Classical Demand Theory

Anuj Bhowmik

Economic Research Unit
Indian Statistical Institute
203 Barackpore Trunk Road
Kolkata 700108
India

Email: anuj.bhowmik@isical.ac.in,
anujbhowmik09@gmail.com

Homepage: http://www.isical.ac.in/~anuj.bhowmik/

Microeconomic Theory I
Semester I, 2013

Outline

1. Basic assumptions
2. Utility maximization problem (UMP)
3. Strong axiom of revealed preference

Outline

Outline

1. Basic assumptions
2. Utility maximization problem (UMP)
3. Strong axiom of revealed preference

Mas-Colell, A., Whinston, M. D. and Green, J. (1995). Microeconomic Theory

Introduction

The analysis of consumer behavior begins by specifying the consumer's preferences over the commodity bundles in the consumption set $X \subseteq \mathbb{R}^L_+$. The consumer's preferences are captured by the preference relation $\succeq$ on X.

Desirability assumptions

It is often reasonable to assume that larger amounts of commodities are preferred to smaller ones.

Monotonicity: The preference relation $\succeq$ on X is *monotone* if $x \in X$ and $y >> x$ (that is, $y_\ell > x_\ell$ for all $1 \leq \ell \leq L$) implies that $y \succ x$.

Strong monotonicity: The preference relation $\succeq$ on X is *strongly monotone* if $x \in X$ and $y \geq x$ (that is, $y_\ell \geq x_\ell$ for all $1 \leq \ell \leq L$) and $y \neq x$ implies that $y \succ x$.

Local non-satiation: The preference relation $\succeq$ on X is *locally non-satiated* if for all $x \in X$ and every $\varepsilon > 0$, there exists $y \in X$ such that $||y - x|| \equiv \sqrt{\sum_{\ell=1}^{L} (y_\ell - x_\ell)^2} \leq \varepsilon$ and $y \succ x$.

Weak Monotonicity: The preference relation $\succeq$ on $X = \mathbb{R}^L_+$ is said to be *weakly monotone* if and only if $y \geq x$ implies $y \succeq x$.

1. If $\succeq$ on X is strongly monotone then it is monotone.
2. If $\succeq$ is complete and strongly monotone then it is also weak monotone.
3. Local non-satiation is a much weaker requirement than monotonicity.

Proposition

If $\succeq$ on $X = \mathbb{R}^L_+$ is locally non-satiated, rational and weak monotone then it is monotone.

Proof: Consider any $x, y \in X$ such that $y >> x$. We will have to show that $y \succ x$. Consider

$$y' = \frac{1}{2}x + \frac{1}{2}y.$$

Since $y' >> x$, by weak monotonicity, $y' \succeq x$. There exists an $\varepsilon > 0$ small enough such that for all $z \in N_\varepsilon(y')$,

$$y >> z >> x.$$

By local non-satiation, there exists a $z' \in N_\varepsilon(y')$ such that $z' \succ y'$. So, $z' \succ y' \succeq x$ and by transitivity, $z' \succ x$. By weak monotonicity, $y \succeq z'$. Finally, by transitivity, we get $y \succ x$.

Consider the following three related sets.

1. The *indifference set* containing point x is the set of bundles that are indifferent to x, that is, $I(x) = \{y \in X : x \sim y\}$.
2. The *upper contour set* of bundle x is the set of bundles that are at least as good as x, that is, $R(x) = \{y \in X : y \succeq x\}$.
3. The *lower contour set* of bundle x is the set of bundles that x is at least as good as, that is, $L(x) = \{y \in X : x \succeq y\}$.

Observe that $R(x) \cap L(x) = I(x)$.

Convexity assumptions

A second significant assumption, that of convexity of $\succeq$, concerns the trade-offs that the consumer is willing to make among different goods.

Convexity: The preference relation $\succeq$ on X is *convex* if for every $x \in X$, $R(x)$ is convex, equivalently, if $y \succeq x$ and $z \succeq x$ then $\alpha y + (1 - \alpha)z \succeq x$ for any $\alpha \in [0, 1]$.

Strict convexity: The preference relation $\succeq$ on X is *strictly convex* if for every x, we have $y \succeq x$, $z \succeq x$ and $y \neq z$ implies $\alpha y + (1 - \alpha)z \succ x$ for all $\alpha \in (0, 1)$.

Preference and utility

For analytical purposes, it is very helpful if we can summarize the consumer's preferences by means of a utility function. Unfortunately,with the assumptions made so far, a rational preference relation need not be representable by a utility function.

Lexicographic preference

Assume that $X = \mathbb{R}^2_+$. We say $x \succeq y$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$".

Consequences

[1] $x \succ y \Leftrightarrow \{x_1 > y_1\}$ or $\{x_1 = y_1$ and $x_2 > y_2\}$.

[2] $x \sim y \Leftrightarrow x = y$.

The lexicographic preference is complete, transitive, strongly monotonic and strictly convex.

Proposition

There is no utility function that can represent lexicographic preferences.

Proof. Suppose that there exists a utility function $U : \mathbb{R}^2_+ \to \mathbb{R}$ that represents lexicographic preferences. For every $x_1 \in \mathbb{R}_+$, we can pick a rational number $r(x_1)$ such that

$$u(x_1, 2) > r(x_1) > u(x_1, 1).$$

For any $0 < x_1' < x_1$, we must have $r(x_1') < r(x_1)$ since, due to lexicographic preference,

$$u(x_1, 2) > r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1') > u(x_1', 1).$$

Therefore, $r : \mathbb{R}_+ \to \mathbb{Q}$ is a one-to-one mapping from $\mathbb{R}_+$ (which is uncountable) to $\mathbb{Q}$ (countable). This is mathematically impossible.

Thus our assumption that there exists a utility function U that represents the lexicographic preferences is false.

Continuity

The preference relation $\succeq$ on X is *continuous* if it is preserved in limits. That is, given any two sequences $\{x_n : n \geq 1\}$ and $\{y_n : n \geq 1\}$ with $x_n \succeq y_n$ for all $n \geq 1$, $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, we have $x \succeq y$.

Observe that the lexicographic preference is not continuous. Let $\succeq$ be lexicographic and let $x_n = (\frac{1}{n}, 0)$ and $y_n = (0, 1)$. Since $\frac{1}{n} > 0$ for all n, $x_n \succeq y_n$. Also observe that $y = \lim_{n\to\infty} y_n = (0, 1)$ and $x = \lim_{n\to\infty} x_n = (0, 0)$. Therefore $y \succ x$ which is a violation of continuity.

The preference relation $\succeq$ on X is continuous if and only if for all $x \in X$, $R(x)$ and $L(x)$ are closed sets.

Continuous preferences and utility functions

The utility function $U : X \to \mathbb{R}$ representing $\succeq$ on X is called continuous at some $x \in X$ if $\{x_n : n \geq 1\}$ converges to x, then $\{U(x_n) : n \geq 1\}$ converges to $U(x)$.

The utility function $U : X \to \mathbb{R}$ is *continuous* if it is continuous at every point in X.

Proposition

If $U : X \to \mathbb{R}^{\ell}_{+}$ is a continuous utility function representing $\succeq$ on X, then $\succeq$ must be continuous.

Proof: Take two sequences $\{x_n : n \geq 1\}$ and $\{y_n : n \geq 1\}$ with $x_n \succeq y_n$ for all $n \geq 1$, $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$.

(1) Since U represents $\succeq$, $U(x_n) \geq U(y_n)$ for all $n \geq 1$.

(2) Since U is continuous,

$$U(x) = \lim_{n \geq \infty} U(x_n) \text{ and } U(x) = \lim_{n \geq \infty} U(x_n).$$

So,

$$U(x) - U(y) = \lim_{n \to \infty} (U(x_n) - U(y_n)) \geq 0,$$

which implies $x \succeq y$. Hence $\succeq$ on X is continuous.

Theorem

If preference relation $\succeq$ on X is rational and continuous then there exists a continuous utility function $U : X \to \mathbb{R}$ that represents $\succeq$.

The proof of this theorem is very technical and hence omitted. Instead we prove a simpler result.

Proposition ($*$)

If preference relation R on $X = \mathbb{R}^L_+$ is rational, continuous and monotonic then there exists a utility function $U : \mathbb{R}^L_+ \to \mathbb{R}$ that represents $\succeq$.

Proof: Let $e = (1, \cdots, 1)$. Given any vector $x \in X$, let $\alpha(x) \in \mathbb{R}_+$ be such that $\alpha(x)e \sim x$. We first show that such a number exists and is unique. To do this, assume

$$B = \{\alpha \in \mathbb{R}_+ : \alpha e \succeq x\} \text{ and } W = \{\alpha \in \mathbb{R}_+ : x \succeq \alpha e\}.$$

(1) Then monotonicity implies that B is non-empty and since $\underline{0} \in W$, W is also non-empty.

(2) Continuity implies that both sets are closed.

(3) By completeness, $\mathbb{R}_+ = B \cup W$ and since $\mathbb{R}_+$ is connected, $B \cap W \neq \emptyset$.

Therefore, there is some $\alpha(x) \in \mathbb{R}_+$ such that $\alpha(x)e \sim x$.

Furthermore, by monotonicity, $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$ and hence, there exists exactly one $\alpha(x)$ such that $\alpha(x)e \sim x$.

For any $x \in X$, define $U : X \to \mathbb{R}$ by $U(x) = \alpha(x)$. We show that U is a utility function representing $\succeq$, that is,

$$x \succeq y \Leftrightarrow U(x) \geq U(y).$$

To see this, note by monotonicity assumption that

$$x \succeq y \Leftrightarrow \alpha(x)e \succeq \alpha(y)e \Leftrightarrow \alpha(x) \geq \alpha(y).$$

The utility function $U : X \to \mathbb{R}$ is *increasing* if $x >> y$ implies $U(x) > U(y)$ for all $x, y \in X$.

Suppose that X is convex. The utility function $U : X \to \mathbb{R}$ is *quasi-concave* if the set $\{y \in X : U(y) \geq U(x)\}$ is convex for all $x \in X$ or equivalently,

$$U(\alpha x + (1 - \alpha)y) \geq \min\{U(x), U(y)\}$$

for all $x, y \in X$ and $\alpha \in (0, 1)$.

Moreover, if the inequality is strict for all $x, y \in X$ such that $x \neq y$ and all $\alpha \in (0, 1)$ then U is *strictly quasi-concave*.

Let $U : X \to \mathbb{R}$ be a utility function representing the preference relation $\succeq$ on X. Then

(i) $\succeq$ is monotone if and only if U is increasing.

(ii) $\succeq$ is convex (strictly convex) if and only if U is quasi-concave (strictly quasi-concave).

Homothetic

In many applications it is common to focus on preferences for which it is possible to deduce the consumer's entire preference relation from a single indifference set.

A monotone preference relation $\succeq$ on $X = \mathbb{R}^L_+$ is *homothetic* if all indifference sets are related by proportional expansion along rays, that is, if $x \sim y$ then $\alpha x \sim \alpha y$ for all $\alpha \geq 0$.

Proposition

A continuous $\succeq$ on $X = \mathbb{R}^L_+$ is homothetic if and only if it admits a utility function U which is increasing and homogeneous of degree one, that is, $U(\delta x) = \delta U(x)$ for all $\delta > 0$.

Proof: Suppose that U is a utility function representing $\succeq$ which is homogeneous of degree one and let $\delta \geq 0$, $x, y \in X$ and $x \sim y$. Then

$$U(x) = U(y) \Rightarrow \delta U(x) = \delta U(y) \Rightarrow U(\delta x) = U(\delta y) \Rightarrow \delta x \sim \delta y.$$

Conversely, suppose that $\succeq$ is homothetic. Let $e = (1, \cdots, 1)$. For each $x \in X$, applying an argument similar to that in the proof of Proposition $(*)$, there exists a unique positive real number $\alpha(x)$ such that $\alpha(x)e \sim x$.

Define $U : X \to \mathbb{R}$ by $U(x) = \alpha(x)$. Thus, $\delta U(x)e \sim \delta x$. On the other hand,

$$U(\delta x)e = \alpha(\delta x)e \sim \delta x.$$

Thus, $U(\delta x) = \delta U(x)$.

We assume that the consumer has a rational, continuous and locally non-satiated preference relation $\succeq$ on $X = \mathbb{R}^\ell_+$ and we take U to be a continuous utility function representing $\succeq$. Given any $(p, w) >> 0$, the utility maximization problem (UMP) of the consumer is the following:

$$\sup\{U(x) : x \in B(p, w)\} \tag{1}$$

where

$$B(p, w) = \{x \in \mathbb{R}^\ell_+ : x \cdot p \leq w\}.$$

Proposition

If $(p, w) >> 0$ and U is continuous, then (1) has a solution.

Proof: For any $\ell \in \{1, \cdots, L\}$, we have $x_\ell \leq \frac{w}{p_\ell}$ for all $x \in B(p, w)$. Thus, $B(p, w)$ is bounded.

To show that $B(p, w)$ is closed, let $\{x_n : n \geq 1\} \subseteq B(p, w)$ and $\{x_n : n \geq 1\}$ converges to x. Since $x_n \cdot p \leq w$, we have $x \cdot p \leq w$. So, $x \in B(p, w)$ and $B(p, w)$ is closed.

Hence, $B(p, w)$ is compact. By the extreme value theorem, U attains it supremum on $B(p, w)$.

The *Walrasian demand correspondence* $D : \mathbb{R}^\ell \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^\ell_+$ is defined by

$$D(p, w) = \{x \in B(p, w) : U(x) \geq U(y) \text{ for all } y \in B(p, w)\}.$$

Throughout, we assume that $p \gg 0$ and $w > 0$.

Proposition

Suppose that U is a utility function representing a preference relation $\succeq$ on $X = \mathbb{R}^L_+$. Then the following properties are satisfied:

(i) If $\succeq$ is locally non-satiated and U is continuous then D is homogeneous of degree zero and it satisfies Walras' Law.

(ii) If $\succeq$ is convex, then $D(p, w)$ is a convex set. If $\succeq$ is strictly convex, then $D(p, w)$ is single-valued.

Proof: (i) Recall that $B(\alpha p, \alpha w) = B(p, w)$ for all $(p, w) \gg 0$ and $\alpha > 0$.

$D(\alpha p, \alpha w) = \{x \in B(\alpha p, \alpha w) : U(x) \geq U(y) \text{ for all } y \in B(\alpha p, \alpha w)\}$.

$D(p, w) = \{x \in B(p, w) : U(x) \geq U(y) \text{ for all } y \in B(p, w)\}$.

Thus, $D(\alpha p, \alpha w) = D(p, w)$.

Assume that $x \cdot p < w$ for some $x \in D(p, w)$. Then by local non-satiation, there exists another consumption bundle y sufficiently close to x with $y \cdot p < w$ and $y \succ x$. This contradicts our assumption that $x \in D(p, w)$. So, $x \cdot p = w$ for all $x \in D(p, w)$.

(ii) Suppose that utility U is quasi-concave and that there are two bundles $x, x' \in D(p, w)$ and $x \neq x'$. We will have to show that

$$x'' = \alpha x + (1 - \alpha)x' \in D(p, w)$$

for all $\alpha \in (0, 1)$. Note that $U(x) = U(x') = u^*$ (say). Quasi-concavity implies that

$$U(\alpha x + (1 - \alpha)x') = U(x'') \geq u^*.$$

In addition, $p.x \leq w$ and $p.x' \leq w$ imply

$$p.x'' = [\alpha x + (1 - \alpha)x'] \cdot p \leq w.$$

Therefore $x'' \in B(p, w)$. Since $U(x'') \geq u^*$ and $x'' \in B(p, w)$, we have $x'' \in D(p, w)$. Thus, $D(p, w)$ is a convex set.

(iii) Suppose U is strictly quasi-concave and that $x, x' \in D(p, w)$ and $x \neq x'$. Consider

$$x'' = \alpha x + (1 - \alpha)x'$$

for any $\alpha \in (0, 1)$. Note that $U(x) = U(x') = u^*$ and by strict quasi-concavity we get $U(x'') > u^*$. Moreover, $x'' \in B(p, w)$ and hence, given

$$U(x'') = U(\alpha x + (1 - \alpha)x') > u^*,$$

we have a contradiction to our assumption that x and x' are elements of $D(p, w)$. Hence D is single-valued.

Indirect utility function

For each $(p, w) >> 0$, the utility value of the UMP is denoted by $v(p, w) \in \mathbb{R}$ and $v : \mathbb{R}^{\ell}_{+} \times \mathbb{R}_{+} \to \mathbb{R}$ is called the *indirect utility function*. Note that $v(p, w) = U(x^*)$ for all $x^* \in D(p, w)$, and v often proves to be a very useful analytical tool.

Proposition

Suppose that U is a utility function representing a preference relation $\succeq$ on $X = \mathbb{R}^{L}_{+}$. Then the following properties are satisfied:

(i) If $\succeq$ is locally non-satiated and U is continuous then v is homogeneous of degree zero.

(ii) $v(p, w)$ is strictly increasing in w and non-increasing in p_ℓ for any $\ell \in \{1, \cdots, L\}$.

(iii) $v(p, w)$ is quasi-convex; that is, $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

Proof: Recall that $D(\alpha p, \alpha w) = D(p, w)$ for all $\alpha > 0$. Therefore $x^* \in D(p, w)$ if and only if $x^* \in D(\alpha p, \alpha w)$. So, for any such x^* we get

$$U(x^*) = v(p, w) = v(\alpha p, \alpha w).$$

Let $w' > w$ and $x^* \in D(p, w)$. Since $\succeq$ is local non-satiated, the Walras' law holds which means $p.x^* = w < w'$. We can select a small enough $\epsilon > 0$ such that $p.z \leq w'$ for all $z \in N_\epsilon(x^*)$. By local non-satiation, there exists $y \in N_\epsilon(x^*)$ such that $y \succ x^*$ which implies that

$$v(p, w') \geq U(y) > U(x^*) = v(p, w).$$

Therefore, $v(p, w)$ is increasing in w.

Consider any $(p, w) >> 0$ and $(p', w) >> 0$ such that $p' \geq p$ and $p' \neq p$. Note that $B(p', w) \subset B(p, w)$ and hence $x^* \cdot p \leq w$ for all $x^* \in D(p', w)$ implying any bundle in $D(p', w)$ is affordable when the price-wealth pair is (p, w).

Moreover, there exists bundles that are affordable for the pair (p, w) but were not affordable for the pair (p', w) (since $p' \geq p$ and $p' \neq p$ implies that $B(p, w) \setminus B(p', w) \neq \emptyset$). Hence,

$$v(p', w) = U(x^*) \leq U(y^*) = v(p, w).$$

Hence, the indirect utility function is non-increasing in p_ℓ for any ℓ.

Suppose that $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$. For any $\alpha \in [0, 1]$, consider the price-wealth pair

$$(p'', w'') = (\alpha p + (1 - \alpha)p', \alpha w + (1 - \alpha)w').$$

To establish quasi-convexity we show that $v(p'', w'') \leq \bar{v}$. Thus we show that for any x such that $p''.x \leq w''$ we have $U(x) \leq \bar{v}$. If $p''.x \leq w''$ then from the construction it follows that

$$\alpha x \cdot p + (1 - \alpha)x \cdot p' \leq \alpha w + (1 - \alpha)w'.$$

Hence, either $x \cdot p \leq w$ or $x \cdot p' \leq w'$ or both. If the former inequality holds, then

$$U(x) \leq v(p, w) \leq \bar{v}$$

and if the latter inequality holds then also we have

$$U(x) \leq v(p', w') \leq \bar{v}.$$

Thus we have $U(x) \leq \bar{v}$ for all x such that $x \cdot p'' \leq w''$. Hence, we get the result.

Expenditure minimization problem (EMP)

The expenditure minimization problem (EMP) for any given $p >> 0$ and any $u > U(\underline{0})$ is the following:

$$\min_{x \geq 0} x \cdot p \text{ subject to } U(x) \geq u.$$

The EMP computes the minimum level of wealth required to achieve a utility level u. It is the "dual" problem to the UMP.

The restriction to $u > U(\underline{0})$ rules out only uninteresting situations.

Proposition

Suppose that U is a continuous utility representation of a locally non-satiated $\succeq$ defined on $X = \mathbb{R}^L_+$ and that the price vector is $p >> 0$.

(i) If x^* is the optimal in the UMP when $w > 0$, then x^* is optimal in the EMP when the required level of utility is $U(x^*)$. Moreover, the minimize expenditure level in this EMP is w.

(ii) If x^* is the optimal in the EMP when $u > U(\underline{0})$, then x^* is optimal in the UMP when the wealth level is $x^* \cdot p$. Moreover, the maximized utility level in this UMP is exactly u.

Proof: (i)To prove the first case, suppose x^* is the solution to the UMP when $w > 0$ but x^* is not the solution to EMP when the utility level is $U(x^*)$. Then there exists $x' \neq x^*$ such that

$$U(x') \geq U(x^*) \text{ and } x' \cdot p < x^* \cdot p \leq w.$$

By local non-satiation, there exists x'' close to x' such that $U(x'') > U(x')$ and $x'' \cdot p < w$. Thus, for x'' we get

$$U(x'') > U(x^*) \text{ and } x'' \cdot p \leq w.$$

This is a contradiction to the fact that x^* is a solution to the UMP. Hence, x^* is also a solution to the EMP.

Moreover, we know that the UMP satisfies Walras' law and hence we have $x^* \cdot p = w$. Since we have already established that x^* is a solution to the EMP it follows that the minimum expenditure level in this EMP is w.

(ii) Given $u > U(\underline{0})$, we must have $x^* \neq 0$. Hence, $x^* \cdot p > 0$. Let x^* be a solution to the EMP but not to the UMP when wealth is $x^* \cdot p$. Then there exists an x' such that

$$U(x') > U(x^*) \text{ and } x' \cdot p \leq x^* \cdot p.$$

By continuity of U if $\alpha \in (0, 1)$ is close enough to 1 then we have

$$U(\alpha x') > U(x^*) \text{ and } \alpha x' \cdot p < x' \cdot p \leq x^* \cdot p.$$

This contradicts the fact that x^* is the optimal solution to the EMP.

Suppose that x^* is a solution to the EMP and $U(x^*) > u$. Consider a bundle αx^* where $\alpha \in (0, 1)$. As α approaches to 1,

$$U(\alpha x^*) \geq u \text{ and } \alpha x^* \cdot p < x^* \cdot p.$$

This is a contradiction to the assumption that x^* is a solution to the EMP with required level u.

Given $p >> 0$ and the required utility level $u > U(\underline{0})$, the value of the EMP is denoted by $e(p, u)$ and the $e : \mathbb{R}^\ell_+ \times \mathbb{R} \to \mathbb{R}_+$ is called the *expenditure function*.

Proposition

Suppose that U is a continuous utility function representing a locally non-satiated $\succeq$ on $X = \mathbb{R}^L_+$. The expenditure function e satisfies the following properties.

(i) $e(\cdot, u)$ is homogeneous of degree one.

(ii) $e(p, \cdot)$ is strictly increasing and $e(\cdot, u)$ is non-decreasing in p_ℓ for all ℓ.

(iii) $e(\cdot, u)$ is concave.

Proof: (i) First note that x^* is a solution to the EMP with utility level u and price vector $p >> 0$ if and only if x^* is also a solution to the EMP with utility level u and price vector αp where $\alpha > 0$. Hence

$$e(\alpha p, u) = \alpha p.x^* = \alpha e(p, u)$$

for any $p >> 0$ and $\alpha > 0$.

(ii) Suppose $e(p, \cdot)$ is not strictly increasing. Then there exists u' and u'' such that $u'' > u'$ and

$$e(p, u') = x' \cdot p \geq x'' \cdot p = e(p, u'').$$

By the continuity of U, there exists $\alpha \in (0, 1)$ close enough to 1 such that $U(\alpha x) > u'$ and $\alpha x \cdot p < x' \cdot p$. This contradicts our assumption that x' is a solution to the the EMP with the required utility level u'.

To show that $e(\cdot, u)$ is non-decreasing in p_ℓ for any ℓ, consider p'' and p' such that $p'' \geq p'$ and $p'' \neq p'$. Given any utility level u, let x'' be a solution to the EMP for the price vector p''. Then clearly,

$$e(p'', u) = x'' \cdot p'' \geq x'' \cdot p' \geq e(p', u).$$

(iii) Fix any utility level u and let $p'' = \alpha p + (1 - \alpha)p'$ for any given $\alpha \in [0, 1]$. Suppose x'' is a solution to the EMP for p'' when the utility level is u. Clearly,

$$\begin{aligned} e(p'', u) = x'' \cdot p'' &= \alpha x'' \cdot p + (1 - \alpha)x'' \cdot p' \\ &\geq \alpha e(p, u) + (1 - \alpha)e(p', u) \end{aligned}$$

and we get concavity of the expenditure function in prices.

Hicksian demand correspondence

The set of optimal commodity vectors in the EMP gives the *Hicksian (or compensated) demand correspondence* $H : \mathbb{R}^L_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$, which means $e(p, u) = x \cdot p$ for all $x \in h(p, u)$.

Proposition

Suppose that U is a continuous utility function representing a locally non-satiated $\succeq$ on $X = \mathbb{R}^L_+$. Let $p >> 0$ and $u > 0$. Then

(i) $H(\cdot, u)$ is homogeneous of degree zero.

(ii) No excess utility: For any $x \in H(p, u)$, $U(x) = u$.

(iii) Convexity/Uniqueness : If $\succeq$ is convex then $H(p, u)$ is a convex set and if $\succeq$ is strictly convex then $H(p, u)$ is unique.

Proof: (i) Let $p \gg 0$ and $u, \alpha > 0$. Observe that

$$H(p,u) = \left\{x \in \mathbb{R}^L_+ : x \cdot p = e(p,u)\right\}$$

and

$$H(\alpha p,u) = \left\{x \in \mathbb{R}^L_+ : x \cdot \alpha p = e(\alpha p,u)\right\}.$$

Since $e(\alpha p,u) = \alpha e(p,u)$, we have $H(\alpha p,u) = H(p,u)$.

(ii) Suppose there exists $x \in H(p,u)$ such that $U(x) > u$. By the continuity of U, for α close enough to 1 we get

$$U(\alpha x) > u \text{ and } \alpha x \cdot p < x \cdot p,$$

which contradicts with our assumption that x is a solution to the EMP.

(iii) If $x \in H(p,u)$ and $x' \in H(p,u)$ then $x \cdot p = x' \cdot p = e(p,u)$,

$$U(x) \geq u \text{ and } U(x') \geq u.$$

Consider an element $x'' = \alpha x + (1-\alpha)x'$ for $\alpha \in [0,1]$. Then

$$x'' \cdot p = \alpha x \cdot p + (1-\alpha)x' \cdot p = \alpha e(p,u) + (1-\alpha)e(p,u) = e(p,u)$$

By the convexity of $\succeq$, $U(x'') \geq u$. Hence, $x'' \in H(p,u)$ and so, $H(p,u)$ is convex.

Suppose that $H(p,u)$ is not unique and let $x, x' \in H(p,u)$. By (ii), $U(x) = U(x') = u$. For any $\alpha \in (0,1)$, consider

$$x'' = \alpha x + (1-\alpha)x'.$$

Since $H(p,u)$ is convex, $x'' \in H(p,u)$. However, by strict convexity of $\succeq$ we get $U(x'') > u$, which contradicts with (ii). Thus, $H(p,u)$ is unique.

For any two price-wealth situations (p, w) and (p', w'), if $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$ then $p' \cdot x(p, w) > w'$.

The following example shows that given a Walrasian demand function having *WARP* there is no consistent rational preference relation.

Example

Let

1. $p^1 = (2, 1, 2)$, $p^2 = (2, 2, 1)$, $p^3 = (1, 2, 2)$;
2. $w^1 = w^2 = w^3 = 8$;
3. $x(p^1, w^1) = x^1 = (1, 2, 2)$, $x(p^2, w^2) = x^2 = (2, 1, 2)$, and $x(p^3, w^3) = x^3 = (2, 2, 1)$.

Example

Note that

- $p^3 \cdot x(p^2, w^2) = w^3 = 8$, $x(p^2, w^2) \neq x(p^3, w^3)$ and $9 = p^2 \cdot x(p^3, w^3) > w^2 = 8$. Thus, $x^3 \succ^* x^2$ which implies that $x^3 \succ x^2$.
- $p^2 \cdot x(p^1, w^1) = w^2 = 8$, $x(p^1, w^1) \neq x(p^2, w^2)$ and $9 = p^1 \cdot x(p^2, w^2) > w^1 = 8$. Thus, $x^2 \succ^* x^1$ which implies that $x^2 \succ x^1$.
- $p^1 \cdot x(p^3, w^3) = w^1 = 8$, $x(p^1, w^1) \neq x(p^3, w^3)$ and $9 = p^3 \cdot x(p^1, w^1) > w^3 = 8$. Thus, $x^1 \succ^* x^3$ which implies that $x^1 \succ x^3$.

Example

Hence we have $x^3 \succ x^2 \succ x^1 \succ x^3$ which is incompatible with the fact that $\succeq$ rational preference.

As a result, we now have the strong axiom of revealed preference and is due to Houthakker (1950).

The market demand function $x : \mathbb{R}^\ell \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$ satisfies the *strong axiom of revealed preference* (or *SARP*) if for any list $(p^1, w^1), \cdots, (p^N, w^N)$ with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n \in \{1, \cdots, N-1\}$, we have $p^N.x(p^1, w^1) > w^N$ whenever $p^n.x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \in \{1, \cdots, N-1\}$.

In words, if $x(p^1, w^1)$ is directly or indirectly revealed preferred to $x(p^N, w^N)$, then $x(p^N, w^N)$ cannot be directly revealed preferred to $x(p^1, w^1)$ (so $x(p^1, w^1)$ cannot be affordable at (p^N, w^N)).

If we apply *SARP* to any list $(p^1, w^1), \cdots, (p^N, w^N)$ with $N = 2$ we get *WARP*. Hence *SARP* implies *WARP*.

Moreover, *WARP* does not necessarily imply *SARP*. In the last example, we had a list (p^3, w^3), (p^2, w^2), (p^1, w^1) with

$$x(p^3, w^3) \neq x(p^2, w^2), x(p^2, w^2) \neq x(p^1, w^1)$$

such that

$$p^3 \cdot x(p^2, w^2) = w^3 \text{ and } p^2 \cdot x(p^1, w^1) = w^2.$$

These conditions along with *SARP* imply $p^1 \cdot x(p^3, w^3) > w^1$, but in the last example, $p^1.x(p^3, w^3) = w^1$.

Proposition

If the Walrasian demand function $x : \mathbb{R}^\ell_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$ is generated by a rational preference, then it satisfies *SARP*.

Proof: Let $\succeq$ be a rational preference relation on X. Consider any list $(p^1, w^1), \cdots, (p^N, w^N)$ with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n \in \{1, \cdots, N-1\}$ and $p^n.x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \in \{1, \cdots, N-1\}$.

Since preference is complete,

$$x(p^1, w^1) \succ x(p^2, w^2) \succ \cdots \succ x(p^N, w^N).$$

By transitivity, we also get $x(p^1, w^1) \succ x(p^N, w^N)$. Therefore, for the budget set $B(p^N, w^N)$ we must have

$$x(p^1, w^1) \notin B(p^N, w^N) \Rightarrow p^N \cdot x(p^1, w^1) > w^N.$$

Hence we have *SARP*.

Proposition

If the Walrasian demand function $x : \mathbb{R}^\ell_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$ satisfies *SARP*, then there is a rational preference relation $\succeq$ that rationalizes x, that is, such that for all (p, w), $x(p, w) \succ y$ for every $y \neq x(p, w)$ with $y \in B(p, w)$.

Proof: Define the "directly revealed preferred to" relation $\succ^1$ on the commodity vectors by letting $x^* \succ^1 y^*$ whenever $x^* \neq y^*$, $x^* = x(p, w)$ and $p \cdot y^* \leq w$ for some (p, w).

From $\succ^1$ define the "directly or indirectly revealed preferred to" relation $\succ^2$ by letting $x^* \succ^2 y^*$ whenever there is a chain $x^1 \succ^1 x^2 \succ^1 \cdots \succ^1 x^N$ such that $x^1 = x^*$ and $x^N = y^*$.

Clearly if $x^* \succ^2 y^*$ and $y^* \succ^2 z^*$ then there is a chain

$$x^1 \succ^1 \ldots \succ^1 x^{N'} \succ^1 x^{N'+1} \succ^1 \ldots \succ^1 x^N$$

where $x^1 = x^*$, $x^{N'} = y^*$ and $x^N = z^*$ and hence we get $x^* \succ^2 z^*$. Thus, by definition, $\succ^2$ is transitive.

Moreover, by *SARP*, $\succ^2$ is irreflexive since *SARP* rules out $x^* \succ^2 x^*$. Therefore, $\succ^2$ is irreflexive and transitive. It follows that $\succ^2$ has a total extension $\succ^3$. The relation $\succ^3$ is irreflexive and transitive such that

(1) $x^* \succ^2 y^* \Rightarrow x^* \succ^3 y^*$ and

(b) whenever $x^* \neq y^*$, either $x^* \succ^3 y^*$ or $y \succ^3 x$.

Finally define $\succeq$ by letting $x^* \succeq y^*$ whenever $x^* = y^*$ or $x^* \succ^3 y^*$. It is obvious that $\succeq$ is complete and transitive and that $x(p, w) \succ y$ whenever $p.y \leq w$ and $y \neq x(p, w)$.

Aggregate Demand and Aggregate Wealth

The aggregate behavior of consumers is more important than the behavior of any single consumer.

One can investigate the extent to which the theory presented till now can be applied to aggregate demand, a suitably defined sum of the demands arising from all the economy's consumers.

There are, in fact, a number of different properties of individual demand that we might hope would hold in the aggregate.

Let there be N consumers with rational preference relations $\succeq_i$ and corresponding Walrasian demand functions $x_i : \mathbb{R}^\ell_+ \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$. Note that for each i,

$$x_i(p, w_i) = \begin{bmatrix} x_{1i}(p, w_i) \\ \vdots \\ x_{Li}(p, w_i) \end{bmatrix}.$$

In general, given a price vector $p >> 0$ and the wealth levels $(w_1, \cdots, w_N)$ for the N consumers, aggregate demand function can be written as

$$x(p, w_1, \cdots, w_N) = \sum_{i=1}^{N} x_i(p, w_i).$$

Thus, the aggregate demand depends on prices and on the specific wealth levels of the various consumers.

Proposition

Suppose that U_i is a utility function representing a preference relation $\succeq_i$ on $X = \mathbb{R}^L_+$ for all $1 \leq i \leq N$. If $\succeq_i$ is locally non-satiated and U_i is continuous for all $1 \leq i \leq N$ then $x(p, w_1, \cdots, w_N)$ is homogeneous of degree zero and it satisfies Walras' Law in the sense that $p \cdot x(p, w_1, \cdots, w_N) = \sum_{i=1}^N w_i$.

Example 1

Consider an economy whose set of agents is $\{1, 2, 3\}$ and commodity space is $\mathbb{R}^2_+$.

Agent 1: Initial endowment $w_1 = (1, 2)$ and utility function $U_1(x, y) = xy$.

Agent 2: Initial endowment $w_2 = (1, 1)$ and utility function $U_2(x, y) = x^2y$.

Agent 3: Initial endowment $w_3 = (2, 3)$ and utility function $U_3(x, y) = xy^2$.

Next, we shall determine the demand functions $x_1(\cdot, w_1)$, $x_2(\cdot, w_2)$ and $x_3(\cdot, w_3)$. To this end, let $p = (p_1, p_2) \gg 0$ be fixed.

The first agent maximizes $U_1(x, y)$ subject to budget constraint

$$p_1 x + p_2 y = p_1 + 2p_2.$$

Let $g(x, y) = p_1 x + p_2 y$. Employing Lagrange multipliers, we see that at the maximizing point we must have

$$\frac{\partial}{\partial x} U_1(x, y) = \lambda \frac{\partial}{\partial x} g(x, y)$$

and

$$\frac{\partial}{\partial y} U_1(x, y) = \lambda \frac{\partial}{\partial y} g(x, y).$$

Thus, we have $y = \lambda p_1$, $x = \lambda p_2$ and $p_1 x + p_2 y = p_1 + 2p_2$. Solving the above system, we obtain

$$x_1(p, w_1) = \left(\frac{p_1 + 2p_2}{2p_1}, \frac{p_1 + 2p_2}{2p_2} \right).$$

The second agent maximizes $U_2(x, y)$ subject to budget constraint

$$p_1 x + p_2 y = p_1 + p_2.$$

Using Lagrange multipliers again, we obtain the system

$$2xy = \lambda p_1, x^2 = \lambda p_2 \text{ and } p_1 x + p_2 y = p_1 + p_2.$$

Solving the above system, we obtain

$$x_2(p, w_2) = \left(\frac{2p_1 + 2p_2}{3p_1}, \frac{p_1 + p_2}{3p_2} \right).$$

Finally, for the third agent we have the system

$$y^2 = \lambda p_1, 2xy = \lambda p_2 \text{ and } p_1 x + p_2 y = 2p_1 + 3p_2.$$

In this case, we have

$$x_3(p, w_3) = \left(\frac{2p_1 + 3p_2}{3p_1}, \frac{4p_1 + 6p_2}{3p_2}\right).$$

The aggregate demand is given by

$$x(p, w_1, w_2, w_3) = \sum_{i=1}^{3} x_i(p, w_i) = \left(\frac{11p_1 + 16p_2}{6p_1}, \frac{13p_1 + 20p_2}{6p_2}\right).$$

Example 2

Example

Find the demand function for the preference relation on $\mathbb{R}^3_+$ represented by a utility function $U(x, y, z) = \min\{x, y, z\}$ given the initial endowment $w = (1, 2, 3)$.

Solution: Fix an arbitrary $p = (p_1, p_2, p_3) \gg 0$. We claim that the vector $x^* = (x_0, x_0, x_0)$ which satisfies $p \cdot x^* = p \cdot w$ is the unique maximizer of U on $B(p, w)$.

From $p \cdot x^* = p \cdot w$, we have

$$x_0 = \frac{p_1 + 2p_2 + 3p_3}{p_1 + p_2 + p_3}.$$

Take any $(x, y, z) \in \mathbb{R}^3_+$ and $(x, y, z) \neq (x_0, x_0, x_0)$ such that

$$U(x, y, z) \geq U(x_0, x_0, x_0).$$

Then $x \geq x_0$, $y \geq x_0$ and $z \geq x_0$ with at least one of these inequalities must be strict. So,

$$\begin{aligned}(p_1, p_2, p_3) \cdot (x, y, z) &= p_1 x + p_2 y + p_3 z \\ &> p_1 x_0 + p_2 x_0 + p_3 x_0 \\ &= p_1 + 2p_2 + 3p_3\end{aligned}$$

Thus, $(x, y, z) \notin B(p, w)$. Hence, x^* is the desired maximal element and

$$x(p, w) = x^* = \frac{p_1 + 2p_2 + 3p_3}{p_1 + p_2 + p_3}(1, 1, 1).$$