Public Services

Macroeconomics 3
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Introduction: These class notes are based on Barro and Sala-i-Martin (2004). Continuing with one-sector endogenous growth models, now we examine the Public Services model. Public Services will eliminate the diminishing returns of capital by entering the production function, therefore causing endogenous growth.

1. Model Structure

This model extends the Ramsey framework by including publicly provided government services G as a productive input in firms' technology. These services are modeled as *pure public goods*, meaning they are non-rival, non-excludable, and affect output positively.

1.1 Decentralized Equilibrium

The representative firm's production function is:

$$Y_i = AL_i^{1-\alpha}K_i^{\alpha}G^{1-\alpha} \tag{1}$$

where:

- A: productivity parameter.
- K_i: private capital of firm i.
- L_i: labor employed by firm i, and L is constant.
- G: government services, provided to all firms for free.
- $0 < \alpha < 1$: capital share.

Assume:

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- Lump-sum taxes finance G, so there are no distortions.
- Government commits to a constant ratio of public expenditure to output: $\gamma = G/Y \in (0,1)$.

In per capita terms, with L constant, the firm's production becomes:

$$y_i = Ak_i^{\alpha} G^{1-\alpha} \tag{2}$$

Firms take G as given and choose k_i to maximize profits. The first-order condition implies:

$$\frac{\partial y_i}{\partial k_i} = \alpha A k_i^{\alpha-1} G^{1-\alpha} = r + \delta \tag{3}$$

In symmetric equilibrium, all firms choose the same $k_i = k$, and aggregate output becomes:

$$Y = ALk^{\alpha} G^{1-\alpha} \tag{4}$$

Using $G = \gamma Y$, we substitute into the production function:

$$Y = ALk^{\alpha} (\gamma Y)^{1-\alpha} \tag{5}$$
$$Y^{\alpha} = AL\gamma^{1-\alpha} k^{\alpha} \tag{6}$$
$$Y = \left(AL\gamma^{1-\alpha}\right)^{1/\alpha} k \tag{7}$$

Output is linear in capital: a hallmark of endogenous growth. Public services remove diminishing returns at the aggregate level.

Substituting into the private marginal product:

$$\frac{\partial y_i}{\partial k_i} = \alpha A^{1/\alpha} \gamma^{\frac{1-\alpha}{\alpha}} L^{\frac{1-\alpha}{\alpha}} \tag{8}$$

Hence, from the Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \left(\alpha A^{1/\alpha} \gamma^{\frac{1-\alpha}{\alpha}} L^{\frac{1-\alpha}{\alpha}} - \delta - \rho \right) \tag{9}$$

This growth rate is constant, implying the economy grows endogenously. However, it also exhibits **scale effects**, as growth increases with L, a result that is no support by empirical evidence.

1.2 Social Planner Solution and Optimal Policy

The planner internalizes the effect of public services on productivity. The planner chooses paths $\{c(t), k(t), G(t)\}$ to maximize household utility:

$$\text{máx} \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{10}$$

subject to the aggregate resource constraint (in per capita terms):

$$\dot{k} = Ak^\alpha G^{1-\alpha} - c - \delta k - G/L \tag{11}$$

The current-value Hamiltonian is:

$$\mathcal{H} = \frac{c^{1-\theta} - 1}{1-\theta} - \lambda \left[Ak^\alpha G^{1-\alpha} - c - \delta k - G/L\right] \tag{12}$$

First-order conditions:

$$\frac{\partial \mathcal{H}}{\partial c} = c^{-\theta} - \lambda = 0 \tag{13}$$

$$\frac{\partial \mathcal{H}}{\partial G} = (1-\alpha)Ak^\alpha G^{-\alpha} - 1/L = 0 \tag{14}$$

$$\frac{\partial \mathcal{H}}{\partial k} = \alpha A k^{\alpha-1} G^{1-\alpha} - \delta = -\dot{\lambda} \tag{15}$$

From the second condition, solving for the optimal G:

$$G = ((1-\alpha)AL)^{1/\alpha} k \tag{16}$$

Substituting into the Euler equation (using (8) and (10)):

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\alpha A k^{\alpha-1} \left((1-\alpha)AL\right)^{\frac{1-\alpha}{\alpha}} k^{1-\alpha} - \delta - \rho\right] \tag{17}$$

$$= \frac{1}{\theta}\left[\alpha A^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}} L^{\frac{1-\alpha}{\alpha}} - \delta - \rho\right] \tag{18}$$

Comparison with decentralized equilibrium: The planner's solution is:

$$\gamma^*_{opt} = 1 - \alpha \tag{19}$$

That is, the optimal ratio of public spending to output equals the elasticity of output with respect to the public input. In the decentralized equilibrium, γ is chosen exogenously or sub-optimally, leading to under- or over-investment in public services it if doesn't equal $1 - \alpha$.

1.3 Financing and Policy Implications

In the baseline model, the government uses lump-sum taxes to finance G. This is ideal because it avoids distortions in private decisions. However, in reality, lump-sum taxes are rarely feasible. Alternative instruments include:

1. Constant Consumption Tax: If the government levies a time-invariant consumption tax, it affects the level of consumption but not its growth rate. The Euler equation remains unchanged, so the dynamic efficiency of the economy is preserved. This makes a constant consumption tax a practical, non-distortionary alternative.

2. Income or Wage Taxes: If public spending is financed by taxing labor income (assuming inelastic labor supply), no distortion arises because L is fixed. Thus, the equilibrium conditions remain valid. However, with elastic labor supply (not modeled here), wage taxes would distort labor-leisure choice and reduce labor effort, potentially lowering Y, G, and growth.

On the contrary, some distortion taxes that may reduce the effectiveness of the public services are:

Capital or Output Taxes: These are distortionary in the Ramsey setup. They reduce the return on saving and lower the incentive to accumulate capital, reducing growth. If capital is taxed to finance G, the marginal benefit from the public input may not compensate for the private disincentive to invest.

Conclusion: The model shows how productive public services can generate sustained growth — but only if financed in a way that preserves incentives for capital accumulation. The optimal public spending ratio is $\gamma = 1 - \alpha$, and to implement it without harming growth, policy must avoid distortionary taxes.

2. Public Goods and Congestion Model

We now extend the public services model by introducing *congestion effects*. That is, although the public service G is non-rival in nature, its productivity declines as output expands. This captures the idea that public infrastructure gets "congested" as more firms attempt to use it simultaneously.

Let the production function of a representative firm be:

$$Y = AK \cdot f\left(\frac{G}{Y}\right)$$

where $f' > 0$, $f'' < 0$, and $\lim_{x \to 0^+} f(x) = 0$. This reflects diminishing returns to the effective use of G per unit of output. This specification introduces congestion: as output increases, the effective public input per unit of output, G/Y, decreases.

2.1 Decentralized Equilibrium

We assume a representative agent with utility:

$$U = \int_0^\infty \frac{c^{1-\theta}-1}{1-\theta} e^{-\rho t} dt, \quad \theta > 0$$

and inelastic labor supply $L = 1$. Public services are financed via lump-sum taxation and provided exogenously as a constant share of output:

$$G(t) = \tau Y(t)$$

where $\tau \in (0,1)$ is the public spending to output ratio.

$$Y_i = AK_i f\left(\frac{G}{Y}\right) \quad \rightarrow \quad \text{Individual production}$$

An increase in Y relative to G reduces the public services available to each producer.

Given a $\frac{G}{Y}$, production exhibits constant returns with respect to private capital K_L. If $\frac{G}{Y}$ is maintained, there will be endogenous growth as in the AK model.

$$R = Af\left(\frac{G}{Y}\right) = r + \delta \quad \rightarrow \quad \dot{c} = \frac{1}{\theta}\left[Af\left(\frac{G}{Y}\right) - \delta - \rho\right] \quad \rightarrow \quad \text{Increasing in } \frac{G}{Y}$$

2.2 Social Planner's Problem

$$\text{máx} \int_0^\infty u(c(t)) e^{-\rho t} dt$$

$$\text{s.t.} \quad Y = C + I + G \rightarrow Akf\left(\frac{G}{Y}\right) = c + \dot{k} + \delta k + \frac{G}{L}$$

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} e^{-\rho t} + \lambda\left[Akf\left(\frac{G}{Y}\right) - c - \delta k - \frac{G}{L}\right]$$

Note that since Y depends on k, the derivative is not so straightforward. It is useful to first calculate the derivative with respect to k and G.

$$\begin{aligned}
\frac{\partial y}{\partial k} &= A \cdot f\left(\frac{G}{Y}\right) + Akf'\left(\frac{G}{Y}\right) \cdot \left(-\frac{G/L}{y^2}\right)\frac{\partial y}{\partial k} \\
&= A \cdot f\left(\frac{G}{Y}\right) + \frac{Akf'\left(\frac{G}{Y}\right)}{Akf\left(\frac{G}{Y}\right)} \cdot \left(-\frac{G/L}{y}\right)\frac{\partial y}{\partial k} \\
&= A \cdot f\left(\frac{G}{Y}\right) + \frac{f'(G/Y)}{f(G/Y)} \cdot \left(\frac{-G}{Y}\right)\frac{\partial y}{\partial k} \\
\frac{\partial y}{\partial k} &= \frac{A \cdot f\left(\frac{G}{Y}\right)}{1 + \left(\frac{G}{Y}\right)\frac{f'(G/Y)}{f(G/Y)}}
\end{aligned}$$

$$\frac{\partial y}{\partial G} = Akf'\left(\frac{G}{Y}\right)\cdot\frac{1}{Y} + Akf'\left(\frac{G}{y}\right)\cdot\left(\frac{-G/L}{y^2}\right)\frac{\partial y}{\partial G}$$
$$\frac{\partial y}{\partial G} = \frac{f'\left(G/Y\right)}{f\left(G/Y\right)}\cdot\frac{1}{L} + \frac{f'\left(G/Y\right)}{f\left(G/Y\right)}\cdot\left(\frac{-G/L}{y}\right)\frac{\partial y}{\partial G}$$
$$\frac{\partial y}{\partial G} = \frac{\frac{1}{L}\frac{f'(G/Y)}{f(G/Y)}}{1+\left(G/Y\right)\frac{f'(G/Y)}{f(G/Y)}}$$

Now, after this "light" math, we proceed to derive the FOCs:

$$\frac{\partial\mathcal{H}}{\partial c} = 0 \rightarrow e^{-pt}c^{-\theta} = 0 \tag{20}$$
$$\frac{\partial\mathcal{H}}{\partial G} = 0 \rightarrow \frac{\partial y}{\partial G} = \frac{1}{L} \tag{21}$$
$$\frac{\partial\mathcal{H}}{\partial k} = -\dot{\lambda} \rightarrow \lambda\left[\frac{\partial y}{\partial k} - \delta\right] = -\dot{\lambda} \tag{22}$$

From (21) and our previous derivatives:

$$\frac{\frac{1}{L}\frac{f'(G/Y)}{f(G/Y)}}{1+\left(G/Y\right)\frac{f'(G/Y)}{f(G/Y)}} = \frac{1}{L}$$
$$\frac{f'\left(G/Y\right)}{f\left(G/Y\right)} = 1 + \left(G/Y\right)\frac{f'\left(G/Y\right)}{f\left(G/Y\right)}$$
$$\frac{f'\left(G/Y\right)}{f\left(G/Y\right)} = \frac{1}{1-\left(G/Y\right)}$$

From (20) and (22):

$$-(-\rho e^{-\rho t}c^{-\theta} + e^{-\rho t}c^{-\theta-1}(-\theta)\dot{c}) = e^{-\rho t}c^{-\theta}\left[\frac{\partial y}{\partial k} - \delta\right]$$
$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\frac{\partial y}{\partial k} - \delta - \rho\right] = \frac{1}{\theta}\left[\frac{A\cdot f\left(\frac{G}{Y}\right)}{1+\left(\frac{G}{Y}\right)\frac{f'(G/Y)}{f(G/Y)}} - \delta - \rho\right]$$

Substituting (21):

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\frac{\partial y}{\partial k} - \delta - \rho\right] = \frac{1}{\theta}\left[\frac{A\cdot f\left(\frac{G}{Y}\right)}{1+\left(\frac{G}{Y}\right)\frac{1}{1-(G/Y)}} - \delta - \rho\right]$$
$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[\left(1-G/Y\right)Af(G/Y) - \delta - \rho\right]$$

Remember that G/Y is between 0 and 1 (proportion of GDP representing expenditure), therefore, in this case the decentralized solution presents a higher rate than the centralized solution. Why?

The individual producer, by increasing their level of private capital, increases their private production, and as each one does this, aggregate production increases, reducing the G/Y ratio, increasing congestion and reducing the ratio for other firms. That is, there is a non-internalized negative externality. Since the externality is negative, individual producers over-produce compared to what they should if they internalized this congestion effect.

2.3 Optimal Policy and Congestion Tax

The congestion externality arises because firms, by increasing output, lower the public capital ratio G/Y and reduce everyone's productivity. This negative externality is not internalized in private decisions.

Optimal Tax: To replicate the planner's allocation, implement an output tax:

$$\tau = \frac{G}{Y}$$

This transforms the private return from $Af(G/Y)$ into the social return $(1-G/Y)Af(G/Y)$. It forces firms to internalize the marginal congestion cost they impose on others, aligning private incentives with the social optimum.

Conclusion: Unlike the pure public goods case where the decentralized economy under- or over- provides G, here the distortion is in overaccumulation of private capital, reducing the effectiveness of public inputs. The corrective policy is a Pigouvian tax on output.