0 & 4.0\times10^{-2} & 1\end{bmatrix},\quad |M_d|=\begin{bmatrix}3.9\times10^{-4} & 3.6\times10^{-3} & 0\\ 3.6\times10^{-3} & -2.0\times10^{-2} & 4.1\times10^{-3}\\ 0 & 4.1\times10^{-3} & 1\end{bmatrix} \tag{79}$$

<u>Solution 12 - $\chi^2=1.43$</u>

$$|M_u|=\begin{bmatrix}7.6\times10^{-5} & 6.8\times10^{-4} & 0\\ 6.8\times10^{-4} & 8.7\times10^{-3} & 3.7\times10^{-2}\\ 0 & 3.7\times10^{-2} & 1\end{bmatrix},\quad |M_d|=\begin{bmatrix}3.6\times10^{-4} & 5.3\times10^{-3} & 0\\ 5.3\times10^{-3} & 1.9\times10^{-2} & 7.1\times10^{-4}\\ 0 & 7.1\times10^{-4} & 1\end{bmatrix} \tag{80}$$

<u>Solution 13 - $\chi^2=1.41$</u>

$$|M_u|=\begin{bmatrix}5.3\times10^{-5} & 7.0\times10^{-4} & 0\\ 7.0\times10^{-4} & 7.3\times10^{-3} & 3.9\times10^{-3}\\ 0 & 3.9\times10^{-3} & 1\end{bmatrix},\quad |M_d|=\begin{bmatrix}4.1\times10^{-4} & 5.4\times10^{-3} & 0\\ 5.4\times10^{-3} & 2.0\times10^{-2} & 3.4\times10^{-2}\\ 0 & 3.4\times10^{-2} & 1\end{bmatrix} \tag{81}$$

Table 4: New solutions

	9	10	11	12	13
λ_1^u	1.31×10^{-5}	-1.33×10^{-5}	1.37×10^{-5}	1.33×10^{-5}	-1.33×10^{-5}
λ_2^u	7.42×10^{-3}	7.35×10^{-3}	7.26×10^{-3}	7.37×10^{-3}	7.36×10^{-3}
r_u	8.82×10^{-3}	7.41×10^{-3}	8.77×10^{-3}	8.71×10^{-3}	7.3×10^{-3}
r_u^2	7.78×10^{-5}	5.49×10^{-5}	7.69×10^{-5}	7.59×10^{-5}	5.34×10^{-5}
$Re(p_u)$	-4.96×10^{-4}	-6.18×10^{-4}	-5.68×10^{-4}	-5.8×10^{-4}	6.05×10^{-4}
$Im(p_u)$	4.79×10^{-4}	-3.39×10^{-4}	-3.64×10^{-4}	3.48×10^{-4}	3.48×10^{-4}
$\|p_u\|$	6.9×10^{-4}	7.05×10^{-4}	6.75×10^{-4}	6.76×10^{-4}	6.98×10^{-4}
$Arg(p_u)$	2.37	3.64	3.71	2.6	5.22×10^{-1}
$Re(q_u)$	3.8×10^{-2}	-1.09×10^{-2}	-1.61×10^{-2}	8.98×10^{-3}	4.63×10^{-4}
$Im(q_u)$	-3.08×10^{-3}	2.55×10^{-4}	-3.61×10^{-2}	3.63×10^{-2}	3.88×10^{-3}
$\|q_u\|$	3.82×10^{-2}	1.09×10^{-2}	3.95×10^{-2}	3.74×10^{-2}	3.91×10^{-3}
$Arg(q_u)$	-8.08×10^{-2}	3.12	4.29	1.33	1.45
λ_1^d	1.05×10^{-3}	-1.02×10^{-3}	1.02×10^{-3}	-1.02×10^{-3}	-1.02×10^{-3}
λ_2^d	-2.02×10^{-2}	2.04×10^{-2}	-2.04×10^{-2}	2.04×10^{-2}	2.04×10^{-2}
r_d	-1.29×10^{-2}	2.13×10^{-2}	-1.98×10^{-2}	1.91×10^{-2}	2.02×10^{-2}
r_d^2	1.66×10^{-4}	4.55×10^{-4}	3.9×10^{-4}	3.63×10^{-4}	4.07×10^{-4}
$Re(p_d)$	-3.06×10^{-3}	-4.77×10^{-3}	-3.05×10^{-3}	-4.52×10^{-3}	4.64×10^{-3}
$Im(p_d)$	-2.95×10^{-3}	2.62×10^{-3}	1.96×10^{-3}	-2.72×10^{-3}	-2.67×10^{-3}
$\|p_d\|$	4.25×10^{-3}	5.44×10^{-3}	3.63×10^{-3}	5.28×10^{-3}	5.36×10^{-3}
$Re(q_d)$	8.09×10^{-2}	-4.81×10^{-2}	-1.68×10^{-3}	1.7×10^{-4}	3.98×10^{-3}
$Im(q_d)$	6.55×10^{-3}	-1.12×10^{-3}	3.78×10^{-3}	-6.87×10^{-4}	-3.34×10^{-2}
$\|q_d\|$	8.11×10^{-2}	4.81×10^{-2}	4.14×10^{-3}	7.07×10^{-4}	3.37×10^{-2}
Min	5.68×10^{-1}	1.56	2.97	1.43	1.41
$\|V_{12}\|$	2.25×10^{-1}	2.25×10^{-1}	2.25×10^{-1}	2.25×10^{-1}	2.25×10^{-1}
$\|V_{13}\|$	3.9×10^{-3}	3.53×10^{-3}	3.94×10^{-3}	3.51×10^{-3}	3.53×10^{-3}
$\|V_{23}\|$	4.18×10^{-2}	3.78×10^{-2}	4.24×10^{-2}	3.81×10^{-2}	3.8×10^{-2}
$\|V_{31}\|$	8.87×10^{-3}	9.72×10^{-3}	7.34×10^{-3}	9.71×10^{-3}	9.67×10^{-3}
$x=\|V_{13}/V_{23}\|$	9.34×10^{-2}	9.33×10^{-2}	9.29×10^{-2}	9.2×10^{-2}	9.3×10^{-2}
$y=\|V_{31}/V_{23}\|$	2.13×10^{-1}	2.57×10^{-1}	1.73×10^{-1}	2.55×10^{-1}	2.55×10^{-1}
$\|p_u\|/\|q_u\|$	1.81×10^{-2}	6.47×10^{-2}	1.71×10^{-2}	1.81×10^{-2}	1.78×10^{-1}
$\|p_d\|/\|q_d\|$	5.24×10^{-2}	1.13×10^{-1}	8.77×10^{-1}	7.46	1.59×10^{-1}
$\|p_u\|/\|p_d\|$	1.62×10^{-1}	1.3×10^{-1}	1.86×10^{-1}	1.28×10^{-1}	1.3×10^{-1}
$\|q_u\|/\|q_d\|$	4.7×10^{-1}	2.27×10^{-1}	9.56	5.29×10^{1}	1.16×10^{-1}
$\sqrt{\|p_u\|\|q_u\|}$	5.13×10^{-3}	2.77×10^{-3}	5.16×10^{-3}	5.03×10^{-3}	1.65×10^{-3}
$\sqrt{\|p_d\|\|q_d\|}$	1.86×10^{-2}	1.62×10^{-2}	3.87×10^{-3}	1.93×10^{-3}	1.34×10^{-2}

Reducing parameters

We now aim to reduce the number of parameters by setting our ansatz to be:

$$M_u=\begin{bmatrix}(z\bar{\lambda}^3)^2 & z\bar{\lambda}^5 & 0\\ z\bar{\lambda}^5 & z\bar{\lambda}^3 & \bar{\lambda}^2\\ 0 & \bar{\lambda}^2 & 1\end{bmatrix},\quad M_d=\begin{bmatrix}(-\bar{\lambda}^3)^2 & -i\bar{\lambda}^4c/z & 0\\ i\bar{\lambda}^4c/z & -\bar{\lambda}^3 & \bar{\lambda}^2/z\\ 0 & \bar{\lambda}^2/z & 1\end{bmatrix} \tag{82}$$

Or equivalently:

$$M_u = \begin{bmatrix} (z\bar{\lambda}^3)^2 & 0 & 0 \\ 0 & z\bar{\lambda}^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \bar{\lambda}^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + z\bar{\lambda}^5 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{83}$$

$$M_d = \begin{bmatrix} (-\bar{\lambda}^3)^2 & 0 & 0 \\ 0 & -\bar{\lambda}^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\bar{\lambda}^2}{z} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{\bar{\lambda}^4 c}{z} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{84}$$

Essentially we have set:

$$r_u = z\bar{\lambda}^3, \quad r_d = -\bar{\lambda}^3, \tag{85}$$

$$|p_u| = z\bar{\lambda}^5, \quad |p_d| = \frac{\bar{\lambda}^4 c}{z}, \tag{86}$$

$$|q_u| = \bar{\lambda}^2, \quad |q_d| = \frac{\bar{\lambda}^2}{z}, \tag{87}$$

$$Arg(p_u) = \frac{\pi}{2}, \quad Arg(p_d) = 0, \tag{88}$$

$$\Rightarrow \Delta\theta = Arg(p_u) - Arg(p_d) = \frac{\pi}{2}, \tag{89}$$

$$Arg(q_u) = 0, \quad Arg(q_d) = 0, \tag{90}$$

$$\Rightarrow \Delta\phi = Arg(q_u) - Arg(q_d) = 0, \tag{91}$$

and reduced the number of parameters from eight to three.

Tests

Now we test this ansatz using the χ^2 as before. But now we test the quantities:

$$\lambda_1^u, \quad \lambda_2^u, \quad \lambda_1^d, \quad \lambda_2^d, \quad \lambda, \quad A, \quad \sin(2\beta), \quad \alpha, \tag{92}$$

where

$$\lambda = |V_{us}|, \tag{93}$$

$$A = \frac{|V_{cb}|}{\lambda^2}, \tag{94}$$

$$\beta = Arg(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}), \tag{95}$$

$$\alpha = Arg(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}). \tag{96}$$

We take the centered values and errors as (from the PDG):

$$\lambda_1^u = (6.3 \pm 0.5) \times 10^{-6}, \tag{97}$$

$$\lambda_2^u = (3.2 \pm 0.1) \times 10^{-3}, \tag{98}$$

$$\lambda_1^d = (1.08 \pm 0.05) \times 10^{-3}, \tag{99}$$

$$\lambda_2^d = (2.2 \pm 0.06) \times 10^{-2}, \tag{100}$$

$$\lambda = 0.2251 \pm 0.0005, \tag{101}$$

$$A = 0.95 \pm 0.05, \tag{102}$$

$$\sin(2\beta) = 0.691 \pm 0.017, \tag{103}$$

$$\alpha = 87.6^\circ \pm 3.5^\circ. \tag{104}$$

So we minimise the χ^2:

$$\chi^2 = \frac{(|\lambda_1^u| - 6.3\times10^{-6})^2}{(0.5\times10^{-6})^2} + \frac{(|\lambda_2^u| - 3.2\times10^{-3})^2}{(0.1\times10^{-3})^2} + \frac{(|\lambda_1^d| - 1.08\times10^{-3})^2}{(0.05\times10^{-3})^2} + \frac{(|\lambda_2^d| - 2.2\times10^{-2})^2}{(0.06\times10^{-2})^2} + \frac{(\lambda - 0.2251)^2}{(0.0005)^2} + \frac{(A - 0.95)^2}{(0.05)^2} + \frac{(|\sin(2\beta)| - 0.691)^2}{(0.017)^2} + \frac{(\alpha - \frac{73\pi}{150})^2}{(\frac{7\pi}{360})^2}, \tag{105}$$

We minimise the χ^2 with the parameters as functions of z, $\bar{\lambda}$, and c to find:

$$\chi^2 = 5.68 \quad \text{at} \quad (z, \quad \bar{\lambda}, \quad c) = (0.504, \quad 0.223, \quad 0.949). \tag{106}$$

For this solution the mass matrices become:

$$M_u = \begin{bmatrix} 3.11\times10^{-5} & 2.77\times10^{-4} & 0 \\ 2.77\times10^{-4} & 5.58\times10^{-3} & 4.97\times10^{-2} \\ 0 & 4.97\times10^{-2} & 1 \end{bmatrix}, \quad M_d = \begin{bmatrix} 1.22\times10^{-4} & 4.64\times10^{-3}i & 0 \\ -4.64\times10^{-3}i & -1.11\times10^{-2} & 9.85\times10^{-2} \\ 0 & 9.85\times10^{-2} & 1 \end{bmatrix}. \tag{107}$$

And the CKM matrix is:

$$|V| = \begin{bmatrix} 0.974 & 0.225 & 4.17\times10^{-3} \\ 0.225 & 0.973 & 4.62\times10^{-2} \\ 1.01\times10^{-2} & 4.53\times10^{-2} & 0.999 \end{bmatrix}, \tag{108}$$

and the eigenvalues become:

$$\lambda_1^u = 6.42\times10^{-6}, \tag{109}$$

$$\lambda_2^u = 3.13\times10^{-3}, \tag{110}$$

$$\lambda_1^d = 1.11\times10^{-3}, \tag{111}$$

$$\lambda_2^d = -2.16\times10^{-2}. \tag{112}$$

Examining Parameter Coefficients

Now we test the mass matrices:

$$M_u = \begin{bmatrix} (a\bar{\lambda}^3)^2 & b\bar{\lambda}^5 & 0 \\ b\bar{\lambda}^5 & a\bar{\lambda}^3 & c\bar{\lambda}^2 \\ 0 & c\bar{\lambda}^2 & 1 \end{bmatrix}, \quad M_d = \begin{bmatrix} (-d\bar{\lambda}^3)^2 & -ei\bar{\lambda}^4 & 0 \\ ei\bar{\lambda}^4 & -d\bar{\lambda}^3 & f\bar{\lambda}^2 \\ 0 & f\bar{\lambda}^2 & 1 \end{bmatrix}, \tag{113}$$

with the extra quantity $\frac{m_b}{m_t} = g\bar{\lambda}^3$.

And we minimise the χ^2 with respect to the quantities:

$$\lambda_1^u, \quad \lambda_2^u, \quad \lambda_1^d, \quad \lambda_2^d, \quad \lambda, \quad A, \quad \sin(2\beta), \quad \alpha, \quad \frac{m_b}{m_t}. \tag{114}$$

We take the centered values and errors as (from the PDG):

$$\lambda_1^u = (7.15 \pm 0.56) \times 10^{-6}, \tag{115}$$

$$\lambda_2^u = (3.49 \pm 0.091) \times 10^{-3}, \tag{116}$$

$$\lambda_1^d = (9.55 \pm 0.42) \times 10^{-4}, \tag{117}$$

$$\lambda_2^d = (1.93 \pm 0.15) \times 10^{-2}, \tag{118}$$

$$\frac{m_b}{m_t} = (1.59 \pm 0.016) \times 10^{-2}, \tag{119}$$

$$\lambda = 0.2251 \pm 0.0005, \tag{120}$$

$$A = 0.823 \pm 0.013, \tag{121}$$

$$\sin(2\beta) = 0.691 \pm 0.017, \tag{122}$$

$$\alpha = 87.6^\circ \pm 3.5^\circ. \tag{123}$$

We now fix $\bar{\lambda} = 0.2251$ and minimse the χ^2 with respect to the coefficients (a, b, c, d, e, f, g) to find:

$$\chi^2 = 1.76 \quad \text{at} \quad (a, b, c, d, e, f, g) = (0.5, 0.51, 0.94, 0.85, 1.58, 1.8, 1.39). \tag{124}$$

Now we note that:

$$a \approx b \approx \sin(\frac{\pi}{6}) = \frac{1}{2}, \tag{125}$$

$$c \approx d \approx \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \approx 0.866, \tag{126}$$

$$e \approx \frac{\cos^2(\frac{\pi}{6})}{\tan(\frac{\pi}{6})} = \frac{3}{4}\sqrt{3} \approx 1.3, \tag{127}$$

$$f \approx \frac{\cos(\frac{\pi}{6})}{\tan(\frac{\pi}{6})} = \frac{3}{2}, \tag{128}$$

$$g\approx\frac{1}{\cos^2(\frac{\pi}{6})}=\frac{2}{\sqrt{3}}\approx1.16, \tag{129}$$

χ ansatz

From these coefficient values we now set the ansatz:

$$M_u=\begin{bmatrix}(\sin(\chi)\bar{\lambda}^3)^2 & \sin(\chi)\bar{\lambda}^5 & 0\\ \sin(\chi)\bar{\lambda}^5 & \sin(\chi)\bar{\lambda}^3 & \cos(\chi)\bar{\lambda}^2\\ 0 & \cos(\chi)\bar{\lambda}^2 & 1\end{bmatrix},\quad M_d=\begin{bmatrix}(-\cos(\chi)\bar{\lambda}^3)^2 & -\frac{\cos^2(\chi)}{\tan(\chi)}i\bar{\lambda}^4 & 0\\ \frac{\cos^2(\chi)}{\tan(\chi)}i\bar{\lambda}^4 & -\cos(\chi)\bar{\lambda}^3 & \frac{\cos(\chi)}{\tan(\chi)}\bar{\lambda}^2\\ 0 & \frac{\cos(\chi)}{\tan(\chi)}\bar{\lambda}^2 & 1\end{bmatrix}, \tag{130}$$

and minimise the χ^2 with respect to the observables from (114) and measured values and errors as in (115) to (123), allowing χ and $\bar{\lambda}$ to vary, to find:

$$\chi^2=12.46 \quad \text{at} \quad (\chi,\,\bar{\lambda})=(0.483,\,0.231). \tag{131}$$

The mass matrices then become:

$$M_u=\begin{bmatrix}3.3\times10^{-5} & 3.07\times10^{-4} & 0\\ 3.07\times10^{-4} & 5.74\times10^{-3} & 4.73\times10^{-2}\\ 0 & 4.73\times10^{-2} & 1\end{bmatrix},\quad M_d=\begin{bmatrix}1.2\times10^{-4} & 4.28\times10^{-3}i & 0\\ 4.28\times10^{-3}i & -1.09\times10^{-2} & 9.03\times10^{-2}\\ 0 & 9.03\times10^{-2} & 1\end{bmatrix}, \tag{132}$$

with the CKM matrix:

$$|V|=\begin{bmatrix}0.974 & 0.225 & 3.62\times10^{-3}\\ 0.225 & 0.974 & 4.07\times10^{-2}\\ 8.89\times10^{-3} & 3.99\times10^{-2} & 0.999\end{bmatrix}, \tag{133}$$

and we have the observable quantities:

$$\lambda_1^u=5.99\times10^{-6}, \tag{134}$$

$$\lambda_2^u=3.52\times10^{-3}, \tag{135}$$

$$\lambda_1^d=1.03\times10^{-3}, \tag{136}$$

$$\lambda_2^d=-1.99\times10^{-2}, \tag{137}$$

$$\frac{m_b}{m_t}=1.58\times10^{-2}, \tag{138}$$

$$A=0.807, \tag{139}$$

$$\sin(2\beta)=0.707, \tag{140}$$

$$\alpha=82.92, \tag{141}$$

Appendices

Proof of invariance of D and V under unitary transformations on M

We have:

$$D_i = U_i M_i U_i^\dagger \quad \Rightarrow \quad M_i = U_i^\dagger D_i U_i, \tag{142}$$

and:

$$V = U_u U_d^\dagger. \tag{143}$$

Suppose we choose some other matrix $M_i^{'} = U^\dagger M_i U$, where U is some arbitrary unitary matrix.

Then

$$(U_i^{'})^\dagger D_i^{'} U_i^{'} = M_i^{'} = U^\dagger M_i U = U^\dagger U_i^\dagger D_i U_i U = (U_i^{'})^\dagger D_i U_i^{'}. \tag{144}$$

Hence $D_i^{'} = D_i$ i.e. the matrix that is obtained by diagonalising $M_i^{'}$ is the same as the matrix obtained by diagonalising M_i. That is, D_i is invariant under the transformation $M_i \to M_i^{'}$.

Now,

$$V = U_u U_d^\dagger. \tag{145}$$

Suppose we have $V^{'} = U_u^{'}(U_d^{'})^\dagger$.

Then

$$V^{'} = U_u^{'}(U_d^{'})^\dagger = U_u U U^\dagger U_d^\dagger = U_u I U_d^\dagger = U_u U_d^\dagger = V. \tag{146}$$

Thus V is also invariant under this change of mass matrices.

Recall

$$M_i = \begin{bmatrix} r_i^2 & p_i & 0 \\ p_i & r_i & q_i \\ 0 & q_i & 1 \end{bmatrix}, \tag{147}$$

and take

$$U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{148}$$

then

$$M_i^{'} = U^\dagger M_i U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_i^2 & p_i & 0 \\ p_i & r_i & q_i \\ 0 & q_i & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_i^2 & -p_i & 0 \\ -p_i & r_i & q_i \\ 0 & q_i & 1 \end{bmatrix}, \tag{149}$$

so V is invarient under the transformation $p_i \to -p_i$ i.e. changing the signs of both p_u and p_d. Similarly for

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \equiv q_i \to -q_i, \tag{150}$$

and

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv p_i \to -p_i \text{ and } q_i \to -q_i. \tag{151}$$

Similarly D and V are invariant under transformations of the form $M_i^{'}=\Phi^\dagger M_i\Phi$ and if we set:

$$\Phi=\begin{bmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_2} \end{bmatrix}, \tag{152}$$

then, with p_i and q_i complex:

$$M_i^{'}=\begin{bmatrix} r_i^2 & p_i e^{i\phi_1} & 0 \\ p_i^* e^{-i\phi_1} & r_i & q_i e^{-i\phi_2} \\ 0 & q_i^* e^{i\phi_2} & 1 \end{bmatrix}=\Phi^\dagger M_i\Phi=\Phi^\dagger U_i^\dagger D_i U_i\Phi=(U_i^{'})^\dagger D_i U_i^{'}. \tag{153}$$

Hence,

$$V^{'}=U_u^{'}(U_d^{'})^\dagger=U_u\Phi\Phi^\dagger U_d^\dagger=V. \tag{154}$$

So we have some choice of phase convention, for example, setting $\phi_1=-\frac{(Arg(p_u)+arg(p_d))}{2}\Rightarrow Arg(p_u)=-Arg(p_d)$. Similarly we can set $Arg(q_u)=-Arg(q_d)$.

Approximate observable quantities

In the complex case:

$$Det(M)=Det(D) \quad\Rightarrow\quad r^3-|q|^2r^2-|p|^2 \quad=\quad \prod_{n=1}^{3}\lambda_n\approx\lambda_1\lambda_2, \tag{155}$$

$$\sum_{ij=11,22,33}[M]_{ij}=\sum_{ij=11,22,33}[D]_{ij} \quad\Rightarrow\quad r^3+r^2+r-|q|^2-|p|^2 \quad=\quad \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3\approx\lambda_1\lambda_2+\lambda_2+\lambda_1. \tag{156}$$

Hence,

$$\sum_{ij=11,22,33}[M]_{ij}-Det(M) \quad=\quad |q|^2(r^2-1)+r^2+r\approx\lambda_1+\lambda_2, \tag{157}$$

and since $\lambda_1\ll\lambda_2$ and $r^2\ll r$ and $1-r^2\approx 1$:

$$\lambda_2\approx r-|q|^2, \tag{158}$$

and substituting λ_2 into (90):

$$\lambda_1\approx r^2-\frac{|p|^2}{r-|q|^2}. \tag{159}$$

And since:

$$V\approx\begin{bmatrix} 1 & \frac{p_d}{\lambda_2^d}-\frac{p_u}{\lambda_2^u} & \frac{p_u(q_u-q_d)}{\lambda_2^u} \\ \frac{p_u^*}{\lambda_2^u}-\frac{p_d^*}{\lambda_2^d} & 1 & q_d-q_u \\ \frac{p_d^*(q_d^*-q_u^*)}{\lambda_2^d} & q_u^*-q_d^* & 1 \end{bmatrix}, \tag{160}$$

we have:

$$V \approx \begin{bmatrix} 1 & \frac{p_d}{r_d-|q_d|^2}-\frac{p_u}{r_u-|q_u|^2} & \frac{p_u(q_u-q_d)}{r_u-|q_u|^2} \\ \frac{p_u^*}{r_u-|q_u|^2}-\frac{p_d^*}{r_d-|q_d|^2} & 1 & q_d-q_u \\ \frac{p_d^*(q_d^*-q_u^*)}{r_d-|q_d|^2} & q_u^*-q_d^* & 1 \end{bmatrix}. \tag{161}$$

Hence:

$$|V_{12}| \approx |\frac{p_d}{r_d-|q_d|^2}-\frac{p_u}{r_u-|q_u|^2}|, \tag{162}$$

$$|V_{13}| \approx |\frac{p_u(q_u-q_d)}{r_u-|q_u|^2}|, \tag{163}$$

$$|V_{23}| \approx |q_d-q_u|, \tag{164}$$

$$|V_{31}| \approx |\frac{p_d^*(q_d^*-q_u^*)}{r_d-|q_d|^2}|. \tag{165}$$

Proof of Commutator/J relation

First let the commutator, C , be defined:

$$C=[M_u,M_d]=M_uM_d-M_dM_u, \tag{166}$$

for any two hermitian matrices, M_u and M_d. Then C is anti-hermitian, i.e. $C=-C^\dagger$.

Proof:

$$-C^\dagger=-(M_uM_d-M_dM_u)^\dagger=-M_d^\dagger M_u^\dagger+M_u^\dagger M_d^\dagger=M_uM_d-M_dM_u=C \quad \Box \tag{167}$$

Now define the Jarlskogg invariant, J, as

$$J=\pm Im(V_{i\alpha}V_{i\beta}^*V_{j\beta}V_{j\alpha}^*). \tag{168}$$

Then,

$$Det(C)=2i\Delta_u\Delta_dJ, \tag{169}$$

where $\Delta_i=(\lambda_1^i-\lambda_2^i)(\lambda_2^i-\lambda_3^i)(\lambda_3^i-\lambda_1^i)$.

Proof

First we perform a unitary transformation on M_u so that:

$$M_u \to M_u^{'}=UM_uU^\dagger=D_u, \tag{170}$$

where D_u is diagonal. To keep $Det(C)$ invariant we perform the same transformation on M_d:

$$M_d \to M_d^{'}=UM_dU^\dagger=UU_d^\dagger D_dU_dU^\dagger=VD_dV^\dagger, \tag{171}$$

as $U=U_u$ is the unitary matrix which diagonalises M_u and $V=U_uU_d^\dagger$.

So C becomes:

$$C=D_uVD_dV^\dagger-VD_dV_\dagger D_u \tag{172}$$

.

Recall: $D_i = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}_i$.

Hence, $C_{ij} = (\lambda_i^u - \lambda_j^u)\sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* \lambda_\alpha^d$.

Now, by the orthogonality of V we have:

$$\sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* = 0 \Rightarrow \sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* \lambda_\beta^d = 0 \tag{173}$$

.

So

$$C_{ij} = (\lambda_i^u - \lambda_j^u)(\sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* \lambda_\alpha^d - \sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* \lambda_\beta^d) = (\lambda_i^u - \lambda_j^u)(\sum_{\alpha=1}^{3} (\lambda_\alpha^d - \lambda_\beta^d) V_{i\alpha}V_{j\alpha}^*) \tag{174}$$

, for any β so we choose $\beta \in 1,2,3$ such that $\beta \neq i, \beta \neq j$. Cleary if $i = j$ or $\alpha = \beta$ then $C_{ij} = 0$.

Hence,

$$Det(C) = C_{12}C_{23}C_{31} + C_{21}C_{32}C_{13} = C_{12}C_{23}C_{31} - (C_{12}C_{23}C_{31})^* \tag{175}$$

as $C = -C^\dagger$. So,

$$Det(C) = 2iIm(C_{12}C_{23}C_{31}) \tag{176}$$

.

From (109):

$$C_{12} = (\lambda_1^u - \lambda_2^u)[(\lambda_1^d - \lambda_3^d)V_{11}V_{21}^* + (\lambda_2^d - \lambda_3^d)V_{12}V_{22}^*], \tag{177}$$

$$C_{23} = (\lambda_2^u - \lambda_3^u)[(\lambda_2^d - \lambda_1^d)V_{22}V_{32}^* + (\lambda_3^d - \lambda_1^d)V_{23}V_{33}^*], \tag{178}$$

$$C_{31} = (\lambda_3^u - \lambda_1^u)[(\lambda_1^d - \lambda_2^d)V_{31}V_{11}^* + (\lambda_3^d - \lambda_2^d)V_{33}V_{13}^*], \tag{179}$$

Hence,

$$\begin{aligned}
Det[C] = 2iIm(C_{12}C_{23}C_{31}) = 2iIm((\lambda_1^u - \lambda_2^u)(\lambda_2^u - \lambda_3^u)(\lambda_3^u - \lambda_1^u)[(\lambda_1^d - \lambda_3^d)V_{11}V_{21}^* \\
+ (\lambda_2^d - \lambda_3^d)V_{12}V_{22}^*][(\lambda_2^d - \lambda_1^d)V_{22}V_{32}^* + (\lambda_3^d - \lambda_1^d)V_{23}V_{33}^*][(\lambda_1^d - \lambda_2^d)V_{31}V_{11}^* + (\lambda_3^d - \lambda_2^d)V_{33}V_{13}^*]) \\
\\
= 2i\Delta_u Im([(\lambda_3^d - \lambda_1^d)(\lambda_1^d - \lambda_2^d)^2|V_{11}|^2 V_{21}^* V_{22}V_{32}^* V_{31} - (\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)V_{11}V_{21}^* V_{22}V_{32}^* V_{33}V_{13}^* \\
- (\lambda_1^d - \lambda_2^d)(\lambda_3^d - \lambda_1^d)^2|V_{11}|^2 V_{21}^* V_{23}V_{33}^* V_{31} \\
+ (\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)^2|V_{33}|^2 V_{11}V_{21}^* V_{23}V_{13}^* - (\lambda_2^d - \lambda_3^d)(\lambda_1^d - \lambda_2^d)^2|V_{22}|^2 V_{12}V_{32}^* V_{31}V_{11}^* \\
+ (\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)^2|V_{22}|^2 V_{12}V_{32}^* V_{33}V_{13}^* + (\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)V_{12}V_{22}^* V_{23}V_{33}^* V_{31}V_{11}^* \\
- (\lambda_3^d - \lambda_1^d)(\lambda_2^d - \lambda_3^d)^2|V_{33}|^2 V_{12}V_{22}^* V_{23}V_{13}^*]), \\
\\
= 2i\Delta_u Im((\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)[V_{12}V_{22}^* V_{23}V_{33}^* V_{31}V_{11}^* - V_{11}V_{21}^* V_{22}V_{32}^* V_{33}V_{13}^*] \\
+ |V_{11}|^2(\lambda_1^d - \lambda_2^d)(\lambda_3^d - \lambda_1^d)[(\lambda_1^d - \lambda_2^d)V_{22}V_{21}^* V_{31}V_{32}^* - (\lambda_3^d - \lambda_1^d)V_{23}V_{21}^* V_{31}V_{33}^*] \\
+ |V_{22}|^2(\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)[(\lambda_2^d - \lambda_3^d)V_{12}V_{32}^* V_{33}V_{13}^* - (\lambda_1^d - \lambda_2^d)V_{12}V_{32}^* V_{31}V_{11}^*] \\
+ |V_{33}|^2(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)[(\lambda_3^d - \lambda_1^d)V_{11}V_{21}^* V_{23}V_{13}^* - (\lambda_2^d - \lambda_3^d)V_{12}V_{22}^* V_{23}V_{13}^*]).
\end{aligned} \tag{180}$$

Ignore the $2i\Delta_u$ for now and split this into four terms:

$$t1 = Im(\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)[V_{12}V_{22}^* V_{23}V_{33}^* V_{31}V_{11}^* - V_{11}V_{21}^* V_{22}V_{32}^* V_{33}V_{13}^*], \tag{181}$$

$$t2 = |V_{11}|^2(\lambda_1^d - \lambda_2^d)(\lambda_3^d - \lambda_1^d)[(\lambda_1^d - \lambda_2^d)V_{22}V_{21}^* V_{31}V_{32}^* - (\lambda_3^d - \lambda_1^d)V_{23}V_{21}^* V_{31}V_{33}^*], \tag{182}$$

$$t3 = |V_{22}|^2(\lambda_1^d - \lambda_2^d)(\lambda_2^d - \lambda_3^d)[(\lambda_2^d - \lambda_3^d)V_{12}V_{32}^* V_{33}V_{13}^* - (\lambda_1^d - \lambda_2^d)V_{12}V_{32}^* V_{31}V_{11}^*], \tag{183}$$

$$t4 = |V_{33}|^2(\lambda_2^d - \lambda_3^d)(\lambda_3^d - \lambda_1^d)[(\lambda_3^d - \lambda_1^d)V_{11}V_{21}^* V_{23}V_{13}^* - (\lambda_2^d - \lambda_3^d)V_{12}V_{22}^* V_{23}V_{13}^*]. \tag{184}$$

Now recall $\sum_{\alpha=1}^{3} V_{i\alpha}V_{j\alpha}^* = 0 \Rightarrow V_{31}V_{33}^* = -V_{21}V_{23}^* - V_{11}V_{13}^*$ and $-V_{11}V_{13}^* = V_{21}V_{23}^* + V_{31}V_{33}^*$. Hence, $t1$ becomes:

$$\begin{aligned} t1 &= \Delta_d Im[-V_{12}V_{22}^* V_{23}V_{11}^*(V_{21}V_{23}^* - V_{11}V_{13}^*) + V_{21}^* V_{22}V_{32}^* V_{33}(V_{21}V_{23}^* + V_{31}V_{33}^*)] \\ &= \Delta_d[-|V_{23}|^2 Im(V_{12}V_{22}^* V_{21}V_{11}^*) - |V_{11}|^2 Im(V_{12}V_{22}^* V_{23}V_{13}^*) + |V_{21}|^2 Im(V_{22}V_{23}^* V_{33}V_{32}^*) + |V_{33}|^2 Im(V_{22}V_{21}^* V_{31}V_{32}^*)] \\ &= \Delta_d[-|V_{23}|^2(-J) - |V_{11}|^2 J + |V_{21}|^2 J + |V_{33}|^2(-J)] \\ &= \Delta_d J[|V_{23}|^2 - |V_{11}|^2 + |V_{21}|^2 - |V_{33}|^2]. \end{aligned} \tag{185}$$

Now since $J = \pm Im(V_{i\alpha}V_{i\beta}^* V_{j\beta}V_{j\alpha}^*)$,

$$\begin{aligned} t2 &= |V_{11}|^2(\lambda_1^d - \lambda_2^d)(\lambda_3^d - \lambda_1^d)[(\lambda_1^d - \lambda_2^d)V_{22}V_{21}^* V_{31}V_{32}^* - (\lambda_3^d - \lambda_1^d)V_{23}V_{21}^* V_{31}V_{33}^*] \\ &= |V_{11}|^2(\lambda_1^d - \lambda_2^d)(\lambda_3^d - \lambda_1^d)[(\lambda_1^d - \lambda_2^d)(-J) - (\lambda_3^d - \lambda_1^d)J] = |V_{11}|^2 \Delta_d J. \end{aligned} \tag{186}$$

Similarly:

$$t3 = |V_{22}|^2 \Delta_d J, \tag{187}$$

and

$$t4 = |V_{33}|^2 \Delta_d J, \tag{188}$$

Hence,

$$Det[C] = 2i\Delta_u \Delta_d J(|V_{23}|^2 - |V_{11}|^2 + |V_{21}|^2 - |V_{33}|^2 + |V_{11}|^2 + |V_{22}|^2 + |V_{33}|^2) = 2\Delta_u \Delta_d J(|V_{21}|^2 |V_{22}|^2 + |V_{23}|^2), \tag{189}$$

and since $\sum_{k=1}^{3} |V_{i\alpha}|^2 = 1$

$$Det[C] = 2i\Delta_u \Delta_d J \quad \square \tag{190}$$

Approximation for J

Recall from (70) we have:

$$V \approx \begin{bmatrix} 1 & -\frac{p}{\lambda_2} & \frac{qp}{\lambda_2} \\ \frac{p^*}{\lambda_2} & 1 & -q \\ p^* q^* & q^* & 1 \end{bmatrix}_u \begin{bmatrix} 1 & \frac{p}{\lambda_2} & qp \\ -\frac{p^*}{\lambda_2} & 1 & q \\ \frac{p^* q^*}{\lambda_2} & -q^* & 1 \end{bmatrix}_d, \tag{191}$$

$$\approx \begin{bmatrix} 1 + \frac{p_u p_d^* + q_u p_u q_d^* p_d^*}{\lambda_2^u \lambda_2^u} & \frac{p_d}{\lambda_2^d} - \frac{p_u}{\lambda_2^u} - \frac{q_u p_u q_d^*}{\lambda_2^u} & \frac{p_u(q_u - q_d)}{\lambda_2^u} + p_d q_d \\ \frac{p_u^*}{\lambda_2^u} - \frac{p_d^*}{\lambda_2^d} - \frac{q_u p_d^* q_d^*}{\lambda_2^d} & 1 + \frac{p_u^* p_d}{\lambda_2^u \lambda_2^d} + q_u q_d^* & q_d - q_u + \frac{p_u^* p_d q_d}{\lambda_2^u} \\ \frac{p_d^*(q_d^* - q_u^*)}{\lambda_2^d} + p_u^* q_u^* & q_u^* - q_d^* + \frac{p_u^* q_u^* p_d}{\lambda_2^d} & 1 + q_u^* q_d(p_u^* p_d + 1) \end{bmatrix}. \tag{192}$$

Hence we can construct J, from (108):

$$J = Im(V_{12}V_{13}^* V_{23}V_{22}^*), \tag{193}$$

$$J \approx \tilde{J} = Im[(\frac{p_d}{\lambda_2^d} - \frac{p_u}{\lambda_2^u} - \frac{q_u p_u q_d^*}{\lambda_2^u})(\frac{p_u(q_u - q_d)}{\lambda_2^u} + p_d q_d)^*(q_d - q_u + \frac{p_u^* p_d q_d}{\lambda_2^u})(1 + \frac{p_u^* p_d}{\lambda_2^u \lambda_2^d} + q_u q_d^*)^*], \tag{194}$$

$$\tilde{J}=Im[\frac{(p_d\lambda_2^u-p_u\lambda_2^d-q_up_uq_d^*\lambda_2^d)(p_u(q_u-q_d)+\lambda_2^up_dq_d)^*((q_d-q_u)\lambda_2^u+p_u^*p_dq_d)(p_u^*p_d+\lambda_2^u\lambda_2^d(1+q_uq_d^*))^*}{(\lambda_2^u)^4(\lambda_2^d)^2}], \quad (195)$$

Multiplying the leading order terms we get:

$$\tilde{J}=Im[\frac{(p_d\lambda_2^u-p_u\lambda_2^d)p_u^*(q_u-q_d)^*(q_d-q_u)}{(\lambda_2^u)^2\lambda_2^d}], \quad (196)$$

$$\tilde{J}=Im[\frac{|q_u-q_d|^2(|p_u|^2\lambda_2^d-p_u^*p_d\lambda_2^u)}{(\lambda_2^u)^2\lambda_2^d}], \quad (197)$$

$$\tilde{J}=Im[\frac{-|q_u-q_d|^2(|p_u||p_d|e^{-i\Delta\theta})}{\lambda_2^u\lambda_2^d}], \quad (198)$$

$$\tilde{J}=\frac{|q_u-q_d|^2|p_u||p_d|\sin\Delta\theta}{\lambda_2^u\lambda_2^d}, \quad (199)$$

And hence,

$$\tilde{J}=\frac{|V_{cb}|^2|p_u||p_d|\sin\Delta\theta}{\lambda_2^u\lambda_2^d}, \quad (200)$$

$$\tilde{J}=|V_{ub}||V_{td}|\sin\Delta\theta, \quad (201)$$

Now, the next highest order terms are:

$$a=Im[\frac{p_u^*p_dq_d(\lambda_2^up_d-\lambda_2^dp_u)\lambda_2^u\lambda_2^d(p_u^*(q_u^*-q_d^*))}{\lambda_2^u\lambda_2^d}], \quad (202)$$

$$b=-Im[\frac{\lambda_2^dp_uq_uq_d^*\lambda_2^u(q_d-q_u)\lambda_2^u\lambda_2^dp_u^*(q_u^*-q_d^*)}{\lambda_2^u\lambda_2^d}], \quad (203)$$

$$c=Im[\frac{(p_up_d^*+\lambda_2^u\lambda_2^dq_u^*q_d)(\lambda_2^u(q_d-q_u))(\lambda_2^up_d-\lambda_2^dp_u)(p_u^*(q_u^*-q_d^*))}{\lambda_2^u\lambda_2^d}], \quad (204)$$

$$d=Im[\frac{\lambda_2^up_d^*q_d^*\lambda_2^u(q_d-q_u)(\lambda_2^up_d-\lambda_2^dp_u)\lambda_2^u\lambda_2^d}{\lambda_2^u\lambda_2^d}]. \quad (205)$$

Which become:

$$a=\frac{|p_u|^2|p_d||q_d|(\lambda_2^u|p_d||q_d|\sin(2\Delta\theta)-\lambda_2^d|p_u||q_d|\sin(\Delta\theta)+\lambda_2^d|p_u||q_d|\sin(\Delta\theta+\Delta\phi)-\lambda_2^u|p_d||q_u|\sin(2\Delta\theta+\Delta\phi))}{(\lambda_2^u)^3\lambda_2^d}, \quad (206)$$

$$b=-\frac{|p_u|^2|q_u||q_d|(|q_u|^2+|q_d|^2-2|q_u||q_d|\cos(\Delta\phi))\sin(\Delta\theta)}{(\lambda_2^u)^2}, \quad (207)$$

$$c=\frac{|p_u|(|q_u|^2+|q_d|^2-2|q_u||q_d|\cos(\Delta\phi))(|p_u|^2|p_d|\sin(\Delta\theta)+\lambda_2^u|q_u||q_d|(\lambda_2^u|p_d|\sin(\Delta\theta+\Delta\phi)-\lambda_2^d|p_u|\sin(\Delta\phi)))}{(\lambda_2^u)^3\lambda_2^d}, \quad (208)$$

$$d=-\frac{|p_d||q_d|(\lambda_2^u|p_d||q_u|\sin(\Delta\phi)+\lambda_2^d|p_u||q_d|\sin(\Delta\theta)-\lambda_2^d|p_u||q_u|\sin(\Delta\theta+\Delta\phi))}{\lambda_2^u\lambda_2^d}. \quad (209)$$

Numerically, for our complex solutions, we can show that these higher order terms are negligible.

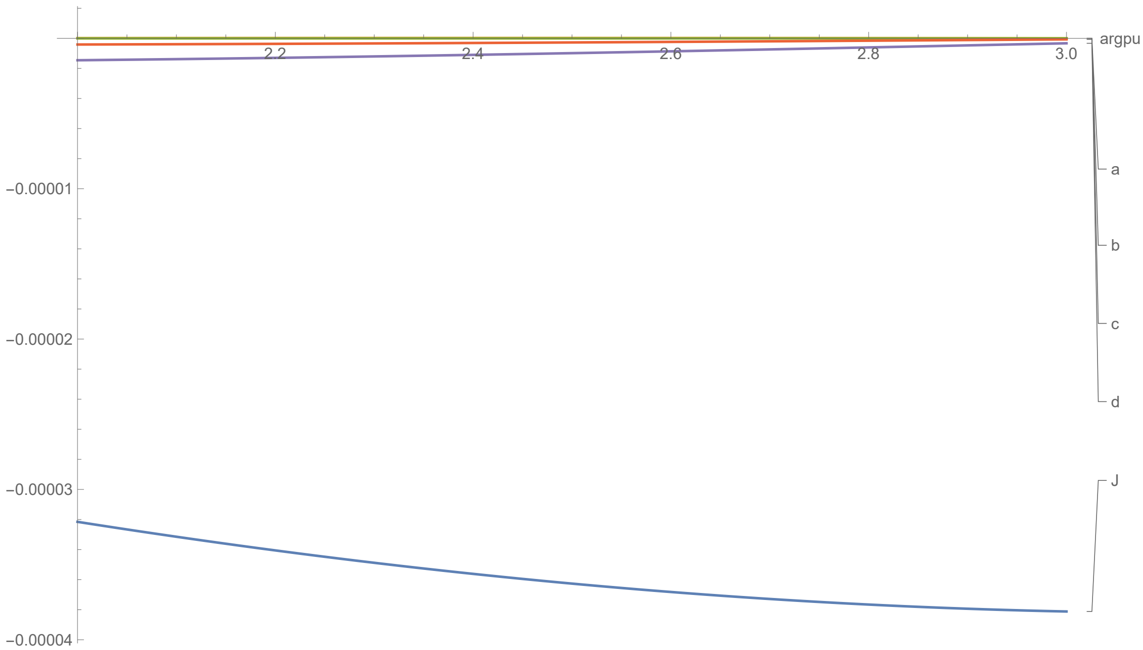
We can also see that the higher order terms in the approximation of J are negligible in the space around our solutions. For example:

Table 5: New solutions

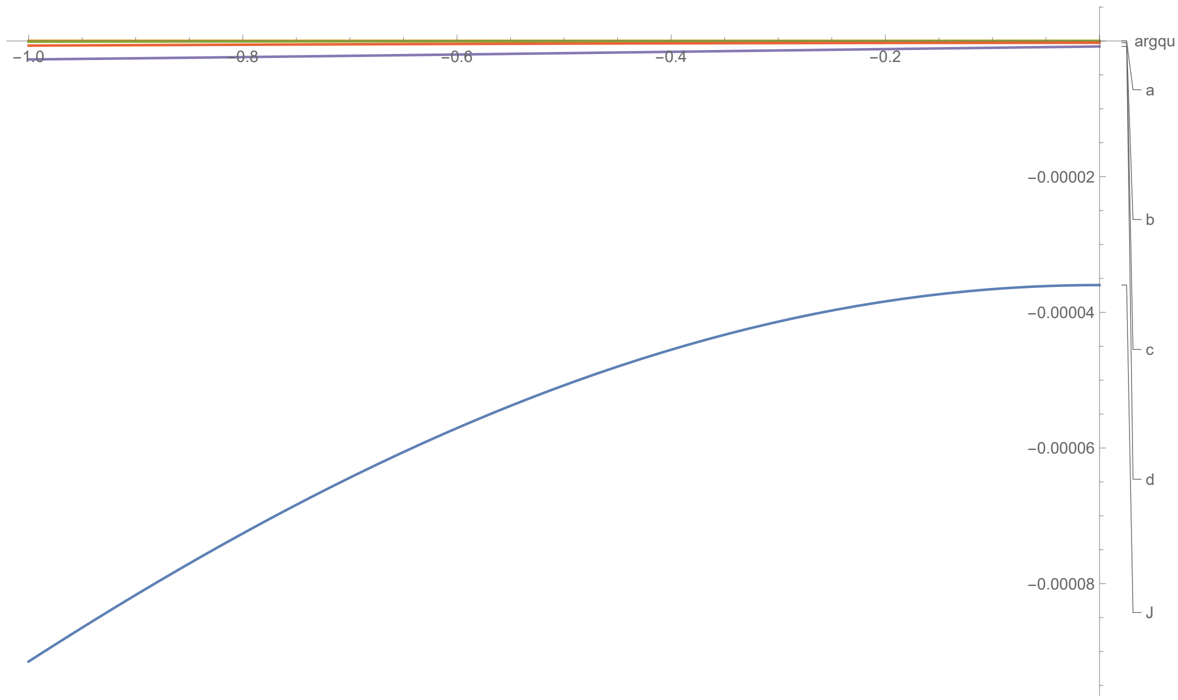
	9	10	11	12	13
J	3.36×10^{-5}	2.89×10^{-5}	-2.57×10^{-5}	-2.9×10^{-5}	2.91×10^{-5}
$\tilde{J}$	3.94×10^{-5}	3.23×10^{-5}	-2.8×10^{-5}	-3.27×10^{-5}	3.27×10^{-5}
$\tilde{J}+a+b+c+d$	4.08×10^{-5}	3.19×10^{-5}	-2.82×10^{-5}	-3.29×10^{-5}	3.24×10^{-5}

<u>Solution 9</u>

J and higher order terms varying $Arg(p_u)$:

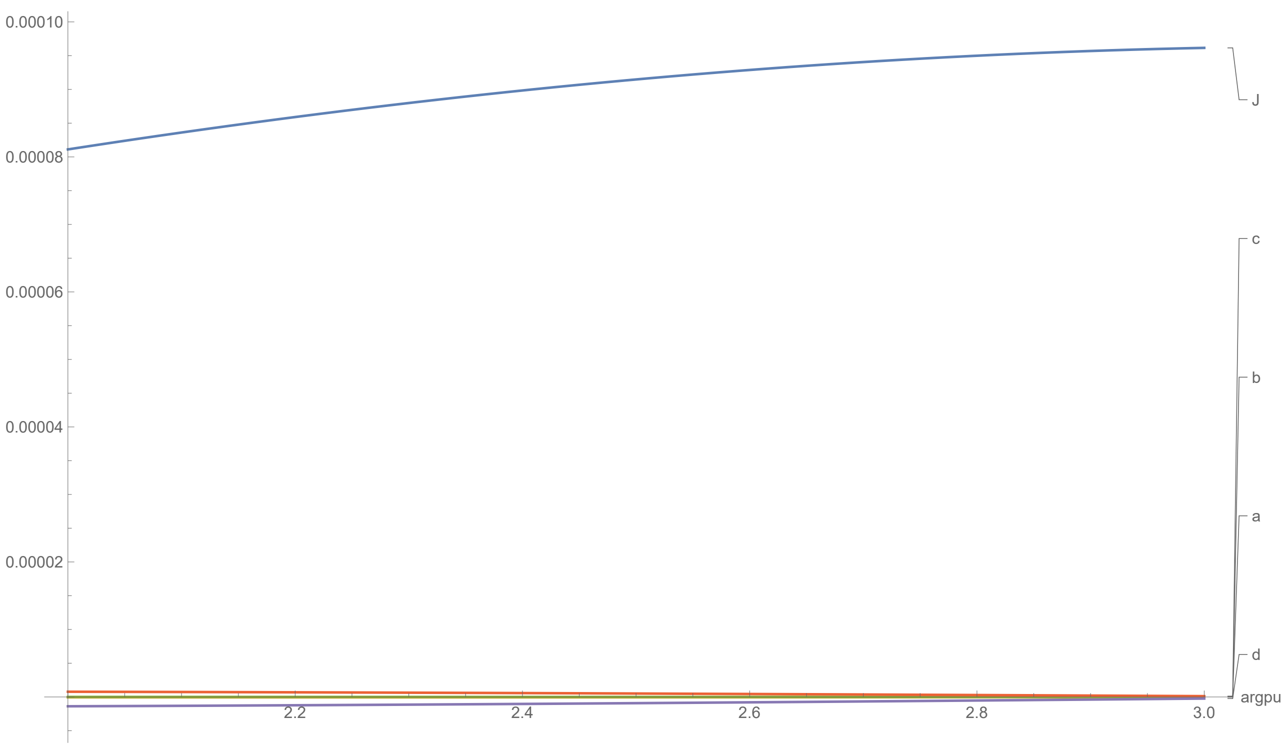


Varying $Arg(q_u)$:



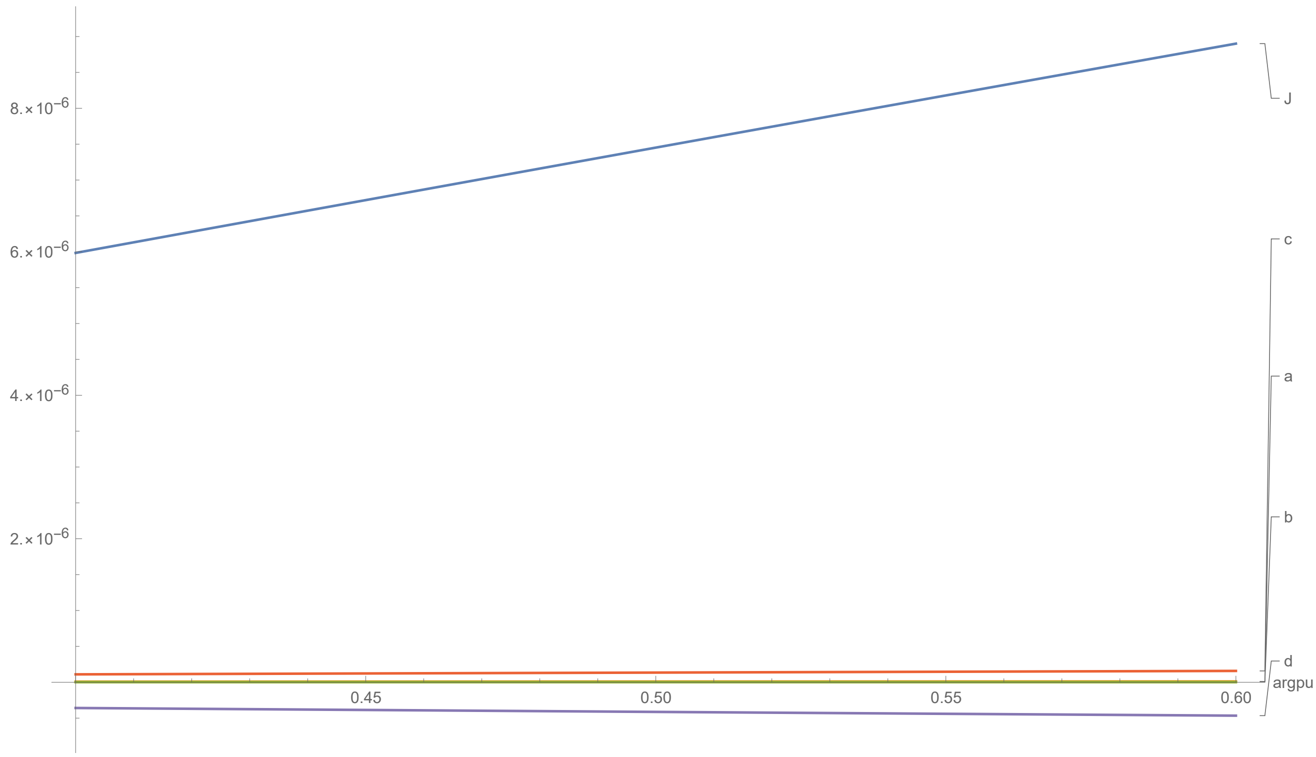
<u>Solution 10</u>

Varying $Arg(p_u)$:

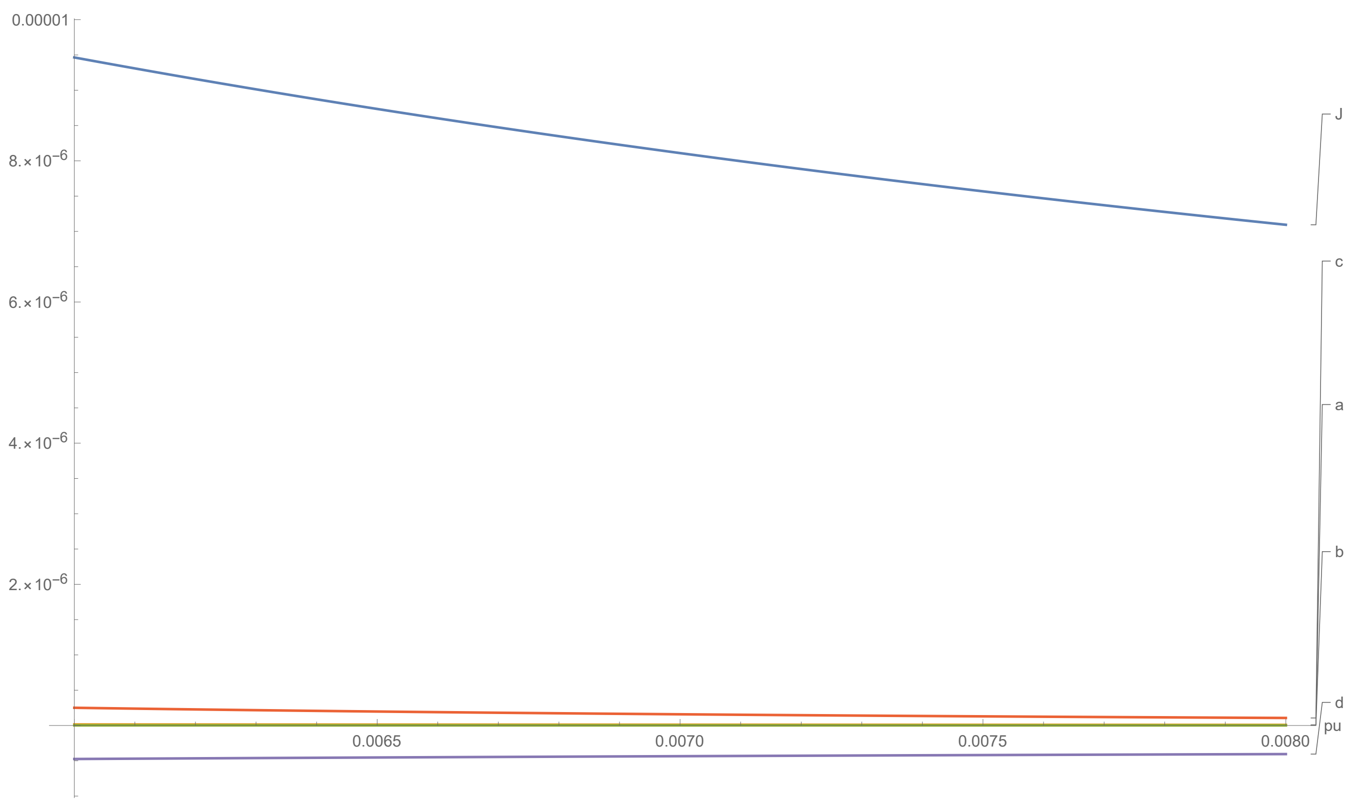


<u>Solution 13</u>

Varying $Arg(p_u)$:


8.×10^-6
6.×10^-6
4.×10^-6
2.×10^-6
0.45
0.50
0.55
0.60
J
c
a
b
d
argpu

Varying $|p_u|$:


0.00001
8.×10^-6
6.×10^-6
4.×10^-6
2.×10^-6
0.0065
0.0070
0.0075
0.0080
J
c
a
b
d
pu

Pseudocode

For the complex case we use Wolfram Mathematica to obtain numerically exact V matrices from the inputted parameters $|p_u|$, $|p_d|$, $Arg(pu)$, $|q_u|$, $|q_d|$, $Arg(qu)$, r_u, and r_d (recall that we set the phase convention $Arg(p_u) = -Arg(p_d)$) as follows:

- Define
$$M(r,\,|p|,\,argp,\,|q|,\,argq) = \begin{bmatrix} r^2 & |p|e^{iargp} & 0 \\ |p|e^{-iargp} & r & |q|e^{iargq} \\ 0 & |q|e^{-iargq} & 0 \end{bmatrix}; \tag{210}$$
- Create a list of the eigenvectors of M, ordering the list by the x elements of the eigenvectors as x1 $\gg$ x2 $\gg$ x3:
$$Egv(r,\,|p|,\,argp,\,|q|,\,argq) = [\upsilon_1,\upsilon_2,\upsilon_3]; \tag{211}$$
- Now create the matrix Ut who's rows are the conjugates of the eigenvectors:
$$Ut(r,\,|p|,\,argp,\,|q|,\,argq) = \begin{bmatrix} \upsilon_1 \\ \upsilon_2 \\ \upsilon_3 \end{bmatrix}^* = \begin{bmatrix} x1^* & y1^* & z1^* \\ x2^* & y2^* & z2^* \\ x3^* & y3^* & z3^* \end{bmatrix}; \tag{212}$$
- Create the transformation matrix Φ:
$$\Phi(Ut) = \begin{bmatrix} e^{-I*Arg(Ut[1,1])} & 0 & 0 \\ 0 & e^{-I*Arg(Ut[2,2])} & 0 \\ 0 & 0 & e^{-I*Arg(Ut[3,3])} \end{bmatrix}, \tag{213}$$
  and the Matrix U with real diagonal elements:
$$U(r,\,|p|,\,argp,\,|q|,\,argq) = \Phi(Ut(r,\,|p|,\,argp,\,|q|,\,argq)).Ut(r,\,|p|,\,argp,\,|q|,\,argq) \tag{214}$$
  Recall that the D_i and observable quantities are invariant under such a transformation.
- Now construct V:
$$V(ru,\,|pu|,\,argp,\,|qu|,\,argq,\,rd,\,|pd|,\,|qd|) = U(ru,\,|pu|,\,argpu,\,|qu|,\,argqu).U(rd,\,|pd|,\,-argpu,\,|qd|,\,-argqu)^\dagger. \tag{215}$$

We now wish to minimise $\chi^2 = \chi^2_v + \chi^2_{\lambda_k}$ from equations (75) and (76) where the λ_k are the numerically exact eigenvalues of M corresponding to the eigenvectors υ_k. To minimise:

- Create a random population of 10 sets of arguments within reasonable bounds for the arguments.
- For each set in the start population create a new population of, for example, 99 sets of arguments taken from the normal distribution with mean centered at the argument start value and variance 5% of the start value. So now we have a list of 100 sets of arguments close to the starting arguments (including the start set).
- Compare the χ^2 value for each set in the new population and replace the starting set in the original population with the set with the smallest χ^2.
- Iterate this process over the population to produce 10 minimised χ^2 values and the minimum points.

References

[1] K. A. Olive *et al.* [Particle Data Group Collaboration], Chin. Phys. C **38** (2014) 090001. doi:10.1088/1674-1137/38/9/090001