AK Model

Macroeconomics 3
Professor: César Salinas
Teaching Assistant: Francisco Arizola*

Introduction: These class notes are based on Barro and Sala-i-Martin (2004). With the Ramsey model, we endogenized the consumption-saving decision. Furthermore, we found that the growth rate of per capita variables equals the growth rate of technological progress, which we assumed exogenous. Therefore, the next logical step is to endogenize this technological progress and understand its sources. In this first endogenous growth model, we eliminate decreasing returns to capital: the famous AK model.

1. Households

The household problem remains the same as in the Ramsey model. They live infinitely and maximize their aggregate utility:

$$U = \int_0^\infty u[c(t)] \cdot e^{nt} \cdot e^{-\rho t}\, dt \tag{1}$$

As before, households receive labor income (wages) which they use to accumulate assets and consume. Their per capita budget constraint is given by:

$$\dot{a} = w + ra - c - na \tag{2}$$

Where a represents assets per person, w wages, r the interest rate on assets, c per capita consumption, and n population growth rate. And as always, we must impose a condition to prevent any Ponzi schemes:

$$\lim_{t\to\infty} \left[a(t) \cdot \exp\left(-\int_0^t [r(v) - n]\, dv\right)\right] \geq 0 \tag{3}$$

Assuming a utility function of the form:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta} \tag{4}$$

*Email: fa.arizolab@up.edu.pe

The optimal conditions are again:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \cdot (r - \rho) \tag{5}$$

And the transversality condition:

$$\lim_{t \to \infty} \left[a(t) \cdot \exp\left(- \int_0^t [r(v) - n] \, dv \right) \right] = 0 \tag{6}$$

2. Firms

The only difference in the model lies in the production function, which eliminates decreasing marginal returns to capital (its exponent is now equal to 1):

$$y = f(k) = Ak \tag{7}$$

Where $A > 0$ is a technological constant. This specification has two crucial characteristics:

- Constant marginal product: $f''(k) = 0$ (no decreasing returns)
- Violation of Inada conditions: $f'(k) = A$ even when $k \to \infty$

It is precisely this violation of the Inada condition $\lim_{k \to \infty} f'(k) = 0$ that enables endogenous growth. While the absence of decreasing returns may seem unrealistic at first glance, the interpretation becomes more reasonable if we consider that K includes not only physical capital but also human capital, public infrastructure, knowledge, etc.

Profit maximization conditions require that the marginal product of capital equals its user cost $R = r + \delta$. The novelty is that now the marginal product is constant:

$$r = A - \delta \tag{8}$$

Additionally, since the marginal product of labor is zero in this basic model (L doesn't appear in the production function), the wage w corresponding to labor would be zero.

3. Equilibrium

We assume, as in previous models, that the economy is closed, so our market clearing condition holds: $a = k$. Substituting this, $r = A - \delta$, and $w = 0$ into equations (2), (5) and (6):

$$\dot{k} = (A - \delta - n)k - c \tag{9}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(A - \delta - \rho) \tag{10}$$

$$\lim_{t\to\infty} \left\{k(t) \cdot e^{-(A-\delta-n)t}\right\} = 0 \tag{11}$$

Equation (9) reveals that the consumption growth rate **does not depend** on the level of per capita capital k. This confirms endogenous growth and implies that the consumption path follows:

$$c(t) = c(0) \cdot e^{\frac{1}{\theta}(A-\delta-\rho)t} \tag{12}$$

Where $c(0)$, the initial consumption level, will be determined later. To ensure positive growth but with bounded utility:

$$A > \rho + \delta > (A - \delta)(1 - \theta) + \theta n + \delta \tag{13}$$

This equation tells us two things. The first part, $A > \rho + \delta$, ensures that the growth rate is positive $\dot{c}/c > 0$. The second part guarantees that utility is bounded, meaning that utility from very distant periods is discounted so much that it contributes almost nothing, and thus aggregate utility converges (is not infinite)[1].

To find the capital growth rate, we divide equation (9) by k:

$$\frac{\dot{k}}{k} = (A - \delta - n) + \frac{c}{k}$$

By definition, in steady state all variables grow at constant rates, so with $\frac{\dot{k}}{k}$ constant and $(A - \delta - n)$ constant, $\frac{c}{k}$ must also be constant. Therefore, we conclude that per capita consumption and per capita capital grow at the same rate. Moreover, since per capita output y is a linear function of k, it also grows at that same rate: $\gamma_k = \gamma_c = \gamma_y$.

4. Transition Dynamics

Now, this is the growth rate of per capita variables in steady state. To calculate the capital growth rate outside steady state, we substitute the expression for $c(t)$ from equation (12) into (9):

[1] To see this, substitute equation (12) and the functional form of utility into:

$$U = \int_0^\infty u[c(t)] \cdot e^{nt} \cdot e^{-\rho t}\, dt$$

And observe that for the integral not to diverge to infinity, the exponent of the discount factor must be negative, which implies:

$$\rho + \delta > (A - \delta)(1 - \theta) + \theta n + \delta$$

$$\dot{k} = (A - \delta - n)k - c(0)e^{\frac{1}{\theta}(A-\delta-\rho)t} \tag{14}$$

This is a first-order linear differential equation in k, whose general solution is[2]:

$$k(t) = \text{constant} \cdot e^{(A-\delta-n)t} + \frac{c(0)}{\varphi}e^{\frac{1}{\theta}(A-\delta-\rho)t} \tag{15}$$

Where the parameter φ is defined as:

$$\varphi \equiv \frac{(A-\delta)(\theta-1)}{\theta} + \frac{\rho}{\theta} - n \tag{16}$$

Which can also be rewritten as:

$$\varphi = (A - \delta - n) - \gamma \tag{17}$$

where $\gamma = \frac{1}{\theta}(A - \delta - \rho)$ is the consumption growth rate (equation 10). Condition (13) guarantees that $\varphi > 0$. Substituting (15) into the transversality condition (11):

$$\lim_{t\to\infty} \left\{ \text{constant} + \frac{c(0)}{\varphi}e^{-\varphi t} \right\} = 0 \tag{18}$$

Since $\varphi > 0$, the second term converges to zero, so the condition requires the constant to be zero. This implies the key relationships:

$$c(t) = \varphi k(t) \tag{19}$$

$$\frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \frac{1}{\theta}(A - \delta - \rho) \tag{20}$$

Moreover, since $y = Ak$, all variables grow at the same rate:

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} = \frac{\dot{c}}{c} = \gamma \tag{21}$$

Absence of transition dynamics: This tells us that all variables grow at the same rate at all times. That is, the AK model exhibits balanced growth from the initial moment - there is no transition phase to steady state. The variables start at:

- $k(0)$: initial capital stock
- $c(0) = \varphi k(0)$: initial consumption
- $y(0) = Ak(0)$: initial output

[2]See Appendix for a more detailed solution

And grow at the constant rate $\gamma = \frac{1}{\theta}(A - \delta - \rho)$ for all $t \geq 0$.

In the AK model, variations in fundamental parameters have differentiated effects on economic variables. A permanent increase in the population growth rate (n) does not alter per capita growth rates (determined by equation 20), but reduces the *level* of per capita consumption, as deduced from equations (16) and (19). Conversely, changes in productivity (A), time preference (ρ), or intertemporal risk aversion (θ) affect both levels and growth rates of c and k.

The gross saving rate in this model exhibits peculiar behavior:

$$s = \frac{\dot{K} + \delta K}{Y} = \frac{1}{A}\left(\frac{\dot{k}}{k} + n + \delta\right) = \frac{A - \rho + \theta n + (\theta - 1)\delta}{\theta A} \tag{22}$$

Where $\frac{\dot{k}}{k} = \frac{1}{\theta}(A - \delta - \rho)$.

Notably, the saving rate is constant over time and depends - except for n - on the same parameters that determine the per capita growth rate. This property reflects the absence of transition dynamics characteristic of the AK model, where all key variables grow at constant rates from the outset.

5. Phase Diagram of the AK Model

The dynamic analysis of the AK model can be illustrated through a phase diagram in (k, c). Unlike neoclassical models, in this case there is no $\dot{c} = 0$ curve because consumption growth is always positive ($A > \rho + \delta$), reflected by arrows permanently pointing upward in the diagram. The $\dot{k} = 0$ condition defines a straight line through the origin with slope $A - \delta - n$, where arrows point rightward for values of k greater than those on this line, and leftward for smaller values.

The stable path (saddle path) of the AK model is another straight line, but with slope $\varphi = (A - \delta - n) - \gamma$, smaller than the slope of the $\dot{k} = 0$ curve. The transversality condition and Euler equation ensure the economy always lies on this stable path, maintaining a constant consumption-capital ratio (c/k). This property reinforces the key characteristic of the AK model: the absence of transition dynamics, as the economy is permanently in a state of balanced growth where all variables grow at the constant rate γ.

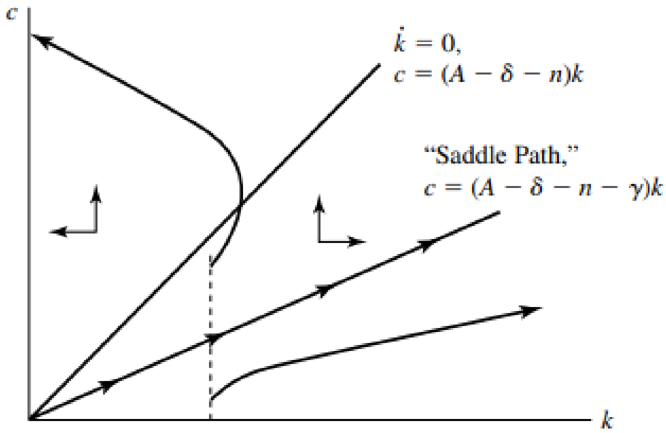

Figura 1: Phase Diagram of the AK Model

6. Key Conclusions of the AK Model

The AK model represents a radical contrast with traditional neoclassical models by generating endogenous growth without needing exogenous technological progress. The per capita growth rate ($\gamma = \frac{1}{\theta}(A - \delta - \rho)$) depends directly on fundamental parameters: it increases with higher capital productivity (A), lower time preference (ρ), and lower intertemporal risk aversion (θ). This key result stems from the absence of decreasing returns to capital ($f''(k) = 0$), allowing policy changes affecting A (such as infrastructure or human capital investment) to have permanent effects on the growth rate, not just on income levels as in the neoclassical model.

Unlike the Ramsey model where long-term growth is determined exogenously (x) and only transition dynamics exist, the AK model has no transition phase - the economy grows at a constant rate from the start. While this theoretical framework maintains the Pareto optimality of competitive equilibrium, its main empirical limitation lies precisely in the assumption of constant returns to scale in capital. Nevertheless, when convergence to steady state is slow (as empirical evidence suggests), the AK model provides a reasonable approximation for prolonged time horizons, being particularly useful for analyzing the role of policies affecting aggregate capital productivity in a broad sense.

7. Appendix: Solution of the Differential Equation

We start from the dynamic equation for capital:

$$\dot{k} = (A - \delta - n)k - c(0)e^{\frac{1}{\theta}(A-\delta-\rho)t}$$

Which can be rewritten as a non-homogeneous linear differential equation:

$$\dot{k} - (A - \delta - n)k = -c(0)e^{\frac{1}{\theta}(A-\delta-\rho)t} \tag{1}$$

Step 1: Integrating Factor

To solve (1), we use the integrating factor method. We define:

$$\mu(t) = e^{\int -(A-\delta-n)dt} = e^{-(A-\delta-n)t}$$

Step 2: Multiply by the Integrating Factor

We multiply both sides of (1) by $\mu(t)$:

$$e^{-(A-\delta-n)t}\dot{k} - (A - \delta - n)e^{-(A-\delta-n)t}k = -c(0)e^{\left[\frac{1}{\theta}(A-\delta-\rho)-(A-\delta-n)\right]t}$$

The left side is now the derivative of a product:

$$\frac{d}{dt}\left(e^{-(A-\delta-n)t}k\right) = -c(0)e^{-\varphi t} \tag{2}$$

where we have defined:

$$\varphi \equiv \left(\frac{\theta - 1}{\theta}\right)(A - \delta) + \frac{\rho}{\theta} - n \tag{3}$$

Step 3: Integrate Both Sides

We integrate equation (2) from 0 to t:

$$\int_0^t \frac{d}{dt}\left(e^{-(A-\delta-n)t}k\right) dt = -c(0)\int_0^t e^{-\varphi t}dt$$

Which gives us:

$$e^{-(A-\delta-n)t}k(t) - k(0) = \frac{c(0)}{\varphi}\left(e^{-\varphi t} - 1\right) \tag{4}$$

Step 4: Solve for k(t)

Solving for $k(t)$:

$$k(t) = k(0)e^{(A-\delta-n)t} - \frac{c(0)}{\varphi}e^{(A-\delta-n)t} + \frac{c(0)}{\varphi}e^{(A-\delta-n-\varphi)t}$$

Substituting φ from (3) and simplifying:

$$k(t) = \left[k(0) - \frac{c(0)}{\varphi}\right]e^{(A-\delta-n)t} + \frac{c(0)}{\varphi}e^{\frac{1}{\theta}(A-\delta-\rho)t} \tag{5}$$

Step 5: Transversality Condition

For the transversality condition to hold:

$$\lim_{t\to\infty} e^{-(A-\delta-n)t}k(t) = 0$$

The first term in (5) diverges unless:

$$k(0) - \frac{c(0)}{\varphi} = 0 \implies c(0) = \varphi k(0)$$

Therefore, the final solution is:

$$k(t) = k(0)e^{\frac{1}{\theta}(A-\delta-\rho)t}$$

Stability Condition

To guarantee convergence, we require:

$$\varphi > 0 \implies (\theta - 1)(A - \delta) + \rho - \theta n > 0$$

This condition ensures that utility is bounded and the transversality condition holds.