

3.10 Correlated Equilibria

Consider the following battle of the sexes game

	C	F
C	(2,5)	(0,0)
F	(0,0)	(5,2)

3.10 Correlated Equilibria

Up to now it has been assumed that the players choose their strategies independently of each other, i.e. there is no communication.

However, in coordination (and anti-coordination games) it would help players to communicate in order to choose an equilibrium which is "favourable to both".

For example, in the game above the players could agree to toss a coin and if the result is heads, then both play C . Otherwise, both play F .

This would be an "equitable" solution to the game. Both players would receive an expected reward of 3.5.

Correlated Equilibria

One important feature of the solution presented above is that given the result of the coin toss, neither player wishes to depart from the agreed action given that the other player keeps to the agreement.

For example, given the result is heads, the proposed action pair is (C, C) . If player 1 changes unilaterally from this action, then she obtains 0 rather than 2.

This is the essence of a correlated equilibrium: given the signal available to each player (here, they jointly observe the result of a coin toss), none of them can gain by ignoring the signal given the others take the appropriate action.

In general, we assume that each player receives his/her own private signal. In certain cases (discussed later), we may assume that the signal is public (observed by all the players).

Definition of a Correlated Strategy Pair in a 2-Player Game

A correlated strategy pair is given by a joint distribution over the set of pure strategy pairs. For example, the correlated strategy described for the battle of the sexes game is

	C	F
C	$\frac{1}{2}$	0
F	0	$\frac{1}{2}$

i.e. with probability 0.5 (after a heads) both play C , otherwise both play F .

Definition of a Correlated Strategy Pair in a 2-Player Game

It should be noted that given the action to be taken by one player under such a correlated strategy pair, the action to be taken by the other is known. In such cases, the strategy pair can be achieved by the players jointly observing a public signal.

Otherwise, it is assumed that players each observe a private signal telling them which action to take. The joint distribution of the signals corresponds to the joint distribution of the actions to be taken.

Relations Between Correlated Strategy Pairs, Pure and Mixed Strategies

Suppose both players have two possible actions. The general form of a correlated strategy pair is

	C	D
A	p_1	p_2
B	p_3	p_4

, where $p_1 + p_2 + p_3 + p_4 = 1$. Such a correlated strategy can be presented as a 4-dimensional vector $\pi = (p_1, p_2, p_3, p_4)$.

Expected Rewards Under a Correlated Strategy Pair

This correlated strategy pair means that (A, C) is played with probability p_1 , (A, D) is played with probability p_2 , etc.

The expected reward of Player i under correlated strategy π is denoted $R_i(\pi)$ and calculated with respect to the joint distribution of the actions to be taken, i.e. the payoff of Player i is given by

$$R_i(\pi) = p_1 R_i(A, C) + p_2 R_i(A, D) + p_3 R_i(B, C) + p_4 R_i(B, D).$$

This is a linear combination of the p_i .

Relations Between Correlated Strategy Pairs, Pure and Mixed Strategies

If $p_i = 1$ for some i , then the correlated strategy pair is a pair of pure strategies.

For example, $\pi = (0, 1, 0, 0)$ corresponds to the pair of pure strategies (A, D) .

If π is of the form $(qr, q[1 - r], [1 - q]r, [1 - q][1 - r])$, then it corresponds to a pair of mixed strategies.

In this case, Player 1 takes action A with probability q and Player 2 takes action C with probability r (independently of the action of Player 2).

Relations Between Correlated Strategy Pairs, Pure and Mixed Strategies

It follows that the set of correlated strategy pairs is an extension of the set of mixed strategy pairs.

In general, communication is required to attain a correlated strategy pair.

It is assumed that the form of a correlated strategy pair and the way in which it is to be achieved is agreed upon before the game is played.

However, it is assumed that this agreement is not binding (and cannot be made binding). Hence, players are free to ignore the action recommended to them.

Conditions for a Correlated Equilibrium in a 2×2 Matrix Game

According to the correlated strategy pair

	C	D
A	p_1	p_2
B	p_3	p_4

Player 1 is advised (by the appropriate signalling procedure) to play A with probability $p_1 + p_2$. Given Player 1 is advised to play A , the probability that Player 2 is advised to play C is $\frac{p_1}{p_1 + p_2}$.

Conditions for a Correlated Equilibrium in a 2×2 Matrix Game

Each player should maximise their expected reward given the signal (recommendation) he/she receives.

Hence, if Player 1 obtains the recommendation to play A , her expected reward under such a correlated strategy pair is

$$\frac{p_1 R_1(A, C)}{p_1 + p_2} + \frac{p_2 R_1(A, D)}{p_1 + p_2}.$$

If she ignores this recommendation (from the linearity of her expected payoff, we may assume that she plays B), her expected reward is

$$\frac{p_1 R_1(B, C)}{p_1 + p_2} + \frac{p_2 R_1(B, D)}{p_1 + p_2}.$$

Conditions for a Correlated Equilibrium in a 2×2 Matrix Game

For stability, we require that

$$\frac{p_1 R_1(A, C)}{p_1 + p_2} + \frac{p_2 R_1(A, D)}{p_1 + p_2} \geq \frac{p_1 R_1(B, C)}{p_1 + p_2} + \frac{p_2 R_1(B, D)}{p_1 + p_2}.$$

This leads to

$$p_1 R_1(A, C) + p_2 R_1(A, D) \geq p_1 R_1(B, C) + p_2 R_1(B, D).$$

Conditions for a Correlated Equilibrium in a 2×2 Matrix Game

It should be noted that if $p_1 = p_2 = 0$, then the above derivation does not make sense (since we divide by 0).

However, in this case Player 1 is never recommended to play A and so we may ignore this condition.

However, in this case the final inequality is simply $0 \geq 0$, which is satisfied (i.e. in practice this condition is ignored when $p_1 = p_2 = 0$).

Conditions for a Correlated Equilibrium in a 2×2 Matrix Game

Arguing in a similar way, the 4 conditions for a correlated equilibrium are

$$p_1 R_1(A, C) + p_2 R_1(A, D) \geq p_1 R_1(B, C) + p_2 R_1(B, D)$$

$$p_3 R_1(B, C) + p_4 R_1(B, D) \geq p_3 R_1(A, C) + p_4 R_1(A, D)$$

$$p_1 R_2(A, C) + p_3 R_2(B, C) \geq p_1 R_2(A, D) + p_3 R_2(B, D)$$

$$p_2 R_2(A, D) + p_4 R_2(B, D) \geq p_2 R_2(A, C) + p_4 R_2(B, C)$$

These conditions correspond in turn to the following recommendations: 1) Player 1 to play A , 2) Player 1 to play B , 3) Player 2 to play C and 4) Player 2 to play D .

Relation Between Nash Equilibria and Correlated Equilibria

Any Nash equilibrium pair of strategies is also a Correlated Equilibrium. A pair of mixed strategies that is not a Nash equilibrium is not a correlated equilibrium.

Any randomisation over Nash equilibria is also a correlated equilibrium. Any randomisation over a set of strong Nash equilibria can be attained by joint observation of a public signal.

For example, in the battle of the sexes game, both (F, F) and (C, C) are strong Nash equilibria. Any correlated strategy pair that picks (F, F) with probability p and otherwise picks (C, C) is a correlated equilibrium.

Relation Between Nash Equilibria and Correlated Equilibria

There is also a mixed Nash equilibrium where Player 1 plays C with probability $\frac{2}{7}$ and Player 2 plays C with probability $\frac{5}{7}$. Presented as a correlated equilibrium, this corresponds to

	C	F
C	$\frac{2}{7} \times \frac{5}{7} = \frac{10}{49}$	$\frac{2}{7} \times \frac{2}{7} = \frac{4}{49}$
F	$\frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$	$\frac{5}{7} \times \frac{2}{7} = \frac{10}{49}$

Choosing a Correlated Equilibrium

Since any Nash equilibrium is a correlated equilibrium and multiple Nash equilibria may occur, it is clear that there may well be multiple correlated equilibria.

Hence, if the problem of the choice of a Nash equilibrium exists, then the problem of the choice of a correlated equilibrium also exists.

However, the concept of correlated equilibrium assumes that communication is possible and hence players may choose an appropriate equilibrium based on some criterion. First we consider the set of attainable payoffs.

The Set of Attainable Payoffs Under a Correlated Strategy Pair

Consider the following game (an example of the chicken game).

	F	C
F	(0,0)	(8,2)
C	(2,8)	(6,6)

The Set of Attainable Payoffs Under a Correlated Strategy Pair

For example, by randomising between the Nash equilibria (C, F) and (F, C) , any vector of expected payoffs on the line between $(8, 2)$ and $(2, 8)$ can be attained.

Similarly, we can attain a vector of expected payoffs anywhere on any of the lines between the payoff vectors given by any two pairs of strategies.

By randomising between 3 or more strategy pairs, we can obtain any payoff vector within the interior of the lines obtained above.

It follows that the set of attainable payoffs is the convex hull of the payoff vectors given in the payoff matrix (the smallest convex set that contains all these vectors).

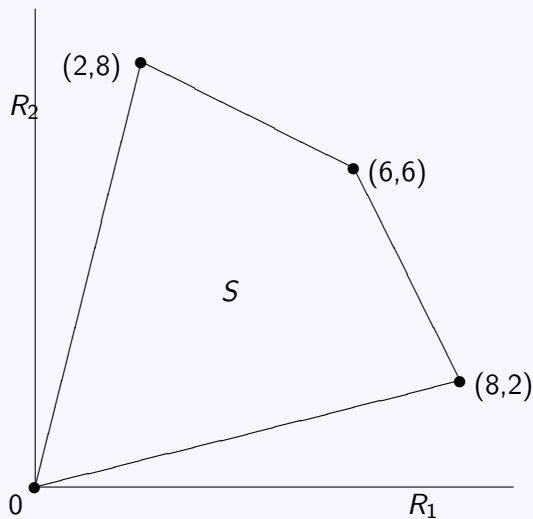
The Set of Attainable Payoffs Under a Correlated Strategy Pair

This set can be obtained by joining the points given by each payoff vector in the payoff matrix.

The external set of lines thus drawn is the boundary of the convex hull (set of attainable payoffs).

The convex hull for the chicken game considered above is illustrated on the following slide.

The Set of Attainable Payoffs Under a Correlated Strategy Pair



The Set of Attainable Payoffs Under a Correlated Strategy Pair

Note that the maximum payoff attainable by a player must occur at one of the vertices of the convex hull (i.e. be attained when a pair of pure strategies is played).

Similarly, the sum of the payoffs must be maximised for some pair of pure strategies.

The Set of Pareto Optimal Payoff Vectors

A payoff vector is Pareto optimal if it is a) attainable and b) there is no attainable payoff vector for which one player attains a greater payoff and the other player obtains at least the same payoff.

For the chicken game, it can be seen that the set of Pareto optimal solutions is the union of the line from $(2,8)$ to $(6,6)$ and the line from $(6,6)$ to $(8,2)$. This is the top right border of the set of attainable payoffs.

Choosing a Correlated Equilibrium

For example, the solution presented for the battle of the sexes game equalises the payoff of the players while maximising the sum of the expected payoffs (the sum of the payoffs cannot be more than 7).

An equilibrium which maximises the sum of the expected payoffs of the players is called a **utilitarian equilibrium**.

An equilibrium which maximises the expected payoff of Player i is called a **Libertarian i equilibrium**.

An equilibrium which maximises the minimum expected payoff of a player is called an **egalitarian equilibrium**. From the argument given above, the coin tossing solution is an egalitarian equilibrium.

Deriving an Appropriate Correlated Equilibrium

Since the expected payoffs of the players are linear combinations of the p_i (probabilities of choosing the particular strategy pairs), the criteria given above involve maximising a linear combination of p_i .

The conditions for a correlated equilibrium are given by a set of linear inequalities involving the p_i .

Equilibria of the types given above can be derived by defining the problem as a linear programming problem.

In many cases we can derive appropriate correlated equilibria using mathematical argumentation, rather than using the more general simplex method.

Deriving an Appropriate Correlated Equilibrium

Consider the battle of the sexes game

	C	F
C	(2,5)	(0,0)
F	(0,0)	(5,2)

Deriving an Appropriate Correlated Equilibrium

The linear programming problem defining the Libertarian 1 equilibrium is given by

$$\max z = 2p_1 + 0p_2 + 0p_3 + 5p_4 = 2p_1 + 5p_4$$

subject to

- 1) the conditions for (p_1, p_2, p_3, p_4) to define a joint distribution, i.e. $p_i \geq 0$, $i = 1, 2, 3, 4$ and $p_1 + p_2 + p_3 + p_4 = 1$,
- 2) the conditions for (p_1, p_2, p_3, p_4) to be a correlated equilibrium (see next slide).

Deriving an Appropriate Correlated Equilibrium

The conditions for a correlated equilibrium are

$$2p_1 + 0p_2 \geq 0p_1 + 5p_2 \Rightarrow p_1 \geq \frac{5p_2}{2}$$

$$0p_3 + 5p_4 \geq 2p_3 + 0p_4 \Rightarrow p_4 \geq \frac{2p_3}{5}$$

$$5p_1 + 0p_3 \geq 0p_1 + 2p_3 \Rightarrow p_1 \geq \frac{2p_3}{5}$$

$$0p_2 + 2p_4 \geq 5p_2 + 0p_4 \Rightarrow p_4 \geq \frac{5p_2}{2}$$

Deriving an Appropriate Correlated Equilibrium

We could find the appropriate solution by solving this linear programming problem. But such problems are often easy to solve if we first derive the Nash equilibria.

The pure Nash equilibria are (C, C) and (F, F) . The strategy pair (F, F) gives the maximum possible payoff to Player 1 (over the set of pure strategy pairs) and is a Nash equilibrium (and thus a correlated equilibrium).

Hence, (F, F) must be the correlated equilibrium which maximises the payoff of Player 1.

Similarly, (C, C) is the correlated equilibrium which maximises the payoff of Player 2.

Deriving an Appropriate Correlated Equilibrium

Now consider the utilitarian equilibrium. The problem in this case is

$$\max z = (2 + 5)p_1 + (0 + 0)p_2 + (0 + 0)p_3 + (5 + 2)p_4 = 7p_1 + 7p_4,$$

subject to the same conditions as before.

Again, knowing the pure Nash equilibria for this problem, we can calculate the (set of) utilitarian equilibria.

(F, F) and (C, C) are Nash equilibria which maximise the sum of the payoffs to the players (over the set of pure strategy pairs).

Deriving an Appropriate Correlated Equilibrium

Any randomisation over these two Nash equilibria is a correlated equilibrium and gives the same sum of payoffs.

Hence, any π of the form $\pi = (p, 0, 0, 1 - p)$ is a utilitarian equilibrium.

Deriving an Appropriate Correlated Equilibrium

Now consider the egalitarian equilibrium. This battle of the sexes is not symmetric, but there is a degree of symmetry in the payoffs that implies that at an egalitarian equilibrium both players have the same expected payoff.

A 2×2 matrix game where both players can choose either action A or action B will be called "quasi-symmetric" if

$$\begin{aligned} R_1(i, j) &= R_2(j, i), \\ R_1(i, i) &= R_2(j, j), \text{ where } i \neq j, i, j \in \{A, B\}. \end{aligned}$$

i.e. a payoff vector on the leading diagonal is the reverse of the other payoff vector on that diagonal. Note that a symmetric game is quasi-symmetric.

Deriving an Appropriate Correlated Equilibrium

Result At an egalitarian equilibrium of a quasi-symmetric game, both players must obtain the same expected payoff.

It follows that to find an egalitarian equilibrium of a quasi-symmetric game, one should maximise the expected sum of the payoffs subject to a) the standard constraints for a correlated equilibrium (stability and the conditions required for a distribution) and b) the constraint that both players should obtain the same payoff.

Deriving an Appropriate Correlated Equilibrium

In this case the problem (as before) is to maximise $z = 7p_1 + 7p_4$, but with the additional constraint that the payoffs of the players are equal, i.e.

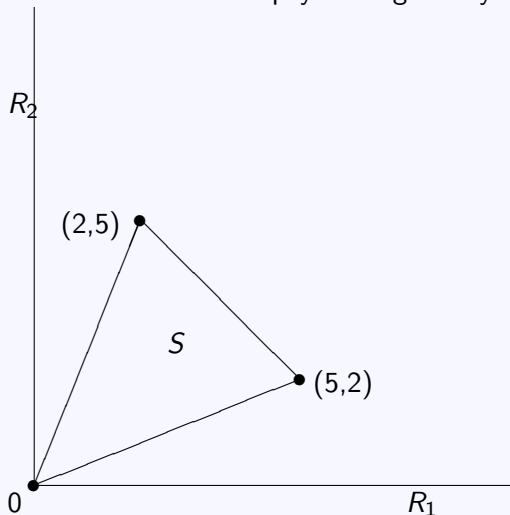
$$2p_1 + 5p_4 = 5p_1 + 2p_4 \Rightarrow p_1 = p_4.$$

Any correlated equilibrium of the form $(p, 0, 0, 1 - p)$ maximises the sum of the expected payoffs.

Setting $p = \frac{1}{2}$, the player's expected payoffs are equalised and hence the egalitarian equilibrium is $(\frac{1}{2}, 0, 0, \frac{1}{2})$.

Graphical Presentation/Interpretation of the Solution

The set of attainable payoffs is given by



Graphical Presentation/Interpretation of the Solution

There are 2 pure Nash equilibria which give payoff vectors of $(2, 5)$ and $(5, 2)$, respectively.

The set of Pareto optimal solutions is the line between these two payoff vectors, i.e. any Pareto optimal solution can be obtained by appropriately randomising between the two Nash equilibria.

Thus any Pareto optimal solution can be obtained at a correlated equilibrium.

In such a case the solution of the equilibrium choice problem can be solved by choosing the appropriate payoff vector from the set of Pareto optimal payoff vectors and defining the corresponding randomisation over the pure Nash equilibria.

Graphical Presentation/Interpretation of the Solution

Hence, the Libertarian 1 equilibrium has to attain a payoff vector of $(5, 2)$ (the Pareto optimal payoff which maximises the payoff of Player 1). The corresponding correlated strategy is (F, F) (in terms of a strategy pair) or $(0, 0, 0, 1)$ (in terms of the standard description of a correlated strategy).

Similarly, the Libertarian 2 equilibrium is (C, C) [or $(1, 0, 0, 0)$] and gives a payoff of $(2, 5)$.

Graphical Presentation/Interpretation of the Solution

In this battle of the sexes game, each Pareto optimal solution gives the same sum of payoffs. Thus, any mixture between (C, C) and (F, F) is a utilitarian equilibrium.

The payoff at the egalitarian equilibrium is given by the element of the set of Pareto optimal payoffs at which the minimum payoff to either of the players is maximised.

This condition is achieved if either a) both players obtain the same expected payoff, or b) if one player always gets more than the other at a Pareto optimal payoff vector, then we maximise the payoff of the player with the minimum reward.

In this case, we should equalise the payoffs of the players. It follows that both players should obtain $\frac{7}{2}$ and the appropriate correlated equilibrium is to choose (F, F) with probability 0.5, otherwise choose (C, C) .

Graphical Presentation/Interpretation of the Solution

Hence, the derivation of appropriate correlated equilibria is relatively straightforward when any Pareto optimal payoff vector can be attained at a correlated equilibrium.

It should be noted that in the Chicken game (C, C) is not a Nash equilibrium (and is hence not a correlated equilibrium), but the corresponding payoff vector, $(6, 6)$, is an element of the set of Pareto optimal payoff vectors.

It follows that not all Pareto optimal payoff vectors can be attained by a correlated equilibrium and the choice of an appropriate correlated equilibrium may be more difficult.

This problem will be considered in the tutorials.