Team 4 - Computational Finance - Assignment 1

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1 INTRODUCTION

Computational systems have become increasingly important in many financial applications including derivatives pricing. A derivative is a financial product whose value depends on the values of other more basic underlying assets such as commodities, precious metals, currency, bonds, stocks, stocks indices, etc. Two of the most well-known examples of derivatives are forward contracts and options, with the main difference being that in a forward contract the buyer is obligated to buy the asset at time T whereas with an option the buyer can choose to exercise the option or not. An important question to ask when discussing forwards and options is how to price them. While pricing of forward contracts is relatively straightforward, the pricing of options is slightly more difficult. Various methods exist, and two of the most common ones are the binomial tree model and the Black-Scholes model. This report introduces the theoretical frameworks behind these models and performs several experiments to compare the two models with each other.

This report will be structured as follows: section 2 provides the necessary theoretical framework, section 3 outlines the methodology, section 4 presents the results and section 5 discusses them.

2 THEORETICAL FRAMEWORK

2.1 Compounding

2.1.1 Discrete compounding. A common assumption for financial models is that money invested in the money-market or bank-account yields interest at a constant, risk-free rate r. The frequency at which interest is compounded and added into the balance is defined as the compounding type. Examples of discrete compounding types are annual compounding, semi-annual compounding or monthly compounding. The general equation for discrete compounding types is as follows:

$$S_t = S_0(1+\frac{r}{n})^{nt} \tag{1}$$

In equation 1, S_t is the the amount after time t , whereas S_0 is the initial amount or principal. r represents the risk-free interest rate in decimal form, n the number of compounding periods in 1 year and t is the time in years.

2.1.2 Continuous compounding. In theory, continuous compounding means that an account will be compounding interest over an infinite number of periods per year. To get the continuous compounding equation, we should consider equation 1 as n approaches ∞.

$$\lim_{n\to\infty} S_t = S_0(1+\frac{r}{n})^{nt} \tag{2}$$

In mathematics, the limit for equation (3) is known to be e:

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = e \tag{3}$$

Assuming $n = rx$, $\frac{r}{n}$ can be rewritten as $\frac{1}{x}$. Performing the change of variables to 2 and subsequently using the information from equation 3 the result is:

$$\lim_{x\to\infty} S_t = S_0(1+\frac{1}{x})^{xrt} \tag{4}$$

$$S_t = S_0 e^{rt} \tag{5}$$

In short, this means that an amount C invested in the money-market at continuous compounding has a value of Ce^{rt} after a period Δt. Δt here refers to the difference in time starting from when the amount was invested until time = t.

2.2 Coupon bonds

A coupon bond B is a debt security under which the issuer owes a dept to the holder. The issuer is obligated to pay periodic interest and must return the principal amount at maturity. In other words, it is a financial product where the holder receives a sequence of constant payments (so called coupons) at fixed intervals from the issuer until the bond matures and the principal is repaid. To determine the fair value of B, add up the discounted value of the principal plus the discounted values of the coupons. The future value S_t of a certain amount S_0 is calculated as in equation 5. Rewriting this equation for S_0 leads to the general equation for the discounted value of a certain amount S_t:

$$S_0 = \frac{S_t}{e^{rt}} \tag{6}$$

For example, consider a coupon bond with a principal of €50,000, a maturity of 2 years and quarterly coupons of €300 (hence 8 coupon payments). Assuming that the money-market has a risk-free interest rate of 1.5% at continuous compounding, the fair value of this bond can be calculated as follows:

$$B = \frac{50000}{e^{0.015*2}} + \sum_{i=1}^{8} \frac{300}{e^{0.015*3*i/12}} = 50882.204 \tag{7}$$

2.3 Arbitrage

In derivatives markets, arbitrage occurs when there are risk-free opportunities, such as for example exploring price differences of an underlying asset in different markets. If one US dollar is worth .5000 British pounds in London, and one US dollar is worth .5001 British pounds in New York, the arbitrageur might want to purchase dollars with pounds in London and then sell the dollars for pounds in New York. Depending on the volume and the transaction costs, this could be a profitable arbitrage opportunity. Due to market efficiency, arbitrage opportunities are hard to find. When they do exist, they are typically small and fleeting. Profiting significantly from arbitrage often requires timely action and large sums of money. If arbitrage opportunities arise, they quickly disappear as traders taking advantage of the arbitrage push the derivative's price until it equals the value of replicating portfolios. Due to market efficiency, the very act of engaging in arbitrage serves to eliminate the arbitrage opportunity. Therefore, it is generally assumed that there are no arbitrage opportunities.

2.4 Forwards

A forward is a contract between two parties to buy or sell an asset S at a specified future date T for a specified future delivery price K. At the entry of the contract ($t = 0$), K is specified to be equal to F_0 in such a way that it has zero initial value:

$$K = F_0 = S_0 e^{rT} \tag{8}$$

To prove this, examine the opposite possibility where K does not equal $S_0 e^{rT}$. Consider a forward contract to buy a stock for price S_0 at time T_0 with risk-free interest rate r. If $K > S_0 e^{rT}$, an arbitrageur can borrow S_0 at interest rate r until T, buy the stock, enter a forward contract to sell it for K at maturity T, and have a profit of $K - S_0 e^{rT}$. If $K < S_0 e^{rT}$, an arbitrageur can short the stock, lend S_0 at interest rate r until maturity T, enter a forward contract to buy the stock for K at T, and have a profit of $S_0 e^{rT} - K$. In short, any forward price F_0 that does not equal $S_0 e^{rT}$ allows for arbitrage, and because no arbitrage is assumed, $K = F_0 = S_0 e^{rT}$.

2.5 Options

Other than a forward contract, call and put options give the buyer the possibility but not the obligation to buy or sell (respectively) an asset at a set price at time T. There are two different types of options: European Style Options, which can be exercised only at maturity T, and American Style Options, which can be exercised at any time leading to maturity T. The payoffs for call and put options refers to the profit or loss that an option buyer or seller makes from a trade.

To further explain payoffs, consider a portfolio which contains a European call option on a stock and an investment of Ke^{-rT}. The call option has a delivery price of K. The price of the stock at $t = 0$ is S_0, and at maturity is S_T. If $S_T < K$, the call option will not be exercised and the payoff for the call option is zero, but the investment continues to give returns. However, if $S_T > K$, the call option will be exercised and the payoff is $(S_T - K)$, and it is important to consider the return on the initial investment as well. At maturity T, the investment of Ke^{-rT} will be K. This means that the payoff for this particular portfolio is defined as $MAX(S_T - K + K, K)$ as can be seen in figure 1.

Secondly, consider a portfolio with a European put option P_t and one share of a stock. If $S_T < K$ at maturity T, the put option will be exercised and the stock will be sold for the strike price K leading to a payoff of $(K - S_T)$. However, it is important to keep in mind that the stock S_t continues to be possessed. This yields a return of $(K - S_t + S_t)$. If $S_T > K$, the put option will not be exercised, and in this case the payoff for the option is 0, but the stock gives returns due to increasing value. Since the portfolio contains both a European put option of a stock and the stock itself, the payoff received on this portfolio is $MAX(K - S_t + S_t, S_t)$. This means that the payoff of this portfolio is defined as $MAX(K, S_t)$ as seen in figure 2.

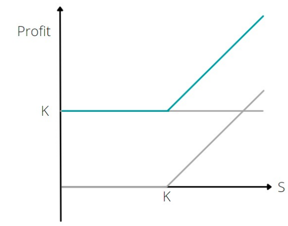

Figure 1: Call + investment

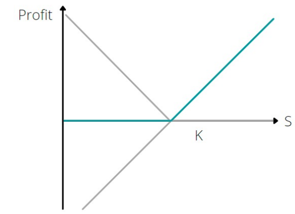

Figure 2: Put + stock

In short: $C+Ke^{-rT}$ is a European call on a stock plus an investment of Ke^{-rT}. $P_t + S_t$ is a European put on the stock plus the stock itself. At the maturity (T) of the options, both are worth $MAX(S_t, K)$. Since both are worth the same amount, they must be worth the same today: If the price of one of these options is out of line, it presents an arbitrage opportunity to put the prices back in line. Therefore, the following equation holds, which is called the *put-call parity*:

$$C_t + Ke^{-r(T-t)} = P_t + S_t \tag{9}$$

2.6 Binomial tree model

The binomial tree is a discrete-time model for the evolution of asset prices. It is a simplified representation of the market, where the price of a stock S after a time period ΔT can make either of two movements: up or down. Obviously, this is not in line with reality, however the binomial tree model seems to be a reasonably good approximation to continuous-time models, and it is much easier to understand and compute.

The probability for the up movement is p, so consequently the probability for the down movement is $1-p$. If the stock price makes the up movement $S_{t+\delta t} = S_t * u$ and if it makes the down movement $S_{t+\delta t} = S_t * d$ where $0 < d < u$. f_0 represents the risk neutral price for the option. The formulas for f_0, p, u and d are given in figure 3.

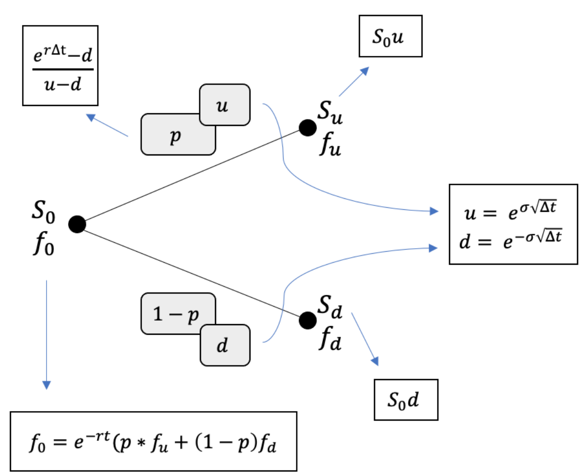

Figure 3: Set up of the binomial tree

2.7 Black-Scholes model

The Black-Scholes model is a continuous-time model for the evaluation of asset prices and probably the most popular tool for financial derivative valuation. The standard version of the model derives the price of a European-style call option. The main assumption of the Black-Scholes model is that the underlying asset price follows a geometric Brownian motion, with constant drift and constant volatility. The standard Black-Scholes formula for the risk neutral price of a European **call** option C_t at $t = 0$ expiring at $t = T$ with constant interest rate r, constant volatility σ, stock price S_t, and strike price K is as follows, where N represents a normal distribution:

$$C_t = S_t N(d_1) - e^{-rT} K N(d_2) \tag{10}$$

$$d_1 = \frac{ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Since the chain rule indicates that the derivative of d_1 and d_2 with respect to S equals $\frac{\delta d_1}{\delta S} = \frac{\delta d_2}{\delta S} = \frac{1}{S}\frac{1}{\sigma\sqrt{T}}$, one finds that the derivative of the Black-Scholes equation in 10 with respect to S is calculated as follows:

$$\frac{\delta C_t}{\delta S_t} = N(d_1) + \frac{1}{S_t}\frac{1}{\sigma\sqrt{T}}[S_t N(d_1) - e^{-rT} K N(d_2)] \tag{11}$$

Simple substitution for d_2 yields:

$$\begin{aligned} e^{-rT} K N(d_2) &= e^{-rT} K N(d_1 - \sigma\sqrt{T}) \\ &= e^{-rT} K \frac{1}{\sqrt{2\pi}} e^{\frac{-(d_1 - \sigma\sqrt{T})^2}{2}} \\ &= e^{-rT} K \frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} e^{\frac{-(-2d_1\sigma\sqrt{T} + \sigma^2 T)}{2}} \\ &= e^{-rT} K N(d_1) e^{d_1 \sigma\sqrt{T}} e^{-\sigma^2 T} \\ &= K N(d_1) e^{-rT - \frac{\sigma^2 T}{2} + \ln(\frac{S_t}{K}) + rT + \frac{\sigma^2 T}{2}} \\ &= K N(d_1) \frac{S_t}{K} \\ &= S_t N(d_1) \end{aligned}$$

Now, using the above substitution in equation 11:

$$\frac{\delta C_t}{\delta S_t} = N(d_1) + \frac{1}{S_t}\frac{1}{\sigma\sqrt{T}}[S_t N(d_1) - S_t N(d_1)] \tag{12}$$

Hence, the derivative of equation 12 with respect to S_t is equal to the normal distribution of d_1, which is a well-known property of the Black-Scholes model:

$$\Delta = \frac{\delta C_t}{\delta S_t} = N(d_1) \tag{13}$$

The Δ constant above is the value of the long position in the stock required to replicate the European call option. The C_t in equation 10 calculates the risk neutral price of a call. The risk neutral price of a European **put** can then be derived through the use of the put-call parity as was previously introduced. Rewriting equation 9 gives:

$$P_t = C_t - S_t + K e^{-rT} \tag{14}$$

Substitution of C_t using equation 10 yields: .

$$\begin{aligned} P_t &= S_t N(d_1) - S_t + K e^{-rT} - K e^{-rT} N(d_2) \\ &= -S_t(1 - N(d_1)) + K e^{rT}(1 - N(d_2)) \\ &= -S_t N(-d_1) + K e^{-rT} N(-d_2) \end{aligned}$$

Using the Black-Scholes formula for C_t and the put-call parity, the risk neutral price of a European put option (P_t) is derived.

$$P_t = -S_t N(-d_1) + K e^{-rT} N(-d_2) \tag{15}$$

3 METHODOLOGY

In this section we will consider several experiments that will cover the topics introduced in section 2. The following experiments were conducted:

(1) **Binomial Tree Model for European Options:** we construct a 50-step binomial tree model to value an European option. In this experiment we consider a maturity of one year ($T = 1$), a strike price of 99 € ($K = 99$), risk-free interest rate of 6% ($r = 0.06$), a current stock price of 100 € ($S_0 = 100$) and a volatility of 20% ($\sigma = 0.2$).
(2) **Black-Scholes Model:** As seen in section 2, it is possible to value an European Option using the Black-Scholes method with equations 10 and 15. In this experiment, for the same setting as before, we perform option pricing with the Black Sholes method.
(3) **Binomial Tree vs Black-Scholes:** We conduct an experiment to evaluate the convergence of the Binomial Tree model. We evaluate how the option pricing varies with the number of steps, as well as the volatility and compare it to the Black-Scholes value. Moreover we compare the hedge parameter Δ for both methods and argue their differences.
(4) **Binomial Tree Model for American Options:** American options provide the possibility of exercising the option earlier before maturity. We adapt the binomial price model used to price European options to accommodate American options too. We compare both methods and discuss the main differences.
(5) **Hedging Simulation with Euler Method:** In this experiment we perform hedging simulation. Each time we perform 20 runs on the stock price simulation. We first vary the frequency of the hedge adjustment (daily/weekly) with matching volatility in the stock price and Δ calculations. We then repeat this process with different volatility values for both components and discuss the differences.

4 RESULTS

(1) **Binomial Tree Model for European Options:**
Given the setting specified in the section 3, the European call option price is:

$$f_0 = 11.5464$$

(2) **Black-Scholes Model:**
For the same setting, the European call option calculated with the Black-Scholes model is:

$$f_0 = 11.5443$$

(3) **Binomial Tree vs Black-Scholes:**
The effect of increasing the number of steps used in the binomial model on the option price is depicted in figure 4.

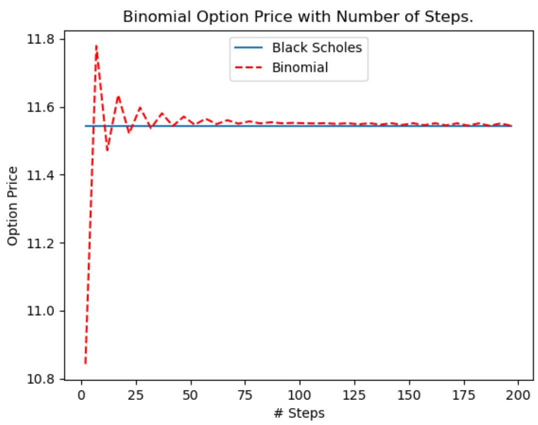

Figure 4: Effect of increasing steps size on binomial tree option pricing.

The option values with respect to varying volatility can be seen in figure 5.

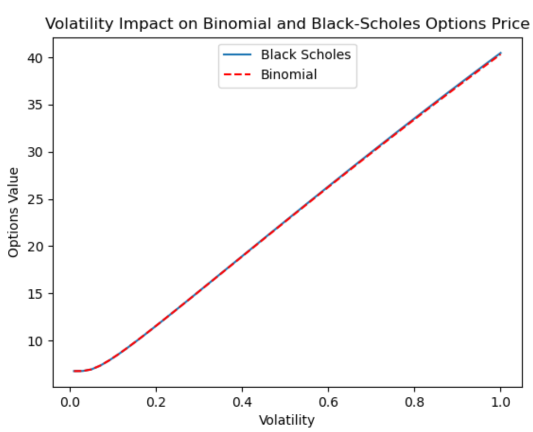

Figure 5: 50-step Binomial Tree and Black-Scholes option value for different volatility levels.

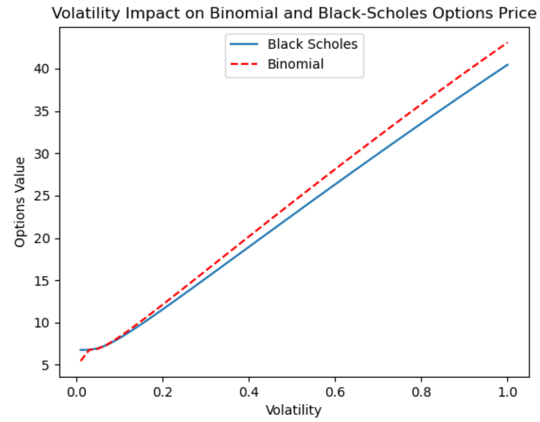

Figure 6: 3-Step Binomial Tree and Black-Scholes option value for different volatility levels.

Number of shares of stocks Δ to hedge the portfolio for a fixed volatility of 20%:

Binomial Tree	Black Scholes
$\Delta = 0.6726$	$\Delta = 0.6737$

Table 1: Binomial Tree vs Black-Sholes hedge parameter value.

In figure 7 we can see how the volatility influences the hedge parameter Δ:

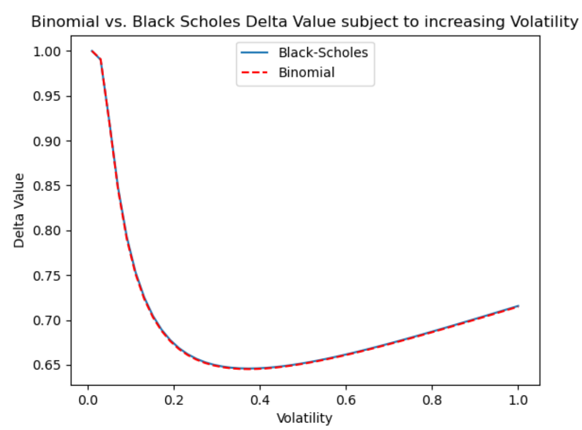

Figure 7: Binomial Tree and Black-Scholes hedge parameter for different volatility levels.

(4) **Binomial Tree Model for American Options:**
The option prices of the American binomial tree model can be seen in table 2 together with the corresponding European option prices.

	Call	Put
European	11.5464	4.7811
American	11.5464	5.3478

Table 2: American Put and Call option prices with Binomial Tree Model.

(5) **Hedging Simulation with Euler Method:** This experiment performs hedging simulation. First, the frequency of the hedge adjustment is varied (daily/weekly) with matching volatility in the stock price and Δ calculations. Then, the process is repeated with different volatility values for both components and the differences are discussed.

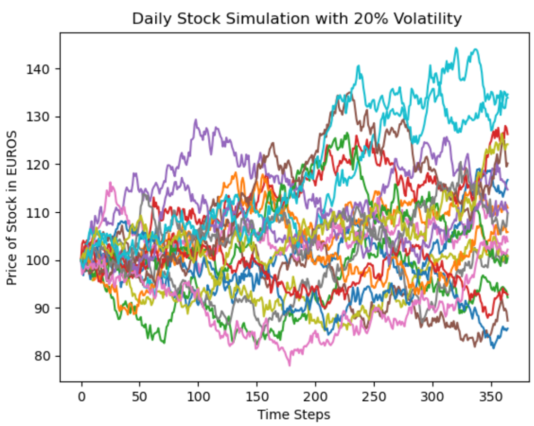

Figure 8: Daily stock simulation with Euler's Method.

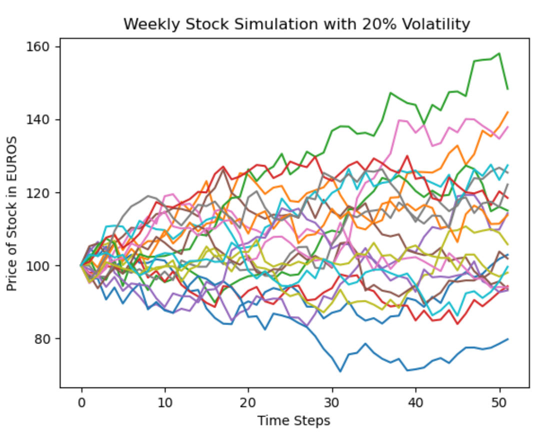

Figure 9: Weekly stock simulation with Euler's Method.

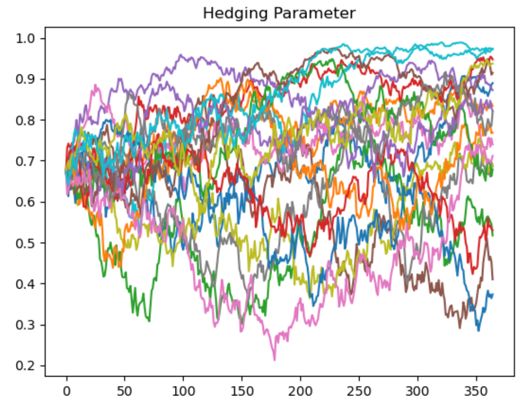

Figure 10: Daily Hedge Simulation.

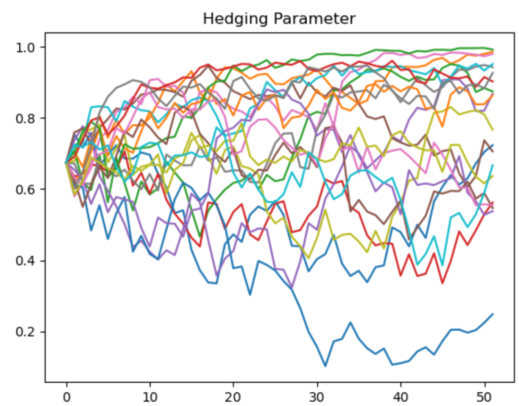

Figure 11: Weekly Hedge Simulation.

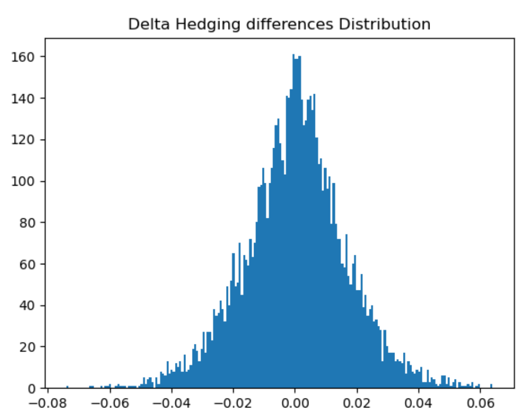

Figure 12: Daily hedge value difference.

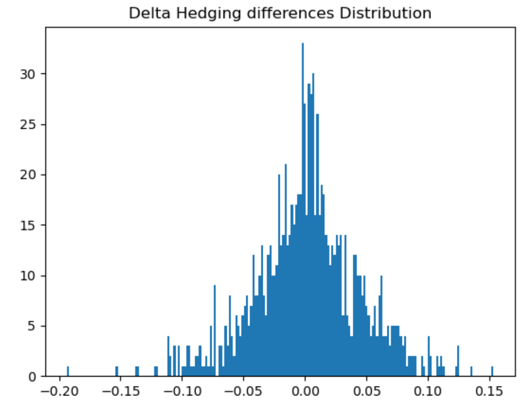

Figure 13: Weekly hedge value difference.

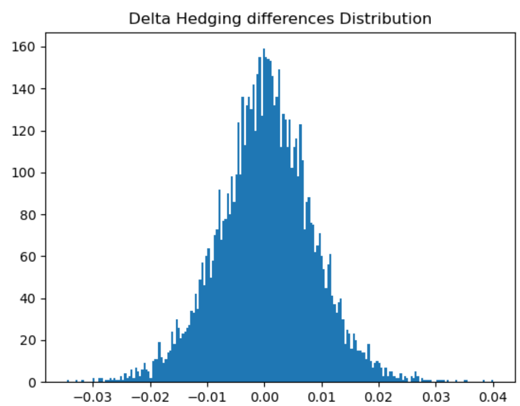

Figure 14: Hedge value difference for stock price volatility of 10%.

5 DISCUSSION AND CONCLUSION

In this section we will comment on the results obtained for each experience. Experience (1) and (2) consisted of implementing the Binomial Tree and Black-Scholes model for the setting described previously. As seen before, the first model is a discrete approximation of the option price. Both results are similar, which indicates that, in this case, a 50-step binomial model is already a good approximation of the theoretically contin value calculated with the Black-S.

In experience (3) we explore this concept of discrete approximation by analysing the convergence of the Binomial Tree model and comparing it to its continuous counterpart, the Black-Scholes option price. In figure 4, it is noticeable how the number of steps influences the option pricing. When the number of steps is lower, the option price differs greatly from the Black-Scholes value. As the steps increase the Binomial Tree Model starts converging to the Black-Scholes method, as expected. After 50 steps, the binomial model

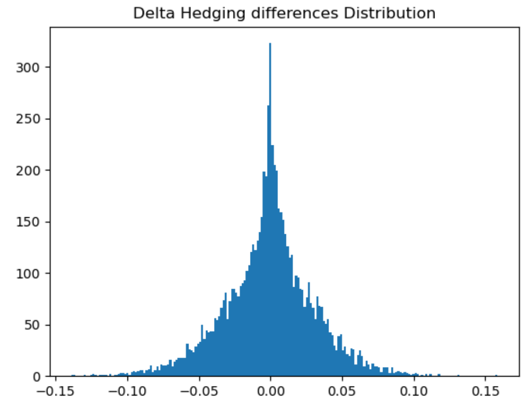

Figure 15: Hedge value difference for stock price volatility of 40%.

already provides a good approximation to the Black-Scholes value. It is also important to analyse both methods in terms of computational complexity. The binomial method can be implemented in two ways: i) recursively ii) iteratively using a matrix representation of the tree. i) has a computational complexity of $O(2^N)$, since it generates 2 recursive calls N times, whereas ii) requires to loop through a $N \times N$ matrix, making it $O(N^2)$. In figure XXX it is possible to visualise computational complexity of both methods. On the other hand, the Black-Scholes method provides a close form solution to the problem, making it $O(1)$.

We experiment with volatility values and observe the changes (figure 5 and figure 6). Since the 50 step binomial model is a good approximation for the Black-Scholes model, we see that the option value approximates for both models are consistent with increasing volatility (figure 5). With increasing volatility, the difference between the stock price at maturity T and the strike price K will be higher. The pay-off increases hence influencing the option price paid for that higher pay-off to increase as well. As seen, the option price is linearly and positively correlated to the volatility. In figure 6 we observe the same relation between volatility and option price, but now considering a 3-step binomial model. The volatility and option prices remain linearly correlated for both models, however, we notice that the prices for different volatility values deviate. This further supports the issue previously raised on the discrete approximation of low-step binomial models: the approximation tends to deviate from the Black-Scholes model results.

In this third experiment, we compare also the values of the hedge parameter of both models. With zero volatility the stock price is perfectly predictable. The payoff is guaranteed to be the increase in the stock price due to interest rate. Therefore, the model suggests to own the entire stock instead of a certain fraction of it (delta) because you are certain to make a profit.

For experience (4) we analyse American options and compare them with European ones. As seen in section 2, these provide the opportunity of early exercise. In table 2 we present the results obtained when valuing a call and put for both American and European options. For call options we notice that the price does not change. This is related to the risk-free interest rate considered. The risk-free interest rate is positive, meaning that, theoretically, the value of my underlying asset should be greater tomorrow that it is today. For this reason, exercising earlier will not yield any benefits and the payoff at time t is lower than the expected payoff at maturity T. However, when considering a put option one is on the other side of this trade. In the nodes where there is a positive payoff at time $t \leq T$, the value of the payoff at that time will be greater than the expected payoff in the future, due to the positive risk free rate. For this reason, the intrinsic value of the option at time t will change, and this change influences the price of the option at time $t = 0$. We can observe this in table 2 as we see the prices for the put options are different.

Experience (5) we simulate the stock price weekly and daily as seen in figures 8 and 9. The differences between prices at each time step is greater when considering weekly hedge adjustment frequency, which makes the hedge parameter difference between time steps to be greater in this case. We can further verify this by looking into the distribution of the differences for the hedge parameter (figure 12 and 13). The tails of the normally distributed differences are wider in the case of weekly hedging, varying between $[-0.20, 0.20]$.

We also experiment with unmatching volatility values of the stock price. We consider a lower stock volatility of 10% (figure 14) and higher stock volatility of 40% (figure 15) that the one assumed in the hedge computation (20%). For a lower volatility of the stock price the calculation of the hedging parameter with the Black-Scholes method leads to lower uncertainty of the hedging parameter differences (as observed in figure 14 where the distribution varies between [-0.04,0.04]). On the other hand, for a higher volatility of the stock price the calculation of the hedging parameter with the Black-Scholes method leads to higher uncertainty of the hedging parameter differences (as observed in figure 15 where the distribution varies between [-0.15,0.15]).