Team 4 - Computational Finance - Assignment 2

Aukje Terpstra
11426705
aukje.terpstra@students.auc.nl

Francisco Pereira
13911376
francisco.pereira@student.uva.nl

Palash Kanwar
13837494
palash@umich.edu

1 INTRODUCTION

Monte Carlo simulation is a mathematical technique used to estimate the possible outcomes of an uncertain event. It was invented by John von Neumann and Stanislaw Ulam during the second World War to improve decision making under uncertain conditions. It was named after the famous Monte Carlo casino in Monaco, since the element of chance is core to the modeling approach, like a game of roulette. Since its introduction, Monte Carlo simulations have assessed the impact of risk in many real-life scenarios, such as in artificial intelligence, sales forecasting, and project management. In finance, Monte Carlo simulations are often used for derivative pricing. This report introduces the theoretical framework behind the Monte Carlo simulation and demonstrates several experiments to illustrate how the simulation works and can be used.

This report will be structured as follows: section 2 provides the necessary theoretical framework, section 3 outlines the methodology, section 4 and 5 present and discuss the results, respectively, and the final section 6 contains a conclusion.

2 THEORETICAL FRAMEWORK

2.1 Options

Options give a buyer the right, but not the obligation, to buy (call) or sell (put) an asset at a set price K at time T. The option payoff refers to the profit or loss that an option buyer or seller makes from a trade.

2.1.1 European Options. European options are a type of options which can be exercised only at maturity T, as opposed to American options, which can be exercised at any time before maturity T. The payoff for a European call option is $\max(0, S_t - K)$, and for a European put $\max(0, K - S_t)$, where S_t is the value of the asset at time T.

2.1.2 Asian Options. An Asian option is a type of option where the payoff depends on the **average** price of the underlying asset over a certain period of time as opposed to standard options (European and American) where the payoff depends on the price of the underlying asset at a given point in time. The payoff of an Asian call option is $\max(0, S_{avg} - K)$, and for an Asian put $\max(0, K - S_{avg})$.

2.1.3 Digital Options. A digital option is a type of option that offers the opportunity of a fixed payout if the underlying market price exceeds the strike price K. They are also called all-or-nothing options, binary options, or fixed return options. Graphically, the payoff is a step function and is denoted by $\mathbb{1}_{\{S_T > K\}}$.

2.2 Monte Carlo Simulation

A Monte Carlo simulation is a model used to predict the probability of different outcomes when the intervention of random variables is present. In finance, Monte Carlo simulation is usually used for derivatives where the payoff is dependent on the history of the underlying variable or where there are several underlying variables. Two theoretical concepts form the foundation of the Monte Carlo simulations: The Law of Large Numbers (LLN) and the Central Limit Theorem (CLT). The LLN states that as sample size n increases, its mean gets closer to the population mean.

The CLT states that the sample mean is a random variable that is normally distributed with a mean value equal to the population mean, and a standard deviation that is given by the population standard deviation divided by the square root of the sample size. Since the sample standard deviation is inversely proportional to the sample size, the precision of the calculated mean value decreases with larger sample sizes.

To further illustrate Monte Carlo simulation, consider the following step-by-step approach to value a derivative dependent on single market variable S:

(1) Sample n random paths for S in a risk-neutral world
(2) Calculate the payoff for each random path
(3) Calculate the mean of the payoffs to get an estimate of the expected payoff
(4) Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative

Monte Carlo simulations support a variety of stochastic processes and can deal relatively easy with options with complex payoffs and path dependent options. Monte Carlo simulations tend to be the most efficient procedure when there are three or more stochastic variables, because the time taken to carry out a Monte Carlo simulation increases approximately linearly with the number of variables, whereas the time taken for most other numerical procedures increases exponentially.

Monte Carlo simulations also have several shortcomings. For example, Monte Carlo can not easily deal with American options as the numerical simulation of stochastic differential equations can be tricky due to nested simulations being required, and the estimates for the Greeks are unstable when discontinuous payoffs are considered.

2.3 Estimating the Hedge Parameter Delta

To measure the sensitivity of the derivative price to its parameters, one can use the derivatives of the derivative. The derivatives of derivative prices are called *Greeks*, because of the Greek letters that are used to refer to the various derivatives. For example, the hedge parameter Delta Δ is the derivative of the option price with respect to the underlying asset's initial value ($\Delta = \frac{\delta}{\delta S(0)} \mathbb{E}[Y]$). It is used in some situations to hedge the risk associated with option trading. There are three methods to estimate Delta:

(1) Finite-difference approximations
(2) Pathwise method
(3) Likelihood ratio method

In this report, only a subset of finite-difference approximations will be considered, namely the forward, which is also called the bump and revalue method.

2.3.1 Bump and Revalue. In the bump and revalue method, the Euler formula is used to approximate Δ, where $V(S)$ is the option price at time T, and $\Delta = \frac{\delta V}{\delta S}$:

$$\Delta = \frac{V(S+\epsilon) - V(S)}{\epsilon} \tag{1}$$

To solve it, choose ϵ as small as possible but not too close to machine precision (e.g. ϵ= 0.01, 0.02, or 0.5), and then run Monte Carlo at two different points, $V(S+\epsilon)$ and $V(S)$, to obtain an approximation of Δ.

It is important to note that when using random numbers for the bumped and unbumped estimates, the results of the bump and revalue method are unstable. Increasing the iterations does not improve accuracy, and the variance increases for smaller ϵ. Covariance can be increased and the variance reduced by making use of the same seeds. After using the same seeds, the results should be much more stable.

2.3.2 Pathwise Method. The main idea of the pathwise method is to differentiate each simulated outcome with respect to the parameter of interest. If Y is the discounted payoff of the option, θ is a parameter that it depends on, and $\alpha(\theta) = \mathbb{E}[Y(\theta)]$ is the price of the option, $\alpha'(\theta)$ can be estimated by taking the derivative of Y and then the expectation, where

$$Y'(\theta) = \lim_{h \to 0} \frac{Y(\theta+h) - Y(\theta)}{h} \tag{2}$$

This has expectation $\mathbb{E}[Y'(\theta)]$ and is an unbiased estimator of $\alpha'(\theta)$ if

$$\mathbb{E}[\frac{\delta}{\delta\theta} Y(\theta)] = \frac{\delta}{\delta\theta}\mathbb{E}[Y(\theta)] \tag{3}$$

Using the Pathwise method to determine the Black-Scholes Delta for options, the following equation holds:

$$\frac{\delta Y}{\delta S(0)} = \frac{\delta Y}{\delta S(T)} \frac{\delta S(T)}{\delta S(0)} \tag{4}$$

Since for a normal call option $Y = e^{-rT}[S(T) - K]^+$,

$$\frac{\delta Y}{\delta S(T)} = e^{-rT} \mathbb{1}_{\{S(T)>K\}}$$
$$\frac{\delta S(T)}{\delta S(0)} = \frac{S(T)}{S(0)}$$
$$\frac{\delta Y}{\delta S(0)} = e^{-rT} \frac{S(T)}{S(0)} \mathbb{1}_{\{S(T)>K\}}$$

A problem arises when trying to use the Pathwise method to find the hedge parameter of a digital option. In this case, the payoffs are $Y = e^{-rT} \mathbb{1}_{\{S(T)>K\}}$. This constitutes a problem in the Pathwise method, as the payoffs are non-differentiable: $\frac{\delta Y}{\delta S(T)} = 0$. A possible solution to this problem is to make use of smoothing, which makes the payoff function differentiable. This can be achieved with the use of the cumulative density function (cdf), for which the derivative is known, and is the probability density function. The results are however dependent on the standard deviation of the cdf. A lower standard deviation will better approximate the cdf to the step function which in turn provide better estimates of the hedge parameter.

2.3.3 Likelihood Ratio Method. In contrast to the pathwise method, the likelihood ratio method differentiates a probability density (rather than a discounted payoff Y) with respect to the parameter of interest, θ. It provides a good potential alternative to the pathwise method when Y is not continuous in θ, as it does not require smoothness. The expected discounted payoff is expressed as an integral:

$$V(\theta) = \mathbb{E}[f(S_T)] = \int f(S_T) g(S_T, \theta) dS_T \tag{5}$$

Taking the derivative of $V(\theta)$ gives $\mathbb{E}[f\frac{\dot{g}}{g}]$. This means that delta is estimated by the following equation:

$$\Delta = e^{-rT} f(S_T) \frac{\dot{g}(S_T, \theta)}{g(S_T, \theta)} \tag{6}$$

Where the score function is:

$$\frac{\dot{g}(S_T, \theta)}{g(S_T, \theta)} = \frac{Z}{S_0 \sigma \sqrt{T}} \tag{7}$$

Finally, the hedge parameter is then determined by:

$$\Delta = e^{-rT} f(S_T) \frac{Z}{S_0 \sigma \sqrt{T}} \tag{8}$$

The main advantage comes from the fact that once we have the score function (which does not depend on the payoff) and the payoff function ($f(x)$), calculating the hedge parameter is straightforward, as opposed to the Pathwise method where one needs to consider smoothing techniques for non-differentiable payoffs.

2.4 Variance Reduction Techniques

The standard error of a Monte Carlo estimate is given by the following equation, where σ represents the standard deviation and N the number of simulations:

$$\frac{\sigma(payoff)}{\sqrt{N}} \tag{9}$$

Another drawback of Monte Carlo that was not mentioned before, and that can be deduced from equation 9 is that a large number of simulations are typically required to obtain accurate results. There are ways to improve the accuracy of the results without actually increasing the number of simulations. One such technique is called the Control Variate technique.

2.4.1 Control Variate Technique. The control variate technique can be used when (1) there are two similar derivatives A and B and (2) one of them has a simpler analytic solution (B) than the other one (A). Derivative A is the one that needs to be valued. Two simulations using the same random number streams and the same δt are carried out in parallel. The first is used to obtain an estimate f_A^* of the value of A; the second is used to obtain an estimate f_B^*, of the value of B. A better estimate f_A of the value of A is then obtained using formula 10, where f_B is the known true value of B calculated analytically.

$$f_A = f_A^* - f_B^* + f_B \tag{10}$$

In this report, this technique is explored with Asian options. Asian options are considered based on Geometric and Arithmetic averages, which are both similar. Monte Carlo estimates of the arithmetic and geometric Asian option are used together with the analytical value of the Asian option based on geometric averages which is known. Finally, equation 10 is used to calculate the value of the Asian option based on arithmetic averages.

3 METHODOLOGY

This section describes a set of experiments conducted in order to better understand and explore the concepts introduced in the Theoretical Framework (2). The goal is to provide the reader with a practical point of view of the topics discussed and showcase the advantages and limitations of the different algorithms and computational methods. During this work we will consider different option settings, which are:

a) *European Put Option*
b) *Digital Call Option*
c) *Asian Call Option*

All settings have $S = 100$, $K = 99$, $T = 1$ year, $r = 6\%$ and $\sigma = 20\%$.

For each experiment we reference the setting being considered by the corresponding letter. In the following points we describe the different experiments executed:

(1) **Monte Carlo for European Put Option Pricing:** this experiment considers the *(a) European Put Option* and consists of analysing how it is affected by different parameters. We first investigate how the number of simulations impacts the option price and compare it with the theoretical Black-Scholes value. Secondly, for a fixed number of simulations we explore the changes of the option price with respect to a varying strike price and volatility.
(2) **Bump and Revalue:** for this exercise, we once again consider *(a) European Put Option* . The goal is now to estimate the δ (hedge parameter) using the Bump and Revalue method. As stated in 2.3.1, the stability of the results is highly dependent on random numbers stream sampled for the bumped and unbumped estimates. We analyse this phenomenon by making use of seeds. We track and compare the Bump and Revalue results with and without using seeds.
(3) **Hedge Parameter (δ) of a Digital Option:** For this third experiment, we now consider *(b) Digital Call Option*. We compute δ according to three methods:
    - Bump and Revalue
    - Pathwise
    - Likelihood Ratio

    We investigate and state the advantages and limitations of each.
(4) **Variance Reduction in Asian Options:** Here we access scenario *(c) Asian Call Option* where we first compare the analytical option value based on the Geometric Averages to the respective Monte Carlo estimate. Moreover, we apply the Control Variate technique to calculate the value of the Asian option based on arithmetic averages. In this case we further investigate how the results depend on the number of simulations, the number of steps considered and the strike price.

4 RESULTS

(1) **Monte Carlo for European Put Option Pricing:**

Figure 1 shows the convergence of the Monte Carlo simulations. Accordingly, Table 1 reports the estimates and standard errors for different numbers of simulations.
Figures 2 and 3 showcase the variation of the Option value with respect to the Strike Price and Volatility, respectively. In both cases, we report the relationship values for the Monte Carlo estimates with 10^1 and 10^5 simulations, as well as, the Black-Scholes values.

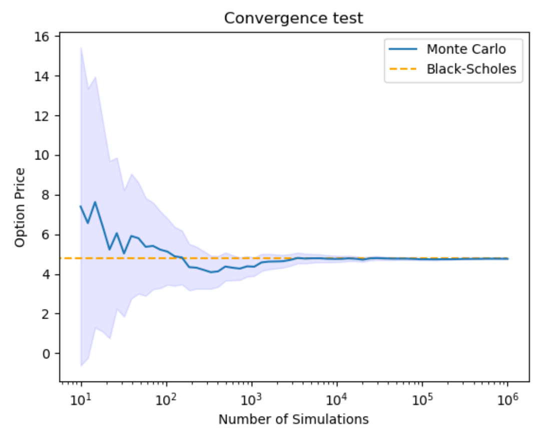

Figure 1: Convergence of the Monte Carlo estimates with respect to the number of simulations.

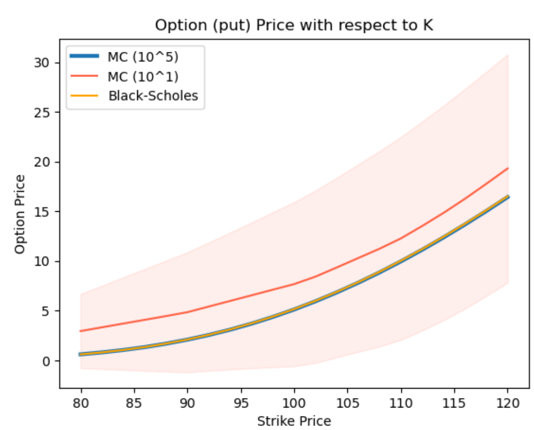

Figure 2: Option Value with respect to the strike price K. In the figure it is visible how the relation Option$\tilde{\text{S}}$trike is impacted by the number of simulations used in the Monte Carlo estimate and how it compares to the Black-Scholes relationship.

Table 1: Monte Carlo results and respective standard errors.

Simulations	Option Value Estimate	Std. Error	Confidence Interval
10^1	7.3998	4.0939	[-0.6243, 15.4240]
10^2	4.8716	0.8582	[3.1895, 6.5536]
10^3	4.3653	0.2471	[3.8810, 4.8495]
10^4	4.7613	0.0800	[4.6046, 4.9181]
10^5	4.7429	0.0252	[4.6935, 4.7923]
10^6	4.7656	0.0080	[4.7500, 4.7813]

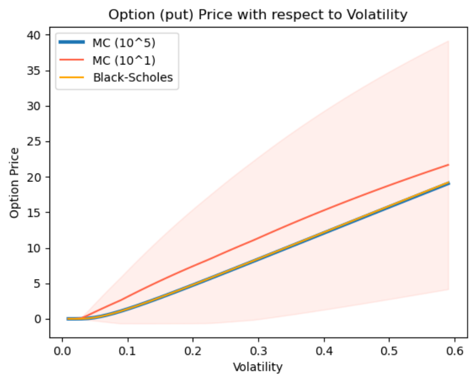

Figure 3: Option Value with respect to the volatility σ. In the figure it is visible how the relation Option$\tilde{V}$olatility is impacted by the number of simulations used in the Monte Carlo estimate and how it compares to the Black-Scholes relationship.

(2) **Bump and Revalue:** Figures 4 and 5 display the unseeded and seeded absolute relative errors obtained for the delta values, respectively. The relative errors are calculated with respect to the delta obtained from the Black-Scholes formula. Furthermore, Figure 6 displays the progression of the delta value relative error for a fixed number of simulations (10^5) for increasing epsilons.

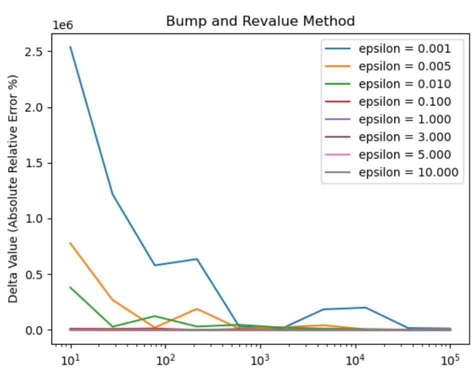

Figure 4: Non seeded bump and revalue method. The plot presents the absolute relative error with respect to the analytical Black-Scholes model.

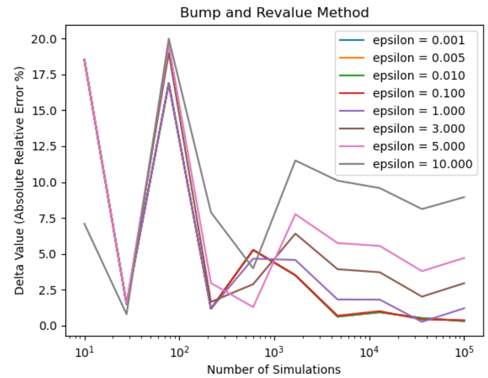

Figure 5: Seeded bump and revalue method. The plot presents the absolute relative error with respect to the analytical Black-Scholes model.

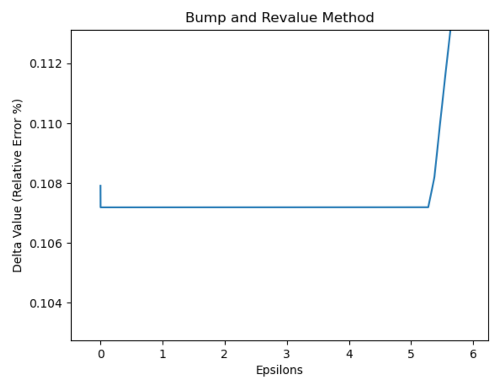

Figure 6: Seeded bump and revalue method with respect to varying epsilons and fixed number of simulations.

(3) **Hedge Parameter (δ) of a Digital Option:** Using a fixed number of simulations, table 2 displays the different delta values obtained for the Bump and Revalue, Pathwise, and Likelihood Ratio methods for a digital option. Figure 7 is used to illustrate the progression of the delta value obtained using the smoothed Pathwise variance reduction technique for a digital call option. To bolster the results obtained in table 2, figure 8 displays the convergence of the delta value with increasing number of simulations using different sophisticated methods for smoothing.

Table 2: Hedge Parameter estimates for 10^6 simulations.

Bump and Revalue	Pathwise	Likelihood Ratio
0.017423	0.017656	0.018234

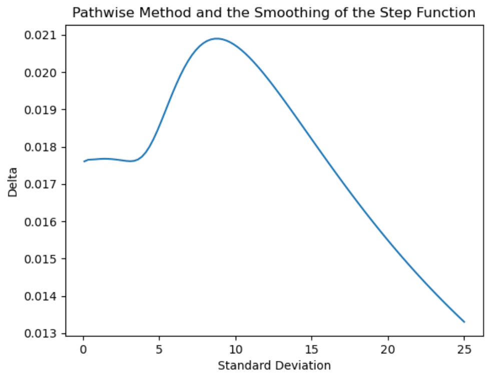

Figure 7: This figure illustrates how the smoothing of the digital option impacts the delta value. The standard deviation deviation of the cumulative density function determines the smoothness.

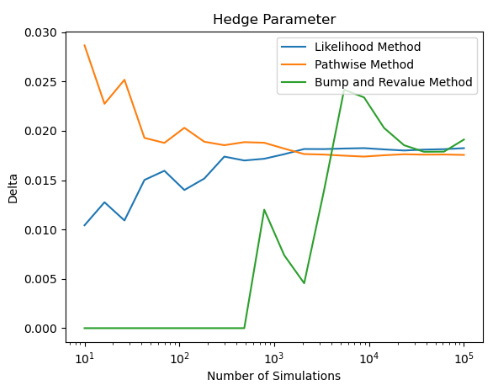

Figure 8: The different methods to estimate the hedge parameter for the digital call option.

(4) **Variance Reduction in Asian Options:**
The value of the Asian Option calculated with the analytical expression based on geometric averages and with Monte Carlo simulations can be seen in table 3. For this we considered 1000 simulations ($N = 1000$) and weekly time steps ($M = 52$) for each path.
Figure 9 displays the convergence of the Monte Carlo estimates for the Asian Option Price with increasing simulations. The figure plots the Monte Carlo estimates using arithmetic averages, the analytical asian option price using geometric averages and the Monte Carlo estimates of the arithmetic averages using the control variate technique.

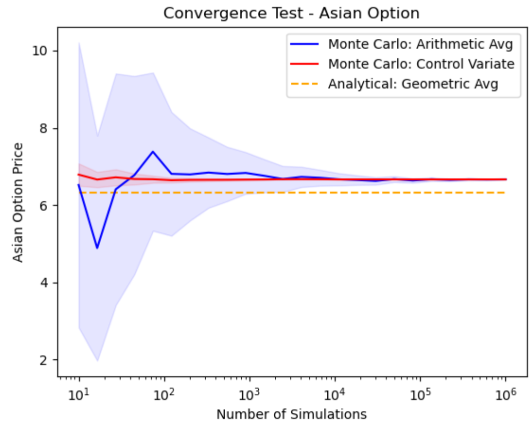

Figure 9: Monte Carlo estimates of the Asian option value with respect to the number of simulations. The reduced variance of the Control Variate method is noticeable.

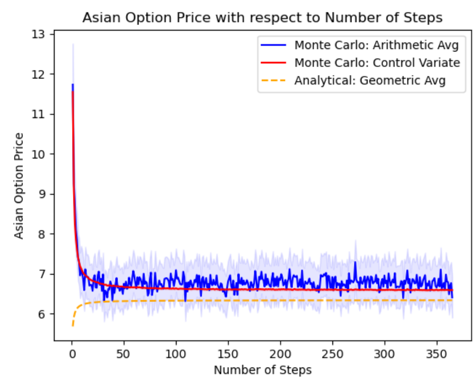

Figure 10: Monte Carlo estimates of the Asian option value with respect to the number of steps considered for the averages.

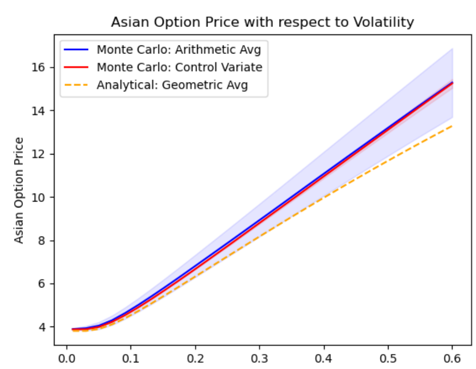

Figure 11: Asian option value with respect to the volatility σ.

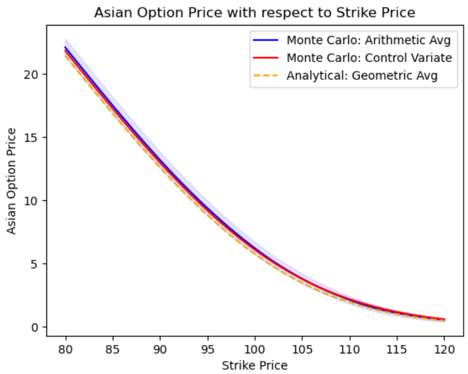

Figure 12: Asian option value with respect to the strike price K.

Table 3: Asian Option prices based on the analytical formula and Monte Carlo (52 t.s. and 1000 simulations).

Analytical Geometric Averages	Monte Carlo
6.312	6.798 ± 0.512

As the last part of the assignment, Figures 10, 11, and 12 display the performance of the control variate technique for different parameter settings such as varying time steps, volatility, and strike price.

5 DISCUSSION

(1) **Monte Carlo for European Put Option Pricing:**
Given *(a) European Put Option* and analysing figure 1, we see the convergence of the Monte Carlo estimates to the analytical Black-Scholes value of the option price.
In this experiment we also compare the option price with varying volatility and strike price, visible in figure 2 and 3. We distinguish between a less and higher reliable Monte Carlo estimate (10^1 vs 10^5 simulations, respectively), and analyse it against the close-form solution obtained by the Black-Scholes model. Further supporting the reliability of the Monte Carlo estimate for a higher number of simulations, we see that the values of the Monte Carlo estimate with a high number of simulations (blue line) are approximately the same as the theoretical value. On the other hand, with a lower number of simulations (red line), the values diverge. The results obtained are intrinsically related to the Law of Large Numbers and the Central Limit Theorem, and we can expect the average of all the simulations to almost surely converge to the expected value. Furthermore, we see how the increasing number of simulations allows one to make a more reliable estimate with a lower standard error and narrower confidence intervals (seen in figure 1 and in table 1).

(2) **Bump and Revalue:**
Figures 4 and 5 present the results for non-seeded and seeded experiments of the bump and revalue method. The non-seeded experiments 4 exhibit unstable and odd results, as we see that the absolute relative error with respect to the analytical Black-Scholes model is of very high order. The contrary is depicted on the seeded values (figure 5). The use of common random streams of numbers allows for greater stability of results and we observe that for a lower epsilon the relative error is also lower. We further investigate this relationship by plotting the relative error with respect to varying epsilons and a fixed number of simulations (figure 6). The stability of the bumped and revalue method is not only affected by high epsilon values, but as these approach very low values (close to machine precision) the stability is affected.

(3) **Hedge Parameter (δ) of a Digital Option:** In this experiment we compare the results from three techniques to estimate hedge parameter. Figure 8 provides a nice overview of how these method vary and converge with respect to the number of simulations. The Pathwise method is theoretically inapplicable to digital options due to the non-differentiable step function, as seen in section 2. Thus, the method relies on the smoothing of this function which was done using the cumulative distribution function and the respective derivative, the probability density function. The results of the hedge parameter will depend on how well the smoothed function describes the step, which is related to the variance of the cumulative density function. Figure 7 shows the relation between the hedge parameter value and the standard deviation. A lower variance corresponds to a steeper c.d.f., which better resembles the step function, and therefore provides stable results. Contrarily, the higher variance provides diverging and unstable results for delta, as the c.d.f. will resemble less the step function.
In contrast, the Likelihood Ratio method, differentiates a probability density with respect to the parameter of interest (instead of the payoff). When the payoff is not continuous (which is the case of the digital option), it presents itself as a good alternative as the value converges without being dependent of any other parameters, which is not the case with the Pathwise and Bump and Revalue methods.
Ultimately we see that all these methods approximate the value of the option around the same value. Table 2 presents the respective estimates.

(4) **Variance Reduction in Asian Options:**
As seen from experience (1), a large number of simulations is usually required to obtain a accurate and reliable estimate. In figure 9 we analyse the convergence of the option price, now considering the Control Variate (in red). As seen, this allows a quicker convergence with a significant lower standard error when compared to the Monte Carlo method without a control variate (blue). Around 10^2 simulations, the value of the Control Variate estimate converges to the expected valued with a narrow confidence interval, whereas it takes

almost 10^5 simulations for the Monte Carlo method without the Control Variate to achieve the same results.
The number of steps also plays a considerable role. Figure 10 illustrates how the number of steps also contribute to the convergence. For lower number of steps the price of the option will be higher when considering arithmetic averages because it introduces a higher variance.
We consider, once again, varying volatility and strike prices. Figures 11 and 12 depict these relationships. The higher volatility increases the option price, as it introduces uncertainty, and the lower strike price, given that we're considering a call option, increases the option value. It is important here to analyse the significant reduction of the confidence intervals by using the Control Variate. Looking at the volatility example (figure 11) the simple Monte Carlo method shows a drastically increase of the confidence intervals once we consider higher volatility levels. The Control Variate method allows one to estimate the option price with high volatility levels and maintain a narrow confidence interval.
Throughout this experiment we have also plotted the analytical Asian Option price based on the geometric averages (yellow). The results of the geometric mean, for all cases, is lower than the price based on arithmetic averages, which was expected. The geometric average is always lower than the arithmetic average as it accounts for the compounding effect.

6 CONCLUSION

In this work we have explored the use of Monte Carlo simulations for option valuation. The experiments carried out investigate the different advantages and disadvantages of this method considering different options types and scenarios. We conclude that Monte Carlo based techniques are powerful and accurate ways to value derivatives and greeks when there is no close form solution available. Despite being computationally expensive, some methods such as the Control Variate technique help to reduce the computational cost necessary to carry out reliable estimates.