Lecture Notes

Auction Theory

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Chapter 1

Introduction

Most people are familiar with auctions involving, art, livestock or real estate, where an auctioneer (a seller) is looking for the highest price he can get. It is also generally known that most government business (military supplies, construction, and most other government-bought goods and services) are contracted for based a procurement auction, in which the auctioneer (buyer) is looking for the lowest price. Publicly-owned assets (such as airwave frequencies, timer rights, oil leases, or public companies on their way to privatization) are also sold off by the government in auctions. In addition, the US government sells treasury bills through an auction process every week.

Auctions, of course are not new. In fact, almost any buying and selling transaction can be viewed as the result of an auction process. A consumer looking at advertisements for automobiles, books or groceries can be viewed an auctioneer evaluating bids from competing suppliers. With the use of internet search engines such price comparisons are now commonplace. Many people are also familiar with Internet-based auction sites such as eBay, Amazon.com, UBid and others, where individuals and corporations sell a multitude of goods following specific auction rules.

Auctions have been used since antiquity for the sale of a variety of objects. Herodotus reports that auctions were used in Babylon as early as 500 b.c. Today both the range and the value of objects sold by auction has grown to staggering proportions. Art objects and antiques have always been sold at the fall of the auctioneer's hammer. But now numerous kinds of commodities ranging from tobacco, fish, and fresh flowers to scrap metal and gold bullion are sold by means of auctions. Bond issues by public utilities are usually auctioned off to investment banking syndicates. Long-term securities are sold in weekly auctions conducted by the U.S. Treasury to finance the borrowing needs of the government. Perhaps the most important use of auctions has been to facilitate the transfer of assets from public to private hands – a worldwide phenomenon in the past two decades. These have included the sale of industrial enterprises in Eastern Europe and the former Soviet Union and transportation systems in Britain and Scandinavia. Traditionally, the rights to use natural resources from public property – such as timber rights and off-shore oil leases – have been sold by means of auctions. In the modern era, auctions of rights to use the electromagnetic spectrum for communication are also a worldwide phenomenon. Finally, there has been a tremendous growth in both the number of Internet auction websites, where individuals can put up items for sale under common auction rules, and the value of goods sold there.

The process of procurement via competitive bidding is nothing but an auction except that in this case the bidders compete for the right to sell their products or services. Billions of dollars of government purchases are almost exclusively made in this way, and the practice is widespread, if not endemic, in business. In what follows, an auction will be understood to include the process of procurement via competitive bidding. Of course, in this case it is the person bidding lowest who wins the contract.

What is the reason that auctions and competitive bidding are so prevalent? Are there situations to which an auction is particularly suited as a selling mechanism as opposed to, say, a fixed, posted price? From the point of view of the bidders, what are good bidding strategies? From the point of view of the sellers, are particular forms of auctions likely to bring greater revenues than others? These and other questions form the subject matter of this book

1.1 What is an Auction

An auction format is a mechanism to allocate resources among a group of bidders. An auction model includes three major parts: a description of the potential bidders, the set of possible resource allocations (describing the number of goods of each type, whether the goods are divisible, and whether there are legal or other restrictions on how the goods may be allocated), and the values of various resource allocations to each participant. Thus, a wide variety of selling institutions fall under the rubric of "an auction."

A common aspect of auction-like mechanisms is that they induce (or elicit) information, in the form of bids, from potential buyers regarding their willingness to pay and the outcome - that is, who wins what and pays how much - is determined solely on the basis of the received information. An implication of this is that auctions are *universal* in the sense that they may be used to sell any good.

A second important aspect of auction-like mechanisms is that they are *anonymous*. By this I mean that the identities of the bidders play no role in determining who wins the object and who pays how much.

1.2 Why Auctions

Both buyers and sellers use auction as a mechanism to accomplish the following:

- Price discovery: In many cases seller (or buyers) do not know what an item or service is worth and how much should they sell or buy it for. An auction serves as a "market test" (in fact, this very term is used by many companies to describe an auction process) to ascertain what are the prevailing prices.

- Winner determination: The auction process is used to determine who the object (contract, item, or whatever) should be allocated to, or who "wins" the auction.

- Payment mechanism: Finally, the process can be used to determine how much the winner should pay. As shown later, the traditional process when participants pay what they bid is only one of many possible pricing mechanisms.

The popularity of government and corporate auctions is also rooted in their perceived fairness. Auction processes have rules which are explained before the auction and thus can avoid "special deals" between certain buyers and sellers, manipulation of the market, and wins by favorite bidders.

1.3 Evaluating Auctions

When deciding between various auction mechanisms, the auctioneer has a very large number of auction designs to choose from. The most important criteria in the choice of auction format are the following:

- Revenue-auctioneers are looking for the auction that will yield the maximum revenue for the item sold. While this is an important criterion of auction theory and will be assumed to be the case in the bulk of this section, other considerations are also important not only to governments but to many corporations.

- Efficiency-an auction is successful if the bidder that values the item most *ex post*-actually gets it. In some contexts, such as government auctions this is important, especially when the government is selling public assets. When the sale involves future delivery of services, as in many procurement auctions, efficiency means that the contract is more likely to be carried out and the service provided at a high level.

- Total surplus maximization or social welfare maximization: Maximize the sum of the sell and bidders.

Beside those criteria mentioned above, some others are also used to evaluate an auction such as time and effort that will involved in an auction, simplicity of the auction rule.

1.4 Some Common Auction Forms

1.4.1 The English Auction

The English auction is the open ascending price auction. In one variant of the English auction, the sale is conducted by an auctioneer who begins by calling out a low price and raises it, typically in small increments, as long as there are at least two interested bidders. The auction stops when there is only one interested bidder.

1.4.2 The Dutch Auction

The Dutch auction is the open descending price counterpart of the English auction. Here the auctioneer begins by calling out a price high enough so that presumably no bidder is interested in buying the object at that price. This price is gradually lowered until some bidder indicates her interest.

1.4.3 The Sealed-Bid First-Price Auction

In this auction form, bidders submit bids in sealed envelopes; the person submitting the highest bid wins the object and pays what he bid.

1.4.4 The Sealed-Bid Second-Price Auction

As its name suggests, once again bidders submit bids in sealed envelopes; the person submitting the highest bid wins the object but pays not what he bid, but the second highest bid.

1.5 Standard Auctions

Traditionally, auction theory deals with the sale of single and multiple items separately. While single item auctions are a special case of multiple items auctions, the theory regarding single items is more developed.

The basic paradigm is that of multiple bidders, each having a value to them for winning the item being auctioned off. Their goal is to buy the item below that value so the difference between the valuation and the price paid leaves the bidders with some profit (or surplus in economic terms). Naturally, that valuation is the maximum bid that any participant is willing to place. In any auction process the seller does not know the value that bidders place on the item auctioned off and bidders do not know with certainty how other bidders value the item. The information available to bidders and the corresponding type of auctions can be classified as follows:

- Private Value (PV) Auctions-where each bidder knows only the value of the item to himself. Even if he would know what other bidders are willing to pay this would not affect his own valuation. Such a model is appropriate when the value of an item to a bidder is derived from its consumption alone and not from later resale.

- Interdependent Value Auctions-where the value of the items sold is not known to the bidders. Each bidder has only an estimate (signal) regarding the value (this may be an expert opinion or a test result). If a given bidder would have known the signals of other bidders, his own estimate of the true value may change.

- Common Value (CV) Auctions-a special case of interdependent values in which the value of the item ex post is the same for all bidders. For example, when oil leases are auctioned off, bidders have only their own test results regarding the actual amount of oil in the tract being leased. After the auction, however, the winner will find out exactly the amount of oil in the ground and this oil has a certain market value.

1.6 Equivalent Auctions

Open auctions requires that the bidders collect in the same place, whereas sealed bids may be submitted by mail, so a bidder may observe the behavior of other bidders in one format and not in another. For rational decision makers, however, some of these differences are superficial.

- The Dutch open descending price auction is strategically equivalent to the first-price sealed-bid auction.

- When values are private, the English open ascending auction is also (not strategically) equivalent to the second-price sealed-bid auction.

Chapter 2

Some Useful Mathematics

2.1 Continuous Distributions

Given a random variable X, which takes on values in $[0, \omega]$, we define its cumulative distribution function $F : [0, \omega] \to [0, 1]$ by

$$F(x) = Prob[X \leq x]$$

the probability that X takes on a value not exceeding x. By definition, the function F is nondecreasing and satisfies $F(0) = 0$ and $F(\omega) = 1$ (if $\omega = \infty$, then $lim_{x\to\infty} F(x) = 1$). In this course, we always suppose that F is increasing and continuously differentiable.

The derivative of F is called the associated probability density function and is usually denoted by the corresponding lowercase letter $f \equiv F'$. By assumption, f is continuous and we will suppose that for all $x \in (0, \omega)$, $f(x)$ is positive. The interval $[0, \omega]$ is called the support of the distribution.

If X is distributed according to F, then the expectation of X is

$$E(X) = \int_0^\omega x f(x) dx$$

and if $\gamma : [0, \omega] \to \mathbb{R}$ is some arbitrary function, then the expectation of $\gamma(X)$ is analogously defined as

$$E[\gamma(X)] = \int_0^\omega \gamma(x) f(x) dx.$$

Sometimes the expectation of $\gamma(X)$ is also written as

$$E[\gamma(X)] = \int_0^\omega \gamma(x) dF(x).$$

The *conditional expectation* of X given that $X < x$ is

$$E[X|X < x] = \frac{1}{F(x)} \int_0^x tf(t)dt$$

and so

$$F(x)E[X|X < x] = \int_0^x tf(t)dt = xF(x) - \int_0^x F(t)dt$$

which is obtained by integrating the right-hand side of the first equality by parts.

2.1.1 Hazard Rates

Let F be a distribution function with support $[0, \omega]$. The hazard rate of F is the function $\lambda : [0, \omega) \to \mathbb{R}_+$ defined by

$$\lambda(x) \equiv \frac{f(x)}{1 - F(x)}$$

If F represents the probability that some event will happen before time x, then the hazard rate at x represents the instantaneous probability that the event will happen at x, given that it has not happened until time x. The event may be the failure of some component-a lightbulb, for instance-and hence it is sometimes also known as the "failure rate." Note that $\lambda(x) \to \infty$ as $x \to \omega$. Observe

$$-\lambda(x) = \frac{d}{dx} \ln(1 - F(x)),$$

if we write

$$F(x) = 1 - \exp(-\int_0^x \lambda(t)dt),$$

This shows that any arbitrary function $\lambda : [0, \omega) \to \mathbb{R}_+$ such that for all $x < \omega$,

$$\int_0^x \lambda(t)dt) < \infty \text{and} \lim_{x \to \omega} \int_0^x \lambda(t)dt = \infty$$

is the hazard rate of *some* distribution. If for all $x \geq 0$, $\lambda(x)$ is constant, say $\lambda(x) \equiv \lambda > 0$, then, the associated distribution results in the *exponential distribution*

$$F(x) = 1 - \exp(-\lambda x)$$

whose expectation $E[X] = 1/\lambda$.

Closely related to the hazard rate is the function $\sigma : (0, \omega] \to \mathbb{R}_+$ defined by

$$\sigma(x) \equiv \frac{f(x)}{F(x)}$$

sometimes known as the *reverse hazard rate* or is referred to as the *inverse of the Mills' ratio*. Since

$$\sigma(x) = \frac{d}{dx} \ln F(x)$$

if we write

$$F(x) = \exp(-\int_x^\omega \sigma(t)dt),$$

This shows that any arbitrary function $\sigma : (0, \omega] \to \mathbb{R}_+$ such that for all $x > 0$,

$$\int_x^\omega \sigma(t)dt) < \infty \text{and} \lim_{x \to 0} \int_x^\omega \sigma(t)dt = \infty$$

is the reverse hazard rate of some distribution.

2.2 Stochastic Dominance

2.2.1 First-Order Stochastic Dominance

Given two distribution functions F and G, we say that F (first-order) stochastically dominates G if for all $z \in [0, \omega]$,

$$F(z) \leq G(z).$$

Suppose $u : [0, \omega] \to \mathbb{R}$ is an increasing and differentiable function. If X stochastically dominates Y , and these have distribution functions F and G, respectively, then

$$\begin{aligned}
&E[u(X)] - E[u(Y)] \\
&= \int_0^\omega u(z)(f(z) - g(z))dz \\
&= [u(z)(F(z) - G(z))]_0^\omega - \int_0^\omega u'(z)[F(z) - G(z)]dz \because \text{integration by parts} \\
&= -\int_0^\omega u'(z)[F(z) - G(z)]dz \\
&\geq 0 (\because u' > 0 \text{ and } F \leq G)
\end{aligned}$$

2.2.2 Second-Order Stochastic Dominance

Suppose X is a random variable with distribution function F. Let Z be a random variable whose distribution conditional on $X = x$, $H(|X = x)$ is such that for all x, $E[Z|X = x] = 0$. Suppose $Y = X + Z$ is the random variable obtained from first drawing X from

F and then for each realization $X = x$, drawing a Z from the conditional distribution $H(|X = x)$ and adding it to X. Let G be the distribution of Y so defined. We will then say that G is a mean-preserving spread of F.

Definition 2.2.1 *$f : D \to \mathbb{R}$ is a **concave** function if for all $x, y \in D$,*

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \forall \alpha \in [0,1].$$

While the random variables X and Y have the same mean-that is, $E[X] = E[Y]$ - the variable Y is "more spread-out" than X since it is obtained by adding a "noise" variable Z to X. Now suppose $u : [0,\omega] \to \mathbb{R}$ is a concave function. Then we have the following Jensen's inequality.

Jensen's Inequality: Suppose that $u : [0,\omega] \to \mathbb{R}$ is concave. Then,

$$E(u(X)) \leq u(EY)).$$

Using Jensen's inequality, we obtain

$$\begin{aligned} E_Y[u(Y)] &= E_X[E_Z[u(X+Z)]|X=x] \\ &\leq E_X[u(E_Z[X+Z|X=x])] \\ &= E_X[u(X)] \end{aligned}$$

Given two distributions F and G with the same mean, we say that F *second-order stochastically dominates* G if for all concave functions $u : [0,\omega] \to \mathbb{R}$,

$$\int_0^\omega u(x)f(x)dx \geq \int_0^\omega u(y)g(y)dy.$$

Second-order stochastic dominance is also equivalent to the statement that for all x,

$$\int_0^x G(y)dy \geq \int_0^x F(y)dy$$

with an equality when $x = \omega$.

2.2.3 Hazard Rate Dominance

Suppose that F and G are two distributions with hazard rates $\lambda_F$and λ_G, respectively. If for all x, $\lambda_F(x) \leq \lambda_G(x)$, then we say that F *dominates* G *in terms of the hazard rate.* This order is also referred in short as hazard rate dominance.

If F dominates G in terms of the hazard rate, then

$$F(x) = 1 - \exp(-\int_0^x \lambda_F(t)dt) \leq 1 - \exp(-\int_0^x \lambda_G(t)dt) = G(x)$$

and hence F stochastically dominates G. Thus, hazard rate dominance implies first-order stochastic dominance.

2.2.4 Reverse Hazard Rate Dominance

Suppose that F and G are two distributions with reverse hazard rates σ_F and σ_G, respectively. If for all x, $\sigma_F(x) \geq \sigma_G(x)$, then we say that F *dominates* G *in terms of the reverse hazard rate.* This order is also referred to as reverse hazard rate dominance, in short.

If F dominates G in terms of the reverse hazard rate, then

$$F(x) = \exp(-\int_x^\omega \sigma_F(t)dt) \leq \exp(-\int_x^\omega \sigma_G(t)dt) = G(x)$$

and hence, again, F stochastically dominates G. Thus, reverse hazard rate dominance also implies first-order stochastic dominance.

2.3 Order Statistics

Let $X_1, X_2, \cdots, X_n$ be n independently draws from a distribution F with associated density f. Let $Y_1^{(n)}, Y_2^{(n)}, ..., Y_n^{(n)}$ be a rearrangement of these so that $Y_1^{(n)} \geq Y_2^{(n)} \geq \ \geq Y_n^{(n)}$. The random variables $Y_k^{(n)}$, $k = 1, 2, \cdots, n$ are referred to as *order statistics.*

Let $F_k^{(n)}$ denote the distribution of $Y_k^{(n)}$, with corresponding probability density function $f_k^{(n)}$.

When the "sample size" n is fixed and there is no ambiguity, I will simply write Y_k instead of $Y_k^{(n)}$, F_k instead of $F_k^{(n)}$ and f_k instead of $f_k^{(n)}$. We will typically be interested in properties of the highest and second highest order statistics, Y_1 and Y_2.

2.3.1 Highest Order Statistic

The distribution of the highest order statistic Y_1 is easy to derive. The event that $Y_1 \leq y$ is the same as the event: for all k, $X_k \leq y$. Since each X_k is an independent draw from the same distribution F, we have that[1]

$$F_1(y) = F(y)^n.$$

The associated probability density function is

$$f_1(y) = nF(y)^{n-1} f(y).$$

Observe that if F stochastically dominates G (for all x, $F(x) \leq G(x)$) and F_1 and G_1 are the distributions of the highest order statistics of n draws from F and G, respectively, then F_1 stochastically dominates G_1 (Can you prove it?).

2.3.2 Second-Highest Order Statistic

The distribution the second-highest order statistic Y_2 can also be easily derived. The event that Y_2 is less than or equal to y is the union of the following disjoint events: (i) all X_k's are less than or equal to y, and (ii) $n-1$ of the X_k's are less than or equal to y and one is greater than y. There are n different ways in which (ii) can occur, so we have that

$$\begin{aligned} F_2(y) &= \underbrace{F(y)^n}_{\text{(i)}} + \underbrace{nF(y)^{n-1}(1 - F(y))}_{\text{(ii)}} \\ &= nF(y)^{n-1} - (n-1)F(y)^n. \end{aligned}$$

The associated probability density function is

$$f_2(y) = n(n-1)(1 - F(y))F(y)^{n-2} f(y).$$

Again, it can be verified that if F stochastically dominates G and F_2 and G_2 are the distributions of the second-highest order statistics of n draws from F and G, respectively, then F_2 stochastically dominates G_2.

Some Relationships

Observe that

$$\begin{aligned} F_2^n(y) &= nF(y)^{n-1} - (n-1)F(y)^n \\ &= nF_1^{n-1}(y) - (n-1)F_1^n(y) \end{aligned}$$

[1] We write $F(y)^n$ to denote $(F(y))^n$.

and so

$$f_2^{(n)}(y) = nf_1^{n-1)}(y) - (n-1)f_1^{(n)}(y),$$

This immediately implies that

$$E[Y_2^{(n)}] = nE[Y_1^{(n-1)}] - (n-1)E[Y_1^{(n)}]$$

Also note that

$$\begin{aligned} f_2^n(y) &= n(n-1)(1-F(y))F(y)^{n-2}f(y) \\ &= n(1-F(y))f_1^{n-1}(y). \end{aligned}$$

2.4 Affiliation

Suppose that the random variables $X_1, X_2, \cdots, X_n$ are distributed on some product of intervals $\mathcal{X} \subset \mathbb{R}^n$ according to the joint density function f. The variables $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ are said to be *affiliated* if for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

$$f(\mathbf{x}' \vee \mathbf{x}'')f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}')f(\mathbf{x}'') \tag{2.1}$$

where

$$\mathbf{x}' \vee \mathbf{x}'' = (\max(x_1', x_1''), \cdots, \max(x_n', x_n''))$$

denotes the component-wise *maximum* of $\mathbf{x}'$ and $\mathbf{x}''$, and

$$\mathbf{x}' \wedge \mathbf{x}'' = (\min(x_1', x_1''), \cdots, \min(x_n', x_n''))$$

denotes the component-wise *minimum* of $\mathbf{x}'$ and $\mathbf{x}''$. If (2.1) is satisfied, then we also say that f is affiliated.[2]

Suppose that the density function $f : \mathcal{X} \to \mathbb{R}_+$ is strictly positive in the interior of $\mathcal{X}$ and twice continuously differentiable. It is "easy" to verify that f is affiliated if and only if, for all $i \neq j$,

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln f \geq 0.$$

In other words, the off-diagonal elements of the Hessian of ln f are nonnegative.

[2]1A function g is said to be *supermodular* if $g(\mathbf{x}' \vee \mathbf{x}'') + g(\mathbf{x}' \wedge \mathbf{x}'') \geq g(\mathbf{x}') + g(\mathbf{x}'')$. Thus, f is affiliated if and only if $\ln f$ is supermodular; in other words, f is *log-supermodular*.

Suppose that the random variables $X_1, X_2, \cdots, X_n$ are symmetrically distributed and, define $Y_1, Y_2, \cdots, Y_{n-1}$ to be the largest, second largest, $\cdots$, smallest from among $X_2, X_3, \cdots, X_n$. From (2.1) it follows that if g denotes the joint density of $X_1, Y_1, Y_2, \cdots, Y_{n-1}$, then

$$g(x_1, y_1, y_2, \cdots, y_{n-1}) = (n-1)! f(x_1, y_1, y_2, \cdots, y_{n-1})$$

if $y_1 \geq y_2 \geq \cdots \geq y_{n-1}$ and 0 otherwise. Now it immediately follows that

- If $X_1, X_2, \cdots, X_n$ are symmetrically distributed and affiliated, then $X_1, Y_1, Y_2, \cdots, Y_{n-1}$ are also affiliated.

2.4.1 Monotone Likelihood Ratio Property

Suppose the two random variables X and Y have a joint density $f : [0, \omega]^2 \to \mathbb{R}$. If X and Y are affiliated, then for all $x' \geq x$ and $y' \geq y$,

$$f(x', y) f(x, y') \leq f(x, y) f(x', y') \Leftrightarrow \frac{f(x, y')}{f(x, y)} \leq \frac{f(x', y')}{f(x', y)} \tag{2.2}$$

Let $F(.|x) \equiv F_Y(.|X = x)$ denote the conditional distribution of Y given $X = x$ and let $f(.|x) \equiv f_Y(.|X = x)$ denote the corresponding density function. Then by Bayes' rule, (2.2) is equivalent to

$$\frac{f(y'|x) f(x)}{f(y|x) f(x)} \leq \frac{f(y'|x') f(x')}{f(y|x') f(x')} \Leftrightarrow \frac{f(y'|x)}{f(y|x)} \leq \frac{f(y'|x')}{f(y|x')}$$

Thus, we determine that if X and Y are affiliated, then for all $x' \geq x$, the *likelihood ratio*

$$\frac{f(.|x')}{f(.|x)}$$

is increasing and this is referred to as the *monotone likelihood ratio property.*

2.4.2 Likelihood Ratio Dominance

The distribution function F dominates G in terms of the *likelihood ratio* if for all $x < y$,

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)},$$

which is equivalent to $\frac{f(x)}{f(y)} \leq \frac{g(x)}{g(y)}$. Thus if X and Y are affiliated, the following properties hold:

- For all $x' \geq x$, $F(.|x')$ dominates $F(.|x)$ in terms of the *hazard rate*; that is,

$$\lambda(y|x') \equiv \frac{f(y|x')}{1 - F(y|x')} \leq \frac{f(y|x)}{1 - F(y|x)} \equiv \lambda(y|x)$$

  or equivalently, for all y, $\lambda(y|.)$ is nonincreasing.

- For all $x' \geq x$, $F(.|x')$ dominates $F(.|x)$ in terms of the *reverse hazard rate*; that is,

$$\sigma(y|x') \equiv \frac{f(y|x')}{F(y|x')} \leq \frac{f(y|x)}{F(y|x)} \equiv \sigma(y|x)$$

  or equivalently, for all y, $\sigma(y|.)$ is nondecreasing.

- For all $x' \geq x$, $F(.|x')$ (first order) *stochastically dominates* $F(.|x)$; that is,

$$F(y|x') \leq F(y|x)$$

  or equivalently, for all y, $F(y|.)$ is nonincreasing.

All of these results extend in a straightforward manner to the case where the number of conditioning variables is more than one. Suppose $Y, X_1, X_2, \cdots, X_n$ are affiliated and let $F_Y(.|\mathbf{x})$ denote the distribution of Y conditional on $X = \mathbf{x}$. Then, using the same arguments as above, it can be deduced that for all $\mathbf{x}' \geq \mathbf{x}$, $F_Y(.|\mathbf{x}')$ dominates $F_Y(.|\mathbf{x})$ in terms of the likelihood ratio. The other dominance relationships then follow as usual.

Chapter 3

Private Value Auctions

We begin the formal analysis by considering equilibrium bidding behavior in the four common auction forms in an environment with independently and identically distributed private values. In the previous chapter we argued that the open descending price (or Dutch) auction is strategically equivalent to the first-price sealed-bid auction. When values are private, the open ascending price (or English) auction is also equivalent to the second-price sealed-bid auction, albeit in a weaker sense. Thus, for our purposes, it is sufficient to consider the two sealed-bid auctions.

This chapter introduces the basic methodology of auction theory. We postulate an informational environment consisting of (i) a valuation structure for the bidders-in this case, that of private values-and (ii) a distribution of information available to the bidders-in this case, it is independently and identically distributed. We consider different auction formats-in this case, first- and second-price sealed-bid auctions. Each auction format now determines a game of incomplete information among the bidders and, keeping the informational environment fixed, we determine a Bayesian-Nash equilibrium for each resulting game.When there are many equilibria, we usually select one on some basis-dominance, perfection, or symmetry-but make sure that the criterion is applied uniformly to all formats. The relative performance of the auction formats on grounds of revenue or efficiency is then evaluated by comparing the equilibrium outcomes in one format versus another.

3.1 The Symmetric Model

There is a single object for sale and N potential buyers are bidding for the object. Bidder i assigns a value of X_i to the object-the maximum amount a bidder is willing to pay for the object. Each X_i is independently and identically distributed on some interval $[0, \omega]$ according to the increasing distribution function F. It is assumed that F admits a continuous density $f \equiv F'$ and has full support. We allow for the possibility that the support of F is the nonnegative real line $[0, \infty)$ and if that is so, with a slight abuse of notation, write $\omega = \infty$. In any case, it is assumed that $E[X_i] < \infty$.

Bidder i knows the realization x_i of X_i and only that other bidders' values are independently distributed according to F. Bidders are risk neutral – they seek to maximize their expected profits. All components of the model other than the realized values are assumed to be commonly known to all bidders. In particular, the distribution F is common knowledge, as is the number of bidders.

Finally, it is also assumed that bidders are not subject to any liquidity or budget constraints-each bidder i has sufficient resources so that, if necessary, he or she can pay the seller up to his or her value x_i. Thus, each bidder is both willing and able to pay up to his or her value. We emphasize that the distribution of values is the same for all bidders and we will refer to this situation as one involving *symmetric* bidders. In this framework, we will examine two major auction formats:

I. A first-price sealed-bid auction, where the highest bidder gets the object and pays the amount he bid.

II. A second-price sealed-bid auction, where the highest bidder gets the object and pays the second highest bid.

Each of these auction formats determines a game among the bidders. A strategy for a bidder is a function $\beta_i : [0, \omega] \to \mathbb{R}+$, which determines his or her bid for any value. We will typically be interested in comparing the outcomes of a symmetric equilibrium-an equilibrium in which all bidders follow the same strategy-of one auction with a symmetric equilibrium of the other. Given that bidders are symmetric, it is natural to focus attention on symmetric equilibria. We ask the following questions:

(i) What are symmetric equilibrium strategies in a first-price auction (I) and a second-price auction (II)?

(ii) From the point of view of the seller, which of the two auction formats yields a higher expected selling price in equilibrium?

3.1.1 Second Price Sealed-Bid Auctions

Although the first-price auction format is more familiar and even natural, we begin our analysis by considering second-price auctions. The strategic problem confronting bidders in second-price auctions is much simpler than that in first-price auctions, so they constitute a natural starting point. Also recall that in the private values framework, second-price auctions are equivalent to open ascending price (or English) auctions. In a second-price auction, each bidder submits a sealed bid of b_i, and given these bids, the payoffs are:

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

We assume that if there is a tie, so that $b_i = \max_{j \neq i} b_j$, the object goes to each winning bidder with equal probability.

Bidding behavior in a second-price auction is straightforward.

Proposition 3.1.1 *In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{II}(x) = x$, reporting her true value x.*

Proof 3.1.1 (Proof of Proposition 3.1.1:) *Consider a bidder i, with no loss of generality, and suppose that $p = \max_{j \neq i} b_j$ is the highest competing bid. By bidding x_i, bidder i will win if $x_i > p$ and not if $x_i < p$. Suppose, by way of contradiction, that he bids $z_i < x_i$. If $x_i > z_i \geq p$, then he still wins and his profit is still $x_i - p$. If $p > x_i > z_i$, he still loses. However, if $x_i > p > z_i$, then he loses. However, if he had bid x_i, he would have made a positive profit. Thus, bidding less than x_i can never increase his profit but in some circumstances may actually decrease it. A similar argument shows that it is not profitable to bid more than x_i.*

It should be noted that the argument in Proposition 3.1.1 relied neither on the assumption that bidders' values were independently distributed nor the assumption that they were identically so. Only the assumption of private values is important and Proposition 3.1.1 holds as long as this is the case.

With Proposition 3.1.1 in hand, let us ask how much each bidder expects to pay in equilibrium. Fix a bidder, say 1, and let the random variable $Y_1 \equiv Y_1^{(N-1)}$ denote the highest value among the $N-1$ remaining bidders. In other words, Y_1 is the highest order statistic of $X_2, X_3, \cdots, X_N$. Let G denote the distribution function of Y_1. Clearly, for all y, $G(y) = F(y)^{N-1}$. In a second-price auction, the expected payment by a bidder with value x can be written as

$$\begin{aligned}
m^{II}(x) &= \text{Prob[Win]} \times E[\text{2nd highest bid}|x \text{ is the highest bid}] \\
&= \text{Prob[Win]} \times E[\text{2nd highest bid}|x \text{ is the highest value}] \\
&= G(x) \times E[Y_1|Y_1 < x] \\
&= G(x)\frac{1}{G(x)}\int_0^x yf(y)dy = \int_0^x yf(y)dy.
\end{aligned}$$

3.1.2 First Price Sealed-Bid Auctions

In a first price auction, each bidder submits a sealed bid of b_i, and given these bids, the payoffs are

$$\Pi_i = \begin{cases} x_i - b_i & \text{if } b_i > \max_{j\neq i} b_j \\ 0 & \text{if } b_i < \max_{j\neq i} b_j \end{cases}$$

If there is more than one bidder with the highest bid the object goes to each such bidder with equal probability. In a first price auction, no bidder would bid an amount equal to his value since this would only guarantee a payoff of 0. Fixing the bidding behavior of others, the bidder faces a simple trade off-an increase in the bid will increase the probability of winning while, at the same time reducing the gains from winning. To get some idea about how these effects balance off, we begin with a heuristic derivation of symmetric equilibrium strategies.

Suppose that all bidders $j \neq 1$ follow the symmetric, increasing and differentiable equilibrium strategy $\beta^I \equiv \beta : [0, \omega] \to [0, \infty)$. Suppose bidder 1 receives a signal, $X_1 = x$, and bids b. We wish to determine the optimal b.

Claim 3.1.1 *$b \leq \beta(\omega)$ and $\beta(0) = 0$.*

Proof 3.1.2 (Proof of Claim 3.1.1:) *It can never be optimal to choose a bid $b > \beta(\omega)$ since in that case, the bidder would win for sure and could do better by reducing his bid slightly so that he still wins for sure but pays less. A bidder with value 0 would never submit a positive bid since he would make a loss if he were to win the auction.*

Bidder 1 wins the auction whenever he submits the highest bid - that is, whenever $\max_{j\neq 1} \beta(X_j) < b$. Since β is increasing, $\max_{j\neq 1} \beta(X_j) = \beta(\max_{j\neq 1} Xj) = \beta(Y_1)$, where as before, $Y_1 \equiv Y_1^{(N-1)}$, the highest of $N-1$ values. Bidder 1 wins whenever $\beta(Y_1) < b$ or equivalently, whenever $Y_1 < \beta^{-1}(b)$. His expected payoff is therefore

$$G(\beta^{-1}(b)) \times (x - b),$$

where, G is the distribution of Y_1. Maximizing this with respect to b yields the first order condition:

$$\frac{g(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(x - b) - G(\beta^{-1}(b)) = 0$$

where $g = G'$ is the density of Y_1[1].

At a symmetric equilibrium, $b = \beta(x)$, and thus the above first order condition yields the differential equation

$$G(x)\beta'(x) + g(x)\beta(x) = xg(x)$$

or equivalently,

$$\frac{d}{dx}(G(x)\beta(x)) = xg(x)$$

and since $\beta(0) = 0$, we have

$$\begin{aligned}\beta(x) &= \frac{1}{G(x)} \int_0^x yg(y)dy \\ &= E[Y_1|Y_1 < x](= \frac{1}{G(x)} m^{II}(x)).\end{aligned}$$

The derivation of β is only heuristic because it is merely a necessary condition – we have not formally established that if the other $N-1$ bidders follow β, then it is indeed optimal for a bidder with value x to bid $\beta(x)$. The next proposition verifies that this is indeed correct.

[1]Define $y = \beta^{-1}(b)$. Then, we have $b = \beta(y)$ and obtain $db/dy = \beta'(y) = \beta'(^{-1}(b))$ after taking differentiation with respect to b.

Proposition 3.1.2 *Symmetric equilibrium strategies in a first price auction are given by*

$$\beta^I(x) = E[Y_1|Y_1 < x$$

, where Y_1 is the highest of $N-1$ independently drawn values, that is, his equilibrium bidding price is the expectation of the second-highest private value under the condition the bidder wins,, which we will show below it is less than his true value.

Proof 3.1.3 (Proof of Proposition 3.1.2:) *Suppose that all but bidder i follow the strategy $\beta^I \equiv \beta$ given above. We will argue that in that case it is optimal for bidder i to follow β also. First, notice that β is an increasing and continuous function. Thus, in equilibrium the bidder with the highest value submits the highest bid and wins the auction. Denote by $z = \beta^{-1}(b)$ the value for which b is the equilibrium bid, that is, $\beta(z) = b$. In other words, bidder i pretends to receive signal z by bidding b even when he receives x as the true signal. Then we can write bidder i's expected payoff from bidding $b = \beta(z)$ when his value is x as follows:*

$$\begin{aligned}
\Pi(b,x) = \Pi(\beta(z),x) &= G(z)(x - \beta(z)) \\
&= G(z)x - G(z)E[Y_1|Y_1 < z] \\
&= G(z)x - \int_0^z yg(y)dy \\
&= G(z)x - G(z)z + \int_0^z G(y)y (\because \text{ integration by parts}) \\
&= G(z)(x - z) + \int_0^z G(y)y
\end{aligned}$$

We thus obtain that

$$\Pi(\beta(x),x) - \Pi(\beta(z),x) = G(z)(z - x) - \int_x^z G(y)dy \geq 0$$

regardless of whether $z \geq x$ or $z \leq x$, which implies that β is a symmetric equilibrium strategy.

Using integration by parts, the equilibrium bid can be rewritten as

$$\beta^I(x) = x - \int_0^x \frac{G(y)}{G(x)}dy < x.$$

Thus, unlike the second price auction, the bidder has no incentive to tell her true value at the first price auction.

However, note that

$$\frac{G(y)}{G(x)} = [\frac{F(y)}{F(x)}]^{N-1},$$

Then, as the number of bidders (i.e., N) increases, the equilibrium bid $\beta^I(x)$ approaches x.

Example 3.1.1 *Values are exponentially distributed on* $[0, \infty)$ *and there are only two bidders.*

If $F(x) = 1 - \exp(-\lambda x)$*, for some* $\lambda > 0$*, and* $N = 2$*, then*

$$\begin{aligned}\beta^I(x) &= x - \int_0^x \frac{F(y)}{F(x)} dy \\ &= \frac{1}{\lambda} - \frac{x \exp(-\lambda x)}{1 - \exp(-\lambda x)}\end{aligned}$$

Expected Payment in First Price Auctions

In a first price auction, the winner pays what he bid and thus the expected payment by a bidder with value x is

$$m^I(x) = \text{Prob[Win]} \times \text{Amount bid} = G(x) \times E[Y_1 | Y_1 < x] = m^{II}(x),$$

which is the same as in a second-price auction.

3.2 Revenue Comparison

Proposition 3.2.1 *With independently and identically distributed private values, the expected revenue in a first price auction is the same as the expected revenue in a second price auction.*

Proof 3.2.1 (Proof of Proposition 3.2.1:) *Let* $G(x) = F(x)^{N-1}$ *be the probability that a particular bidder wins the auction and let* $g(x) = G'(x) = (N-1)F(x)^{n-2}f(x)$ *be the associated density function. The ex ante expected payment of a particular bidder in either auction is*

$$\begin{aligned}E[m^A(X)] &= \int_0^\omega m^A(x) f(x) dx \\ &= \int_0^\omega (\int_0^x y g(y) dy) f(x) dx,\end{aligned}$$

where $A = I$ or II. Interchanging the order of integration, we obtain that

$$\begin{aligned} E[m^A(X)] &= \int_0^\omega (\int_y^\omega f(x)dx)yg(y)dy \\ &= \int_0^\omega y(1-F(y))g(y)dy. \end{aligned}$$

The expected revenue accruing to the seller $E[R^A]$ is just N(i.e., the number of bidders) times the ex ante *expected payment of an individual bidder, so*

$$\begin{aligned} E[R^A] &= NE[m^A(X)] \\ &= N\int_0^\omega y(1-F(y))g(y)dy \\ &= \int_0^\omega yf_2^N(y)dy \\ &= E[Y_2^N] \end{aligned}$$

where $f_2^N(y) = N(N-1)(1-F(y))F(y)^{N-2}f(y)$ and $g(y) = (n-1)F(y)^{n-2}f(y)$. In either case, the expected revenue is just the expectation of the second highest value.

The fact that the expected selling prices in the two auctions are equal is all the more striking because in specific realizations of the values the price at which the object is sold may be greater in one auction or the other. With positive probability, the revenue R^I in a first-price auction exceeds R^{II}, the revenue in a second-price auction, and vice versa. For instance, when values are uniformly distributed and there are only two bidders, the equilibrium strategy in a first-price auction is $\beta^I(x) = \frac{1}{2}x$. If the realized values are such that$\frac{1}{2}x_1 > x_2$, then the revenue in a first-price auction is greater than that in a second-price auction. On the other hand, if $\frac{1}{2}x_1 < x_2 < x_1$, the opposite is true. Thus, while the revenue may be greater in one auction or another depending on the realized values, we have argued that *on average* the revenue to the seller will be the same.

Actually, we can say more about the distribution of prices in the two auctions. It is clear that the revenues in a second-price auction are more variable than in its first-price counterpart. In the former, the prices can range between 0 and ω; in the latter, they can only range between 0 and $E[Y1]$. A more precise result can be formulated along the following lines. Let L^I denote the distribution of the equilibrium price in a first-price auction and likewise, let L^{II} be the distribution of prices in a second-price auction. Then L^{II} is a mean-preserving spread of L^I-from the perspective of the seller, a second-price auction is riskier than a first-price auction. Every risk-averse seller prefers the latter to the former (assuming, of course, that bidders are risk-neutral).[2] The next proposition shows that the risk averse, revenue maximizing seller prefers first price auction to second price auction.

Proposition 3.2.2 *With independently and identically distributed private values, the distribution of equilibrium prices in a second price auction is a mean-preserving spread of the distribution of equilibrium prices in a first price auction.*

Proof 3.2.2 (Proof of Proposition 3.2.2:) *The revenue in a second price auction is just the random variable $R^{II} = Y_2^{(N)}$; the revenue in a first price auction is the random variable $R^I = \beta(Y_1^{(N)})$, where $\beta \equiv \beta^I$ is the symmetric equilibrium strategy. Now we can write*

$$E[R^{II}|R^I = p] = E[Y_2^{(N)}|Y_1^{(N)} = \beta^{-1}(p)].$$

But for all y, we know

$$E[Y_2^{(N)}|Y_1^{(N)} = y] = E[Y_1^{(N-1)}|Y_1^{(N-1)} < y].$$

This is because the only information regarding the second-highest of N values, $Y_2^{(N)}$, that the event that the highest of N values $Y_1^{(N)} = y$ provides is that the highest of $N-1$ values, $Y_1^{(N.1)}$, is less than y. Or formally,

$$\begin{aligned} Y_1^N &\geq \overbrace{Y_2^N \geq \cdots \geq Y_N^N}^{N-1} \\ y &\geq \underbrace{Y_1^{N-1} \geq \cdots \geq Y_{N-1}^{N-1}}_{N-1} \end{aligned}$$

Using this, we can write

$$\begin{aligned} E[R^{II}|R^I = p] &= E[Y_1^{(N-1)}|Y_1^{(N-1)} < \beta^{-1}(p)] \\ &= \beta(\beta^{-1}(p))(\because \beta^I(x) = E[Y_1|Y_1 < x]) \\ &= p \end{aligned}$$

Since $E[R^{II}|R^I = p] = p$, there exists a random variable Z such that the distribution of R^{II} is the same as that of $R^I + Z$ and $E[Z|R^I = p] = 0$. Therefore, the distribution of R^{II} is a mean-preserving spread of that of R^I.

[2] 1This is also equivalent to the statement that L^I dominates L^{II} in the sense of secondorder stochastic dominance.

3.3 Reserve Price

In the analysis so far, we have implicitly assumed that the seller cab sell the object at any price. In many instances, sellers reserve the right to not sell the object if the price determined in the auction is lower some threshold amount, say $r > 0$. Such a price is called the *reserve price*. What we will show here is to extend the revenue equivalence to the equivalence between the first and second price auctions with reserve price.

3.3.1 Reserve Prices in Second Price Auctions

Since the price at which the object is sold can never be lower than r, no bidder with a value $x < r$ can make a positive profit in the auction. In a second price auction, a reserve price makes no difference to the behavior of the bidders. bidders – it is still a weakly dominant strategy to bid one's value. The expected payment of a bidder with value r is now just $rG(r)$, and the expected payment of a bidder with value $x \geq r$ is

$$m^{II}(x, r) = rG(r) + \int_r^x yg(y)dy$$

since the winner pays the reserve price r whenever the second-highest bid is below r.

3.3.2 Reserve Prices in First Price Auctions

If β^I is a symmetric equilibrium of the first price auction with reserve price r, it must be that $\beta^I(r) = r$. This is because a bidder with value r wins only if all other bidders have values less than r and, in that case, can win with a bid of r itself. In all other respects, the analysis of a first-price auction is unaffected, and in a manner analogous to Proposition 3.1.2 we obtain that a symmetric equilibrium bidding strategy for any bidder with value $x \geq r$ is

$$\begin{aligned} \beta^I(x) &= E[\max\{Y_1, r\} | Y_1 < x] \\ &= r\frac{G(r)}{G(x)} + \frac{1}{G(x)} \int_r^x yg(y)dy \end{aligned}$$

The expected payment of a bidder with value $x \geq r$ is

$$\begin{aligned} m^I(x, r) &= G(x)\beta^I(x) \\ &= rG(r) + \int_r^x yg(y)dy \end{aligned}$$

which is the same as in the second price auctions with reserve price. Thus, we conclude that the revenue equivalence theorem can be generalized to the reserve price auctions.

3.3.3 Revenue Effects of Reserve Prices

Let A denote either the first ($A = I$) or second ($A = II$) price auction. In both, the expected payment of a bidder with value r is $rG(r)$. Recall that the expected payment of a bidder with value $x \geq r$ is

$$m^A(x,r) = rG(r) + \int_r^x yg(y)dy$$

The ex ante expected payment of a bidder is then

$$\begin{aligned} E[m^A(x,r)] &= \int_r^\omega m^A(x,r)f(x)dx \\ &= r(1 - F(r))G(r) + \int_r^\omega y(1 - F(y))g(y)dy \end{aligned}$$

Suppose that the seller attaches a value $x_0 \in [0, \omega)$. This means that if the object is left unsold, the seller would derive a value x_0 from its use. Clearly, the seller would not set a reserve price r that is lower than x_0. Then the overall expected payoff of the seller from setting a reserve price $r \geq x_0$ is

$$\begin{aligned} \Pi_0 &= N \times E[m^A(x,r)] + x_0 F(r)^N \\ &= Nr(1 - F(r))G(r) + N \int_r^\omega y(1 - F(y))g(y)dy + F(r)^N x_0 \end{aligned}$$

Differentiating this with respect to r, we obtain

$$\frac{d\Pi_0}{dr} = N[1 - F(r) - rf(r)]G(r) + NG(r)f(r)x_0$$

Now recall that the hazard rate function associated with the distribution F is defined as $\lambda(x) = f(x)/(1 - F(x))$. Thus, we can write

$$\frac{d\Pi_0}{dr} = N[1 - (r - x_0)\lambda(x)](1 - F(r))G(r)$$

First, notice that if $x_0 > 0$, then the derivative of Π_0 at $r = x_0$ is positive, implying that the seller should set a reserve price $r > x_0$. If $x_0 = 0$, then derivative of Π_0 at $r = 0$ is 0, but as long as $\lambda(r)$ is bounded, the expected payment attains a local minimum at 0, so a small reserve price leads to an increase in revenue. Thus, *a revenue maximizing seller should always set a reserve price that exceeds his or her value.*

The relevant first-order condition implies that the optimal reserve price r^*. must satisfy

$$r^*\lambda(r^*) = 1 \Leftrightarrow r^* = \frac{1}{\lambda(r^*)}$$

If $\lambda(.)$ is increasing, this condition is also sufficient and it is remarkable that the optimal reserve price does not depend on the number of bidders.

3.3.4 The Optimal Reserve Price in a Second Price Auction

Consider a second price auction with two bidders and suppose $x_0 = 0$. By setting a positive reserve price r, the seller faces a trade-off between the following two events:

1. (LOSS) if the highest value among the bidders, Y_1, is smaller than r, the object will remain unsold.

2. (GAIN) while the highest value Y_1 exceeds r, the second-highest value, Y_2, is smaller than r.

The probability of the first event is $F(r)^2$ and the loss is at most r. So for small r, the expected loss is at most $rF(r)^2$. The probability of the second event is $2F(r)(1 - F(r))$, and for small r, the gain is of order r, so the expected gain is of order $2rF(r)(1 - F(r))$. Thus, the expected gain from setting a small reserve price exceeds the expected loss.

3.3.5 Entry Fees as a Substitute of Reserve Prices

An alternative instrument that the seller can also use to exclude buyers with low values is an *entry fee*-a fixed and non-refundable amount that bidders must pay the seller prior to the auction in order to be able to submit bids. An entry fee is the price of admission to the room of bidders can be excluded by asking each bidder to pay an entry fee e such that

$$e = \int_0^r G(y)dy$$

Notice that e equals the expected payoff of a bidder with value r in either a first- or second-price auction, so a bidder with value $x < r$ would not find it worthwhile to pay e

in order to participate in the auction. Then, the exclusion effect of a reserve price r can be replicated with an entry fee of e. Conversely, the exclusion effect of an entry fee e can be duplicated with a reserve price of r.

A closely related topic to entry fees in auctions is participation costs in auctions. The fundamental structure of auctions with participation costs is one through which an indivisible object is allocated to one of many potential buyers, and in order to participate in the auction, buyers must incur some costs After the cost is incurred, a bidder can submit a bid. The bidder who submits the highest bid wins the object and pays a price – highest or second highest – according to the auction formats. There are many sources for participation costs. For instance, sellers may require that those who submit bids have a certain minimum amount of bidding funds which may compel some bidders to borrow; bidders themselves may have transportation costs to go to an auction place; or they need spend some money to learn the rules of the auction and how to submit bids. Bidders even have opportunity costs to attend an auction. Cao and Tian (2008, 2009b) study the equilibrium of auctions with participation costs under different specific auction formats, assuming the values of bidders are private information while participation costs are common knowledge. Kaplan and Sela (2006) studies this problem while assuming the values are common knowledge and participation costs are private information. Cao and Tian (2009a) study the equilibrium of second price auctions when both values and participation costs are private information.

3.3.6 Efficiency versus Revenue

Although the revenue under both first and second price auction in the presence of reserve price is the same, it may result in inefficient outcome. A reserve price (or equivalently, an entry fee) raises the revenue to the seller but may have a detrimental effect on efficiency. Suppose that the value that the seller attaches to the object is 0. In the absence of a reserve price, the object will always be sold to the highest bidder and in the symmetric model studied here, that is also the bidder with the highest value. Thus, both the first- and second-price auctions allocate efficiently in the sense that the object ends up in the hands of the person who values it the most. If the seller sets a reserve price $r > 0$, however, there is a positive probability that the object will remain in the hands of seller and this is inefficient. This simple observation implies that there may be a trade-off between efficiency and revenue.

3.4 The Revenue Equivalence Principle

In the previous sections we saw that regardless of the distribution of values, the expected selling price in a symmetric first-price auction is the same as that in a second-price auction. As a result a risk-neutral seller is indifferent between the two formats. The fact that the expected selling prices in the two auctions are equal is quite remarkable. The two auctions are not strategically equivalent , and in particular instances, the price in one or the other auction may be higher. This section explores the reasons underlying the equality of expected revenues in Proposition 3.2.1. In the process, we will discover that this equality extends beyond first- and second-price auctions to a whole class of auction forms.

3.4.1 Main Results

The auction forms we consider all have the feature that buyers are asked to submit bids-amounts of money they are willing to pay. These bids alone determine who wins the object and how much the winner pays. We will say that an auction is standard if a person who bids the highest amount is awarded the object. An example of a nonstandard method is a lottery in which the chances that a particular bidder wins is the ratio of his or her bid to the total amount bid by all. Such a lottery is nonstandard since the person who bids the most is not necessarily the one who is awarded the object.

Given a standard auction form, A, and a symmetric equilibrium β^A of the auction, let $m^A(x)$ be the equilibrium expected payment by a bidder with value x. It turns out, quite remarkably, that provided that the expected payment of a bidder with value 0 is 0, the expected payment function $m^A(.)$ does not depend on the particular auction form A. As a result, the expected revenue in any standard auction is the same, a fact known as the revenue equivalence principle.

Proposition 3.4.1 (The Revenue Equivalence Principle) *Suppose that values are independently and identically distributed and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment* *of a bidder with value zero is zero, yields the same expected revenue to the seller.*

Proof 3.4.1 (Proof of Proposition 3.4.1:) *Consider a standard auction form, A, and fix a symmetric equilibrium β of A. Let $m^A(x)$ be the equilibrium expected payment in auction A by a bidder with value x. Suppose that β is such that $m^A(0) = 0$.*

Consider a particular bidder i and suppose other bidders are following the equilibrium strategy β. Suppose that bidder i with value x contemplates a different bid $\beta(z)$ instead of the equilibrium bid $\beta(x)$. This means that bidder i with value x pretends that he received a different value z in order to seek a profitable deviation. Bidder i wins when his bid $\beta(z)$ exceeds the highest competing bid $\beta(Y_1)$, or when $z > Y_1$. His expected payoff is

$$\Pi_i^A(z, x) = G(z)x - m_i^A(z)$$

where $G(z) \equiv F(z)^{N-1}$ is the distribution of Y_1, i.e., the probability that bidder i will be the winner. Differentiating Π_i with respect to z, we obtain the first order condition for maximizing profit.

$$\frac{\partial}{\partial z}\Pi_i^A(z, x) = g(z)x - \frac{d}{dz}m_i^A(z) = 0.$$

At an equilibrium it is optimal to report $z = x$, so we obtain that for all y,

$$\frac{d}{dy}m_i^A(y) = g(y)y.$$

Solving the above differential equation with the condition that $m_i^A(0) = 0$, we have

$$\begin{aligned} m_i^A(x) &= \int_0^x yg(y)dy \\ &= G(x) \times E[Y_1|Y_1 < x]. \end{aligned}$$

Since m_i^A does not depend upon the particular auction form A, this complete the proof.

3.4.2 Applications of the Revenue Equivalence Principle

The revenue equivalence principle is a powerful and useful tool. In this subsection we show how it can be used to derive equilibrium bidding strategies in alternative, unusual auction forms. We then show how it can be extended and applied to situations in which bidders are unsure as to how many other, rival bidders they face.

Equilibrium of All-Pay Auctions

In an *all-pay* auction, each bidder submits a bid and need to say what he bids, but only the highest bidder wins the object. Hence, the all-pay auction is a standard auction. The "non-standard" aspect of an all-pay auction is, however, that all bidders pay what they bid. Why should we be bothered by the all-pay auction? Because it is a useful model of lobbying activity. In such models, different interest group spend money - their "bids" - in order to influence government policy and the group spending the most - the highest "bidder" - is able to tilt policy in its favored direction, thereby, "winning the auction." Since money spent on lobbying is a sunk cost borne by all groups regardless of which group is successful in obtaining its preferred policy, such situations have a natural all-pay aspect.

Suppose for the moment that there is a symmetric, increasing equilibrium $\beta = \beta^{AP}$ of the all-pay auction with the property that $\beta(0) = 0$. Now in an all-pay auction, the expected payment of a bidder with value x is the same as his bid - he forfeits his bid regardless of whether he wins or not - and so the equilibrium of the all-pay auction must be

$$\begin{aligned}\beta^{AP}(x) &= m^A(x) \\ &= \int_0^x yg(y)dy.\end{aligned}$$

To verify that this indeed constitutes an equilibrium of the all-pay auction, suppose that all bidders but i are following the strategy $\beta = \beta^{AP}$. If he bids an amount $\beta(z)$, the expected payoff of a bidder i with value x is

$$\Pi_i^{AP}(\beta(z), x) = G(z)x - \beta(z) = G(z)x - G(z)x \int_0^z yg(y)dy.$$

Integrating the second term of the right hand side of the above equation we obtain

$$\Pi_i^{AP}(\beta(z), x) = G(z)(x - z) + \int_0^z G(y)dy$$

which is the same as the payoff obtained in a first-price auction by bidding $\beta^I(z)$ against other bidders who are following β^I. Therefore, this is maximized by choosing z = x. Namely, we have

$$\Pi_i^{AP}(\beta(x), x) = \int_0^x G(y)dy = \Pi_i^I(\beta(x), x).$$

Equilibrium of Third Price Auctions

Suppose that there are at least three bidders. Consider a sealed-bid auction in which the highest bidder wins the object but pays a price equal to the third-highest bid. A third price auction, as it is called, is a purely theoretical construct: there is no known instance of such a mechanism actually being used. It is an interesting construct nevertheless equilibria of such an auction display some unusual properties and leads to a better understanding of the workings of the standard auction forms. Here we show how the revenue equivalence principle can once again be used to derive equilibrium bidding strategies.

Proposition 3.4.2 *Suppose that there are at least three bidders and $F()$ is log-concave. Symmetric equilibrium strategies in a third price auction are given by*

$$\beta^{III}(x) = x + \frac{F(x)}{(N-2)f(x)} \tag{3.1}$$

An important feature of the equilibrium in a third price auction is worth noting: the equilibrium bid *exceeds* the value. To better understand this phenomenon, first, notice that for much the same reason as in a second price auction, it is dominated for a bidder to bid below his value in a third-price auction. Unlike in a second-price auction, however, it is not dominated for a bidder to bid above his value. Fix some equilibrium bidding strategies of the third-price auction, say β, and suppose that all bidders except 1 follow β. Suppose bidder 1 with value x bids an amount $b > x$. If $\beta(Y_2) < x < \beta(Y_1) < b$, this is better than bidding x since it results in a profit, whereas bidding x would not. If, however, $x < \beta(Y_2) < \beta(Y_1) < b$, then bidding b results in a loss. When $b - x = \epsilon$ is small, the gain in the first case is of order ϵ^2, whereas the loss in the second case is of order ϵ^3. Thus, it is optimal to bid higher than one's value in a third-price auction.

Comparing equilibrium bids in first, second, and third price auctions in case of symmetric private values, we have seen that

$$\beta^{I}(x) < \beta^{II}(x) = x < \beta^{III}(x)$$

(assuming, of course, that the distribution of values is log-concave).

Proof 3.4.2 (Proof of Proposition 3.4.2:) *Suppose that there is a symmetric, increasing equilibrium of the third price auction, say β^{III} with $m^{III}(0) = 0$, which is needed to* *apply Proposition 3.4.1. Then, we have that for all x, the expected payment in a third price auction is*

$$m^{III}(x) = \int_0^x yg(y)dy \tag{3.2}$$

Uncertain Number of Bidders

Let $\mathcal{N} = 1, 2, \cdots, N$ denote the set of *potential* bidders and let $\mathcal{A} \subset \mathcal{N}$ be the set of actual bidders, that is, those that participate in the actual auction. All potential bidders draw their values independently from the same distribution $F(.)$.

Consider an actual bidder $i \in \mathcal{A}$ and let p_n denote the probability that any participating bidder assigns to the event that he is facing n other bidders. Thus, bidder i assigns the probability p_n that the number of actual bidder is $n + 1$. What is important is that the exact process by which the set of actual bidders be symmetric so that every actual bidder holds the *same* beliefs about how many other bidders he faces. It is also important that the set of actual bidders does not depend on the realized values.

Consider a standard auction A and a symmetric and increasing equilibrium β of the auction. Consider the expected payoff of a bidder with value x who holds $\beta(z)$ instead of the equilibrium bid $\beta(x)$. When he faces n other bidders, he wins if $Y_1^{(n)}$, the highest of n values drawn from $F(.)$, is less than z and the probability of this event is $G^(n)(z) = F(z)^n$. The overall probability that he will win when he bids $\beta(z)$ is therefore

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z).$$

His expected payoff from bidding $\beta(z)$ when his value is x is then

$$\Pi^A(z, x) = G(z)x - m^A(z)$$

and the remainder of the argument is the same as in Proposition 3.4.1. Thus, we conclude that the *revenue equivalence principle holds even if there is uncertainty about the number of bidders.*

Suppose that the object is sold using a second-price auction. Even though the number of rival buyers that a particular bidder faces is uncertain, it is still a dominant strategy for him to bid his value. The expected payment in a second-price auction of an actual bidder with value x is therefore

$$m^{II}(x) = \sum_{n=0}^{N-1} p_n G^{(n)}(x) E[Y_1^{(n)} | Y_1^{(n)} < x]$$

Now suppose that the object is sold using a first-price auction and that β is a symmetric and increasing equilibrium. The expected payment of an actual bidder with value x is

$$m^I(x) = G(x)\beta(x)$$

where $G(x)$ is as defined earlier. The revenue equivalence principle implies that for all x, $m^I(x) = m^{II}(x)$, so

$$\begin{aligned}
\beta(x) &= \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} E[Y_1^{(n)} | Y_1^{(n)} < x] \\
&= \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} \beta^{(n)}(x).
\end{aligned}$$

where $\beta^{(n)}$ is the equilibrium bidding strategy in a first-price auction in which there are exactly $n+1$ bidders for sure (see Proposition 3.1.2). Thus, *the equilibrium bid for an actual bidder with value x when he is unsure about the number of rivals he faces is a weighted average of the equilibrium bids in auctions when the number of bidders is known to all.*

Chapter 4

Extensions of Independent Private Values Auctions

The revenue equivalence principle is a very powerful result but based on the following key assumptions:

1 *Independence* – the values of different bidders are independently distributed.

2 *Risk Neutrality* – all bidders seek to maximize their expected profits.

3 *No budget constraints* – all bidders have the ability to pay up to their respective values.

4 *Symmetry* – the values of all bidders are distributed according to the same distribution function F.

In this chapter we will relax each assumption while keep others to see how the revenue equivalence principle breaks down.

4.1 Auctions with Risk-Averse Bidders

We argue that if bidders are risk-averse, but all other assumptions are retained, the revenue equivalence principle no longer holds. Suppose that each bidder has a von- Neumann-Morgenstern utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that satisfies $u(0) = 0$, $u' > 0$ and $u'' < 0$. The next proposition says that if bidders are risk-averse, the revenue maximizing seller should use a first price auction over a second price auction.

Proposition 4.1.1 *Suppose that bidders are risk-averse with the same utility function. With symmetric, independent private values, the expected revenue in a first price auction is greater than in a second price auction.*

Proof 4.1.1 (Proof of Proposition 4.1.1:) *Notice that risk aversion makes no difference in a second price auction. So, let us examine a first price auction. Suppose that when bidders are risk averse and have the utility function u, the equilibrium strategies are given by an increasing and differentiable function $\gamma : [0, \omega] \to \mathbb{R}_+$ satisfying $\gamma(0) = 0$. If all other bidders follow this strategy, then bidder i will never bid more than $\gamma(\omega)$ (Do you see why?). Given a value x, bidder i's problem is summarized as follows:*

$$\max_{z \in [0,\omega]} G(z)u(x - \gamma(z)),$$

where $G \equiv F^{(n-1)}$ is the distribution of the highest of $n - 1$ values. The first-order condition for this maximization problem is

$$g(z) \times u(x - \gamma(z)) - G(z) \times \gamma'(z) \times u'(x - \gamma(z)) = 0.$$

In a symmetric equilibrium, it must be optimal to choose $z = x$. Hence we get

$$\gamma'(x) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \times \frac{g(x)}{G(x)}.$$

With risk neutrality, $u(x) = x$, the counterpart of the above condition is given

$$\beta'(x) = (x - \beta(x)) \times \frac{g(x)}{G(x)}$$

where β denotes the equilibrium strategy with risk-neutral bidders.

Next notice that if u is strictly concave and $u(0) = 0$, for all $y > 0$, $[u(y)/u'(y)] > y$. Using this fact, we can derive that

$$\gamma'(x) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \times \frac{g(x)}{G(x)} > (x - \gamma(x)) \times \frac{g(x)}{G(x)}$$

Now if $\beta(x) > \gamma(x)$, we have $(x - \gamma(x)) \times \frac{g(x)}{G(x)} > (x - \beta(x))\gamma(x)) \times \frac{g(x)}{G(x)}$ which implies that $\gamma'(x) > \beta'(x)$. It is easy to check that $\beta(0) = \gamma(0) = 0$. So for all $x > 0$, we have

$$\gamma(x) > \beta(x),$$

a contradiction. Thus, in a first-price auction, risk aversion causes an increase in equilibrium bids. Since bids have increased, the expected revenue has also increased. Using

Proposition 3.4.1 and the fact that the expected revenue in a second-price auction is unaffected by risk aversion, we deduce that the expected revenue in a first-price auction is higher than that in a second-price auction.

A key feature of the standard auction model with risk-neutral bidders is that the payoff functions are separable in money. In particular, they are *quasi-linear* – linear in the payments that bidders make – and bidders maximize their expected profits, which are just

Expected Value – Expected Payment

This separation between the expected value and the expected payment is crucial for revenue equivalence principle. Specifically, in the proof of Proposition 3.4.1, this separation leads to the conclusion that the expected payments are the same in any standard auction. Risk-averse bidders, on the other hand, maximize

Expected Utility of (Value – Payment)

and since utility is nonlinear - it is concave the maximum is no longer linear in the payments that bidders make. The fact that bidders' objective functions are no longer linear in their payments is the reason for the failure of the revenue equivalence principle.

4.2 Auction with Budget Constrained Bidders

Until now we have implicitly assumed that bidders are able to pay the seller up to amounts equal to their values. We continue with the basic symmetric independent private value setting in which there is a single object for sale and n potential buyers are bidding for the object. As before, bidder i assigns a value of X_i to the object. In addition, each bidder is subject to an absolute budget of W_i. We suppose that there is no circumstance in which a bidder with value-budget pair (x_i, w_i) pay more than w_i. So, if bidder i bids more than w_i and he becomes the winner, then he is no longer eligible to obtain the object (because he cannot pay the money he is supposed to pay) and a small penalty would be imposed.

Each bidder's value-budget pair (X_i, W_i) is independently and identically distributed on $[0, 1] \times [0, 1]$ according to the density function f.[1] Bidder i knows the realized value-budget pair (x_i, w_i) and only that other bidders' value-budget pairs are independently

[1]The independence holds only across bidders. The possibility that for each bidder the values and

distributed according to the same f. Bidders are assumed to be risk neutral. I will refer to the pair (x_i, w_i) as the type of bidder i. In any auction form A, (either a first $(A = I)$ or second price $(A = II)$ auction), a bidder's strategy is a function of the form $B^A : [0,1] \times [0,1] \rightarrow \mathbb{R}_+$ that determines the amount bid depending on both his value and his budget.

4.2.1 Second Price Auctions

Proposition 4.2.1 *In a second price auction, it is a dominant strategy to bid according to $B^{II}(x,w) = \min\{x,w\}$, that is, the person bids either his true value or its budget dependent on if his value is less than his budget.*

Proof 4.2.1 (Proof of Proposition 4.2.1:) *First, notice that it is dominated to bid above one's budget. Set $b_i > w_i$. What we want to show is that such b_i is dominated by w_i. Suppose that $b_i > \max_{j \neq i} b_j > w_i$. Then, he is the winner and has to pay more than w_i for the object. In this case, he does not obtain the object and pays some penalty so that he is worse off than he bids w_i. In all other cases, no such b_i cannot be better than w_i.*

Second, if $x_i \leq w_i$, then the budget constraint does not bind and the same argument goes through as if he faces no financial constraint. If $x_i > w_i$, the previous argument equally applies here so that any bid more than w_i is dominated by w_i.

The strategy we adopt here to analyze the behavior of financially constrained bidders is to transform such constrained bidders into financially "unconstrained" bidders who still make the same bid. This approach is valid. After all, what we can observe is not their types but their bids.

For every type (x,w), define $x'' = \min\{x,w\}$ and consider the type $(x'',1)$. Notice that a bidder of type $(x'',1)$ effectively never faces a financial constraint. Since $\min\{x'',1\} = x'' = \min\{x,w\}$, we have that $B^{II}(x,w) = B^{II}(x'',1)$. Thus, in a second-price auction the type $(x'',1)$ would submit a bid identical to that submitted by type (x,w) . The type $(x'',1)$ is, as it were, the richest member of the family with types (x,w) such that $\min\{x,w\} = x''$. Figure 4.1 depicts the set of types who bid the same in a second-

budgets are correlated is admitted. For example, the less value he assigns to the object, the more severely budget constrained he can be.

price auction as does type (x, w). This consists of all types on the thin-lined right angle "Leontief iso-bid curve whose corner lies on the diagonal.

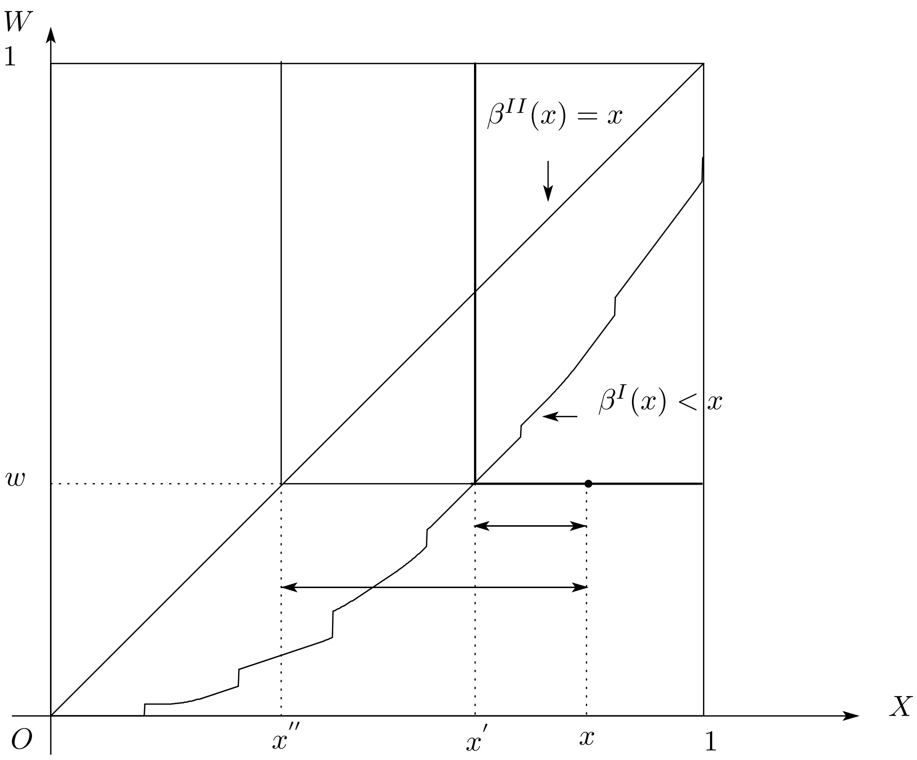

Figure 4.1: First- and Second-Price Auctions with Budget Constraint

Let $m^{II}(x, w)$ denote the expected payment of a bidder of type (x, w) in a second price auction. Since $B^{II}(x, w) = B^{II}(x'', 1)$, we have

$$m^{II}(x, w) = m^{II}(x'', 1).$$

Now define

$$\mathcal{L}^{II}(x'') = \{(X, W) | B^{II}(X, W) < B^{II}(x'', 1)\}$$

to be the set of types who bid less than type $(x'', 1)$ in a second price auction. Define

$$F^{II}(x'') = \int_{\mathcal{L}^{II}(x'')} f(X, W) dX dW$$

to be the probability that a type $(x'', 1)$ will out-bid *one* other bidder. Note that this is indeed the distribution function of the random variable $X'' = \min\{X, W\}$. The probability that a type $(x'', 1)$ will actually win the auction is just $(F^{II}(x''))^{n-1} \equiv G^{II}(x'')$. We can write the expected utility of a type $(x'', 1)$ when bidding $B^{II}(z, 1)$ as

$$G^{II}(z)x'' - m^{II}(z, 1).$$

In equilibrium, it is optimal to bid $B^{II}(x'', 1)$ when the true type is $(x'', 1)$ and so that we have

$$m^{II}(x'', 1) = \int_0^{x''} yg^{II}(y)dy$$

where g^{II} is the density function associated with G^{II} . The *ex ante* expected payment of a bidder in a second price auction with financial constraints can be written as

$$\begin{aligned} E[R^{II}] &= \int_0^1 m^{II}(x'', 1)f^{II}(x'')dx'' \\ &= E[Y_2^{II(n)}] \end{aligned}$$

where $Y_2^{II(n)}$ is the second highest of N draws from the distribution F^{II}.

4.2.2 First Price Auctions

Suppose that in a first price auction, the equilibrium strategy is of the form

$$B^I(x, w) = \min\{\beta(x), w\}$$

for some increasing function $\beta(x)$. Recall that $\beta(x) < x$, otherwise a bidder of type $x < w$ would deviate by bidding slightly less. Although sufficient conditions in terms of the primitives of the model that guarantee the existence of such an equilibrium can be provided, here we content ourselves with directly assuming that such an equilibrium exists.

As in the case of a second-price auction, for every type (x, w) define x' to be a value such that $\beta(x') = \min\{\beta(x), w\}$ and consider the type $(x', 1)$. As before, a bidder of type $(x', 1)$ effectively never faces a financial constraint. But since $\min\{\beta(x'), 1\} = \beta(x') = \min\{\beta(x), w\}$, we have that $B^I(x, w) = B^I(x', 1)$. Thus, in a first-price auction the type $(x', 1)$ would submit a bid identical to that submitted by type (x, w). Now define

$$\mathcal{L}^I(x') = \{(X, W)|B^I(X, W) < B^I(x'', 1)\}$$

and define m^I, F^I , and G^I in an analogous way. The ex ante expected payment of a bidder in a first price auction with financial constraints can be written as

$$E[R^I] = E[Y_2^{I(n)}]$$

where $Y_2^{I(n)}$ is the second highest of N draws from the distribution F^I.

4.2.3 Revenue Comparison

Proposition 4.2.2 *Suppose that bidders are subject to financial constraints. If the first price auction has a symmetric equilibrium of the form $B^I(x, w) = \min\{\beta(x), w\}$, then the expected revenue in a first price auction is greater than the expected revenue in a second price auction.*

Proof 4.2.2 (Proof of Proposition 4.2.2:) *To compare the expected payments in the two auctions, notice that since $\beta(x) < x$ for all x, the definition of $\mathcal{L}^{II}(x)$ and $\mathcal{L}^I(x)$ imply that $\mathcal{L}^I(x) \subset \mathcal{L}^{II}(x)$. See Figure 4.2. Then, by definition, $F^I(x) \leq F^{II}(x)$ and a strict inequality holds for all $x \in (0, 1)$. Then we conclude that F^I first-order stochastically dominates F^{II} . Let $G^I \equiv (F^I)^{n-1}$ and $G^{II} \equiv (F^{II})^{n-1}$. This implies that G^I first-order stochastically dominates G^{II}. Hence, for any $x \in [0, 1]$, we obtain*

$$m^I(x, 1) \equiv \int_0^x yg^I(y)dy \geq \int_0^x yg^{II}(y)dy \equiv m^{II}(x, 1).$$

In effect, $m^I(x, 1) \geq m^{II}(x, 1)$ for any $x \in [0, 1]$. This implies that

$$E[Y_2^{I(n)}] \equiv n \times \int_0^1 m^I(x, 1)f^I(x)dx > n \times \int_0^1 m^{II}(x, 1)f^{II}(x)dx \equiv [E[Y_2^{II(n)}].$$

This completes the proof.

4.3 Auctions with Asymmetric Bidders

4.3.1 Asymmetric First Price Auctions with Two Bidders

Suppose there are two bidders with values X_1 and X_2, which are independently distributed according to the functions F_1 on $[0, \omega_1]$ and F_2 on $[0, \omega_2]$, respectively. Assume further that there is an equilibrium of the first price auction in which the two bidders follow the

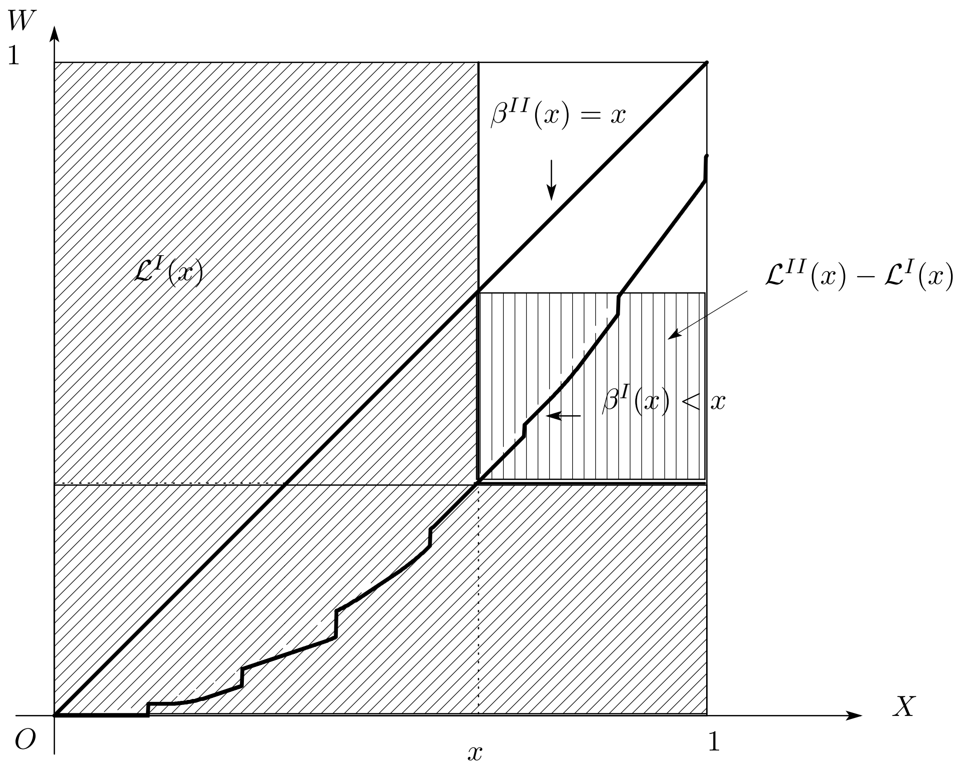

Figure 4.2: $\mathcal{L}^I(x) \subset \mathcal{L}^{II}(x)$

strategies β_1 and β_2, respectively. As usual, these are increasing and differentiable and have inverses $\phi_1 \equiv \beta_1^{-1}$ and $\phi_2 \equiv \beta_2^{-1}$.

It is clear that $\beta_1(0) = 0 = \beta_2(0)$, since it would be dominated for a bidder to bid more than the value. Moreover, $\beta_1(\omega_1) = \beta_2(\omega_2)$ since otherwise, if say, $\beta_1(\omega_1) > \beta_2(\omega_2)$, then bidder 1 would win with probability 1 when his value is ω_1 and would pay more than he needs to – he could increase his payoff by bidding slightly less than $\beta_1(\omega_1)$. Let $\overline{b} \equiv \beta_1(\omega_1) = \beta_2(\omega_2)$ be the common highest bid submitted by either bidder. Given that bidder $j = 1, 2$ is following the strategy β_j , the expected payoff of bidder $i \neq j$ when his value is x_i and he bids an amount $b < \overline{b}$ is

$$\begin{aligned}\Pi_i(b, x_i) &= F_j(\phi_j(b))(x_i - b) \\ &= H_j(b)(x_i - b)\end{aligned}$$

where $H_j(.) \equiv F_j(\phi_j(.))$ denotes the distribution of bidder j's bids. Since it is optimal to bid $b = \beta_i(x_i)$, equivalently, $x_i = \beta_i^{-1}(b) = \phi_i(b)$, the first order condition for bidder i

requires that

$$h_j(\phi_i(b) - b) = H_j(b), \forall b < \bar{b}$$

where $j \neq i$ and $h_j(b) \equiv H_j'(b) = f_j(\phi_j(b))\phi_j'(b)$ is the density of j's bids. This can be rearranged as

$$\phi_j'(b) = \frac{F_j(\phi_j(b))}{f_j(\phi_j(b))} \cdot \frac{1}{\phi_i(b) - b} \tag{4.1}$$

A solution to the system of differential equations in (4.1)-one for each bidder-together with the relevant boundary conditions constitutes an equilibrium of the first-price auction. Unfortunately, an explicit solution can be obtained only in some special cases, and so instead, we deduce some properties of the equilibrium strategies indirectly. To do this, we make some assumptions regarding the specific nature of the asymmetries.

4.3.2 Weakness Leads to Aggressive Behaviors

Suppose that bidder 1's values are "first order stochastically higher" than those of bidder 2. Namely, bidder 1 is more likely to be a "stronger" bidder than bidder 2. The assumption that F_1 dominates F_2 in terms of the reverse hazard rate implies that $\omega_1 \geq \omega_2$ and for all $x \in (0, \omega_2)$,

$$\sigma_1 = \frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)} = \sigma_2 \tag{4.2}$$

Proposition 4.3.1 *Suppose that the value distributions of bidder 1 dominates that of bidder 2 in terms of the reverse hazard rate. Then in a first price auction, the "weak" bidder 2 bids more aggressively than the "strong" bidder 1-that is,*

$$\beta_1(x) < \beta_2(x), \forall x \in (0, \omega_2).$$

Proof 4.3.1 (Proof of Proposition 4.3.1:) *First, notice that if there exists a c such that $0 < c < \bar{b}$ and $\phi_1(c) = \phi_2(c) \equiv z$, then (4.1) and (4.2) imply that*

$$\phi_2'(c) = \frac{F_2(z)}{f_2(z)} \frac{1}{(z - c)} > \frac{F_1(z)}{f_1(z)} \frac{1}{(z - c)} = \phi_1'(c).$$

Since $\phi_i(c) = 1/\beta_i'(z)$, this is equivalent to saying that if there exists a z such that $\beta_1(z) = \beta_2(z)$, then $\beta_1'(z) > \beta_2'(z)$. In other words, if the curves β_1 and β_2 ever intersect, the former is steeper than the latter and this implies that they intersect at most once.

We will argue by contradiction. So suppose that there exists an $x \in (0, \omega_2)$ such that $\beta_1(x) \geq \beta_2(x)$. Then either β_1 and β_2 do not intersect at all so that $\beta_1 > \beta_2$ everywhere; or they intersect only once at some value $z \in (0, \omega_2)$ and for all x such that $z < x < \omega_2$, $\beta_1(x) > \beta_2(x)$. In either case, for all x close to ω_2, $\beta_1(x) > \beta_2(x)$.

Next suppose $\omega_1 = \omega_2 \equiv \omega$. If we write $\beta_1(\omega) = \beta_2(\omega) = \overline{b}$, then in terms of the inverse bidding strategies we have that for all b close to $\overline{b}$, $\phi_1(b) < \phi_2(b)$. This implies that for all b close to $\overline{b}$,

$$H_1(b) = F_1(\phi_1(b)) \leq F_2(\phi_2(b)) = H_2(b)$$

and since $H_1(\overline{b}) = 1 = H_2(\overline{b})$, it must be that $h_1(b) > h_2(b)$. Now using the first order condition we derived before we obtain that for all b close to $\overline{b}$,

$$\phi_1(b) = \frac{H_2(b)}{h_2(b)} + b > \frac{H_1(b)}{h_1(b)} + b = \phi_2(b)$$

which is a contradiction.

Why is it that the weak bidder bids more aggressively than does the strong bidder? To gain some intuition, it is useful to see why the opposite is impossible-that is, it cannot be that the strong bidder bids more aggressively than does the weak bidder. If for all x, $\beta_1(x) > \beta_2(x)$, then certainly the distribution H_1 of competing bids facing the weak bidder is stochastically higher than the distribution H_2 of competing bids facing the strong bidder. It is easy to see that all else being equal, a bidder who faces a stochastically higher distribution of bids – in the sense of reverse hazard rate dominance-will bid higher. It is also true that for a particular bidder, all else being equal, a higher realized value will lead to a higher bid. Now consider a particular bid b and suppose that $\beta_1(x_1) = \beta_2(x_2) = b$. Since by assumption, the strong bidder bids more aggressively, it must be that the value at which the strong bidder bids b is lower than the value at which the weak bidder bids b-that is, $x_1 < x_2$. This means that, relative to the strong bidder, the weak bidder faces both a stochastically higher distribution of competing bids-H_1 versus H_2-and has a higher value – x_2 versus x_1. Since both forces cause bids to be higher, if it were optimal for the strong bidder to bid b when his value is x_1, it cannot be optimal for the weak bidder to bid b when his value is x_2. Thus, we have a contradiction.

Put another way, in equilibrium the two forces must balance each other. The weak bidder faces a stochastically higher distribution of competing bids than does the strong bidder, but the value at which any particular bid b is optimal for the weak bidder is lower than it is for the strong bidder.

4.3.3 Efficiency Comparison

An auction is said to be *efficient* if the object always ends up in the hands of the person who values it the most *ex post*. In a second price auction, it is a weakly dominant strategy for a bidder to bid his value - recall that this is true even when bidders are asymmetric - so the winning bidder is also the one with the highest value. Thus, the second price auction is always efficient under the assumption of private values. First, we remark the proposition below with no proof.

Proposition 4.3.2 *With independently and identically distributed private values, efficiency is achieved as a symmetric equilibrium in both first price and second price auctions.*

This must be straightforward if you understand the materials in environments with private independent values. If bidders are asymmetric, however, this is not true any more for first price auctions. Next, we claim the following. You need not follow the proof below unless you are interested in it.

Proposition 4.3.3 *With asymmetrically distributed private values, a second price auction always allocates the object efficiently, whereas with positive probability, a first price auction does not.*

Proof 4.3.2 (Proof of Proposition 4.3.3:) *First, we should note that it is a weakly dominant strategy for a bidder to bid his value in a second price auction even when bidders are asymmetric. To see this, we just try to remember that the proof for the truth telling as a dominant strategy does not rely on the symmetry assumption at all.*

In contrast, asymmetries inevitably lead to inefficient allocations in a first price auction. Suppose that there are two bidders and (β_1, β_2) is an equilibrium of the first price auction such that both strategies are continuous and increasing. Assume without loss of generality that $\beta_1(x) < \beta_2(x)$ for some x. Since both strategies are continuous and increasing, there exists $\epsilon > 0$ small enough so that

$$\beta_1(x + \epsilon) < \beta_2(x - \epsilon).$$

This means that with positive probability the allocation is inefficient since bidder 2 would win the auction even though he has a lower value than bidder 1.

4.4 Resale

In the previous section, we confirmed that asymmetries among bidders lead to inefficient allocations in first price auctions. Achieving an efficient allocation may well be an important policy goal of the seller, especially if the seller is a government undertaking the privatization of some public asset. This seems to imply that such a seller should use an efficient auction - with private values, say a second price auction. An argument against this point of view, in the Chicago school vein, is that even if the outcome of the auction is inefficient, post-auction transactions among buyers – resale - will result in an efficient final allocation.

To illustrate the role of resale in auctions, we discuss "Asymmetric Auctions with Resale," by Isa Hafalir and Vijay Krishna (2007). A model of the first price auction with resale is the following: There are only two bidders. The bidders first participate in a standard sealed-bid first price auction. The winning bid is publicly announced. We assume - as is common in real-world auctions - that the losing bid is not announced. In the second stage, the winner of the auction - say j - may, if he wishes to, offer to sell the object to the other bidder $i \neq j$ at some price p. If the offer is accepted by i, a sale ensues. If the offer is rejected, the original owner j retains the object. Thus, resale takes place via take-it-or-leave-it offer by the winner of the auction. In the following we present their main results without the proofs.

Theorem 4.4.1 *The first-price auction with resale has a unique increasing equilibrium.*

Proposition 4.4.1 *There is an ex post equilibrium of the second price auction with resale in which both bidders bid their values and the outcome is efficient.*

Although there might be other ex post equilibria, we restrict our attention from now on to the unique ex post equilibrium outcome of the second price auction with resale in which bidders bid their values and the outcome is efficient. The model is the same as the first price auction with resale except for the change in the auction format - that is, there is a second price auction and then the winner, if he so wishes, can resell the object to the other bidder via a *take-it-or-leave-it* offer. There is one important difference, however. Under second price rules, the winner of the auction inevitably knows the losing bid - after all this is the price he pays in the auction. This, of course, considerably simplifies the inference problem faced by a winning bidder and puts the losing bidder in a weak position during resale.

Theorem 4.4.2 *The seller's revenue from a first price auction with resale is at least as great as that from a second price auction with resale.*

We usually think that the possibility of resale makes our analysis very complicated. Paradoxically, the above result shows that the possibility of resale makes our analysis much simpler and the revenue comparison possible. However, there is no clear way of extending these results to the environments where there are more than two bidders.

4.5 The Discrete Valuation Case

In this section, we relax another assumption from the standard IPV model. We allow valuations to be drawn from a discrete distribution. We will use an example to illustrate that with discrete types we will have to look for a Bayesian Nash equilibrium in mixed strategies as a pure strategy equilibrium will not exist.

Example 4.5.1 *Suppose we have two bidders with valuations $x \in \{0, 1\}$ with equal probabilities. Then, there is no equilibrium of the first-price auction in pure strategies. To see this, suppose bidder 2 bids 0 if his valuation is 0 (a bidder with zero valuation will never bid higher than zero). And if his valuation is 1 his bid is $b \geq 0$. Similarly, bidder 1 bids zero if his valuation is zero. What is his bid if his valuation is 1? The expected utility of bidder 1 is*

$$\Pi_1 = \frac{1}{2}(1-0) + \frac{1}{2}(1-b_1), \textit{if } b_1 > b.$$

Thus if $b \geq 1$, bidder 1 bids $b_1 = 0$. If $b < 1$ and $b_1 > b$, $\Pi_1 = \frac{1}{2} + \frac{1}{2}(1 - b_1)$. Thus, if $b \geq 1$ the best reply of bidder 1 is to bid 0 and if $b < 1$ there is no best reply since the *expected utility increases as $b_1 > b$ decreases. That is, there is no pure strategy Bayesian Nash equilibrium.*

Let us find a mixed strategy equilibrium. First, note that if bidder 2 has a zero valuation, then he bids zero for sure. If bidder 2 has valuation 1, suppose that his bid belongs to $[0, x]$ with probability $G(x)$. For G to define an equilibrium strategy, we need that $Pr(\{x\}) = 0$ whenever $x > 0$-given that if $Pr(\{x\}) > 0$ whenever bidder 1 intends to bid x, he may slightly increase his bid increasing his probability of winning by $Pr(\{x\})$. What is the best reply for bidder 1? His expected utility is

$$\Pi_1 = \frac{1}{2}(1-b) + \frac{1}{2}(1-b)G(b).$$

Thus, bidder 1 will bid b such that b maximizes $(1-b)(1+G(b))$. Thus if F is the distribution of bidder's 1 bids we have that

$$F(\{b; b \textit{ does not maximize}(1-b)(1+G(b))\}) = 0$$
$$G(\{b; b \textit{ does not maximize}(1-b)(1+G(b))\}) = 0$$

Suppose $k = \max_{b\geq 0}(1-b)(1+G(b))$. Then if x, x' are such that $F(x)F(x') > 0$,

$$(1-x)(1+G(x)) = k = (1-x')(1+G(x')).$$

The infimum of such x must satisfy $(1-x) = k$ or $x = 1-k$. The supremum of $x' : (1-x')2 = k$. Thus the support of the distribution is $[1-k, 1-k/2]$. If $k = 1$,

$$F(x) = G(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$
$$x \in [0, \frac{1}{2}]$$

These probability distributions represent the mixed strategy equilibrium for this example. They are depicted in the diagram above. The interpretation is that a bidder who receives a type 1 will submit a bid in $[0, \alpha]$ with probability $\alpha/(1-\alpha)$ if $\alpha \leq \frac{1}{2}$ and will never submit a bid higher than $\frac{1}{2}$.

4.6 Sealed-Bid Auctions with Complete Information

In this section, we study a special case where the values of bidders are common information. An object is to be assigned to a player in the set $\{1, \cdots, n\}$ in exchange for a payment. Player i's valuation of the object is v_i, and $v_1 > v_2 > \cdots > v_n > 0$. The mechanism used to assign the object is a sealed-bid auction: the players simultaneously submit bids (nonnegative numbers), and the object is given to the player with the lowest index among those who submit the highest bid, in exchange for a payment.

4.6.1 The First Price Sealed-Bid Auction

In a *first-price* auction, the payment that the winner makes is the price that he bids. We formulate the first price auction as a game with complete information.

- $N = \{1, \cdots, n\}$: The set of players (bidders).
- $B_i = [0, \infty)$ for each $i \in N$: The set of possible bids by player i. A generic bid by player i is denoted $b_i \in B_i$.
- $u_i(b) = u_i(b_1, \cdots, b_n) = v_i - b_i$ if player i is the lowest index among those who submit $b_i = \max_{j \in N} b_j$; and $u_i(b) = 0$ otherwise: player i's payoff function.

Claim 4.6.1 *Player 1 obtains the object in all Nash equilibria.*

Proof 4.6.1 (Proof of Claim 4.6.1:) *Fix a Nash equilibrium $b^* \in B_1 \times \cdots \times B_n$. Suppose, on the contrary, that player 1 does not obtain the object. Assume that player $j \neq 1$ is the winner. Because of rationality of player j, we have $b_j^* \in [0, v_j]$. Set $\widetilde{b}_1 = b_j^* + \epsilon$ for $\epsilon > 0$ small enough so that $v_1 > \widetilde{b}_1 = b_j^* + \epsilon$. Then, we have*

$$u(\widetilde{b}_1, b_{-1}^*) = v_1 - \widetilde{b}_1 > u_1(b^*) = 0.$$

This contradicts the hypothesis that b^* is a Nash equilibrium.

4.6.2 The Second Price Sealed-Bid Auction

In a *second price* auction, the payment that the winner makes is the highest bid among those submitted by the players who do not win (so that if only one player submits the highest bid then the price paid is the *second* highest bid). We formulate the second price auction.

- $N = \{1, \cdots, n\}$: The set of players (bidders).

- $B_i = [0, \infty)$ for each $i \in N$: The set of possible bids by player i. A generic bid by player i is denoted $b_i \in B_i$.

- $u_i(b) = u_i(b_1, \cdots, b_n) = v_i - \max_{j \neq i} b_j$ if $b_i > \max_{j \neq i} b_j$ and $u_i(b) = 0$ otherwise: player i's payoff function.

Claim 4.6.2 *In a second price auction, the bid v_i of any player i is a weakly dominant action.*

Proof 4.6.2 (Proof of Claim 4.6.2:) *Let $b_i^* = v_i$. Let $\widetilde{b}_i$ be any other bid than b_i^*. Consider two cases: (Case 1) $\widetilde{b}_i > v_i$ and (Case 2) $\widetilde{b}_i < v_i$.*

Case 1. *$u_i(b_i^*, b_{-i}) = 0 > u_i(\widetilde{b}_i, b_{-i})$ for $b_{-i} \in B_{-i}$ with the property that $\widetilde{b}_i > \max_{j=i} b_j > v_i$. For any other b_{-i}, we have $u_i(b_i^* i, b_{-i}) \geq u_i(\widetilde{b}_i, b_{-i})$.*

Case 2. *$u_i(b_i^*, b_{-i}) = v_i - \max_{j \neq i} b_j > 0 = u_i(\widetilde{b}_i, b_{-i})$ for any $b_{-i} \in B_{-i}$ with the property that $v_i > \max_{j \neq i} b_j > \widetilde{b}_i$. For any other b_{-i}, we have $u_i(b_i^*, b_{-i}) \geq u_i(\widetilde{b}_i, b_{-i})$.*

With the consideration of cases 1 and 2, we can conclude that $b_i^* = v_i$ is a weakly dominant action.

Claim 4.6.3 *In a second price auction, there is a(n) ("inefficient) Nash equilibrium in which the winner is not player 1.*

Proof 4.6.3 (Proof of Claim 4.6.3:) *We construct the following action profile b^*:*

- $b_j^* > v_1$;

- $b_1^* < v_j$;

- $b_i^* = 0$ *for any* $i \notin \{1, j\}$.

It remains to show that b^* is indeed a Nash equilibrium. It is relatively easy to check that no player has any profitable deviation.

Chapter 5

Mechanism Design

The mechanism design approach distinguishes sharply between the apparatus under the control of the designer, which we call a *mechanism*, and the world of things that are beyond the designer's control, which we call the *environment*. A mechanism consists of rules that govern what the participants are permitted to do and how these permitted actions determine *outcomes*. An environment comprises three lists: a list of the participants or potential participants, another of the possible outcomes, and another of the participants' possible *types* - that is, their capabilities, preferences, information, and beliefs.

The theory of mechanism design evaluates alternative designs based on their comparative *performance*. Formally, performance is the function that maps environments into outcomes. The goal of mechanism design is to determine what performance is possible and how mechanisms can best be designed to achieve the designer's goals. Mechanism design addresses three common questions:

1. Is it possible to achieve a socially optimal outcome, for instance, it picks an efficient (yet to be defined) allocation for every possible environment in some class?;

2. What is the complete set of social optimal goals that are implementable by some mechanism?; and

3. What mechanism can implement a social choice goal (optimizes performance according to the mechanism designer's performance criterion)?

A seller has one indivisible object, to sell and there are N risk-neutral potential buyers from the set $\mathcal{N} = 1, 2, \cdots, N$. Buyers have private values and these are independently distributed. Buyer i's value X_i is distributed over the interval $\mathcal{X}_i = [0, \omega_i]$ according to the distribution function F_i with associated density function f_i. We allow for asymmetries among the buyers: the distributions of values need not be the same for all buyers.

For the sake of simplicity, we suppose that the value of the object to the seller is 0. Let $\mathcal{X} = \times_{j=1}^{N} \mathcal{X}_j$ denote the product of the sets of buyers' values, and for all i, let $\mathcal{X}_{-i} = \times_{j \neq i} \mathcal{X}_j$. Define $f(\mathbf{x})$ to be the joint density of $\mathbf{x} = (x_1, x_2, \cdots, x_N)$. Since the values are independently distributed, $f(\mathbf{x}) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_N(x_N)$. Similarly, define $f_{-i}(\mathbf{x}_{-i})$ to be the joint density of $\mathbf{x}_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N)$.

5.1 Mechanisms

A selling *mechanism* $(\mathcal{B}, \Pi, \mu)$ has the following components: a set of possible **messages** B_i for each buyer; an *allocation rule* $\Pi : \mathcal{B} \to [0,1]^n$, where for any $\mathbf{b} \in \mathcal{B}$, $(\Pi_1(\mathbf{b}), \cdots, \Pi_n(\mathbf{b})) \in [0,1]^n$ satisfies the property that $\sum_{i \in N} \Pi_i(\mathbf{b}) \leq 1$ and $\Pi_i(\mathbf{b})$ denotes the probability that buyer i obtains the object when the message profile is $\mathbf{b}$.[1]; and a *payment rule* $\mu : \mathcal{B} \to \mathbb{R}^n$. An allocation rule determines, as a function of all n messages, the probability $\Pi_i(\mathbf{b})$ that i will get the object. A payment rule determines, as a function of all n messages, for each buyer i, the expected payment $\mu_i(\mathbf{b})$ that i must make.

Every mechanism defines a game of incomplete information among the buyers. An n-tuple of strategies $\beta_i : \mathcal{X}_i \to \mathcal{B}_i$ is an *equilibrium* of a mechanism if, for all $i \in N$ and for all $x_i \in \mathcal{X}_i$, given the strategies β_{-i} of other buyers, $\beta_i(x_i)$ maximizes i's expected payoff.

Notice that both first- and second-price auctions are mechanisms. The set of possible bids $\mathcal{B}_i$ in both can be safely assumed to be $\mathcal{X}_i$. Assuming that there is no reservation price, the allocation rule for both is $\Pi_i(\mathbf{b}) = 1$ if $b_i > \max_{j \neq i} b_j$ and $\Pi_j(\mathbf{b}) = 0$ for $j \neq i$. They differ only in the associated payment rules. For a first-price auction, $\mu_i^I(\mathbf{b}) = b_i$ if $b_i > \max_{j \neq i} b_j$ and $\mu_j^I(\mathbf{b}) = 0$ for $j \neq i$. For a second-price auction, $\mu_i^{II}(\mathbf{b}) = \max_{j \neq i} b_j$ if $b_i > \max_{j \neq i} b_j$ and $\mu_j^{II}(\mathbf{b}) = 0$ for $j \neq i$. If there are ties, each winning bidder has an equal likelihood of being awarded the object, so the Π_j have to take account of this.

[1]This formulation allows the seller to keep the object with positive probability

5.1.1 The Revelation Principle

A mechanism could, in principle, be quite complicated since we have made no assumptions on the sets $\mathcal{B}_i$ of "bids" or "messages." A smaller and simpler class consists of those mechanisms for which the set of messages is the same as the set of values - that is, for all i, $\mathcal{B}_i = \mathcal{X}_i$. Such mechanisms are called direct since, in effect, every buyer is asked to directly report a value. Formally, a *direct mechanism* $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ consists of the space of valuations $\mathcal{X}$ and a pair of functions $\mathbf{Q} : \mathcal{X} \to \Delta$ and $\mathbf{M} : \mathcal{X} \to \mathbb{R}^N$ where $Q_i(\mathbf{x})$ is the probability that i will get the object and $M_i(\mathbf{x})$ is the expected payment by i. If it is an equilibrium for each buyer to reveal his or her true value, then the direct mechanism is said to have a truthful equilibrium. We will refer to the pair $(\mathbf{Q}(\mathbf{x}), \mathbf{M}(\mathbf{x}))$ as the *outcome* of the mechanism at $\mathbf{x}$.

The following result, referred to as the *revelation principle*, shows that the outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism. In this sense, there is no loss of generality in restricting attention to direct mechanisms.

Proposition 5.1.1 (Revelation Principle) *Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (1) it is an equilibrium for each buyer to report his value truthfully and (2) the truthful equilibrium outcomes of the direct mechanism are the same as in the given equilibrium of the original mechanism.*

Proof 5.1.1 (Proof of Proposition 5.1.1:) *Fix an arbitrary mechanism* $(\mathcal{B}, \Pi, \mu)$ *and an equilibrium* β *of that mechanism. Let* $\boldsymbol{Q} : \mathcal{X} \to \Delta$ *and* $\boldsymbol{M} : \mathcal{X} \to \mathbb{R}^N$ *be defined as follows:* $\boldsymbol{Q}(\boldsymbol{x}) = \Pi(\beta(\boldsymbol{x}))$ *and* $\boldsymbol{M}(\boldsymbol{x}) = \mu(\beta(\boldsymbol{x}))$. *Conclusions (1) and (2) can now be verified routinely.*

The idea underlying the revelation principle is very simple. Fix a mechanism and an equilibrium β of the mechanism. Now instead of having the buyers submit messages $b_i = \beta_i(x_i)$ and then applying the rules of the mechanism in order to determine the outcome - who gets the object and who pays what - we could directly ask the buyers to "report" their values x_i and then make sure that the outcome is the same as if they had submitted bids $\beta_i(x_i)$. Put another way, a direct mechanism does the "equilibrium calculations" for the buyers automatically. Now suppose that some buyer finds it profitable to be untruthful and report a value of z_i when his true value is x_i. Then in the original mechanism the same buyer would have found it profitable to submit a bid of $\beta_i(z_i)$ instead of $\beta_i(x_i)$. But since the β_i constitute an equilibrium, this is impossible.

5.1.2 Incentive Compatibility

Given a direct mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$, define

$$q_i(z_i) = \int_{\mathcal{X}_{-i}} Q_i(z_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) d\mathbf{x}_{-i} \tag{5.1}$$

to be the probability that i will get the object when he reports his value to be z_i and all other buyers report their values truthfully. Similarly, define

$$m_i(z_i) = \int_{\mathcal{X}_{-i}} M_i(z_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) d\mathbf{x}_{-i} \tag{5.2}$$

to be the expected payment of i when his report is z_i and all other buyers tell the truth. It is important to note that because the values are independently distributed, both the probability of getting the object and the expected payment depend only on the *reported* value z_i and not on the true value, say x_i.The expected payoff of buyer i when his true value is x_i and he reports z_i, assuming that all other buyers tell the truth, can then be written as

$$q_i(z_i)x_i - m_i(z_i) \tag{5.3}$$

The direct mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ is said to be *incentive compatible* (**IC**) if for all i, for all x_i and for all z_i,

$$U_i(x_i) \equiv q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i). \tag{5.4}$$

We will refer to U_i as the equilibrium payoff function.

5.1.3 Characterizations of Incentive Compatible Direct Mechanisms

Thanks to the revelation principle, we can focus only on incentive compatible direct mechanisms, without loss of generality. Thus, in this subsection, we investigate the implications for studying incentive compatible direct mechanisms.

1. (**Convexity of Value Functions**): For each reported value z_i, the expected payoff $q_i(z_i)x_i - m_i(z_i)$ is an affine function of the true value x_i. Then, incentive compatibility implies that

$$U_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\},$$

that is, U_i is a maximum of a family of affine functions, therefore U_i is a convex function. This is due to the envelope theorems.

2. (**IC**$\Rightarrow$ q_i **is nondecreasing in** x_i): For all x_i and z_i, we can write

$$\begin{aligned} q_i(x_i)z_i - m_i(x_i) &= q_i(x_i)x_i - m_i(x_i) + q_i(x_i)(z_i - x_i) \\ &= U_i(x_i) + q_i(x_i)(z_i - x_i). \end{aligned}$$

Thus, incentive compatibility is equivalent to the requirement that for all x_i and z_i,

$$U_i(z_i) \geq U_i(x_i) + q_i(x_i)(z_i - x_i). \tag{5.5}$$

This implies that for all x_i, $q_i(x_i)$ is the slope of a line that supports the function U_i at the point x_i. A convex function is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Thus, at every point that U_i is differentiable,

$$U_i'(x_i) = q_i(x_i). \tag{5.6}$$

Since U_i is convex, this implies that q_i is a nondecreasing function.[2]If U_i is twice differentiable and q_i is differentiable, it follows that $U_i''(x_i) = q_i'(x_i) \geq 0$.

3. (q_i **is nondecreasing in** x_i $\Rightarrow$ **IC**)

$$\begin{aligned} \textbf{IC} &\Leftrightarrow U_i(z_i) \geq U_i(x_i) + q_i(x_i)(z_i - x_i) \\ &\Leftrightarrow U_i(z_i) - U_i(x_i) \geq q_i(x_i)(z_i - x_i) \\ &\Leftrightarrow [U_i(0) + \int_0^{z_i} q_i(t_i)dt_i] - [U_i(0) + \int_0^{z_i} q_i(x_i)dt_i] \geq q_i(x_i)(z_i - x_i) \\ &\Leftrightarrow \int_{x_i}^{z_i} q_i(t_i)dt_i \geq q_i(x_i)(z_i - x_i), \end{aligned}$$

[2] U_i is convex if and only if, for all z_i, $x_i \in \mathcal{X}_i$,

$$U_i(z_i) - U_i(x_i) \geq U_i'(x_i)(z_i - x_i).$$

which certainly holds if q_i is nondecreasing.

Thus, by (2) and (3), **IC** and q_i **is nondecreasing in** x_i are equivalent.

4. (**Payoff Equivalence**):Since every absolutely continuous function is the definite integral of its derivative, we have

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)dt_i, \tag{5.7}$$

which implies that up to an additive constant, the expected payoff to a buyer in an incentive compatible direct mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ depends only on the allocation rule $\mathbf{Q}$. If $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ and $(\mathcal{X}, \mathbf{Q}, \overline{\mathbf{M}})$ are two incentive compatible mechanisms with the same allocation rule $\mathbf{Q}$ but different payment rules, then the expected payoff functions associated with the two mechanisms, U_i and $\overline{U}_i$, respectively, differ by at most a constant - the two mechanisms are payoff equivalent. Put another way, the "shape" of the expected payoff function is completely determined by the allocation rule $\mathbf{Q}$ alone. The payment rule $\mathbf{M}$ only serves to determine the constants $U_i(0)$.

5.1.4 Revenue Equivalence

Proposition 5.1.2 (Revenue Equivalence) *If the direct mechanism $(\mathcal{X}, \boldsymbol{Q}, \boldsymbol{M})$ is incentive compatible, then for all i and x_i, the expected payment is*

$$m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i \tag{5.8}$$

Thus, the expected payments in any two incentive compatible mechanisms with the same allocation rule are equivalent up to constant.

Proof 5.1.2 (Proof of Proposition 5.1.2:) *Since $U_i(x_i) = q_i(x_i)x_i - m_i(x_i)$ and $U_i(0) = -m_i(0)$, the equality in (5.7) can be rewritten as (5.8).*

5.1.5 Individual Rationality

The direct mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ is said to be *individually rational* (**IR**) if for all i and x_i, the equilibrium expected payoff $U_i(x_i) \geq 0$. We are implicitly assuming here that by not participating, a buyer can guarantee himself a payoff of zero.

If the mechanism is incentive compatible, $U_i(.)$ is nondecreasing in x_i. This means that $U_i(x_i) \geq U_i(0)$ for any $x_i \in [0, \omega_i]$. Then individual rationality is equivalent to the requirement that $U_i(0) \geq 0$, and since $U_i(0) = -m_i(0)$, this is equivalent to the requirement that $m_i(0) \leq 0$.

5.2 Optimal Mechanisms

In this section we view the seller as the designer of the mechanism and examine mechanisms that maximize the expected revenue - the sum of the expected payments of the buyers - among all mechanisms that are incentive compatible and individually rational. We reiterate that when carrying out this exercise, the revelation principle guarantees that there is no loss of generality in restricting attention to direct mechanisms.

We will refer to a mechanism that maximizes expected revenue, subject to the incentive compatibility and individual rationality constraints, as an *optimal mechanism*.

5.2.1 Setup

Suppose that the seller uses the direct mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$. The expected revenue of the seller is

$$E[R] = \sum_{i \in \mathcal{N}} E[m_i(X_i)],$$

where the *ex ante* expected payment of buyer i is

$$\begin{aligned}
E[m_i(X_i)] &= \int_0^{\omega_i} m_i(x_i) f_i(x_i) dx_i \\
&= m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i) dx_i - \int_0^{\omega_i} \int_0^{x_i} q_i(t_i) f_i(x_i) dt_i dx_i \\
&= m_i(0) + \int_0^{\omega_i} q_i(x_i) x_i f_i(x_i) dx_i - \int_0^{\omega_i} (1 - F_i(t_i)) q_i(t_i) dt_i \\
&= m_i(0) + \int_0^{\omega_i} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) q_i(x_i) f_i(x_i) dx_i \\
&= m_i(0) + \int_{\mathcal{X}} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) Q_i(X) f(X) dX \\
(\because q_i(x_i) &= \int_{\mathcal{X}_{-i}} Q_i(x_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) d\mathbf{x}_{-i})
\end{aligned}$$

The seller's objective therefore is to find a mechanism that maximizes

$$\sum_{i\in\mathcal{N}} m_i(0) + \sum_{i\in\mathcal{N}} \int_{\mathcal{X}} (x_i - \frac{1-F_i(x_i)}{f_i(x_i)}) Q_i(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$$

subject to the constraint that the mechanism is incentive compatible (**IC**) - q_i is nondecreasing and individually rational (**IR**) - $m_i(0) \leq 0$.

5.2.2 Solution

Define

$$\psi_i(x_i) \equiv x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$$

to be the *virtual valuation* of a buyer with value x_i. Appealing to integration by parts, we obtain

$$\begin{aligned} E[\psi_i(x_i)] &= \int_0^{\omega_i} x_i f_i(x_i) dx_i - \int_0^{\omega_i} (1 - F_i(x_i)) dx_i \\ &= \int_0^{\omega_i} x_i f_i(x_i) dx_i - \{[x_i(1-F_i(x_i))]_0^{\omega_i} + \int_0^{\omega_i} x_i f_i(x_i) dx_i\} \\ &= 0 \end{aligned}$$

The design problem is said to be *regular* if for all i, the virtual valuation $\psi_i(x_i)$ is an increasing function of the true value x_i. Since

$$\psi_i(x_i) = x_i - \frac{1}{\lambda_i(x_i)}$$

where $\lambda_i(x_i) \equiv f_i(x_i)/(1 - F_i(x_i))$ is the hazard rate function associated with F_i, a sufficient condition for regularity is that for all i, $\lambda_i(.)$ is increasing. In what follows, we assume that the design problem is regular.

The seller should choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i\in\mathcal{N}} m_i(0) + \int_{\mathcal{X}} (\sum_{i\in\mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x})) f(\mathbf{x}) d\mathbf{x} \tag{5.9}$$

Temporarily neglect the **IC** and **IR** constraints, and consider the expression

$$\sum_{i\in\mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \tag{5.10}$$

from the second term in (5.9). The function $\mathbf{Q}$ is then like a weighting function, and clearly it is best to give weight only to those $\psi_i(x_i)$ that are maximal, provided they are positive. This would maximize the function in (5.10) at *every* point X and so also maximize its integral.

With this in mind, consider a mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ where

- the allocation rule $\mathbf{Q}$ is that the object goes to buyer i with positive probability if and only if $\psi_i(x_i) = \max_{j\in\mathcal{N}} \psi_j(x_j)$; thus,

$$Q_i(\mathbf{x}) > 0 \Leftrightarrow \psi_i(x_i) = \max_{j\in\mathcal{N}} \psi_j(x_j) \geq 0 \tag{5.11}$$

- the payment rule $\mathbf{M}$ is

$$M_i(\mathbf{x}) = Q_i(\mathbf{x})x_i - \int_0^{x_i} Q_i(z_i, \mathbf{x}_{-i})dz_i \tag{5.12}$$

We claim that (5.10) and (5.11) define an optimal mechanism. First, notice that the resulting q_i is a nondecreasing function, which means that the incentive compatibility condition is satisfied. To see that the individual rationality condition is also satisfied, suppose $z_i < x_i$. Then by the regularity condition, $\psi_i(z_i) < \psi_i(x_i)$ and thus for all $\mathbf{x}_{-i}$, it is also the case that $Q_i(z_i, \mathbf{x}_{-i}) \leq Q_i(x_i, \mathbf{x}_{-i})$. By construction of the payment rule $\mathbf{M}$, it is the case that $M_i(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i}$ and hence $m_i(0) = 0$. Therefore, the proposed mechanism is both incentive compatible and individually rational. It is also optimal since it separately maximizes the two terms in (5.9) over all $\mathbf{Q}(\mathbf{x}) \in \Delta$. In other words, the maximized value of the expected revenue is

$$E[\max\{\psi_1(X_1), \psi_2(X_2), \cdots, \psi_N(X_N), 0\}]$$

A more intuitive formula may be obtained by writing

$$y_i(\mathbf{x}_{-i}) = \inf\{z_i : \psi_i(z_i) \geq 0 \text{ and } \forall j \neq i, \psi_i(z_i) \geq \psi_j(x_j)\}$$

defined as the smallest value for i that "wins" against $\mathbf{x}_{-i}$. Then, we can rewrite the proposed mechanism $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ as

$$Q_i(x_i, \mathbf{x}_{-i}) = \begin{cases} 1 & \text{if } z_i > y_i(\mathbf{x}_{-i}) \\ 0 & \text{if } z_i < y_i(\mathbf{x}_{-i}) \end{cases}$$

which results in

$$\int_0^{x_i} Q_i(z_i, \mathbf{x}_{-i}) dz_i = \begin{cases} x_i - y_i(\mathbf{x}_{-i}) & \text{if } z_i > y_i(\mathbf{x}_{-i}) \\ 0 & \text{if } z_i < y_i(\mathbf{x}_{-i}) \end{cases}$$

and thus by (5.12), we have

$$M_i(\mathbf{x}) = \begin{cases} y_i(\mathbf{x}_{-i}) & \text{if } Q_i(\mathbf{x}) = 1 \\ 0 & \text{if } Q_i(\mathbf{x}) = 0 \end{cases}$$

Thus, only the "winning" buyer pays anything; he pays the smallest value that would result in his winning. Thus, we obtain the main result of this section:

Proposition 5.2.1 (Revenue Equivalence) *Suppose the design problem is regular. Then, the following is an optimal mechanism:*

$$Q_i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \psi_i(x_i) > \max_{j \neq i} \psi_j(x_j) \text{ and } \psi_i(x_i) \geq 0 \\ 0 & \text{if } \psi_i(x_i) < \max_{j \neq i} \psi_j(x_j) \end{cases}$$

and

$$M_i(\boldsymbol{x}) = \begin{cases} y_i(\boldsymbol{x}_{-i}) & \text{if } Q_i(\boldsymbol{x}) = 1 \\ 0 & \text{if } Q_i(\boldsymbol{x}) = 0 \end{cases}$$

Suppose that $f_i = f$ and $\psi_i = \psi$ for all i. Then, we have

$$y_i(\mathbf{x}_{-i}) = \max\{\psi^{-1}(0), \max_{j \neq i} x_j\}.$$

Thus, the optimal mechanism is a second price auction with a reserve price $r^* = \psi^{-1}(0)$.

5.2.3 Discussion and Interpretation of the Optimal Mechanisms

Why is it optimal to allocate the object on the basis of virtual valuations? The argument below follows Bullow and Roberts (1989). Suppose the seller makes a take-it-or-leave-it offer to a buyer at a price of p. The probability that the buyer will accept the offer is just $1 - F(p)$, the probability that his value exceeds p. We can think of the probability of purchase as the "quantity" demanded by i and thus write the buyer's implied "demand curve" as $q(p) \equiv 1 - F(p)$. The inverse demand curve is then $p(q) \equiv F^{-1}(1-q)$. The resulting "revenue function" facing the seller is

$$p(q) \times q = qF^{-1}(1-q)$$

and differentiating the revenue with respect to q

$$\frac{d}{dq}(p(q) \times q) = F^{-1}(1-q) - \frac{q}{F'(F^{-1}(1-q))}$$

Since $F^{-1}(1-q) = p$ we have that

$$MR(p) \equiv p - \frac{1-F(p)}{f(p)} = \psi(p)$$

the virtual valuation of i at $p(q) = p$. Thus, the virtual valuation of a buyer $\psi(p)$ can be interpreted as a *marginal revenue*, and recall that we have assumed that ψ is strictly increasing. Facing this buyer in isolation, the seller would set a "monopoly price" of r^* by setting $MR(p) = MC$, the marginal cost. Since the latter is assumed to be zero, $MR(r^*) = \psi(r^*) = 0$, or $r^* = \psi^{-1}(0)$.

When facing many buyers, the optimal mechanism calls for the seller to set discriminatory reserve prices of $r_i^* = \psi_i^{-1}(0)$ for the buyers. If no buyer's value x_i exceeds his reserve price r^* , the seller keeps the object. Otherwise, it is allocated to the buyer with the *highest marginal revenue* and this "winning" buyer is asked to pay $p_i = y_i(X_{-i})$, the smallest value such that he would still win.

5.3 Efficient Mechanisms

The optimal mechanism derived in the previous section is typically inefficient, and there are two separate sources of inefficiency. First, the optimal mechanism calls on the seller to retain the object if the highest virtual valuation is negative. Since buyers' values are always nonnegative and the value to the seller is 0, this means that with positive probability, the object is not allocated to one of the buyers even though there would be social gains from doing so. Second, even when the object is allocated, it is allocated to the buyer with the highest *virtual valuation* $\psi_i(x_i) \equiv x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$ and, in the asymmetric case, this need not be the buyer with the highest value.

In the context of a sale of an object to many potential buyers, we have argued that a second price auction (without a reserve price) will always allocate the object efficiently. This section concerns a generalization of the second price auction that is applicable to other contexts.

We generalize our setup very slightly to allow the values of agents to lie in some interval $\mathcal{X}_i = [\alpha_i, \omega_i] \subset \mathbb{R}$, thereby allowing, when $\alpha_i < 0$, for the possibility of negative values.

An allocation rule $\mathbf{Q}^* : \mathcal{X} \to \Delta$ is said to be *efficient* if it maximizes "social welfare". that is, for all $\mathbf{x} \in \mathcal{X}$,

$$\mathbf{Q}^*(\mathbf{x}) \in \arg\max_{\mathbf{Q} \in \Delta} \sum_{j \in \mathcal{N}} Q_j x_j.$$

When there are no ties, an efficient rule allocates the object to the person who values it the most.[3] Any mechanism with an efficient allocation rule is said to be efficient. Given an efficient allocation rule $\mathbf{Q}^*$, define the maximized value of social welfare by

$$W(\mathbf{x}) \equiv \sum_{j \in \mathcal{N}} Q_j^*(\mathbf{x}) x_j$$

when the values are $\mathbf{x}$. Similarly, define

$$W_{-i}(\mathbf{x}) \equiv \sum_{j \neq i} Q_j^*(\mathbf{x}) x_j$$

as the welfare of agents other than i.

5.3.1 The VCG Mechanism

The Vickrey-Clarke-Groves, or VCG *mechanism* $(\mathcal{X}, \mathbf{Q}^*, \mathbf{M}^V)$, is an efficient mechanism with the payment rule $\mathbf{M}^V : \mathcal{X} \to \mathbb{R}^N$ given by

$$M_i^V(\mathbf{x}) = W(\alpha_i, \mathbf{x}_{-i}) - W_{-i}(\mathbf{x}).$$

$M_i^V(\mathbf{x})$ is thus the difference between social welfare at i's lowest possible value α_i and the welfare of other agents at i's reported value x_i; assuming in both cases that the efficient allocation rule $\mathbf{Q}^*$ is employed.

In the context of auctions, $\alpha_i = 0$ and it is straightforward to see that the VCG mechanism is the same as a second price auction. In the auction context, $M_i^V(\mathbf{x}) =$

[3] 3There may be more than one efficient rule depending on how ties are resolved.

$W_{-i}(0, \mathbf{x}_{-i}) - W_{-i}(\mathbf{x})$, and this is positive if and only if $x_i \geq \max_{j \neq i} x_j$. In that case, $M_i^V(\mathbf{x})$ is equal to $\max_{j \neq i} x_j$, the second highest value.

The VCG mechanism is incentive compatible. Indeed, truth-telling is a weakly dominant strategy in the VCG mechanism. We sometimes say that the VCG mechanism is dominant strategy incentive compatible (DSIC). If the other buyers report values $\mathbf{x}_{-i}$, then by reporting a value of z_i, agent i's payoff is

$$\mathbf{Q}^*(z_i, \mathbf{x}_{-i})x_i - M_i^V(z_i, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{N}} Q_j^*(z_i, \mathbf{x}_{-i})x_j - W(\alpha_i, \mathbf{x}_{-i})$$

The definition of $\mathbf{Q}^*$ implies that for all $\mathbf{x}_{-i}$, the first term is maximized by choosing $z_i = x_i$; and since the second term does not depend on z_i, it is optimal to report $z_i = x_i$. Thus, i's equilibrium payoff when the values are $\mathbf{x}$ is

$$\mathbf{Q}_i^*(\mathbf{x})x_i - M_i^V(\mathbf{x}) = W(\mathbf{x}) - W(\alpha_i, \mathbf{x}_{-i})$$

which is just the difference in social welfare induced by i when he reports his true value x_i as opposed to his lowest possible value α_i.

Since the VCG mechanism is incentive compatible, the equilibrium expected payoff function U_i^V associated with the VCG mechanism,

$$U_i^V(x_i) = E[W(x_i, \mathbf{X}_{-i}) - W(\alpha_i, \mathbf{X}_{-i})]$$

is convex and increasing. Clearly, $U_i^V(\alpha_i) = 0$ and the monotonicity of U_i^V now implies that the VCG mechanism is also individually rational. The next proposition shows that the VCG mechanism is the optimal mechanism among those satisfying IC, IR and Eff. This implies that if one requires a mechanism to be individually rational and efficient, one will not obtain extra flexibility for the class of mechanisms by weakening DSIC into the usual IC.

Proposition 5.3.1 (Revenue Equivalence) *Among all mechanisms for allocating a single object that are efficient, incentive compatible, and individually rational, the VCG mechanism maximizes the expected payment of each agent.*

Proof 5.3.1 (Proof of Proposition 5.3.1:) *If $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M})$ is some other efficient mechanism that is also incentive compatible, then by the payoff equivalence we know that for* *all i, the expected payoff functions for this mechanism, say U_i, differ from U_i^V by at most an additive constant, say c_i. If $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M})$ is also individually rational, then this constant must be nonnegative - that is,*

$$c_i = U_i(x_i) - U_i^V(x_i).$$

This is because otherwise we would have $U_i(\alpha_i) < U_i^V(\alpha_i) = 0$, contradicting that $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M})$ was individually rational. Since the expected payoffs in $(\boldsymbol{Q}^*, \boldsymbol{M})$ are greater than in the VCG mechanism, and the two have the same allocation rule, the expected payments must be lower.

5.3.2 Budget Balance

A mechanism is said to *balance the budget* if for every realization of values, the net payments from agents sum to zero - that is,

$$\sum_{i \in \mathcal{N}} M_i(\mathbf{x}) = 0, \forall x \in \mathcal{X}.$$

This means that the seller keeps no surplus. In other words, all the surplus the seller can generate should be re-distributed to the buyers who did not obtain the object. Think about the government as the seller. Note that the VCG mechanism does not always satisfy budget balance. For example, it is easy to construct an example in which there exists a value realization $x \in \mathcal{X}$ such that

$$\sum_{i \in \mathcal{N}} M_i^V(\mathbf{x}) > 0.$$

Green and Laffont (1979) showed that "generally," there is no direct mechanism that is DSIC, IR, Eff, and further budget balanced (B-B).

Theorem 5.3.1 *Suppose that $\mathcal{X}$ for each buyer i N. Then, there is no direct mechanism that is DSIC, IR, Eff, and B-B.*

With the above impossibility result in mind, we seek for some possibility results by relaxing DSIC into the usual incentive compatibility (IC) but keeping IR, Eff, and B-B.

The Arrow-d'Aspremont-Gerard-Varet or AGV mechanism (also called the "expected externality" mechanism) $(\mathcal{X}, \mathbf{Q}^*, \mathbf{M}^A)$ is defined by

$$M_i^A(\mathbf{x}) = \frac{1}{n-1} \sum_{j \neq i} \{E_{\mathbf{x}_{-j}}[W_{-j}(x_j, \mathbf{x}_{-j})]\} - E_{\mathbf{x}_{-i}}[W_{-i}(x_i, \mathbf{x}_{-i})]$$

so that for all $\mathbf{x}$, (Can you prove the following equation?)

$$\sum_{i \in \mathcal{N}} M_i^A(\mathbf{x}) = 0.$$

To see that the AGV mechanism is incentive compatible, suppose that all other agents are reporting their values $\mathbf{x}_{-i}$ truthfully. The expected payoff to i from reporting z_i when his true value is x_i is

$$\begin{aligned} & E_{\mathbf{x}_{-i}}[Q_i^*(z_i, \mathbf{x}_{-i})x_i + W_{-i}(z_i, \mathbf{x}_{-i})] \\ & - E_{\mathbf{x}_{-i}}[\frac{1}{N-1}\sum_{j \neq i} E_{\mathbf{x}_{-j}}[W_{-j}(x_j, \mathbf{x}_{-j})]] \end{aligned}$$

and since the second term is independent of z_i, this is maximized by setting $z_i = x_i$.

It is easy to see that the AGV mechanism may not satisfy the individual rationality constraint. The question of whether there are efficient, incentive compatible, individually rational mechanisms that, at the same time, balance the budget can also be answered by means of the VCG mechanism.

Proposition 5.3.2 *There exists an efficient, incentive compatible, and individually rational mechanism that balances the budget if and only if the VCG mechanism results in an expected surplus.*

Proof 5.3.2 (Proof of Proposition 5.3.2:) *(only if part): This follows from Proposition 5.3.1: If the VCG mechanism runs a deficit, then any efficient, incentive compatible, and individually rational mechanism must run a deficit. Because the VCG mechanism is the revenue-maximizing (optimal) one among all those mechanisms that are IC, IR, and Eff.*

(if part): First, consider the VCG mechanism $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^V)$. *Buyer i's equilibrium payoff function in the VCG mechanism is*

$$\begin{aligned} U_i^V(x_i) &= E[W(x_i, \boldsymbol{x}_{-i}) - W(\alpha_i, \boldsymbol{x}_{-i})] \\ &= E[W(x_i, \boldsymbol{x}_{-i})] - E[W(\alpha_i, \boldsymbol{x}_{-i})] \\ &= E[W(x_i, \boldsymbol{x}_{-i})] - c_i^V \end{aligned}$$

where $c_i^V \equiv E[W(\alpha_i, \boldsymbol{x}_{-i})]$, *which is a constant. Next consider the AGV mechanism* $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^A)$. *From the payoff equivalence, we know that there exists a constant* c_i^A *such* *that*

$$U_i^A(x_i) = E[W(x_i, \boldsymbol{x}_{-i})] - c_i^A.$$

Suppose that the VCG mechanism runs an expected surplus - that is,

$$E[\sum_{i\in\mathcal{N}} M_i^V(X)] \geq 0.$$

Then

$$E[\sum_{i\in\mathcal{N}} M_i^V(X)] \geq \underbrace{E[\sum_{i\in\mathcal{N}} M_i^A(X)] = 0}_{\sum_{i\in\mathcal{N}} M_i(\boldsymbol{x})=0, \forall \boldsymbol{x}}$$

Equivalently,

$$\sum_{i\in\mathcal{N}} c_i^V \geq \sum_{i\in\mathcal{N}} c_i^A \tag{5.13}$$

For all $i > 1$*, define* $d_i = c_i^A - c_i^V$ *and let* $d_1 = -\sum_{j=2}^N d_j$*. Consider the mechanism* $(\mathcal{X}, \boldsymbol{Q}^*, \overline{\boldsymbol{M}})$ *defined by*

$$\overline{M}_i(\boldsymbol{x}) = M_i^A(\boldsymbol{x}) + d_i, \forall i \in \mathcal{N}.$$

Then, the equilibrium payoff functions for the two mechanisms are given as follows:

$$\overline{U}_i(\boldsymbol{x}) = E[Q_i^*(x_i, \boldsymbol{x}_{-i})x_i] - E[\overline{M}_i(x_i, \boldsymbol{x}_{-i})]$$
$$U_i^A(\boldsymbol{x}) = E[Q_i^*(x_i, \boldsymbol{x}_{-i})x_i] - E[M_i^A(x_i, \boldsymbol{x}_{-i})]$$

By construction, $\overline{\boldsymbol{M}}$ *balances the budget.* $(\mathcal{X}, \boldsymbol{Q}^*, \overline{\boldsymbol{M}})$ *is also incentive compatible since the payoff to each agent in the mechanism* $(\mathcal{X}, \boldsymbol{Q}^*, \overline{\boldsymbol{M}})$ *differs from the payoff from an incentive compatible mechanism* $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^A)$ *by an additive constant. Thus, it remains to verify that* $(\mathcal{X}, \boldsymbol{Q}^*, \overline{\boldsymbol{M}})$ *is individually rational. For all* $i \neq 1$,

$$\begin{aligned}\overline{U}_i(x_i) &= U_i^A(x_i) + d_i \\ &= U_i^A(x_i) + c_1^A - c_1^V \\ &= E(W(x_i, \boldsymbol{x}_{-i})) - c_1^V \\ &= U_i^V(x_i) \geq 0.\end{aligned}$$

By construction $\sum_{i=1}^N d_i = 0$*. Observe from (5.13) that*

$$d_1 = -\sum_{i>1} d_i = \sum_{i>1}(c_i^V - c_i^A) \geq c_1^A - c_1^V.$$

Thus

$$\begin{aligned}\overline{U}_1(x_1) &= U_1^A(x_1) + d_1\\ &= U_1^A(x_1) + c_1^A - c_1^V\\ &= E(W(x_i, \boldsymbol{x}_{-i})) - c_1^V\\ &= U_1^V(x_1) \geq 0.\end{aligned}$$

so that $(\mathcal{X}, \boldsymbol{Q}^*, \overline{\boldsymbol{M}})$ *is also individually rational.*

5.4 An Application to Bilateral Trade

Suppose that there is a seller with a privately known cost $C \in [\underline{c}, \overline{c}]$ of producing a single indivisible good. Suppose also that there is a buyer with a privately known value $V \in [\underline{v}, \overline{v}]$ of consuming the good. The cost C and value V are *independently* distributed, and the prior distributions are commonly known and have full support on the respective intervals. Thus, there is incomplete information on both sides of the market. Finally, suppose that $\underline{v} < \overline{c}$ and $\overline{v} \geq \underline{c}$, so that the supports overlap and sometimes it is efficient not to trade. Is there some way to guarantee that trade will take place whenever it should? To answer this question, it is natural to adopt a mechanism design perspective.

A mechanism decides whether or not the good is traded. It also decides the amount P the buyer pays for the good and the amount R the seller receives. If the good is traded, the net gain to the buyer is $V - P$, and the net gain to the seller is $R - C$. At the moment, we do not restrict P or R to be positive or negative, nor do we assume that the budget is balanced-that is, $P = R$. A mechanism is efficient if whenever $V > C$, the object is produced and allocated to the buyer.

Proposition 5.4.1 *In the bilateral trade problem, there is* ***no*** *mechanism that is efficient, incentive compatible, individually rational, and at the same balances the budget.*

Proof 5.4.1 (Proof of Proposition 5.4.1:) *The proof is based on Krishna and Perry (2000). Similar proof is also found in Williams (1999). First, consider the VCG mechanism, whose operation in this context is as follows:*

The buyer announces a valuation V *and the seller announces a cost* C*.*

1. *If $V \leq C$, the object is not exchanged and no payments are made.*

2. *If $V > C$, the object is exchanged. The buyer pays* $\max C, \underline{v}$ *and the seller receives* $\min V, \bar{c}$.

It is easy to verify that it is a weakly dominant strategy for the buyer to announce $V = v$ and the seller to announce $C = c$. This mechanism is also efficient since, in equilibrium, the object is transferred whenever $v > c$.

A buyer with value $\underline{v}$ has an expected payoff of 0, and any buyer with value $v > \underline{v}$ has a positive expected payoff. Similarly, a seller with cost $\bar{c}$ has an expected payoff of 0, and any seller with cost $c < \bar{c}$ has a positive expected payoff. Thus, the mechanism is individually rational.

Whenever $V > C$, so there is trade, the fact that $\underline{v} < \bar{c}$ implies that the amount the seller receives $R = \min V, \bar{c}$ is greater than the amount buyer pays $P = \max C, \underline{v}$. Thus, in this context, the VCG mechanism always runs a deficit, i.e., violates budget balance. Indeed, for any realization of V and C such that $V > C$, the deficit $R - P = V - C$, which is exactly equal to the ex post *gains that result from trade.*

Now suppose that we have some other mechanism that is IC, IR, and Eff. Let $m_B^V(v)$ and $m_S^V(c)$ be the equilibrium expected payments of buyer with value v and seller with cost c in the VCG mechanism, respectively. Since the VCG mechanism is IR with the condition that $\underline{v} < \bar{c}$,

$$m_B^V(\underline{v}) = 0 \text{ and } m_S^V(\bar{c}) = 0$$

Let also $m_B(v)$ and $m_S(c)$ be the equilibrium expected payments of buyer with value v and seller with cost c in the proposed mechanism, respectively. By the revenue equivalence principle, there are constants $K, L \in \mathbb{R}$ such that for any v, c,

$$m_B(v) = m_B^V(v) + K$$
$$m_S(c) = m_S^V(c) + L$$

Since the proposed mechanism is individually rational, we have

$$m_B(\underline{v}) \leq 0 \text{ and } m_s(\bar{c}) > 0.$$

This implies that $K \leq 0$ and $L \geq 0$.

The expected deficit under the proposed mechanism is just the expected deficit under the VCG mechanism plus $L - K \geq 0$*. But if the VCG mechanism runs a deficit, we have argued in Proposition 5.3.1 that every other IC, IR, and Eff mechanism also runs a deficit. Since the choice of the proposed mechanism is arbitrary other than the requirements of incentive compatibility, individual rationality, and efficiency, there exists* ***no*** *mechanism that is IC, IR, Eff, and B-B.*

Chapter 6

Auctions with Interdependent Values

There are two fundamental assumptions we have maintained so far:

1. Each bidder's valuation for the object depends solely upon his private information (Private Values)

2. With his private information, each bidder knows nothing about other bidders' private information. (Independence)

From now on, we will relax these two assumptions all together. First, we focus on the first assumption. We assume that each bidder has some (but not necessarily all) private information concerning the value of the object. Bidder i's private information is summarized as the realization of the random variable $X_i \in [0, \omega_i]$, called i's signal. It is assumed that the value of the object to bidder i, V_i, can be expressed as a function of all bidders' signals and we will write

$$V_i = v_i(X_1, X_2, \cdots, X_N)$$

where the function v_i is bidder i's *valuation* and is assumed to be nondecreasing in all its variables and twice continuously differentiable. In addition, it is assumed that v_i is strictly increasing in X_i.[1]

This specification presupposes that the value is completely determined by the signals. In a more general setting, suppose that $V_1, V_2, \cdots, V_N$ denote the N (unknown) values

[1]This implies that for any $i, j \in \mathcal{N}$ and $\mathbf{x}_{-j}$, we have $v_i(x_j, \mathbf{x}_{-j}) \geq v_i(x_j', \mathbf{x}_{-j})$ whenever $x_j > x_j'$ with strict inequality for $j = i$.

to the bidders; and S denotes a signal available only to the seller. Then, we denote $v_i = V_i(X_1, X_2, \cdots, X_N, S)$. In this case, we can define

$$v_i(x_1, x_2, \cdots, v_N) \equiv E[V_i | X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N]$$

as the expected value to bidder i conditional on all the information available to bidders. With either specification, I suppose that $v_i(0, 0, ..., 0) = 0$ and that $E[V_i] < \infty$. We continue to assume that bidders are risk neutral - each bidder maximizes the expectation of $V_i - p_i$, where p_i is the price paid.

This specification of the values includes, as a special case, the private values model of earlier chapters in which $v_i(X_1, \cdots, X_N) = X_i$. Another special case is a pure *common value* in which all bidders assign the same value

$$V = v(X_1, X_2, \cdots, X_N)$$

to the object - the valuations of the bidders are identical. Bidders' information consists only of their own signals of course, so while the *ex post* value is common to all, it is unknown to any particular bidder. A special case that is of both analytic and practical interest entails first specifying a distribution for the common value V and then assuming that conditional on the event $V = v$, bidders' signals X_i are independently distributed. Typically, it is also assumed that each X_i is an unbiased estimator of V , so that $E[X_i | V = v] = v$. This particular specification has been used to model the information structure associated with auctions of oil-drilling leases and is sometimes called the "mineral rights" model.

The interdependence of values complicates the decision problem facing a bidder. In particular, since the exact value of the object is unknown and depends also on other bidders' signals, an *a priori* estimate of this value may need to be revised as a result of events that take place during, and even after, the auction. The reason is that these events may convey valuable information about the signals of other bidders. One such event is the announcement that the bidder has won the auction.

The Winner's Curse

Prior to the auction the only information available to a bidder, say 1, is that his own signal $X_1 = x$. Based on this information alone, his estimate of the value is $E[V | X_1 = x]$. Now suppose that the object is sold using a sealed-bid first-price auction and consider what happens when and if it is announced that bidder 1 is, in fact, the winner. If all bidders are symmetric and follow the same strategy β, then this fact reveals to bidder 1 that the highest of the other $N-1$ signals is less than x. As a result, his estimate of the value upon learning that he is the winner is $E[V|X_1 = x, Y_1 < x]$, which is less than $E[V|X_1 = x]$. The announcement that he has won leads to a decrease in the estimated value; in this sense, winning brings "bad news." A failure to foresee this effect and take it fully into account when formulating bidding strategies will result in what has been called the *winner's curse* - the possibility that the winner pays more than the value.

We emphasize that the winner's curse arises only if bidders do not calculate the value of winning correctly and overbid as a result – it does not arise in equilibrium.

Nonequivalence of English and Second-Price Auctions

A second consequence of interdependent values is that it is no longer the case that the English (or open ascending) auction is strategically equivalent to the sealed-bid second-price auction. The difference in the two auction formats is that in an English auction active bidders get to know the prices at which the bidders who have dropped out have done so. This allows the active bidders to make inferences about the information that the inactive bidders had and in this way to update their estimates of the true value. A sealed-bid second-price auction, by its very nature, makes no such information available.

There are two cases in which this information is irrelevant. First, if there are only two bidders, then the English auction is always equivalent to a second-price auction; in this case, when one of the bidders drops out in an English auction, the auction is over. The second case arises if the bidders have private values; in this case, the information gleaned from others is irrelevant.

Affiliation

We also relax the assumption that bidders' information is independently distributed by allowing for the possibility that bidders' signals are correlated. Thus, the joint density of the bidders' signals, $f(\mathbf{X})$, need not be a product of densities of individual signals, $f_i(X_i)$. In fact, we will assume that the signals $X_1, X_2, \cdots, X_N$ are positively affiliated. Affiliation is a strong form of positive correlation and roughly means that if a subset of the X_i's are all large, then this makes it more likely that the remaining X_j's are also large. While a formal definition and a more detailed discussion may be found in Chapter 2, for the purposes of this chapter, the following three implications of affiliation are sufficient.

First. define, as usual, the random variables $Y_1, Y_2, \cdots, Y_{N-1}$ to be the largest, second largest, $\cdots$, smallest from among $X_2, X_3, \cdots, X_N$. If the variables $X_1, X_2, \cdots, X_N$ are affiliated, then the variables $X_1, Y_1, \cdots, Y_{N-1}$ are also affiliated.

Second. let $G(.|x)$ denote the distribution of Y_1 conditional on $X_1 = x$. Then the fact that X_1 and Y_1 are affiliated implies that if $x' > x$, then $G(.|x')$ dominates $G(.|x)$ in terms of the reverse hazard rate - that is, for all y,

$$\frac{g(y|x')}{G(y|x')} \geq \frac{g(y|x)}{G(y|x)}$$

Third. if γ is any increasing function, then $x' > x$ implies that

$$E[\gamma(Y_1)|X_1 = x'] \geq E[\gamma(Y_1)|X_1 = x].$$

6.1 The Symmetric Model

With interdependent values and affiliated signals, there are two aspects to symmetry. The first concerns the symmetry of the valuations v_i and the second concerns the symmetry of the distribution of signals.

It is assumed that all signals X_i are drawn from the same interval $[0, \omega]$ and that the valuations of the bidders are symmetric in the following sense. For all i, I can write these in the form

$$u_i(X) = u(X_i, X_{-i})$$

and the function u, which is the same for all bidders, is symmetric in the last $N - 1$ components. This means that from the perspective of a particular bidder i, the signals of other bidders can be interchanged without affecting the value. when $N = 3$, for all x, y, and z the symmetry implies that $u(x, y, z) = u(x, z, y)$.

It is also assumed that the joint density function of the signals f, defined on $[0, \omega]^n$, is a symmetric function of its arguments and the signals are affiliated.

Define the function

$$v(x, y) = E[V_1|X_1 = x, Y_1 = y]$$

to be the expectation of the value to bidder 1 when his signal is x and the highest signal among the other bidders, Y_1, is y. Given the assumptions we have made so far, v is a nondecreasing function of x and y. In fact, we will assume that v is strictly increasing in x. Moreover, since $u(0, 0, \cdots, 0) = 0$, we know that $v(0, 0) = 0$.

6.2 Second Price Auctions with Interdependent Values

Proposition 6.2.1 *Symmetric equilibrium strategies in a second price auction are given by:*

$$\beta^{II}(x) = v(x, x)$$

Proof 6.2.1 (Proof of Proposition 6.2.1:) *Suppose all other bidders $j \neq i$ follow the strategy $\beta \equiv \beta^{II}$. Bidder i's expected payoff when his signal is x and he bids an amount b is*

$$\begin{aligned}\Pi(b, x) &= \int_0^{\beta^{-1}(b)} (v(x, y) - \beta(y))g(y|x)dy \\ &= \int_0^{\beta^{-1}(b)} (v(x, y) - v(y, y))g(y|x)dy.\end{aligned}$$

where $g(.|x)$ is the density of $Y_1 \equiv \max_{j\neq i} X_j$ conditional on $X_1 = x$. Since v is increasing in the first argument, we know that

- *$v(x, y) - v(y, y) > 0$ for all $y < x$ and*
- *$v(x, y) - v(y, y) < 0$ for all $y > x$.*

Thus, Π is maximized by choosing b so that $\beta^{-1}(b) = x$.

Proposition 6.2.1 applies, of course, to the special case of private values (where $v(x, x) = x$) and in those circumstances the equilibrium strategy is weakly dominant. With general interdependent values, however, the strategy β^{II} identified above is not a dominant strategy.

Claim 6.2.1 *$\beta^{II}(x) = v(x, x)$ is the* ***unique*** *symmetric equilibrium strategy.*

Proof 6.2.2 (Proof of Claim 6.2.1:) *Let β be an increasing symmetric equilibrium strategy. For an arbitrary bid b, one can define z such that $b = \beta(z)$. Define the expected payoff of bidder i as follows:*

$$\Pi(z, x) = \int_0^z (v(x, y) - \beta(y))g(y|x)dy.$$

Differentiating Π with respect to z gives

$$\frac{d\Pi(z, x)}{dz} = (v(x, z) - \beta(z))g(z|x)$$

Our equilibrium hypothesis requires that the first-order condition be satisfied at $z = x$. Thus, the solutions must satisfy the following equality:

$$v(x, x) - \beta(x) = 0.$$

Since v is strictly increasing in the first argument, the solution is unique.

We will see later, however, that even symmetric second-price auctions may have other, asymmetric equilibria.

It is instructive to find the equilibrium bidding strategies explicitly in an example. In the example that follows, there is a common value and conditional on that value, bidders' signals are independently distributed. In other words, it is an instance of the "mineral rights" model.

Example 6.2.1 *Suppose that there are three bidders with a common value V for the object that is uniformly distributed on $[0, 1]$. Given $V = v$, bidders' signals X_i are uniformly and independently distributed on $[0, 2v]$.*

Let $X = (X_1, X_2, X_3)$ and $Z \equiv \max\{X_1, X_2, X_3\}$. The density of X_i conditional on $V = v$ is $1/2v$ on the interval $[0, 2v]$, so the joint density of (V, X) is $1/8v^3$ on the set

$$\{(V, X)|X_i \leq 2V, \forall i = 1, 2, 3\}.$$

Note that the only information about V that knowledge of X_1, X_2, X_3 provides is that $V \geq Z/2$. Thus, the joint density of $X = (X_1, X_2, X_3)$ is

$$\begin{aligned} f(x_1, x_2, x_3) &= \int_{z/2}^1 \frac{1}{8v^3} dv \\ &= \frac{4 - z^2}{16z^2} \end{aligned}$$

where $z = \max\{x_1, x_2, x_3\}$. Thus, the density of V conditional on $X = x$ is the same as the density of V conditional on $Z = z$, so

$$\begin{aligned} f(v|X = x) &= f(v|Z = z) \\ &= \frac{1}{f(x_1, x_2, x_3)} \times \frac{1}{8v^3} \\ &= \frac{1}{8v^3} \times \frac{4 - z^2}{16z^2} \end{aligned}$$

on the interval $[z/2, 1]$. Thus,

$$\begin{aligned} E(V|X = x) &= E(v|Z = z) \\ &= \int_{1/2z}^{1} vf(v|X = x)dv \\ &= \frac{2z}{2 + z}. \end{aligned}$$

Notice that since $Y_1 = \max\{X_2, X_3\}$ and $Z = \max\{X_1, X_2, X_3\}$, $Z = \max\{X_1, Y_1\}$.

$$\begin{aligned} v(x, y) &= E(v|X_1 = x, Y_1 = y) \\ &= E(v|Z = \max\{x, y\}) \\ &= \frac{2\max\{x, y\}}{2 + \max\{x, y\}}. \end{aligned}$$

Thus we obtain

$$\beta^{II}(x) = v(x, x) = \frac{2x}{2 + x}.$$

6.3 English Auctions

In an English auction,

- an auctioneer sets the price at zero and gradually raises it.
- The current price is observed by all and bidders signal their willingness to buy by pushing a button that controls light.
- At any time, the set of active bidders is commonly known.
- Bidders may drop out at any time, but once they do so, they cannot reenter the auction at a higher price.

- The auction ends when there is only one active bidder.

A symmetric equilibrium strategy in an English auction is thus a collection $\beta = (\beta^N, \beta^{N-1}, \ldots, \beta^2)$ of $N-1$ functions $\beta^k : [0,1] \times \mathbb{R}_+^{N-k} \to \mathbb{R}_+$, for $1 < k \leq N$, where $\beta^k(x, p_{k+1}, \ldots, p_N)$ is the price at which bidder 1 will drop out if the number of bidders who are still active is k, his own signal is x, and the prices at which the other $N-k$ bidders dropped out were $p_{k+1} \geq p_{k+2} \geq \ldots \geq p_N$.

6.3.1 Constructing the Symmetric Equilibrium in an English Auction

When all bidders are active, let

$$\beta^N(x) = u(\underbrace{x, x, \ldots, x}_{N}) \tag{6.1}$$

and notice that $\beta^N(x)$ is a continuous and increasing function.

Suppose that bidder N, say, is the first to drop out at some price p_N and let x_N be the unique signal such that $\beta^N(x_N) = p_N$ (since $\beta^N(.)$ is continuous and increasing there exists a unique such x_N). When some bidder drops out at a price p_N, let the remaining $N-1$ bidders who are still active follow the strategy

$$\beta^{N-1}(x) = u(\underbrace{x, x, \ldots, x}_{N-1}, x_N)$$

where $\beta^N(x_N) = p_N$. The function $\beta^{N-1}(., p_N)$ is also continuous and increasing.

Proceeding recursively in this way, for all k such that $2 \leq k < N$ suppose that bidders $N, N-1, \ldots, k+1$ have dropped out of the auction at prices $p_N, p_{N-1}, \ldots, p_{k+1}$, respectively. Let the remaining k bidders who are still active follow the strategy

$$\beta^k(x, \underbrace{p_{k+1}, \ldots, p_N}_{N-k}) = u(\underbrace{x, \ldots, x}_{k}, \underbrace{x_{k+1}, \ldots, x_N}_{N-k}) \tag{6.2}$$

We will argue that these strategies constitute an equilibrium of the English auction, but before doing so formally it is worthwhile to understand the nature of the bidding strategies. Suppose that bidders $k+1, k+2, \ldots, N$ have dropped out, so only k bidders are still active. Because the strategies are revealing, the signals $x_{k+1}, x_{k+2}, \ldots, x_N$ of the bidders who have dropped out become known to the other bidders. Consider a particular bidder, say 1, with signal x, and suppose the other bidders are following β^k. Bidder 1 evaluates whether or not he should drop out at the current price p and does the following "mental calculation." He asks what would happen if he were to win the good at the current price p. Now the only way this can happen is if all the other $k-1$ bidders drop out at p. In that case, bidder 1 would infer that each of their signals were equal to a y such that $\beta^k(y, p_{k+1}, \ldots, p_N) = p$. The value of the object would then be inferred to be

$$u(x, \underbrace{y, \ldots, y}_{k-1}, \underbrace{x_{k+1}, x_{k+2}, \ldots, x_N}_{N-k}).$$

It is worth continuing in the auction if and only if the inferred value of the object exceeds the current price p. Thus, the strategy calls for bidder 1 to continue until the price is such that if he were to win the object at that price he would just break even.

Proposition 6.3.1 *Symmetric equilibrium strategies in an English auction are given by β defined in (6.1) and (6.2).*

Proof 6.3.1 (Proof of Proposition 6.3.1:) *Consider bidder 1 with signal $X_1 = x$ and suppose that all other bidders follow the strategy β. Define $Y_1, Y_2, \ldots, Y_{N-1}$ to be the largest, second-largest,..., smallest of $X_2, X_3, \ldots, X_N$, respectively.*

Suppose that the realizations of $Y_1, Y_2, \ldots, Y_{N-1}$, denoted by $y_1, y_2, \ldots, y_{N-1}$, respectively, are such that bidder 1 wins the object if he also follows the strategy β. Then it must be that $x > y_1$. Because the strategy is increasing. The price that bidder 1 pays is the price at which the bidder with the second highest signal, y_1, drops out. From (6.2), the price is $u(y_1, y_1, y_2, \ldots, y_{N-1})$. Since $x > y_1$ and u is strictly increasing in the first argument, bidder 1's payoff upon winning is

$$u(x, y_1, y_2, \ldots, y_{N-1}) - u(y_1, y_1, y_2, \ldots, y_{N-1}) > 0.$$

Bidder 1 cannot affect the price he pays and winning yields a positive profit. Thus, he cannot do better than to follow β.

Next, suppose that the realizations of $Y_1, Y_2, \ldots, Y_{N-1}$ are such that bidder 1 does not win the object by following β. Then it must be that $x < y_1$. Again, because the strategy is increasing. The only way he can change the outcome is that he does not drop out and wins the auction. Then, from the price he pays is $u(y_1, y_1, y_2, \ldots, y_{N-1})$. But since u is *strictly increasing in the first argument, we have*

$$u(x, y_1, y_2, \ldots, y_{N-1}) - u(y_1, y_1, y_2, \ldots, y_{N-1}) < 0,$$

which makes bidder 1 worse off than dropping out of the auction. So, bidder 1 cannot do better than to drop out by following β.

6.3.2 Ex Post Equilibrium

The equilibrium strategy β defined by (6.1) and (6.2) depends only on the valuation functions $u(.)$ and not on the underlying distribution of signals f. In other words, the strategies form an *ex post* equilibrium. This means that for any realization of the signals, the bidders have no cause to regret the outcome even if, all signals were to become publicly known. This also means that the equilibrium strategy β has an important "no regret" feature – that is, for any realization of the signals the bidders have no cause to regret the outcome even if, after the fact, all signals were to become publicly known. In sharp contrast, once there are three or more bidders, bidders playing the symmetric equilibrium of the second-price auction, identified in the previous sections, may suffer from regret after the fact. The equilibrium of the second-price auction is not an *ex post* equilibrium. Nor is the symmetric equilibrium of the first-price auction an *ex post* equilibrium.

The symmetric equilibrium of the English auction has the strong no regret property because, in fact, in the course of the auction the signals of all other bidders are revealed to the winner, so he does not regret winning. On the other hand, bidders who drop out do not regret losing because if they were to win, it would be at a price that is too high.

6.4 First Price Auctions with Interdependent Values

Suppose all other bidders $j \neq i$ follow the increasing and differentiable strategy β. Let $G(.|x)$ denote the distribution of $Y_1 \equiv \max_{j \neq i} X_j$ conditional on $X_i = x$ and let $g(.|x)$ be the associated conditional density function. The expected payoff to bidder i when his signal is x and he bids $\beta(z)$ is

$$\begin{aligned} \Pi_i(z, x) &= \int_0^z (v(x, y) - \beta(z)) g(y|x) dy \\ &= \int_0^z v(x, y) g(y|x) dy - \beta(z) G(z|x) \end{aligned}$$

The first-order condition is

$$(v(x,z) - \beta(z))g(z|x) - \beta'(z)G(z|x) = 0.$$

By our equilibrium hypothesis, the above first-order condition must be satisfied when $z = x$. Thus, we obtain the following differential equation:

$$\beta'(x) = (v(x,x) - \beta(x))\frac{g(x|x)}{G(x|x)} \tag{6.3}$$

Claim 6.4.1 *$\beta(0) = 0$.*

Proof 6.4.1 (Proof of Claim 6.4.1:) *We argue that $v(x,x) - \beta(x) \geq 0$ for any x. Suppose not, then, to bid 0 is better. Since by assumption, $v(0,0) = 0$, we must have $\beta(0) = 0$.*

Proposition 6.4.1 *Symmetric equilibrium strategies in a sealed-bid first price auction are given by*

$$\beta^I(x) = \int_0^x v(y,y)dL(y|x)$$

where

$$L(y|x) = \exp(-\int_y^x \frac{g(t|t)}{G(t|t)}dt)$$

Proof 6.4.2 (Proof of Proposition 6.4.1:) *The proof consists of four steps.*

- ***Step 1**:* $L(.|x)$ is a distribution function with support $[0,x]$.

  To see this recall that, because of affiliation, for all $t > 0$,

  $$\frac{g(t|t)}{G(t|t)} \geq \frac{g(t|0)}{G(t|0)}$$

  and so

  $$\begin{aligned} -\int_0^x \frac{g(t|t)}{G(t|t)}dt &\leq -\int_0^x \frac{g(t|0)}{G(t|0)}dt \\ &= -\int_0^x \frac{d}{dt}(\ln G(t|0))dt \\ &= \ln G(0|0) - \ln G(x|0) \\ &= -\infty(\because \ln G(0|0) = \ln 0 = -\infty \text{ and } 0 < \ln G(x|0) < \infty) \end{aligned}$$

  Applying the exponential function to both sides implies that $L(0|x) = 0$. Moreover, $L(x|x) = 1$ and $L(.|x)$ is nondecreasing. Therefore, $L(.|x)$ is a distribution function.

- ***Step 2***: $\beta(x) = \int_0^x v(y,y)dL(y|x)$

  We execute the following computation:

$$\begin{aligned}
\int_0^x v(y,y)dL(y|x) &= \int_0^x v(y,y)\frac{dL(y|x)}{dy}dy \\
&= \int_0^x v(y,y)\frac{g(y|y)}{G(y|y)}L(y|x)dy \\
&= \int_0^x [\beta'(y)L(y|x) + \beta(y)\frac{g(y|y)}{G(y|y)}L(y|x)]dy \\
&= \int_0^x \frac{\partial}{\partial y}[\beta(y)L(y|x)]dy(\because \frac{\partial L(y|x)}{\partial y} = L'(y|x) = \frac{g(y|y)}{G(y|y)}L(y|x)) \\
&= [\beta(y)L(y|x)]_0^x \\
&= \beta(x)L(x|x) - \beta(0)L(0|0) \\
&= \beta(x)(\because L(x|x) = 1 \textit{ and } \beta(0) = 0.)
\end{aligned}$$

- ***Step 3***: (FOSD) $L(y|x') \leq L(y|x)$ for all $y \in [0,x]$ if $x' > x \Leftrightarrow \beta^I(.)$ is increasing.

  Assume that $x' > x$. Then, one can execute a series of computations as follows:

$$\int_y^{x'} \frac{g(t|t)}{G(t|t)}dt \geq \int_y^{x} \frac{g(t|t)}{G(t|t)}dt$$
$$\exp(-\int_y^{x'} \frac{g(t|t)}{G(t|t)}dt) \leq \exp(-\int_y^{x} \frac{g(t|t)}{G(t|t)}dt)L(y|x') \leq L(y|x)$$

- ***Step 4***: $\beta^I \equiv \beta$ constitutes an equilibrium.

  Since β is increasing, the expected payoff of a bidder with signal x who bids $\beta(z)$ can be written as

$$\begin{aligned}
\Pi(z,x) &= \int_0^z (v(x,y) - \beta(z))g(y|x)dy \\
&= \int_0^z v(x,y)g(y|x)dy - \beta(z)G(z|x)
\end{aligned}$$

  Differentiating this with respect to z yields

$$\begin{aligned}
\frac{\partial \Pi}{\partial z} &= (v(x,z) - \beta(z))g(z|x) - \beta'(z)G(z|x) \\
&= G(z|x)[(v(x,z) - \beta(z))\frac{g(z|x)}{G(z|x)} - \beta'(z)]
\end{aligned}$$

  If $z < x$, then since $v(x,z) > v(z,z)$ and because of affiliation,

$$\frac{g(z|x)}{G(z|x)} > \frac{g(z|z)}{G(z|z)}$$

we obtain

$$\frac{\partial \Pi}{\partial z} > G(z|x)[(v(x,z) - \beta(z))\frac{g(z|z)}{G(z|z)} - \beta'(z)] = 0.$$

Similarly, if $z > x$*, then* $\frac{\partial \Pi}{\partial z} < 0$*. Thus,* $\Pi(z,x)$ *is maximized when* $z = x$*.*

6.5 Revenue Ranking

6.5.1 English versus Second Price Auctions

In this section, we will show that under the assumption that signals are affiliated, the English auction out-performs the second price auction. The formality of this is given as a proposition below.

Proposition 6.5.1 *The expected revenue from an English auction is at least as great as the expected revenue from a second price auction.*

Proof 6.5.1 (Proof of Proposition 6.5.1:) *Recall that symmetric equilibrium strategies in a second price auction are given by* $\beta^{II}(x) = v(x,x)$*. If* $x > y$*, we have*

$$\begin{aligned}
v(y,y) &= E[u(X_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 = y, Y_1 = y] \\
&= E[u(Y_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 = y, Y_1 = y] \\
&\leq E[u(Y_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 = x, Y_1 = y]
\end{aligned}$$

where the last inequality follows from that fact that $x > y$*,* $u(.)$ *is increasing in all its arguments and signals are affiliated. The expected revenue in a second price auction can be written as*

$$\begin{aligned}
E[R^{II}] &= E[\beta^{II}(Y_1)|X_1 > Y_1] \\
&= E[v(Y_1, Y_1)|X_1 > Y_1] \\
&= E[E[u(Y_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 = y, Y_1 = y]|X_1 > Y_1] \\
&\leq E[E[u(Y_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 = x, Y_1 = y]|X_1 > Y_1] \\
&= E[u(Y_1, Y_1, Y_2, \ldots, Y_{N-1})|X_1 > Y_1] \\
&= E[\beta^2(Y_1, Y_2, \ldots, Y_{N-1})] \\
&= E[R^{Eng}]
\end{aligned}$$

where β^2 is the strategy used in an English auction when only two bidders remain. The price at which the second-to-last bidder drops out, $\beta^2(Y_1, Y_2, \ldots, Y_{N-1})$, is, of course, the price paid by the winning bidder.

We postpone discussion of this result, and the reasons underlying it. For the moment, notice that the English auction yields a strictly higher revenue than a second-price auction only if values are interdependent and signals are affiliated. With private values, the two are equivalent. The same is true if signals are independent.

6.5.2 Second Price versus First Price Auctions

Proposition 6.5.2 *The expected revenue from a second price auction is at least as great as the expected revenue from a first price auction.*

Proof 6.5.2 (Proof of Proposition 6.5.2:) *The payment of a bidder with signal x upon winning the object in a first-price auction is just his bid $\beta^I(x)$, where β^I is defined in Proposition 6.7.1. The expected payment of a bidder with signal x upon winning the object in a second-price auction is $E[\beta^{II}(Y_1)|X_1 = x, Y_1 < x]$, where β^{II} is defined in Proposition 6.2.1. We will show that the former is no greater than the latter. Since in both auctions the probability that a bidder with signal x will win the auction is the same – it is just the probability that x is the highest signal – this will establish the proposition.*

$$\begin{aligned} E[\beta^{II}(Y_1)|X_1 = x, Y_1 < x] &= E[v(Y_1, Y_1)|X_1 = x, Y_1 < x] \\ &= \int_0^x v(y, y) dK(y|x) \end{aligned}$$

and where for all $y < x$,

$$K(y|x) \equiv \frac{1}{G(x|x)} G(y|x).$$

Note that $K(.|x)$ is a distribution function with support $[0, x]$. Recall that

$$\beta^I(x) = \int_0^x v(y, y) dL(y|x)$$

where $L(.|x)$ is also a distribution function with support $[0, x]$. Now, we claim the following.

Claim 6.5.1 *(FOSD) $K(y|x) \leq L(y|x)$ for all $y < x$.*

If this claim is correct, we conclude

$$\int_0^x v(y,y)dK(y|x) \geq \int_0^x v(y,y)dL(y|x) \Rightarrow E[R^{II}] \geq E[R^{II}]$$

Proof 6.5.3 (Proof of Claim 6.5.1:) *Because of affiliation, for all $t < x$,*

$$\frac{g(t|t)}{G(t|t)} \leq \frac{g(t|x)}{G(t|x)}$$

Hence, for all $y < x$, we execute a series of computations:

$$\begin{aligned}
-\int_y^x \frac{g(t|t)}{G(t|t)}dt &\geq -\int_y^x \frac{g(t|x)}{G(t|x)}dt \\
&= -\int_y^x \frac{d}{dt}(\ln G(t|x))dt \\
&= \ln G(y|x) - \ln G(x|x) = \ln(\frac{G(y|x)}{G(x|x)})
\end{aligned}$$

This implies that

$$\exp(-\int_y^x)\frac{g(t|t)}{G(t|t)}dt \geq \frac{1}{G(x|x)}G(y|x) \Rightarrow L(y|x) \geq K(y|x),$$

and this completes the proof.

The conclusions of Propositions 6.5.1 and 6.5.2 are summarized as follows:

Proposition 6.5.3 *In the symmetric model with interdependent values and affiliated signals, the English, second-price, and first-price auctions can be ranked in terms of expected revenue as follows:*

$$E[R^{Eng}] \geq E[R^{II}] \geq E[R^I].$$

When the symmetric equilibria of English, the second-price, and the first price auctions, respectively, are only considered, the above proposition shows that, among three auction forms, most information will be revealed in English auctions and least information will be revealed in the first price auction. The second price auction lies somewhere between these two. One important implication of this revenue raking is that the more information is revealed, the more expected revenue is generated.

6.6 An Example

Suppose that S_1, S_2, and T are uniformly and independently distributed on $[0, 1]$. There are two bidders. Bidder 1 receives the signal $X_1 = S_1 + T$, and bidder 2 receives the signal $X_2 = S_2 + T$. The object has a common value

$$V = \frac{1}{2}(X_1 + X_2)$$

for both bidders.

Even though S_1, S_2, and T are independently distributed, the random variables X_1 and X_2 are affiliated. Moreover, since there are only two bidders, $Y_1 = X_2$. The joint density of X_1 and Y_1 is given in Figure 6.1

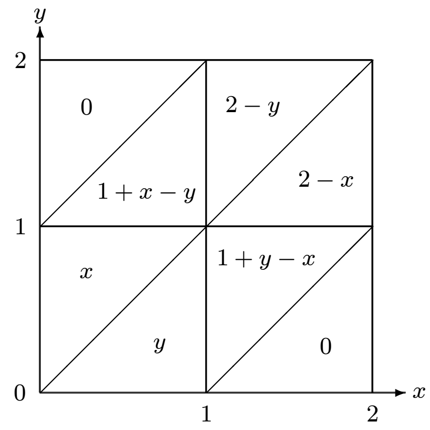

Figure 6.1: Joint Density of X_1 and Y_1

From this it may be calculated that for all $x \in [0, 2]$,

$$\frac{g(x|x)}{G(x|x)} = \frac{2}{x}.$$

and for all $y \in [0, x]$,

$$L(y|x) = \frac{y^2}{x^2}$$

Note that $v(x, y) = (x + y)/2$. Therefore, we obtain

$$\begin{aligned}
\beta^I(x) &= \int_0^x v(y, y) dL(y|x) \\
&= \int_0^x y \frac{\partial L(y|x)}{\partial y} dy \\
&= \int_0^x y \frac{2y}{x^2} dy \\
&= \frac{2}{3} x < x = v(x, x) = \beta^{II}(x)
\end{aligned}$$

Revenue Ranking In a second price auction the equilibrium bidding strategy is $\beta^{II}(x) = x$. The expected revenue in a second price auction is

$$\begin{aligned}
E[R^{II}] &= E[\min\{X_1, X_2\}] \\
&= E[\min\{T_1, T_2\}] + E[T] \\
&= \underbrace{\int_0^1 (\int_0^x y dy) dx)}_{x<y} + \underbrace{\int_0^1 (\int_0^y x dx) dy)}_{x>y} + \frac{1}{2} \\
&= \int_0^1 \frac{x^2}{2} dx + \int_0^1 \frac{y^2}{2} dy + \frac{1}{2} \\
&= 5/6.
\end{aligned}$$

The expected revenue in a first price auction is

$$\begin{aligned}
E[R^I] &= E[\max\{\frac{2}{3} X_1, \frac{2}{3} X_2\}] \\
&= \frac{2}{3} E[\max\{S_1, S_2\}] + \frac{2}{3} E[T] \\
&= 7/9.
\end{aligned}$$

Thus, we conclude $E[R^{II}] > E[R^I]$.

6.7 Efficiency

An auction is said to allocate *efficiently* if the bidder with the highest *value* is awarded the object. In the context of the symmetric model with interdependent values, the winning bidder is the one with the highest *signal* in all three of the auction forms. It is important to note that the bidder with the highest signal need *not* be the one with the highest value.

Example 6.7.1 (Symmetric equilibria may be inefficient) If the valuations in a two-bidder symmetric situation are

$$v_1(x_1, x_2) = \frac{1}{3}x_1 + \frac{2}{3}x_2$$
$$v_2(x_1, x_2) = \frac{2}{3}x_1 + \frac{1}{3}x_2$$

Thus

$$v_1 > v_2 \Leftrightarrow x_2 > x_1 \text{ and } v_1 < v_2 \Leftrightarrow x_2 < x_1,$$

which implies that the bidder with the higher signal is the one with the *lower* value, so all three auctions forms, almost always, allocate the object inefficiently. The reason for this inefficiency follows from the fact that each bidder's signal has a greater influence on the other bidder's valuation than it does on his own valuation.

Here we introduce a condition which turns out to be sufficient for all three auction forms to possess symmetric equilibria that are efficient.

Definition 6.7.1 *The valuations satisfy the **single crossing condition** if for all i and $j \neq i$ and for all x,*

$$\frac{\partial v_i}{\partial x_i}(\boldsymbol{x}) \geq \frac{\partial v_j}{\partial x_i}(\boldsymbol{x})$$

The single crossing condition implies that, keeping all other signals fixed, i's valuation as a function of i's signal x_i is steeper than j's valuation. That is, the two cross at most *once*.

Proposition 6.7.1 *With symmetric, interdependent values and affiliated signals, suppose the single crossing condition is satisfied. Then the second price, English, and first price auctions all have symmetric equilibria that are efficient.*

Proof 6.7.1 (Proof of Proposition 6.7.1:) *In the symmetric model with interdependent values, the value to bidder i is written as*

$$v_i(\boldsymbol{x}) = u(x_i, \boldsymbol{x}_{-i}).$$

Let u'_1 denote the partial derivative of u with respect to its first argument and let u'_j denote the partial derivative of u with respect to its j-th argument. In the symmetric case, the

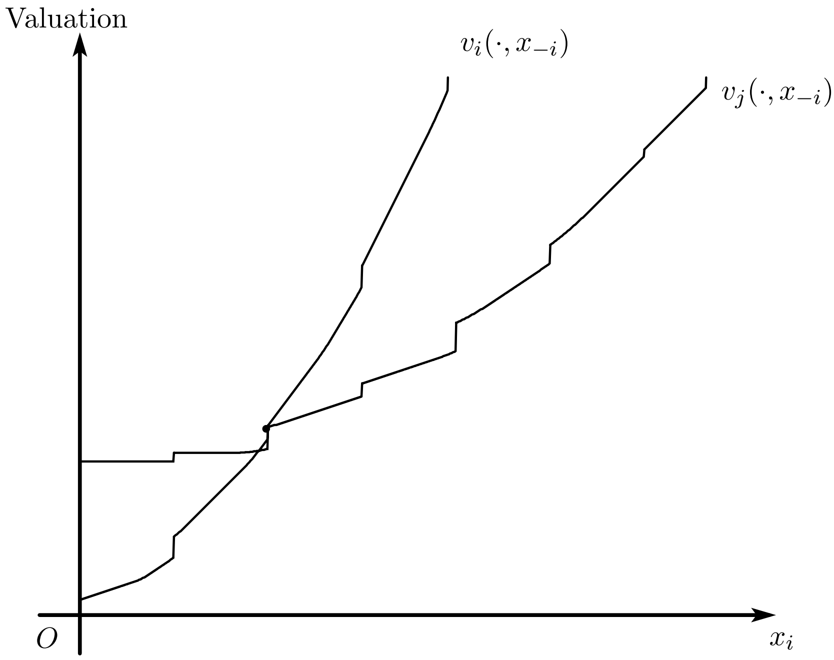

Figure 6.2: The Single Crossing Condition

single crossing condition reduces that $u_1' \geq u_j'$ for all $j \neq 1$. Since $u(.)$ is symmetric in the last $N-1$ arguments and strictly increasing its first argument, the single crossing condition implies that $u_1' > u_2'$.

The single crossing condition ensures that the ex post values of different bidders will be ordered in the same way as their signals. To see this, suppose that $x_i > x_j$, and define $\alpha(t) = (1-t)(x_j, x_i, \boldsymbol{x}_{-ij}) + t(x_i, x_j, \boldsymbol{x}_{-ij})$ to be the line joining the points $(x_j, x_i, \boldsymbol{x}_{-ij})$ and $(x_i, x_j, \boldsymbol{x}_{-ij})$. Using the fundamental theorem of calculus for line integrals, we can write

$$u(x_i, x_j, \boldsymbol{x}_{-ij}) = u(x_j, x_i, \boldsymbol{x}_{-ij}) + \int_0^1 \nabla u(\alpha(t))\alpha'(t)dt$$

where

$$\nabla u(\alpha(t))\alpha'(t) = u_1'(\alpha(t))(x_i - x_j) + u_2'(\alpha(t))(x_j - x_i) \geq 0$$

with

$$\alpha'(t) = (x_i - x_j, x_j - x_i, \underbrace{0, \ldots, 0}_{N-2})$$

since $x_i > x_j$ and $u_1' \geq u_2'$.

Then we conclude

$$u(x_i, x_j, \boldsymbol{x}_{-ij}) > u(x_j, x_i, \boldsymbol{x}_{-ij}).$$

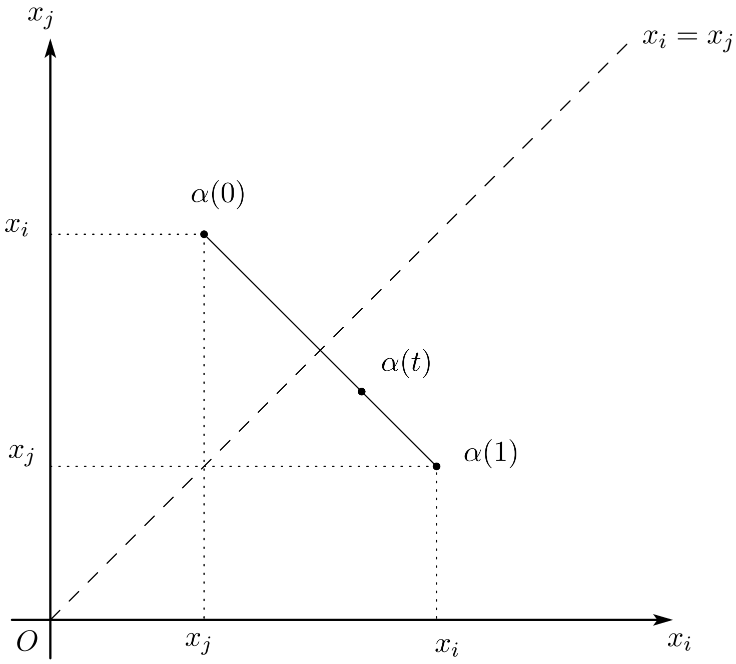

Figure 6.3: The Construction of $\alpha(t)$

This completes the proof.

6.8 The Revenue Ranking ("Linkage") Principle

Suppose A is a standard auction in which the highest bid wins the object and that it has a symmetric equilibrium, β^A. Consider bidder 1 and suppose that all other bidders follow the symmetric equilibrium strategy. Let $W^A(z, x)$ denote the expected price paid by bidder 1 if he is the *winning* bidder when his signal is x but bids as if his signal were z, i.e., bids $\beta^A(z)$.

In a first price auction, the winning bidder pays exactly what he bid, so

$$W^I(z, x) = \beta^I(z)$$

where β^I is the symmetric equilibrium strategy in the auction. In a second price auction, the amount he will have to pay is uncertain, so the expected payment upon winning is

$$W^{II}(z, x) = E[\beta^{II}(Y_1)|X_1 = x, Y_1 < z]$$

where β^{II} is the symmetric equilibrium strategy in the second price auction.

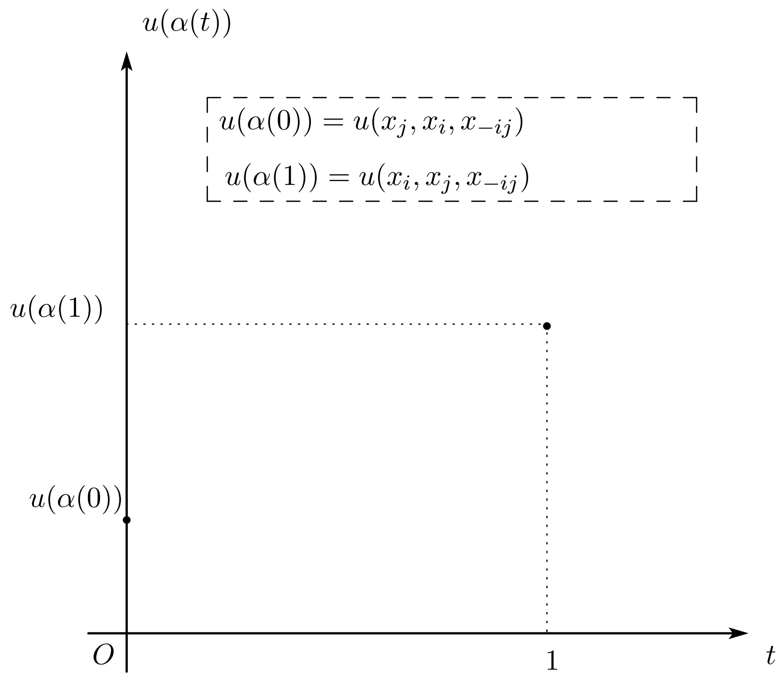

Figure 6.4: $u(x_i, x_j, \mathbf{x}_{-ij}) > u(x_j, x_i, \mathbf{x}_{-ij})$ if $x_i > x_j$.

Let $W_2^A(z, x)$ denote the partial derivative of the function $W^A(.,.)$ with respect to its second argument, evaluated at the point (z, x). We propose the revenue ranking principle below.

Proposition 6.8.1 (The Revenue Ranking Principle) *Let A and B be two auctions in which the highest bidder wins and only he pays a positive amount. Suppose that each has a symmetric and increasing equilibrium such that (i) for all x, $W_2^A(x, x) \geq W_2^B(x, x)$; (ii) $W^A(0, 0) = 0 = W^B(0, 0)$. Then the expected revenue in A is at least as large as the expected revenue in B.*

Proof 6.8.1 (Proof of the Revenue Ranking Principle:) *Consider auction A and suppose that all bidders $j \neq 1$ follow the symmetric equilibrium strategy β^A. The probability that bidder 1 with signal x who bids $\beta^A(z)$ will win is*

$$G(z|x) \equiv Prob\{Y_1 < z | X_1 = x\}.$$

Thus, each bidder in auction A maximizes

$$\int_0^z v(x, y) g(y|x) dy - G(z|x) W^A(z, x).$$

In equilibrium it is optimal to choose $z = x$, so the relevant first-order condition is

$$g(x|x)v(x,x) - g(x|x)W^A(x,x) - G(x|x)W_1^A(x,x) = 0,$$

where W_1^A denotes the partial derivative of W^A with respect to its first argument. This can be rearranged so that

$$W_1^A(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^A(x,x)$$

Similarly,

$$W_1^B(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^B(x,x)$$

and hence

$$W_1^A(x,x) - W_1^B(x,x) = -\frac{g(x|x)}{G(x|x)}[W^A(x,x) - W^B(x,x)] \ (*)$$

Define

$$\Delta(x) = W^A(x,x) - W^B(x,x)$$

so that

$$\Delta'(x) = [W_1^A(x,x) - W_1^B(x,x)] + [W_2^A(x,x) - W_2^B(x,x)] \ (**)$$

Using (*) and (**) yields

$$\Delta'(x) = -\frac{g(x|x)}{G(x|x)}\Delta(x) + \underbrace{[W_2^A(x,x) - W_2^B(x,x)]}_{\geq 0} \ (***)$$

Observe that equation $(***)$ says that, for any x, $\Delta(x) < 0 \Rightarrow \Delta'(x) \geq 0$. What we want to show is $\Delta(x) \geq 0$ for all x. We argue by contradiction. Suppose not, there exists $x > 0$ such that $\Delta(x) < 0$ because we know that $\Delta(0) = 0$, which follows from our hypothesis that $W^A(0,0) = W^B(0,0)$. Since $\Delta(.)$ is continuous (because $\Delta(.)$ is differentiable), by the mean value theorem, there must exist x' with $0 < x' < x$ for which $\Delta(x') < 0$ and $\Delta'(x') < 0$, which contradicts the confirmed condition that $\Delta(x) < 0 \Rightarrow \Delta'(x) \geq 0$.

6.8.1 First Price versus Second Price Auctions

For the first price auction,

$$W^I(z,x) = \beta^I(z)$$

which is a function of z not x, where β^I is the symmetric equilibrium bidding strategy, and thus

$$W_2^I(x, x) = 0.$$

Also,

$$\begin{aligned} W^{II}(z, x) &= E[\beta^{II}(Y_1)|X_1 = x, Y_1 < z] \\ &= E[v(Y_1, Y_1)|X_1 = x, Y_1 < z] \end{aligned}$$

where β^{II} is the symmetric equilibrium bidding strategy in the second price auction. Since β^{II} is increasing, affiliation implies that

$$W_2^{II}(x, x) \geq 0$$

for all x. Thus, from the revenue ranking principle, we conclude that the revenue from the second price auction is no less than that of the first price auction. Namely,

$$E[R^{II}] \geq E[R^I]$$

6.9 The Extended Revenue Ranking Principle

The revenue ranking principle we have established applies to auctions in which only the winner pays a positive amount. In particular, it does not apply to all-pay auctions and war of attrition auctions. Here we shall extend the revenue ranking principle into such auctions.

Let $M^A(z, x)$ be the expected payment by a bidder with signal x who bids as if his signal were z in an auction A. Recall that for an all-pay auction, when $v(., y)g(y|.)$ is increasing for all y, $M^{AP}(z, x) = \beta^{AP}(z)$ with

$$\beta^{AP}(x) = \int_0^x v(y, y)g(y|y)dy$$

is a symmetric and increasing equilibrium bidding strategy.

For auctions in which only the winner pays, we define $M^A(z, x) = G(z|x)W^A(z, x)$, where $G(.|x)$ is a probability distribution of Y_1 conditional on x. So, in a first price auction, $M^I(z, x) = G(z|x)\beta^I(z)$.

Proposition 6.9.1 *Let A and B be two standard auctions in which the highest bidder wins. Suppose that each has a symmetric and increasing equilibrium such that (i) $M_2^A(x,x) \geq M_2^B(x,x)$ for all x; (ii) $M^A(0,0) = 0 = M^B(0,0)$. Then the expected revenue in A is at least as large as the expected revenue in B.*

Proof 6.9.1 (Proof of Proposition 6.9.1:) *The expected payoff of a bidder with signal x who bids $\beta^A(z)$ is*

$$\int_0^z v(x,y)g(y|x)dy - M^A(z,x).$$

In equilibrium, it is optimal to choose $z = x$ and the resulting first-order conditions imply

$$M_1^A(x,x) = v(x,x)g(x|x) \ (*)$$

It is important to notice that the above expression of $M_1^A(x,x)$ does not depend upon the auction form A. Define $\Delta(x) = M^A(x,x) - M^B(x,x)$. Using $(*)$, we deduce

$$\begin{aligned}
\Delta'(x) &= [M_1^A(x,x) - M_1^B(x,x)] + [M_2^A(x,x) - M_2^B(x,x)] \\
&= v(x,x)g(x|x) - v(x,x)g(x|x)] + [M_2^A(x,x) - M_2^B(x,x)] \\
&= M_2^A(x,x) - M_2^B(x,x) \geq 0 (\because \textit{by our hypothesis})
\end{aligned}$$

Since $\Delta(0) = 0$ again, by our hypothesis, we conclude that $\Delta(x) \geq 0$ for all x, as desired.

6.9.1 Ranking All-Pay Auctions

Note that in a first price auction

$$M^I(z,x) = G(z|x)\beta^I(x)$$

so

$$M_2^I(z,x) = \frac{\partial}{\partial x}[G(z|x)\beta^I(x)] < 0$$

since affiliation implies that the first order stochastic dominance (FOSD), which thus implies that $G(z|x)$ is decreasing in x for all z.

Suppose that there is a symmetric, increasing equilibrium in the all-pay auction β^{AP}. Then,

$$M^{AP}(z,x) = \beta^{AP}(z)$$

so that

$$M_2^{AP}(z,x) = \frac{\partial}{\partial x}\beta^{AP}(z) = 0.$$

Since $M_2^{AP}(x,x) > M_2^I(x,x)$, the extended revenue ranking principle implies that the expected revenue from an all-pay auction is greater than that from a first price auction, *provided that the all-pay auction has an increasing equilibrium.*

Proposition 6.9.2 *Suppose that $v(.,y)g(y|.)$ is increasing for all y. Then, the expected revenue from an all-pay auction is at least as great as that from a first price auction.*

Chapter 7

Asymmetries and Other Complications

7.1 The Symmetry Assumption

So far we have discussed auctions with interdependent values and affiliated signals in which the symmetry assumption appears in one form or another in a number of places. In the next step we would like to extend all the analysis into asymmetric situations. Before doing that, we summarize a set of various symmetric assumptions.

- All bidders have the same value function: $v_i(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ for all $i \in \mathcal{N}$.

- Other bidders' signals matter in terms of payoff only up to its distribution: Define $\pi : \mathcal{N} \setminus \{i\} \to \mathcal{N} \setminus \{i\}$ as a permutation of all the bidders' names but i's. For any $i \in \mathcal{N}$ and $(x_i, x_{-i}) \in [0, \omega]^N$, we have $v_i(x_i, x_{-i}) = v_i(x_i, x_{-\pi(i)})$ for any permutation π. Here $x_{-\pi(i)} = \times_{j \neq i} x_{\pi(j)}$.

- Each bidder's belief about other bidders' signals matters only up to its distribution: Let $f : [0, \omega]^N \to \mathbb{R}$ be a joint density function of all bidders' signals. Define $\sigma : \mathcal{N} \to \mathcal{N}$ as a permutation of all bidders' names. Then $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any permutation σ.

- All bidders use the same strategy: Let $\beta_i : [0, \omega] \to \mathbb{R}$ be bidder i's equilibrium bidding strategy. Then, there exists $\beta : [0, \omega] \to \mathbb{R}$ such that $\beta_i(x) = \beta(x)$ for any $x \in [0, \omega]$ and $i \in \mathcal{N}$.

7.2 Asymmetric Uniform Distributions

Suppose bidder 1's value X_1 is uniformly distributed on $[0, 1/(1-\alpha)]$ and bidder 2's value X_2 is uniformly distributed on $[0, 1/(1+\alpha)]$ and $\alpha \in [0, 1)$. Then $F_1(x) = (1-\alpha)x$ and $F_2(x) = (1+\alpha)x$ and $f_1(x) = 1-\alpha$ and $f_2(x) = 1+\alpha$. Note that F_1 first order stochastically dominates (FOSD) F_2. To see why, you should draw the graph of the distribution functions. In other words, bidder 1 is stronger than bidder 2. Recall that $\alpha = 0$ corresponds to the symmetric case.

7.2.1 Second Price Auctions

It is still a dominant strategy for each bidder to bid his valuation, i.e., $\beta_i(x) = x$ for $i = 1, 2$. Given the assumptions we have made, it is easy to check that (please do it as an exercise)

$$E[R^{II}_{\alpha=0}] > E[R^{II}_{\alpha>0}].$$

7.2.2 First Price Auctions

Let β_1 and β_2 be the equilibrium bidding strategy of bidder 1 and bidder 2 in the first price auction, respectively. Assume further that these are increasing and differentiable and have inverses $\phi_1 \equiv \beta_1^{-1}$ and $\phi_2 \equiv \beta_2^{-1}$, respectively. Note that in Chapter 3 we established the two claims:

- $\beta_1(0) = \beta_2(0) = 0$.
- $\beta_1(\omega_1) = \beta_2(\omega_2) = \bar{b}$.

The analysis developed in Krishna's book (pp. 49-51) reveals that the equilibrium bidding strategy of a first price auction is as follows:

$$\phi_i(b) = \frac{2b}{1 + k_i b^2}, i = 1, 2$$

with

$$k_i = \frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}$$

and

$$\overline{b} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}.$$

7.2.3 Revenue in the First Price Auction

Since $\omega_1 = 1/(1-\alpha)$ and $\omega_2 = 1/(1+\alpha)$, the highest amount that either bidder bids is $\overline{b} = 1/2$. Moreover, we have that $k_1 = -4\alpha$ and $k_2 = 4\alpha$. Please check this yourself. The inverse equilibrium bidding strategies are: for all $b \in [0, 1/2]$,

$$\phi_1(b) = \frac{2b}{1 - 4\alpha b^2}$$
$$\phi_2(b) = \frac{2b}{1 + 4\alpha b^2}$$

The distribution of the equilibrium prices in a first price auction is

$$L_\alpha^I(p) = Prob\{\max\{\beta_1(X_1), \beta_2(X_1)\} \leq p\}$$

where $p \in [0, 1/2]$. We execute the following computation.

$$\begin{aligned}
L_\alpha^I(p) &= Prob\{\beta_1(X_1) \leq p\} \times Prob\{\beta_2(X_2) \leq p\} \\
&= F_1(\phi_1(p)) \times F_2(\phi_2(p)) \\
&= (1-\alpha)\frac{2p}{1 - 4\alpha p^2} \times (1+\alpha)\frac{2p}{1 + 4\alpha p^2} \\
&= \frac{(1-\alpha^2)(2p)^2}{1 - \alpha^2(2p)^4}
\end{aligned}$$

It can be checked that

$$\frac{\partial L_\alpha^I(p)}{\partial \alpha} < 0, \forall p$$

So $L_{\alpha>0}^I$ first order stochastically dominates (FOSD) $L_{\alpha=0}^{II}$, which thus implies

$$E[R_{\alpha>0}^I] > E[R_{\alpha=0}^I].$$

With the help of the revenue equivalence principle, we conclude

$$E[R_{\alpha>0}^I] > E[R_{\alpha=0}^I] = E[R_{\alpha=0}^{II}] > E[R_{\alpha>0}^{II}]$$

This shows that the expected revenue from an asymmetric first price auction is greater than that from an asymmetric second price auction.

7.3 Failure of the Revenue Ranking between English and Second Price Auctions

Recall that the revenue comparison between the English auctions and the second price auctions:

$$E[R^{Eng}] \geq E[R^{II}]$$

This is true in environments with affiliated signals in which the symmetric signal, valuation, and bidding strategy are considered. This revenue comparison does not extend to the case of asymmetric bidders. Since the two auctions are strategically equivalent when there are only two bidders or/and values are private, we necessarily have to consider the auctions with interdependent values in which there are at least three bidders so as to derive the failure of the revenue ranking between English and second price auctions.

Example 7.3.1 *With asymmetric bidders, the expected revenue in a second price auction may exceed that in an English auction.*

Suppose that there are three bidders. Bidder 1 and 2 attach a common value to the object, whereas bidder 3 has private values. Specifically,

$$\begin{aligned} v_1(x_1, x_2, x_3) &= \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ v_2(x_1, x_2, x_3) &= \frac{1}{2}x_1 + \frac{1}{2}x_2 \\ v_3(x_1, x_2, x_3) &= x_3 \end{aligned}$$

Assume further that X_1,X_2, and X_3 are independently and uniformly distributed on $[0, 1]$.

7.3.1 Equilibrium and Revenues in a Second Price Auction

Since he has private values, it is a weakly dominant strategy for bidder 3 to bid her value. Let β denote the bidding strategy for bidders 1 and 2 (by symmetry we can suppose that they use the same strategy) and suppose that β is increasing and continuous. Armed with some foreknowledge, let us suppose that for $i = 1, 2$, $\beta(x_i) = kx_i$ where $k > 0$ is a constant (You might ask: How do I know that the equilibrium strategy is linear? The answer is we don't know. Here I just propose the linear strategy as a candidate for the equilibrium and thereafter confirm that it is indeed the equilibrium. So, hang on for the moment.).

Given that bidder 2 bids according to $\beta(x_2) = kx_2$ and bidder 3 bids her value x_3, the price that bidder 1 pays upon winning is $\max\{kX_2, X_3\}$. The expected payoff of bidder 1 when his signal is x_1 and he bids b is

$$\Pi_1(b, x_1) = \int_0^{b/k} [\underbrace{\int_0^{kx_2} (\frac{x_1 + x_2}{2} - kx_2)dx_3}_{kX_2 > X_3} + \underbrace{\int_{kx_2}^{b} (\frac{x_1 + x_2}{2} - x_3)dx_3]}_{kX_2 < X_3}$$

It may be verified that

$$\Pi_1(b, x_1) = \frac{1}{12k^2}(6x_1 b^2 k + 3b^3 - 8b^3 k).$$

Maximizing this with respect to b yields the first-order condition below:

$$12x_1 bk + 9b^2 - 24b^2 k = 0$$

which results $b = \frac{4k}{8k-3}x_1$. Hence, the optimal bidding strategy of bidder 1 is a linear function in x_1. When we set $k = 7/8$, both bidder 1 and 2 use the same strategy $\beta(x) = 7x/8$. This means that there was no loss of generality to assume that bidder 1 and 2 uses the same "linear" strategy for establishing the symmetric equilibrium.

Thus, it is an equilibrium for both bidders 1 and 2 to bid k times their values and for bidder 3 to bid her value. The price is then the second highest of kX_1, kX_2,X_3, and its distribution is easily computed to be as follows:

First, we calculate the following conditional probability shown in the table below.

Price	The probability conditional on the 2nd highest bid
kx_1	$Prob\{kx_2 > kx_1\}Prob\{x_3 < kx_1\} + Prob\{kx_2 < kx_1\}Prob\{x_3 > kx_1\}$
kx_2	$Prob\{kx_1 > kx_2\}Prob\{x_3 < kx_2\} + Prob\{kx_1 < kx_2\}Prob\{x_3 > kx_2\}$
x_3	$Prob\{kx_1 > x_3\}Prob\{kx_2 < x_3\} + Prob\{kx_1 < x_3\}Prob\{kx_2 > x_3\}$

Then, the distribution of price is computed as follows: for any $p \leq k$,

$$
\begin{aligned}
L^{II}(p) &\equiv Prob\{R^{II} \leq p\} \\
&= \int_0^{p/k} \underbrace{(\int_{x_1}^1 dx_2 \int_0^{kx_1} dx_3)}_{kX_2>kX_1>X_3} dx_1 + \int_0^{p/k} \underbrace{(\int_0^{x_1} dx_2 \int_{kx_1}^1 dx_3)}_{X_3>kX_1>kX_2} dx_1 \\
&+ \int_0^{p/k} \underbrace{(\int_{x_2}^1 dx_1 \int_0^{kx_2} dx_3)}_{kX_1>kX_2>X_3} dx_2 + \int_0^{p/k} \underbrace{(\int_0^{x_2} dx_1 \int_{kx_2}^1 dx_3)}_{X_3>kX_2>kX_1} dx_2 \\
&+ \int_0^{p} \underbrace{(\int_{x_3/k}^1 dx_1 \int_0^{x_3/k} dx_2)}_{kX_1>X_3>kX_2} dx_3 + \int_0^{p} \underbrace{(\int_0^{x_3/k} dx_1 \int_{x_3/k}^1 dx_3)}_{kX_2>X_3>kX_1} dx_3
\end{aligned}
$$

With algebra calculation we can get

$$L^{II}(p) = \frac{p^2 + 2kp^2 - 2p^3}{k^2}.$$

Thus the expected revenue

$$E[R^{II}] = \int_0^k p dL^{II}(p) = \frac{175}{384}.$$

7.3.2 Equilibrium and Revenues in an English Auction

Since he has private values, it is a weakly dominant strategy for bidder 3 to drop out at his value regardless of the history and who else is active. The following strategies constitute an *ex post* equilibrium:

active bidders	$\{1,2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
1	x_1	x_1	$\frac{1}{2}x_1 + \frac{1}{2}x_2$	NA
2	x_2	x_2	NA	$\frac{1}{2}x_1 + \frac{1}{2}x_2$
3	x_3	NA	x_3	x_3

The table indicates the price at which each bidder should drop out given the set of active bidders and the previous history of exits. Thus, for instance, bidder 1 should drop out at a price $p = x_1$ if the set of active bidders is $\{1,2,3\}$; at $p = x_1$ if the set of active bidders is $\{1,2\}$; and at $p = 1/2x_1 + 1/2x_2$ if the set of active bidders is $\{1,3\}$ and bidder 2 dropped out at a price $p_2 = x_2$, so that his equilibrium strategy revealed his signal to bidders 1 and 3. This equilibrium is also completely analogous to the equilibrium of the English auction in the symmetric setting of the previous chapter. It is an ex post equilibrium and, given any history, each bidder stays in until the break-even price p such that if all other active bidders were to drop out at p, the value of the object to i would be exactly p.

Figure 7.2 depicts the resulting equilibrium outcomes. Bidder 3's signal is held fixed at some $X_3 = x_3 < 1/2$ and in the different regions of (x_1, x_2) space, both the winner's identity - the circled numbers - and the price he or she pays is indicated. Thus, for instance, in the right-hand quadrilateral region, bidder 1 is a winner. When $x_2 > x_3$ - above the dotted line - the price is x_2 and when $x_2 < x_3$ - below the dotted line - the price is x_3. A similar figure can be drawn for the case where $X_3 = x_3 \geq 1/2$.

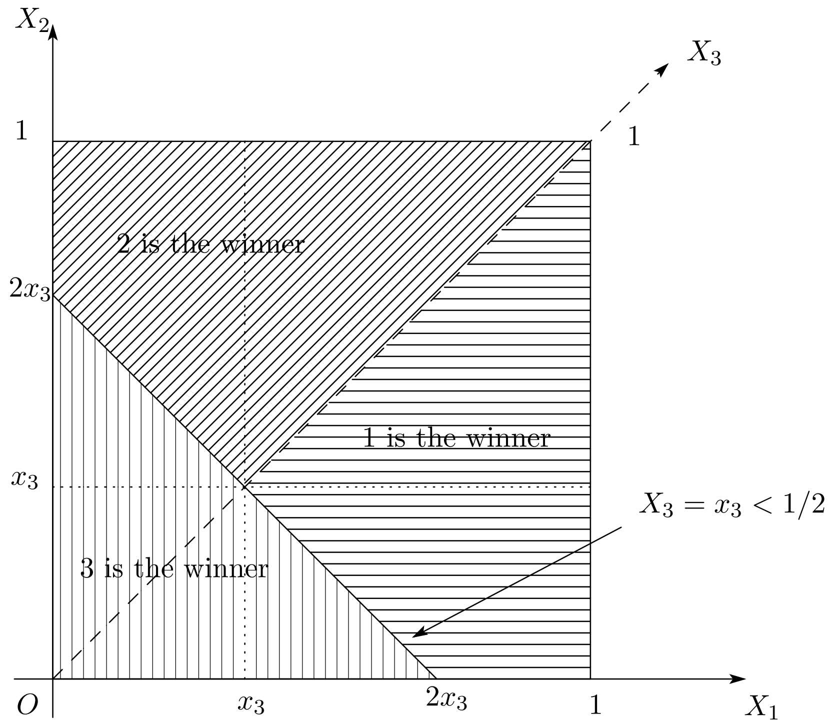

Figure 7.1: Who the Winner is When $X_3 = x_3 < \frac{1}{2}$

To compute the expected revenue in the English auction, it is convenient to derive the expected payments of each bidder separately and then find the expected revenue as the sum of these payments. Bidder 1's expected payment is computed as follows. Bidder 1 wins if and only if

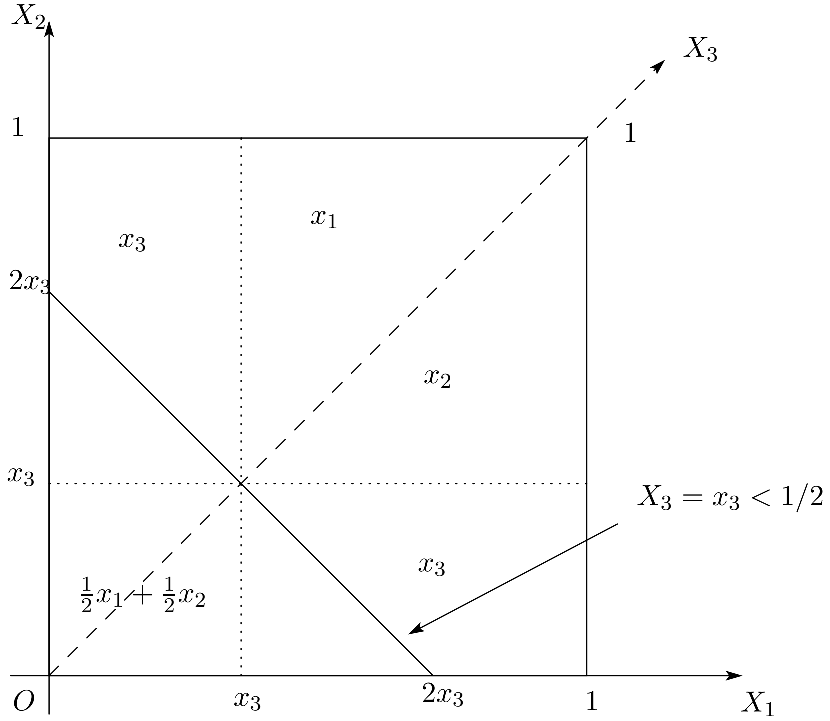

Figure 7.2: The Equilibrium Price when $X_3 = x_3 < \frac{1}{2}$

1. he is not the first to drop out when all the bidders are active, i.e., $x_1 > \min\{x_2, x_3\}$;

2. if $x_2 > x_3$, so that bidder 3 drops out first, then $x_1 > x_2$ and the price is x_2; and

3. if $x_2 < x_3$, so that bidder 2 drops out first, then $x_1/2 + x_2/2 > x_3$ and the price is x_3.

From the seller's point of view, bidder 1's expected payment is

$$m_1 = \int_0^1 \int_0^{x_1} [\underbrace{\int_0^{x_2} x_2 dx_3}_{\{1,2\}} + \underbrace{\int_{x_2}^{\frac{1}{2}x_1+\frac{1}{2}x_2} x_3 dx_3}_{\{1,3\}}] dx_2 dx_1 = \frac{11}{96}.$$

By symmetry, bidder 2's ex ante expected payment is the same as bidder 1's. So, $m_2 = \frac{11}{96}$.

Finally, consider Bidder 3. Bidder 3 wins if and only if $x_3 > x_1/2 + x_2/2$, and in that case, the price is $x_1/2 + x_2/2$. Bidder 3's ex ante expected payment when his signal is x_3 is

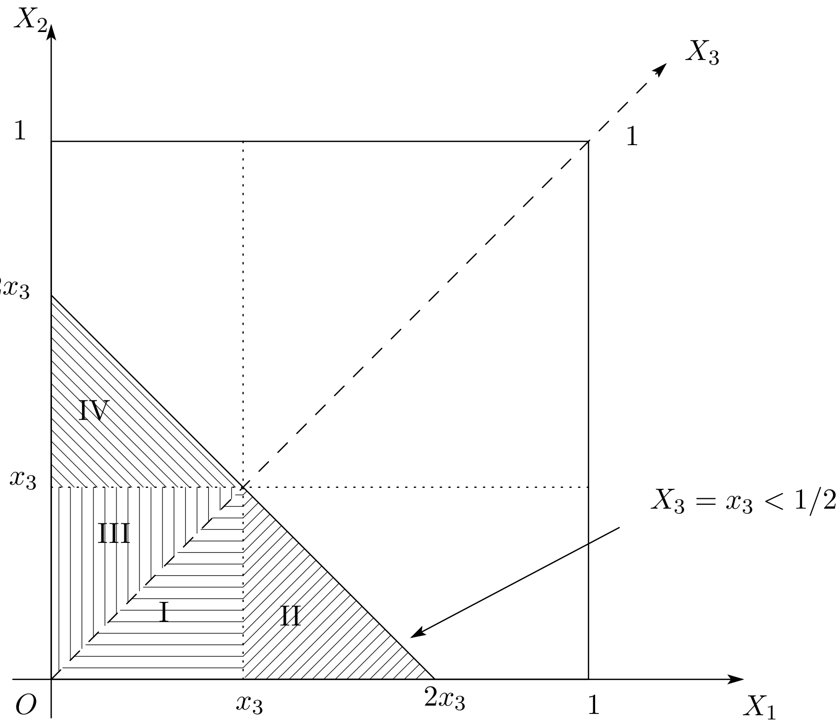

Figure 7.3: The region where Bidder 3 is the winner

$$\begin{aligned} m_3(x_3) &= \int_0^{x_3} \overbrace{\int_0^{x_1} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_2}^{\{1,3\}} dx_1 + \int_{x_3}^{\min\{2x_3,1\}} \overbrace{\int_0^{2x_3-x_1} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_2}^{\{1,3\}} dx_1 \\ &+ \int_0^{x_3} \overbrace{\int_0^{x_2} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_1}^{\{2,3\}} dx_2 + \int_{x_3}^{\min\{2x_3,1\}} \overbrace{\int_0^{2x_1-x_2} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_1}^{\{2,3\}} dx_2 \\ &= 2\int_0^{x_3} \int_0^{x_1} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_2 dx_1 + 2\int_{x_3}^{\min\{2x_3,1\}} \int_0^{2x_3-x_1} (\frac{1}{2}x_1 + \frac{1}{2}x_2)dx_2 dx_1 \end{aligned}$$

and so

$$m_3 = \int_0^1 m_3(x_3)dx_3 = \frac{5}{24}.$$

The total revenue in an English auction is

$$m_1 + m_2 + m_3 = \frac{7}{16},$$

and so

$$E[R^{Eng}] < E[R^{II}].$$

Chapter 8

Efficiency and the English Auction

With private values, of course, the English auction – and, in that setting, the second-price auction as well – always allocates efficiently. If every bidder adopts the equilibrium strategy calling for him to drop out when the price reaches his privately known value, the bidder with the highest value will win the object. With interdependent values, however, the question of efficiency is more delicate. In this setting, bidders cannot behave in the same manner as noted earlier - dropping out when the price reaches the value - since they have only partial information about their own values. At best, they can drop out when the price reaches the *estimated* value. Efficiency requires that the person with the highest *actual* (realized) value obtain the object; and when each bidder is only partly informed, this is a strong requirement.

We examine circumstances under which in the general model with interdependent and possibly asymmetric values, the English auction, nevertheless, allocates efficiently. It is assumed that the valuations $v_i(\mathbf{x})$ are continuously differentiable functions of all the signals and that $v_i(\mathbf{0}) = 0$. Furthermore, we have assumed that for all i and j,

$$\frac{\partial v_i}{\partial x_j} \geq 0$$

with a strict inequality when $i = j$. In this chapter, the joint distribution of bidders' signals, f, will play no further role since we will be concerned with *ex post* equilibria and these depend only on the valuation functions v_i.

8.1 The Single Crossing Condition

The single crossing condition embodies the notion that a bidder's own information has a greater influence on his own value than it does on some other bidder's value.

Definition 8.1.1 *For a given profile of signal $\boldsymbol{x}$, the winners circle $\mathcal{I}(\boldsymbol{x})$ is the set of bidders satisfying the following property:*

$$i \in \mathcal{I}(\boldsymbol{x}) \Leftrightarrow V_i(\boldsymbol{x}) = \max_{j \in \mathcal{N}} V_j(\boldsymbol{x}).$$

Thus, the object is allocated efficiently at $\mathbf{x}$, if the person it goes to - the winner - belongs to the winners circle $\mathcal{I}(\mathbf{x})$. The valuations $v(.)$ satisfy the pairwise *single crossing condition* if at any $\mathbf{x}$ with $\#\mathcal{I}(\mathbf{x}) \geq 2$, for all $i, j \in \mathcal{I}(\mathbf{x})$,

$$\frac{\partial v_j}{\partial x_j}(\mathbf{x}) > \frac{\partial v_i}{\partial x_j}(\mathbf{x})$$

at every $\mathbf{x}$ such that $v_i(\mathbf{x}) = v_j(\mathbf{x}) = \max_{k \in \mathcal{N}} v_k(\mathbf{x})$, so that the values of i and j are equal and maximal. It will be convenient to denote the partial derivative of bidder i's valuation with respect to bidder j's signal by v'_{ij}, that is,

$$v'_{ij}(\mathbf{x}) \equiv \frac{\partial v_i}{\partial x_j}(\mathbf{x}).$$

In this notation, the *single crossing condition* can be written as $v'_{jj}(\mathbf{x}) > v'_{ij}(\mathbf{x})$.

The single crossing condition is depicted in Figure 8.1. In the left-hand panel, the slope of v_j with respect to x_j is greater than that of v_i whenever they have the same value. The right-hand panel depicts the iso-value or "indifference curves of the two bidders, both at the same level, p. Whenever they intersect, bidder j's indifference curve is steeper, so they can cross at most once.

Note that the condition above is weaker than the condition introduced in Chapter 6 since the inequality above is required to hold only at points $\mathbf{x}$ where the values v_i and v_j are the same and maximal.

8.2 Two-Bidder Auctions

When there are only two bidders, the single crossing condition is sufficient to guarantee the existence of an efficient equilibrium in an English auction.

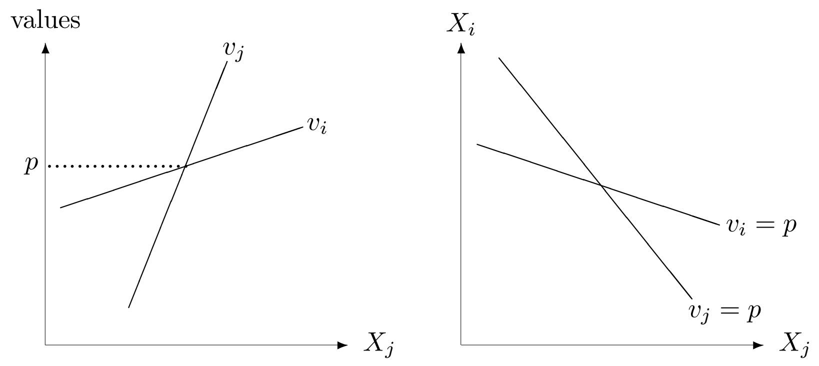

Figure 8.1: Single Crossing Condition

Proposition 8.2.1 *Suppose that the valuations v satisfy the single crossing condition. Then, there exists an ex post equilibrium of the two-bidder English auction that is efficient.*

Proof 8.2.1 (Proof of Proposition 8.2.1:) *Suppose that there exist continuous and increasing functions ϕ_1 and ϕ_2 such that for all $p \leq \min_i \phi_i(\omega_i)$, these solve the following pair of equations:*

$$v_1(\phi_1(p), \phi_2(p)) = p$$
$$v_2(\phi_1(p), \phi_2(p)) = p \ (*)$$

If we define $\beta_i : [0, \omega_i] \to \mathbb{R}_+$, setting $\beta_i = \phi_i^{-1}$ guarantees that ϕ_1^{-1} and ϕ_2^{-1} form an equilibrium.

Step 1: Existence of Ex Post Equilibrium

Suppose that there exists such a solution to the equations $(*)$ and, without loss of generality, suppose that $\beta_1(x_1) = p_1 > p_2 = \beta_2(x_2)$. Then, $(*)$ implies

$$v_1(\phi_1(p_2), \phi_2(p_2)) = p_2$$

and since $x_1 = \phi_1(p_1) > \phi_1(p_2)$ and $\phi_2(p_2) = x_2$,

$$v_1(x_1, x_2) > p_2$$

because, by assumption, $v'_{11} > 0$. This implies that the winning bidder makes an ex post *profit when he wins and since he cannot affect the price he pays, he cannot do better. It is also the case that*

$$v_2(\phi_1(p_1), \phi_2(p_1)) = p_1$$

and since $\phi_2(p_1) > \phi_2(p_2) = x_2$ *and* $\phi_1(p_1) = x_1$,

$$v_2(x_1, x_2) < p_1$$

because $v'_{22} > 0$. *This implies that the losing bidder has no incentive to raise his bid since if he were to do so and win the auction, it would be at a price that is too high. Thus, if there is an increasing solution to* $(*)$, *there exists an* ex post *equilibrium*

Step 2: Efficiency

The equilibrium constructed here is efficient because from $(*)$

$$v_1(\phi_1(p_2), \phi_2(p_2)) = v_2(\phi_1(p_2), \phi_2(p_2))$$

and again since $x_1 = \phi_1(p_1) > \phi_1(p_2)$ *and* $\phi_2(p_2) = x_2$,

$$v_1(x_1, x_2) > v_2(x_1, x_2)$$

because $v'_{11} > v'_{21}$ *by the single crossing condition.*

In two-bidder auctions, the single crossing condition guarantees that there is a pair of continuous and increasing functions (ϕ_1, ϕ_2) *satisfying* $(*)$ *and in that case,* $(\beta_1, \beta_2) = (\phi_1^{-1}, \phi_2^{-1})$ *constitutes an efficient ex post equilibrium. We omit a proof of the existence of such functions as this is implied by a more general result to follow.*

Note that $(*)$ asks a bidder, say 1, to stay in until a price $\beta_1(x_1)$ such that if bidder 2 were to drop out at $\beta_1(x_1)$, and his signal $x_2 = \phi_2(\beta_1(x_1))$ were inferred, bidder 1 would just break even since

$$v_1(x_1, \phi_2(\beta_1(x_1))) = \beta_1(x_1).$$

The equations $(*)$ will thus be referred to as the break-even conditions.

Maskin (1992) also show that the single crossing condition is *necessary* - in a certain sense - for the existence of efficient ex post equilibrium in the English auction with two bidders.

Claim 8.2.1 *Suppose that the pairwise single-crossing condition is violated at some interior signal profile. Then, the English auction with* $n \geq 2$ *bidders does not possess an efficient ex post equilibrium.*

This claim is indicated in Maskin (1992). The following example illustrates that efficient equilibria may exists even when the single crossing condition is violated on the *boundary* of the signals' domain.

Example 8.2.1 *Consider the English auction with two bidders with value functions of the form*

$$V_1 = \frac{2}{3}x_1 + \frac{1}{3}x_2$$
$$V_2 = x_1 + x_2$$

There exists an efficient ex post equilibrium.

At the point $x_1 = x_2 = 0$, $V_1 = V_2$, the pairwise single crossing condition is violated, while at any other $\mathbf{x}$, it is vacuously satisfied. Strategies $\beta_1(x_1) = x_1$ and $\beta_2(x_2) = \infty$ (bidder 2 never drops out first) form an ex post equilibrium, which is efficient. The next example illustrates the following: When there are *three* or *more* bidders, the single crossing condition by itself is not sufficient to guarantee that the English auction is efficient.

Example 8.2.2 *With three or more bidders, there may not exists an efficient equilibrium of the English auction even if the single crossing condition is satisfied.*

$$v_1(x_1, x_2, x_3) = x_1 + 2x_2x_3 + \alpha(x_2 + x_3)$$
$$v_2(x_1, x_2, x_3) = \frac{1}{2}x_1 + x_2$$
$$v_3(x_1, x_2, x_3) = x_3$$

where $\alpha < 1/18$ is a parameter.

Let us verify that the single crossing condition is satisfied. First, consider changes in x_2. Now $v_1 = v_2$ implies that $(v'_{22} - v'_{12})x_2 = (1 - 2x_3 - \alpha)x_2 = 1/2x_1 + \alpha x_3$. If either $x_1 > 0$ or $x_3 > 0$, then $v'_{22} > v'_{12}$. On the other hand, if both $x_1 = 0$ and $x_3 = 0$, then again $v'_{22} = 1 > \alpha = v'_{12}$. Thus, whenever $v_1 = v_2$, we have $v'_{22} > v'_{12}$. Likewise, whenever $v_1 = v_3$, $v'_{33} > v'_{13}$. All other comparisons are straightforward.

Suppose, by way of contradiction, that there is an efficient equilibrium in the English auction and let β denote the strategies when all bidders are active. If x_2 and x_3 are both greater than $\frac{1}{2}$, then for all x_1, v_1 is greater than both v_2 and v_3. Efficiency requires, therefore, that when all bidders are active, bidder 1 is *never* the first to drop out.

But now consider signals x_1, x_2 and x_3 such that bidders 2 and 3 have the same value and bidder 1 has a lower value (for example, let $x_1 = \frac{1}{8}$, $x_2 = \frac{1}{4}$, and $x_3 = \frac{5}{16}$). Clearly, $\beta_2(x_2) = \beta_3(x_3)$ is impossible since then bidder 1 would win the object and that is inefficient. Suppose that $\beta_2(x_2) < \beta_3(x_3)$, so that bidder 2 drops out first. For small $\epsilon > 0$, if bidder 1's signal is $x_1 + \epsilon$, bidder 2 has the highest value and it is inefficient for her to drop out. On the other hand, suppose $\beta_2(x_2) > \beta_3(x_3)$, so that bidder 3 drops out first. Now if bidder 1's signal is $x_1 - \epsilon$, bidder 3 has the highest value and it is inefficient for him to drop out. This is a contradiction, so there cannot be an efficient equilibrium.

8.3 The Average Crossing Condition

The single crossing condition is a *bilateral* condition - it is separately applied to pairs of bidders - and, as we have seen, is not sufficient to guarantee that the English auction has an efficient equilibrium once there are three or more bidders. We now introduce a *multilateral* extension of the single crossing condition, called the *average crossing condition*, that links the valuations of all the bidders more closely and guarantees the existence of an efficient equilibrium in the English auction for any number of bidders.

For any subset of bidders $\mathcal{A} \subset \mathcal{N}$, define

$$\overline{v}_{\mathcal{A}}(\mathbf{x}) = \frac{1}{|\mathcal{A}|} \sum_{i \in \mathcal{A}} v_i(\mathbf{x})$$

to be the average of the values of the bidders in $\mathcal{A}$ when the signals are $\mathbf{x}$. The average crossing condition is just a single crossing condition between a bidder's value v_i and the average value $\overline{v}_{\mathcal{A}}$ with respect to signals x_j of other bidders $j \in \mathcal{A}$.

The valuations v are said to satisfy the *average crossing condition* if for all $\mathcal{A} \subset \mathcal{N}$, for all $i, j \in \mathcal{A}$ with $i \neq j$,

$$\frac{\partial \overline{v}_{\mathcal{A}}}{\partial x_j}(\mathbf{x}) > \frac{\partial v_i}{\partial x_j}(\mathbf{x})$$

at every $\mathbf{x}$ such that for all $\ell \in \mathcal{A}$, $v_\ell(\mathbf{x}) = \max_{k \in \mathcal{N}} v_k(\mathbf{x})$, so that the values of bidders in $\mathcal{A}$ are maximal.

The average crossing condition requires that the influence of any bidder's signal on some other bidder's value be smaller than its influence on the average of all the bidder's

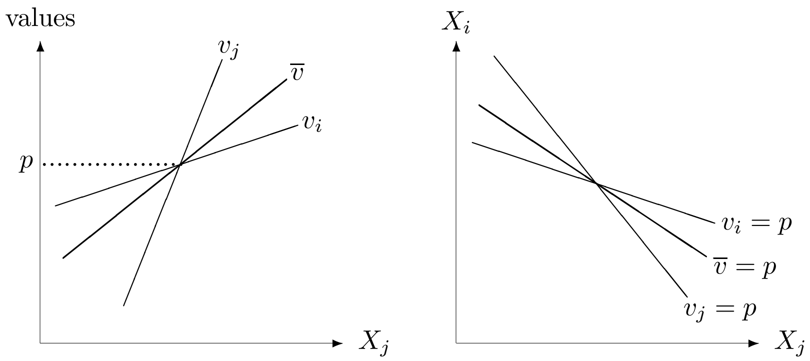

Figure 8.2: Average Crossing Condition

values. Since all influences cannot be below average, it must be that for all $i, j \in \mathcal{A}$ with $i \neq j$,

$$\frac{\partial v_j}{\partial x_j}(\mathbf{x}) > \frac{\partial \overline{v}_{\mathcal{A}}}{\partial x_j}(\mathbf{x}) > \frac{\partial v_i}{\partial x_j}(\mathbf{x})$$

at every $\mathbf{x}$ such that for all $\ell \in \mathcal{A}$, $v_\ell(\mathbf{x}) = \max_{k \in \mathcal{N}} v_k(\mathbf{x})$. Thus, the average crossing condition implies the single crossing condition and is equivalent to it when there are only two bidders.

8.4 Three or More Bidders

With three bidders, the single crossing and average crossing conditions can be conveniently represented as Figure 8.3 below. For $j = 1, 2, 3$, define $\mathbf{v}'_j = (v'_{1j}, v'_{2j}, v'_{3j})$ to be the vector of influences of bidder j's signal x_j on all three bidders. Suppose further that these are re-scaled so that they lie in the unit simplex Δ (the labels on the vertices, e^1, e^2 and e^3, denote the three unit vectors).

The single crossing condition requires that each $\mathbf{v}'_j \in S_j = \{\mathbf{t} \in \Delta | t_i < t_j \forall i \neq j\}$. Because of the re-scaling, the average of the elements of $\mathbf{v}'_j$ is just $1/3$ and the average crossing condition requires that each $\mathbf{v}'_j \in A_j = \{\mathbf{t} \in \Delta | t_i < 1/3, \forall i \neq j\}$.

The average crossing condition is flexible enough to accommodate both pure *private values* - that is, for all i, $v_i(\mathbf{x}) = u_i(x_i)$ - and, as a limiting case, pure *common values* - that is, for all i, $v_i(\mathbf{x}) = w(\mathbf{x})$. In the figure, these correspond to the vertices and the center of the simplex, respectively. More generally, if the valuations are additively *separable* into a

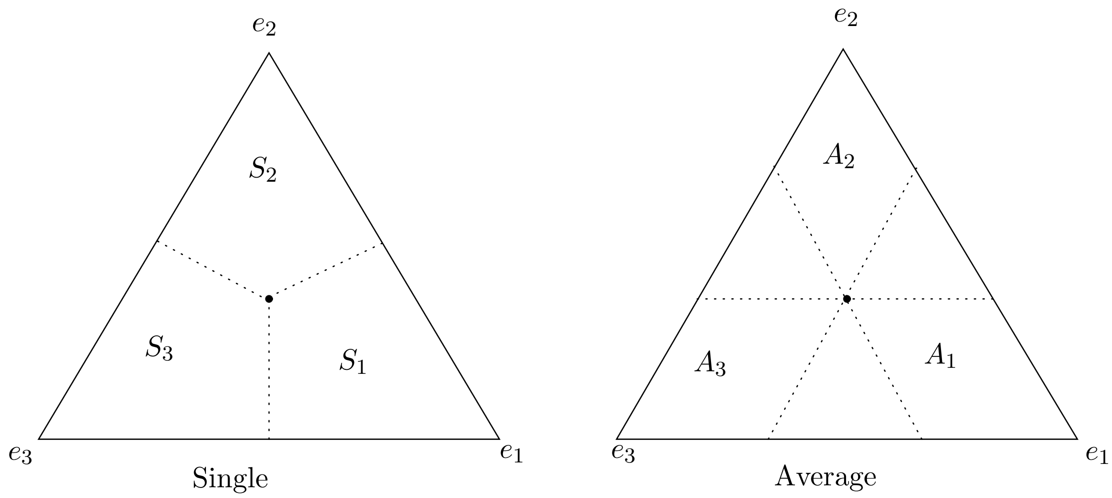

Figure 8.3: A Comparison of the Crossing Conditions

private value and a common value component - that is, for all i, $v_i(\mathbf{x}) = u_i(x_i) + w(\mathbf{x})$, where $u_i' > 0$, then the average crossing condition is satisfied.

Formally, a bidding strategy for bidder i is a collection of functions

$$\beta_i^{\mathcal{A}} : [0, \omega_i] \times \mathbb{R}_+^{\mathcal{N}\setminus\mathcal{A}} \to \mathbb{R}_+$$

where $i \in \mathcal{A} \subset \mathcal{N}$ and $|\mathcal{A}| > 1$. The function $\beta_i^{\mathcal{A}}$ determines the price $\beta_i^{\mathcal{A}}(x_i, \mathbf{P}_{\mathcal{N}\setminus\mathcal{A}})$ at which i will drop out when the set of active bidders, including i, is $\mathcal{A}$; his own signal is x_i; and the bidders in $\mathcal{N} \setminus \mathcal{A}$ have dropped out at prices $\mathbf{P}_{\mathcal{N}\setminus\mathcal{A}} = (p_j)_{j\in\mathcal{N}\setminus\mathcal{A}}$. We will require that

$$\beta_i^{\mathcal{A}}(x_i, \mathbf{P}_{\mathcal{N}\setminus\mathcal{A}}) > \max\{p_j | j \in \mathcal{N} \setminus \mathcal{A}\}.$$

Let $\beta = ((\beta_i^{\mathcal{A}})_{i\in\mathcal{A}})_{\mathcal{A}\subset\mathcal{N}}$ be the collection of all bidders' strategies. If there is an equilibrium β such that the functions $\beta_i^{\mathcal{A}}$ are increasing in x_i and bidder i drops out at some price $p_i = \beta_i^{\mathcal{A}}(x_i, \mathbf{P}_{\mathcal{N}\setminus\mathcal{A}})$, the all remaining bidders $j \neq i$ would deduce that $X_i = x_i$. In that case, with a slight abuse of notation, we will write

$$\beta_i^{\mathcal{A}}(x_i, \mathbf{x}_{\mathcal{N}\setminus\mathcal{A}}) \equiv \beta_i^{\mathcal{A}}(x_i, \mathbf{P}_{\mathcal{N}\setminus\mathcal{A}}).$$

Finally, let $\Gamma(\mathcal{A}, \mathbf{x}_{\mathcal{N}\setminus\mathcal{A}})$ denote the *sub-auction* in which the set of active bidders is $\mathcal{A} \subset \mathcal{N}$ and the signals of the bidders in the set $\mathcal{N} \setminus \mathcal{A}$, who have dropped out, are $\mathbf{x}_{\mathcal{N}\setminus\mathcal{A}}$.

Proposition 8.4.1 *Suppose that the valuations $v(.)$ satisfy the average crossing condition. Then, there exists an ex post equilibrium of the English auction that is efficient.*

Proof 8.4.1 (Proof of Proposition 8.4.1:) *The proof consists of three lemmas:*

Lemma 8.4.1 *Suppose that for all $\mathcal{A} \subset \mathcal{N}$ and for all $\boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}$, there exists a unique set of continuous and increasing functions $\phi_i : \mathbb{R}_+ \to [0, \omega_i]$ for each $i \in \mathcal{A}$ such that for all $p \leq \min_{i\in\mathcal{A}} \phi_i^{-1}(\omega_i)$ and for all $j \in \mathcal{A}$,*

$$v_j(\phi_{\mathcal{A}}(p), \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}) = p \ (*).$$

Define $\beta_i^{\mathcal{A}} : [0, \omega_i] \times \prod_{j\in\mathcal{N}\setminus\mathcal{A}}[0, \omega_j] \to \mathbb{R}_+$ by

$$\beta_i^{\mathcal{A}}(x_i, \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}) = \phi_i^{-1}(x_i).$$

Then β is an ex post equilibrium of the English auction.

Proof 8.4.2 (Proof of Lemma 8.4.1:) *Consider bidder 1, say, and suppose that all other bidders $i \neq 1$ are following the strategies $\beta_i^{\mathcal{A}}$ as specified above. Suppose that bidder 1 gets the signal x_1 but deviates and decides to drop out at some price other than $\beta_i^{\mathcal{A}}(x_i)$. We will argue that no such deviation is profitable. For purpose of exposition, the arguments that follow assume that it is never the case that two bidders drop out simultaneously at the same price.*

First, suppose that bidder 1 gets the signal x_1 and wins the object by following the strategy β_1 as prescribed above. Bidder 1 cannot affect the price he pays for the object. So, the only way that a deviation could be profitable is if winning leads to a loss for bidder 1 and the deviation causes him to drop out. Suppose that he wins the object when the set of active bidders is $\mathcal{A} = \{1, 2\}$ and bidder 2 drops out at price

$$p^* = \beta_2^{\mathcal{A}}(x_2)$$

Since the equilibrium strategies in every sub-auction are increasing, the signals of the inactive bidders can be perfectly inferred from the prices at which they dropped out. The break-even conditions $(*)$ imply that

$$v_1(\phi_1(p^*), \phi_2(p^*), \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}) = p^*$$

By definition, $\phi_2(p^*) = x_2$ and since $\beta_1^{\mathcal{A}}(x_1) > p.$ and $x_1 > \phi_1(p^*)$. Now since $v'_{11} > 0$, this implies that

$$v_1(x_1, \boldsymbol{x}_{-1}) > p^*$$

showing that in equilibrium, bidder 1 makes an ex post profit whenever he wins with a bid of $\beta_1^{\mathcal{A}}(x_1)$. Thus, any deviation that causes him to drop out is not profitable.

Second, suppose that the strategy β_1 calls on bidder 1 to drop out at some price p_1^* but bidder 1 deviates and remains active longer than $\beta_1^{\mathcal{A}}(x_1) = p_1^*$ in some sub-auction $\Gamma(\mathcal{A}, \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}})$. This makes a difference only if he stays active until all other bidders have dropped out and he actually wins the object. So, suppose this is the case and suppose, without loss of generality, that the bidders in $\mathcal{A} = \{1, 2, ..., A\}$ drop out in the order $A, (A-1), \ldots, 2$ at prices $p_A \leq p_{A-1} \leq \ldots \leq p_2$, respectively, so that bidder 1 wins the object at a price of p_2. We will argue that such a deviation cannot be profitable for him.

When bidder $j+1 \in \mathcal{A}$ drops out at price p_{j+1}, $(*)$ implies that

$$v_1(\phi_1^{j+1}(p_{j+1}), \phi_2^{j+1}(p_{j+1}), \ldots, \phi_{j+1}^{j+1}(p_{j+1}), x_{j+2}, \ldots, x_N) = p_{j+1}$$

where ϕ_i^{j+1} are the inverse bidding strategies being played when the set of active bidders is $\{1, 2, \ldots, j+1\}$, and since bidder $j+1$ drops out at p_{j+1} for which $\phi_{j+1}^{j+1}(p_{j+1}) = x_{j+1}$. But the break-even conditions when the set of active bidders is $\{1, 2, \ldots, j\}$ imply that

$$v_1(\phi_1^{j}(p_{j+1}), \phi_2^{j}(p_{j+1}), \ldots, \phi_{j}^{j}(p_{j+1}), x_{j+1}, x_{j+2}, \ldots, x_N) = p_{j+1}.$$

Thus, for all $j < A$ and $i = 1, 2, \ldots, j$,

$$\phi_i^j(p_{j+1}) = \phi_i^{j+1}(p_{j+1})$$

and since $p_j \geq p_{j+1}$, this implies that for all $j < A$ and $i = 1, 2, \ldots, j$,

$$\phi_i^j(p_j) \geq \phi_i^{j+1}(p_{j+1}) \ (**)$$

Similarly, at the last stage, when bidder 2 drops out it must be that

$$v_1(\phi_1^2(p_2), x_2, x_3, \ldots, x_N) = p_2$$

and now applying $(**)$ repeatedly when $i = 1$ results in

$$\phi_1^2(p_2) \geq \phi_1^3(p_3) \geq \ldots \phi_1^A(p_A)$$

But $p_A > p_1^*$, so $\phi_1^A(p_A) > \phi_1^A(p_1^*) = x_1$. Thus $\phi_1^2(p_2) > x_1$ and since $v_{11}' > 0$,

$$v_1(x_1, x_2, \ldots, x_N) < p_2$$

and by staying in and winning the object at a price p_2, bidder 1 makes a loss. Thus, bidder 1 cannot benefit by remaining active longer than $\beta_1^{\mathcal{A}}(x_1)$.

Finally notice that none of these arguments would be affected if the signals $\boldsymbol{x}$ were common knowledge. Thus, we have shown that β is an ex post *equilibrium.*

Lemma 8.4.2 *Suppose that the valuations $v(.)$ satisfy the average crossing condition. Then, for all $\mathcal{A} \subset \mathcal{N}$ and for all $\boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}$, there exists a unique set of differentiable and increasing functions $\phi_i : \mathbb{R}_+ \to [0, \omega_i]$ for each $i \in \mathcal{A}$ such that for all $p \leq \min_{i\in\mathcal{A}} \phi_i^{-1}(\omega_i)$ and for all $j \in \mathcal{A}$,*

$$v_j(\phi_{\mathcal{A}}(p), \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}}) = p \ (*).$$

Proof 8.4.3 (Proof of Lemma 8.4.2:) *First, consider $\mathcal{A} = \mathcal{N}$. Then, the break-even conditions $(**)$ may be compactly written as*

$$\boldsymbol{v}(\phi(p)) = p\boldsymbol{e} \ (***)$$

where $e \in \mathbb{R}^N$ is a vector of 1's. Recall that $\boldsymbol{v}(\boldsymbol{0}) = \boldsymbol{0}$, so when $p = 0$, it is possible to set $\phi(0) = \boldsymbol{0}$. Differentiating $(***)$ with respect to p results in

$$D\boldsymbol{v}'(\phi(p))\phi'(p) = \boldsymbol{e}$$

where $D\boldsymbol{v} \equiv [v_{ij}']$ is the $N \times N$ matrix of partial derivatives of $\boldsymbol{v}$ and

$$\phi'(p) \equiv (\phi_i'(p))_{i=1}^N.$$

A differentiable and increasing solution ϕ to $(***)$ exists if and only if there is an increasing solution to the system of differential equations

$$D\boldsymbol{v}(\phi)\phi' = \boldsymbol{e}$$
$$\phi(0) = \boldsymbol{0}.$$

The fundamental theorem of differential equations guarantees that there exists a unique solution ϕ to this system for all $p \leq \min_{i\in\mathcal{A}} \phi_i^{-1}(\omega_i)$.

The same argument can be applied in a sub-auction $\Gamma(\mathcal{A}, \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}})$ once the initial conditions are chosen with some care. As an example, consider the sub-auction where one of the bidders, say N, with signal x_N has dropped out. Let $\mathcal{A} = \mathcal{N} \setminus \{N\}$ and consider the sub-auction $\Gamma(\mathcal{A}, \boldsymbol{x}_{\mathcal{N}})$. From the solution to the game $\Gamma(\mathcal{N})$ as above, this must have been at a price p_N such that $\phi_N^{\mathcal{N}}(p_N) = x_N$. For all $i \in \mathcal{A}$, let $x_i = \phi_i^{\mathcal{N}}(p_N)$, where $\phi_i^{\mathcal{N}}$ are the inverse bidding strategies in $\Gamma(\mathcal{N})$. Then in the sub-auction $\Gamma(\mathcal{A}, \boldsymbol{x}_{\mathcal{N}})$, a solution to the system

$$D\boldsymbol{v}_{\mathcal{A}}(\phi_{\mathcal{A}})\phi'_{\mathcal{A}} = \boldsymbol{e}$$
$$\phi_{\mathcal{A}}(p_N) = \boldsymbol{x}_{\mathcal{A}}.$$

determines the inverse bidding strategies. This has a solution and we will show in the appendix that the average crossing condition guarantees that $\phi'_{\mathcal{A}} \gg 0$. Proceeding recursively in this way results in strategies satisfying $(***)$ in all sub-auctions.

Lemmas 8.4.1 and 8.4.2 together imply that under the average crossing condition, there exists an ex post *equilibrium satisfying the break-even condition $(**)$. To complete the proof, we now show that the equilibrium is efficient.*

Lemma 8.4.3 *Suppose that the valuations $v(.)$ satisfy the average crossing condition and β is an equilibrium of the English auction such that $\beta_i^{\mathcal{A}}$ are continuous and increasing functions whose inverses satisfy the break-even conditions $(**)$. Then, β is efficient.*

Proof 8.4.4 (Proof of Lemma 8.4.3:) *Consider the case when all bidders are active. To economize on notation, let $\beta_i^{\mathcal{N}} \equiv \beta_i$ and $\phi_i \equiv \beta_i^{-1}$. Suppose that the signals are $x_1, x_2, \ldots, x_N$ and that $\beta_i(x_i) = p_i$. Without loss of generality, suppose that $p_1 \geq p_2 \geq \ldots \geq p_{N-1} > p_N$ so that bidder N is the first to drop out. Now $(*)$ implies that for all i,*

$$v_i(\phi_1(p_N), \phi_2(p_N), \ldots, \phi_N(p_N)) = \overline{v}(\phi_1(p_N), \phi_2(p_N), \ldots, \phi_N(p_N)).$$

Since $\phi_N(p_N) = x_N$ and for all $i \neq N$, $\phi_i(p_i) > \phi_i(p_N)$, the average crossing condition implies that

$$v_N(x_1, x_2, \ldots, x_N) < \overline{v}(x_1, x_2, \ldots, x_N)$$

Since the ex post value of bidder N is less than the average ex post *value of all the bidders, it must be that*

$$v_N(x_1, x_2, \ldots, x_N) < \max_i v_i(x_1, x_2, \ldots, x_N)$$

Thus, the person who is the first to drop out does not have the highest value. The same argument can be made in every sub-auction $\Gamma(\mathcal{A}, \boldsymbol{x}_{\mathcal{N}\setminus\mathcal{A}})$, *so that at no state does the bidder with the highest value drop out. Thus, the equilibrium is efficient.*

This completes the proof of the proposition.

Example 8.4.1 *In an efficient equilibrium of the English auction, bidders need not drop out in order of their ex post values.*

Suppose that there are three bidders with valuations

$$\begin{aligned} v_1(x_1, x_2, x_3) &= x_1 + \frac{1}{3}x_3 \\ v_2(x_1, x_2, x_3) &= \frac{1}{3}x_1 + x_2 \\ v_3(x_1, x_2, x_3) &= \frac{1}{3}x_2 + x_3 \end{aligned}$$

and all signals lie in $[0, 1]$.

The average crossing condition is satisfied, so there exists an efficient equilibrium satisfying the break-even conditions. When all bidders are active, the equilibrium prescribes that bidders follow the strategy $\beta_i^{\mathcal{N}}(x_i) = \frac{4}{3}x_i$.

Suppose that $(x_1, x_2, x_3) = (\epsilon^2, \epsilon, 1 - \epsilon)$ where $\epsilon < \frac{1}{2}$. Then bidder 1 is the first to drop out. But for small enough ϵ, $v_2 < v_1 < v_3$. Thus, while the equilibrium is efficient, bidders do not necessarily drop out in order of increasing values.

Example 8.4.2 *There may be an efficient equilibrium of the English auction that does not satisfy the break-even conditions.*

Suppose that there are three bidders with valuations

$$\begin{aligned} v_1(x_1, x_2, x_3) &= x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 \\ v_2(x_1, x_2, x_3) &= x_2 \\ v_3(x_1, x_2, x_3) &= x_3 \end{aligned}$$

and all signals lie in $[0, 1]$.

In this case, the average crossing condition is not satisfied. In fact, there is no increasing solution (ϕ_1, ϕ_2, ϕ_3) to the break-even conditions. But the following constitutes an efficient equilibrium. Bidders 2 and 3 have private values and drop out when the price reaches their values regardless of the set of active bidders. Bidder 1 adopts a "wait-and-see" strategy; if all bidders are active, bidder 1 remains active no matter what his signal. Formally, his strategy is: for all x_1, $\beta_1^{\mathcal{N}}(x_1) = 1$; and if bidder 3, say, drops out at a price p_3 and $\mathcal{A} = \{1, 2\}$, then $\beta_1^{\mathcal{A}}(x_1, p_3) = \min\{3x_1 + 2p_3, 1\}$. If bidder 2 drops out, then an analogous strategy is followed.

This is an efficient equilibrium because if all bidders are active, then between bidders 2 and 3, the one who drops out first has the lower value. Once bidder 3, say, has dropped out, the winning bidder has the higher value.

Example 8.4.3 *With three or more bidders, the second-price auction may not have an efficient equilibrium.*

The valuations are the same as in Example 8.4.1, that is,

$$\begin{aligned} v_1(x_1, x_2, x_3) &= x_1 + \frac{1}{3}x_3 \\ v_2(x_1, x_2, x_3) &= \frac{1}{3}x_1 + x_2 \\ v_3(x_1, x_2, x_3) &= \frac{1}{3}x_2 + x_3 \end{aligned}$$

and we make no specific assumptions regarding the distribution of signals. Notice that the example does not satisfy the assumptions of the symmetric model-bidder 2's signal and bidder 3's signal do not affect bidder 1's value in the same way.

Suppose, by way of contradiction, that the second-price auction has an efficient equilibrium. Let $x_1 = \frac{1}{2}$, $x_2 = \frac{5}{12}$, and $x_3 = \frac{1}{4}$. Then $v_1 = v_2 > v_3$. Fix x_1 and x_2. For small $\epsilon > 0$, when $x_3 = \frac{1}{4} + \epsilon$, $v_1 > v_2 > v_3$ and bidder 1 should win. On the other hand, when $x_3 = \frac{1}{4} - \epsilon$, $v_2 > v_1 > v_3$ and bidder 2 should win. But since 1 and 2's signals are unchanged, in a second-price auction, the same bidder wins in both cases. This is a contradiction.

This example satisfies the average crossing condition, so the English auction does have an efficient equilibrium.

Chapter 9

Mechanism Design with Interdependent Values

As a preliminary observation, note that the revelation principle from Chapter 5 applies equally well to the setting of interdependent values and affiliated signals. Proposition 5.1.1 continues to hold in this general setting without amendment: Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report his signal truthfully, and (ii) the outcomes are the same as in the original mechanism. A direct mechanism asks buyers to report their private information - in this case, their signals - and replicates the equilibrium outcomes of the original mechanism.

As before, denote by $\mathcal{X}_i$ the set of signals that buyer i can receive and let $\mathcal{X} = \times_j \mathcal{X}_j$. Let Δ denote the set of probability distributions over the set of buyers $\mathcal{N}$. The revelation principle allows us to restrict attention to mechanisms of the form $(\mathcal{X}, \mathbf{Q}, \mathbf{M})$ consisting of a pair of functions $\mathbf{Q} : \mathcal{X} \to \Delta$ and $\mathbf{M} : \mathcal{X} \to \mathbb{R}^N$, where $Q_i(\mathbf{x})$ is the probability that i will get the object and $M_i(\mathbf{x})$ is the payment that i is asked to make.

9.1 Efficient Mechanisms

Example 9.1.1 *If the single crossing condition does not hold, then there may be no mechanism that allocates the object efficiently.* Suppose

$$v_1(x_1, x_2) = x_1$$
$$v_2(x_1, x_2) = x_1^2$$

Suppose that buyer 1's signal X_1 lies in $[0, 2]$. Notice that buyer 2's signal does not affect the value of either buyer, so there is no loss in supposing that it is a constant. The valuations do not satisfy the single crossing condition since $v_1(1, x_2) = v_2(1, x_2)$ but

$$\frac{\partial v_1}{\partial x_1}(1, x_2) < \frac{\partial v_2}{\partial x_1}(1, x_2)$$

Clearly, $v_1(x_1, x_2) > v_2(x_1, x_2)$ if and only if $x_1 < 1$, so it is efficient to allocate the object to buyer 1 when his signal is low and to buyer 2 when it is high.

Suppose there is a mechanism that is efficient and has the payment rule $M_1 : [0, 2] \to \mathbb{R}$ for buyer 1. Since buyer 2 has no private information that is relevant, her signal is assumed to be a constant, so buyer 1's payment can only depend on his own reported signal.

Now if $y_1 < 1 < z_1$, then efficiency and incentive compatibility together require that when his true signal is z_1,

$$0 - M_1(z_1) \geq z_1 - M_1(y_1)$$

and likewise, when his true signal is y_1,

$$y_1 - M_1(y_1) \geq 0 - M_1(z_1)$$

Together these imply that $y_1 \geq z_1$, which is a contradiction.

Suppose that there exists an efficient mechanism with an *ex post* equilibrium. Then, by a version of the revelation principle, there exists an efficient direct mechanism in which truth-telling is an *ex post* equilibrium. We will now argue that the valuation functions must satisfy the single crossing condition. Denote by $\mathbf{x}_{-i}$ the signals of all buyers other than i. If regardless of his signal x_i, buyer i either always wins or always loses, then the single crossing condition holds vacuously for x_i. Otherwise, we will say that buyer i is **pivotal** at $\mathbf{x}_{-i}$ if there exist signals y_i and z_i such that

$$v_i(y_i, \mathbf{x}_{-i}) > \max_{j \neq i} v_j(y_i, \mathbf{x}_{-i}) \text{ and } v_i(z_i, \mathbf{x}_{-i}) > \max_{j \neq i} v_j(z_i, \mathbf{x}_{-i})$$

Ex post incentive compatibility requires that when his signal is y_i, it be optimal for i to report y_i rather than z_i, so that

$$v_i(y_i, \mathbf{x}_{-i}) - M_i(y_i, \mathbf{x}_{-i}) \geq -M_i(z_i, \mathbf{x}_{-i}).$$

Likewise, when his signal is z_i, it is optimal to report z_i rather than y_i, so that

$$-M_i(z_i, \mathbf{x}_{-i}) \geq v_i(z_i, \mathbf{x}_{-i}) - M_i(y_i, \mathbf{x}_{-i})$$

Combining the two conditions results in

$$v_i(y_i, \mathbf{x}_{-i}) \geq M_i(y_i, \mathbf{x}_{-i}) - M_i(z_i, \mathbf{x}_{-i}) \geq v_i(z_i, \mathbf{x}_{-i})$$

Thus, a necessary condition for ex post incentive compatibility is

$$v_i(y_i, \mathbf{x}_{-i}) \geq v_i(z_i, \mathbf{x}_{-i}).$$

Keeping others' signals fixed, an increase in buyer i's value that results from a change in his own signal cannot cause him to lose if he were winning earlier. Thus, *ex post* incentive compatibility implies that the mechanism must be *monotonic* in values.

Efficiency now requires that if i has the highest value, and he wins the object, they should still have the highest value, and win the object, if his signal increases. This in turn requires that at any x_i such that $v_i(x_i, \mathbf{x}_{-i}) = v_j(x_i, \mathbf{x}_{-i})$, we must have

$$\frac{\partial v_i}{\partial x_i}(x_i, \mathbf{x}_{-i}) > \frac{\partial v_j}{\partial x_i}(x_i, \mathbf{x}_{-i}).$$

Thus, the single crossing condition is necessary for efficiency.

9.2 The Generalized VCG Mechanism

We now show that the single crossing condition is also sufficient to guarantee efficiency. If it is satisfied, then a generalization of the Vickrey-Clarke-Groves (VCG) mechanism to the interdependent values environment accomplishes the task.

Consider the following *direct* mechanism. Each buyer is asked to report his signal. The object is then awarded efficiently relative to these reports - it is awarded to the buyer whose value is the highest when evaluated at the reported signals. Formally,

$$Q_i^*(\mathbf{x}) = \begin{cases} 1 & \text{if } v_i(\mathbf{x}) > \max_{j \neq i} v_j(\mathbf{x}) \\ 0 & \text{if } v_i(\mathbf{x}) < \max_{j \neq i} v_j(\mathbf{x}) \end{cases}$$

and if more than one buyer has the highest value, the object is awarded to each of these buyers with equal probability. The buyer who gets the object pays an amount

$$M_i^*(\mathbf{x}) = v_i(y_i(\mathbf{x}_{-i}), \mathbf{x}_{-i})$$

where

$$y_i(\mathbf{x}_{-i}) = \inf\{z_i | v_i(z_i, \mathbf{x}_{-i}) \geq \max_{j \neq i} v_j(z_i, \mathbf{x}_{-i})\}$$

is the smallest signal such that given the reports $\mathbf{x}_{-i}$ of the other buyers, it would still be efficient for buyer to get the object. A buyer who does not obtain the object does not pay anything.

9.2.1 The Working of the Generalized VCG Mechanism

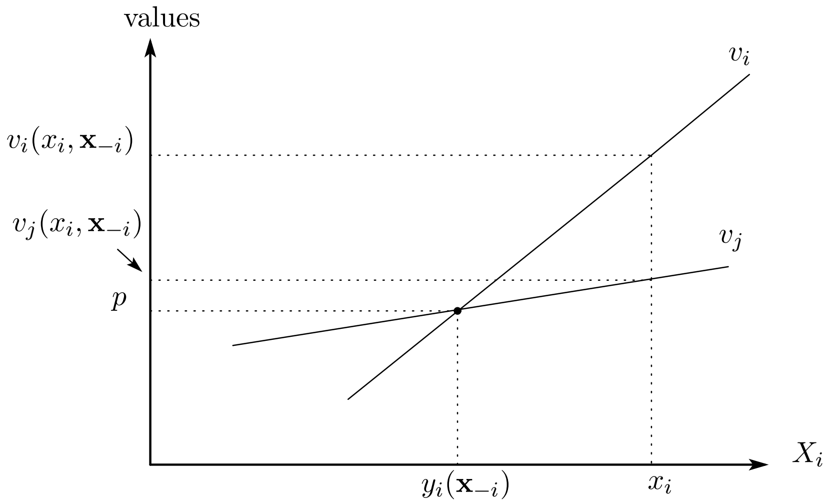

Figure 9.1: The Generalized VCG Mechanism

The workings of the mechanism are illustrated in Figure 9.1, which depicts the values of buyers i and j as functions of i's signal (the reported signals of the others, $\mathbf{x}_{-i}$, are held fixed). At the signal x_i, buyer i has the highest value. In particular, it exceeds that of buyer j. The signal $y_i(\mathbf{x}_{-i}) < x_i$ is the smallest signal such that his value is at least as large as that of another buyer, i.e., buyer j and buyer i is asked to pay an amount $p = v_i(y_i(\mathbf{x}_{-i}), \mathbf{x}_{-i})$.

The generalized VCG mechanism adapts the workings of a second price auction with private values to the interdependent values setting. If a buyer i, who obtains the object were asked to pay the second highest value at the reported signals, say $v_j(x_i, \mathbf{x}_{-i})$, then he would have the incentive to report a lower signal in order to lower the price paid. The generalized VCG mechanism restores the incentive to tell the truth by asking the winning buyer to pay $v_i(y_i(\mathbf{x}_{-i}), \mathbf{x}_{-i})$, in stead of $v_j(x_i, \mathbf{x}_{-i})$. The key point is that, as in a second price auction with private values, the reports of a buyer influence whether or not he obtains the object but do not influence the price paid if indeed he does so.

Proposition 9.2.1 *Suppose that the valuations $v(.)$ satisfy the single crossing condition. Then, truth-telling is an efficient ex post equilibrium of the generalized VCG mechanism $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^*)$.*

Proof 9.2.1 (Proof of Proposition 9.2.1:) *Suppose that when all buyers report their signals truthfully, buyer i's value is the highest when evaluated at the reported signals so that*

$$v_i(x_i, \boldsymbol{x}_{-i}) \geq_{j \neq i} v_j(x_i, \boldsymbol{x}_{-i}).$$

Buyer i pays an amount $v_i(y_i(\boldsymbol{x}_{-i}), \boldsymbol{x}_{-i})$, which is no greater than the true value of the object, so he makes a nonnegative surplus. If buyer i reports a z_i such that $z_i > y_i(\boldsymbol{x}_{-i})$, then by the single crossing condition, $v_i(z_i, \boldsymbol{x}_{-i}) > \max_{j \neq i} v_j(z_i, \boldsymbol{x}_{-i})$, so he would still obtain the object and pay the same amount as if he had reported x_i. If he reports a $z_i < y_i(\boldsymbol{x}_{-i})$, then he is no longer the winner so that his surplus is zero. This also cannot be a profitable deviation. Finally, consider the case where $z_i = y_i(\boldsymbol{x}_{-i})$. By construction, we have $x_i \geq y_i(\boldsymbol{x}_{-i})$. Thus, $z_i \neq x_i$ requires that $x_i > z_i = y_i(\boldsymbol{x}_{-i})$. Then, by the single crossing condition, buyer i is strictly worse off by announcing such z_i. Thus, no $z_i \neq x_i$ can be a profitable deviation in the circumstances that it is efficient for i to win the object.

Now suppose that when all buyers report their signals truthfully, there exists buyer i for whom

$$v_i(x_i, \boldsymbol{x}_{-i}) \leq_{j \neq i} v_j(x_i, \boldsymbol{x}_{-i})$$

and so that buyer i's payoff is zero. This means that $x_i < y_i(\boldsymbol{x}_{-i})$ and for him to win, the single crossing condition ensures that buyer i would have to report a $z_i \geq y_i(\boldsymbol{x}_{-i}) > x_i$. In that case, he would pay an amount

$$M_i^*(z_i, \boldsymbol{x}_{-i}) = v_i(y_i(\boldsymbol{x}_{-i}), \boldsymbol{x}_{-i}) > v_i(\boldsymbol{x})$$

and so this would not be profitable either.

In the case of private value, the generalized VCG mechanism reduces to the ordinary second price auction and in that case, truth-telling is a dominant strategy. Furthermore, when there are only two buyers, the generalized VCG mechanism is the direct mechanism that corresponds to the efficient equilibrium of the English auction. In the generalized VCG mechanism, the mechanism designer is assumed to have knowledge of the valuation functions v_i and the mechanism is then able to elicit information regarding the signals x_i that is privately held by the buyers.

9.3 Optimal Mechanisms

In this section we assume that each buyer's signal X_i is drawn at random from a finite set

$$\mathcal{X}_i = \{0, \Delta, 2\Delta, \dots, (t_i - 1)\Delta\}$$

with t_i possible signals. Buyers' values are determined by the joint signal via the valuation functions $v_i : \mathcal{X} \to \mathbb{R}_+$ satisfying $v_i(\mathbf{0}) = 0$. We suppose that

$$v_i(x_j + \Delta, \mathbf{x}_{-j}) \geq v_i(x_j, \mathbf{x}_{-j})$$

with strict inequality if $i = j$. The discrete version of the *single crossing condition* is as follows: for all i, j with $i \neq j$,

$$v_i(x_i, \mathbf{x}_{-i}) \geq v_j(x_i, \mathbf{x}_{-i}) \Rightarrow v_i(x_i + \Delta, \mathbf{x}_{-i}) \geq v_j(x_i + \Delta, \mathbf{x}_{-i})$$

and if the former is a strict inequality, then so is the latter.

9.3.1 Full Surplus Extraction

The optimal (=revenue maximizing) mechanism when values are interdependent and buyers' signals are statistically correlated is a modification of the generalized VCG mechanism. Although the optimal mechanism shares many important features with the generalized VCG mechanism, it depends critically on the distribution of signals unlike the generalized VCG mechanism. Recall that the efficiency properties of the generalized VCG mechanism did not depend upon the distribution of signals but only upon the valuation functions v_i.

Let Π denote the joint probability distribution of buyers' signals: $\Pi(\mathbf{x})$ is the probability that $\mathbf{X} = \mathbf{x}$. Let Π_i be a matrix with t_i rows and $\times_{j \neq i} t_j$ columns whose elements are the conditional probabilities $\pi(\mathbf{x}_{-i}|x_i)$. Each row of Π_i corresponds to a signal x_i of buyer i, whereas each column corresponds to a vector of signals $\mathbf{x}_{-i}$ of the other buyers. The entry $\pi(\mathbf{x}_{-i}|x_i)$ then represents the *beliefs* of buyer i regarding the signals of the other buyers conditional on his own information.

We will refer to $\Pi(\mathbf{x})$ as the matrix of beliefs of buyer i. If the signals are independent, buyer i's own signal provides no information about the signals of the other buyers. As a result, with independent signals, the rows of $\Pi(\mathbf{x})$ are identical and hence $\Pi(\mathbf{x})$ is of rank one.

Proposition 9.3.1 *Suppose that signals are discrete and the valuations $v(.)$ satisfy the single crossing condition. If for every i, the matrix of beliefs Π_i is of full rank, then there exists a mechanism in which truth-telling is an efficient ex post equilibrium in which the expected payoff of every buyer is exactly zero.*

Proof 9.3.1 (Proof of Proposition 9.3.1:) *Consider the generalized VCG mechanism $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^*)$. Define*

$$U_i^*(x_i) = \sum_{\boldsymbol{x}_{-i}} \pi(\boldsymbol{x}_{-i}|x_i)[Q_i^*(\boldsymbol{x})v_i(\boldsymbol{x}) - M_i^*(\boldsymbol{x})]$$

to be the expected payoff of buyer i with signal x_i in the truth-telling equilibrium of the generalized VCG mechanism. Let $\boldsymbol{u}_i^*$ denote the t_i sized column vector $(U_i^*(x_i))_{x_i \in \mathcal{X}_i}$.

Since the matrix Π_i is of full row rank t_i, there exists a column vector $\boldsymbol{c}_i = (c_i(\boldsymbol{x}_{-i}))_{\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}}$ of size $\times j \neq i t_j$ such that

$$\Pi_i \boldsymbol{c}_i = \boldsymbol{u}_i^*.$$

Equivalently, for all x_i,

$$\sum_{\boldsymbol{x}_{-i}} \pi(\boldsymbol{x}_{-i}|x_i) c_i(\boldsymbol{x}_{-i}) = U_i^*(x_i)$$

Consider the Cremer-McLean (CM) mechanism $(\mathcal{X}, \boldsymbol{Q}^*, \boldsymbol{M}^C)$ defined by

$$M_i^C(\boldsymbol{x}) = M_i^*(\boldsymbol{x}) + c_i(\boldsymbol{x}_{-i})$$

Now observe that truth-telling is also an ex post *equilibrium of the CM mechanism. This is because the allocation rule $\boldsymbol{Q}^*$ is the same as in the generalized VCG mechanism* *and the payment rule M_i^C for buyer i differs from M_i^* by an amount that does not depend on his own report. In this equilibrium, the expected payoff of buyer i with signal x_i is*

$$U_i^C(x_i) = \sum_{\boldsymbol{x}_{-i}} \pi(\boldsymbol{x}_{-i}|x_i)[Q_i^*(\boldsymbol{x})v_i(\boldsymbol{x}) - M_i^C(\boldsymbol{x})] = 0$$

by construction.

9.3.2 Comments on the CM mechanism

1 If there are private values but these are correlated, then the payment in the CM mechanism is a just the payment in a second price auction plus the terms $c_i(\mathbf{x}_{-i})$. In that case, truth-telling is a dominant strategy in the optimal auction as well.

2 The CM mechanism $(\mathcal{X}, \mathbf{Q}^*, \mathbf{M}^C)$ has two separate components: the generalized VCG mechanism and the additional "lottery" $c_i(\mathbf{x}_{-i})$ that buyer i faces and the outcomes of this lottery - the amounts he is asked to pay - are determined by the reports of the other buyers. How buyer i evaluates this lottery depends on his own signal since, given the statistical dependence among signals, for different realization of X_i, the expected payment implicit in the lottery is different.

3 For some realizations of all the signals, a buyer's payoffs may be negative. Therefore, the CM mechanism is not *ex post* individually rational. Of course, by construction, the CM mechanism is interim individually rational.

4 While the conditions of the full surplus extraction result require only that the matrix of beliefs be of full rank, when buyers' signals are "almost independent," the lottery $c_i(\mathbf{x}_{-i})$ may involve, with small probabilities, very large payments.

Chapter 10

Multiple Object Auctions

For simplicity, in this chapter we assume all bidders have independently private values.

10.1 An Introduction to Multiple Object Auctions

When multiple objects are to be sold, many options are open to the seller. First, the seller must decide whether to sell the objects separately in multiple auctions or jointly in a single auction. In the former case, the objects are sold one at a time in separate auctions – conducted sequentially, say - in a way that the bids in the auction for one of the objects do not directly influence the outcome of the auction for another. In the latter case, the objects are sold at one go in a single auction, but not necessarily all to the same bidder, and the bids on the various objects collectively influence the overall allocation.

Second, the seller must choose among a variety of auction formats, and there is a wide range of possibilities to choose from. For instance, if the seller decides to sell the objects one at a time in a sequence of single-object auctions, there is still the question of the particular auction form - first price, second-price, or some other format - to adopt. If the seller decides to sell the objects at one go in a single auction, there are also many possibilities. We begin by outlining the workings of a few auction forms for the sale of multiple units of the same good at one go, returning to study multiple one at a time, sequential, or simultaneous auctions later.

10.1.1 Sealed-Bid Auctions for Selling Identical Units

Three sealed-bid auction formats for the sale of K identical objects are of particular interest.

D. The *discriminatory* (or "pay-your-bid") auction.

U. The *uniform-price* auction.

V. The *Vickrey* auction.

In each of these auctions, a bidder is asked to submit K bids b_k^i, satisfying $b_1^i \geq b_2^i \geq \ldots \geq b_K^i$, to indicate how much he is willing to pay for each additional unit. Thus, b_1^i is the amount i is willing to pay for one unit, $b_1^i + b_2^i$ is the amount he is willing to pay for two units and so on. We will refer to $\mathbf{b}^i = (b_1^i, b_2^i, \ldots, b_K^i)$ as a *bid vector.*

A bid vector $\mathbf{b}^i$ can be usefully thought of as an "inverse demand function" and can be inverted to obtain i's demand function $d^i : \mathbb{R}_+ \rightarrow \{1, 2, \ldots, K\}$:

$$d^i(p) \equiv \max\{k : p \leq b_k^i\}$$

Remark 10.1.1 *In particular, if $b_k^i > b_{k+1}^i$, then at any price p lying between b_k^i and b_{k+1}^i, bidder i is willing to buy exactly k units. A bidder's demand is clearly nonincreasing in the price. Since the demand function is just the "inverse" of the bid vector, and vice versa, submitting the bid vector $\boldsymbol{b}^i$ is equivalent to submitting the demand function d^i. We will thus use these interchangeably.*

In all three of the auction formats considered here, a total of $N \times K$ bids $\{b_k^i : i = 1, 2, \ldots, N; k = 1, 2, \ldots, K\}$ are collected and the K units are awarded to the K highest of these bids - that is, if bidder i has $k \leq K$ of the K highest bids, then i is awarded k units.

We will refer to an auction in which the K highest bids are deemed winning and awarded objects as a *standard auction.* The three auctions introduced next are all standard but differ in terms of their pricing rules - how much each bidder is asked to pay for the units he is awarded.

Discriminatory Auctions

In a discriminatory auction, each bidder pays an amount equal to the sum of his bids that are deemed to be winning-that is, the sum of his bids that are among the K highest of the $N \times K$ bids submitted in all. Formally, if exactly k^i of the ith bidder's K bids b^i_k are among the K highest of all bids received, then i pays

$$\sum_{k=1}^{k^i} b^i_k$$

This amounts to perfect price discrimination relative to the submitted demand functions; hence the name of the auction.

The discriminatory pricing rule can also be framed in terms of the *residual supply function* facing each bidder. At any price p the residual supply facing bidder i, denoted by $s^{-i}(p)$, is equal to the total supply K less the sum of the amounts demanded by other bidders, provided that this is nonnegative. Formally,

$$s^{-i}(p) \equiv \max\{K - \sum_{j \neq i} d^j(p), 0\}$$

and this is clearly a nondecreasing function of the price. The discriminatory auction asks each bidder to pay an amount equal to the area under his own demand function up to the point where it intersects the residual supply curve.

The discriminatory auction is the natural multi-unit extension of the first price sealed-bid auction. In particular, if there is only a single unit for sale ($K = 1$), then the discriminatory auction reduces to a first-price auction.

Uniform-Price Auctions

In a uniform-price auction all K units are sold at a "market-clearing" price such that the total amount demanded is equal to the total amount supplied. In the discrete model studied here, there is some leeway in defining the price that clears the market - any price lying between the highest losing bid and the lowest winning bid equates demand and supply. We adopt the rule that the *market-clearing* price is the same as the highest losing bid.

Denote by $\mathbf{c}^{-i}$ the K-vector of *competing bids* facing bidder i. This is obtained by rearranging the $(N-1)K$ bids b^j_k of bidders $j \neq i$ in decreasing order and selecting the first K of these. Thus, $\mathbf{c}_1^{-i}$ is the highest of the other bids, $\mathbf{c}_2^{-i}$ is the second-highest, and so on. The number of units that bidder i wins is just the number of competing bids he defeats. For instance, in order for i to win exactly one unit it must be the case that $b_1^i > \mathbf{c}_K^{-i}$ and $b_2^i < \mathbf{c}_{K-1}^{-i}$; that is, he must defeat the lowest competing bid but not the second lowest. Similarly, in order to win exactly two units, bidder i must defeat the two lowest competing bids but not the third lowest. More generally, bidder i wins exactly $k^i > 0$ units if and only if

$$b_{k^i}^i > \mathbf{c}_{K-k^i+1}^{-i} \text{ and } b_{k^i+1}^i < \mathbf{c}_{K-k^i}^{-i}$$

Remark 10.1.2 *Observe that the residual supply function s^{-i} facing bidder i can also be obtained from the vector of competing bids $\mathbf{c}^{-i}$ since*

$$s^{-i}(p) = K - \max\{k : c_k^{-i} \geq p\}$$

The highest losing bid - the market-clearing price - is then just

$$p = \max\{b_{k^i+1}^i, \mathbf{c}_{K-k^i}^{-i}\}$$

and in a uniform-price auction, if bidder i wins k_i units, then he pays k_i times p. The market-clearing price can also be written as

$$p = \max_i\{b_{k^i+1}^i\}$$

The uniform-price auction reduces to a second-price sealed-bid auction when there is only a single unit for sale ($K = 1$). It thus seems that it is a natural extension of the second-price auction to the multiunit case. As we will see, however, it does not share many important properties with the second-price auction, so the analogy is imperfect.

Vickrey Auctions

In a Vickrey auction, a bidder who wins k^i units pays the k^i highest losing bids of the other bidders – that is, the k^i highest losing bids not including his own. As before, denote by $\mathbf{c}^{-i}$ the K-vector of competing bids facing bidder i, so that c_1^{-i} is the highest of the other bids, c_2^{-i} is the second highest, and so on.

To win one unit, bidder i's highest bid must defeat the lowest competing bid – that is, $b_1^i > c_K^{-i}$. To win a second unit, i's second highest bid must defeat the second lowest competing bid - that is, $b_2^i > c_{K-1}^{-i}$. To win the kth unit, i's kth highest bid must defeat the kth lowest competing bid. The Vickrey pricing rule is the following. Bidder i is asked to pay c_K^{-i} for the first unit he wins, c_{K-1}^{-i} for the second unit, c_{K-2}^{-i} for the third unit, and so on. Thus, if bidder i wins k_i units, then the amount he pays is

$$\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i}$$

The basic principle underlying the Vickrey auction is the same as the one underlying the Vickrey-Clarke-Groves mechanism: Each bidder is asked to pay an amount equal to the externality he exerts on other competing bidders. In the example, had bidder 1 been absent, the three units allocated to him would have gone to the other bidders: two to bidder 2 and one to bidder 3. According to the demand function submitted by him, bidder 2 is willing to pay b_2^2 and b_3^2 , respectively, for two additional units. Similarly, bidder 3 is willing to pay b_3^3 for one additional unit. Bidder 1 is asked to pay the sum of these amounts. The amounts that bidders 2 and 3 are asked to pay are determined in similar fashion.

Like the uniform-price auction, the Vickrey auction also reduces to a second-price sealed-bid auction when there is only a single unit for sale ($K = 1$). Unlike the uniform-price auction, however, it shares many important properties with the second-price auction and is, as we will argue, the appropriate extension of the second-price auction to the case of multiple units.

An Illustrative Example

Consider a situation in which there are six units ($K = 6$) to be sold to three bidders and the submitted bid vectors are

$$\mathbf{b}^1 = (50, 47, 40, 32, 15, 5)$$
$$\mathbf{b}^2 = (42, 28, 20, 12, 7, 3)$$
$$\mathbf{b}^3 = (45, 35, 24, 14, 9, 6)$$

Then the six highest bids are

$$(b_1^1, b_2^1, b_1^3, b_1^2, b_3^1, b_2^3) = (50, 47, 45, 42, 40, 35)$$

so that bidder 1 is awarded three units, bidder 2 is awarded one unit, and bidder 3 is awarded two units.

- In a discriminatory auction, bidder 1 pays $b_1^1 + b_2^1 + b_3^1 = 50 + 47 + 40 = 137$.
- In a uniform-price auction, $\mathbf{c}^{-1} = \{45, 42, 35, 28, 24, 20\}$. Now since $b_3^1 > c_4^{-1}$ and $b_4^1 < c_3^{-1}$, bidder 1 wins three units and the market-clearing price is $\max\{b_4^1, c_4^{-1}\} = 32$ so bidder 1 pays a total of 96.
- In a Vickrey Auction, bidder 1's payment is $c_6^{-1} + c_5^{-1} + c_4^{-1} = 20 + 24 + 28 = 72$

10.1.2 Some Open Auctions

Each of the three sealed-bid auction formats introduced here has a corresponding open format

Dutch Auctions

In the *multi-unit* Dutch (or open descending price) auction, as in its single unit counterpart, the auctioneer begins by calling out a price high enough so that no bidder is willing to buy any units at that price. The price is then gradually lowered until a bidder indicates that he is willing to buy a unit at the current price. This bidder is then sold an object at that price and the auction continues - the price is lowered further until another unit is sold, and so on. This continues until all K units have been sold.

The multi-unit Dutch auction is *outcome equivalent* to the discriminatory auction in the sense that if each bidder behaves according to a bid vector $\mathbf{b}^i$, indicating his interest in purchasing one unit when the price reaches b_1^i, another when the price reaches b_2^i, and so on, then the outcome is the same as when each bidder submits the bid vector b_i in a discriminatory auction.

English Auctions

In the *multi-unit* English (or open ascending-price) auction, the auctioneer begins by calling out a low price and then gradually raises it. Each bidder indicates - by using hand signals, by holding up numbered cards, or electronically - how many units he is willing to buy at that price - in other words, his demand at that price. As the price rises, bidders naturally reduce the number of units they are willing to buy. The auction ends when the total number of units demanded is exactly K and all units are sold at the price where the total demand changes from $K+1$ to K.

The multi-unit English auction bears the same relation to the uniform price auction as the ordinary English auction does to the second-price sealed-bid auction - the two are outcome equivalent. The equivalence between the two multi-unit auctions is weak for the same reason that the equivalence between the single unit auctions was weak - potentially useful information is available in the open auctions that is not available in the sealed-bid formats. Once again, with private values, this information is irrelevant.

Ausubel Auctions

The Ausubel auction is an alternative ascending-price format that is outcome equivalent to the Vickrey auction. As in the English auction, the auctioneer begins by calling out a low price and then raises it. Each bidder indicates his demand $d^i(p)$ at the current price p and the quantity demanded is reduced as the price rises.

10.2 Equilibrium and Efficiency with Private Values

10.2.1 The Basic Model

There are K identical objects for sale and N potential buyers are bidding for these. Bidder i's valuation for the objects is given by a *private value vector* $X^i = (X_1^i, X_2^i, \ldots, X_K^i)$, where X_k^i represents the *marginal value* of obtaining the kth object. The total value to the bidder of obtaining exactly $k \leq K$ objects is then the sum of the first k marginal values: $\sum_{l=1}^{k} X_l^i$. It is assumed that the marginal values are declining in the number of units obtained so that $X_1^i \geq X_2^i \geq \ldots \geq X_K^i$. Bidders are assumed to be risk neutral.

Bidders are symmetric - each X^i is independently and identically distributed on the set

$$\mathcal{X} = \{x \in [0, \omega]^K : \forall k, x_k \geq x_{k+1}\}$$

according to the density function f.

We can invert each bidder's valuation vector x_i to obtain his "true" demand function

δ^i defined by

$$\delta^i(p) \equiv \max\{k : p \leq x_k^i\}$$

Some special cases of the preceding model - involving restrictions on the probability distribution that values are drawn from - are of interest.

Limited Demand Model

It may be that even though K units are being sold, each bidder has used for at most $L < K$ units. In that case, the support of f is the set

$$\mathcal{X}(L) = \{\mathbf{x} \in \mathcal{X} : \forall k > L, x_k = 0\}$$

so that there is no value derived from obtaining more than L units. Value vectors are of the form $(x_1, x_2, \ldots, x_L, 0, 0, \ldots, 0)$ and we then suppose that each bidder submits a bid $\mathbf{b}^i$, which is also an L vector. In an extreme instance, each bidder has used for only one unit ($L = 1$), and we will refer to this case as one of *single-unit demand.* If bidders value more than one unit with positive probability, then we will refer to that as the case of *multi-unit demand.* The single-unit demand model is of interest because equilibrium behavior there is analogous to equilibrium behavior in auctions where only a single object is sold and demanded. As we will see, this is not true with multi-unit demand.

Multiuse Model

A second, analytically useful, restriction on the form of the density function f occurs if the value vector X consists of order statistics of independent draws from some underlying distribution. Specifically, suppose that each bidder draws $L \leq K$ values $Z_1, Z_2, \ldots, Z_L$ independently from some distribution F, and it is useful to think of these as the values derived from the object in different uses. If he obtains only one unit, then it is used in the best way possible, so his value for the first unit is $X_1 = \max\{Z_1, Z_2, \ldots, Z_L\}$. If he obtains a second unit, it is put to the second-best use possible, so the marginal value of the second unit, X_2, is the second-highest of $\{Z_1, Z_2, \ldots, Z_L\}$. The marginal value of the third unit, X_3, is the third highest of $\{Z_1, Z_2, \ldots, Z_L\}$, and so on.

For a private values environment, it is the case that any equilibrium in the sealed-bid environment is outcome equivalent to an equilibrium of the corresponding open auction.

Thus, any equilibrium in the discriminatory auction is equivalent to an equilibrium in the multi-unit Dutch auction, any equilibrium in the uniform-price auction to one in the multi-unit English auction, and any equilibrium in the Vickrey auction to one in the Ausubel auction. The Vickrey auction is the simplest from a strategic standpoint, so we begin our analysis there. Next we turn to the uniform-price and discriminatory formats.

10.2.2 Vickrey Auctions

Denote $p_k^i \equiv c_{K-k^i+k}^{-i}$ as the price bidder i pays for the kth unit. Notice that, by definition, $p_1^i \leq p_2^i \leq \ldots \leq p_{k^i}^i$. Just as it is a weakly dominant strategy to bid one's value in a second price auction of a single object, it is a weakly dominant strategy to "bid one's true demand function" in a multi-unit Vickrey auction.

Proposition 10.2.1 *In a Vickrey auction, it is a weakly dominant strategy to bid according to $\beta^V(\boldsymbol{x}) = \boldsymbol{x}$.*

Proof 10.2.1 (Proof of Proposition 10.2.1:) *Consider bidder i and the bids $\boldsymbol{b}^{-i}$ submitted by the other bidders. As before, let $\boldsymbol{c}^{-i}$ be a vector consisting of the K highest bids of the other bidders. Suppose further that when bidder i submits a bid $\boldsymbol{b}^i = \boldsymbol{x}^i$, he is awarded k^i units. According to the Vickrey pricing rule, his payment is given by $\sum_{k=1}^{k^i} c_{K-k^i+k}^{-i}$ and it is the case that for all $k \leq k^i$, $x_k^i \geq c_{K-k^i+k}^{-i} = p_k^i$, whereas for all $k > k^i$, $x_k^i \leq c_{K-k^i+k}^{-i} = p_k^i$.*

Now suppose bidder i were to submit a bid vector $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he is awarded the same number of units as when he submitted his true value vector $\boldsymbol{x}^i$; then the prices he pays for these units would be unaffected, as would his overall surplus - the total value less the sum of the prices paid.

If bidder i were to submit a $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he is awarded a greater number of units, say $l^i > k^i$, than if he were to submit his true value vector $\boldsymbol{x}^i$, then the prices he would pay for the first k^i units would be unchanged and, therefore, so would the surplus derived from these. For any unit $k > k^i$, however, the price p_k^i exceeds (or, at best equals) the kth marginal value x_k^i, so the surplus from these $l^i - k^i$ units would be negative (or at best, zero). As a result, the overall surplus would be lower (or at best, the same) than that if he were to bid truthfully.

Finally, if bidder i were to submit a $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ such that he is awarded a smaller number of units, say $l^i < k^i$, as when he submitted his true value vector $\boldsymbol{x}^i$, then the prices he would pay for the first l^i units would be unchanged and therefore so would the surplus derived from these. But the surplus from any unit any unit $k < k^i$ was positive and is now forgone. Thus, by winning fewer units bidder 1's overall surplus would be lower than if he were to bid truthfully.

It is important to observe that nowhere in the proof did we make use of the assumption that bidders were symmetric - Proposition 10.2.1 continues to hold even if bidders are asymmetric. An immediate consequence of the dominant strategy nature of the Vickrey auction is that the K objects are awarded in an efficient manner - they are awarded to the K highest values x_k^i. For future reference we record this observation as follows:

Proposition 10.2.2 *The Vickrey auction allocates the objects efficiently.*

The efficiency property of the Vickrey auction also extends to its open ascending-price counterpart, the Ausubel auction: It is an equilibrium strategy for a bidder to reduce his demand according to his true demand function δ^i, obtained by inverting his value vector $\mathbf{x}^i$. The resulting allocation is always efficient.

10.2.3 Efficiency in Multiunit Auctions

Consider any multiunit auction format in which bidders submit bid vectors $\mathbf{b}^i$ and the objects are awarded to the K highest bids - that is, a *standard auction*.

Consider an equilibrium of some standard auction $(\beta^1, \beta^2, \ldots, \beta^N)$ and a particular realization of bidders' values $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N$. In a standard auction, the K units will be awarded to the K highest of the $N \times K$ bids $\beta_k^i(\mathbf{x}^i)$, whereas efficiency demands that the K units be awarded to the K highest of the $N \times K$ marginal values x_k^i. For the equilibrium to allocate efficiently for *every* realization of the values, the ranking of the $N \times K$ bids $\beta^i(\mathbf{x}^i)$ must agree with the ranking of the $N \times K$ values x_k^i. In other words, efficiency requires that for all i, j and k, l,

$$x_k^i > x_l^j \text{ if and only if } \beta_k^i(\mathbf{x}^i) > \beta_l^j(\mathbf{x}^j),$$

which has two implications:

(i) It must be that bidder i's bid on the kth object $\beta_k^i(\mathbf{x}^i)$ cannot depend on the value of, say, the lth object, x_l^i where $l \neq k$. That bidding strategies must be *separable* - the bid on the kth object can only depend on the kth marginal valuation.

(ii) The different components of the bidding strategy must be symmetric across both bidders and objects – that is, for all i, j and k, l, $\beta_k^i(.) > \beta_l^j(.)$. Otherwise, with positive probability there are situations in which the allocation will be inefficient;

Proposition 10.2.3 *An equilibrium of a standard auction is efficient if and only if the bidding strategies are separable and symmetric across both bidders and objects-that is, there exists an increasing function β such that for all i and k,*

$$\beta_k^i(\boldsymbol{x}^i) = \beta(x_k^i)$$

10.2.4 Uniform-Price Auctions

This section explores the strategic properties of the uniform-price auction and finds that, in general, the uniform-price auction does not inherit the dominant strategy property of the second-price auction. In fact, the conditions required by Proposition 10.2.3 fail; as a result, the uniform-price auction is generally inefficient.

We begin by noting that in the independent private values setting studied here the uniform-price auction is known to have a pure strategy equilibrium. But a closed form expression for the strategies is not available, so we proceed indirectly. Rather than explicitly calculating equilibrium strategies - a difficult task even in specific examples - we will instead deduce some structural features that any equilibrium must have.

Recall that in a uniform price auction, bidder i wins exactly $k^i > 0$ units if and only if

$$b_{k^i}^i > c_{K-k^i+1}^{-i} \text{ and } b_{k^i+1}^i < c_{K-k^i}^{-i}$$

and the highest losing bid-the price at which all units are sold - is then

$$p = \max\{b_{k^i+1}^i, c_{K-k^i+1}^{-i}\}$$

Claim 10.2.1 $\forall i, k$, $b_k^i \leq x_k^i$; *i.e., the bids cannot exceed marginal values.*

Proof 10.2.2 (Proof of Claim 10.2.1:) *Suppose that some bidder i bids an amount $b^i_k > x^i_k$. We claim that this is weakly dominated by the strategy of bidding $b^i_k = x^i_k$ (and if there is another bid, say b^i_{k+1}, such that $b^i_k \geq b^i_{k+1} > x^i_k$, then this is also reduced to x^i_k). If $b^i_k = p$, the price at which the units are sold, then bidder i is winning exactly $k-1$ units and reducing his bid to x^i_k can only improve his profits by possibly decreasing the price. If $b^i_k < p$, then reducing this bid to x^i_k makes no difference. If $b^i_k > p > x^i_k$, then bidder i is making a loss on at least one unit and decreasing bi k to x^i_k will reduce his loss – he will no longer win the units for which the price exceeded the marginal value. If $b^i_k > x^i_k > p$, then reducing his bid to x^i_k again makes no difference since he would still win the kth unit at the same price.*

Claim 10.2.2 *For all i, $b^i_1 = x^i_1$; i.e., the bid on the first unit must be the same as its value.*

Proof 10.2.3 (Proof of Claim 10.2.2:) *Suppose that $b^i_1 < x^i_1$. If $p \geq x^i_1 > b^i_1$, then bidder i is not winning any objects - all his bids are below the market-clearing price – and this would not change if he were to raise b^i_1 to x^i_1. If $x^i_1 > p \geq b^i_1$, then again bidder i is not winning any objects, but if he were to raise b^i_1 to x^i_1, then he may win a unit and at a price that would be profitable. Finally, if $x^i_1 > b^i_1 > p$, then raising his bid to x^i_1 makes no difference. Thus, we have argued that in a uniform-price auction it is a weakly dominant strategy for a bidder to bid truthfully for the first unit. Put another way, bidders do not have any incentive to shade their bids b^i_1 for the first unit.*

Remark 10.2.1 *Bidders do have the incentive to shade their bids $b^i_2, b^i_3, \ldots, b^i_K$ for additional units, however, and this feature distinguishes the uniform-price auction from the Vickrey auction; in the latter, there is no incentive to shade bids for any of the units. Submitting a vector $\boldsymbol{b}^i \leq \boldsymbol{x}^i$, $\boldsymbol{b}^i \neq \boldsymbol{x}^i$ is equivalent to submitting a demand function d^i such that for some prices p the amount demanded $d^i(p)$ is lower than the true demand $\delta^i(p)$. Bid shading is thus sometimes referred to as demand reduction.*

Demand Reduction

Suppose $K = 2$ and that the density of values f has full support on the set $\mathcal{X}$. Fix a symmetric equilibrium of the uniform-price auction $\beta = (\beta_1, \beta_2)$ that satisfies $\beta(\mathbf{0}) = \mathbf{0}$.

Suppose that all bidders other than bidder 1, say, follow β. Suppose further that the marginal values bidder 1 assigns to the two units are given by $\mathbf{x} = (x_1, x_2)$ and bidder 1 bids $\mathbf{b} = (b_1, b_2)$. (Bidder indices are omitted so that we write b_k instead of b_k^1, and so on.) Let $\mathbf{c} = (c_1, c_2)$ be the competing bids facing bidder 1 and suppose that the distribution of the random variable $\mathbf{C}$ has a density on the set $\mathcal{X}$ given by $h(.)$. Bidder 1's expected payoff is then given by

$$\Pi(\mathbf{b}, \mathbf{x}) = \int_{\{\mathbf{c}:c_1<b_2\}} (x_1 + x_2 - 2c_1)h(\mathbf{c})d\mathbf{c} + \int_{\{\mathbf{c}:c_2<b_1 \text{ and } c_1>b_2\}} (x_1 - \max\{b_2, c_2\})h(\mathbf{c})d\mathbf{c}$$

The first term is bidder 1's payoff when he wins both units and the second term is his payoff when he wins only one unit. (See Figure 10.1)

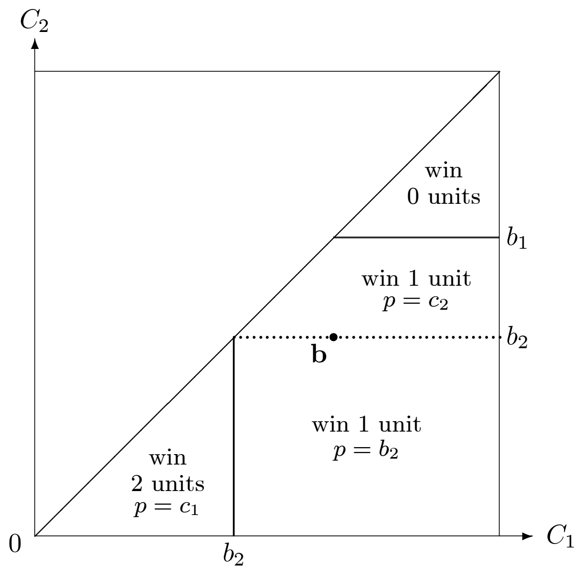

Figure 10.1: Outcomes in a Uniform-Price Auction with Two Units for Sale

Let H_1 denote the marginal distribution of the higher competing bid C_1 and H_2 that of the lower competing bid C_2 with densities h_1 and h_2, respectively. Thus, $H_1(b_2) = Prob[C_1 < b_2]$ is the probability that bidder 1 will defeat both competing bids and win two units. Similarly, $H_2(b_1) = Prob[C_2 < b_1]$ is the probability that he will defeat the lower competing bid, so win *at least* one unit. The probability that he will win *exactly* one unit is then the difference $H_2(b_1) - H_1(b_2)$. Also, $H_2(b_2) - H_1(b_2) = Prob[C_2 < b_2 < C_1]$ is the probability that the highest losing bid - the price at which the units are sold - is

b_2. Using these facts, bidder 1's expected payoff can be rewritten as

$$\begin{aligned}\Pi(\mathbf{b}, \mathbf{x}) = & H_1(b_2)(x_1 + x_2) - 2\int_0^{b_2} c_1 h_1(c_1) dc_1 \\ & + [H_2(b_1) - H_1(b_2)]x_1 \\ & - [H_2(b_2) - H_1(b_2)]b_2 - \int_{b_2}^{b_1} c_2 h_2(c_2) dc_2\end{aligned}$$

It can be checked that

$$\frac{\partial \Pi}{\partial b_2}|_{b_2=x_2} = -[H_2(x_2) - H_1(x_2)] < 0$$

since H_1 stochastically dominates H_2.

We have thus argued that a bidder can increase his payoff by shading his bid for the second unit - that is, the equilibrium bid for the second unit must be such that $b_2 < x_2$.

In a uniform price auction, the shading of bids for units other than the first-demand reduction - results from the fact that, with positive probability, every bid other than that for the first unit may determine the price paid on all units. In other words, a bidder's own bids influence the price he pays. By contrast, in a Vickrey auction, a bidder's own bids determine how many units he wins but have no influence on the prices paid - each unit is purchased at a competing bid.

Two aspects of our analysis were special. First, we examined only the case when the number of units was two. The argument for demand reduction is quite general – considering more units only adds notational complexity – and applies no matter how many units are sold. Thus, no matter what K is, $\beta_1^i(x_1^i) = x_1^i$ and for all $k > 1$, $\beta_k^i(x_k^i) < x_k^i$. Figure 10.2 is a schematic portrayal of demand reduction when the number of units for sale is greater than two. Second, we assumed that the distribution of the competing bids facing a bidder admitted a density $h(c)$, so the distribution of bids did not have any mass points. This is not entirely innocuous since it rules out the possibility that for some open set of value vectors, the bids on some unit are constant. It turns out, however, that the demand reduction occurs even if the bidding strategies have mass points.

Proposition 10.2.4 *Every undominated equilibrium of the uniform-price auction has the property that the bid on the first unit is equal to the value of the first unit. Bids on other units are lower than the respective marginal values.*

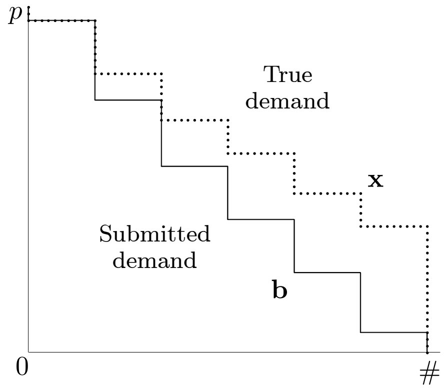

Figure 10.2: Demand Reduction in a Uniform-Price Auction

Proposition 10.2.5 *Every undominated equilibrium of the uniform price auction is inefficient.*

Demand reduction (or bid shading) can be severe when the number of bidders is small relative to the number of units for sale. An extreme case of this phenomenon occurs in the following example.

Example 10.2.1 *There are two units for sale and two bidders with value vectors $\boldsymbol{X}$ that are identically and independently distributed according to the density function $f(x) = 2$ on $\mathcal{X} = \{x \in [0,1]^2 : x_1 \geq x_2\}$.*

With this distribution of values, a symmetric equilibrium of the uniform-price auction is $\beta_1(x_1, x_2) = x_1$ and $\beta_2(x_1, x_2) = 0$. In every realization, each bidder wins one unit and the price is zero!

To see this, suppose that bidder 2 is following the strategy β as specified and bidder 1 with value vector $\mathbf{x}$ submits a bid vector $\mathbf{b} = (b_1, b_2) \gg 0$. As usual, let $\mathbf{c}$ denote the competing bids facing bidder 1-in this case, these are just the bids submitted by bidder 2-and we know that $\mathbf{c} = (y_1, \mathbf{0})$ where y is bidder 2's value vector. Since $c_2 = 0$, bidder 1 is sure to win at least one unit. He wins only one unit if his low bid is a losing bid – that is, if $y_1 > b_2$ and in that case he pays b_2. He wins both units if his low bid of b_2 exceeds the high bid of bidder 2, that is, if $y_1 < b_2$, and in that case the price he pays for each unit is y_1.

Let F_1 be the marginal distribution of X_1 with corresponding density f_1. Since $f(x) = 2$ on $\mathcal{X}$, we know that $F_1(x_1) = (x_1)^2$ and $f_1(x_1) = 2x_1$ on the interval $[0,1]$.

Bidder 1's expected payoff from bidding $\mathbf{b}$ when his values are $\mathbf{x}$ is simply

$$\begin{aligned}\Pi(\mathbf{b},\mathbf{x}) = F_1(b_2)(x_1+x_2) - 2\int_0^{b_2} y_1 h_1(y_1)dy_1 \\ + (1-F_2(b_2))(x_1-b_2)\end{aligned}$$

where the first term is the payoff from winning both units and the second from winning only one unit.

Differentiating with respect to b_2 we obtain

$$\begin{aligned}\frac{\partial \Pi}{\partial b_2} &= f_1(b_2)(x_2-b_2) - 1 + F_1(b_2) \\ &= 2b_2(x_2-b_2) - 1 - (b_2)^2 \\ &= -(x_2-b_2)^2 \\ &\leq 0\end{aligned}$$

and this is strictly negative whenever $b_2 < x_2$. So it is optimal to set $b_2 = 0$ whatever the value of x_2.

In the example, demand reduction is so extreme that the equilibrium price is always zero and the two units are always split between the two bidders regardless of their values. The resulting inefficiency is clear. Finally, note that low revenue equilibria of this sort cannot arise if the number of bidders exceeds the number of units for sale.

Single-Unit Demand

The inefficiency of the uniform-price auction does not result from the fact that multiple units are sold per se but rather from the fact that multiple units are demanded. Consider a situation in which $K > 1$ units are up for sale but each bidder has used for at most one unit - the case of single-unit demand. This is equivalent to supposing that the value vectors are drawn from the set

$$\mathcal{X}(1) = \{\mathbf{x} \in [0,\omega]^K : \forall k > 1, x_k = 0\}$$

Thus, f no longer has full support on $\mathcal{X}$.We already know that in a uniform price auction it is weakly dominant to bid $b_1 = x_1$, and since the value of all additional units is zero, each bidder is bidding truthfully. Put another way, if all bidders have unit demand, there is no possibility that a winning bidder will influence the price paid, and hence there is no incentive for demand reduction. The upshot of this is that with single-unit demand the uniform-price auction is efficient.

10.2.5 Discriminatory Auctions

Recall that in a discriminatory or "pay-your-bid" auction, a bidder who is awarded $k^i \leq K$ units pays the sum of his first k^i bids, $b_1^i + b_2^i + \ldots + b_{k^i}^i$. Once again we note that pure strategy equilibria are known to exist in a discriminatory auction when bidders have independent private values. When bidders are symmetric, as assumed in this chapter, a symmetric equilibrium is known to exist. No explicit characterization of the strategies is available, however, so, as in the previous section, we proceed indirectly by deducing properties that any equilibrium must satisfy.

To further understand the nature of equilibrium bids, let us look more closely at the two unit and two bidder case. Fix a symmetric equilibrium (β_1, β_2) of the discriminatory auction. First, notice that if the highest amount ever bid on the second unit is $\bar{b} = \max \beta_2(\mathbf{x})$, then it makes no sense for a bidder to bid more than $\bar{b}$ on the first unit. This is because any bid b_1 on the first unit that is greater than $\bar{b}$ will win with probability 1 and the bidder could do better by reducing it slightly. Thus, we have that in equilibrium

$$\max_{\mathbf{x}} \beta_1(\mathbf{x}) = \bar{b} = \max_{\mathbf{x}} \beta_2(\mathbf{x})$$

Second, consider a particular bidder and let the random variable $\mathbf{C} = (C_1, C_2)$ denote the competing bids - that is, the bids of the other bidder. Let H_1 denote the marginal distribution of a bidder's high bid C_1 and let H_2 denote the marginal distribution of the other bidder's low bid C_2. Thus,

$$H_k(c) = Prob[\beta_k(\mathbf{X}) \leq c]$$

Since, for all $\mathbf{x}$, $\beta_1(\mathbf{x}) \geq \beta_2(\mathbf{x})$, it is clear that the distribution H_1 stochastically dominates the distribution H_2. As usual, let h_1 and h_2 denote the corresponding densities.

Suppose a bidder has values (x_1, x_2) and bids (b_1, b_2). He wins both units if $C_1 < b_2$ and the probability of this event is $H_1(b_2)$. He wins exactly one unit if $C_2 < b_1$ and

$C_1 > b_2$ and the probability of this event is $H_2(b_1) - H_1(b_2)$. Thus, the expected payoff is

$$\begin{aligned}\Pi(\mathbf{b}, \mathbf{x}) &= H_1(b_2)(x_1 + x_2 - b_1 - b_2) \\ &\quad + [H_2(b_1) - H_1(b_2)](x_1 - b_1) \\ &= H_2(b_1)(x_1 - b_1) + H_1(b_2)(x_2 - b_2)\end{aligned}$$

The bidder's optimization problem is choose $\mathbf{b}$ to maximize $\Pi(\mathbf{b}, \mathbf{x})$ subject to the constraint that $b_1 \geq b_2$. When the constraint $b_1 \geq b_2$ does not bind at the optimum, so that $b_1 > b_2$, the first-order conditions for an optimum are

$$\begin{aligned}h_2(b_1)(x_1 - b_1) &= H_2(b_1) \\ h_1(b_2)(x_2 - b_2) &= H_1(b_2)\end{aligned}$$

Thus, we deduce that whenever $b_1 > b_2$, the bids are completely separable in the values - that is, β_1 does not depend on x_2 and β_2 does not depend on x_1.

When the constraint $b_1 \geq b_2$ binds at the optimum, so that $b_1 = b_2 \equiv b$, the first-order condition is

$$h_2(b)(x_1 - b) + h_1(b)(x_2 - b) = H_2(b) + H_1(b)$$

In this case, the bidder submits a "flat demand" function - bidding the same amount for the each of the two units. If it is optimal to submit the flat demand bid b for the value vector $\mathbf{x} = (x_1, x_2)$ and also for the value vector $\mathbf{z} = (z_1, z_2)$, then it is optimal to submit the same flat demand bid b for any convex combination of the values $\lambda x + (1 - \lambda) z$, where $0 \leq \lambda \leq 1$.

10.3 Revenue Equivalence in Multi-unit Auctions

The basic setup is the same as in the previous section. Specifically, there are K identical objects for sale and N potential buyers are bidding for these. Bidder i's valuation for the objects is given by a K-vector $\mathbf{X}^i = (X_1^i, X_2^i, \cdots, X_K^i)$, where X_k^i represents the marginal value of obtaining the kth object and these are declining. We do allow for asymmetries among bidders. Thus, bidders' value vectors, while drawn independently from

$$\mathcal{X} = \{\mathbf{x} \in [0, \omega]^K : \forall k, x_{k+1} \geq x_k\}$$

need not be identically distributed. Let bidder i's value vector $\mathbf{X}^i$ be distributed on $\mathcal{X}$ according to the density function f_i.

Fix an auction form A for allocating multiple units and fix an equilibrium $\beta = (\beta^1, \beta^2, \cdots, \beta^N)$ of A. Consider a particular bidder, say i, and suppose that other bidders $j \neq i$ follow the equilibrium strategies β^j. Suppose that bidder i's value vector is $\mathbf{x}^i$ but he reports that it is $\mathbf{z}^i$-that is, he bids $\beta^i(\mathbf{z}^i)$ instead of $\beta^i(\mathbf{x}^i)$. Let $q_1^i(\mathbf{z}^i)$ denote the probability that bidder i will win the first unit when he bids $\beta^i(\mathbf{z}^i)$, let $q_2^i(\mathbf{z}^i)$ denote the probability that he will win a second unit, and so on. In general, $q_k^i(\mathbf{z}^i)$ denotes the probability that he will win the kth unit.[1] If his value vector is $\mathbf{x}^i$, a bidder's gain from reporting that it is $\mathbf{z}^i$ can be written as

$$\sum_{k=1}^{K} q_k^i(\mathbf{z}^i) x_k^i = \mathbf{q}^i(\mathbf{z}^i)\mathbf{x}^i$$

where $\mathbf{q}^i(\mathbf{z}^i)$ is the K-vector of probabilities $q_k^i(\mathbf{z}^i)$.

Consider two different auction forms and fix equilibrium strategies in each. We will say that the two auction forms have the same *allocation rule* if the resulting probabilities $q_k^i(\mathbf{z}^i)$ in equilibrium are the same.

Let $m^i(\mathbf{z}^i)$ be the equilibrium expected payment in auction A by bidder i when he reports that his value vector is $\mathbf{z}^i$. Suppose that the auction is such that in equilibrium $m^i(\mathbf{0}) = 0$. Bidder i's expected payoff is

$$\mathbf{q}^i(\mathbf{z}^i)\mathbf{x}^i - m^i(\mathbf{z}^i)$$

In equilibrium it is optimal to bid $\beta^i(\mathbf{x}^i)$, so for all $\mathbf{z}^i$,

$$\mathbf{q}^i(\mathbf{x}^i)\mathbf{x}^i - m^i(\mathbf{x}^i) \geq \mathbf{q}^i(\mathbf{z}^i)\mathbf{x}^i - m^i(\mathbf{z}^i). \tag{10.1}$$

Now define

$$U^i(\mathbf{x}^i) \equiv \max_{\mathbf{z}^i}\{\mathbf{q}^i(\mathbf{z}^i)\mathbf{x}^i - m^i(\mathbf{z}^i)\} \tag{10.2}$$

to be the maximized payoff function and since U^i is the maximum of a family of affine functions-one for each $\mathbf{z}^i$)-it is convex.

[1]For $k < K$, $q_k^i - q_{k+1}^i$ is the probability that bidder i will win exactly k units; q_K^i is the probability that bidder i will win all K units.

Notice that (10.1) can be rewritten as follows: for all $\mathbf{x}^i$ and $\mathbf{z}^i$,

$$U^i(\mathbf{z}^i) \geq U^i(\mathbf{z}^i) + \mathbf{q}^i(\mathbf{x}^i)(\mathbf{z}^i - \mathbf{x}^i) \tag{10.3}$$

The inequality in (10.3), a consequence of equilibrium play, implies for all $\mathbf{x}^i$, the probability vector $\mathbf{q}^i(\mathbf{x}^i)$ is a subgradient of the payoff function U^i, which is convex, at the point $\mathbf{x}^i$. In other words, the vector $\mathbf{q}^i(\mathbf{x}^i)$ is perpendicular to the hyperplane that supports the function U^i at $\mathbf{x}^i$-the graph of the function U^i lies above the hyperplane.

So far we argued in a manner that is parallel to the case for only one object to sale, the only difference being that we now have multidimensional values-that is, value vectors. In the one-dimensional problem we were able to integrate q_i, the probability of winning the object, to obtain U_i. We wish to carry out a similar exercise here, and our next step serves to reduce the multidimensional problem to a single dimension.

Fix an arbitrary point $\mathbf{x}^i$ and define a function $V^i : [0,1] \to \mathbb{R}$ by

$$V^i(t) = U^i(t\mathbf{x}^i)$$

so that $V^i(0) = U^i(\mathbf{0})$ and $V^i(1) = U^i(\mathbf{x}^i)$. Since $U^i : \mathcal{X} \to \mathbb{R}$ is convex and continuous, $V^i : [0,1] \to \mathbb{R}$ is also convex and continuous. The function V^i is just a restriction of U^i to the line joining $\mathbf{0}$ to $\mathbf{x}^i$, so, while also convex, V^i is a function of only one variable.

A convex function of one variable is absolutely continuous and thus it is differentiable almost everywhere in the interior of its domain. Furthermore, every absolutely continuous function is the integral of its derivative, so we have

$$V^i(1) = V^i(0) + \int_0^1 \frac{dV^i(t)}{dt} dt \tag{10.4}$$

Now suppose $t \in (0,1)$ is such that V^i is differentiable at t. From (10.3),

$$V^i(t+\Delta) - V^i(t) = U^i((t+\Delta)\mathbf{x}^i) - U^i(t\mathbf{x}^i) \geq \mathbf{q}^i(t\mathbf{x}^i)\Delta\mathbf{x}^i$$

If $\Delta > 0$, then we get

$$\frac{V^i(t+\Delta) - V^i(t)}{\Delta} \geq \mathbf{q}^i(t\mathbf{x}^i)\mathbf{x}^i$$

and taking the limit as $\Delta \downarrow 0$ we obtain that

$$\frac{dV^i(t)}{dt} \geq \mathbf{q}^i(t\mathbf{x}^i)\mathbf{x}^i.$$

On the other hand, if $\Delta < 0$, then taking the limit as $\Delta \uparrow 0$ we get the opposite inequality. Thus, if V^i is differentiable at $t \in (0, 1)$,

$$\frac{dV^i(t)}{dt} = \mathbf{q}^i(t\mathbf{x}^i)\mathbf{x}^i.$$

Now substituting in (10.4) we obtain that for all $\mathbf{x}^i \in \mathcal{X}$:

$$U^i(\mathbf{x}^i) = U^i(\mathbf{0}) + \int_0^1 \mathbf{q}^i(t\mathbf{x}^i)\mathbf{x}^i dt \tag{10.5}$$

Thus, at any point $\mathbf{x}^i$, the payoff U^i is determined by the probabilities $\mathbf{q}^i$ up to an additive constant.

Since

$$m^i(\mathbf{x}^i) = \mathbf{q}^i(\mathbf{x}^i)\mathbf{x}^i - U^i(\mathbf{x}^i)$$

the expected payments in any two auctions with the same allocation rule are also the same. We have thus shown that the revenue equivalence principle holds for multi-unit auctions as well.

Proposition 10.3.1 *The equilibrium payoff (and payment) functions of any bidder in any two multi-unit auctions that have the same allocation rule differ at most by an additive constant.*

When each bidder, has used for at most one unit-the case of single-unit demand-the discriminatory and uniform-price auctions allocate efficiently. In this case, the revenue equivalence principle derived above can be applied, so the two auctions are revenue equivalent to the Vickrey auction.

With multi-unit demands, only the Vickrey auction is generally efficient. In this case, the revenue equivalence principle provides little insight in determining the revenue from the discriminatory and uniform-price auctions relative to that from the Vickrey auction. Indeed, it is known that no general ranking of the revenues can be obtained-depending on the distribution of values one or the other auction may be superior. In specific instances, however, it is possible to argue on a *priori* grounds that all three auctions are efficient and in these instances the power of the revenue equivalence principle can be brought to bear.

10.3.1 Revenue Equivalence with Multi-unit Demand: An Example

Suppose that there are three units for sale ($K = 3$) and two bidders, each of whom wants at most two units ($L = 2$). Bidders' value vectors $\mathbf{X} = (X_1, X_2)$ are two-dimensional and are identically and independently distributed according to the density function f with support $\{\mathbf{x} \in [0,1]^2 : x_1 \geq x_2\}$.

Notice that the environment specified here has the feature that in any standard auction each bidder is assured of winning at least one unit. The Vickrey auction is, of course, efficient. In what follows, we will first argue, on a *priori* grounds, that the uniform-price auction also has an efficient equilibrium and we will then use the revenue equivalence principle to explicitly derive the equilibrium bidding strategies. We will analyze the discriminatory auction in the same way. The special structure given above is interesting because it represents one of the few known instances in which the equilibrium bidding strategies in the three multi-unit auctions can be derived explicitly.

Let F_1 and F_2 denote the marginal distributions of X_1 and X_2, respectively, and let f_1 and f_2 be the corresponding marginal densities.

Vickrey Auction

In a Vickrey auction it is a dominant strategy for each bidder to bid truthfully – that is, to submit a bid vector $\mathbf{b}^i = \mathbf{x}^i$. The vector of competing bids that bidder i faces is

$$\mathbf{c}^{-i} = (x_1^j, x_2^j, 0)$$

where $j \neq i$. He is sure to win at least one unit, so he pays the third highest competing bid $c_3^{-i} = 0$ for this unit. He wins another unit if his value for the second unit exceeds bidder j's value for the second unit-that is, if $x_2^i > x_2^j$ and in that case pays the second-highest competing bid, $c_2^{-i} = x_2^j$, for this unit.

A bidder's expected payment when his value vector is $\mathbf{x} = (x_1, x_2)$ is therefore

$$\begin{aligned} m^V(\mathbf{x}) &= Prob[X_2 < x_2]E[X_2|X_2 < x_2] \\ &= \int_0^{x_2} y f_2(y)dy \end{aligned} \tag{10.6}$$

and notice that this is independent of x_1.

Example 10.3.1 *Suppose that X_1 and X_2 are the highest and second-highest order statistics, respectively, of two independent draws from a uniform distribution on* $[0,1]$.

In a Vickrey auction, the expected revenue of the seller is

$$E[R^V] = E[Y_4^{(4)}] = \frac{1}{5}$$

where $Y_4^{(4)}$ is the lowest of four independent draws from the uniform distribution on $[0,1]$.

Uniform-Price Auction

Recall that in a uniform price auction it is weakly dominant for each bidder to bid the value on the first unit. Thus, in any undominated equilibrium, $b_1^i = x_1^i$. It remains to determine the equilibrium bids on the second unit. Suppose that the bids on the second unit are determined by the increasing function $\beta_2 : [0,1] \to \mathbb{R}+$ and in the interest of notational simplicity let us write $\beta \equiv \beta_2$. (Here we are postulating that the bids on the second unit are independent of the value of the first unit, and as we will see, this is indeed the case.) Now notice that if such an equilibrium exists, it will allocate efficiently. The reason is that once again each bidder is guaranteed to win at least one unit. A bidder will win a second unit only if $\beta(x_2^i) > \beta(x_2^j)$ and since β is increasing, $x_2^i > x_2^j$, so it is efficient for him to do so.

We can now use the revenue equivalence principle to deduce that in any such equilibrium of the uniform-price auction the expected payment of a bidder with value vector x is the same as in a Vickrey auction, so using (10.6) we obtain

$$\begin{aligned} m^U(\mathbf{x}) &= m^V(\mathbf{x}) \\ &= \int_0^{x_2} y f_2(y) dy. \end{aligned}$$

On the other hand, it is easy to see that the equilibrium expected payment of bidder 1 when his value vector is $\mathbf{x} = (x_1, x_2)$ is

$$m^U(\mathbf{x}) = \int_0^{x_2} 2\beta(y) f_2(y) dy + (1 - F_2(x_2))\beta(x_2).$$

The first term is the expected payment in the event that bidder i wins both units so that the highest losing bid, and hence the price, is $\beta(x_2^j)$. The second term derives from the event that bidder i wins only one unit, so the highest losing bid is his own bid for the second unit-that is, $\beta(x_2)$.

The revenue equivalence principle implies that if β is the equilibrium bidding strategy for the second unit, then it must satisfy: for all $z \in [0, \omega]$,

$$\int_0^z y f_2(y) dy = \int_0^z 2\beta(y) f_2(y) dy + (1 - F_2(z))\beta(z)$$

Differentiating with respect to z, we deduce that β must satisfy the differential equation

$$z f_2(z) = \beta(z) f_2(z) + (1 - F_2(z))\beta'(z)$$

together with the initial condition $\beta(0) = 0$. This can be rearranged as

$$\beta'(z) = (z - \beta(z))\lambda_2(z)$$

where

$$\lambda_2(z) \equiv \frac{f_2(z)}{1 - F_2(z)}$$

is the hazard rate function associated with F_2. The solution to the differential equation above is

$$\beta(z) = \int_0^z y\lambda_2(y) dL(y|z)$$

is the relevant integrating factor. The strategy β is increasing and the argument that this constitutes an equilibrium is straightforward.

We have thus shown that a symmetric equilibrium bidding strategy in the uniform-price auction is

$$\beta^U(x_1, x_2) = (x_1, \beta(x_2)).$$

Example 10.3.2 (Example 10.3.1 continued) *X_1 and X_2 are the highest and second-highest order statistics from a uniform distribution on* $[0, 1]$.

It may be verified that $\beta(z) = z^2$. The selling price is the lowest of the four bids submitted in total. Since three units are for sale, the expected revenue is

$$\begin{aligned} E[R^U] &= 3E[\beta(Y_4^{(4)})] \\ &= 3E[(Y_4^{(4)})^2] = \frac{1}{5}. \end{aligned}$$

where $Y_4^{(4)}$ is the lowest of four independent draws from the uniform distribution on $[0, 1]$.

Discriminatory Auction

Once again, each bidder is assured of winning at least one unit. Now suppose that a bidder bids (b_1, b_2) such that $b_1 > b_2$. Reducing the bid on the first unit to $b_1 - \epsilon > b_2$ does not affect a bidder's chances of winning the first unit-he wins it regardless of his bid – but increases his payoff since he pays less for this unit. Thus, it cannot be optimal to bid in a way that $b_1 > b_2$ and in equilibrium we must have $b_1 = b_2$. In other words, in any equilibrium of the discriminatory auction bidders always submit flat demand functions. Moreover, the amount bid is determined solely by the marginal value of the second unit, x_2.

Suppose that the bids on the second unit are determined by the increasing function $\beta_2 : [0, 1] \to \mathbb{R}+$ and in the interest of notational simplicity let us once again write $\beta \equiv \beta_2$. Now notice that in this example, any equilibrium of the discriminatory auction with an increasing β is efficient. Suppose bidder i follows the equilibrium strategy of bidding the flat demand function $(\beta(x_2^i), \beta(x_2^i))$. He wins one unit for sure and wins a second unit if $\beta(x_2^i) > \beta(x_2^j)$. Since β is increasing, $x_2^i > x_2^j$, so it is efficient for i to win a second unit.

Since the equilibrium is efficient we can invoke the revenue equivalence principle to deduce that the expected payment of a bidder with value vector x in a discriminatory auction is the same as in a Vickrey auction. Using (10.6) we obtain

$$m^D(\mathbf{x}) = m^V(\mathbf{x}) = \int_0^{x_2} y f_2(y) dy$$

On the other hand, we know that the expected payment in a discriminatory auction is

$$m^D(\mathbf{x}) = \beta(x_2) + F_2(x_2)\beta(x_2)$$

The first term is the sure payment that the bidder will make for the first unit. The second term is the expected payment for the second unit, which he wins only if his value for the second unit x_2 exceeds the other bidder's value for the second unit. Since bidders submit flat demands, the amount paid for each unit is the same. From the above two equations we have

$$\beta(x_2) = \frac{1}{1 + F_2(x_2)} \int_0^{x_2} y f_2(y) dy$$

The function β is increasing and the argument that this constitutes as equilibrium is straightforward.

Thus, we have shown that a symmetric equilibrium in the discriminatory auction is

$$\beta^D(x_1, x_2) = (\beta(x_2), \beta(x_2)).$$

Example 10.3.3 (Example 10.3.1 continued) *X_1 and X_2 are the highest and second-highest order statistics from a uniform distribution on* $[0, 1]$.

The bidding strategy in the discriminatory auction is given by

$$\beta(z) = \frac{z^2 - \frac{2}{3}z^3}{1 + 2z - z^2}.$$

The expected payment of a bidder with value x_2 for the second unit is

$$m^D(\mathbf{x}) = (x_2)^2 - \frac{2}{3}(x_2)^3$$

and the *ex ante* expected payment of a bidder is

$$m^D = \int_0^1 m^D(\mathbf{x})dx_2 = \frac{1}{10}$$

Since there are two bidders, the expected revenue of the seller in a discriminatory auction is twice the *ex ante* payment of each bidder and equals, as it must, $\frac{1}{5}$

10.4 Sequential Sales

10.4.1 Sequential First Price Auctions

Consider a situation in which the K identical items are sold to $N > K$ bidders using a series of first-price sealed-bid auctions. Specifically, one of the items is auctioned using the first-price format, and the price at which it is sold - the winning bid - is announced. The second item is then sold and again the price at which it is sold - the winning bid in the second auction - is announced. The third item is then sold, and so on.

We restrict attention to situations in which each bidder has use for at most one unit - the case of *single-unit demand*. We also suppose that bidders have private values and that each bidder's value X_i is drawn independently from the same distribution F on $[0, \omega]$. It is then natural to look for a symmetric equilibrium.

A bidding strategy for a bidder consists of K functions $\beta_1^I, \beta_2^I, \ldots, \beta_K^I$, where $\beta_k^I(x, p_1, p_2, \ldots, p_{k-1})$ denotes the bid in the kth auction given that the bidder's value is x and the prices in the $k-1$ previous auctions were $p_1, p_2, \ldots, p_{k-1}$, respectively. All this assumes, of course, that the particular bidder has not already won an object and so is still active in the kth auction. (From now on, if there is no ambiguity, the superscript "I" identifying the first-price format will be omitted.)

Notice that if the equilibrium strategies β^K are increasing functions of the value x, then the items will be sold in order of decreasing values. The first item will go to the bidder with the highest value, the second to the bidder with the second-highest value, and so on. In that case the K units will be allocated efficiently.

In what follows it will be convenient to think of the auctions as being held in different periods. Moreover, the auctions are assumed to be held in a short enough time - say, the same day - so that bidders do not discount payoffs from later periods.

Two Units

We begin by looking at a situation in which only two units are sold ($K = 2$), so a symmetric equilibrium consists of two functions (β_1, β_2), denoting the bidding strategies in the first and second periods, respectively. We conjecture that these are increasing and differentiable.

Since the first-period strategy β_1 is assumed to be invertible, the value of the winning bidder in the first period is commonly known; it is just $y_1 = \beta_1^{-1}(p_1)$. Thus, the second-period strategy can be thought of as a function $\beta_2 : [0, \omega] \times [0, \omega] \to \mathbb{R}_+$ so that a bidder with value x bids an amount $\beta_2(x, y_1)$ if $Y_1 = y_1$.

Proposition 10.4.1 *Suppose bidders have single-unit demand and two units are sold by means of sequential first-price auctions. Symmetric equilibrium strategies are*

$$\beta_1^I(x) = E[Y_2 | Y_1 < x]$$
$$\beta_2^I(x) = E[Y_2 | Y_2 < x < Y_1],$$

where $Y_1 \equiv Y_1^{(N-1)}$ *is the highest, and* $Y_2 \equiv Y_2^{(N-1)}$ *is the second highest, of* $N-1$ *independently drawn values*

Proof 10.4.1 (Proof of Proposition 10.4.1:) *Let* F_1 *and* F_2 *be the distributions of* Y_1 *and* Y_2*, respectively, and let* f_1 *and* f_2 *be the corresponding densities. We prove the above proposition by backward induction.*

Second-Period Strategy

Consider the second-period auction and the decision problem facing a particular bidder, say 1, whose value is x*. Suppose all other bidders follow the equilibrium strategy* $\beta)_2(,y_1)$ *and bidder 1 bids* $\beta_2(z,y_1)$ *in the second auction. Since the bidders competing against bidder 1 in the second auction have values* $Y_2, Y_3, \ldots, Y_{N-1}$ *and in equilibrium* $Y_2 < y_1$*, it makes no sense for bidder 1 to bid an amount greater than* $\beta_2(y_1,y_1)$*. His expected payoff in the second auction if he bids* $\beta_2(z,y_1)$ *for some* $z \leq y_1$ *is*

$$\Pi(z,x;y_1) = F_2(z|Y_1 = y_1) \times [x - \beta_2(z,y_1)]$$

Differentiating $\Pi(z,x;y_1)$ *with respect to* z *we obtain the first-order condition that in equilibrium, for all* x*,*

$$\beta_2'(x,y_1) = \frac{f_2(x|Y_1 = y_1)}{F_2(x|Y_1 = y_1)}[x - \beta_2(x,y_1)]$$

together with the boundary condition $\beta_2(0,y_1) = 0$*.*

It can be easily shown that

$$F_2(z|Y_1 = y_1) = F_1^{(N-2)}(z|Y_1^{(N-2)}) < y_1 = \frac{F(z)^{N-2}}{F(y_1)^{N-2}}$$

and so

$$\beta_2'(x,y_1) = \frac{(N-2)f(x)}{F(x)}[x - \beta_2(x,y_1)]$$

or equivalently,

$$\frac{\partial}{\partial x}(F(x)^{N-2})\beta_2(x,y_1) = (N-2)F(x)^{N-3}f(x)x$$

Notice that the right-hand side of the preceding equation is independent of y_1 *and hence of the price announcement. This means that* β_2 *is, in fact, independent of* y_1*; in other words, the* price announcements have no effect *on the equilibrium bids in the second period. The solution to the differential equation is*

$$\beta_(x) = E[Y_2|Y_2 < x < Y_1].$$

Thus, the complete bidding strategy for the second period is to bid $\beta_2(x)$ *if* $x \leq y_1$ *and to bid* $\beta_2(y_1)$ *if* $x > y_1$*. The latter may occur if bidder 1 himself underbid in the first auction, say by mistake, causing someone else, with a lower value, to win. Even though this represents "off-equilibrium" behavior on the part of bidder 1 himself, recall that a strategy must prescribe actions in all contingencies.*

First-Period Strategy

For bidder 1 with value x, suppose that all other bidders are following the first-period strategy β_1. Further, suppose that all bidders, including bidder 1, will follow β_2 in the second period, regardless of what happens in the first period.

The equilibrium calls on bidder 1 to bid $\beta_1(x)$ in the first stage, but consider what happens if he decides to bid $\beta_1(z)$ instead. If $z \geq x$, his payoff is

$$\Pi(z, x) = F_1(z)[x - \beta_1(z)] + (N-1)(1 - F(z))F(x)^{N-2}[x - \beta_2(x)]$$

where the first term results from the event $Y_1 < z$, so that he wins the first auction with a bid of $\beta_1(z)$. The second term results from the event $Y_2 < x \leq z \leq Y_1$, so that he loses the first auction but wins the second. On the other hand, if $z < x$, his payoff is

$$\Pi(z, x) = F_1(z)[x - \beta_1(z)] + [F_2(x) - F_1(x)][x - \beta_2(x)] + \int_z^x [x - \beta_2(y_1)]f_1(y_1)dy_1$$

where the first term again results from the event $Y_1 < z$. The second term results from the event $Y_2 < x < Y_1$, so that he loses the first auction but wins the second with a bid of $\beta_2(x)$. The third term results from the event $z < Y_1 < x$ so that he loses the first auction but wins the second with a bid of $\beta_2(Y_1)$.

The first-order conditions in the two cases are

$$\begin{aligned} 0 &= f_1(z)[x - \beta_1(z)] - F_1(z)\beta_1'(z) \\ &\quad - (N-1)f(z)F(x)^{N-2}[x - \beta_2(x)]. \end{aligned}$$

and

$$\begin{aligned} 0 &= f_1(z)[x - \beta_1(z)] - F_1(z)\beta_1'(z) \\ &\quad - f_1(z)[x - \beta_2(z)]. \end{aligned}$$

respectively. In equilibrium it is optimal to bid $\beta_1(x)$ and setting $z = x$ in either first-order condition results in the differential equation

$$\beta_1'(x) = \frac{f_1(x)}{F_1(x)[\beta_2(x) - \beta_1(x)]}$$

together with the boundary condition $\beta_1(0) = 0$ and we can get the solution

$$\beta_1(x) = E[Y_2 | Y_1 < x]$$

This completes the proof.

More than Two Units

The analysis for more than two units is the same as the analysis for two units and we will not provide the details here only give the following results

Proposition 10.4.2 *Suppose bidders have single-unit demand and K units are sold by means of sequential first-price auctions. Symmetric equilibrium strategies are given by*

$$\beta_k^I(x) = E[Y_k | Y_k < x < Y_{k-1}]$$

where $\beta_k^I(x)$ denotes the bidding strategy in the kth auction and $Y_k \equiv Y_k^{(N-1)}$ is the kth highest of $N-1$ independently drawn values.

Example 10.4.1 *Values are uniformly distributed on* $[0, 1]$.

In the last period, the equilibrium bidding strategy is

$$\beta_K(x) = \frac{N-K}{N-K+1}x.$$

Proceeding inductively, it may be verified that the bidding strategy in the kth first-price auction is

$$\beta_k(x) = \frac{N-K}{N-k+1}x.$$

Figure 10.3 depicts the bidding strategies for the case of three objects ($K = 3$) and five bidders ($N = 5$).

Equilibrium Bids and Prices

Some features of the equilibrium strategies are worth noting. First, for all k, $\beta_{k+1}(x) > \beta_k(x)$, that is, a bidder with value x who is active in the kth auction and fails to win, bids higher in the $k+1$st auction. Informally, this is due to the deterioration of available supply relative to current demand. The higher bids from those who did not win in the previous period is, however, mitigated by the fact that there is one fewer bidder in this period. Indeed, the remaining bidders have smaller values than the winner of the previous round. Remarkably, the equilibrium is such that the two effects exactly offset each other and the prices in successive auctions show no trend. Precisely, the equilibrium price path is a *martingale* - at the end of the kth auction, the expected price in the $k+1$st auction is the same as the realized price in the kth auction.

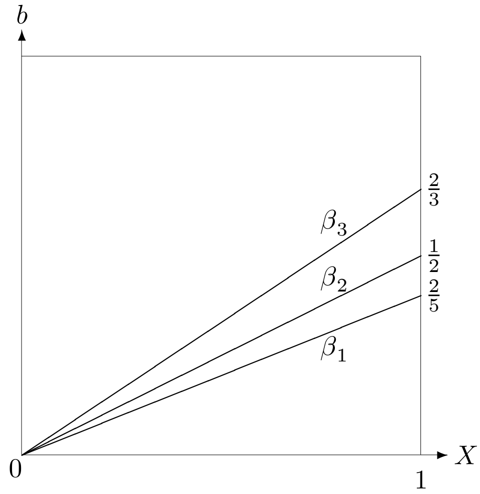

Figure 10.3: Equilibrium of Sequential First-Price Auction

To see this, suppose that in equilibrium, bidder 1 with value x wins the kth auction. Then, absent any ties, it must be that

$$Y_{K-1} < \ldots < Y_k < x < Y_{k-1} < \ldots < Y_1$$

Now let the random variables P_k and P_{k+1} denote the prices in periods k and $k+1$, respectively. We know that the realized price in period k, $p_k = \beta_k(x)$. Moreover, the price in the $k+1$st period is the random variable $P_{k+1} = \beta_{k+1}(Y_k)$ and

$$\begin{aligned} E[P_{k+1}|P_k = p_k] &= E[\beta_{k+1}|Y_k < x < Y_{k-1}] \\ &= \beta_k(x) = p_k. \end{aligned}$$

This establishes that the price path is a martingale. An implication of this property of the price path is that there are no opportunities for intertemporal arbitrage. For instance, if $E[P_{k+1}|P_k = p_k] < p_k$, then it would benefit bidders to decrease their bids on the current item with a view to waiting for the next item.

10.4.2 Sequential Second Price Auctions

Revenue Equivalence

If $\beta_1^{II}, \beta_2^{II}, \ldots, \beta_K^{II}$ is a symmetric increasing equilibrium, then, as in the case of sequential first-price auctions, the K units will be allocated efficiently. Indeed, the first unit will go to the bidder with the highest value, the second to the bidder with the second-highest value, and so on. This means that the two mechanisms - the sequential first - and second-price formats - are revenue equivalent. Specifically, if $m^I(x)$ and $m^{II}(x)$ denote the expected payment by a bidder with value x in K sequential first- and second-price formats, respectively, then for all x,

$$m^I(x) = m^{II}(x)$$

Now define $m_k^I(x)$ to be the expected payment made in the kth auction by a bidder with value x when the items are sold by means of K first-price auctions. Define $m_k^{II}(x)$ in analogous fashion for the sequential second-price format. Clearly,

$$m^I(x) = \sum_{k=1}^{K} m_k^I(x) \text{ and } m^{II}(x) = \sum_{k=1}^{K} m_k^{II}(x)$$

While the revenue equivalence principle as such only guarantees that the overall expected payments in the two formats are the same, we claim that, in fact, for all k,

$$m_k^I(x) = m_k^{II}(x)$$

that is, the expected payment in the kth first-price auction is the same as the expected payment in the kth second-price auction. In other words, we claim that the two auctions are payment equivalent period by period. We argue by induction, starting with the Kth auction. Prior to the last auction, the information available to the remaining $N - K + 1$ bidders in either format is the same. For instance, bidder 1 knows his own value x, that his competitors have values $Y_{K+1}, Y_{K+2}, \ldots, Y_N$, and that $Y_K = y_K$. The revenue equivalence principle implies that $m_K^I(x) = m_K^{II}(x)$. Now consider the start of auction $K - 1$ and think of the remaining two formats as mechanisms for allocating two units. Once again the information available to the remaining $N - K + 2$ bidders is the same. For instance, bidder 1 knows his own value x, that his competitor have values $Y_K, Y_K, \ldots, Y_N$ and that $Y_{K-1} = y_{K-1}$. Once again, the revenue equivalence principle implies that

$$m_{K-1}^I(x) + m_K^{II} = m_{K-1}^{II}(x) + m_k^I(x)$$

and since $m_K^I(x) = m_K^{II}(x)$, we have $m_{K-1}^I(x) = m_{K-1}^{II}(x)$. Proceeding inductively in this way establishes that for all k, $m_k^I(x) = m_k^{II}(x)$.

Equilibrium Bids

Proposition 10.4.3 *Suppose bidders have single-unit demand and K units are sold by means of sequential second-price auctions. Symmetric equilibrium strategies are given by*

$$\beta_K^{II}(x) = x$$

and for all $k < K$,

$$\beta_k^{II}(x) = \beta_{k+1}^{I}(x)$$

where $\beta_{k+1}^{I}(x)$ is the $k+1$st period equilibrium bidding strategy in the sequential first-price auction format, derived in Proposition 10.4.2.

Proof 10.4.2 (Proof of Proposition 10.4.3:) *Clearly, in the last period it is a dominant strategy to bid one's value - that is,*

$$\beta_K^{II}(x) = x$$

and this is for the same reason that it is a dominant strategy in a single unit second-price auction. Now notice that for any $k < K$, if bidder 1 with value x wins the kth auction, then it must be that

$$Y_{K-1} < \ldots < Y_k < x < Y_{k-1} < \ldots < Y_1$$

and the price he pays - the highest competing bid-is $\beta_k^{II}(Y_k)$. Thus,

$$m_k^{II} = Prob[Y_k < x < Y_{k-1}] \times E[\beta_k^{II}(Y_k)|Y_k < x < Y_{k-1}]$$

On the other hand, in the first-price format a winning bidder pays his own bid, so

$$m_k^{I} = Prob[Y_k < x < Y_{k-1}] \times \beta_k^{I}(x).$$

But since $\beta_k^{I}(x) = E[\beta_{k+1}^{I}(Y_k)|Y_k < x < Y_{k-1}]$, the fact that the kth period expected payments are equal implies that

$$\begin{aligned} &Prob[Y_k < x < Y_{k-1}] \times E[\beta_k^{II}(Y_k)|Y_k < x < Y_{k-1}] \\ &= Prob[Y_k < x < Y_{k-1}] \times \beta_k^{I}(x). \end{aligned}$$

Differentiating both sides of the equality with respect to x results in the identity

$$\beta_k^{II}(x) = \beta_{k+1}^{I}(x)$$

Example 10.4.2 *Values are uniformly distributed on* $[0,1]$.

In the last period of a sequential second-price auction, it is a weakly dominant strategy to bid one's value, so

$$\beta_K(x) = x$$

Bidding strategies in earlier periods can be found using the strategies for the sequential first-price auction derived in Example 15.1 and applying the characterization obtained in Proposition 10.4.3. This results in

$$\beta_k(x) = \frac{N-K}{N-k}x$$

The bidding strategies are portrayed in Figure 10.4 for the case of three objects ($K = 3$) and five bidders ($N = 5$) and should be compared with the strategies for the sequential first-price auction in Figure 10.3.

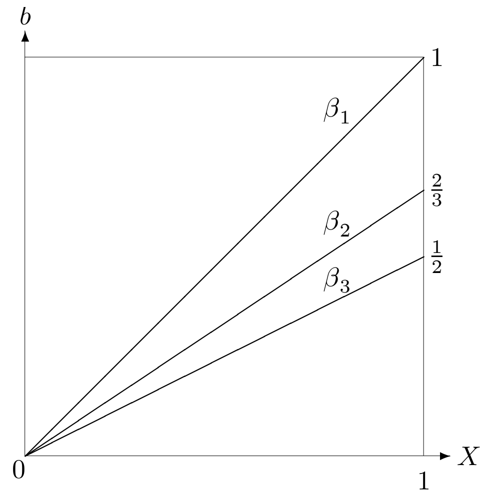

Figure 10.4: Equilibrium of Sequential Second-Price Auction

Again, some properties of the equilibrium bidding strategies are worth noting.

- First, for all k, $\beta_k^{II}(x) > \beta_k^{I}(x)$, that is, every bidder bids more in a second-price sequential auction than in its first-price counterpart.

- Second, while it is a dominant strategy to bid one's value in the last period, this is not the case in earlier periods. This is because in any period $k < K$, there is an "option value" associated with not winning the current auction - the expected payoff arising from the possibility of winning an auction in some later period. In contrast to the case of a single-object, the strategies in a sequential second-price auction are optimal only if other bidders also adopt them.

- Third, the equilibrium price process in a sequential second price auction is also a martingale. Suppose bidder 1 with value x wins in period $K-1$. Then $Y_{K-1} < x < Y_{K-2}$ and the price in period $K-1$ is the realization of the random variable $P_{K-1} = \beta^{II}_{K-1}(Y_{K-1})$. Let the realized price be $p_{K_1} = \beta^{II}_{K-1}(Y_{K-1})$, where y_{K-1} is the realized value of Y_{K-1}. In the last period the bidder with value $Y_{K-1} = y_{K-1}$ will win and the price in the last period will be $P_K = \beta^{II}_K(Y_K) \equiv Y_K$ since it is weakly dominant to bid one's value in the last auction.

$$E[P_K | P_{K-1} = p_{K-1}] = p_{K-1}$$

  In earlier periods, the martingale property of prices in a sequential second price auction is a consequence of the corresponding property in a sequential first-price auction and the relationship between the two equilibrium strategies in Proposition 10.4.3.

Bibliography

Amann, E., and W. Leininger (1995). Expected Revenue of All-Pay and First-Price Sealed-Bid Auctions with Affiliated Signals. *Zeitschrift fur Nationalokonomie*, **61**, 273-279.

Ashenfelter, O. (1989). How Auctions Work for Wine and Art. *Journal of Economic Perspectives*, **3**, 23-36.

Ausubel, L. (1997). An Efficient Ascending-Bid Auction for Multiple Objects. Mimeo, University of Maryland, June.

Ausubel, L. (2001). A Generalized Vickrey Auction. *Econometrica*, forthcoming.

Avery, C. (1998). Strategic Jump Bidding in English Auctions. *Review of Economic Studies*, **65**, 185-210.

Back, K., and J. Zender (1993). Auctions of Divisible Goods: On the Rationale for the Treasury Experiment. *Review of Financial Studies*, **6**, 733-764.

Baye, M., Kovenock, D., and Vries, C. de. (1993). Rigging the lobbying process: An application of all – pay auctions., *American Economic Review*, **83**, 289-294.

Bikhchandani, S., and J. Riley (1991). Equilibria in Open Common Value Auctions. *Journal of Economic Theory*, **53**, 101-130.

Bresky, M. (2000). Equilibria in Multi-Unit Auctions. Mimeo, CERGE, Czech Republic, October.

Bulow, J., and P. Klemperer (1996). Auctions versus Negotiations. *American Economic Review*, **86**, 180-194.

Bulow, J., and J. Roberts (1989). The Simple Economics of Optimal Auctions. *Journal of Political Economy*, **97**, 1060-1090.

Cao, X., and G. Tian (2008). Second Price Auctions with Differentiated Participation Costs. *Working Paper*.

Cao, X., and G. Tian (2009a). Second Price Auctions with Two-Dimensional Private Information on Values and Participation Costs. *Working Paper*.

Cao, X., and G. Tian (2009b). Equilibrium in First Price Auctions with Participation Costs. *Working Paper*.

Che, Y-K., and I. Gale (1998). Standard Auctions with Financially Constrained Bidders. *Review of Economic Studies*, **65**, 1-22.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public Choice*, **2**, 19-33.

Cremer, J., and R. McLean (1985). Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist when Demands are Interdependent. *Econometrica*, **53**, 345-361.

Cremer, J., and R. McLean (1988). Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions. *Econometrica*, **56**, 1247-1257.

Engelbrecht-Wiggans, R. (1994). Auctions with price-proportional benefits to bidders. *Games and Economic Behavior*, **6**, 339-346.

Engelbrecht-Wiggans, R., and Kahn, C. (1998). Low revenue equilibria in simultaneous auctions (Working Paper). University of Illinois.

Engelbrecht-Wiggans, R., and C. Kahn (1998a). Multi-Unit Auctions with Uniform Prices. *Economic Theory*, **12**, 227-258.

Engelbrecht-Wiggans, R., and C. Kahn (1998b). Multi-Unit Pay-Your-Bid Auctions with Variable Rewards. *Games and Economic Behavior*, **23**, 25-42.

Engelbrecht-Wiggans, R., P. Milgrom, and R. Weber (1983). Competitive Bidding and Proprietary Information. *Journal of Mathematical Economics*, **11**, 161-169.

Griesmer, J., R. Levitan and M. Shubik (1967). Toward a Study of Bidding Processes, Part IV - Games with Unknown Costs. *Naval Research Logistics Quarterly*, **14**, 415-434.

Groves, T. (1973). Incentives in teams. Econometrica, 41, 617-613.

Gupta, M., and B. Lebrun (1999). First Price Auctions with Resale. *Economics Letters*, **64**, 181-185.

Haile, P. (1999). Auctions with Resale. Mimeo, University of Wisconsin, October.

Hafalir, I. and Krishna, V. (2006). Asymmetric Auctions with Resale. Working Paper

Harstad, R. M., and Levin, D. (1990). A class of dominance solvable common - value auctions. *Review of Economic Studies*, **52**, 525-528.

Harstad, R., J. Kagel, and D. Levin (1990). Equilibrium Bid Functions for Auctions with an Uncertain Number of Bidders. *Economics Letters*, **33**, 35-40.

Holt, C. (1980). Competitive Bidding for Contracts under Alternative Auction Procedures. *Journal of Political Economy*, **88**, 433-445.

Jeitschko, T. (1999). Equilibrium Price Paths in Sequential Auctions with Stochastic Supply. *Economics Letters*, **64**, 67-72.

Kagel, J. H., and Levin, D. (1993). Independent private value auctions: bidder behavior in first-, second-, and third- price auctions with varying numbers of bidders. *Economic Journal*, **103**, 868-879.

Kaplan, T.R. and Sela, A. (2006). Second Price Auctions with Private Entry Costs. *Working paper.*

Katzman, B. (1999). A Two Stage Sequential Auction with Multi-Unit Demands. *Journal of Economic Theory*, **86**, 77-99.

Krishna, V., and J. Morgan (1997). An Analysis of the War of Attrition and the All-Pay Auction. *Journal of Economic Theory*, **72**, 343-362.

Krishna, V., and Perry, M. (1998). Efficient mechanism design (Discussion Paper). Center for Rationality, Hebrew University Jerusalem.

Landsberger, M., and B. Tsirelson (2000). Correlated Signals against Monotone Equilibria. Mimeo, University of Haifa, April.

Levin, D., and J. Smith (1996). Optimal Reservation Prices in Auctions. *Economic Journal*, **106**, 1271-1283.

Marshall, R., M. Meurer, J.-F. Richard, and W. Stromquist (1994). Numerical Analysis of Asymmetric First Price Auctions. *Games and Economic Behavior*, **7**, 193-220.

Maskin, E. (1992). Auctions and Privatization. in H. Siebert (ed.), *Privatization*, Kiel: Institut fur Weltwirtschaften der Universiat Kiel, 115-136.

Maskin, E., and Riley, J. G. (1984). Optimal auctions with risk averse buyers. Econometrica, 52, 1473-1518.

Maskin, E., and Riley, J. G. (1985). Auction theory with private values. *American Economic Review, Papers and Proceedings*, **75**, 150-155.

Maskin, E., and J. Riley (2000a). Asymmetric Auctions. *Review of Economic Studies*, **67**, 413-438.

Maskin, E., and J. Riley (2000b). Equilibrium in Sealed High Bid Auctions. *Review of Economic Studies*, **67**, 439-454.

Matthews, S. (1987). Comparing auctions for risk averse buyers: A buyer's point of view. *Econometrica*, **55**, 633-646.

McAfee, P., J. McMillan, and P. Reny (1989). Extracting the Surplus in a Common Value Auction. *Econometrica*, **57**, 1451-1460.

McAfee, R., and McMillan, J. (1987a). Auctions and bidding. *Journal of Economic Literature*, **25**, 699-738.

McAfee, R., and McMillan, J. (1987b). Auctions with a stochastic number of bidders. *Journal of Economic Theory*, **43**, 1-19.

McAfee, P., and D. Vincent (1993). The Declining Price Anomaly. *Journal of Economic Theory*, **60**, 191-212.

Milgrom, P., and Weber, R. J. (1982). A Theory of Auctions and Competitive Bidding. *Econometrica*, **50**, 1089-1122.

Milgrom, P. R. (1987). Auction theory. In T. F. Bewley (Ed.), *Advances in economic theory* (p. 1-32). Cambridge: Econometric Society Monographs No. 12, Cambridge University Press.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research*, **6**, 58-73.

Noussair, C. (1995). Equilibria in a Multi-Object Uniform Price Sealed Bid Auction with Multi-Unit Demands. *Economic Theory*, **5**, 337-351.

Ortega Reichert, A. (1968). Models for Competitive Bidding under Uncertainty. Ph.D. Dissertation (Technical Report No. 8), Department of Operations Research, Stanford University.

Perry, M., and P. Reny (1999). On the Failure of the Linkage Principle in Multi-Object Auctions. *Econometrica*, **67**, 885-890.

Plum, M. (1992). Characterization and computation of Nash - equilibria for auctions with incomplete information. *International Journal of Game Theory*, **20**, 393-418.

Reny, P. (1999). On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica*, **67**, 1029-1056.

Riley, J. G., and Samuelson, W. F. (1981). Optimal Auctions. *American Economic Review*, **71**, 381-392.

Swinkels, J. (1999). Asymptotic Efficiency for Discriminatory Private Value Auctions. *Review of Economic Studies*, **66**, 509-528.

Swinkels, J. (2001). Efficiency of Large Private Value Auctions. *Econometrica*, **69**, 37-68.

Vickery, W. (1961). Counterspeculation, Auctions and Competitive Sealed Tenders. *Journal of Finance*, **16**, 8-37.

Vickrey, W. (1962). Auctions and Bidding Games. *Princeton Conference Series*, **29**, Princeton, NJ: Princeton University Press, 15-27.

Weber, R. (1983). Multiple Object Auctions. in R. Engelbrecht-Wiggans, M. Shubik, and R. Stark (eds.), Auctions, Bidding and Contracting: Uses and Theory, New York, NY: New York University Press, 165-191.

Wilson, R. (1979). Auctions of shares. *Quarterly Journal of Economics*, **93**, 675-689.

Wilson, R. (1992). Strategic analysis of auctions. In R. Aumann and S. Hart (Eds.), *Handbook of game theory* (Vol. I, pp. 227-280). Elsevier Science.

Wilson, R. B. (1977). A bidding model of perfect competition. *Review of Economic Studies*, **44**, 511-518.

Wolfstetter, E. (2001). Third- and higher-price auctions. In M. Braulke and S. Berninghaus (Eds.), *Beitrage zur Mikro- und zur Makrookonomik: Festschrift fur Hans-Jurgen Ramser* (p. 495-503). Springer Verlag.