

Universal Quantum Gate Sets & the T-Operator

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Through the 6.s089 final project, I expanded my understanding of quantum gates and circuits by reading papers and articles, as well as implementing two circuits on the IBM Quantum Experience. In particular, I explored gates not discussed over the course of the class (T, S, Deutsch) and the definition of universal gate sets. Through my experiments on the IBM quantum computer I not only verified the universality of the $\{C_{NOT}, T, H\}$ set, but was able to show the affect of T-depth on actual quantum circuits.

A Universal Gate Set is defined as a finite set of gates that can approximate any unitary matrix arbitrarily well. Any possible operation on a quantum computer must belong to this set or, in other words, any unitary operator can be expressed as a finite sequence of gates from the set. This, however, creates a contradiction. We want to be able to create an uncountable number of quantum gates (all possible) using a finite number of operations (the number of gates in the set). An uncountable set cannot be created from a countable set. Thus, for a set to be regarded as universal we only require that any quantum operation can be approximated by a sequence of gates from this finite set. In fact, the Solovay-Kitaev Theorem guarantees that quantum operations for unitaries on a constant number of qubits can be approximated efficiently. The desired accuracy of the operation is arbitrary, meaning we can create sets that better approximate our system than others.

Through my readings, I found out that the operators we learned about in class fell under the Clifford Group. This includes the single-qubit Pauli X, Y, and Z operators and Hadamard operator for superpositions, as well as the two-qubit C_{NOT} operator. The final element of the Clifford Group, which we did not cover in class, is the S-Operator, which is a C_{NOT} phase shift on the Bloch Sphere that creates a superposition relating to the spins of electrons. The significance of the Clifford Group is that it has been shown to be simulated effectively on a classical computer, thus meaning it cannot be universal. This proof comes from the Gottesman-Knill Theorem, which states that stabilizer circuits (composed of Clifford operators) and even some highly entangled states can be efficiently simulated on a classical computer. In order to harness the full power of quantum computers, we must include at least one non-Clifford gate in our circuit.

This non-Clifford gate is often the T-Operator, a $\pi/4$ phase shift on the Bloch Sphere (in fact $S = T^2$). The T-Operator makes it possible to reach all the different points on the Bloch Sphere, through a property called T-depth. Essentially, by increasing the number of T-gates in our circuit, we cover the Bloch Sphere more densely with states we can reach. In my IBM Quantum Experience experiment, I explored this T-depth property and used a simple universal gate set, consisting of the Hadamard, C_{NOT} , and T operators. Although simple, this is a known universal gate set through the work of Barenco et. al in 1995, that any unitary matrix can be written as a combination of single- and two-qubit gates. In fact a generic interaction between two qubits (that can accurately be implemented between any two qubits) can be used to calculate anything in the quantum realm. This differs from the classical realm, in which reversible computing requires three-bit gates.

This three-bit classical gate is the Toffoli (CC_{NOT}) gate. If the first two bits are 1, the third bit is inverted. Otherwise, all three bits remain the same. It is the key to reversible computing, which requires that all calculations are time-invertible and the mapping from states to successors is one-to-one. A Toffoli gate can be used to build systems that perform any desired Boolean function,

in a reversible manner. The quantum equivalent of this is the Deutsch Gate, which is a single-gate set of universal quantum gates (like the Toffoli is composed of XOR, AND, etc.). In fact, the classical Toffoli gate is reducible to a quantum $D(\pi/2)$ gate. This implies that all classical logic operations can be performed on universal quantum computers, but quantum computers have much further capabilities ($D(\pi/3)$, $D(\pi/4)$, etc.).

For my experiment on the IBM quantum computer, I decided to tie this altogether by implementing a Toffoli (or $D(\pi/2)$) gate using the $\{CNOT, T, H\}$ universal gate set, with input $|1\rangle, |1\rangle, |0\rangle$ [Figure 1]. Using the IBM notation, this should produce output 00111 (first two qubits remain 1, third qubit is flipped from 0 to 1). Before running the experiment, I wanted to verify that the circuit was correct and ran a simulation, in which IBM assumes ideal conditions. The output of the simulation was as expected [Figure 2]. However, when I actually ran the experiment on the IBM 5-qubit quantum computer with 1024 shots, I was surprised by how significantly noise affected the results [Figure 3]. Upon finding a paper that explains how reducing T-depth should improve the results of a quantum circuit, I reimplemented my circuit by rearranging the locations of the T gates in addition to using an S operator (before I was using 2 T operators instead, since $S = T^2$) [Figure 4]. By making all these changes I was able to reduce the T-depth from 7 to 4 (if I had not used the S operator, it would have been 5). I ran this circuit using 1024 shots and was amazed to find how significantly the results improved [Figure 5]. This allowed me to mentally connect how, by decreasing T-depth, we are actually limiting the regions on the Bloch sphere to which the qubit state can reach, thus reducing the likelihood of noise or error pushing the qubit into a wrong state.

In all, through this experiment, I was able to show the affect of T-depth on actual quantum computers and illustrate why the $\{CNOT, T, H\}$ is a universal quantum gate set (by implementing the Toffoli gate).

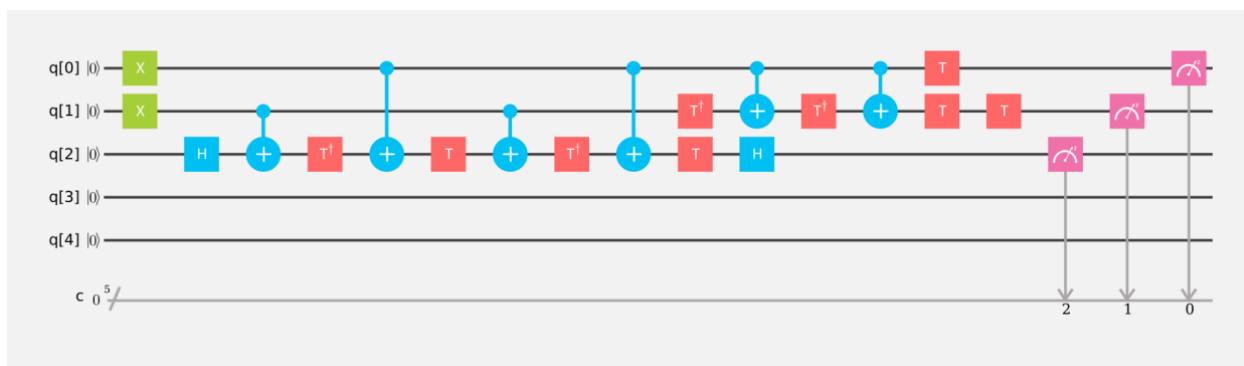


Figure 1: IBM Quantum Experience Experiment #1

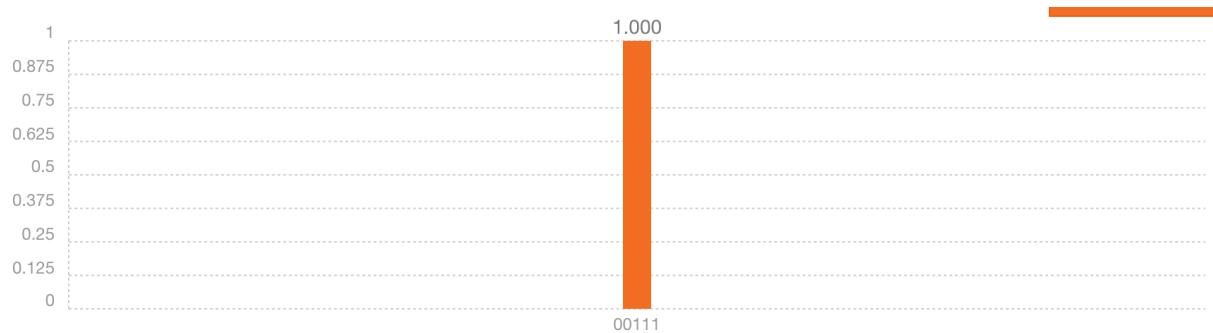


Figure 2: Simulation Results of Experiment #1

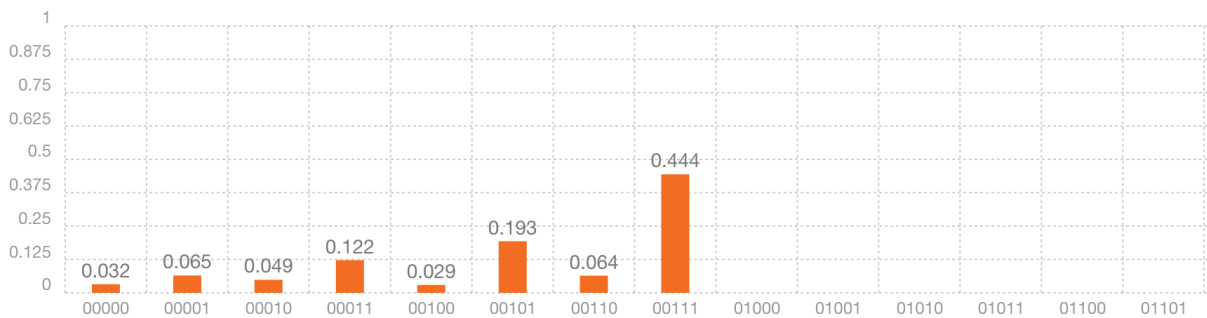


Figure 3: IBM Quantum Experience Experiment #1 Results

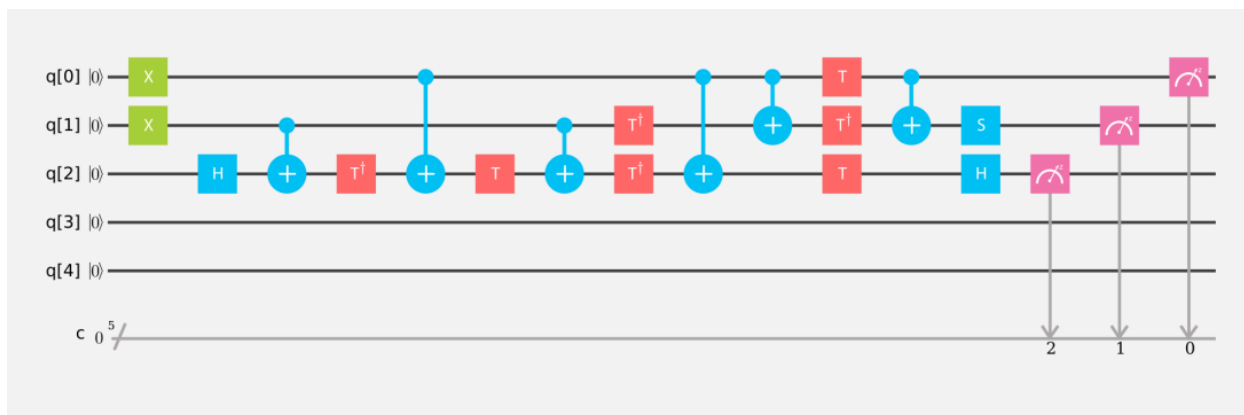


Figure 4: IBM Quantum Experience Experiment #2

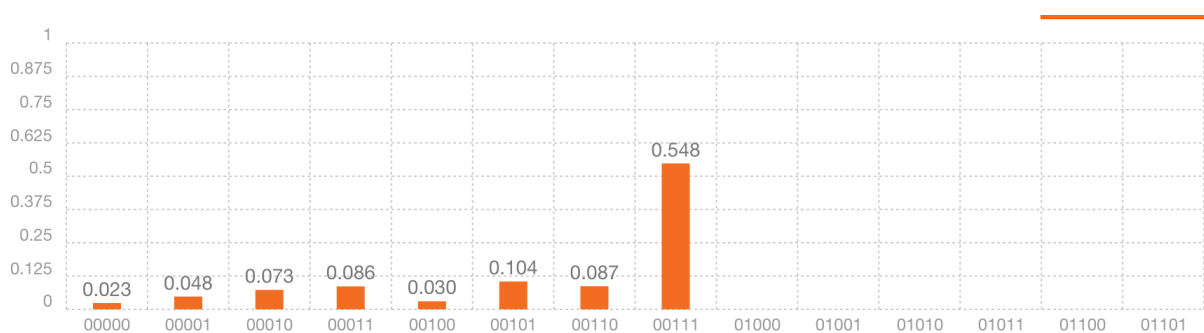


Figure 5: IBM Quantum Experience Experiment #2 Results