General Equilibrium Theory

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Chapter 1

Mathematical Preliminaries

In this chapter, some mathematical terminologies and preliminaries are introduced. These include basic notations, definitions and many important facts, which will be used in the subsequent chapters. Most of these are taken from [1, 2].

1.1 Set Theory

A *set* is a collection of objects, and objects constituting a set are called *elements* of the set. Typically, the uppercase letters $X, Y, Z, ...$ are used to denote sets and those representing elements are the lowercase letters $x, y, z, ...$ The symbols $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ represent the sets of *positive integers*, *rational numbers* and *real numbers* respectively. In addition, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ denotes the set of non-negative real numbers and $\mathbb{R}^\star = \mathbb{R} \cup \{\infty, -\infty\}$ is the set of *extended real numbers*, where ∞ and $-\infty$ can be interpreted as $-\infty < x < \infty$ for any real number x. The symbol ∞ is called the *infinity*. For any $a, b \in \mathbb{R}$ with $a < b$, define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \text{ and } (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Here $[a, b]$ and (a, b) are called the *closed interval* and the *open interval*. In some instance, the term *family* is used instead of set. As usual, $\emptyset$ refers to the set containing no element and is known as the *empty set*. The notation $x \in X$ indicates that x is an element of X. If x is not an element of X, the notation $x \notin X$ is employed. For any two sets X and Y, let $X \setminus Y = \{x \in X : x \notin Y\}$. The expression $X \subseteq Y$ means that $x \in X$ implies $x \in Y$. In this case, X is called a *subset* of Y. The term *subfamily* is applied in an appropriate place. If $Y \subseteq X$, then $X \setminus Y$ is termed as the *complement* of Y in X. If $X \subseteq Y$ and $Y \subseteq X$, then X and Y are said to be *identical* and written as

$X = Y$. Further, if X and Y are not identical, then the notation $X \neq Y$ is used. In addition, $X \subset Y$ denotes the situation "$X \subseteq Y$ and $X \neq Y$".

The *power set* of a set X, denoted by $\mathscr{P}(X)$, is the family of all subsets of X. For any $\{A_j : j \in J\} \subseteq \mathscr{P}(X)$, define

$$\bigcup_{j \in J} A_j = \{x \in X : x \in A_j \text{ for some } j \in J\},$$

and

$$\bigcap_{j \in J} A_j = \{x \in X : x \in A_j \text{ for all } j \in J\}.$$

The notations $\bigcup_{j \in J} A_j$ and $\bigcap_{j \in J} A_j$ are sometimes written as $\bigcup\{A_j : j \in J\}$ and $\bigcap\{A_j : j \in J\}$ respectively. Here $\bigcup_{j \in J} A_j$ and $\bigcap_{j \in J} A_j$ are termed as the *union* and the *intersection* of the family $\{A_j : j \in J\}$. The notation $\prod_{j \in J} A_j$ refers to the *Cartesian product* of $\{A_j : j \in J\}$, which is defined by

$$\prod_{j \in J} A_j = \{(x_j : j \in J) : x_j \in A_j \text{ for all } j \in J\}.$$

In particular, in the case of two sets A and B, notations $A \cup B$, $A \cap B$ and $A \times B$ are utilized instead to denote the union, the intersection and the Cartesian product of A and B, respectively. Two sets A and B are *disjoint* if $A \cap B = \emptyset$, and a family $\{A_j : j \in J\}$ is called *pairwise disjoint* if A_i and A_j are disjoint for all $i, j \in J$ with $i \neq j$. The *symmetric difference* between two sets A and B is defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. A *partition* of a non-empty set X is a family $\{A_j : j \in J\}$ of non-empty pairwise disjoint subsets of X satisfying $\bigcup_{j \in J} A_j = X$. Let $\{\mathscr{B}_i : 1 \leq i \leq \ell\}$ be a finite family of partitions of X, then

$$\left\{\bigcap_{i=1}^{\ell} B_i : \bigcap_{i=1}^{\ell} B_i \neq \emptyset,\ B_i \in \mathscr{B}_i \text{ for all } 1 \leq i \leq \ell\right\}$$

is also a partition of X. This is called the *refinement* of $\{\mathscr{B}_i : 1 \leq i \leq \ell\}$.

Let X and Y be two sets. A *relation* between elements of X and Y is a subset of $X \times Y$. If $X = Y$, then such a relation is also termed as the *binary relation* on X. A binary relation $\succeq$ on X is said to be *reflexive* if $(x, x) \in \succeq$ for all $x \in X$, *symmetric* if $(x, y) \in \succeq$ then $(y, x) \in \succeq$, and *transitive* if $(x, y) \in \succeq$ and $(y, z) \in \succeq$ imply $(x, z) \in \succeq$. An *equivalence relation* on X is a binary relation on X which is reflexive, symmetric, and transitive. Further, a binary relation $\succeq$ on X is called *complete* if for any $x, y \in X$, either $(x,y) \in \succeq$ or $(y,x) \in \succeq$ or both. The notation $(x,y) \in \succeq$ is also written as $x \succeq y$.

Axiom of Choice. If $\{A_j : j \in J\}$ is a non-empty family of non-empty sets, then there is a function $f : J \to \bigcup_{j\in J} A_j$ satisfying $f(j) \in A_j$ for each $j \in J$. In other words, the Cartesian product of a non-empty family of non-empty sets is non-empty.

1.2 Calculus on $\mathbb{R}^\ell$

Let

$$\mathbb{R}^\ell = \left\{ \left(x^1, \cdots, x^\ell \right) : x^i \in \mathbb{R} \text{ for all } 1 \leq i \leq \ell \right\}$$

Define

$$\varrho(x,y) = \sqrt{\sum_{i=1}^{\ell} (x^i - y^i)^2}.$$

In particular, when $\ell = 1$ then $\varrho(x,y)$ gives the *absolute difference* between x and y, and it is usually denoted by a special natation $|x-y|$. For a non-empty subset E of $\mathbb{R}^\ell$, the *diameter* of E is defined by

$$\text{diam}E = \sup\{\varrho(x,y) : x,y \in E\}.$$

A set E is *bounded* if $\text{diam}E < \infty$, and is *unbounded* if $\text{diam}E = \infty$. The *open ball* of radius $\varepsilon > 0$ centered at a point x in $\mathbb{R}^\ell$ is

$$B_\varepsilon(x) = \{y \in \mathbb{R}^\ell : \varrho(x,y) < \varepsilon\}.$$

A set U in $\mathbb{R}^\ell$ is called an *open set* if for every $x \in U$ there is some $\varepsilon(x) > 0$ such that $B_\varepsilon(x) \subseteq U$. A set E is called a *closed set* if $\mathbb{R}^\ell \setminus E$ is open. Suppose that E is a non-empty subset of $\mathbb{R}^\ell$. The *interior* of E, denoted by $\text{int}E$, is the largest (with respect to "$\subseteq$") open set contained in E. Thus, if E is an open set, then $\text{int}E = E$. The *closure* of E, denoted by $\text{cl}E$, is the smallest closed set containing E. Thus, if E is a closed set, then $\text{cl}E = E$. Note that a point $x \in \text{cl}E$ is equivalent to the fact that $U \cap E \neq \emptyset$ for every neighborhood U of x. A *neighborhood* of a point x in $\mathbb{R}^\ell$ is any set E such that $x \in \text{int}E$. In such a situation, x is called an *interior point* of E. A point x is said to be a *limit point* of a set E if for any neighborhood U of x, $(U \setminus \{x\}) \cap E \neq \emptyset$. Let $A \subseteq X \subseteq \mathbb{R}^\ell$. The set A is called *open in* X if $A = B \cap X$ for some open set B in $\mathbb{R}^\ell$ and it is called *closed in* X if $X \setminus A$ is open in X. Recall that a set in $\mathbb{R}^\ell$ is called a *compact set* if it is closed and bounded. The $(\ell-1)$-simplex of $\mathbb{R}^\ell$ is defined as

$$\Delta^\ell = \left\{ x = (x^1, \cdots, x^\ell) \in \mathbb{R}^\ell : x^i \geq 0 \text{ for all } 1 \leq i \leq \ell \text{ and } \sum_{i=1}^{\ell} x^i = 1 \right\}.$$

Given $X, Y \subseteq \mathbb{R}^\ell$, a relation $f \subseteq X \times Y$ is called a *function* if for every $x \in X$ there is a $y \in Y$ such that $(x, y) \in f$ and $(x, y_1), (x, y_2) \in f$ implies $y_1 = y_2$. The unique y is called the *image of* x *under* f, denoted as $y = f(x)$. A function f from X to Y is usually written as $f : X \to Y$ instead of $f \subseteq X \times Y$. A function $f : X \to Y$ is said to be *one-one* if $f(x) \neq f(y)$ for $x \neq y$, $x, y \in X$ and *onto* if for each $y \in Y$ there is some $x \in X$ such that $f(x) = y$. A *bijection* is a one-one and onto function. A function $f : X \to Y$ is *continuous* if for any open set V in Y, the set $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X. A function $f : X \to \mathbb{R}$ is called the *real-valued*. The *support* of a real-valued function $f : X \to \mathbb{R}$ is defined by $\mathrm{supp}(f) = \{x \in X : f(x) \neq 0\}$. Let $A \subseteq X$ and $f : A \to X$ be a function. A point $x \in A$ is termed a *fixed point of* f if $x = f(x)$.

Brouwer Fixed Point Theorem. Any continuous function $f : \Delta^\ell \to \Delta^\ell$ has a fixed point.

A sequence $\{x_n : n \geq 1\}$ converges to x in $X \subseteq \mathbb{R}^\ell$ if $\{\varrho(x_n, x) : n \geq 1\}$ converges to 0. The notation $\lim_{n\to\infty} x_n = x$ is used to represent that $\{x_n : n \geq 1\}$ converges to x. A *Cauchy sequence* in X is a sequence $\{x_n : n \geq 1\}$ such that for each $\varepsilon > 0$ there is some $N \geq 1$ such that $\varrho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Note that a sequence is convergent if and only if it is a Cauchy sequence. A very important fact of $\mathbb{R}^\ell$ is that a set A is compact if and only if every sequence in A has a convergent subsequence. The following conditions are equivalent:

(i) A function $f : X \to Y$ is continuous.

(ii) If $\{x_n : n \geq 1\}$ converges to x, then $\{f(x_n) : n \geq 1\}$ converges to $f(x)$.

A sequence $\{f_n : n \geq 1\} : X \to Y$ converges *pointwise* to a function $f : X \to Y$ if $\{f_n(x) : n \geq 1\}$ converges to $f(x)$ for all $x \in X$. In this case, f is named as the *pointwise limit* of $\{f_n : n \geq 1\}$. A sequence $\{f_n : n \geq 1\} : X \to Y$ converges *uniformly* to a function $f : X \to Y$ if for each $\varepsilon > 0$ there is some $N \geq 1$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and $n \geq N$.

1.3 Correspondences

A *correspondence* F from X to Y is defined as associating to each $x \in X$ a subset $F(x)$ of Y and is denoted by $F : X \rightrightarrows Y$. The *graph* of F, denoted by Gr_F, is defined as

$$\mathrm{Gr}_F = \{(x, y) \in X \times Y : y \in F(x),\ x \in X\}.$$

By identifying F with its graph, one can treat F as a relation between elements of X and Y. Here $F(x)$ is called the *image of* F *at* x. The *domain* of F is defined by $\mathrm{Dom}(F) = \{x \in X : F(x) \neq \emptyset\}$ and F is called *non-empty valued* if $\mathrm{Dom}(F) = X$. There are two ways to define the inverse image by F of a subset U of Y:

$$F^-(U) = \{x \in X : F(x) \cap U \neq \emptyset\} \text{ and } F^+(U) = \{x \in X : F(x) \subseteq U\}.$$

Here $F^-(U)$ and $F^+(U)$ are called the *lower and upper inverses of* U *by* F. If Z is a set and $G : Y \rightrightarrows Z$ then the *composition correspondence* $G \circ F : X \rightrightarrows Z$ is defined as

$$(G \circ F)(x) = \bigcup\{G(y) : y \in F(x)\}.$$

If $F(x)$ is a singleton for each $x \in X$, then F is called a *function*. The lower case letters such as $f, g, h, ...$ are employed to denote functions. A correspondence $F : X \rightrightarrows Y$ can also be written in the form of a function as $F : X \to \mathscr{P}(Y)$. A correspondence $F : X \rightrightarrows Y$ is called *closed at* x if a sequence $\{(x_n, y_n) \in X \times Y : n \geq 1\}$ converges to $(x, y) \in X \times Y$ and $y_n \in F(x_n)$ for every n then $y \in F(x)$. It is *closed* (has *closed graph*) if it is closed at every point of X.

Definition 1.3.1. Let $X, Y \subseteq \mathbb{R}^\ell$. A correspondence $F : X \rightrightarrows Y$ is *upper hemicontinuous at* x (resp. *lower hemicontinuous at* x) if for any open subset V of Y with

$$F(x) \subseteq V (\text{resp. } F(x) \cap V \neq \emptyset)$$

there is an open set $U \subseteq X$ containing x such that

$$F(x') \subseteq V (\text{resp. } F(x') \cap V \neq \emptyset)$$

for all $x' \in U$. The correspondence $F : X \rightrightarrows Y$ is *upper hemicontinuous* (resp. *lower hemicontinuous*) if it is upper hemicontinuous at every point of X (resp. lower hemicontinuous at every point of X). A correspondence is *continuous* if it is both upper and lower hemicontinuous.

Example 1.3.1. In this example, it is shown that the concepts of closedness, lower (upper) hemicontinuity of F are different from each other.

(i) Define a correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} \left\{\frac{1}{x}\right\}, & \text{if } x \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Note that F is closed at 0, but it is not upper (lower) hemicontinuous at 0.

(ii) Define a correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} [0,1], & \text{if } x \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

In this case, F is lower hemicontinuous at 0, but it is neither upper hemicontinuous nor closed at 0.

(iii) Define a correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} \{0\}, & \text{if } x \neq 0; \\ [0,1], & \text{otherwise.} \end{cases}$$

In this case, F is upper hemicontinuous at 0, but it is not lower hemicontinuous at 0.

(iv) Define a correspondence $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$F(x) = \begin{cases} \{|x|\}, & \text{if } x \neq 0; \\ (0,1], & \text{otherwise.} \end{cases}$$

In this case, F is upper hemicontinuous at 0, but it is neither lower hemicontinuous nor closed at 0.

Theorem 1.3.2. *Let $X, Y \subseteq \mathbb{R}^\ell$. Suppose $F : X \rightrightarrows Y$ is a upper hemicontinuous at x and that $F(x)$ is a closed set. Then F is closed at x. Moreover, if Y is compact and F is closed at x, then F is upper hemicontinuous at x.*

Theorem 1.3.3. *If $F : X \rightrightarrows Y$ is non-empty compact valued, then F is upper hemicontinuous at x if and only if for every sequence $\{x_n : n \geq 1\}$ converging to x and $y_n \in F(x_n)$ there is a convergent subsequence of $\{y_n : n \geq 1\}$ with limit in $F(x)$.*

Theorem 1.3.4. *A correspondence $F : X \rightrightarrows Y$ is lower hemicontinuous at x if and only if for every sequence $\{x_n : n \geq 1\}$ converging to x and $y \in F(x)$ there is a sequence $\{y_n : n \geq 1\}$ such that $y_n \in F(x_n)$ and $\lim_{n\to\infty} y_n = y$.*

Example 1.3.5. The correspondence $F : \mathbb{R}^\ell_+ \rightrightarrows \mathbb{R}^\ell_+$, defined by $F(x) = \{y \in \mathbb{R}^\ell_+ : 0 \leq y \leq x\}$, is continuous.

Theorem 1.3.6. *Suppose that $F : X \rightrightarrows Y$ is a correspondence. The following statements are equivalent.*

(i) $F : X \rightrightarrows Y$ *is upper hemi-continuous.*

(ii) *For each open subset V of Y, $F^+(V)$ is open.*

(iii) *For each closed subset C of Y, $F^-(C)$ is closed.*

Theorem 1.3.7. *Suppose that $F : X \rightrightarrows Y$ is a correspondence. The following statements are equivalent.*

(i) $F : X \rightrightarrows Y$ *is lower hemi-continuous.*

(ii) *For each open subset V of Y, $F^-(V)$ is open.*

(iii) *For each closed subset C of Y, $F^+(C)$ is closed.*

Proposition 1.3.8. *Let $F : X \rightrightarrows Y$ be compact-valued and upper hemicontinuous. If K is compact, then $F(K)$ is compact.*

It can checked that if $F : X \rightrightarrows Y$ has open graph, then it is lower hemi-continuous. Note that if $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are lower(resp. upper) semicontinuous at x, so is $G \circ F$. Given $F : X \rightrightarrows Y$ and $F_i : X \rightrightarrows Y$ $(i \geq 1)$, define

(i) $\mathrm{cl}F : X \rightrightarrows Y$ by $(\mathrm{cl}F)(x) = \mathrm{cl}F(x)$.

(ii) $\mathrm{co}F : X \rightrightarrows Y$ by $(\mathrm{co}F)(x) = \mathrm{co}F(x)$.

(iii) $\bigcup\{F_i : i \geq 1\} : X \rightrightarrows Y$ by $\left(\bigcup\{F_i : i \geq 1\}\right)(x) = \bigcup\{F_i(x) : i \geq 1\}$.

(iv) $\bigcap\{F_i : i \geq 1\} : X \rightrightarrows Y$ by $\left(\bigcap\{F_i : i \geq 1\}\right)(x) = \bigcap\{F_i(x) : i \geq 1\}$.

(v) $\sum_{i\geq 1} F_i : X \rightrightarrows Y$ by $\left(\sum_{i\geq 1} F_i\right)(x) = \sum_{i\geq 1} F_i(x)$.

(vi) $\prod\{F_i : i \geq 1\} : X \rightrightarrows Y$ by $\left(\prod\{F_i : i \geq 1\}\right)(x) = \prod\{F_i(x) : i \geq 1\}$.

It can be checked that $\mathrm{Gr}_{\bigcup\{F_i:i\geq 1\}} = \bigcup\{\mathrm{Gr}_{F_i} : i \geq 1\}$, $\mathrm{Gr}_{\bigcap\{F_i:i\geq 1\}} = \bigcap\{\mathrm{Gr}_{F_i} : i \geq 1\}$. Note that $\mathrm{clGr}_F \neq \mathrm{Gr}_{\mathrm{cl}F}$ and $\mathrm{coGr}_F \neq \mathrm{Gr}_{\mathrm{co}F}$, which are follows from the following examples.

Example 1.3.9. Consider a correspondence $F : (0,1) \rightrightarrows \mathbb{R}$ defined by $F(x) = (0,1)$ for all $x \in (0,1)$. Then $\mathrm{Gr}_F = (0,1) \times (0,1)$ and $\mathrm{clGr}_F = (0,1) \times [0,1]$.

Example 1.3.10. Consider a correspondence $F : X \rightrightarrows Y$ defined by

$$F(x) = \begin{cases} \left\{0, \frac{1}{x}\right\}, & \text{if } x \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Then Gr_F is closed, but coGr_F is not closed.

Theorem 1.3.11. *Suppose that $F : X \rightrightarrows Y$ and $F_i : X \rightrightarrows Y$ $(i \geq 1)$ are upper hemicontinuous at x.*

(i) Then $\mathrm{cl}F : X \rightrightarrows Y$ upper hemicontinuous at x.

(ii) If F is compact-valued, then $\mathrm{co}F : X \rightrightarrows Y$ is upper hemicontinuous at x.

(iii) Then $\bigcup\{F_i : 1 \leq i \leq m\} : X \rightrightarrows Y$ is upper hemicontinuous at x.

(iv) If $G : X \rightrightarrows Y$ is closed at x and $F(x)$ is compact, then $F \cap G : X \rightrightarrows Y$ is upper hemicontinuous at x.

(v) If $G : X \rightrightarrows Y$ is upper hemicontinuous at x, and F and G are closed valued, then $F \cap G : X \rightrightarrows Y$ is upper hemicontinuous at x.

(vi) If F_i is closed-valued for all $i \geq 1$ with at least one F_i is compact valued, then $\bigcap\{F_i : i \geq 1\} : X \rightrightarrows Y$ is upper hemicontinuous at x.

(vii) If F_i is compact-valued for all $1 \leq i \leq m$, then $\sum_{i=1}^m F_i : X \rightrightarrows Y$ is upper hemicontinuous at x.

(viii) If F_i is compact-valued for all $i \geq 1$, then $\prod\{F_i : i \geq 1\} : X \rightrightarrows Y$ is upper hemicontinuous at x.

Theorem 1.3.12. *Suppose that $F : X \rightrightarrows Y$ and $F_i : X \rightrightarrows Y$ $(i \geq 1)$ are lower hemicontinuous at x.*

(i) Then $\mathrm{cl}F : X \rightrightarrows Y$ lower hemicontinuous at x. In addition, if $\mathrm{cl}F : X \rightrightarrows Y$ lower hemicontinuous at x, then so is F.

(*ii*) *Then* co$F : X \rightrightarrows Y$ *is lower hemicontinuous at* x.

(*iii*) *Then* $\bigcup\{F_i : i \geq 1\} : X \rightrightarrows Y$ *is lower hemicontinuous at* x.

(*iv*) *If* $G : X \rightrightarrows Y$ *has open graph, then* $F \cap G : X \rightrightarrows Y$ *is lower hemicontinuous at* x.

(*v*) *Then* $\sum_{i=1}^{m} F_i : X \rightrightarrows Y$ *is lower hemicontinuous at* x.

(*vi*) *Then* $\prod\{F_i : 1 \leq i \leq m\} : X \rightrightarrows Y$ *is lower hemicontinuous at* x.

Example 1.3.13. Consider two correspondences $F, G : [0,1] \rightrightarrows [0,1]$ defined by $F(x) = \{x\}$ and $G(x) = \{1 - x\}$. Clearly, F and G are lower hemicontinuous. In this case, $F \cap G : [0,1] \rightrightarrows [0,1]$ is defined by

$$(F \cap G)(x) = \begin{cases} \emptyset, & \text{if } x \neq \frac{1}{2}; \\ \left\{\frac{1}{2}\right\}, & \text{if } x = \frac{1}{2}. \end{cases}$$

Note that $F \cap G$ is not lower hemicontinuous at $x = \frac{1}{2}$.

Let $F : X \rightrightarrows X$ be a correspondence. A *fixed point of* F is a point $x_0 \in X$ such that $x_0 \in F(x_0)$.

Theorem 1.3.14 (Kakatuni's fixed point theorem)**.** *Let* X *be a nonempty compact convex subset of* $\mathbb{R}^\ell$ *and* $F : X \rightrightarrows X$ *be a correspondence such that* $F(x)$ *is non-empty and convex for each* $x \in X$. *If* F *has closed graph (or is closed-valued and upper hemi-continuous), then there exists a fixed point of* F.

A *selection* of $F : X \rightrightarrows X$ is a single-valued function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$. If a selection f of F is continuous, then it is called a *continuous selection.* It follows from the Michael selection theorem that if X is a compact subset of $\mathbb{R}^\ell$ and F is a lower hemicontinuous correspondence such that $F(x)$ is non-empty, closed and convex for all $x \in x$, then there exists a continuous selection of F.

Chapter 2

Walrasian Equilibia: the Case of Finite Exchange Economies

Consider a pure exchange economy $\mathscr{E}$ whose set of *agents* is denoted by $N = \{1, \cdots, n\}$. The *consumption* set of agent $i \in N$ is denoted by $X_i \subseteq \mathbb{R}^\ell_+$. The *preference* of an agent $i \in N$ is a reflexive, complete and transitive relation on X_i, denoted by $\succeq_i$. In this chapter, some properties of budget sets and demand sets of $\mathscr{E}$ are studies in finite economies. These properties allow to obtain the existence of a Walrasian equilibrium.

2.1 The Case When $X_i = \mathbb{R}^\ell_+$

In this section, the consumption set of each agent is taken as $\mathbb{R}^\ell_+$. The section is decomposed into the following two subsections. The first one deals with some crucial properties of the budget and demand sets of each agent, and the last one contains the existence theorem.

2.1.1 Budget and Demand Sets

Given a price vector $p = (p^1, \cdots, p^\ell)$, the *budget set* of an agent $i \in N$ corresponding to an initial endowment bundle w_i is defined by

$$B_i(p, w_i) = \left\{ x \in \mathbb{R}^\ell_+ : p \cdot x \leq p \cdot w_i \right\}.$$

The budget line of a budget set $B_i(p, w_i)$ is the set $\{x \in B_i(p, w_i) : p \cdot x = p \cdot w_i\}$. Clearly, the budget set $B_i(p, w_i)$ is closed. The condition for boundedness or unboundedness of the budget sets is included in the next theorem.

Theorem 2.1.1. *Fix an initial endowment vector $w_i \in \mathbb{R}_+^\ell$. All budget sets $B_i(p, w_i)$ for p are bounded if and only if $p \gg 0$.*

Proof. Suppose that every budget set $B_i(p, w_i)$ for a price p is bounded. It is claimed that $p^h > 0$ for each h. Indeed, if some $p^h = 0$, then $\{ne(h) : n \geq 1\} \subseteq B_i(p, w_i)$, where $e(h)$ denotes the standard unit vector in the i^{th} direction. So, each $B_i(p, w_i)$ is unbounded.

Conversely, let $p \gg 0$. Suppose that $r = \min\{p^1, \cdots, p^\ell\}$. Note that $r > 0$. If $x \in B_i(p, w_i)$, then for $1 \leq h \leq \ell$, one has

$$rx^h \leq p \cdot x \leq p \cdot w_i.$$

Thus, $x^h \leq \frac{p \cdot w_i}{r} < \infty$, and $B_i(p, w_i)$ is bounded. □

Corollary 2.1.2. *A budget set $B_i(p, w_i)$ is compact if and only if $p \gg 0$.*

Corollary 2.1.3. *A budget set $B_i(p, w_i)$ is unbounded if and only if some component of p is 0.*

Theorem 2.1.4. *The budget correspondence $B_i : \left(\mathbb{R}_+^\ell \setminus \{0\}\right) \times \mathbb{R}_+^\ell \rightrightarrows \mathbb{R}_+^\ell$ is closed.*

Proof. Let $\left(p_n, w_{(i,n)}, x_n\right) \to (p, w_i, x)$ and $x_n \in B_i\left(p_n, w_{(i,n)}\right)$ for all $n \geq 1$. Then $p_n \cdot x_n \leq p_n \cdot w_{(i,n)}$. Passing through a limit, one has $p \cdot x \leq p \cdot w_i$. So, $x \in B_i(p, w_i)$ and B_i is closed. □

Corollary 2.1.5. *The budget correspondence $B_i(\cdot, w_i) : \left(\mathbb{R}_+^\ell \setminus \{0\}\right) \rightrightarrows \mathbb{R}_+^\ell$ is closed for all $w_i \in \mathbb{R}_+^\ell$ and the budget correspondence $B_i(p, \cdot) : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+^\ell$ is closed for all $p \in \mathbb{R}_+^\ell \setminus \{0\}$.*

Exercise 2.1.6. Show that the budget correspondence $B_i : \left(\mathbb{R}_+^\ell \setminus \{0\}\right) \times \mathbb{R}_+^\ell \rightrightarrows \mathbb{R}_+^\ell$ is not lower hemicontinuous.

Solution. Let $w_i = 0$ and $p^h = 0$ for some $1 \leq h \leq \ell$. Take $w_{(i,n)} = \frac{1}{n}e(h)$ and $p_n = p + \frac{1}{n}e(h)$ for all $n \geq 1$. Note that if $x \in B(p_n, w_{(i,n)})$ then $x^h \leq \frac{1}{n}$. So, if $x = e(h)$ then there is no $x_n \in B(p_n, w_{(i,n)})$ such that $\{x_n : n \geq 1\}$ converges to x.

Theorem 2.1.7. *The budget correspondence $B_i : \left(\mathbb{R}_+^\ell \setminus \{0\}\right) \times \mathbb{R}_{++}^\ell \rightrightarrows \mathbb{R}_+^\ell$ is lower hemicontinuous.*

Proof. Define $\text{int}B_i : \left(\mathbb{R}_+^\ell \setminus \{0\}\right) \times \mathbb{R}_{++}^\ell \rightrightarrows \mathbb{R}_+^\ell$ by

$$\text{int}B_i(p, w_i) = \left\{x \in \mathbb{R}_+^\ell : p \cdot x < p \cdot w_i\right\}.$$

It is claimed that $\mathrm{int}B_i$ is lower hemicontinuous. To see this, suppose that the sequence $\{(p_n, w_{(i,n)}) : n \geq 1\} \subseteq (\mathbb{R}^\ell_+ \setminus \{0\}) \times \mathbb{R}^\ell_{++}$ converges to $(p, w_i) \in (\mathbb{R}^\ell_+ \setminus \{0\}) \times \mathbb{R}^\ell_{++}$ and $x \in \mathrm{int}B_i(p, w_i)$. This means $p \cdot x < p \cdot w_i$. Thus, $p_n \cdot x < p \cdot w_{(i,n)}$ and $x \in B_i(p_n, w_{(i,n)})$. So, $\mathrm{int}B_i$ is lower hemicontinuous at (p, w_i). Since (p, w_i) is an arbitrary point of $(\mathbb{R}^\ell_+ \setminus \{0\}) \times \mathbb{R}^\ell_{++}$, $\mathrm{int}B_i$ is lower hemicontinuous. It follows that B_i is lower hemicontinuous. $\square$

Figure 2.1: Budget Set

Given a price vector p, the *demand set* of an agent i corresponding to his initial endowment bundle w_i and preference $\succeq_i$ is defined by

$$D_i(p, w_i, \succeq_i) = \{x \in B_i(p, w_i) : x \succeq_i y \text{ for all } y \in B_i(p, w_i)\}.$$

Definition 2.1.1. A preference relation $\succeq$ defined on a convex subset X of $\mathbb{R}^\ell_+$ is said to be

(a) *convex* if $y \succeq x$ and $z \succeq x$ imply $\alpha y + (l - \alpha)z \succeq x$ for all $0 < \alpha < 1$;

(b) *strictly convex* if $y \succeq x$, $z \succeq x$ and $y \neq z$ imply $\alpha y + (1-\alpha)z \succ x$ for all $0 < \alpha < 1$.

Definition 2.1.2. A preference relation $\succeq$ on X_i is said to be

(a) *upper semicontinuous* if for each $x \in X_i$, the set $\{y \in X_i : y \succeq_i x\}$ is closed;

(b) *lower semicontinuous* if for each $x \in X_i$, the set $\{y \in X_i : x \succeq_i y\}$ is closed; and

(c) *continuous* whenever $\succeq_i$ is both upper and lower semicontinuous, i. e., whenever for each $x \in X_i$, the sets $\{y \in X_i : y \succeq_i x\}$ and $\{y \in X_i : x \succeq_i y\}$ are closed.

Definition 2.1.3. A preference relation $\succeq_i$ on X_i is said to be

(a) *monotone* whenever $x, y \in X_i$ and $x \gg y$ imply $x \succ_i y$; and

(b) *strictly monotone* whenever $x, y \in X_i$ and $x > y$ imply $x \succ_i y$.

Theorem 2.1.8. *For a price $p \gg 0$ and a continuous preference $\succeq_i$ on $\mathbb{R}^\ell_+$, the following statements hold.*

(i) $D_i(p, w_i, \succeq_i) \neq \emptyset$.

(ii) *If $\succeq_i$ is strictly convex, then $D_i(p, w_i, \succeq_i)$ has exactly one element.*

(iii) *If $\succeq_i$ is strictly convex and strictly monotone, then $D_i(p, w_i, \succeq_i)$ has exactly one element lying on the budget line.*

Proof. (i) For each $x \in B_i(p, w_i)$, let

$$C_x = \{y \in B_i(p, w_i) : y \succeq_i x \text{ for all } x \in B_i(p, w_i)\}.$$

Note that $C_x \neq \emptyset$ for all $x \in B_i(p, w_i)$. Since $\succeq_i$ is upper semicontinuous, C_x is closed for all $x \in B_i(p, w_i)$. Note that

$$D_i(p, w_i, \succeq_i) = \bigcap \{C_x : x \in B_i(p, w_i)\}.$$

It is claimed that $\bigcap \{C_x : x \in B_i(p, w_i)\} \neq \emptyset$. To this end, let $x_1, \cdots, x_n \in B_i(p, w_i)$ and $x_1 \succeq_i \cdots \succeq_i x_n$. So, $C_{x_1} \subseteq \cdots \subseteq C_{x_n}$ and thus, $\bigcap \{C_{x_i} : 1 \leq i \leq n\} \neq \emptyset$. This implies that the collection $\{C_x : x \in B_i(p, w_i)\}$ of non-empty closed sets has the finite intersection property. Since $B_i(p, w_i)$ is compact, $\bigcap \{C_x : x \in B_i(p, w_i)\} \neq \emptyset$.

(ii) It is easy to verify and is left as an exercise for the reader.

(iii) It is easy to verify and is left as an exercise for the reader. $\square$

Example 2.1.9. Consider an economy whose set of agents is $N = \{1, 2, 3\}$ and commodity space is $\mathbb{R}^2_+$.

(i) *Agent 1:* Initial endowment $w_1 = (1, 2)$ and utility function $U_1(x, y) = xy$.

(ii) *Agent 2:* Initial endowment $w_2 = (1,1)$ and utility function $U_2(x,y) = x^2y$.

(iii) *Agent 3:* Initial endowment $w_3 = (2,3)$ and utility function $U_3(x,y) = xy^2$.

Let $p = (p_1, p_2) \gg 0$. The first agent maximizes $U_1(x,y)$ subject to the budget constraint

$$p_1x + p_2y = p_1 + 2p_2.$$

Let $g(x,y) = p_1x + p_2y$. Employing Lagrange multipliers, at the maximizing point, one has

$$\frac{\partial}{\partial x}U_1(x,y) = \lambda\frac{\partial}{\partial x}g(x,y)$$

and

$$\frac{\partial}{\partial y}U_1(x,y) = \lambda\frac{\partial}{\partial y}g(x,y).$$

Thus, one obtains $y = \lambda p_1$, $x = \lambda p_2$ and $p_1x + p_2y = p_1 + 2p_2$. Solving the above system, one concludes

$$D_1(p, w_1, U_1) = \left\{\left(\frac{p_1 + 2p_2}{2p_1}, \frac{p_1 + 2p_2}{2p_2}\right)\right\}.$$

The second agent maximizes $U_2(x,y)$ subject to the budget constraint

$$p_1x + p_2y = p_1 + p_2.$$

Using Lagrange multipliers again, one obtains

$$2xy = \lambda p_1, x^2 = \lambda p_2 \text{ and } p_1x + p_2y = p_1 + p_2.$$

Solving the above system, one has

$$D_2(p, w_2, U_2) = \left\{\left(\frac{2p_1 + 2p_2}{3p_1}, \frac{p_1 + p_2}{3p_2}\right)\right\}.$$

Finally, for the third agent, one can show that

$$y^2 = \lambda p_1, 2xy = \lambda p_2 \text{ and } p_1x + p_2y = 2p_1 + 3p_2.$$

In this case, one has

$$D_3(p, w_3, U_3) = \left\{\left(\frac{2p_1 + 3p_2}{3p_1}, \frac{4p_1 + 6p_2}{3p_2}\right)\right\}.$$

Theorem 2.1.10. *For a price $p \in \partial\mathbb{R}^\ell_+$ and a preference relation $\succeq_i$ on $\mathbb{R}^\ell_+$, the following statements hold.*

(i) If $\succeq_i$ is strictly monotone, then $D_i(p, w_i, \succeq_i) = \emptyset$.

(ii) If $\succeq_i$ is strictly monotone on $\mathbb{R}^\ell_{++}$ such that everything in the interior is preferred to anything on the boundary and if an element $w \in \mathbb{R}^\ell_+$ satisfies $p \cdot w > 0$, then $D_i(p, w_i, \succeq_i) = \emptyset$.

Proof. Assume that $p^h = 0$ for some $1 \leq h \leq \ell$.

(i) Suppose that $\succeq_i$ is strictly monotone and let $x \in B_i(p, w_i)$. Consider an element

$$y = \left(x^1, \cdots, x^{h-1}, x^h + 1, x^{h+1}, \cdots, x^\ell\right).$$

Then $y \in B_i(p, w_i)$ and $y > x$. The strict monotonicity of $\succeq_i$ implies $y \succ_i x$. Since x is an arbitrary point in $B_i(p, w_i)$, one has $D_i(p, w_i, \succeq_i) = \emptyset$.

(ii) Suppose $\succeq_i$ satisfies the stated properties and that $p \cdot w > 0$. It follows from $p \cdot w > 0$ that $B_i(p, w_i)$ contains strictly positive elements and so $D_i(p, w_i, \succeq_i) \subseteq \mathbb{R}^\ell_{++}$. However, if x is any strictly positive element in $B_i(p, w_i)$, then

$$y = \left(x^1, \cdots, x^{h-1}, x^h + 1, x^{h+1}, \cdots, x^\ell\right)$$

is also a strictly positive element in $B_i(p, w_i)$ satisfying $y > x$. Since $\succeq_i$ is strictly monotone on $\text{int}\mathbb{R}^\ell_+$, $y \succ_i x$ must hold. Thus, $D_i(p, w_i, \succeq_i) = \emptyset$. $\square$

The following three propositions are easy to verify and are left as exercises for the reader.

Proposition 2.1.11. *If $(p, w_i, \succeq_i) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_+ \times (\mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+)$ and $\succeq_i$ is convex and upper semicontinuous, then the set $D_i(p, w_i, \succeq_i)$ is convex and compact.*

Proposition 2.1.12. *If $(p, w_i, \succeq_i) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_+ \times (\mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+)$ and $\succeq_i$ is monotone, then the equality $p \cdot x = p \cdot w_i$ holds for all $x \in D_i(p, w_i, \succeq_i)$.*

Proposition 2.1.13. *The demand correspondence is homogeneous of degree zero.*

Theorem 2.1.14. *If $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \gg 0$, then there exists a bounded subset B of $\mathbb{R}^\ell_+$ such that $D_i(p, w_i, \succeq_i) \subseteq B$ holds for each $n \geq 1$.*

Proof. Let $p_n = (p_n^1, \cdots, p_n^\ell) \to p$ in $\mathbb{R}^\ell_{++}$. Put $e = (1, \cdots, 1)$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\varepsilon_1 e \leq p_n \leq \varepsilon_2 e$ and $w_i \leq \varepsilon_2 e$. Pick an element $x \in D_i(p, w_i, \succeq_i)$. Then

$$\varepsilon_1 x^h \leq p_n^h x \leq p_n \cdot x \leq p_n \cdot w_i \leq \varepsilon_2^2 \ell.$$

Take $B = [0, d]^\ell \subset \mathbb{R}^\ell_+$, where $d = \frac{\varepsilon_2^2 \ell}{\varepsilon_1}$. Note that B is bounded and $D_i(p_n, w_i, \succeq_i) \subseteq B$ for each $n \geq 1$. □

Theorem 2.1.15. *Let $\succeq_i$ be strictly monotone and upper semicontinuous. Suppose that $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \in \partial\mathbb{R}^\ell_+ \setminus \{0\}$ and $p \cdot w_i > 0$. If $\{x_n : n \geq 1\}$ satisfies $x_n \in D_i(p_n, w_i, \succeq_i)$ for all $n \geq 1$, then $\{x_n : n \geq 1\}$ is unbounded above.*

Proof. Assume the contrary. That is, there is a bounded sequence $\{x_n : n \geq 1\}$ such that $x_n \in D_i(p_n, w_i, \succeq_i)$ for all $n \geq 1$. Then there exists a subsequence $\{z_n : n \geq 1\}$ of $\{x_n : n \geq 1\}$ such that $\{z_n : n \geq 1\}$ converges to z. It follows from $p_n \cdot z_n = p_n \cdot w_i$ that $p \cdot z = p \cdot w_i$. It is claimed that $z \in D_i(p, w_i, \succeq_i)$. Pick an element $y \in B_i(p, w_i)$. Since $p \cdot w_i > 0$, one has $p \cdot \lambda y < p \cdot w_i$ for all $0 < \lambda < 1$. Choose an increasing sequence $\{\lambda_m : m \geq 1\} \subset (0, 1)$ converging to 1. So, there exists an increasing sequence $\{n_m : m \geq 1\}$ of positive integers such that $p_n \cdot \lambda_m y < p_n \cdot w_i$ for all $n \geq n_m$ which means $\lambda_m y \in B_i(p_n, w_i)$ for all $n \geq n_m$. Thus, $z_n \succeq_i \lambda_m y$ for all $n \geq n_m$. Taking $m \to \infty$, one obtains $z \succeq_i y$. Since y is an arbitrary element of $B_i(p, w_i)$, one has $z \in D_i(p, w_i, \succeq_i)$, which contradicts with the conclusion of Theorem 2.1.10. Consequently, every sequence $\{x_n : n \geq 1\}$ with $x_n \in D_i(p_n, w_i, \succeq_i)$ is unbounded above. □

Theorem 2.1.16. *Let $\succeq_i$ be monotone and upper semicontinuous. Suppose that $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \in \mathbb{R}^\ell_{++}$ and $x_n \in D_i(p_n, w_i, \succeq_i)$ for all $n \geq 1$. Then there exists a subsequence $\{z_n : n \geq 1\}$ of $\{x_n : n \geq 1\}$ such that $\{z_n : n \geq 1\}$ converges to some $z \in D_i(p, w_i, \succeq_i)$.*

Proof. By Theorem 2.1.14, there is a bounded set B such that $\{x_n : n \geq 1\} \subseteq B$. So, there exists a subsequence $\{z_n : n \geq 1\}$ of $\{x_n : n \geq 1\}$ such that $\{z_n : n \geq 1\}$ converges to some $z \in \mathbb{R}^\ell_+$. Applying an argument similar to that in Theorem 2.1.15, one can show that $z \in D_i(p, w_i, \succeq_i)$. □

Theorem 2.1.17. *The demand correspondence $D_i(\cdot, w_i, \succeq_i) : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ is closed for all $w_i \geq 0$.*

Proof. Let $\{(p_n, x_n) : n \geq 1\} \subseteq \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_+$ be a sequence converging to $(p, x) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_+$. By Theorem 2.1.16, there exists a subsequence $\{z_n : n \geq 1\}$ of $\{x_n : n \geq 1\}$

satisfying $z_n \to z \in D_i(p, w_i, \succeq_i)$. Since $x_n \to x$, one has $x = z$. So, $x \in D_i(p, w_i, \succeq_i)$ and $D_i(\cdot, w_i, \succeq_i) : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_+$ is closed. $\square$

Exercise 2.1.18. Show that the demand correspondence $D_i(\cdot, w_i, \succeq_i) : \mathbb{R}^\ell_+ \to \mathbb{R}^\ell_+$ is closed for all $w_i \geq 0$.

2.1.2 The Existence Theorem

Define the *excess demand correspondence* $\zeta : \mathbb{R}^\ell_{++} \rightrightarrows \mathbb{R}^\ell_+$ by

$$\zeta(p) = \sum_{i \in N} D_i(p, w_i, \succeq_i) - \sum_{i \in N} w_i.$$

A price p is said to be an *equilibrium price* whenever $0 \in \zeta(p)$. Throughout the subsection, it is assumed that $\succeq_i$ is continuous, convex and strictly monotone preference for all $i \in N$, and $\sum_{i \in N} w_i \in \mathbb{R}^\ell_{++}$.

Exercise 2.1.19. Show that the excess demand correspondence is non-empty, convex and compact valued.

Exercise 2.1.20. Show that $p \cdot z = 0$ for each $z \in \zeta(p)$.

Theorem 2.1.21. *Suppose that $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \in \partial\mathbb{R}^\ell_+ \setminus \{0\}$. If $\{z_n : n \geq 1\}$ satisfies $z_n \in \zeta(p_n)$ for all $n \geq 1$, then $\{z_n : n \geq 1\}$ is unbounded above.*

Proof. Since $\sum_{i \in N} w_i \in \mathbb{R}^\ell_{++}$ and $p \in \partial\mathbb{R}^\ell_+ \setminus \{0\}$, one has $\sum_{i \in N} p \cdot w_i \in \mathbb{R}^\ell_{++}$. This implies that $p \cdot w_{i_0} > 0$ for some $i_0 \in N$. Let

$$z_n = \sum_{i \in N} x_{(i,n)} - \sum_{i \in N} w_i$$

for some $x_{(i,n)} \in D_i(\cdot, w_i, \succeq_i)$ for all $i \in N$ and $n \geq 1$. By Theorem 2.1.15, one has $\left\{x_{(i_0,n)} : n \geq 1\right\}$ is unbounded above. Thus, $\{z_n : n \geq 1\}$ is unbounded above. $\square$

Theorem 2.1.22. *Suppose that $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \in \mathbb{R}^\ell_{++}$ and $z_k \in \zeta(p_n)$ for all $n \geq 1$. Then there exists a subsequence $\{y_k : k \geq 1\}$ of $\{z_k : k \geq 1\}$ such that $\{y_k : k \geq 1\}$ converges to some $y \in \zeta(p)$.*

Proof. Define

$$z_k = \sum_{i \in N} x_{(i,k)} - \sum_{i \in N} w_i$$

for some $x_{(i,k)} \in D_i(\cdot, w_i, \succeq_i)$ for all $i \in N$ and $k \geq 1$. Then one has the following conclusion.

(i) There exists a subsequence $\left\{\left(x^{(1)}_{(i,k)} : i \in N\right) : k \geq 1\right\}$ of $\left\{\left(x_{(1,k)} : i \in N\right) : k \geq 1\right\}$ such that $\left\{x^{(1)}_{(1,k)} : k \geq 1\right\}$ converges to some $x_1 \in D_1(\cdot, w_1, \succeq_1)$.

(ii) There exists a subsequence $\left\{\left(x^{(2)}_{(i,k)} : i \in N\right) : k \geq 1\right\}$ of $\left\{\left(x^{(1)}_{(1,k)} : i \in N\right) : k \geq 1\right\}$ such that $\left\{\left(x^{(2)}_{(1,k)}, x^{(2)}_{(2,k)}\right) : k \geq 1\right\}$ converges to an element $(x_1, x_2) \in D_1(p, w_1, \succeq_1)$ $\times D_2(p, w_2, \succeq_2)$.

Applying these arguments successively n times, one can obtain a subsequence $\left\{\left(x^{(n)}_{(i,k)} : i \in N\right) : k \geq 1\right\}$ of $\left\{\left(x^{(n-1)}_{(i,k)} : i \in N\right) : k \geq 1\right\}$ such that

$$\lim_{k\to\infty} \left(x^{(n)}_{(i,k)} : i \in N\right) = (x_i : i \in N) \in \prod_{i\in N} D_i(p, w_i, \succeq_i).$$

Let $y_k = \sum_{i\in N} x^{(n)}_{(i,k)} - \sum_{i\in N} w_i$. Then $\{y_k : k \geq 1\}$ is a subsequence of $\{x_k : k \geq 1\}$ converging to $y = \sum_{i\in N} x_i - \sum_{i\in N} w_i$. □

Theorem 2.1.23. *For each $0 < \varepsilon < 1$ there exists a closed ball $\hat{B}^\varepsilon$ such that $\zeta(p) \subseteq \hat{B}^\varepsilon$ holds for all $p \in \mathbb{R}^\ell_{++}$ satisfying $\varepsilon \leq p^h \leq 1$ for all $1 \leq h \leq \ell$.*

Proof. Fix an $\varepsilon \in (0,1)$ and put $\delta = \varrho\left(0, \sum_{i\in N} w_i\right)$. Take an element $p \in \mathbb{R}^\ell_{++}$ such that $\varepsilon \leq p^h \leq 1$ for all $1 \leq h \leq \ell$. If $x \in D_i(p, w_i, \succeq_i)$, then

$$\varepsilon x^h \leq p^h x^h \leq p \cdot x = p \cdot w_i \leq \sum_{h=1}^{\ell} w_i^h \leq \delta.$$

So, $0 \leq x^h \leq \frac{\delta}{\varepsilon}$ for all $1 \leq h \leq \ell$ and hence, $\varrho\left(0, x\right) \leq \frac{\delta\sqrt{\ell}}{\varepsilon}$. Thus, if $y = \sum_{i\in N} x_i - \sum_{i\in N} w_i \in \zeta(p)$ then

$$\varrho\left(0, y\right) \leq \sum_{i\in N} \varrho\left(0, x_i\right) + \varrho\left(0, \sum_{i\in N} w_i\right) \leq \frac{n\delta\sqrt{\ell}}{\varepsilon} + \delta.$$

So, the closed ball centered at zero with radius $\frac{n\delta\sqrt{\ell}}{\varepsilon} + \delta$ has the desired properties. □

To state the next theorem, define

$$S = \left\{p \in \mathbb{R}^\ell_{++} : \sum_{h=1}^{\ell} (p^h)^2 = 1\right\}.$$

Theorem 2.1.24. *For each $0 < \varepsilon \leq \frac{1}{\ell}$, let $S_\varepsilon = \left\{p \in S : p^h \geq \varepsilon \text{ for all } 1 \leq h \leq \ell\right\}$. Then there exists a $p \in S_\varepsilon$ such that $z \in \zeta(p)$ implies $q \cdot z \leq 0$ for all $q \in S_\varepsilon$.*

Proof. Fix an $\varepsilon \in \left(0, \frac{1}{\ell}\right]$. By Theorem 2.1.23, there is a closed ball $\hat{B}^\varepsilon$ containing $\zeta(p)$ for all $p \in S_\varepsilon$. It is easy to verify that S_ε is a non-empty, convex and compact subset of S. Consider a correspondence $F_\varepsilon : \mathbb{R}^\ell \rightrightarrows S_\varepsilon$ defined by

$$F_\varepsilon(z) = \{p \in S_\varepsilon : p \cdot z \geq q \cdot z \text{ for all } q \in S_\varepsilon\}.$$

Note that F_ε is non-empty, convex and compact valued. Now, define another correspondence $G_\varepsilon : S_\varepsilon \times \mathbb{R}^\ell \rightrightarrows S_\varepsilon \times \mathbb{R}^\ell$ by $G_\varepsilon(p, z) = F_\varepsilon(z) \times \zeta(p)$. It is claimed that G_ε is closed. Indeed, let $\{((p_k, z_k), (q_k, y_k)) : k \geq 1\}$ be a sequence converging to $((p, z), (q, y))$ in $(S_\varepsilon \times \mathbb{R}^\ell)^2$ and $(q_k, y_k) \in G_\varepsilon(p_k, z_k)$ for all $k \geq 1$. Since $q_k \in F_\varepsilon(z_k)$ for all $k \geq 1$, one has $q_k \cdot z_k \geq q' \cdot z_k$ for all $z_k \in S_\varepsilon$. Passing through a limit, one obtains $q \cdot z \geq q' \cdot z$. So, $q \in F_\varepsilon(z)$. Note that $p_k \to p$, $y_k \in \zeta(p_k)$ and $y_k \to y$. By Theorem 2.1.22, one has $y \in \zeta(p)$. Thus, $(q, y) \in G_\varepsilon(p, z)$ and G_ε is closed. By Kakutani's Fixed Point Theorem, there is a point $(p, z) \in G_\varepsilon(p, z)$. Thus, $z \in \zeta(p)$ and $p \in F_\varepsilon(z)$. These imply $p \cdot z = 0$ and $q \cdot z \leq 0$ for all $q \in S_\varepsilon$. $\square$

Theorem 2.1.25. *There is an equilibrium price.*

Proof. By Theorem 2.1.24, for all $k \geq 1$, there is a $p_k \in S_{\frac{1}{k+\ell}}$ such that $z_k \in \zeta(p_k)$ implies $p_k \cdot z_k \leq 0$ for all $q \in S_{\frac{1}{k+\ell}}$. Then there is a subsequence $\{\pi_k : k \geq 1\}$ of $\{p_k : k \geq 1\}$ such that $\pi_k \to \pi \in \mathrm{cl}S$. It is claimed that $\{z_k : k \geq 1\}$ is bounded. To see this , let $z_k = y_k - \sum_{i \in N} w_i$ for some $y_k \in \sum_{i \in N} D_i(\cdot, w_i, \succeq_i)$. Since $\left(\frac{1}{\ell}, \cdots, \frac{1}{\ell}\right) \in S_{\frac{1}{n+\ell}}$, one concludes

$$0 \leq \sum_{h=1}^{\ell} y_n^h \leq \sum_{i \in N} \sum_{h=1}^{\ell} w_i^h.$$

Thus, $\{y_k : k \geq 1\}$ is bounded. Hence, $\{z_k : k \geq 1\}$ is bounded. By passing to a subsequence if necessary, one can assume that $z_k \to z$. Then $p \in \mathbb{R}^\ell_{++}$ and $z \in \zeta(p)$. It follows that $q \cdot z \leq 0$ for all $q \in S$. Then $q \cdot z \leq 0$ for all $q \in \mathrm{cl}S$. So, $z \leq 0$ and since $p \cdot z = 0$, one has $z = 0$. Thus, $0 = z \in \zeta(p)$ and this completes the proof. $\square$

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