Log Linearization

Macroeconomics 3
Professor: César Salinas
Teaching Assistant: Francisco Arizola*

Introduction: In macroeconomics, many dynamic models are based on nonlinear equations that more realistically reflect the behavior of agents and the evolution of the economy. However, these equations are often difficult to solve analytically, which complicates their analysis and use for making predictions or studying economic policy effects. In this context, log-linearization emerges as a useful tool that allows us to simplify these models by approximating them with linear versions around a steady state or long-run equilibrium. The main goal of this technique is to facilitate the study of the responses of economic variables to small shocks, while preserving the basic structure of the original model.

1. Basic algebra of log-linearization

It may sound complex, but in summary, log-linearization is a technique that consists of converting a nonlinear equation into a linear equation expressed in terms of logarithmic deviations from a steady state or equilibrium value. The main mathematical tool used in log-linearization is the natural logarithm. The idea of a *logarithmic deviation* involves taking the natural logarithm of a variable, for example $\ln(X)$, and then evaluating a small change in that variable, such as $\ln(X_1) - \ln(X_2)$. When this difference is small, it can be interpreted as the percentage change between X_1 and X_2. This is useful because in economics we are often interested in understanding how variables respond in relative or percentage terms to small perturbations.

The second tool we will use in log-linearization is the **Taylor approximation**. This tells us that any sufficiently smooth function can be represented as a power series around a particular point x^* (usually an equilibrium value or steady state). Intuitively, this expansion allows us to approximate a nonlinear function with a polynomial function that is much easier to work with. In the economic context, this technique is especially useful because it allows us to analyze how a variable responds to small changes in its determinants, while staying close to the equilibrium.

To formalize this approximation, consider an arbitrary univariate function $f(x)$. Taylor's theorem tells us that this function can be expressed as a power series around the point x^*:

*Author of these lecture notes and inspired by the notes of Eric Sims. You can contact me through my Email: fa.arizolab@up.edu.pe

$$f(x) = f(x^*) + f'(x^*) \cdot \frac{1}{1!}(x - x^*) + f''(x^*) \cdot \frac{1}{2!}(x - x^*)^2 + f^{(3)}(x^*) \cdot \frac{1}{3!}(x - x^*)^3 + \cdots$$

If the function $f(x)$ is sufficiently smooth, the higher-order terms (that is, the second and higher derivatives) will be small, and the function can be well-approximated in a neighborhood of x^* using only the first few terms of the expansion. Thus, the first-order linear approximation of $f(x)$ around x^* is:

$$f(x) \approx f(x^*) + f'(x^*) \cdot (x - x^*)$$

This approximation is the algebraic foundation of log-linearization and allows us to transform complex models into simpler expressions that capture economic behavior around a point x^*, which in our analysis usually represents the steady state.

2. Procedure

The general procedure for log-linearizing a function consists of the following steps:

1. Take natural logarithms of both sides of the nonlinear expression.
2. Apply a first-order Taylor expansion around the steady state.
3. Simplify and express the result in terms of percentage deviations from the steady state.

Suppose we have the following nonlinear function:

$$f(x) = \frac{g(x)}{h(x)}$$

The first step to log-linearize this expression is to take natural logarithms on both sides of the equation:

$$\ln f(x) = \ln g(x) - \ln h(x)$$

Next, we apply a first-order Taylor expansion to each of the terms around a point x^*:

$$\ln f(x) \approx \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*)$$
$$\ln g(x) \approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*)$$

$$\ln h(x) \approx \ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

This is based on the fact that the derivative of a natural logarithm satisfies:

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

Substituting the expansions into the original logarithmic equation:

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) \approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*) - \left[\ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)\right]$$

Grouping terms:

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*) \approx \ln g(x^*) - \ln h(x^*) + \left(\frac{g'(x^*)}{g(x^*)} - \frac{h'(x^*)}{h(x^*)}\right)(x - x^*)$$

Since by definition $\ln f(x^*) = \ln g(x^*) - \ln h(x^*)$, these terms cancel out and we obtain:

$$\frac{f'(x^*)}{f(x^*)}(x - x^*) \approx \frac{g'(x^*)}{g(x^*)}(x - x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

To express everything in percentage terms, we multiply and divide each term by x^*:

$$\frac{x^* f'(x^*)}{f(x^*)} \cdot \frac{(x - x^*)}{x^*} \approx \frac{x^* g'(x^*)}{g(x^*)} \cdot \frac{(x - x^*)}{x^*} - \frac{x^* h'(x^*)}{h(x^*)} \cdot \frac{(x - x^*)}{x^*}$$

To simplify notation, we define the percentage deviation of x from its steady-state value as:

$$\hat{x} \equiv \frac{x - x^*}{x^*}$$

Then, we can rewrite the previous expression in a more compact form:

$$\frac{x^* f'(x^*)}{f(x^*)} \cdot \hat{x} \approx \frac{x^* g'(x^*)}{g(x^*)} \cdot \hat{x} - \frac{x^* h'(x^*)}{h(x^*)} \cdot \hat{x}$$

This technique can be easily extended to the multivariable case, where each variable is approximated around its steady-state value.

3. Examples

3.1. Cobb-Douglas Production Function

Consider the Cobb-Douglas production function of the form:

$$y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$$

The first step is to take natural logarithms:

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1-\alpha) \ln n_t$$

Now, we expand each term around its steady-state value using a first-order Taylor expansion. For the left-hand side:

$$\ln y_t \approx \ln y^* + \frac{1}{y^*}(y_t - y^*)$$

For the right-hand side:

$$\ln a_t \approx \ln a^* + \frac{1}{a^*}(a_t - a^*), \quad \ln k_t \approx \ln k^* + \frac{1}{k^*}(k_t - k^*), \quad \ln n_t \approx \ln n^* + \frac{1}{n^*}(n_t - n^*)$$

Substituting everything into the original expression:

$$\begin{aligned}
\ln y^* + \frac{1}{y^*}(y_t - y^*) \approx{}& \ln a^* + \frac{1}{a^*}(a_t - a^*) \\
&+ \alpha \left(\ln k^* + \frac{1}{k^*}(k_t - k^*) \right) \\
&+ (1-\alpha) \left(\ln n^* + \frac{1}{n^*}(n_t - n^*) \right)
\end{aligned}$$

Since the steady-state values satisfy:

$$\ln y^* = \ln a^* + \alpha \ln k^* + (1-\alpha) \ln n^*$$

The constant terms cancel out, yielding:

$$\frac{1}{y^*}(y_t - y^*) = \frac{1}{a^*}(a_t - a^*) + \frac{\alpha}{k^*}(k_t - k^*) + \frac{1-\alpha}{n^*}(n_t - n^*)$$

Finally, we express everything in terms of percentage deviations from the steady state, using the notation:

$$\tilde{y}_t = \frac{y_t - y^*}{y^*}, \quad \tilde{a}_t = \frac{a_t - a^*}{a^*}, \quad \tilde{k}_t = \frac{k_t - k^*}{k^*}, \quad \tilde{n}_t = \frac{n_t - n^*}{n^*}$$

We thus obtain the log-linearized form:

$$\tilde{y}_t = \tilde{a}_t + \alpha \tilde{k}_t + (1-\alpha)\tilde{n}_t$$

Intuition: the (percentage) deviations of output from its steady-state value depend on deviations of productivity, capital, and labor from their respective steady-state values. The magnitude of these percentage deviations is given by the coefficients in the last expression.

3.2. Euler equation for consumption

Consider the Euler equation that arises from the household's intertemporal optimization problem under a CRRA (Constant Relative Risk Aversion) utility function:

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta(1 + r_t)$$

where $\sigma > 0$ is the coefficient of relative risk aversion. We take natural logarithms on both sides:

$$\sigma \ln c_{t+1} - \sigma \ln c_t = \ln \beta + \ln(1 + r_t)$$

We expand both sides around the steady state using a first-order Taylor expansion. For the left-hand side:

$$\sigma \ln c^* + \frac{\sigma}{c^*}(c_{t+1} - c^*) - \Big(\sigma \ln c^* + \frac{\sigma}{c^*}(c_t - c^*)\Big) = \ln \beta + \ln(1 + r^*) + \frac{1}{1+r^*}(r_t - r^*)$$

Canceling out constant terms:

$$\frac{\sigma}{c^*}(c_{t+1} - c^*) - \frac{\sigma}{c^*}(c_t - c^*) = \ln \beta + \ln(1 + r^*) + \frac{1}{1+r^*}(r_t - r^*)$$

We know that in the steady state, consumption is constant, so:

$$1 + r^* = \frac{1}{\beta} \quad \Rightarrow \quad \ln(1 + r^*) = -\ln \beta$$

Substituting this in, we get:

$$\frac{\sigma}{c^*}(c_{t+1} - c^*) - \frac{\sigma}{c^*}(c_t - c^*) = \frac{1}{1+r^*}(r_t - r^*)$$

Now we define percentage deviations from the steady state for consumption as:

$$\tilde{c}_t = \frac{c_t - c^*}{c^*}, \quad \tilde{c}_{t+1} = \frac{c_{t+1} - c^*}{c^*}$$

And the deviation of the interest rate, which is already a percentage variable, in levels as:

$$\tilde{r}_t = r_t - r^*$$

Therefore, we can rewrite the equation as:

$$\sigma(\tilde{c}_{t+1} - \tilde{c}_t) = \frac{1}{1+r^*}\tilde{r}_t$$

Finally, if we assume $\frac{1}{1+r^*} \approx 1$, a good approximation when r^* is small, we get:

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\sigma}\tilde{r}_t$$

This expression indicates that the growth rate of consumption is **approximately** proportional to the deviation of the real interest rate from its steady-state value, with $\frac{1}{\sigma}$ interpreted as the intertemporal elasticity of substitution.

In the following lecture notes, we will delve deeper into Real Business Cycle (RBC) models and use the log-linearization method to better understand the importance of this approach.