Learning by Doing & Knowledge Spillovers Model

Macroeconomics 3
Professor: César Salinas
Teaching Assistant: Francisco Arizola*

Introduction: These class notes are based on Barro and Sala-i-Martin (2004). Continuing with one-sector endogenous growth models, today we examine the Learning by Doing & Knowledge Spillovers model. Instead of simply "blowing up"the exponent like the AK model, this model offers a more interesting response: by accumulating capital, not only does output increase due to greater capital, but it also increases due to the greater knowledge generated from investment. These notes study Romer's (1986) model, which rethinks economic growth through two central mechanisms: learning by doing in investment and knowledge externalities.

1. Model Structure

The household problem remains the same as in the Ramsey model. Households live infinitely and maximize their aggregate utility, so the optimal conditions are given by:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \cdot (r - \rho) \tag{1}$$

And the transversality condition:

$$\lim_{t\to\infty} \left[a(t) \cdot \exp\left(-\int_0^t [r(v) - n]\, dv \right) \right] = 0 \tag{2}$$

Again, the change will come from the production function. We start with a labor-augmenting production function for firm i:

$$Y_i = F(K_i, A_i L_i) \tag{3}$$

where K_i and L_i are conventional production factors and A_i is the knowledge index available to the firm. Additionally, the function F satisfies neoclassical properties: positive and diminishing marginal returns for each factor, constant returns to scale, and the Inada

*Email: fa.arizolab@up.edu.pe

conditions. However, an additional change to the production function will be assuming a constant population, meaning its growth rate equals 0.

Now, the technology A, which in this model is called the knowledge index, does not grow at an exogenous rate. Instead, its growth stems from two key assumptions:

- *Learning by doing* works through each firm's net investment. That is, when a firm increases its capital stock, it simultaneously increases its knowledge index.
- Each firm's knowledge is a **public good**, meaning other firms can access it without any cost (knowledge spillovers).

Combining both assumptions, it follows that firm i's knowledge index is proportional to the economy's total capital level, as it benefits not only from its own net investment (*learning by doing*) but also from other firms' knowledge (*spillovers*). Since $\dot{A}_i$ is proportional to $\dot{K}$, then A_i is proportional to K. Therefore, we substitute:

$$Y_i = F(K_i, KL_i) \tag{4}$$

In this equation, endogenous growth can already be seen intuitively. From the neoclassical production function assumptions, we know that if we increase K_i while keeping KL_i constant, the function exhibits diminishing marginal returns. However, in this model, increasing K_i also increases K, albeit to a lesser extent. If other firms also increase their private capital level through investment, then the growth in K will be proportional to that in K_i, and due to constant returns to scale, the production function will exhibit continuous growth.

Let's proceed with the firms' problem. The profits of firm i can be written as:

$$\pi_i = L_i[F(k_i, K) - (r + \delta)k_i - w] \tag{5}$$

Assuming each firm is very small, K is taken as given. The first-order conditions are given by:

$$\frac{\partial y_i}{\partial k_i} = F_1(k_i,\, K) = r + \delta \tag{6}$$

$$\frac{\partial Y_i}{\partial L_i} = F(k_i,\, K) - k_i \cdot F_1(k_i, K) = w \tag{7}$$

where $F_1(\cdot)$, the partial derivative of $F(k_i,\, K)$ with respect to its first argument k_i, is the **private marginal product of capital**. This concept is "private"because it omits the contribution of k_i to the aggregate knowledge stock K.

To characterize $F_1(k_i, K)$, we start from the symmetric equilibrium where all firms are identical ($k_i = k$, so $K = kL$). Using the fact that $F(k_i, K)$ is homogeneous of degree 1, we express the average product of capital as:

$$\frac{F(k_i, K)}{k_i} = f\left(\frac{K}{k_i}\right) = f(L) \quad (4.26) \tag{8}$$

with $f'(L) > 0$ and $f''(L) < 0$, showing that the average product increases with L but is independent of k (due to knowledge spillovers).

Now, differentiating $F(k_i, K) = k_i f\left(\frac{K}{k_i}\right)$ with respect to k_i, and substituting (8), we obtain the explicit form of the private marginal product:

$$\begin{aligned}
F_1(k_i, K) &= \frac{\partial}{\partial k_i}\left[k_i f\left(\frac{K}{k_i}\right)\right] && (9)\\
&= f\left(\frac{K}{k_i}\right) + k_i f'\left(\frac{K}{k_i}\right)\left(-\frac{K}{k_i^2}\right) && (10)\\
&= f\left(\frac{K}{k_i}\right) - \frac{K}{k_i} f'\left(\frac{K}{k_i}\right) && (11)\\
&= f(L) - Lf'(L) && (12)
\end{aligned}$$

This derivation reveals two important points:

- The private marginal product $F_1(k_i, K)$ is less than the average product $f(L)$ (since $f'(L) > 0$)
- It is constant in k but increasing in L (since $f''(L) < 0$)

2. Equilibrium

Again, in our closed economy with households and firms, we combine their optimal conditions to find the equilibrium. Using the condition $r = F_1(k_i, K) - \delta$ and the form of the private marginal product of capital from (12), we have:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f(L) - Lf'(L) - \delta - \rho\right] \tag{13}$$

As in the AK model, this rate is constant (if L is constant). Additionally, we assume parameters that guarantee positive growth with finite utility:

$$f(L) - Lf'(L) > \rho + \delta > \frac{(1-\theta)}{\theta}\left[f(L) - Lf'(L) - \delta - \rho\right] + \delta \tag{14}$$

This condition is analogous to the equation derived from the AK model (See NC3 for details).

Substituting $a = k$ and the first-order conditions from the firms' problem into the household budget constraint (or the simpler procedure of setting up the resource constraint in the economy[1]), we obtain the accumulation equation for k:

$$\dot{k} = f(L)k - c - \delta k \tag{15}$$

[1] The economy's output must equal consumption plus investment. See NC3 for details.

Using this equation along with the transversality condition, we can show that the model has no transitional dynamics: k and y always grow at the rate $\dot{c}/c$ from (13). The analysis is analogous to that of the AK model, so the detailed proof is omitted.

3. Pareto Optimality and Policy Implications

To assess optimality, we compare the decentralized solution with the social planner's problem. The planner maximizes household utility (with $n = 0$) subject to the accumulation constraint, internalizing the knowledge *spillovers* that decentralized agents ignore. Recall that the social planner allocates all resources in the economy and does not face prices, so they simply maximize utility subject to a resource constraint. The planner's Hamiltonian is:

$$J = e^{-\rho t} \frac{c^{1-\theta} - 1}{1 - \theta} + \nu \left[f(L)k - c - \delta k \right] \tag{16}$$

The optimality conditions ($J_c = 0$, $\dot{\nu} = -J_k$, and the transversality condition) yield the optimal growth rate:

$$\frac{\dot{c}}{c}^{planner} = \frac{1}{\theta} \left[f(L) - \delta - \rho \right] \tag{17}$$

3.1. Gap Between Equilibrium and Optimum

The model shows a gap between the consumption growth rate in the decentralized solution and the Pareto optimal social planner's solution:

- **Planner**: Uses the *average product* of capital $f(L)$ (constant social returns).
- **Decentralized equilibrium**: Uses the *private marginal product* $f(L) - Lf'(L)$ (underestimates externalities).

The difference between them, $Lf'(L) > 0$, quantifies the inefficiency from non-internalized externalities, leading to: $\frac{\dot{c}}{c}^{\text{decentralized}} < \frac{\dot{c}}{c}^{\text{optimal}}$

3.2. Policy Implementation

To achieve the social optimum, some incentive is needed to raise private returns to the social level. For example, an investment subsidy equivalent to a tax credit that increases the private return on capital. Clearly, if this subsidy is to be implemented without causing other distortions, it should be financed with *lump-sum taxes*. However, such taxes are unrealistic. Therefore, in our model without leisure-consumption decisions, a constant consumption tax over time could serve as a non-distortionary funding source for this subsidy, providing a more realistic alternative.

4. Scale Effects

The model predicts that higher L increases growth (via $f(L)$ and $f(L) - Lf'(L)$), but this contradicts empirical evidence. Moreover, this implies that with growing L, the consumption growth rate would become increasingly higher.

Frankel (1962) and Lucas (1988) Alternative: If $A_i = K/L$ (knowledge is proportional not to aggregate capital but to *average* capital), growth becomes independent of L. This specification eliminates scale effects while preserving the endogenous growth model dynamics.