Ramsey Model

Macroeconomics 3
Professor: César Salinas
Teaching Assistant: Francisco Arizola*

Introduction: These class notes are based on Barro and Sala-i-Martin (2004). An important limitation of previous models was the exogeneity of the savings rate and, consequently, the fraction of income consumed. Now, savings will be treated as an endogenous variable resulting from the households' optimization process.

1. Households

1.1. Model Setup

Families provide labor services in exchange for wages, receive interest income on assets, purchase goods for consumption, and save by accumulating assets. The basic model assumes identical households: each has the same preference parameters, faces the same wage rate (since all workers are equally productive), starts with the same assets per person, and has the same population growth rate. Given these assumptions, the analysis can use the usual representative agent framework, in which equilibrium is derived from the decisions of a single family.

Each household contains one or more working adults. When making plans, these adults take into account the welfare and resources of future generations. Then, $C(t)$ is total consumption at time t, and $c(t) \equiv \frac{C(t)}{L(t)}$ is consumption per adult.

Each household aims to maximize total utility, U, which includes that of their future descendants (so we work with an infinite horizon). Current adults expect their family size to grow at a rate n due to the net influences of fertility and mortality. If we normalize the number of adults at time 0 to one, the family size at time t—corresponding to the adult population—is:

$$L(t) = e^{nt}$$

$$U = \int_0^\infty u[c(t)] \cdot e^{nt} \cdot e^{-\rho t}\, dt \tag{1}$$

We assume that $u(c)$ is increasing in c and concave—$u'(c) > 0, u''(c) < 0$. The concavity assumption generates a desire to smooth consumption over time. We also assume that $u(c)$ satisfies the Inada conditions: $u'(c) \to \infty$ when $c \to 0$, and $u'(c) \to 0$ when $c \to \infty$.

*Author of these lecture notes, you can contact me through my Email: fa.arizolab@up.edu.pe

The multiplication of $u(c)$ by the family size, $L = e^{nt}$, adds the utilities of all living members at time t. The factor $e^{-\rho t}$ reflects the time preference rate, $\rho > 0$, which discounts utilities the later they are received. We assume $\rho > n$, which ensures that U in equation (1) is finite if c is constant over time.

Households own assets in the form of capital shares (to be introduced later) or as loans. Negative loans represent debts. We denote the household's net assets per person as $a(t)$, where $a(t)$ is measured in real terms, that is, in units of consumption goods. Additionally, we assume that each adult inelastically supplies one unit of labor services per unit of time.

The total income received by the set of households is, therefore, the sum of labor income, $w(t) \cdot L(t)$, and asset income, $r(t) \cdot (\text{Assets})$. Households use the income they do not consume to accumulate more assets:

$$\frac{d(\text{Assets})}{dt} = r \cdot (\text{Assets}) + wL - C \tag{2}$$

where we omit time subscripts whenever there is no ambiguity. Since a is per capita assets, we have

$$\dot{a} = \frac{d(A/L)}{dt} = \frac{1}{L} \cdot \left(\frac{d(\text{Assets})}{dt} \right) - na \tag{3}$$

Therefore, if we divide equation (2) by L and substitute (3), we obtain the budget constraint in per capita terms:

$$\dot{a} = w + ra - c - na \tag{4}$$

To rule out Ponzi schemes, we assume that the credit market imposes a constraint on the amount of borrowing. The appropriate constraint turns out to be that the present value of assets must be asymptotically non-negative, that is:

$$\lim_{t\to\infty} \left[a(t) \cdot \exp\left(-\int_0^t [r(v) - n]\, dv \right) \right] \geq 0 \tag{5}$$

Therefore, we formulate the household's problem as:

$$\max U = \int_0^\infty u[c(t)] \cdot e^{(n-\rho)t}\, dt$$

subject to:

$$\dot{a} = w + ra - c - na$$

$$\lim_{t\to\infty} \left[a(t) \cdot \exp\left(-\int_0^t [r(v) - n]\, dv \right) \right] \geq 0$$

The Hamiltonian for this optimization problem is:

$$H = u(c) \cdot e^{-(\rho-n)t} + \lambda\,(w + ra - c - na)$$

1.2. Solution to the Household Problem

The first-order conditions are:

$$\frac{\partial H}{\partial c} = 0 \implies \lambda = u'(c)e^{-(\rho-n)t} \tag{6}$$

$$\frac{\partial J}{\partial a} = -\dot{\lambda} \implies -\dot{\lambda} = \lambda(r-n) \tag{7}$$

And the transversality condition:

$$\lim_{t\to\infty} [\lambda(t)\cdot a(t)] = 0 \tag{8}$$

If we differentiate equation (6) with respect to time and substitute $\dot{\lambda}$ in equation (7), we obtain the basic condition for choosing consumption over time:

$$\begin{aligned} e^{-(\rho-n)t}\left[u''(c)\dot{c} - (\rho-n)u'(c)\right] &= -u'(c)e^{-(\rho-n)t}(r-n) \\ u''(c)\dot{c} - (\rho-n)u'(c) &= -u'(c)(r-n) \\ u''(c)\dot{c} &= -u'(c)(r-n) + (\rho-n)u'(c) \\ u''(c)\dot{c} &= u'(c)(\rho-r) \\ r &= \rho - \frac{u''(c)\dot{c}}{u'(c)} \end{aligned} \tag{9}$$

This is the Euler equation. This condition states that households choose their consumption such that the return rate, r, equals the time preference rate, ρ, adjusted by the rate of change in the marginal utility of consumption, $\frac{u''(c)\dot{c}}{u'(c)}$. The interest rate, r, on the left-hand side of equation (9) is the return rate on savings. The right-hand side of the equation can be seen as the return rate on consumption.

The term $\frac{u''(c)\dot{c}}{u'(c)}$ represents the rate of change of the marginal utility of consumption over time. If $\dot{c} > 0$, consumption is growing, and since $u''(c) < 0$, the term $\frac{u''(c)\dot{c}}{u'(c)}$ will be negative. This implies that the return r will be greater than ρ, incentivizing saving for higher future consumption.

The magnitude of the elasticity of marginal utility, $\left[-\frac{u''(c)\dot{c}}{u'(c)}\right]$, is sometimes called the reciprocal of the intertemporal elasticity of substitution. Equation (9) shows that to find a steady state where r and $\dot{c}/c$ are constant, this elasticity must be asymptotically constant[1]. Therefore, we follow the common practice of assuming the functional form

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta} \tag{10}$$

[1]This can be seen more easily by multiplying the last term by c/c:

$$r = \rho - \frac{u''(c)c}{u'(c)}\frac{\dot{c}}{c}$$

Where $\theta > 0$, so that the elasticity of marginal utility equals the constant $-\theta$. The elasticity of substitution for this utility function is the constant $\sigma = 1/\theta$. Therefore, this form is called the *constant intertemporal elasticity of substitution* (CIES) utility function. The higher θ, the faster the proportional decrease in $u'(c)$ in response to increases in c and, thus, the less willing households are to accept deviations from a uniform pattern of c over time. As θ approaches 0, the utility function approaches a linear form in c; linearity means households are indifferent to the timing of consumption if $r = \rho$ holds.

The form of $u(c)$ in equation (2.10) implies that the optimality condition in equation (2.9) simplifies to:

$$\frac{\dot{c}}{c} = \frac{1}{\theta} \cdot (r - \rho) \tag{11}$$

Therefore, the relationship between r and ρ determines whether households choose a per capita consumption pattern that increases over time, remains constant, or decreases over time. A lower willingness to substitute intertemporally (a higher value of θ) implies a smaller response of $\dot{c}/c$ to the difference between r and ρ.

Transversality Condition: The transversality condition in equation (8) states that the value of the household's per capita assets—the quantity $a(t)$ multiplied by the shadow price $\lambda(t)$—must approach 0 as time tends to infinity. If we think of infinity as the end of the planning horizon, the intuition is that optimizing agents do not wish to have any remaining assets at the end, as utility would increase if the assets were used for consumption within the finite horizon.

2. Firms

Firms produce goods, pay wages for labor, and make payments for renting capital. Each firm has access to the production technology:

$$Y(t) = F[K(t), L(t), T(t)]$$

where Y is the flow of output, K is the capital input (in units of goods), L is the labor input, and $T(t)$ is the level of technology, which is assumed to grow at a constant rate $x \geq 0$. Thus, $T(t) = e^{xt}$, where we normalize the initial level of technology, $T(0)$, to 1. The function $F(\cdot)$ satisfies the neoclassical properties. In particular, Y exhibits constant returns to scale in K and L, and each input has a positive and diminishing marginal product.

In the Solow-Swan model, it was shown that a steady state coexists with technological progress at a constant rate only if this progress takes the *labor-augmenting* form:

$$Y(t) = F[K(t), L(t) \cdot T(t)] \tag{12}$$

If we again define *effective labor* as the product of raw labor and the level of technology:

$$\hat{L} \equiv L \cdot T(t) \tag{13}$$

The production function can be written as:

$$Y = F(K, \hat{L}) \tag{14}$$

It will be convenient for us to work with variables that are constant in the steady state. In the previous chapter, it was shown that the steady state of the model with exogenous technological progress was such that per capita variables grew at the rate of technological progress, x. This property will continue to hold in the present model. Therefore, we will again work with quantities per unit of effective labor:

$$\hat{y} \equiv \frac{Y}{\hat{L}} \quad \text{and} \quad \hat{k} \equiv \frac{K}{\hat{L}} \tag{15}$$

The production function can then be rewritten in intensive form:

$$\frac{Y}{TL} = F\left(\frac{K}{LT}, \frac{LT}{LT}\right) = \hat{y} = f(\hat{k}) \tag{16}$$

We consider that firms rent capital services from households that own the capital. If we let $R(t)$ be the rental rate of a unit of capital, the total cost of capital for a firm is RK, which is proportional to K. Since capital stocks depreciate at a constant rate $\delta \geq 0$, the net rate of return for a household that owns a unit of capital is $R - \delta$. Recall that households can also receive the interest rate r on funds lent to other households. Since capital and loans are perfect substitutes as stores of value, we must have the market-clearing condition[2] $r = R - \delta$ or, equivalently, $R = r + \delta$.

The flow of profits of the representative firm at any point in time is given by:

$$\pi = F(K, \hat{L}) - (r + \delta) \cdot K - wL \tag{17}$$

The problem of maximizing the present value of profits reduces here to a problem of maximizing profits in each period without considering outcomes in other periods. Profits can be rewritten as:

$$\pi = \hat{L} \cdot \left[f(\hat{k}) - (r + \delta) \cdot \hat{k} - we^{-xt}\right] \tag{18}$$

A competitive firm, which takes r and w as given, maximizes profits for a given $\hat{L}$ by setting

$$f'(\hat{k}) = r + \delta \tag{19}$$

As is known, in a complete market equilibrium, w equals the marginal product of labor corresponding to the value of $\hat{k}$. We then have:

[2]In equilibrium, the return on capital must compensate for both the interest rate on loans and the depreciation of capital. If this condition is not met, households would have additional incentives to switch between capital and loans as stores of value.

$$\frac{\partial Y}{\partial L} = w$$

$$\frac{(\partial \hat{y} LT)}{\partial L} = w$$

$$\frac{\partial \hat{y}}{\partial \hat{k}}\frac{\partial \hat{k}}{\partial L} LT + \hat{y}T = \frac{\partial \hat{y}}{\partial \hat{k}}(-1)\frac{K}{TL^2}TL + \hat{y}T = f'(\hat{k})(-1)\frac{K}{TL}T + f(\hat{k})T = w$$

$$\left[f(\hat{k}) - \hat{k} \cdot f'(\hat{k})\right] e^{xt} = w \tag{20}$$

3. Equilibrium

We begin with the behavior of competitive households facing an interest rate, r, and a wage rate, w. We then introduced competitive firms that also faced given values of r and w. Now we can combine the behavior of households and firms to analyze the structure of a competitive market equilibrium.

Since the economy is closed, all debts within the economy must cancel out. Therefore, the assets per adult, a, are equal to the capital per worker, k. The equality between k and a is due to the fact that all the capital stock must be owned by someone in the economy; in particular, in this closed economy model, all the domestic capital stock must be owned by domestic residents.

From equation (4), and using $a = k$, $\hat{k} = ke^{-xt}$, and the conditions for r and w in equations (19) and (20), we obtain:

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta) \cdot \hat{k} \tag{21}$$

Where $\hat{c} \equiv C/\hat{L} = ce^{-xt}$, and $\hat{k}(0)$ is given. Equation (21) is the resource constraint for the entire economy: the change in the capital stock equals output minus consumption and depreciation, and the change in $\hat{k} \equiv K/\hat{L}$ also accounts for the growth of $\hat{L}$ at the rate $x + n$.

The differential equation (21) is the key relationship that determines the evolution of $\hat{k}$ and, therefore, of $\hat{y} = f(\hat{k})$ over time. The missing element, however, is the determination of $\hat{c}$. If we knew the relationship of $\hat{c}$ to $\hat{k}$ (or $\hat{y}$), or if we had another differential equation determining the evolution of $\hat{c}$, we could study the complete dynamics of the economy.

In the Solow-Swan model of Chapter 1, the missing relationship was provided by the assumption of a constant savings rate. This assumption implied the linear consumption function, $\hat{c} = (1 - s) \cdot f(\hat{k})$. In the present context, the behavior of the savings rate is not as simple, but we know from household optimization that c grows according to equation (11).

If we use the conditions $r = f'(\hat{k}) - \delta$ and $\hat{c} = ce^{-xt}$, we obtain:

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{c}}{c} - x = \frac{1}{\theta} \cdot [f'(\hat{k}) - \delta - \rho - \theta x] \tag{22}$$

We can write the transversality condition in terms of $\hat{k}$ by substituting $a = k$ and $\hat{k} = ke^{-xt}$ into equation (5) to obtain:

$$\lim_{t\to\infty} \hat{k} \cdot \exp\left(- \int_0^t [f'(\hat{k}) - \delta - x - n]\, dv \right) = 0 \tag{23}$$

Equations (21) and (22) form a system of two differential equations in $\hat{c}$ and $\hat{k}$. This system, together with the initial condition $\hat{k}(0)$ and the transversality condition, determines the time paths of $\hat{c}$ and $\hat{k}$.

4. Steady State

The steady-state values for $\hat{c}$ and $\hat{k}$ are determined by setting equations (21) and (22) to zero. The phase diagram resulting from this system of equations is shown in Figure 1.

The curve corresponds to $\hat{c} = f(\hat{k}) - (x + n + \delta) \cdot \hat{k}$, showing pairs of $(\hat{k}, \hat{c})$ that satisfy $\dot{\hat{k}} = 0$ in equation (21). Note that the peak in the curve occurs when $f'(\hat{k}) = \delta + x + n$, so that the interest rate, $f'(\hat{k}) - \delta$, equals the steady-state growth rate of output, $x + n$. This equality between the interest rate and the growth rate corresponds to the golden rule level of $\hat{k}$, because it leads to a maximum of $\hat{c}$ in the steady state. We denote by $\hat{k}_{\text{gold}}$ the value of $\hat{k}$ corresponding to the golden rule.

On the other hand, the vertical line corresponds to the condition $\dot{\hat{c}} = 0$, which implies:

$$f'(\hat{k}^*) = \delta + \rho + \theta x \tag{24}$$

This equation states that the steady-state interest rate, $f'(\hat{k}) - \delta$, equals the effective discount rate, $\rho + \theta x$. Note that $\dot{\hat{c}}/\hat{c} = 0$ holds at this value of $\hat{k}$ regardless of the value of $\hat{c}$. The key to determining $\hat{k}^*$ in equation (24) is the diminishing returns to capital, which makes $f'(\hat{k}^*)$ a monotonically decreasing function of $\hat{k}^*$.

The determination of the steady-state values, $(\hat{k}^*, \hat{c}^*)$, occurs at the intersection of the vertical line with the curve. In particular, with $\hat{k}^*$ determined from equation (24), the value of $\hat{c}^*$ is obtained by setting the expression in equation (21) to zero as:

$$\hat{c}^* = f(\hat{k}^*) - (x + n + \delta) \cdot \hat{k}^* \tag{25}$$

Additionally, note that $\hat{y}^* = f(\hat{k}^*)$ is the steady-state value of $\hat{y}$.

Consider the transversality condition in equation (23). Since $\hat{k}$ is constant in the steady state, this condition holds if the steady-state return rate, $r^* = f'(\hat{k}^*) - \delta$, exceeds the steady-state growth rate, $x + n$. Equation (24) implies that this condition can be written as:

$$\rho > n + (1 - \theta)x \tag{26}$$

If ρ is not high enough to satisfy equation (26), the household optimization problem is not well-defined because infinite utility would be achieved if c grew at the rate x. Therefore, we assume from now on that the parameters satisfy equation (26).

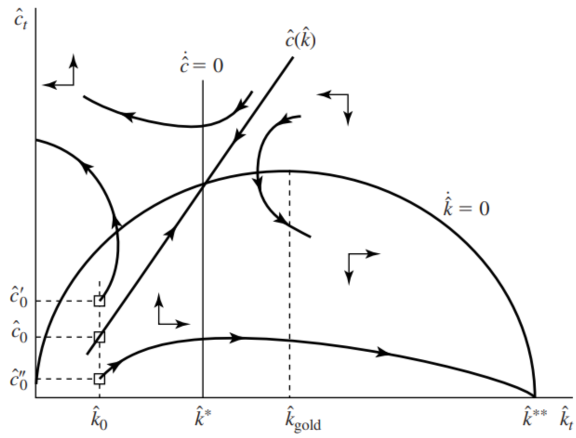

Figura 1: Phase diagram of the Ramsey model

The steady-state value, $\hat{k}^*$, will lie to the left of $\hat{k}_{\text{gold}}$ as long as the transversality condition, equation (26), is satisfied. The steady-state value is determined from $f'(\hat{k}^*) = \delta + \rho + \theta x$, while the golden rule value comes from $f'(\hat{k}_{\text{gold}}) = \delta + x + n$. The inequality in equation (26) implies that $\rho + \theta x > x + n$ and, therefore, $f'(\hat{k}^*) > f'(\hat{k}_{\text{gold}})$ and $\hat{k}^* < \hat{k}_{\text{gold}}$.

Note that the optimizing household does not save enough to reach the golden rule value, $\hat{k}_{\text{gold}}$. The reason is that the impatience reflected in the effective discount rate, $\rho + \theta x$, makes it not worthwhile to sacrifice more current consumption to achieve the maximum of $\hat{c}$ (the golden rule value, $\hat{c}_{\text{gold}}$) in the steady state.

5. Transition Dynamics

5.1. Phase Diagram

First, let's look at the locus $\dot{\hat{c}} = 0$. Since $\dot{\hat{c}} = \hat{c} \cdot \left(\frac{1}{\theta}\right) \cdot \left[f(\hat{k}) - \delta - \rho - \theta x\right]$, there are two ways in which $\dot{\hat{c}}$ can be zero: $\hat{c} = 0$ and $f(\hat{k}) = \delta + \rho + \theta x$, which is a vertical line at $\hat{k}^*$, the capital-labor ratio defined in equation (24). We note that $\hat{c}$ is increasing for $\hat{k} < \hat{k}^*$ (so the arrows point upward in this region) and is decreasing for $\hat{k} > \hat{k}^*$ (where the arrows point downward).

Now, let's look at the locus $\dot{\hat{k}} = 0$. This equation also implies that $\hat{k}$ is decreasing for values of $\hat{c}$ above the solid curve (so the arrows point to the left in this region) and is increasing for values of $\hat{c}$ below the curve (where the arrows point to the right).

Since the loci where $\dot{\hat{c}} = 0$ and $\dot{\hat{k}} = 0$ intersect three times, there are three steady states: the first is the origin ($\hat{c} = \hat{k} = 0$), the second steady state corresponds to $\hat{k}^*$ and $\hat{c}^*$, and the third implies a positive capital stock, $\hat{k}^{**} > 0$, but zero consumption. The second steady state is the one of interest. In particular, note that the pattern of arrows in Figure 1 is such that the economy can converge to this steady state if it starts in two of the four quadrants into which the two curves divide the space.

The dynamic equilibrium follows the stable saddle path shown by the solid curve with arrows. Suppose, for example, that the initial factor ratio satisfies $\hat{k}(0) < \hat{k}^*$, as shown in Figure 1. If the initial consumption ratio is $\hat{c}(0)$, as shown, the economy follows the stable path toward the steady-state pair, $(\hat{k}^*, \hat{c}^*)$.

5.2. The Importance of the Transversality Condition

The transversality condition is crucial for determining the unique equilibrium in the Ramsey model. Suppose a case where it is known that the world will end at a time $T > 0$. In this scenario, the utility function becomes:

$$U = \int_0^T u[c(t)] \cdot e^{nt} \cdot e^{-\rho t} \, dt$$

The no-Ponzi condition is expressed as:

$$a(T) \cdot \exp\left(- \int_0^T [r(v) - n] \, dv \right) \geq 0$$

Here, the only difference from the traditional model is the existence of a terminal date T. This modifies the transversality condition, which now is:

$$a(T) \cdot \exp\left(- \int_0^T [r(v) - n] \, dv \right) = 0$$

Since the exponential term cannot be zero in finite time, the condition implies that assets at the end of the horizon must be zero:

$$a(T) = 0$$

Since $a(t) = k(t)$, the transversality condition can also be expressed as:

$$\hat{k}(T) = 0$$

This means that, at time T, the capital stock must be equal to zero. Therefore, the initial choice of $\hat{c}(0)$ must be such that the capital is completely depleted by T. The agent derives utility only from consumption, so it is in their interest to exhaust all resources by the end of the horizon.

5.3. Dynamics of the Stable Arm and Capital Convergence

The stable arm shown in Figure 1 indicates that, if the initial level of capital per capita is less than its steady-state value, that is, if

$$\hat{k}(0) < \hat{k}^*,$$

Then both capital per capita ($\hat{k}$) and consumption per capita ($\hat{c}$) increase monotonically until they reach their steady-state values.

The increase in $\hat{k}$ implies that the return rate (r) decreases monotonically from its initial value

$$r(0) = f'\big(\hat{k}(0)\big) - \delta,$$

To its steady-state value:

$$r^* = \rho + \theta x.$$

According to equation (22) and the decreasing trajectory of r, the growth rate of consumption per capita ($\frac{\dot{c}}{c}$) also decreases monotonically. That is, the lower $\hat{k}(0)$ and, therefore, $\hat{y}(0)$, the higher the initial value of $\frac{\dot{c}}{c}$, but this rate will fall as convergence to the steady state is achieved.

5.4. The Behavior of the Savings Rate

The gross savings rate, s, is defined as:

$$s = 1 - \frac{\hat{c}}{f(\hat{k})}$$

In the Solow-Swan model (NC1), it was assumed that s was constant at an arbitrary level. However, in the Ramsey model, where consumers optimize, s can follow a more complex trajectory, with phases of increase and decrease as the economy develops and converges to the steady state.

The behavior of the savings rate is ambiguous because it results from the interaction of two opposing effects:

- **Substitution effect:** As $\hat{k}$ increases, the marginal productivity of capital $f'(\hat{k})$ decreases, which reduces the return rate r on savings. This reduces the incentive to save, causing the savings rate to tend to decrease.
- **Income effect:** In poor economies, the income per effective worker $f(\hat{k})$ is much lower than the long-run permanent income. Since households seek to smooth consumption, they tend to consume a larger proportion of their income when they are poor, implying a low savings rate. As $\hat{k}$ grows, the gap between current income and permanent income narrows, leading to a lower proportion of consumption relative to income and an increase in the savings rate.

The evolution of the savings rate during the transition depends on which of these effects dominates. As we know from previous courses, the substitution effect is usually stronger than the income effect, favoring consumption smoothing. However, although increases in the savings rate generate higher growth temporarily, as in Solow, this effect is eventually outweighed by diminishing marginal returns, and the economy converges to a new steady state. In this steady state, the level of capital per capita is higher, but per capita growth remains zero.

Conclusion: Although the savings rate may increase during the transition, this increase is not enough to eliminate the inverse relationship between the growth rate of capital ($\frac{\dot{k}}{k}$) and the level of capital per capita ($\hat{k}$). Therefore, the endogenous determination of the savings rate does not eliminate the convergence property of $\hat{k}$, and $\hat{c}$ in the long run.