Solow-Swan Model

Macroeconomics 3
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Introduction: These lecture notes are based on Barro and Sala-i-Martin (2004). In this first part, we will study growth models with exogenous saving rates. The objective is to evaluate whether an economy can achieve positive growth rates indefinitely through saving and investment in its capital stock.

1. Setup

It is convenient to begin our analysis using a simplified approach that excludes markets and firms. We can think of a unit, a household/producer like Robinson Crusoe, who owns the inputs and also manages the technology that transforms inputs into outputs. There are only three inputs: physical capital $K(t)$, labor $L(t)$, and knowledge $T(t)$. The production function takes the following form:

$$Y(t) = F[K(t), L(t), T(t)] \tag{1}$$

We assume that there is a single sector in which output can either be consumed (C) or invested (I). Investment is used to create new units of physical capital or to replace depreciated capital. Additionally, we assume that there is no government spending and that the economy is closed. Thus, the economy's resource constraint can be stated as:

$$Y(t) = C(t) + I(t) \tag{2}$$

Where $I(t)$ represents gross investment. Each period, a fraction of capital depreciates and becomes unproductive. This depreciation rate is represented by δ. The net increase in the physical capital stock at any given moment is equal to gross investment minus depreciation:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{3}$$

Rearranging (2), we can define saving (S) as $S = I = Y - C$. Then, the saving rate, (s), is given by $s = S/Y$, where $(0 < s < 1)$. From this, we can also conclude that saving

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equals investment (and that the saving rate equals the investment rate). In other words, the saving rate represents the fraction of output allocated to investment. In equation (3):

$$\dot{K}(t) = s \cdot F[K(t), L(t), T(t)] - \delta K(t) \tag{4}$$

The labor input, L, varies over time due to population growth, changes in participation rates, variations in the amount of time worked by the typical worker, and improvements in worker skills and quality. In this chapter, we simplify by assuming that everyone works the same amount of time and has the same constant skill level, which we normalize to one. Thus, we identify changes in the labor input solely with changes in the total population.

Additionally, we assume that the population grows at a constant and exogenous rate, $\dot{L}/L = n \geq 0$, without using resources. If we normalize the number of people at time 0 to 1 and the labor intensity per person also to 1, then the population and labor force at time t are equal to:

$$L(t) = e^{nt} \tag{5}$$

To highlight the importance of capital accumulation, we also assume that the level of technology $T(t)$ is constant over time.

2. The Neoclassical Solow-Swan Model

2.1. The Neoclassical Production Function

We say that a production function, $F[K(t), L(t), T(t)]$, is neoclassical if it satisfies the following properties:

- **Constant returns to scale:** The function $F(\cdot)$ exhibits constant returns to scale. That is, if we multiply capital and labor by the same positive constant, λ, we obtain λ times the amount of output:

  $$F(\lambda K, \lambda L, T) = \lambda \cdot F(K, L, T) \quad \text{for all } \lambda > 0$$

- **Positive and diminishing returns to private inputs:** For all $K > 0$ and $L > 0$, $F(\cdot)$ exhibits positive and diminishing marginal products with respect to each input:

  $$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0$$

  $$\frac{\partial F}{\partial L} > 0, \quad \frac{\partial^2 F}{\partial L^2} < 0$$

  Therefore, keeping technology and labor levels constant, each additional unit of capital contributes positively to output, but these increments decrease as the number of machines increases.

- **Inada Conditions:** The marginal product of capital (or labor) tends to infinity when capital (or labor) approaches zero and converges to zero when capital (or labor) approaches infinity.

$$\lim_{K\to 0}\left(\frac{\partial F}{\partial K}\right) = \lim_{L\to 0}\left(\frac{\partial F}{\partial L}\right) = \infty$$

$$\lim_{K\to \infty}\left(\frac{\partial F}{\partial K}\right) = \lim_{L\to \infty}\left(\frac{\partial F}{\partial L}\right) = 0$$

- **Essentiality:** An input is essential if a strictly positive amount of it is required for positive production.

$$F(0, L) = F(K, 0) = 0$$

Per capita variables: It is logical to determine a country's wealth in terms of per capita output. To capture this property, we define the model in per capita variables and study their behavior.

Since the definition of constant returns to scale applies to all values of λ, it also applies to $\lambda = 1/L$. Therefore, output can be written as:

$$Y = F(K, L, T) = L \cdot F(K/L, 1, T) = L \cdot f(k) \tag{6}$$

Where $k \equiv K/L$ is capital per worker, $y \equiv Y/L$ is output per worker, and the function $f(k)$ is defined as $F(k, 1, T)$. This result means that the production function can be expressed in intensive form (i.e., per worker or per capita) as:

$$y = f(k) \tag{7}$$

Output per person is determined by the amount of physical capital each person has access to.

We can differentiate this condition $Y = L \cdot f(k)$ with respect to K, for fixed L, and then with respect to L, for fixed K, to verify that the marginal products of the factors are given by:

$$\frac{\partial Y}{\partial K} = f'(k)\frac{1}{L}L = f'(k) \tag{8}$$

$$\frac{\partial Y}{\partial L} = f(k) - k \cdot f'(k) \tag{9}$$

Cobb-Douglas Example: A simple production function that is often considered a reasonable description of real economies is the Cobb-Douglas function:

$$Y = AK^{\alpha}L^{1-\alpha} \tag{10}$$

Where $A > 0$ is the technology level and α is a constant with $0 < \alpha < 1$. The Cobb-Douglas function can be written in intensive form as:

$$y = Ak^{\alpha} \tag{11}$$

Note that $f'(k) = A\alpha k^{\alpha-1} > 0$, $f''(k) = -A\alpha(1-\alpha)k^{\alpha-2} < 0$, $\text{lím}_{k\to\infty} f'(k) = 0$, and $\text{lím}_{k\to 0} f'(k) = \infty$. Therefore, the Cobb-Douglas form satisfies the properties of a neoclassical production function[1].

2.2. The Fundamental Equation of the Solow-Swan Model

The change in the capital stock over time is given by equation (3). If we divide both sides of this equation by L, we obtain:

$$\dot{K}/L = s \cdot f(k) - \delta k \tag{12}$$

The right-hand side contains only per capita variables, but the left-hand side does not. Therefore, it is not an ordinary differential equation that can be easily solved. To transform it into a differential equation in terms of k, we differentiate $k \equiv K/L$ with respect to time to obtain

$$\dot{k} \equiv \frac{d(K/L)}{dt} = \dot{K}/L - K\frac{1}{L^2}\dot{L} = \dot{K}/L - nk \tag{13}$$

Where $n = \dot{L}/L$. If we substitute this result into the expression for $\dot{K}/L$, we can rearrange the terms to obtain

$$\dot{k} = s \cdot f(k) - (n+\delta) \cdot k \tag{14}$$

Equation (14) is the fundamental differential equation of the Solow-Swan model. This nonlinear equation depends only on k.

The term $n + \delta$ on the right-hand side of the equation can be considered the effective depreciation rate for the capital-labor ratio, $k \equiv K/L$. If the savings rate, s, were 0, per capita capital would decrease partly due to capital depreciation at rate δ and partly due to the increase in the number of people at rate n.

2.3. Steady State

We define a steady state as a situation in which variables (per capita in our model) grow at constant rates (possibly zero). In the Solow-Swan model, the steady state corresponds to $\dot{k} = 0$ in equation (14), that is, an intersection of the curve $s \cdot f(k)$ with the line $(n+\delta) \cdot k$ in Figure 1. The corresponding value of k is denoted as k^*. (We focus here on

[1] Additionally, the key property of the Cobb-Douglas production function is the behavior of factor income shares. In a competitive economy, capital and labor are paid according to their marginal products. Thus, each unit of capital is paid at $R = f'(k) = \alpha Ak^{\alpha-1}$, and each unit of labor is paid at $w = f(k) - k \cdot f'(k) = (1-\alpha) \cdot Ak^{\alpha}$. The capital income share is then $Rk/f(k) = \alpha$, and the labor income share is $w/f(k) = 1-\alpha$. Hence, in a competitive setting, the factor income shares remain constant—independent of k—when the production function is Cobb-Douglas.

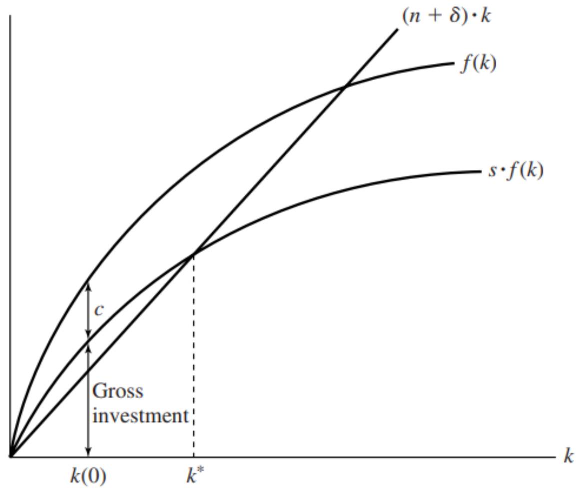

Figura 1: Solow-Swan Model

the intersection at $k > 0$ and disregard the one at $k = 0$). Algebraically, k^* satisfies the condition:

$$s \cdot f(k^*) = (n + \delta) \cdot k^* \tag{15}$$

Since k is constant in the steady state, y and c are also constant at the values $y^* = f(k^*)$ and $c^* = (1 - s) \cdot f(k^*)$, respectively. The constancy of per capita magnitudes means that the levels of the variables—K, Y, and C—grow in the steady state at the population growth rate, n.

Using the Cobb-Douglas production function as an example, we have:

$$k^* = \left(\frac{sA}{n + \delta} \right)^{\frac{1}{1-\alpha}} \tag{16}$$

Therefore, one-time changes in the savings rate, the level of technology, the population growth rate, and the depreciation rate affect the per capita levels of the variables in the steady state. However, in the steady state, it still holds that the variables (Y, K, C) continue to grow at the rate n, and consequently, the per capita variables maintain a growth rate of 0.

2.4. Golden Rule of Capital Accumulation

For a given level of A and given values of n and δ, there is a unique steady-state value $k^* > 0$ for each savings rate s. We denote this relationship as $k^*(s)$, with $dk^*(s)/ds > 0$. The steady-state level of per capita consumption is $c^* = (1 - s) \cdot f[k^*(s)]$.

We know from equation (15) that $s \cdot f(k^*) = (n + \delta) \cdot k^*$. Therefore, we can write an expression for c^* as:

$$c^*(s) = f[k^*(s)] - (n + \delta) \cdot k^*(s) \tag{17}$$

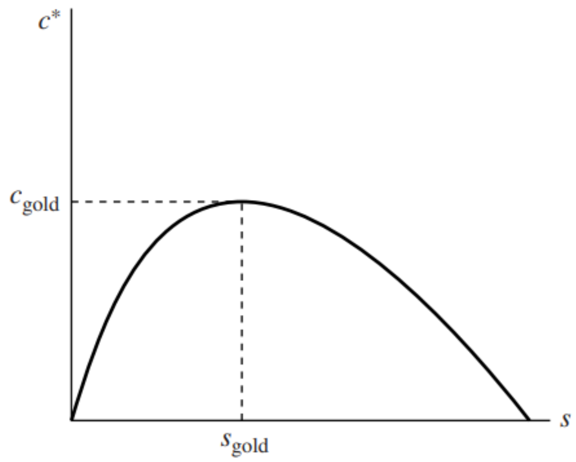

Figura 2: Golden Rule of Capital Accumulation

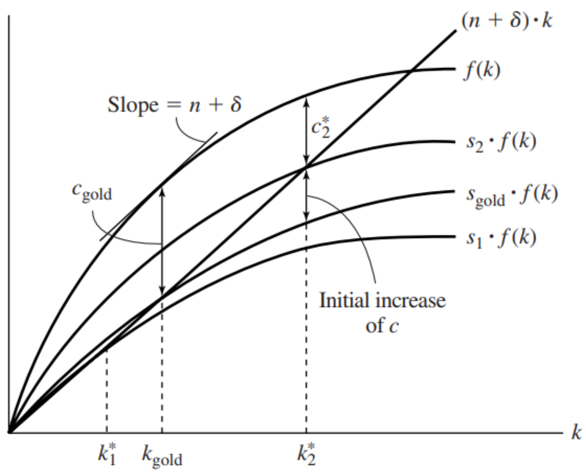

Figura 3: Golden Rule and Dynamic Inefficiency

Figure 2 shows the relationship between c^* and s as implied by equation (17). The quantity c^* increases with s for low levels of s and decreases with s for high values of s. The quantity c^* reaches its maximum when the derivative is zero, that is, when $[f'(k^*) - (n+\delta)] \cdot \frac{dk^*}{ds} = 0$. Since $\frac{dk^*}{ds} > 0$, the term in brackets must be equal to 0. If we denote the value of k^* that corresponds to the maximum of c^* as k_{gold}, then the condition that determines k_{gold} is:

$$f'(k_{gold}) = n + \delta \tag{18}$$

The corresponding savings rate can be denoted as s_{gold}, and the associated level of per capita consumption in the steady state is given by $c_{gold} = f(k_{gold}) - (n + \delta) \cdot k_{gold}$.

The condition in equation (18) is called the golden rule of capital accumulation. In economic terms, the result of the golden rule can be interpreted as: “If we provide the same amount of consumption to the members of each present and future generation, that is, if we do not give less to future generations than to ourselves, then the maximum per capita consumption amount is c_{gold}.”

Consider an economy, as described by the savings rate s_2 in Figure 3, for which $s_2 > s_{gold}$, so that $k_2^* > k_{\text{gold}}^*$ and $c_2^* < c_{\text{gold}}$. Imagine that, starting from the steady state, the savings rate is permanently reduced to s_{gold}. Figure 3 shows that per capita consumption, c—given by the vertical distance between the curves $f(k)$ and $s_{\text{gold}} \cdot f(k)$—initially increases by a discrete amount. Then, the level of c monotonically decreases during the transition toward its new steady-state value, c_{gold}. Therefore, when $s > s_{\text{gold}}$, the economy is saving excessively in the sense that per capita consumption at all points in time could increase by reducing the savings rate. An economy that oversaves is said to be **dynamically inefficient**, because the per capita consumption path lies below other feasible alternative paths.

2.5. Transitional Dynamics

Now, let's see how the per capita income of an economy converges toward its own steady-state value.

Dividing both sides of equation (14) by k implies that the growth rate of k is given by

$$\gamma_k{}^2 \equiv \frac{\dot{k}}{k} = s \cdot \frac{f(k)}{k} - (n + \delta) \tag{19}$$

Note that, at all points in time, the growth rate of the level of a variable is equal to the per capita growth rate plus the exogenous population growth rate n, for example,

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + n \tag{20}$$

Equation (19) states that $\frac{\dot{k}}{k}$ is equal to the difference between two terms. The first term, $s \cdot \frac{f(k)}{k}$, is called the savings curve, and the second term, $(n+\delta)$, is the depreciation curve.

We plot the two curves as a function of k in Figure 4. The savings curve is decreasing; it asymptotes to infinity at $k = 0$ and approaches 0 as k tends to infinity. The depreciation curve is a horizontal line at $n + \delta$. The vertical distance between the savings curve and the depreciation line is equal to the per capita capital growth rate (from equation 19), and the intersection point corresponds to the steady state. Since $n + \delta > 0$ and $s \cdot \frac{f(k)}{k}$ falls monotonically from infinity to 0, the savings curve and the depreciation line intersect once and only once. Therefore, the steady-state capital-labor ratio $k^* > 0$ exists and is unique.

Figure 4 shows that, to the left of the steady state, the curve $s \cdot \frac{f(k)}{k}$ lies above $n + \delta$. Therefore, the growth rate of k is positive, and k increases over time. As k increases, $\frac{\dot{k}}{k}$ decreases and approaches 0 as k approaches k^*. Conversely, if the economy starts above the steady state, $k(0) > k^*$, then the growth rate of k is negative, and k decreases over time. (Note from Figure 4 that, for $k > k^*$, the line $n + \delta$ lies above the curve $s \cdot \frac{f(k)}{k}$, and therefore, $\frac{\dot{k}}{k} < 0$.) The growth rate increases and approaches 0 as k approaches k^*.

We can also study the behavior of output during the transition. The growth rate of per

[2]This notation will be used henceforth to denote the growth rate of a variable

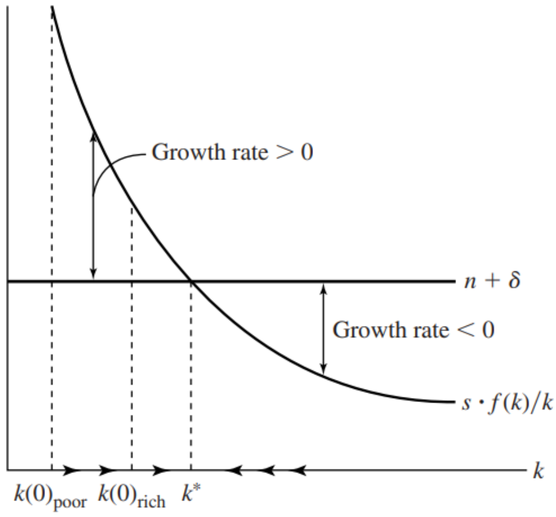

Figura 4: Transitional Dynamics of the Solow-Swan Model

capita output is given by

$$\frac{\dot{y}}{y} = f'(k) \cdot \frac{\dot{k}}{k} = \left[k \cdot \frac{f'(k)}{f(k)} \right] \cdot \frac{\dot{k}}{k} \tag{21}$$

The expression in brackets on the right-hand side is the capital share, that is, the share of capital income in total income.

Equation (21) shows that the relationship between $\frac{\dot{y}}{y}$ and $\frac{\dot{k}}{k}$ depends on the behavior of the capital share. In the Cobb-Douglas case, the capital share is the constant α, and $\frac{\dot{y}}{y}$ is the fraction α of $\frac{\dot{k}}{k}$. Therefore, the behavior of $\frac{\dot{y}}{y}$ mimics that of $\frac{\dot{k}}{k}$.

Thus, $\frac{\dot{y}}{y}$ necessarily decreases as k increases (and therefore as y increases) in the region where $\frac{\dot{k}}{k} \geq 0$, that is, if $k \leq k^*$. In the Solow-Swan model, which assumes a constant savings rate, the level of consumption per person is given by $c = (1 - s) \cdot y$. Therefore, the growth rates of consumption and per capita income are identical at all points in time, $\frac{\dot{c}}{c} = \frac{\dot{y}}{y}$. Consumption, therefore, exhibits the same dynamics as output.

2.6. Policy Experiments

- Suppose the economy is initially in a steady-state position with capital per person equal to k_1^*. **Imagine that the savings rate increases permanently** from s_1 to a higher value s_2, possibly because the government introduces some policy that raises the savings rate. Figure 5 shows that the curve $s \cdot \frac{f(k)}{k}$ shifts to the right. Therefore, the intersection with the line $n + \delta$ also shifts to the right, and the new steady-state capital stock, k_2^*, exceeds k_1^*.

  At $k = k_1^*$, savings are more than sufficient to generate an increase in k. As k increases, its growth rate decreases and approaches 0 as k approaches k_2^*. The result, therefore, is that a permanent increase in the savings rate generates temporarily positive per capita growth rates. In the long run, the levels of k and y are permanently higher, but the per capita growth rates return to zero.

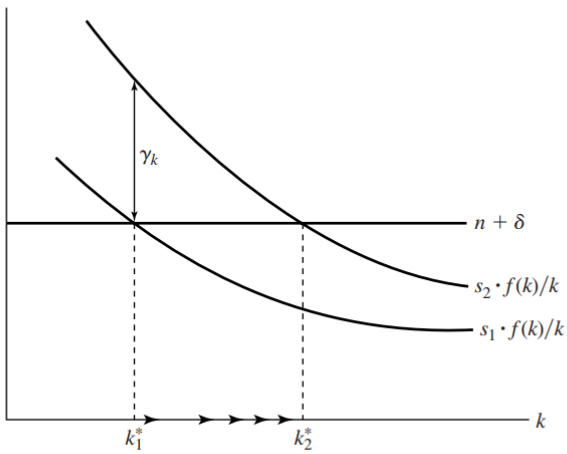

Figura 5: Effects of an Increase in the Savings Rate

The positive transient growth rates might suggest that the economy could grow forever by increasing the savings rate over and over. A problem with this line of reasoning is that the savings rate is a fraction, a number between zero and one. Therefore, we can now answer the question that motivated the beginning of this chapter: "Can per capita income grow forever simply by saving and investing in physical capital?" If the production function is neoclassical, the answer is "no."

- We can also evaluate **permanent changes in the population growth rate**, n. A decrease in n shifts the depreciation line downward, so that the steady-state level of capital per worker would be higher. However, the long-run growth rate of capital per person would still be zero.

- A permanent, one-time improvement in the level of technology has similar temporary effects on per capita growth rates. If the production function $f(k)$ shifts upward proportionally, then the savings curve shifts upward, as in Figure 5. Therefore, $\frac{\dot{k}}{k}$ becomes positive again temporarily. In the long run, the permanent improvement in technology generates higher levels of k and y but no changes in per capita growth rates.

The key difference between improvements in knowledge and increases in the savings rate is that improvements in knowledge have no limits. That is, the production function can shift over and over because, in principle, there are no limits to human knowledge. Therefore, if we want to generate growth in per capita income and consumption in the long run within the neoclassical framework, growth must come from technological progress rather than the accumulation of physical capital.

2.7. Relative and Absolute Convergence

The fundamental equation of the Solow-Swan model (equation 19) implies that the derivative of $\frac{\dot{k}}{k}$ with respect to k is negative:

$$\frac{\partial(\frac{\dot{k}}{k})}{\partial k} = s \cdot \left[\frac{f'(k) - \frac{f(k)}{k}}{k}\right] < 0 \tag{22}$$

Other things being equal, smaller values of k are associated with larger values of $\frac{\dot{k}}{k}$. An important question arises: Does this result mean that economies with less capital per person tend to grow faster in per capita terms? In other words, is there a tendency for convergence among economies?

- **Absolute Convergence:** The hypothesis that poor economies tend to grow faster per capita than rich ones—keeping the other parameters s, n, δ and the production function the same—is referred to as absolute convergence. The key here is that the economies must be homogeneous, which is why when comparing different economies, we observe that this hypothesis does not hold.
- **Relative Convergence:** We can adapt the theory to empirical observations on convergence if we allow for heterogeneity among economies, particularly if we abandon the assumption that all economies have the same parameters and, therefore, the same steady-state positions. If the steady states differ, we must modify the analysis to consider a concept of conditional convergence. The main idea is that an economy grows faster the farther it is from its own steady-state value. This hypothesis, therefore, does apply to different economies.

2.8. Technological Progress

We relax the assumption that the level of technology is constant and allow it to evolve over time. This allows us to escape diminishing returns to capital and sustain per capita growth in the long run.

We first consider the case where technology increases exogenously. Technological progress can take various forms. We will use *labor-augmenting* technological progress.

$$Y = F[K, L \cdot T(t)] \tag{23}$$

where $T(t)$ is the technology index, and $\dot{T}(t) \geq 0$. This form is called *labor-augmenting* technological progress because it increases output in the same way as an increase in the labor stock. (Note that the technology factor, $T(t)$, appears in the production function as a multiple of L.)

The Solow-Swan Model with *Labor-Augmenting* Technology

We now assume that the production function includes labor-augmenting technological progress, as shown in equation (23), and that the technology term, $T(t)$, grows at a constant rate x. The condition for the change in the capital stock is:

$$\dot{K} = s \cdot F[K, L \cdot T(t)] - \delta K \tag{24}$$

If we divide both sides of this equation by L, we can derive an expression for the change in k over time:

$$\dot{k} = s \cdot F[k, T(t)] - (n + \delta) \cdot k \tag{25}$$

The only difference is that output per person now depends on the level of technology, $T(t)$.

We divide both sides of equation (25) by k to calculate the growth rate:

$$\frac{\dot{k}}{k} = s \cdot \frac{F[k, T(t)]}{k} - (n + \delta) \tag{26}$$

As before, $\frac{\dot{k}}{k}$ is equal to the difference between two terms, where the first term is the product of s and the average product of capital, and the second term is $n + \delta$. The only difference is that now, for a given k, the average product of capital, $F[k, T(t)]/k$, increases over time due to the growth of $T(t)$ at rate x. In terms of Figure 4, the downward-sloping curve, $s \cdot F(\cdot)/k$, continuously shifts to the right, and thus the level of k corresponding to the intersection between this curve and the line $n + \delta$ also continuously shifts to the right. We now calculate the growth rate of k in the steady state.

By definition, the steady-state growth rate, $(\dot{k}/k)^*$, is constant. Since s, n, and δ are also constant, equation (26) implies that the average product of capital, $F[k, T(t)]/k$, is constant in the steady state. Due to constant returns to scale, the expression for the average product is $F[1, T(t)/k]$, and therefore it is constant only if k and $T(t)$ grow at the same rate, that is, $(\dot{k}/k)^* = x$.

Output per person is given by:

$$y = F[k, T(t)] = k \cdot F[1, T(t)/k] \tag{27}$$

Since k and $T(t)$ grow in the steady state at rate x, the steady-state growth rate of y is also x. Furthermore, since $c = (1 - s) \cdot y$, the steady-state growth rate of c is also x.

Transition Dynamics with Technological Progress

To analyze the transition dynamics of the model with technological progress, it will be convenient to work with variables that remain constant in the steady state. Since k and $T(t)$ grow in the steady state at the same rate, we can work with the ratio $\hat{k} = k/T(t) = K/[L \cdot T(t)]$. The variable $L \cdot T(t) \equiv \bar{L}$ is often called the ***effective amount of labor***, the physical quantity of labor, L, multiplied by its efficiency, $T(t)$.

The *effective labor* is appropriate because the economy operates as if its labor input were $\bar{L}$. The variable $\hat{k}$ is then the amount of capital per unit of effective labor.

The amount of output per unit of effective labor, $\hat{y} = Y/[L \cdot T(t)]$, is given by:

$$\hat{y} = F(\hat{k}, 1) \equiv f(\hat{k}) \tag{28}$$

Therefore, we can rewrite the production function in intensive form by replacing y and k with $\hat{y}$ and $\hat{k}$, respectively. If we proceed as we did earlier to derive equations (14) and (19), but now use the condition that $A(t)$ grows at rate x, we can derive the dynamic equation for $\hat{k}$:

$$\frac{\dot{\hat{k}}}{\hat{k}} = s \cdot \frac{f(\hat{k})}{\hat{k}} - (x + n + \delta) \tag{29}$$

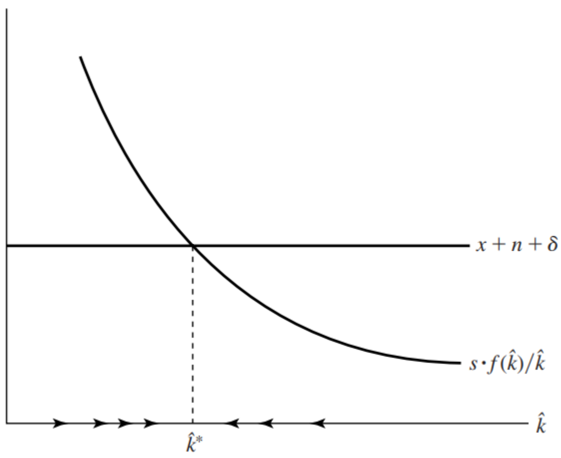

Figura 6: Transition Dynamics with Technological Progress

The only difference between equations (29) and (19) is that the last term on the right-hand side includes the parameter x. The term $x + n + \delta$ is now the effective depreciation rate for $\frac{\dot{\hat{k}}}{\hat{k}} = K/[L \cdot \hat{L}]$. If the savings rate s were zero, $\hat{k}$ would decrease partly due to the depreciation of K at rate δ and partly due to the growth of $\hat{L}$ at rate $x + n$.

Following an argument similar to the previous cases, we can show that the steady-state growth rate of $\hat{k}$ is zero. The steady-state value $\hat{k}^*$ satisfies the condition:

$$s \cdot f(\hat{k}^*) = (x + n + \delta) \cdot \hat{k}^* \tag{30}$$

The transition dynamics of $\hat{k}$ are qualitatively similar to those of k in the previous model. In particular, we can construct a graph similar to Figure 4 in which the horizontal axis involves $\hat{k}$, the downward-sloping curve is now $s \cdot f(\hat{k})/\hat{k}$, and the horizontal line is the level $x+n+\delta$, as shown in Figure 5. We can use this new graph to evaluate the relationship between the initial value $\hat{k}(0)$ and the growth rate $\frac{\dot{\hat{k}}}{\hat{k}}$.

In the steady state, the variables with hats—$\hat{k}$, $\hat{y}$, $\hat{c}$—are now constant. Therefore, the capital variables—k, y, c—now grow in the steady state at the exogenous rate of technological progress, x. The level variables—K, Y, C—grow accordingly in the steady state at the rate $n + x$, which is the sum of the population growth rate and the rate of technological change. It should be noted that, as in the previous analysis that did not consider technological progress, this is because the savings rate or the level of the production function affect the long-run levels $\hat{k}^*$, $\hat{y}^*$, $\hat{c}^*$ but not the steady-state growth rate. Therefore, such disturbances influence the growth rates during the transition from an initial position, represented by $\hat{k}(0)$, to the steady-state value, $\hat{k}^*$.

2.9. A Quantitative Measure of the Speed of Convergence

Finally, we can perform a quantitative assessment of how quickly the economy approaches its steady state for the case of a Cobb-Douglas production function. We can use equation

(29) to determine the growth rate of $\hat{k}$ in the Cobb-Douglas case as:

$$\frac{\dot{k}}{k} = s \cdot A \cdot k^{-(1-\alpha)} - (x + n + \delta) \tag{31}$$

The speed of convergence, β, is measured by how much the growth rate decreases as the capital stock increases proportionally, that is,

$$\beta = -\frac{\partial(\frac{\dot{k}}{k})}{\partial \log k} \tag{32}$$

Note that we define β with a negative sign because the derivative is negative, so β is positive.

To calculate β, we rewrite the growth rate in equation (31) as a function of $\log(\hat{k})$:

$$\frac{\dot{\hat{k}}}{\hat{k}} = s \cdot A \cdot e^{-(1-\alpha)\cdot\log(\hat{k})} - (x + n + \delta) \tag{33}$$

Then, we take the derivative of equation (33) with respect to $\log(\hat{k})$ to obtain an expression for β:

$$\beta = (1 - \alpha) \cdot s \cdot A \cdot \hat{k}^{-(1-\alpha)} \tag{34}$$

Note that the speed of convergence is not constant but decreases monotonically as the capital stock approaches its steady-state value. In the steady state, $s \cdot A \cdot (\hat{k}^*)^{-(1-\alpha)} = (x + n + \delta)$ holds. Therefore, near the steady state, the speed of convergence is equal to:

$$\beta^* = (1 - \alpha) \cdot (x + n + \delta) \tag{35}$$

It can also be shown that this speed of convergence is the same for $\hat{y}$.

The term $\beta^* = (1 - \alpha) \cdot (x + n + \delta)$ in equation (35) indicates how quickly output per effective worker, $\hat{y}$, approaches its steady-state value, $\hat{y}^*$, near the steady state. For example, if $\beta^* = 0{,}05$ per year, 5 percent of the gap between $\hat{y}$ and $\hat{y}^*$ disappears in one year.