Microeconomic Theory: problems and solutions

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Contents

Chapter 1

Preferences and choices

Proposition 1 *Suppose that X is countable and $\succeq$ is a rational preference over X. Then there exists a utility function $U : X \to \mathbb{R}$ that represents $\succeq$.*

Proof: Let $X = \{x_1, x_2, \cdots\}$. Define

$$\delta_{ij} = \begin{cases} 1, & \text{if } x_i, x_j \in X \text{ and } x_i \succ x_j; \\ 0, & \text{otherwise.} \end{cases}$$

For each $x_i \in X$, define

$$U(x_i) = \sum_{j \geq 1} \frac{1}{2^j} \delta_{ij}.$$

Since $\sum_{j \geq 1} \frac{1}{2^j} < \infty$, U is well defined. We show that U is a utility function representing $\succeq$, that is, for any $x_m, x_k \in X$, $x_m \succeq x_k \Leftrightarrow U(x_m) \geq U(x_k)$. To see the implication "$\Rightarrow$", let $x_m, x_k \in X$ and $x_m \succeq x_k$. Define $A = \{j : x_k \succ x_j\}$. The rest of the proof of "$\Rightarrow$" is completed by considering the following two cases.

Case 1. $A = \emptyset$. Then $\delta_{kj} = 0$ for all $j \geq 1$ and so, $U(x_k) = 0$. Thus, $U(x_m) \geq U(x_k)$.

Case 2. $A \neq \emptyset$. Then

$$\delta_{kj} = \begin{cases} 1, & \text{if } j \in A; \\ 0, & \text{if } j \notin A. \end{cases}$$

So, $U(x_k) = \sum_{j \in A} \frac{1}{2^j}$. Since $x_m \succeq x_k$ and $x_k \succ x_j$ for all $j \in A$, applying the transitivity of $\succeq$, we can show that $x_m \succ x_j$ for all $j \in A$. So, $\delta_{mj} = 1$ for all $j \in A$. Hence,

$$U(x_m) \geq \sum_{j \in A} \frac{1}{2^j} = U(x_k).$$

To see the implication "$\Leftarrow$", let $x_m, x_k \in X$ and $U(x_m) \geq U(x_k)$. If $x_k \succ x_m$, then $\delta_{km} = 1$. So, applying an argument similar to that in the case of "$\Rightarrow$", we can show that

$$U(x_k) \geq \frac{1}{2^m} + U(x_m) > U(x_m),$$

which is a contradiction. By the completeness of $\succeq$, we have $x_m \succeq x_k$.

Exercise 1 *Show that a choice structure $(\mathscr{B}, C(.))$ for which a rationalizing preference relation exists satisfies the path-invariance property: For every pair $B_1, B_2 \in \mathscr{B}$ such that $B_1 \cup B_2 \in \mathscr{B}$ and $C(B_1) \cup C(B_2) \in \mathscr{B}$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, that is, the decision problem can be safely sub-divided.*

Solution: Let $\succeq$ be a rational preference relation that rationalize $C(.)$ relative to $\mathscr{B}$ for the choice structure $(\mathscr{B}, C(.))$. Suppose that $x \in C(B_1 \cup B_2) = C^*(B_1 \cup B_2; \succeq)$. This implies that for all $y \in B_1 \cup B_2$, $x \succeq y$. Since $C(B_1) \subseteq B_1$ and $C(B_2) \subseteq B_2$, $C(B_1) \cup C(B_2) \subseteq B_1 \cup B_2$. So, $x \succeq y'$ for all $y' \in C(B_1) \cup C(B_2)$ and we have $x \in C^*(C(B_1) \cup C(B_2); \succeq) = C(C(B_1) \cup C(B_2))$. Thus, we have proved that $C(B_1 \cup B_2) \subseteq C(C(B_1) \cup C(B_2))$.

Consider any $x \in C(C(B_1) \cup C(B_2)) = C^*(C(B_1) \cup C(B_2); \succeq)$. This implies that for all $y \in C(B_1) \cup C(B_2)$, $x \succeq y$. Take any $y_1 \in C(B_1) = C^*(B_1; \succeq)$. Since $y_1 \in C^*(B_1; \succeq)$, $y_1 \succeq z_1$ for all $z_1 \in B_1$. Therefore, we get $x \succeq y_1 \succeq z_1$ for all $z_1 \in B_1$ and using transitivity of $\succeq$, it follows that $x \succeq z_1$ for all $z_1 \in B_1$. Similarly, take any $y_2 \in C(B_2) = C^*(B_2; \succeq)$. Since $y_2 \in C^*(B_2; \succeq)$, $y_2 \succeq z_2$ for all $z_2 \in B_2$. Therefore, we get $x \succeq y_2 \succeq z_2$ for all $z_2 \in B_2$ and using transitivity, we get $x \succeq z_2$ for all $z_2 \in B_2$. It follows that $x \succeq z$ for all $z \in B_1 \cup B_2$ implying that $x \in C^*(B_1 \cup B_2; \succeq) = C(B_1 \cup B_2)$. Hence, we have proved that $C(C(B_1) \cup C(B_2)) \subseteq C(B_1 \cup B_2)$.

Exercise 2 *Let $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{z, x\}\}$. Suppose that the choice structure is stochastic, that is, for every $B \in \mathscr{B}$, $C(B)$ is a frequency distribution over alternatives in B. For example, if $B = \{x, y\}$ then $C(B) = (C_x(B), C_y(B))$ is such that $C_x(B) \geq 0$,*

$C_y(B) \geq 0$ and $C_x(B) + C_y(B) = 1$. We say that the stochastic choice function can be **rationalized by preferences** *if we can find a probability distribution $Pr(\cdot)$ over the six possible strict preference relations on X such that for all $B \in \mathscr{B}$, $C(B)$ is precisely the frequency of choices induced by $Pr(\cdot)$. For example, if $B = \{x, y\}$, then $C_x(B) = Pr(\{\succ : x \succ y\})$.*

(A) Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{2}, \frac{1}{2})$ can be rationalizes by preferences.

(B) Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{4}, \frac{3}{4})$ cannot be rationalizes by preferences.

(C) Determine the $\alpha \in (0, 1)$ at which the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ switches from rationalizability to non-rationalizability.

Solution: Let $\mathscr{P} = \{\succ^1, \cdots, \succ^6\}$ be the set of all possible strict preference orderings with the set of alternatives $X = \{x, y, z\}$. In particular, let

$$\succ^1: x \succ y \succ z, \ \succ^2: x \succ z \succ y, \ \succ^3: y \succ x \succ z,$$

$$\succ^4: y \succ z \succ x, \ \succ^5: z \succ x \succ y \text{ and } \succ^6: z \succ y \succ x.$$

Also let $Pr(\succ^k) = p_k$ for all $k = \{1, \cdots, 6\}$. Therefore, a probability distribution on $\mathscr{P}$ is a vector $p = (p_1, \cdots, p_6) \in [0, 1]^6$ such that $\sum_{k=1}^{6} \succ_k = 1$. To rationalize a stochastic choice structure, we need to find a probability distribution on $\mathscr{P}$ that rationalizes it. We start from part (C) of this question, that is we try to rationalize the stochastic choice structure $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$.

1. $C_x(\{x, y\}) = Pr(\succ : x \succ y) = p_1 + p_2 + p_5 = \alpha$,

2. $C_y(\{x, y\}) = Pr(\succ : y \succ x) = p_3 + p_4 + p_6 = 1 - \alpha$,

3. $C_y(\{y, z\}) = Pr(\succ : y \succ z) = p_1 + p_3 + p_4 = \alpha$,

4. $C_z(\{y, z\}) = Pr(\succ : z \succ y) = p_2 + p_5 + p_6 = 1 - \alpha$,

5. $C_z(\{z, x\}) = Pr(\succ : z \succ x) = p_4 + p_5 + p_6 = \alpha$, and

6. $C_x(\{z, x\}) = Pr(\succ : x \succ z) = p_1 + p_2 + p_3 = 1 - \alpha$.

From (1) and (6), (2) and (3), and (4) and (5) we get

$$p_5 - p_3 = p_1 - p_6 = p_4 - p_2 = 2\alpha - 1. \tag{1.1}$$

From (1.1), we get $[p_1 + p_4 + p_5] - [p_2 + p_3 + p_6] = 6\alpha - 3 \Rightarrow 2[p_1 + p_4 + p_5] = 6\alpha - 2 \Rightarrow p_1 + p_4 + p_5 = 3\alpha - 1$ and since $p_1 + p_4 + p_5 \in [0,1]$ we get $3\alpha - 1 \in [0,1] \Rightarrow \alpha \in [1/3, 2/3]$. Thus the stochastic choice structure $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha, 1-\alpha)$ is rationalizable only if α lies in the interval $[1/3, 2/3]$.

Part (A) of this question is a special case of the stochastic choice structure $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha, 1-\alpha)$ where $\alpha = 1/2 \in [1/3, 2/3]$ and is hence rationalizable. Finally part (B) of this question is a special case of the stochastic choice structure $C(\{x,y\}) = C(\{y,z\}) = C(\{z,x\}) = (\alpha, 1-\alpha)$ where $\alpha = 1/4 \notin [1/3, 2/3]$ and is hence non-rationalizable.

Chapter 2

Production

Proposition 2 *Suppose that $c(\cdot,\cdot)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $Z(\cdot,\cdot)$ is the associated conditional factor demand correspondence.*

(i) If f is homogeneous of degree one then $c(w,\cdot)$ and $Z(w,\cdot)$ are homogeneous of degree one.

(ii) If f is concave, then $c(w,\cdot)$ is a convex function (in particular, marginal costs are non-decreasing in q).

Proof: (i) Let $\alpha > 0$ and $q \geq 0$. Define $Y(q) = \left\{z \in \mathbb{R}^L_+ : f(z) \geq q\right\}$. We first show that

$$\alpha Z(w,q) \subseteq Z(w,\alpha q).$$

Pick an element $z \in Z(w,q)$. Then $f(\alpha z) = \alpha f(z) \geq \alpha q$. For all $z' \in Y(\alpha q)$,

$$f(z') \geq \alpha q \Rightarrow f\left(\frac{1}{\alpha}z'\right) = \frac{1}{\alpha}f(z') \geq q.$$

Since $z \in Z(w,q)$ and $\frac{1}{\alpha}z' \in Y(q)$ for all $z' \in Y(\alpha q)$,

$$w \cdot \left(\frac{1}{\alpha}z'\right) \geq w \cdot z.$$

Hence, for all $z' \in Y(\alpha q)$, $w \cdot z' \geq w \cdot (\alpha z)$ implying that

$$\alpha z \in Z(w, \alpha q).$$

Next, we show that

$$Z(w, \alpha q) \subseteq \alpha Z(w, q).$$

Let $z \in Z(w, \alpha q)$. Then $f(z) \geq \alpha q$ and for all $z' \in Y(q)$, $f(z') \geq q$ implying that $f(\alpha z') \geq \alpha q$. Since $z \in Z(w, \alpha q)$, $w \cdot z \leq \alpha w \cdot z'$ for all $z' \in Y(q)$ implying $z \in \alpha Z(w, q)$. Thus, we conclude that $Z(w, \alpha q) = \alpha Z(w, q)$.

Note that $c(w, \alpha q) = w \cdot z$ for all $z \in Z(w, \alpha q)$. Take an element $z \in Z(w, \alpha q)$. Since $Z(w, \alpha q) = \alpha Z(w, q)$, $z = \alpha z'$ for some $z' \in Z(w, q)$. Thus,

$$c(w, \alpha q) = \alpha w \cdot z' = \alpha c(w, q).$$

(i) Let $q \geq 0$, $q' \geq 0$ and $\alpha \in [0, 1]$. Define $Y(q) = \left\{z \in \mathbb{R}^L_+ : f(z) \geq q\right\}$. Choose $z \in Y(q)$ and $z' \in Y(q')$. Since f is concave,

$$f(\alpha z + (1-\alpha)z') \geq \alpha f(z) + (1-\alpha)f(z') \geq \alpha q + (1-\alpha)q'.$$

So, $\alpha z + (1-\alpha)z' \in Y(\alpha q + (1-\alpha)q')$. Hence,

$$c(w, \alpha q + (1-\alpha)q') \leq w \cdot (\alpha z + (1-\alpha)z') = \alpha w \cdot z + (1-\alpha)w \cdot z',$$

which further implies

$$c(w, \alpha q + (1-\alpha)q') \leq \alpha c(w, q) + (1-\alpha)c(w, q').$$

In the proof of next result, we use the following fact: If A is a closed convex subset of $\mathbb{R}^L$ and $x \notin A$, then there exists some non-zero element $p \in \mathbb{R}^L$ such that $p \cdot x > \sup\{p \cdot y : y \in A\}$. This result is a consequence of the *separating hyperplane theorem*.

Proposition 3 Suppose that $\pi(\cdot)$ is the profit function of the production set Y. Assume that Y is closed, convex, and satisfies the free disposal property. Then

$$Y = \left\{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p >> 0\right\}.$$

Proof: From the definition of $\pi(\cdot)$, $y \in Y$ implies $p \cdot y \leq \pi(p)$ for all $p >> 0$. Thus,

$$Y \subseteq \left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\}.$$

Pick an element $y \in \mathbb{R}^L$ such that $p \cdot y \leq \pi(p)$ for all $p \gg 0$. We claim that $y \in Y$. Then there exists a non-zero element $p \in \mathbb{R}^L$ such that $p \cdot y > \pi(p)$. First, we show that $p \geq 0$. Assume that $p_\ell < 0$ for some $1 \leq \ell \leq L$. Since Y satisfies the free disposal condition,

$$Y - (0, \cdots, 0, n, 0, \cdots, 0) \subset Y$$

for all $n \geq 1$, where n is corresponding to the ℓ^{th}-coordinate of $(0, \cdots, 0, n, 0, \cdots, 0)$. Let $z \in Y$ and define $z_n = z - (0, \cdots, 0, n, 0, \cdots, 0)$ for all $n \geq 1$. Then the sequence $\{p \cdot z_n : n \geq 1\}$ is unbounded from above which is a contradiction with the fact that $\pi(p) < p \cdot y$. So, we conclude that $p \geq 0$. Let $\varepsilon = p \cdot y - \pi(p)$. Consider an element $p_0 \gg 0$ and then choose an $\frac{1}{2} < \alpha < 1$ such that

$$|\pi(p_0) - p_0 \cdot y| < \frac{\varepsilon}{2(1-\alpha)}.$$

Since $\pi(\cdot)$ is convex,

$$\begin{aligned}
\pi(\alpha p + (1-\alpha)p_0) &\leq \alpha\pi(p) + (1-\alpha)\pi(p_0) \\
&< \alpha(p \cdot y - \varepsilon) + (1-\alpha)\left(p_0 \cdot y + \frac{\varepsilon}{2(1-\alpha)}\right) \\
&= (\alpha p + (1-\alpha)p_0) \cdot y + \left(\frac{1}{2} - \alpha\right)\varepsilon \\
&< (\alpha p + (1-\alpha)p_0) \cdot y
\end{aligned}$$

Since $\alpha p + (1-\alpha)p_0 \gg 0$, we arrived at a contradiction. Thus, $y \in Y$ and hence,

$$\left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\} \subseteq Y.$$

So, we conclude that

$$Y = \left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\}.$$

Chapter 3

Competitive Markets

3.1 Introduction

In this chapter we consider an entire economy in which consumers and firms interact through markets. The two goals of this chapter are the following. First, to formally introduce and study two key concepts-the notion of *Pareto optimality* and *competitive equilibrium*. Second, to develop a somewhat special but analytically very tractable context for the study of market equilibrium-the partial equilibrium approach.

3.2 Pareto Optimality and Competitive Equilibrium

Consider an economy consisting of I consumers (indexed $i = 1, \ldots, I$), J firms (indexed $j = 1, \ldots, J$) and L goods (indexed $l = 1, \ldots, L$). Consumer i's preferences over consumption bundles $x_i = (x_{1i}, \ldots, x_{Li})$ in his consumption set $X_i \subseteq \mathfrak{R}^L_+$ are represented by the utility function $u_i(.)$. The total amount of each good $l = 1, \ldots, L$ initially available in the economy, called the *total endowment* of good l, is denoted by $\omega_l \geq 0$ for $l = 1, \ldots, L$. It is also possible, using the production technologies of the firms, to transform some of the initial endowment of a good into additional amounts of other goods. Each firm j has available to it the production possibilities summarized by the production set $Y_j \subset \mathfrak{R}^L$. An element of Y_j is the production vector $y_j = (y_{1j}, \ldots, y_{Lj}) \in \mathfrak{R}^L$. Thus if $(y_1, \ldots, y_J) \in \mathfrak{R}^{LJ}$ are the production vector of the J firms, the total (net) amount of good l available to the economy is $\omega_l + \sum_{j=1}^{J} y_{lj}$.

DEFINITION 1 An *economic allocation* $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$. The allocation $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is *feasible* if

$$\sum_{i=1}^{I} x_{li} \leq \omega_l + \sum_{j=1}^{J} y_{lj} \quad \forall \;\; l = 1, \ldots, L.$$

Thus an economic allocation is feasible if the total amount of each good consumed does not exceed the total amount available from both the initial endowment and production.

DEFINITION 2 A *feasible allocation* $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is said to be *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x'_1, \ldots, x'_I; y'_1, \ldots, y'_J)$ such that $U_i(x'_i) \geq U_i(x_i)$ for all $i = 1, \ldots, I$ and $U_k(x'_k) > U_k(x_k)$ for some consumer k.

An allocation that is Pareto optimal uses society's initial resources and technological possibilities efficiently in the sense that there is no alternative way to organize the production and distribution of goods that makes some consumer better off without making some other consumer worse off.

Note that the criterion of Pareto optimality does not insure that an allocation is in any sense equitable. It serves as a minimal test for the desirability of an allocation; it simply says that there is no waste in the allocation of resources.

The notion of Pareto optimality in Definition 2 is sometimes referred to as strong Pareto efficiency to differentiate it from weak Pareto optimality.

DEFINITION 3 A *feasible allocation* $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is said to be *weak Pareto optimal* (or *weak Pareto efficient*) if there is no other feasible allocation $(x'_1, \ldots, x'_I; y'_1, \ldots, y'_J)$ such that $U_i(x'_i) > U_i(x_i)$ for all $i = 1, \ldots, I$.

Proposition 4 Prove that Pareto efficiency implies weak Pareto efficiency. Also show that if $X_i = \Re^L_+$ for all $i = 1, \ldots, I$, and all consumers' preferences are continuous and strongly monotonic then weak Pareto efficiency implies Pareto efficiency.

Proof: (a) Suppose that a feasible allocation $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is Pareto efficient but is not weak Pareto efficient. Then there exists a feasible allocation $(x'_1, \ldots, x'_I; y'_1, \ldots, y'_J)$

for which $U_i(x_i') > U_i(x_i)$ for all $i = 1, \ldots, I$. But by Pareto optimality of $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ it follows that the allocation $(x_1', \ldots, x_I'; y_1', \ldots, y_J')$ is not feasible. Hence $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ must also be weak Pareto efficient.

(b) Suppose that a feasible allocation $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is weak Pareto efficient but is not Pareto efficient. Hence there exists a feasible allocation $(x_1', \ldots, x_I'; y_1', \ldots, y_J')$ for which $U_i(x_i') \geq U_i(x_i)$ for all i and $U_k(x_k') > U_k(x_k)$ for some k. Since $X_i = \Re_+^L$ for all $i = 1, \ldots, I$ and preferences are strongly monotonic, we must have $U_k(x_k') > U_k(x_k) \geq U_k(0)$ and therefore, $x_k' \geq 0$ and $x_k' \neq 0$. Therefore, we must have at least one commodity s such that $x_{sk}' > 0$. Consider the new allocation $(x_1'', \ldots, x_I''; y_1', \ldots, y_J')$ such that (i) $x_{li}'' = x_{li}'$ for all i and all $l \neq s$, (ii) $x_{si}'' = x_{si}' + \left(\frac{1}{I-1}\right) \epsilon$ for all $i \neq k$ and (iii) $x_{sk}'' = x_{sk}' - \epsilon$. Observe that as long as $x_{sk}' > \epsilon > 0$, the allocation $(x_1'', \ldots, x_I''; y_1', \ldots, y_J')$ is feasible. By strong monotonicity of preferences $U_i(x_i'') > U_i(x_i') \geq U_i(x_i)$ for all $i \neq k$, for any $\epsilon \in (0, x_{sk}')$. By continuity of $U_k(.)$ we have $U_k(x_k'') > U_k(x_k)$ for small enough ϵ. Therefore, we can find an ϵ small enough such that the corresponding allocation $(x_1'', \ldots, x_I''; y_1', \ldots, y_J')$ is feasible and makes every consumer strictly better off in comparison to the original bundle $(x_1, \ldots, x_I; y_1, \ldots, y_J)$. Hence $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ is not weak Pareto optimal which contradicts our assumption. ■

The assumption of strong monotonicity is crucial for Proposition 4 (b). For example, consider $I = 1, 2$, $L = 1$, $X_1 = X_2 = \Re_+$, $U_1(x_1) = 0$ for all $x_1 \in X_1$ and $U_2(x_2) = x_2$ for all $x_2 \in X_2$, $w > 0$ and $Y = -\Re_+$ (no production). Then $(x_1^*, x_2^*) = (\frac{w}{2}, \frac{w}{2})$ is a weakly Pareto efficient allocation but it is not Pareto efficient. Note that consumer 1's preferences are not strongly monotonic.

In competitive market economies, the society's initial endowments and technological possibilities (that is, the firms) are owned by consumers. We suppose that consumer i initially owns ω_{li} of good l where $\sum_{i=1}^{I} \omega_{li} = \omega_l$. We denote consumer i's vector of endowments by $\omega_i = (\omega_{1i}, \ldots, \omega_{Li})$. In addition consumer i owns a share θ_{ij} of firm j giving him a claim to fraction θ_{ij} of firm j's profit. Note that for each $j = 1, \ldots, J$, $\sum_{i=1}^{I} \theta_{ij} = 1$.

In a competitive economy, a market exists for each of the L goods and all consumers and producers act as price takers. The idea behind price-taking assumption is that if consumers and producers are small relative to the size of the market, they will regard market prices as unaffected by their own actions. We denote the vector of prices by $p = (p_1, \ldots, p_L)$.

DEFINITION 4 The allocation $(x_1^*, \ldots, x_I^*; y_1^*, \ldots, y_J^*)$ and price vector $p^* >> 0$ constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied:

(i) *Profit maximization*: For each firm j, y_j^* solves $\max\limits_{y_j \in Y_j} p^*.y_j$.

(ii) *Utility maximization*: For each consumer i, x_i^* solves $\max\limits_{x_i \in X_i} U_i(x_i)$ subject to $p^*.x_i \leq p^*.\omega_i + \sum\limits_{j=1}^{J} \theta_{ij}(p^*.y_j^*)$.

(iii) *Market clearing*: For each good l, $\sum\limits_{i=1}^{I} x_{li}^* = \omega_l + \sum\limits_{j=1}^{J} y_{lj}^*$.

Definition 4 delineates three sorts of conditions that must be met for a competitive economy to be in equilibrium. Condition (i) states that each firm must choose a production plan that maximizes its profits taking as given the equilibrium vector of prices of its outputs and inputs. Condition (ii) requires that each consumer select a consumption bundle that maximizes his utility given the budget constraint imposed by the equilibrium prices and by his wealth. Note that wealth is a function of prices. This dependence is due to the following two reasons: (1) Prices determine the value of the consumer's initial endowments. (2) The equilibrium price affect firms' profits and hence the value of the consumer's shareholdings. Condition (iii) requires that, at equilibrium prices, the desired consumption and production levels identified in conditions (i) and (ii) are in fact compatible; that is, the aggregate supply of each commodity (its total endowment plus its net production) equals the aggregate demand for it. If excess supply or demand existed for a good at the going price, the economy could not be at a point of equilibrium.

From Definition 4 it follows that *if an allocation* $(x_1^*, \ldots, x_I^*; y_1^*, \ldots, y_J^*)$ *and price vector* $p^* >> 0$ *constitute a competitive equilibrium, then so do the allocation* $(x_1^*, \ldots, x_I^*; y_1^*, \ldots, y_J^*)$ *and price vector* αp^* *for any scalar* $\alpha > 0$. First consider condition (i) of Definition 4, that is profit maximization. If y_j^* solves $\max_{y_j \in Y_j} p^*.y_j$ then y_j^* solves $\max_{y_j \in Y_j} \alpha p^*.y_j = \alpha \max_{y_j \in Y_j} p^*.y_j$ (recall that profit function is homogeneous of degree one). Consider condition (ii) of Definition 4, that is utility maximization. Note that the consumer's new budget constraint is $\alpha p^*.x_i \leq \alpha p^*.\omega_i + \sum_{j=1}^{J} \theta_{ij}(\alpha p^*.y_j^*) \Leftrightarrow p^*.x_i \leq p^*.\omega_i + \sum_{j=1}^{J} \theta_{ij}(p^*.y_j^*)$ which is the old budget constraint. Therefore, x_i solves $\max_{x_i \in X_i} U_i(x_i)$ subject to the new budget constraint $\Leftrightarrow$ x_i solves $\max_{x_i \in X_i} U_i(x_i)$ subject to the old budget constraint. Finally, condition (iii) that is, market clearing does not depend on prices. Hence the result follows implying that we can normalize prices without loss of generality, that is, we can always set the price of one good (say good l) at unity (by selecting $\alpha = \frac{1}{p_l} > 0$).

LEMMA 1 If an allocation $(x_1, \ldots, x_I; y_1, \ldots, y_J)$ and price vector $p >> 0$ satisfy the market clearing condition (that is, condition (iii) of Definition 4) for all $l \neq k$ and if every consumer's budget constraint is satisfied with equality, so that $p.x_i = p.\omega_i + \sum_{j=1}^{J} \theta_{ij}(p.y_j)$ for all $i = 1, \ldots, I$, then the market for good k also clears.

Proof: Adding up the consumers' budget constraints over the I consumers we get

$$\sum_{i=1}^{I} p.x_i = \sum_{i=1}^{I} p.\omega_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \theta_{ij}(p.y_j)$$

$$\Rightarrow \sum_{i=1}^{I}\left\{\sum_{l=1}^{L} p_l x_{li}\right\} = \sum_{i=1}^{I}\left\{\sum_{l=1}^{L} p_l \omega_{li}\right\} + \sum_{j=1}^{J}\left\{(p.y_j)\left(\sum_{i=1}^{I}\theta_{ij}\right)\right\}$$

$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I}\omega_{li}\right\} + \sum_{j=1}^{J}\{(p.y_j)(1)\}$$

$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\omega_l + \sum_{j=1}^{J}\sum_{l=1}^{L} p_l y_{lj}$$

$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\omega_l + \sum_{l=1}^{L} p_l\left\{\sum_{j=1}^{J} y_{lj}\right\}$$

$$\Rightarrow \sum_{l=1}^{L} p_l\left(\sum_{i=1}^{I} x_{li} - \omega_l - \sum_{j=1}^{J} y_{lj}\right) = 0$$

$$\Rightarrow \sum_{l\neq k} p_l\left(\sum_{i=1}^{I} x_{li} - \omega_l - \sum_{j=1}^{J} y_{lj}\right) = -p_k\left(\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{j=1}^{J} y_{kj}\right).$$

By market clearing in goods $l \neq k$, the left hand side of the last equation is equal to zero. Thus, the right hand side must also be zero. Since $p_k > 0$, $\sum_{i=1}^{I} x_{ki} - \omega_k - \sum_{j=1}^{J} y_{kj} = 0 \Leftrightarrow \sum_{i=1}^{I} x_{ki} = \omega_k + \sum_{j=1}^{J} y_{kj}$ implying market clearance for good k. ■

If consumers' budget constraints hold with equality then the rupee value of each consumer's planned purchases equals the rupee value of what he plans to sell plus the rupee value of his share (θ_{ij}) of the firms' (net) supply and so the total value of planned purchase in the economy must equal the total value of planned sales. If these values are equal to each other in all markets but one then equality must hold in the remaining market as well.

3.3 Partial equilibrium competitive analysis

In this section we assume that the consumers have quasi-linear preferences. Recall that a continuous $\mathcal{R}$ on $(-\infty, \infty) \times \Re^{L-1}$ is quasi-linear with respect to the first commodity if and only if *it admits* a continuous utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$. It is easy to show that if the continuous utility function is of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$ then preferences are quasi-linear.

The partial equilibrium approach, which originated in Marshall [1], introduces market for a single good (or group of goods) for which each consumer's expenditure is a small portion of his overall budget. When this is so, it is reasonable to assume that changes in the market for this good will leave the prices of all other commodities approximately unaffected and that there will be, in addition, negligible wealth effects in the market under study. We capture these features in the simplest possible way by considering a two-good model in which the expenditure on all commodities other than that under consideration is treated as a single composite commodity (called numeraire commodity) and in which consumers' utility functions take a quasi-linear form with respect to this numeraire. We let x_i and m_i denote consumer i's consumption of good l and the numeraire respectively. Each consumer $i = 1, \ldots, I$ has a utility function that takes the quasilinear form: $u_i(m_i, x_i) = m_i + \phi_i(x_i)$. Each consumer's consumption set is $X = (-\infty, \infty) \times \Re_+$. We assume that ϕ_i is bounded, for all $x_i \geq 0$, $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ and $\phi_i(0) = 0$. We normalize the price of the numeraire to equal one and let p denote the price of good l. We assume that $\omega_{li} = 0$ and $\omega_{mi} > 0$ for all $i = 1, \ldots, I$ and we denote $\omega_m = \sum_{i=1}^{I} \omega_{mi}$.

Each firm $j = 1, \ldots, J$ in this two-good economy is able to produce good l from m. The amount of the numeraire good required by firm j to produce $q_j \geq 0$ units of good l is given by the cost function $c_j(q_j)$ (recall that the price of the numeraire good is 1). Therefore, $Y_j = \{(-z_j, q_j) : q_j \geq 0 \text{ and } z_j \geq c_j(q_j)\}$. We assume that $c_j'(q_j) > 0$ and $c_j''(q_j) \geq 0$ at all $q_j \geq 0$.

Applying Definition 4 we consider first the implications of profit and utility maximization. Given the price p^* for good l, firm j's equilibrium output level q_j^* must solve $\max_{q_j \geq 0} p^* q_j - c_j(q_j)$ which has a necessary and sufficient first-order condition $p^* \leq c_j'(q_j^*)$ with equality if $q_j^* > 0$. On the other hand, consumer i's equilibrium consumption vector (m_i^*, x_i^*) must solve $\max_{m_i \in \Re, x_i \in \Re_+} m_i + \phi(x_i)$ subject to $m_i + p^* x_i \leq \omega_{mi} + \sum_{j=1}^J \theta_{ij}(p^* q_j^* - c_j(q_j^*))$. In any solution to the consumer's problem, the budget constraint holds with equality. Substituting for m_i from this constraint, we can rewrite consumer i's optimization problem such that x_i^* must solve the problem $\max_{x_i \geq 0} \phi_i(x_i) - p^* x_i + \left[\omega_{mi} + \sum_{j=1}^J \theta_{ij}(p^* q_j^* - c_j(q_j^*))\right]$ and the first order condition is $\phi_i'(x_i^*) \leq p^*$ with equality if $x_i^* > 0$.

We conclude that the allocation $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ and the price p^* constitute a competitive equilibrium if and only if

(C1) $p^* \leq c_j'(q_j^*)$ with equality if $q_j^* > 0 \quad j = 1, \ldots, J$.

(C2) $\phi_i'(x_i^*) \leq p^*$ with equality if $x_i^* > 0 \quad i = 1, \ldots, I$.

(C3) $\sum\limits_{i=1}^{I} x_i^* = \sum\limits_{j=1}^{J} q_j^*$.

At any interior solution, condition (C1) says that firm j's marginal benefit from selling an additional unit of good l, p^*, exactly equals its marginal cost $c_j'(q_j^*)$. Condition (C2) says that consumer i's marginal benefit from consuming an additional unit of good l, $\phi_i'(x_i^*)$, exactly equals its marginal cost p^*. Condition (C3) is the market clearing equation. Together, these $I + J + 1$ conditions characterize the $(I + J + 1)$ equilibrium values $(x_1^*, \ldots, x_L^*; y_1^*, \ldots, y_J^*)$ and p^*. Note that as long as $\max_i \phi_i'(0) > \min_j c_j'(0)$, the aggregate consumption and production of good l must be strictly positive in a competitive equilibrium. Observe that *due to quasi-linearity, the equilibrium allocation and price are independent of the distribution of endowments and ownership shares.*

We can derive the aggregate demand function of good l using condition (C2). Since $\phi_i''(.) < 0$ and $\phi(.)$ is bounded, $\phi_i'(.)$ is a strictly decreasing function of x_i taking all the values in the set $(0, \phi_i'(0)]$. Therefore, for each price level $p > 0$, we can solve for a unique level of x_i, denoted by $x_i(p)$ that satisfies condition (C2). Note that if $p \geq \phi_i'(0)$ then $x_i(p) = 0$. The demand function $x_i(p)$ is continuous and non-increasing in p at all $p > 0$ and is strictly decreasing at any $p < \phi_i'(0)$ (at any such p, $x_i'(p) = \frac{1}{\phi_i''(x_i(p))} < 0$). Therefore, the *aggregate demand function* for good l is then the function $x(p) = \sum_i x_i(p)$ which is continuous and non-increasing at all $p > 0$, is strictly decreasing at any $p < \max_i \phi_i'(0)$ and $x(p) = 0$ whenever $p \geq \max_i \phi_i'(0)$. We can similarly derive the aggregate supply function of good l using condition (C1). Suppose that $c_j(.)$ is strictly convex and that $c_j'(q_j) \to \infty$ as $q_j \to \infty$. Then, for any price level $p > 0$, we can solve for a unique level of q_j, denoted by $q_j(p)$ that satisfies condition (C2). For $p \leq c_j'(0)$ we have $q_j(p) = 0$. The supply function $q_j(p)$ is continuous and nondecreasing in p at all $p > 0$ and is strictly increasing at any $p > c_j'(0)$ (at any such p, $q_j'(p) = \frac{1}{c_j''(q_j(p))} > 0$). Therefore, the *aggregate supply function* for good l is then the function $q(p) = \sum_j q_j(p)$ which is continuous and non-increasing at all $p > 0$, is strictly increasing at any $p > \min_j c_j'(0)$ and $q(p) = 0$ whenever $p \leq \min_j c_j'(0)$.

To find the equilibrium price of good l, we need only find the price p^* at which $x(p^*) = q(p^*)$. The existence of $p^* \in (\min_j c_j'(0), \max_i \phi_i'(0))$ follows from the continuity properties of $x(p)$ and $q(p)$. Since $x(p)$ is strictly decreasing and $q(p)$ is strictly increasing, the equilibrium is unique. The individual consumption and production levels of good l are then given by $x_i^* = x_i(p^*)$ for $i = 1, \ldots, I$ and $q_j^* = q_j(p^*)$ for $j = 1, \ldots, J$. If some cost function $c_j(.)$ is merely convex then $q_j(.)$ is a convex valued correspondence and it may be well defined only for a subset of prices. Nevertheless, the basic features of the analysis do not change.

The inverse of the aggregate demand and supply functions also have interpretations that are of interest. At any given level of aggregate output of good l, say $\bar{q}$, the inverse of the industry supply function $q^{-1}(\bar{q})$, gives the price that brings forth aggregate supply $\bar{q}$. That is, when each firm chooses its optimal output level facing the price $p = q^{-1}(\bar{q})$, aggregate supply is exactly equal to $\bar{q}$. As a result, the marginal cost of producing an additional unit of good l at $\bar{q}$ is precisely $q^{-1}(\bar{q})$ regardless of which active firm produces it. Thus $q^{-1}(.)$, the inverse of the industry supply function can be viewed as the *industry marginal cost function*. Likewise, at any given level of aggregate demand $\bar{x}$, the inverse aggregate demand function $P(\bar{x}) = x^{-1}(\bar{x})$, gives the price at which aggregate demand is $\bar{x}$. As a result, the marginal benefit $\phi_i'(x_i)$ in terms of the numeraire from an additional unit of good l is exactly equal to $P(\bar{x})$. The value of the inverse demand function at quantity $\bar{x}$, $P(\bar{x})$, can be viewed as the *marginal social benefit* of good l.

Given these interpretations for the inverse aggregate demand and supply function, we can view competitive equilibrium as involving an aggregate level at which the marginal social benefit of good l is equal to its marginal cost. This suggests a social optimality property which we will study in the next section.

Example 1: Assume that each consumer $i = 1, \ldots, I$ have quasi-linear utility function $U_i(m_i, x_i) = m_i + \phi_i(x_i)$ where $\phi_i(x_i) = 2(a_i x_i)^{\frac{1}{2}}$ with $a_i > 0$. For each firm $j = 1, \ldots, J$, $c_j(q_j) = \frac{q_j^2}{2b_j}$ with $b_j > 0$. Assume that $\omega_{li} = 0$ and $\omega_{mi} > 0$ for all $i = 1, \ldots, I$. In this example, $\infty = \phi_i'(0) > c_j'(0) = 0$ for all i and j. Moreover, for all i and all $x_i > 0$, $\phi_i'(x_i) > 0$ and $\phi_i''(x_i) < 0$ and for all j and all $q_j > 0$, $c_j'(q_j) > 0$ and $c_j''(q_j) > 0$.

1. Profit maximization gives $q_j(p) = b_j p$ for all $j = 1, \ldots, J$. Therefore, aggregate market supply is $q(p) = \sum_{j=1}^{J} q_j(p) = Bp$ where $B = \sum_{j=1}^{J} b_j$.

2. Utility maximization gives $x_i(p) = \frac{a_i}{p^2}$ for all $i = 1, \ldots, I$. Therefore, aggregate market demand is $x(p) = \sum_{i=1}^{I} x_i(p) = \frac{A}{p^2}$ where $A = \sum_{i=1}^{I} a_i$.

3. Market clearance requires that p^* satisfies $x(p^*) = q(p^*) \Rightarrow p^* = \left(\frac{A}{B}\right)^{\frac{1}{3}}$.

4. Profit of each firm $j = 1, \ldots, J$ is $\pi_j(p^*) = p^* q_j(p^*) - c_j(q_j(p^*)) = (b_j - \frac{b_j}{2})(p^*)^2 = \frac{b_j}{2}(p^*)^2 = \frac{b_j}{2}\left(\frac{A}{B}\right)^{\frac{2}{3}} > 0$. Conditional factor demand for money in firm j is $m_j(1, q_j(p^*)) = \frac{b_j}{2}(p^*)^2 = c_j(q_j(p^*))$. Therefore, the aggregate conditional factor demand for money is $\sum_{j=1}^{J} m_j(1, q_j(p^*)) = \frac{B}{2}(p^*)^2$

5. For each $i = 1, \ldots, I$, the demand for money is $m_i(\omega_{mi}, p^*) = \omega_{mi} + (p^*)^2 \sum_{j=1}^{J} \theta_{ij} \frac{b_j}{2} - \frac{a_i}{p^*}$. Therefore, the aggregate demand for money is $\sum_{i=1}^{I} m_i(\omega_{mi}, p^*) = \omega_m + (p^*)^2 \frac{B}{2} - \frac{A}{p^*} = \omega_m - (p^*)^2 \frac{B}{2} + \left((p^*)^2 B - \frac{A}{p^*}\right) = \omega_m - \sum_{j=1}^{J} m_j(1, q_j(p^*)) + \frac{B}{p^*}\left((p^*)^3 - \frac{A}{B}\right) = \omega_m - \sum_{j=1}^{J} m_j(1, q_j(p^*))$. Thus, at $(x_1(p^*), \ldots, x_I(p^*); q_1(p^*), \ldots, q_J(p^*))$ with price vector p^*, the total amount of the numeraire available to the consumers is
$$\sum_{i=1}^{I} m_i(\omega_{mi}, p^*) = \omega_m - \sum_{j=1}^{J} m_j(1, q_j(p^*)) = \omega_m - \sum_{j=1}^{J} c_j(q_j(p^*)).$$

6. Observe that $\omega = (\omega_{m1}, \ldots, \omega_{mI})$ such that $\sum_{i=1}^{I} \omega_{mi} = \omega_l$ played no role in the optimal choice $(x_1(p^*), \ldots, x_I(p^*); q_1(p^*), \ldots, q_J(p^*))$ in the following sense: If the initial endowment is $\omega' = (\omega'_{m1}, \ldots, \omega'_{mI})$ instead of $\omega = (\omega_{m1}, \ldots, \omega_{mI})$ such that $\sum_{i=1}^{I} \omega'_{mi} = \sum_{i=1}^{I} \omega_{mi} = \omega_m$ then the same allocation $(x_1(p^*), \ldots, x_I(p^*); q_1(p^*), \ldots, q_J(p^*))$ continues to be the optimal choice. Note that by changing the endowments of the consumers we have changed their optimization problem (which is reflected

in their budget constraints). However, the aggregate production and aggregate market demand for good l remains unchanged and hence

$$\sum_{i=1}^{I} m_i(\omega'_{mi}, p^*) = \omega_m - \sum_{j=1}^{J} m_j(1, q_j(p^*)) = \omega_m - \sum_{j=1}^{J} c_j(q_j(p^*)).$$

3.4 Partial equilibrium and the fundamental theorems of welfare economics

We study the properties of the Pareto optimal allocations in the framework of the two-good quasilinear economy introduced in the previous section. When consumer preferences are quasilinear, the boundary of the utility possibility set is linear and all points in this boundary are associated with consumption allocations that differ only in the distribution of the numeraire among consumers. Suppose that we fix the consumption and production levels of good l at $(\bar{x}_1, \ldots, \bar{x}_I; \bar{q}_1, \ldots, \bar{q}_J)$. The total amount of the numeraire available for distribution among consumers is $\omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j)$. Quasilinear utility functions allows for an unlimited unit-to-unit transfer of utility across consumers through transfers of the numeraire, the set of utilities that can be attained for the I consumers by appropriately distributing the available amounts of the numeraire is given by

$$\left\{ (u_1, \ldots, u_I) : \sum_{i=1}^{I} u_i \leq \sum_{i=1}^{I} \phi_i(\bar{x}_i) + \omega_m - \sum_{j=1}^{J} c_j(\bar{q}_j) \right\} \tag{3.1}$$

The boundary of this set is a hyperplane with normal $(1, \ldots, 1)$. Note that by altering the consumption and production levels of good l, we necessarily shift the boundary of this set in a parallel manner. Thus, every Pareto optimal allocation must involve the quantities $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ that extend this boundary as far out as possible. *As long as these optimal consumption and production levels for good l is uniquely determined*, Pareto optimal allocations can differ only in the distribution of the numeraire among consumers. Under our assumption that $\phi_i(.)$ is strictly concave for all i and $c_j(.)$ are convex for all j, the consumption allocations in two different Pareto optimal allocations can differ only in the distribution of numeraire among consumers. Note that the optimal individual production levels need not be unique if firms' cost functions are convex but not strictly convex. Indeterminacy of optimal individual production levels arises, for example, when all firms have identical constant returns to scale technologies (for example, if we have $c_j(q_j) = bq_j$ for all $j = 1, \ldots, J$ in Example 1). However, under our assumption that the $\phi_i(.)$ functions are strictly concave and that the $c_j(.)$ functions are convex, the optimal individual consumption levels of good l are necessarily unique and hence so is the aggregate production level $\sum_{j=1}^{J} q_j^*$ of good l. Using these observations one can now prove the First Fundamental Theorem of Welfare Economics. The first fundamental theorem establishes conditions under which market equilibrium are necessarily Pareto optimal. It is a formal expression of Adam Smith's "invisible hand" and is a result that holds with considerable generality.

THEOREM 1 If the price p^* and allocation $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal.

Proof: From condition (3.1) it follows that the optimum consumption and production levels of good l can be obtained as a solution to

$$(\mathbf{PO}) \qquad \max_{(x_1,\ldots,x_I)\geq 0 \; (q_1,\ldots,q_J)\geq 0} \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j) + \omega_m$$

$$\text{subject to} \quad \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} q_j = 0$$

Note that in this optimization exercise we have replaced inequality constraint with equality constraint simply because in any Pareto optimal allocation we cannot have strict inequality given $\phi_i'(x_i) > 0$ for all $x_i > 0$ and for all $i = 1, \ldots, I$.[1] If we let μ be the multiplier on the constraint of the problem **(PO)**, the $I + J$ optimal values $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ and the multiplier μ satisfy the following $I + J + 1$ conditions.

PO(1) $\mu \leq c_j'(q_j^*)$ with equality if $q_j^* > 0 \quad j = 1, \ldots, J$.

PO(2) $\phi_i'(x_i^*) \leq \mu$ with equality if $x_i^* > 0 \quad i = 1, \ldots, I$.

PO(3) $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$.

[1]The function $\sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$ in the objective function of the constraint optimization problem **(PO)** is known as the *Marshallian aggregate surplus*. The optimal consumption and production levels of good l maximize this aggregate surplus measure.

Conditions PO(1)-PO(3) are exactly parallel to (C1)-(C3) with μ replacing p^*. This implies that any competitive equilibrium allocation has consumption and production levels of good l, $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ that satisfy conditions PO(1)-PO(3) when we set $\mu = p^*$. Hence if p^* and allocation $(x_1^*, \ldots, x_I^*; q_1^*, \ldots, q_J^*)$ constitute a competitive equilibrium, then this allocation is Pareto optimal. ■

The correspondence between p and μ in the equilibrium conditions (C1)-(C3) and PO(1)-PO(3) implies that competitive price is exactly equal to the shadow price on the resource constraint for good l in the Pareto optimal problem **(PO)**. In this sense a good's price in a competitive equilibrium reflects precisely its marginal social value. We also develop a converse of Theorem 1, known as the Second Fundamental Theorem of Welfare Economics.

THEOREM 2 For any Pareto optimal levels of utility $(u_1^*, \ldots, u_I^*)$, there are transfers of the numeraire commodity $(T_1, \ldots, T_I)$ satisfying $\sum_{i=1}^{I} T_i = 0$, such that a competitive equilibrium reached from the endowments $(\omega_{m1} + T_1, \ldots, \omega_{mI} + T_I)$ yields precisely the utilities $(u_1^*, \ldots, u_I^*)$.

In our set up, a transfer of one unit of the numeraire from consumer i to consumer i' will cause each of these consumers' equilibrium consumption of the numeraire to change by exactly the amount of the transfer and will cause no other changes. Thus, by appropriately transferring endowments of the numeraire commodity, the resulting competitive equilibrium can be made to yield any utility vector in the boundary of the utility possibility set.

3.5 Free-entry and the long run competitive equilibrium

We consider the case in which an infinite number of firms can potentially be formed, each with access to the most efficient production technology. Moreover, we assume a situation of free entry where firms may enter or exit the market in response to profit opportunities. This is a reasonable approximation when we think of long-run outcome in the market. Suppose that each of an infinite number of potential firms has access to a technology for producing good l with cost function $c(q)$, where q is the individual firm's output of good l. We assume no sunk cost in the long run, that is, $c(0) = 0$. The aggregate demand is $x(.)$, with inverse demand function $P(.)$.

In a long-run competitive equilibrium, we would like to determine price, output levels and the number of firms that are active in the industry. Given our assumption of identical firms, we focus on equilibria in which all active firms produce the same output level, so that a long-run competitive equilibrium can be described by a triple (p, q, J) formed by a price p, an output per firm q and an integer number of active firms J (hence the total industry output is $Q = Jq$).[2] A firm's supply correspondence can then include at most one positive output at any given price level p. The central assumption determining the number of active firms is one of free entry and exit: A firm will enter the market if it can earn nonnegative profits at the going market price and will exit if it can make only negative profits at any positive production level given this price. If all firms, active and potential, take prices as unaffected by their own actions, this implies that active firms must earn exactly zero profits in any long-run competitive equilibrium; otherwise, we would have either no firms willing to be active in the market (if profits were negative) or an infinite number of firms entering the market (if profits were positive).

DEFINITION 5 Given an aggregate demand function $x(p)$ and a cost function $c(q)$ for each potentially active firm having $c(0) = 0$, a triple (p^*, q^*, J^*) is a *long-run competitive equilibrium* if

LC(1) *Profit maximization*: q^* solves $\max_{q \geq 0} p^* q - c(q)$.

LC(2) *Demand equals supply*: $x(p^*) = J^* q^*$.

LC(3) *Free entry condition*: $p^* q^* - c(q^*) = 0$.

The long-run competitive equilibrium price can be thought of as equating demand with the long-run supply, where the long-run supply takes into account firm's entry and exit decisions. We can define a long-run aggregate supply correspondence by

$$Q(p) = \begin{cases} \infty & \text{if } \pi(p) > 0 \\ \{Q \geq 0 : Q = Jq \text{ for some integer } J \geq 0 \text{ and } q \in q(p)\} & \text{if } \pi(p) = 0 \\ 0 & \text{if } \pi(p) < 0 \end{cases}$$

[2] The assumption that all active firms produce the same output level is without loss of generality whenever $c(.)$ is strictly convex on the set $(0, \infty]$.

If $\pi(p) > 0$ then every firm wants to supply an amount strictly bounded away from zero and hence aggregate supply is infinite. If $\pi(p) = 0$ and $Q = Jq$ for some integer J and $q \in q(p)$ then we have J active firms supplying q amount and the rest remain inactive. Note that since $c(0) = 0$, staying inactive is also a profit maximization choice for a firm. Finally, it is quite easy to verify that p^* is a long-run equilibrium price if and only if $x(p^*) \in Q(p^*)$. On the one side, if (p^*, q^*, J^*) is a long-run competitive equilibrium, then condition LC(1) implies that $q^* \in q(p^*)$ and condition LC(3) implies that $\pi(p^*) = 0$. Hence by condition LC(2), $x(p^*) \in Q(p^*)$. On the other side, if $x(p^*) \in Q(p^*)$, then $\pi(p^*) = 0$ and there exists $q^* \in q(p^*)$ and an integer J^* with $x(p^*) = J^* q^*$. In what follows we analyze the long-run competitive equilibrium under specific restrictions on the cost function $c(.)$ and on the aggregate demand function $x(.)$.

1. $c(q) = cq$ for some $c > 0$ and $x(c) > 0$. In this case condition LC(1) requires that $p^* \leq c$. Given $x(c) > 0$, condition LC(2) requires that $q^* > 0$. From condition LC(3) we know that $(p^* - c)q^* = 0 \Rightarrow p^* = c$ and aggregate consumption is $x(c)$. Note that J^* and q^* are indeterminate (any J^* and q^* such that $J^* q^* = x(c)$ satisfies conditions LC(1) and LC(2)). Here the long-run aggregate supply correspondence is

$$Q(p) = \begin{cases} \infty & \text{if } p > c \\ [0, \infty) & \text{if } p = c \\ 0 & \text{if } p < c \end{cases}$$

2. $c(q)$ is increasing and strictly convex and $x(c'(0)) > 0$. In this case *no long-run equilibrium exists*. If $p > c'(0)$, then $\pi(p) > 0$ and supply is infinity. If $p \leq c'(0)$, then long-run supply is zero while $x(p) > 0$. Therefore, the long-run aggregate supply correspondence is

$$Q(p) = \begin{cases} \infty & \text{if } p > c'(0) \\ 0 & \text{if } p \leq c'(0) \end{cases}$$

   Therefore, the demand function $x(p)$ has no intersection with the long-run aggregate supply correspondence.

3. To generate the existence of an equilibrium with determinate number of firms, the

long-run cost function must exhibit a strictly positive quantity $\bar{q}$ at which a firm's average cost of production is minimized. Let $\bar{c} = \frac{c(\bar{q})}{\bar{q}}$ where $\bar{q} > 0$ is the efficient scale. Assume that $x(\bar{c}) > 0$. At any long-run equilibrium (p^*, q^*, J^*), $p^* = \bar{c}$. Suppose that $p^* > \bar{c}$, then $\pi(p^*) > 0$ implying infinite output which cannot be an equilibrium. If $p^* < \bar{c}$ then though $x(p^*) > 0$, $p^*q - c(q) = p^*q - \left(\frac{c(q)}{q}\right) q \leq (p^* - \bar{c})q < 0$ for any $q > 0$. Thus, the firm earns negative profit. Thus, $p^* = \bar{c}$. Moreover, if $p^* = \bar{c}$ then $q^* = \bar{q}$ and $J^* = \frac{x(c)}{\bar{q}}$. Here the long-run aggregate supply correspondence is

$$Q(p) = \begin{cases} \infty & \text{if } p > \bar{c} \\ \{Q \geq 0 : Q = J\bar{q} \text{ for some integer } J \geq 0\} & \text{if } p = \bar{c} \\ 0 & \text{if } p < \bar{c} \end{cases}$$

Observe that the equilibrium price and aggregate output are exactly the same as if the firm had a constant returns to scale technology with unit cost $\bar{c}$. Therefore, long-run competitive equilibrium, if it exists in our two-good quasilinear set up, is always Pareto efficient it achieves the efficient scale of production.

The two main drawbacks of this analysis are-(a) if J^* is small ($J^* = 1$ say) then clearly price taking is a bad assumption to make and (b) if we have nonconvexity, demand may not equal supply (for example, we can have a situation where $x(\bar{c})$ is not an integer multiple of $\bar{q}$).

Bibliography

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[2] Mas-Colell, A., Whinston, M. D. and Green, J. (1995). Microeconomic Theory-Chapter 5, Pages-127-166.