AN INTRODUCTION TO GENERAL EQUILIBRIUM THEORY

LECTURE NOTES: MANIPUSHPAK MITRA

1. INTRODUCTION

We analyze the notion of equilibrium in economies where agents act as price takers. We consider a world with L commodities in which consumers and firms interact through a market system. In this market system, a price is quoted for every commodity, and economic agents take these prices as independent of their actions. After specifying the basic model we define the normative notion of a Pareto optimal allocation. Then we present two notions of price-taking equilibrium: Walrasian competitive equilibrium, and its generalization, a price equilibrium with transfers. The Walrasian equilibrium concept applies to the case of a private ownership economy, in which a consumer's wealth is derived from the consumer's ownership of endowments and from claims to profit shares of firms. The more general notion of a price equilibrium with transfers allows instead for an arbitrary distribution of wealth among consumers. We then explore the relationships between these equilibrium concepts and Pareto optimality.

We first focus on the statement of the (very weak) conditions implying that every price equilibrium with transfers (and, hence, every Walrasian equilibrium) results in a Pareto optimal allocation. This is the *first fundamental theorem of welfare economics*, a formal expression for competitive market economies of Adam Smith's claimed "invisible hand" property of markets. We then state conditions (mainly, convexity assumptions) under which every Pareto optimal allocation can be supported as a price equilibrium with transfers. This result is known as the *second fundamental theorem of welfare economics*. It tells us that if its assumptions are satisfied, then through the use of appropriate transfers, a welfare authority can, in principle, implement any desirable Pareto optimal allocation as a price-taking equilibrium.

Finally, we reexamine the Pareto optimality concept and associated results by making differentiability assumptions and analyzing first-order conditions. There we see how equilibrium prices can be interpreted as the Lagrange multipliers, or shadow prices, that arises in the associated Pareto optimality problem. Before starting our analysis of the general equilibrium model, we revise some concepts of consumer behavior and production.

2. Consumer behavior: the preference based approach

In consumer behavior we deal with the individual's decision problem of making choices from a set of mutually exclusive alternatives X. The set X can be something like a student's career decision problem, that is $X = \{$go to a law school, go to a graduate school and study economics, go to a business school, go to the army,..., be a pop star$\}$. The set X can also be a combo pack of fruits, that is $X = \{$one orange and one banana, one orange and one apple,..., two bananas, ...$\}$. Other examples are $X = \{(a, b) \mid a \in [0, 10], b \in [0, 20]\}$, $X = \{(\text{apple}, \text{money}) \mid \text{apple} \in \{1, 2, \ldots, 20\}, \text{money} \in [0, 10]\}$. In general, the set X can be many things and therefore we can write $X = \{x, y, \ldots\}$ where x, y etc. are elements in the set that can be scalar or vector and the set X can be both finite or infinite. Given the set X, there are two approaches to modelling individual choice behavior-the preference based approach and the choice based approach.

(1) In the preference based approach tastes of an agent are summarized in the agent's preference relation ("at least as good as" relation). Preference relation is the primitive characteristic of an individual or decision maker. The central consistency assumption is rationality of preferences. By imposing the rationality axiom on the decision maker's preference we analyze the consequences of these preferences on the agent's choice behavior.
(2) In the choice based approach agent's choice behavior is the primitive feature and the analysis proceeds by making assumptions directly concerning this behavior. The central consistency axiom is the weak axiom of revealed preference which is a behavioral assumption that guarantees an element of consistency on choice behavior.

In the preference based approach the starting point is the set of possible alternatives that are available to the agent. Let this set be $X = \{x, y, z, \ldots\}$. The preference ordering of the decision maker (or agent or individual) is summarized by R. Technically, R is a binary relation on X allowing the comparison of pairs of alternatives $x, y \in X$. Here xRy means that x is 'at least as good as' y. From R one can derive two important relations on X.

(1) The strict preference relation P is defined as $xPy \Leftrightarrow xRy \,\&\, \sim (yRx)$. This is read as x is preferred to y.
(2) The indifference relation I is defined as $xIy \Leftrightarrow xRy$ and yRx. This is read as x is indifferent to y.

The central consistency axiom is rationality of preference.

Definition 1. The preference relation R on X is *rational* if it satisfies the following properties:

(1) *Completeness:* For all $x, y \in X$, either xRy or yRx or both. Completeness says that the agent has a well-defined preference between any pair of alternatives.

(2) *Transitivity:* For all $x, y, z \in X$, if xRy and yRz, then xRz. Transitivity says that if x is at least as good as y and y is at least as good as z then x is at least as good as z (or $\sim (zPx)$, that is z is not strictly preferred to x).

The assumption that R on X is rational has implications for P on X and for I on X. Specifically, if R on X is rational, then we have the following:

(1) P is both *irreflexive* (xPx never holds) and *transitive* (if xPy and yPz, then xPz).
(2) I is *reflexive* (xIx for all $x \in X$), *transitive* (if xIy and yIz, then xIz) and *symmetric* (if xIy then yIx).
(3) If $xPyRz$ then xPz. Similarly, if $xRyPz$ then xPz.

2.1. **Consumer behavior in a market economy.** The analysis of consumer behavior begins by specifying the consumer's preferences over the commodity bundles in the consumption set $X \subseteq \Re^L$. We study the consumer demand in the context of a market economy. Market economy is an economy where the goods and services (that the consumer may acquire) are either available for purchase at known prices or are available for trade for other goods at known rates of exchange. The basic elements of the consumer's decision problem are the following:

(1) *Commodities:* The decision problem faced by the consumer in a market economy is to choose consumption levels of the various *goods and services or commodities* that are available for purchase in the market. We denote the number of commodities available in the market by L (and a typical element is $l \in \{1, \dots, L\}$). We assume that L is finite. A *commodity vector* (or commodity bundle) is a list of the different commodities, $x = (x_1, \dots, x_L)_{L\times 1} \in \Re^L$ where $x_l < 0$ implies debits or net outflows of goods.
(2) *The consumption set:* The consumption set is $X \subseteq \Re^L$ whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by the consumer's environment (for example, at most 24 hours of leisure). To keep things straightforward $X = \Re^L_+ = \{x \in \Re^L \mid x_l \geq 0 \; \forall \; l \in \{1, \dots, L\}\}$. A special feature of $X = \Re^L_+$ is that it is convex, that is, if $x \in \Re^L_+$ and $x' \in \Re^L_+$ then $\alpha x + (1-\alpha)x' \in \Re^L_+$ for any $\alpha \in [0, 1]$. Recall that, in general, a set $X \subseteq \Re^L$ is *convex* if whenever two vectors x and x' belongs to X, it contains the entire line segment connecting them.
(3) *Competitive budgets:* An individual's consumption choice is limited to those commodity bundles that he can afford. To formalize this we use two assumptions. The L-commodities are traded in the market at rupee prices that are publicly quoted (principle of completeness or universality of markets). In general, a price vector is

$p = (p_1, \ldots, p_L) \in \Re^L$ where $p_l < 0$ simply means that the buyer is paid to consume the commodity. The two assumptions are the following:

(A1) $p >> 0$, that is $p_l > 0$ for all $l \in \{1, \ldots, L\}$.

(A2) Individuals are *price takers*, that is, the prices are beyond the influence of the consumers.

A consumption bundle $x = (x_1, \ldots, x_L) \in X = \Re^L_+$ is *affordable* if its total cost does not exceed the consumer's wealth $w > 0$, that is $p.x = p^T x = \sum_{l=1}^{L} p_l x_l \leq w$. The *Walrasian budget set* is $B_{p,w} = \{x \in \Re^L_+ \mid p.x \leq w\}$. It is the set of all feasible consumption bundles for the consumer who faces market price p and has wealth w. The set $\{x \in \Re^L_+ \mid p.x = w\}$ is called the *budget hyperplane* and for $L = 2$, it is called the *budget line*. The slope of the budget line captures the rate of exchange between the two commodities. The Walrasian budget set is convex, that is if $x \in B_{p,x}$ and $x' \in B_{p,w}$ then for any $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)x' \in B_{p,w}$. What is a sufficient restriction on X that leads to convexity of the Walrasian budget set $B_{p,w}$? If X is a convex set, then $B_{p,w}$ is convex as well.

2.2. **Assumptions on preferences.** It is often reasonable to assume that larger amounts of commodities are preferred to smaller ones.

(1) *Monotonicity:* The preference relation R on X is *monotone* if $x, y \in X$ and $y >> x$ (that is, $y_l > x_l$ for all l) implies that yPx.
(2) *Strong monotonicity:* The preference relation R on X is *strongly monotone* if $x, y \in X$, $y \geq x$ (that is, $y_l \geq x_l$ for all l) and $y \neq x$ implies that yPx.
(3) *Local non-satiation:* The preference relation R on X is *locally non-satiated* if for all $x \in X$ and every $\epsilon > 0$, there exists $y \in X$ such that $||y - x|| \equiv \sqrt{\sum_{l=1}^{L} (y_l - x_l)^2} \leq \epsilon$ and yPx.[1]

Monotonicity requires that commodities we are talking about are goods rather than bad. If R on X is complete and strongly monotone, then it is monotone. Local non-satiation is a much weaker requirement than monotonicity. Local non-satiation says that for any consumption bundle $x \in \Re^L_+$ and any arbitrarily small distance away from x, denoted by $\epsilon > 0$, there is another bundle $y \in \Re^L_+$ within this distance from x (in short, $y \in N_\epsilon(x)$) that is preferred to x. Note that the bundle y may even have less of every commodity. When $X = \Re^L_+$ local non-satiation rules out the extreme situation in which all commodities are bad, since in that case no consumption at all (the point $x = \underline{0}$) would be a satiation point.

Proposition 1. If R on $X = \Re^L_+$ is complete and monotone, then it is locally non-satiated.

[1] Here $||x - y||$ is the Euclidean distance between points x and y, that is $||x - y|| = \sqrt{\sum_{l=1}^{L} (x_l - y_l)^2}$. Therefore, for $L = 1$, $||x - y|| = \sqrt{(x_1 - y_1)^2} = |x_1 - y_1|$, for $L = 2$, $||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and so on.

Proof: Consider any $x \in X$ and any $\tilde{\epsilon} > 0$. Define $y = x + (\epsilon e)/\sqrt{L}$ where $e = (1, \ldots, 1)$ and $\epsilon \in (0, \tilde{\epsilon}]$. Observe that $y >> x$ and $||y - x|| = \sqrt{\sum_{l=1}^{L}(y_l - x_l)^2} = \sqrt{\sum_{l=1}^{L}\left(\epsilon/\sqrt{L}\right)^2} = \epsilon \leq \tilde{\epsilon}$ which implies that $y \in N_{\tilde{\epsilon}}(x)$. Given monotonicity, $y >> x$ implies yPx. Hence, $y \in N_{\tilde{\epsilon}}(x)$ and yPx. Since the selection of x and $\tilde{\epsilon}$ were arbitrary, the result follows. □

We have strong monotone implies monotone and monotone implies local non-satiation. We consider the following three related sets of consumption bundles.

(1) The *indifference set* containing point x is the set of all bundles that are indifferent to x, that is $I(x) = \{y \in X \mid xIy\}$. One implication of local non-satiation is that it rules out "thick" indifference sets.
(2) The *upper contour set* of bundle x is the set of all bundles that are at least as good as x, that is $UC(x) = \{y \in X \mid yRx\}$.
(3) The *lower contour set* of bundle x is the set of all bundles that x is at least as good as, that is $L(x) = \{y \in X \mid xRy\}$.

Observe that $UC(x) \cap L(x) = I(x)$. A second significant assumption, that of convexity of R, concerns the trade-offs that the consumer is willing to make among different goods.

(1) *Convexity:* The preference relation R on X is *convex* if for every $x \in X$, $UC(x)$ is convex, that is if yRx and zRx then $(\alpha y + (1-\alpha)z)\ Rx$ for any $\alpha \in [0,1]$.
(2) *Strict convexity:* The preference relation R on X is *strictly convex* if for every x, we have yRx, zRx and $y \neq z$ implies $(\alpha y + (1-\alpha)z)\ Px$ for all $\alpha \in (0,1)$.

Convexity can also be viewed as a formal expression of a basic inclination of economic agents to diversify. If xIy then $\left(\frac{1}{2}x + \frac{1}{2}y\right) Rx(\text{or } y)$.

2.3. **Utility functions.** Preference relations of the individual is represented by a utility function (whenever possible). A utility function $u(x)$ assigns a numerical value to each element in X, ranking the elements of X in accordance with the individual's preferences.

Definition 2. A function $u : X \to \mathfrak{R}$ is a utility function representing preference relation R on X if, for all $x, y \in X$, $xRy \Leftrightarrow u(x) \geq u(y)$.

It is possible to avoid the notion of utility representation and study economics using only the notion of preferences. Nevertheless, we use utility functions rather than preferences because we find it more convenient to talk about the maximization of a numerical function than a preference relation. A utility function $u(x)$ that represents a preference relation R on X is not unique.

Exercise 1. Show that if $f : \Re \to \Re$ is a increasing function and $u : X \to \Re$ is a utility function representing the preference relation R on X, then $v : X \to \Re$ defined by $v(x) = f(u(x))$ for all $x \in X$ is also a utility function representing the same R on X.

Proof: If $u(.)$ is a utility function representing R on X then for all $x, y \in X$, $xRy \Leftrightarrow u(x) \geq u(y)$. Since f is a strictly increasing function and $v(x) = f(u(x))$ we have for all $x, y \in X$ such that xRy, $u(x) \geq u(y) \Leftrightarrow f(u(x)) \geq f(u(y)) \Leftrightarrow v(x) \geq v(y)$. Hence $v(.)$ is also a valid utility function representing the same R on X. □

Properties of utility functions that are invariant for any increasing transformation are called *ordinal* properties. *Cardinal* properties are those not preserved under all such transformations. Thus, the preference relation R on X associated with a utility function is an ordinal property. On the other hand, the numerical values associated with the alternatives in X and hence the magnitude of any differences in the utility measure between alternatives, are cardinal properties.

2.4. **Utility function and the rationality axiom.**

Proposition 2. A preference relation R on X can be represented by a utility function *only if* it is rational.

Proof: We show that if there is a utility function that represents R on X, then R must be complete and transitive. Since $u(.)$ is a real-valued function defined on X, it must be that for any $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. Since $u(.)$ is a utility function representing R on X, this implies either xRy or yRx. Hence, R on X is complete. Suppose that xRy and yRz. Since $u(.)$ represents R on X, we have $u(x) \geq u(y)$ and $u(y) \geq u(z)$ and hence it follows that $u(x) \geq u(z) \Leftrightarrow xRz$. Thus, we also have transitivity. □

Unfortunately, with the assumptions made so far, a rational preference relation need not be representable by a utility function. For example, consider the lexicographic preference. For simplicity assume that $X = \Re^2_+$. Under a lexicographic preference, we say xRy if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$". What is xPy? Recall that $xPy \Leftrightarrow xRy$ and $\sim (yRx)$. Now $xRy \Leftrightarrow \{x_1 > y_1\}$ or $\{x_1 = y_1$ and $x_2 \geq y_2\}$ and $\sim (yRx) \Leftrightarrow \sim \{y_1 > x_1\}$ and $\sim \{y_1 = x_1$ and $y_2 \geq x_2\} \Leftrightarrow \{x_1 \geq y_1\}$ and $\{y_1 \neq x_1$ or $x_2 > y_2\} \Leftrightarrow \{x_1 \geq y_1$ and $y_1 \neq x_1\}$ or $\{x_1 \geq y_1$ and $x_2 > y_2\} \Leftrightarrow \{x_1 > y_1\}$ or $\{x_1 \geq y_1$ and $x_2 > y_2\}$. Thus, $xPy \Leftrightarrow xRy$ and $\sim (yRx) \Leftrightarrow \{x_1 > y_1\}$ or $\{x_1 = y_1$ and $x_2 > y_2\}$. What is xIy? Recall that $xIy \Leftrightarrow xRy$ and $yRx \Leftrightarrow [\{x_1 > y_1\}$ or $\{x_1 = y_1$ and $x_2 \geq y_2\}]$ and $[\{y_1 > x_1\}$ or $\{y_1 = x_1$ and $y_2 \geq x_2\}] \Leftrightarrow \{x_1 = y_1\}$ and $\{x_2 \geq y_2\}$ and $\{y_2 \geq x_2\} \Leftrightarrow \{x_1 = y_1\}$ and $\{x_2 = y_2\} \Leftrightarrow x = y$.

Even though the lexicographic preference is rational, strongly monotonic and strictly convex, we cannot find a utility function that represents it. With lexicographic ordering no two distinct bundles are indifferent (because indifference sets are singletons). Therefore, we have two dimensions of distinct indifferent sets. Yet, each of these indifference sets must be assigned, in an order preserving way, a different utility number from the one-dimensional real line. This is mathematically impossible.

Proposition 3. There is no utility function that can represent the lexicographic preference.

Proof: Suppose that there exists a utility function $u(.)$ that represents the lexicographic preference. For every $x_1 \in \Re_+$, we can pick a *rational* number $r(x_1)$ such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$.[2] For any $x_1' < x_1$ we must have $r(x_1') < r(x_1)$ since, due to lexicographic preference, $u(x_1, 2) > r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1') > u(x_1', 1)$. Therefore, $r : \Re_+ \to \mathcal{Q}$ where $\mathcal{Q}$ is the set of all rational numbers. Thus, if the lexicographic preference has a utility representation, then we can find a function r which provides a one-to-one mapping from the set of all non-negative real numbers which is *uncountable* to the set of rational numbers which is *countable*. This is a mathematical impossibility and the result follows. $\square$

2.5. **Continuity of preferences.** First we define continuity in terms of sequence. Before doing that we define some mathematical concepts. A sequence in $\Re^L$ is an ordered list of elements of $\Re^L$: $s^1; s^2; \ldots s^n; \ldots$ or equivalently $\{s^n\}_{n=1}^{\infty}$ (or simply $\{s^n\}$). It is a countable subset of $\Re^L$ in which the order matters. A sequence $\{s^n\}$ in $\Re^L$ converges to a limit s if for any $\epsilon > 0$, there is an integer N such that $||s^n - s|| < \epsilon$ for all $n > N$, that is $s = \lim_{n\to\infty} s^n$. For example the sequence $\{s^n\}$ in $\Re^2$ with $s^n = \big(1/n, (2-n^2)/(3n^2+n)\big)$ (for all n) converges to $s = (0, -1/3)$. A set S in $\Re^L$ is *closed* if and only if $s \in S$ for any sequence $\{s^n\}$ such that $s^n \in S$ for all n and $\lim_{n\to\infty} s^n = s$. In other words, S is a closed set if every convergent sequence in S converges to a point in S.

Definition 3. The preference relation R on X is *continuous* if it is preserved in limits. That is, for any sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ with $x^n R y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have xRy. That is the preference cannot exhibit "jumps".

Observe that the lexicographic preference is not continuous. Let the preference be lexicographic and let $x^n = (\frac{1}{n}, 0)$ and $y^n = (0, 1)$. Since $\frac{1}{n} > 0$ for all n, $x^n P y^n$. Also observe that $y = \lim_{n\to\infty} y^n = (0, 1)$, $x = \lim_{n\to\infty} x^n = (0, 0)$ and yPx. But yPx is a violation of continuity. Observe that in this construction the sequence of bundles x^n were all preferred to the bundle y, that is $x^n \in UC(y)$ for all n however the limit point of $\{x^n\}$, that is $x \notin UC(y)$. Hence the upper contour set $UC(y)$ is not closed since we have found a convergent sequence $\{x^n\}$ whose limit point $x \notin UC(y)$. An equivalent definition of continuity is in terms of closed contour sets.

[2] Between any two real numbers we can always find a rational number.

Definition 4. The preference relation R on X is *continuous* if and only if for all $x \in X$, $UC(x)$ and $L(x)$ are closed, that is, they include their boundaries.

Proposition 4. For any rational preference relation R on X, Definition 3 and Definition 4 are equivalent.

Proof: We first show that Definition 3 implies Definition 4. Fix $x^n = x$ for all n. Then by Definition 3 it follows that for all $\{(x, y^n)\}_{n=1}^{\infty}$ such that xRy^n for all n and $\lim_{n\to\infty} y^n = y$ we have xRy. This implies that the lower contour set of x, that is $L(x)$, is closed. Similarly fix $x^n = x$ for all n. Then by Definition 3 it follows that for all $\{(y^n, x)\}_{n=1}^{\infty}$ such that $y^n Rx$ for all n and $\lim_{n\to\infty} y^n = y$ we have yRx. This implies that the upper contour set of x, that is $UC(x)$, is closed.

To prove the converse, suppose there exists two sequences $\{x^n\}$ and $\{y^n\}$ such that $x^n Ry^n$ for all n, $\lim_{n\to\infty} x^n = x$, $\lim_{n\to\infty} y^n = y$ and yPx. Since $\overline{L}(y)$ is open, (A) there exists N_1 such that yPx^n for every $n > N_1$. Since $\overline{UC}(x)$ is open, (B) there exists N_2 such that $y^n Px$ for every $n > N_2$. We can have two cases and for each case we derive a contradiction.
Case (1): If $\{y^n\}$ is such that there exists N_3 such that $y^n Ry$ for all $n > N_3$, then (given (A) and transitivity of preference) $y^n Px^n$ for all $n > \max\{N_1, N_3\}$ contradicting $x^n Ry^n$ for all n.
Case (2): If $\{y^n\}$ is such that there exists a subsequence $\{y^{k(n)}\}$ such that $yPy^{k(n)}$ for every n, then there exists a positive integer m such that $k(m) > N_2$. Since $\overline{UC}(y^{k(m)})$ is open, there exists N_4 such that $y^n Py^{k(m)}$ for all $n > N_4$. Given $x^n Ry^n$ for all n and given $y^n Py^{k(m)}$ for all $n > N_4$, we have $x^n Py^{k(m)}$ for all $n > N_4$. Since $UC(y^{k(m)})$ is closed, $xRy^{k(m)}$. Since $k(m) > N_2$ and since from (B) we already know that $y^n Px$ for all $n > N_2$, $xRy^{k(m)}$ is not possible. □

Let $\overline{L}(x) = L(x) \setminus I(x) = [UC(x)]^c = \{y \in X \mid xPy\}$ be the complement of the upper contour set which is the *strict lower contour set* and let $\overline{UC}(x) = UC(x) \setminus I(x) = [L(x)]^c = \{y \in X \mid yPx\}$ be the complement of the lower contour set which is the *strict upper contour set*. From continuity of R on X it follows that for each $x \in X$, the strict lower contour set $\overline{L}(x)$ and the strict upper contour set $\overline{UC}(x)$ are both open. Thus, if y is strictly preferred to z and if x is a bundle that is close enough to y, then x must be strictly preferred to z. This follows from the fact that the strict upper contour set is open.

Definition 5. The preference relation R on X is *continuous* if whenever $x, y \in X$ and xPy, there exists $\epsilon > 0$ and $\delta > 0$ such that for all $x' \in N_\epsilon(x)$ and all $y' \in N_\delta(y)$, $x'Py'$.

Proposition 5. For any rational R on X, Definition 3 and Definition 5 are equivalent.

Proof: Assume that R on X is continuous according to Definition 5 but not according to Definition 3. Then, there exists $\{x^n, y^n\}_{n=1}^{\infty}$ with $x^n Ry^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y =$

$\lim_{n\to\infty} y^n$ such that yPx. Then, by Definition 5, there exists $\epsilon > 0$ and $\delta > 0$ such that for all $x' \in N_\epsilon(x)$ and all $y' \in N_\delta(y)$, $y'Px'$. Then there is an N large enough such that for all $n > N$, both $x^n \in N_\epsilon(x)$ and $y^n \in N_\delta(y)$. Therefore, for all $n > N$ y^nPx^n (by Definition 5) which contradicts x^nRy^n for all n.

Assume that R on X is continuous according to Definition 3 but not according to Definition 5. Then, there exists $x, y \in X$ such that xPy and for all $\epsilon > 0$ and all $\delta > 0$ there exists $x' \in N_\epsilon(x)$ and there exists $y' \in N_\delta(y)$ such that $y'Rx'$. In particular, for each n, there exists $x^n \in N_{1/n}(x)$ and $y^n \in N_{1/n}(y)$ such that y^nRx^n. Since $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, by Definition 3, yRx which contradicts xPy. □

2.6. **Continuous preferences and utility functions.** The utility function $u : X \to \mathfrak{R}$ representing R on X is continuous at some $x \in X$ if every sequence $\{x^n\}$ that converges to x, the sequence $\{u(x^n)\}$ converges to $u(x)$.

Definition 6. The utility function $u : X \to \mathfrak{R}$ is *continuous* if it is continuous at every point in X.

Proposition 6. If $u(.)$ is a continuous utility function representing R on X, then R must be continuous.

Proof: Take a sequence of two commodity bundles $\{(x^n, y^n)\}_{n=1}^\infty$ such that x^nRy^n for all n, $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$. Since $u(.)$ represents the preference relation R on X, $u(x^n) \geq u(y^n)$ for all n. Define $T^n \equiv u(x^n) - u(y^n) \geq 0$ for all n. By continuity of $u(.)$, $T = \lim_{n\to\infty} T^n = \lim_{n\to\infty} [u(x^n) - u(y^n)] = \lim_{n\to\infty} u(x^n) - \lim_{n\to\infty} u(y^n) = u(x) - u(y) < \infty$. Since $T^n \geq 0$ for all n and $\mathfrak{R}_+$ is closed, the limit point T of the converging sequence $\{T^n\}_{n=1}^\infty$ is non-negative and hence $T = u(x) - u(y) \geq 0$. Thus $u(x) \geq u(y)$ implying xRy. Hence R on X is continuous. □

One can show that if R on X is rational and continuous, then there exists a continuous utility function $u : \mathfrak{R}_+^L \to \mathfrak{R}$ that represents it. But the proof of this result is very technical and hence omitted. Instead we prove a simpler result.

Proposition 7. If preference relation R on $X = \mathfrak{R}_+^L$ is rational, continuous and monotonic, then there exists a utility function $u : \mathfrak{R}_+^L \to \mathfrak{R}$ that represents it.

Proof: Let e be the vector in $\mathfrak{R}_+^L$ consisting of all ones. Then given any vector $x \in X$, let $u(x)$ be such that $xI(u(x)e)$. We first show that such a number exists and is unique. Let $B = \{\alpha \in \mathfrak{R}_+ \mid (\alpha e)Rx\}$ and $W = \{\alpha \in \mathfrak{R}_+ \mid xR(\alpha e)\}$. Then monotonicity implies that B is non-empty and since $\underline{0} \in W$, W is also non-empty. Continuity implies that both sets are closed. By completeness $\mathfrak{R}_+ \subseteq B \cup W$ and since the non-negative orthant of the real line

$\Re_+$ is connected $B \cap W \neq \emptyset$.[3] Therefore, there is some $\alpha(x) \in \Re_+$ such that $(\alpha(x)e)Ix$. Furthermore, by monotonicity $(\alpha_1 e)P(\alpha_2 e)$ whenever $\alpha_1 > \alpha_2$ and hence there can be at most one $\alpha(x)$ such that $(\alpha(x)e)Ix$. For any $x \in X$, let $u(x) = \alpha(x) \in \Re_+$ such that $(\alpha(x)e)Ix$. We show that this utility function represents R, that is $\alpha(x) \geq \alpha(y) \Leftrightarrow xRy$. If $\alpha(x) \geq \alpha(y)$ then $(\alpha(x)e) \geq (\alpha(y)e)$ and hence by monotonicity $xI(\alpha(x)e)R(\alpha(y)e)Iy \Rightarrow xRy$. Similarly, if xRy, then $xI(\alpha(x)e)R(\alpha(y)e)Iy$ and by monotonicity it is necessary that $\alpha(x) \geq \alpha(y)$. Hence $xRy \Rightarrow \alpha(x) \geq \alpha(y)$. □

2.7. **Utility representation of some common preferences.**

Definition 7. The utility function $u(.)$ is *increasing* if the following condition holds: If $x >> y$ then $u(x) > u(y)$.

Definition 8. The utility function $u(.)$ is *quasi-concave* if the set $\{y \in X \mid u(y) \geq u(x)\}$ is convex $\forall\, x \in X$ or equivalently $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in X$ and all $\alpha \in [0,1]$. Moreover, if the inequality is strict for all $x, y \in X$ such that $x \neq y$ and all $\alpha \in (0,1)$, then $u(.)$ is *strictly quasi-concave*.

Proposition 8. Let R be a preference relation defined on X and let $u(.)$ be a utility function representing it.

(i) R is monotone if and only if the utility function $u(.)$ representing it is increasing.
(ii) R is convex (strictly convex) if and only if utility function $u(.)$ representing it is quasi-concave (strictly quasi-concave).

Proof:
Proof of (i): If R is monotone then for any $x >> y$, $xPy \Rightarrow u(x) > u(y)$ implying that any utility function $u(.)$ that represents R is increasing. If a u is increasing then for any $x >> y$, $u(x) > u(y) \Rightarrow xPy$ implying monotonicity of R.
Proof of (ii): We first prove that if R on X is convex (strictly convex) then the utility function $u(.)$ representing this preference R on X is quasi-concave (strictly quasi-concave). Convexity (strict convexity) of R means $\forall\, x, y, z \in X$ ($\forall\, x, y, z$ with $x \neq y$), if xRz and yRz then $(\alpha x + (1-\alpha)y)Rz$ for all $\alpha \in [0,1]$ (then $(\alpha x + (1-\alpha)y)Pz$ for all $\alpha \in (0,1)$). By substituting $z = x$ and using the fact that R on X is representable by a utility function it follows that (a) if $u(y) \geq u(x)$ then $u(\alpha x + (1-\alpha)y) \geq u(x)$ (then $u(\alpha x + (1-\alpha)y) > u(x)$). Similarly by substituting $z = y$ and using the fact that R on X is representable by a utility function it follows that (b) if $u(x) \geq u(y)$ then $u(\alpha x + (1-\alpha)y) \geq u(y)$ (then $u(\alpha x + (1-\alpha)y) > u(y)$). From (a) and (b) we get if R on X is convex then $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$

[3]A connected set is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

(if R on X is strictly convex then $u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$) and hence we get quasi-concavity (strict quasi-concavity) of the utility function representing R on X. Next we prove that if $u(.)$ on X is a quasi-concave (strictly quasi-concave) utility function then R on X is convex (strictly convex). Since $u(.)$ is quasi-concave (strictly quasi-concave) we get $\forall\, x, y, z \in X$ ($\forall\, x, y, z \in X$ with $x \neq y$) such that $u(x) \geq u(z)$ and $u(y) \geq u(z)$, we must have $u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\} \geq u(z)$ ($u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\} \geq u(z)$). Because $u(.)$ represents R, we get $\forall\, x, y, z \in X$ ($\forall\, x, y, z \in X$ with $x \neq y$) we must have xRz and yRz implies $(\alpha x + (1-\alpha)y)Rz$ ($(\alpha x + (1-\alpha)y)Pz$) and hence R on X is convex (strictly convex). □

Note that convexity (strict convexity) of R does not imply that utility $u(.)$ is concave (strictly concave). The utility function $u(.)$ is *concave* if $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$ for any $x, y \in X$ and for all $\alpha \in [0,1]$. The utility function $u(.)$ is *strictly concave* if $u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y)$ for any $x, y \in X$ such that $x \neq y$ and for all $\alpha \in (0,1)$. If a function is concave (strictly concave) then $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y) \geq \min\{u(x), u(y)\}$ for all $\alpha \in [0,1]$ ($u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y) > \min\{u(x), u(y)\}$ for all $\alpha \in (0,1)$ and $x \neq y$) implying quasi-concavity (strict quasi-concavity). Hence, if a function is concave (strictly concave), then it is quasi-concave (strictly quasi-concave).

3. Production

In the production theory we deal with the supply side of the economy and study the process by which the goods and services consumed by individuals are produced. The supply side is composed of a number of production units or 'firms'. Firms are corporations or other legally organized businesses. More importantly firms represent the production possibilities of individuals or households. More formally, a firm is an economic organization where a group of individuals coordinate their skills in order to produce goods and services. There are three main categories of a firm-(1) Proprietorship (e.g. most shopkeepers, garages and restaurants) (2) Partnership (e.g. most audit, law firms, hedge funds) (3) Most large-scale economic activity is organized as limited liability firms. The set of all firms may include some potential production units that are never actually organized. The theory we develop will be able to accommodate both active production processes and potential but inactive production processes.

The full description of a firm includes many aspects: (1) Who owns a firm? (2) Who manages it? (3) How is it organized? (4) What can it do? We are interested in the last question because we want a minimal conceptual apparatus that allows us to analyze market behavior. The firm is viewed as a "black box", able to transform inputs into outputs.

3.1. **Firm's Technology.** The firm's technology is given by the production set. Consider an economy with L commodities. A production vector (also known as input-output or net output or production plan) is a vector $y = (y_1, \ldots, y_L) \in \Re^L$ that describes the (net) outputs of the L commodities from the production process. We adopt the convention that positive numbers denote outputs and negative numbers denote inputs. Some elements of a production vector may be zero meaning that the process has no net output of the commodity. For example, $L = 5$ and $y = (y_1 = -5, y_2 = 2, y_3 = -6, y_4 = 3, y_5 = 0) \in \Re^5$ means that 2 and 3 units of goods 2 and 4 respectively are produced while 5 and 6 units of goods 1 and 3 respectively are used as inputs. Good 5 is neither produced nor used as an input in this production vector. For example, for $L = 2$ a typical element $y = (y_1, y_2) \in \Re^2 \setminus \Re^2_{++}$ allows for (a) $y_1 > 0$ and $y_2 < 0$, (b) $y_1 < 0$ and $y_2 > 0$, (c) $y_1 = 0$ and $y_2 = 0$ and, (d) $y_1 < 0$ and $y_2 < 0$. Observe that the last situation (that is, (d)) is wasteful since given the option of doing nothing (that is, (c)), no profit maximizing firm would ever choose to use inputs and incur cost without producing outputs (assuming prices of y_1 and y_2 are both positive). Putting it generally, (given $p_l > 0$ for all $l = 1, \ldots, L$) if $y \in Y$ and $y' \geq y$ such that $y' \neq y$ and $y' \in Y$, then a profit maximizing firm will always prefer y' to y because the firm can either produce more output using same amount of input or can produce same of output with less input. Either way the firm would earn higher profits. Because of this it is useful to have a mathematical representation for the frontier of Y. The tool we have for this is called the transformation function, $F(y)$.

3.2. **Technological Possibility.** The set of all production vectors that constitute feasible plans for the firm is known as the *production set* and is denoted by $Y \subset \Re^L$. Any $y \in Y$ is possible and $y \notin Y$ is not. The production set is taken as datum of the theory. It is sometimes convenient to describe the production set Y using a function $F(.)$ called the *transformation function*. The transformation function has the property that $Y = \{y \in \Re^L \mid F(y) \leq 0\}$ and $F(y) = 0$ if and only if y is an element of the boundary of Y. The set of boundary points of Y, $\{y \in \Re^L : F(y) = 0\}$ is known as the *transformation frontier*. Therefore, we have (i) $F(y) = 0$ if y is on the frontier, (ii) $F(y) < 0$ if y is in the interior of Y and (iii) $F(y) > 0$ if y is outside the frontier. Thus, if $F(y) < 0$, then y represents some sort of waste, although $F(,)$ tells us neither the form of the waste nor the magnitude. If $F(.)$ is differentiable and if the production vector $\bar{y}$ satisfies $F(\bar{y}) = 0$, then for any commodity pair l and k, $MRT_{lk}(\bar{y}) = \frac{\partial F(\bar{y})}{\partial y_l} / \frac{\partial F(\bar{y})}{\partial y_k}$ assuming $\frac{\partial F(\bar{y})}{\partial y_k} \neq 0$ is called the *marginal rate of transformation* (MRT) of good l for good k at $\bar{y}$. The marginal rate of transformation is a measure of how much the (net) output of good k *can increase* if the firm decreases the (net) output of good l by one marginal unit. Indeed for any boundary point $\bar{y}$ (that is for any point $\bar{y}$ such that

$F(\bar{y}) = 0$) we get $\left(\frac{\partial F(\bar{y})}{\partial y_k}\right) dy_k + \left(\frac{\partial F(\bar{y})}{\partial y_l}\right) dy_l = 0$ and hence the slope of the transformation frontier at $\bar{y}$ is precisely $-MRT_{lk}(\bar{y})$.

3.2.1. *Technologies with distinct inputs and outputs.* In many actual production processes, the set of goods that can be outputs is distinct from the set of goods that can be inputs, hence it is sometimes convenient to distinguish the firm's inputs and outputs. We could for example let $q = (q_1, \ldots, q_M) \geq 0$ denote the production levels of the firm's M outputs and $z = (z_1, \ldots, z_{L-M}) \geq 0$ denote the amounts of the firm's $L - M$ inputs with the convention that the amount of input z_l used is measured in non-negative numbers.

A *single-output technology* is described by means of a production function $f(z)$ that gives the maximum amount q of output that can be produced using input amounts $(z_1, \ldots, z_{L-1}) \geq 0$. For example, if the output is good L, then (assuming that output can be disposed of at no cost) the production function $f(z)$ gives rise to the production set $Y = \{(-z_1, \ldots, -z_{L-1}, q) : q - f(z_1, \ldots, z_{L-1}) \leq 0 \text{ and } (z_1, \ldots, z_{L-1}) \geq 0\}$. Holding the level of output fixed, we can define the *marginal rate of technical substitution* (MRTS) of input l for input k at $\bar{z}$ as $MRTS_{lk}(\bar{z}) = \frac{\partial f(\bar{z})}{\partial z_l} / \frac{\partial f(\bar{z})}{\partial z_k}$. The number $MRTS_{lk}(\bar{z})$ measures the additional amount of input k that can be used to keep output at level $\bar{q} = f(\bar{z})$ when the amount of input l is decreased marginally. It is the production theory analog to the consumer's marginal rate of substitution. In the consumer theory, we look at the trade-off between goods that keeps the utility constant, here; we examine the trade-off between inputs that keeps the amount of output constant. Note that $MRTS_{lk}(\bar{z})$ is simply a renaming of the marginal rate of transformation of input l for input k in the special case of a single-output many input technology. *Why?* For example, with two inputs and single output we have the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$ where $\alpha > 0$ and $\beta > 0$. The marginal rate of technical substitution is $MRTS_{12}(\bar{z}) = \frac{\partial f(\bar{z})}{\partial z_1} / \frac{\partial f(\bar{z})}{\partial z_2} = \frac{\alpha z_2}{\beta z_1}$.

3.3. **Properties of the production set.** Some commonly assumed properties of the production set are the following. Some of them are mutually exhaustive.

(1) *Y is non-empty.* This assumption simply says that *the firm has something it can plan to do* (otherwise there is no need to study the behavior of the firm in question).
(2) *Y is closed.* The set Y *includes its boundary*. Thus, the limit of a sequence of technologically feasible input-output vectors is also feasible. Formally, $y^n \to y$ and $y^n \in Y$ for all n imply that $y \in Y$. This assumption is primarily technical.
(3) *No free lunch.* Suppose that $y \in Y$ and $y \geq 0$, so that the vector y does not use any inputs. The no-free-lunch property is satisfied if this production vector cannot produce output either. That is, whenever, $y \in Y$ and $y \geq 0$, $y = 0$; *it is not possible to produce something out of nothing*. Geometrically, $Y \cap \Re^L_+ \subseteq \{0\}$.

(4) *Possibility of inaction.* This property says that $0 \in Y$: *complete shutdown is possible*. The point in time at which production possibilities are being analyzed is often important for the validity of this assumption. If we are contemplating a firm that could access a set of technological possibilities but that has not yet been organized, then inaction is clearly possible. But if some production decisions have already been made, or if irrevocable contracts for the delivery of some inputs have been signed, inaction is not possible. In this case we say that some costs are sunk.

(5) *Free disposal.* The property of free disposal holds if the absorption of any additional amounts of inputs without any reduction in output is always possible. That is, if $y \in Y$ and $y' \leq y$ (so that y' produces at most the same amount of outputs using at least the same amount of inputs), then $y' \in Y$. Define $Y - \Re^L_+ = \{w \in \Re^L \mid w = y - v$ for some $y \in Y$ and some $v \in \Re^L_+\}$. Then, a formal statement of free disposal is $Y - \Re^L_+ \subseteq Y$. The interpretation is that *extra amount of inputs (or outputs) can be disposed of or eliminated at no technological cost*.

(6) *Irreversibility.* Suppose that $y \in Y$ and $y \neq 0$ then $-y \notin Y$. *It is impossible to reverse a technologically possible production vector to transform an amount of output into the same amount of input that was used to generate it*. If the description of commodity includes the time of its availability, then irreversibility follows from the requirement that inputs be used before outputs emerge.

(7) *Non-increasing returns to scale.* The production technology Y exhibits non-increasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for all scalars $\alpha \in [0, 1]$. *Any feasible input-output vector can be scaled down*. Note that this property imply that inaction is possible.

(8) *Non-decreasing returns to scale.* The production technology Y exhibits non-decreasing returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for any scalar $\alpha \geq 1$. *Any feasible input-output vector can be scaled up*.

(9) *Constant returns to scale.* The production technology Y exhibits constant returns to scale if for any $y \in Y$, we have $\alpha y \in Y$ for any scalar $\alpha \geq 0$. This property is a conjunction of properties (7) and (8).

Exercise 2. Suppose that $f(.)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if $f(.)$ is homogeneous of degree one.

Proof: Suppose that Y exhibits constant returns to scale. Let $z = (z_1, \ldots, z_{L-1}) \in \Re^{L-1}_+$ and $\alpha > 0$. If $(-z, f(z)) \in Y$, then by constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$ and hence $\alpha f(z) \leq f(\alpha z)$. Again, since $(-\alpha z, f(\alpha z)) \in Y$, $\frac{1}{\alpha}(-\alpha z, f(\alpha z)) = (-z, \frac{1}{\alpha} f(\alpha z)) \in Y$ which implies that $\frac{1}{\alpha}f(\alpha z) \leq f(z) \Rightarrow f(\alpha z) \leq \alpha f(z)$. Therefore, $f(\alpha z) = \alpha f(z)$ for all $\alpha > 0$ which implies that $f(.)$ is homogeneous of degree one.

Suppose conversely that $f(.)$ is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha \geq 0$, then $q \leq f(z)$ and hence $\alpha q \leq \alpha f(z) = f(\alpha z)$. Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain that $(-\alpha z, \alpha q) \in Y$. Thus, Y satisfies constant returns to scale. □

For the Cobb-Douglas production function, $f(tz_1, tz_2) = t^{\alpha+\beta} f(z_1, z_2)$ for any $t > 0$. Thus, when $\alpha + \beta = 1$ we have constant returns to scale; when $\alpha + \beta < 1$ we have decreasing returns to scale and when $\alpha + \beta > 1$ we have increasing returns to scale.

(j) *Additivity (or free entry).* If $y \in Y$ and $y' \in Y$, then $y + y' \in Y$. Additivity implies that $ky \in Y$ for any positive integer k. The economic interpretation of additivity is that if y and y' are both possible, then one can set up two plants that do not interfere with each other and carry out production plans y and y' independently. The result is then a product vector $y + y'$. Additivity is also related to the idea of entry. If $y \in Y$ is being produced by a firm and another firm enters and produces $y' \in Y$, then the net result is the vector $y + y'$. Hence, the aggregate production set (the production set describing feasible production plans for the economy as a whole) must satisfy additivity whenever unrestricted entry (or free entry) is possible.

(k) *Convexity.* The production set is convex, that is if $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$. If inaction is possible (that is, if $0 \in Y$), then convexity implies that Y has non-increasing returns to scale. For any $\alpha \in [0, 1]$, we can write $\alpha y = \alpha y + (1 - \alpha).0$. Therefore, if $y \in Y$ and $0 \in Y$, then convexity of Y implies that $\alpha y \in Y$ for all $\alpha \in [0, 1]$. If production plans y and y' produce exactly the same amount of output but uses different input combinations, then a production vector that uses a level of each input that is the average of the levels used in these two plans can do at least as well as either y or y'. Therefore, "unbalanced" input combinations are not more productive than balanced ones (or "unbalanced" output combinations are not least costly to produce than balanced ones).

(l) *Strict convexity.* The production set Y is *strictly convex* if $y, y' \in Y$, $y \neq y'$ and $\alpha \in (0, 1)$ implies $\alpha y + (1 - \alpha)y' \in int(Y)$.

Exercise 3. Show that for a single-output technology, Y is convex if and only if the production function $f(z)$ is concave.

Proof: Suppose Y is convex. Let $z, z' \in \Re_+^{L-1}$ and $\alpha \in [0, 1]$. Then $(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$ and, by convexity of Y, $(-(\alpha z + (1 - \alpha)z'), \alpha f(z) + (1 - \alpha)f(z')) \in Y$. Thus, $\alpha f(z) + (1 - \alpha)f(z') \leq f(\alpha z + (1 - \alpha)z')$ implying concavity of $f(z)$.

Suppose conversely that $f(z)$ is concave. If $(-z, q) \in Y$ and $(-z', q') \in Y$ and $\alpha \in [0,1]$, then $q \leq f(z)$ and $q' \leq f(z')$. Hence $\alpha q + (1-\alpha)q' \leq \alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha)z')$. The last inequality follows from concavity of $f(.)$. Hence $(-(\alpha z + (1-\alpha)z'), \alpha q + (1-\alpha)q') \in Y$ implying convexity of Y. □

Proposition 9. If Y is closed and convex and $-\Re^L_+ \subset Y$, then free disposal holds.

Proof: Let $y \in Y$ and $v \in -\Re^L_+$. Then, for every positive integer n, $nv \in -\Re^L_+ \in Y$ since $-\Re^L_+ \subset Y$. Since Y is convex, $(1 - \frac{1}{n})y + \frac{1}{n}(nv) = (1-\frac{1}{n})y + v \in Y$. Since Y is closed, $y + v = \lim_{n\to\infty}((1-\frac{1}{n})y + v) \in Y$. □

Suppose that Y is a production set, interpreted now as the technology of a single production unit. Denote by Y^+ the *additive closure of* Y, that is, the smallest production set that is additive and contains Y. In other words, Y^+ is the total production set if technology Y can be replicated an arbitrary number of times. If the production set Y itself is additive then the additive closure of Y is Y itself, that is $Y^+ = Y$. If Y is not additive, then Y^+ is equal to the set of vectors of $\Re^L$ that can be represented as the sum of finitely many vectors of Y. If the production set is convex, then we have the following stronger property: Y^+ consists of all "multiplied " production plans.

Proposition 10. If Y is convex, then $Y^+ = \cup_{n=1}^{\infty} nY$ where for any positive integer n, $nY = \{ny \in \Re^L : y \in Y\}$.

Proof: We first show that $\cup_{n=1}^{\infty} nY \subseteq Y^+$. If $y \in nY$ for some positive integer n, then $\frac{1}{n}y \in Y$. By the definition of additive closure of Y it follows that $\frac{1}{n}y \in Y^+$ and since Y^+ is additive we have $y = n\frac{1}{n}y \in Y^+$. We have proved that if $y \in \cup_{n=1}^{\infty} nY$ then $y \in Y^+$ and hence we have established that $\cup_{n=1}^{\infty} nY \subseteq Y^+$.

To show that $Y^+ \subseteq \cup_{n=1}^{\infty} nY$, it is sufficient to show that $\cup_{n=1}^{\infty} nY$ is additive.[4] Let $y \in \cup_{n=1}^{\infty} nY$ and $y' \in \cup_{n=1}^{\infty} nY$. Then there exists positive integers n and n' such that $y \in nY$ and $y' \in n'Y$. Thus, $\frac{1}{n}y, \frac{1}{n'}y' \in Y$. Consider $\alpha = \frac{n}{n+n'} \in (0,1)$. By convexity of Y, $\alpha\frac{1}{n}y + (1-\alpha)\frac{1}{n'}y' = \frac{1}{n+n'}(y + y') \in Y$ and hence $y + y' \in (n+n')Y \subset \cup_{n=1}^{\infty} nY$. Thus $\cup_{n=1}^{\infty} nY$ is an additive set that includes Y and hence $Y^+ \subseteq \cup_{n=1}^{\infty} nY$. □

3.4. **Profit maximization and the supply correspondence.** We assume that there is a vector of prices quoted for the L goods, denoted by $p = (p_1, \ldots, p_L) >> 0$ and that these prices are independent of the production plans of the firm (price taking assumption). We assume that the production set satisfies non-emptiness, closedness and free disposal. Given $p >> 0$ and

[4]By definition Y^+ is the smallest additive set that includes Y and hence if we can show that $\cup_{n=1}^{\infty} nY$ is additive then it has to be a set that includes the smallest additive set of Y which is Y^+.

a production vector $y \in \Re^L$, the profit generated by implementing y is $p.y = \sum_{l=1}^{L} p_l y_l$. By the sign convention, this is precisely the total revenue minus total cost. The firm's profit maximization problem is $\max_y p.y$ subject to $y \in Y \Leftrightarrow \max_y p.y$ subject to $F(y) \leq 0$. Given Y, the firm's profit function $\pi(p)$ associates to every p the amount $\pi(p) = \max\{p.y \mid y \in Y\}$, the value of the solution to the profit maximization problem. Correspondingly, we define the firm's supply correspondence (or firm's net supply to the market with negative entries representing demand for inputs) at p as the set of profit maximizing production vectors $y(p) = \{y \in Y \mid p.y = \pi(p)\}$. In general $y(p)$ may be a set rather than a single vector. It is also possible that no profit maximizing production plan exists.

Exercise 4. Prove that if the production set Y exhibits non-decreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = \infty$.

Proof: Suppose not. That is, there exists $p >> 0$ such that $\pi(p) \in (0, \infty)$. Let $y \in y(p)$ implying that $p.y = \pi(p)$. Consider any $\alpha > 1$. Non-decreasing returns to scale implies that $\alpha y \in Y$. Consider the profit for the bundle αy. Since $\alpha > 1$, $p.\alpha y = \alpha p.y = \alpha\pi(p) > \pi(p) > 0$ contradicting the fact that y is profit maximizing. □

Proposition 11. Suppose that $\pi(.)$ is the profit function of the production set Y and that $y(.)$ is the associated supply correspondence. Assume that Y is closed and satisfies the free disposal property. Then we have the following:

(a) $\pi(.)$ is homogeneous of degree one.
(b) $\pi(.)$ is convex.
(c) $y(.)$ is homogeneous of degree zero.
(d) If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single valued (if not empty).

Proof:
(a) From the definition of $\pi(.)$ and $y(.)$ it follows that $\pi(p) = p.y$ for all $y \in y(p)$ and $\pi(p) > p.y'$ for all $y' \in Y \setminus \{y(p)\}$. Hence, for any $\alpha > 0$, $\pi(\alpha p) = \alpha p.y = \alpha\pi(p) > \alpha p.y'$ for all $y \in y(p)$ and all $y' \in Y \setminus \{y(p)\}$. Therefore, for any $\alpha > 0$, $\pi(\alpha p) = \alpha\pi(p)$ implying that the profit function is homogeneous of degree one in prices.

(b) Consider any pair $p, p' >> 0$, any $\alpha \in [0,1]$ and $\alpha p + (1-\alpha)p'$. Note that $\pi(\alpha p + (1-\alpha)p') = (\alpha p + (1-\alpha)p').y = \alpha p.y + (1-\alpha)p'.y$ for all $y \in y(\alpha p + (1-\alpha)p')$. By definition $\pi(p) \geq p.y$ and $\pi(p') \geq p'.y$ for all $y \in Y$. Hence, $\pi(\alpha p + (1-\alpha)p') \leq \alpha\pi(p) + (1-\alpha)\pi(p')$ implying convexity of the profit function in prices.

(c) Consider any $\alpha > 0$. Using the definition of $y(.)$ and the fact that the profit function is homogeneous of degree one in prices we get $y(\alpha p) = \{y \in Y \mid \alpha p.y = \pi(\alpha p)\} = \{y \in Y \mid \alpha p.y = \alpha\pi(p)\} = \{y \in Y \mid p.y = \pi(p)\} = y(p)$. Therefore, for all $\alpha > 0$, $y(\alpha p) = y(p)$ implying that $y(p)$ is homogeneous of degree zero in prices.

(d) Consider $y, y' \in y(p)$. By definition $p.y = p.y' = \pi(p)$. Since Y is convex, $\alpha y + (1-\alpha)y' \in Y$ for all $\alpha \in [0,1]$. Consider any $\alpha \in [0,1]$. Observe that $p.(\alpha y + (1-\alpha)y') = \alpha p.y + (1-\alpha)p.y' = \alpha\pi(p) + (1-\alpha)\pi(p) = \pi(p)$ implying $\alpha y + (1-\alpha)y' \in y(p)$. Therefore, from convexity of Y it follows that if $y, y' \in y(p)$, then $\alpha y + (1-\alpha)y' \in y(p)$ for any $\alpha \in [0,1]$. Hence, $y(p)$ is a convex set for any given $p >> 0$.

Recall that the production set Y is *strictly convex* if $y, y' \in Y$, $y \neq y'$ and $\alpha \in (0,1)$ implies $\alpha y + (1-\alpha)y' \in int(Y)$. To show that if Y is strictly convex, then $y(p)$ is a singleton, assume that $y, y' \in y(p)$ and $y \neq y'$. Then $p.y = p.y' = \pi(p)$. Consider any $\alpha \in (0,1)$ and $y'' := \alpha y + (1-\alpha)y'$. By the definition of strict convexity, $y'' \in int(Y)$. Note that $p.y'' = \pi(p)$. However, since $y'' \in int(Y)$, there exists a $\bar{y} \in Y$ such that $\bar{y} > y''$ and hence $p.\bar{y} > \pi(p)$ contradicting our assumption that $y, y' \in y(p)$. □

The general mathematical expression of the *law of supply* is that *quantities respond in the same direction as price changes*. Given our sign convention, this means that if the price of an output increases (all other prices remaining the same), then the supply of output increases and if the price of an input increases), then the demand for the input decreases. Note that the law of supply holds for any price change since there is no wealth effect just substitution effect. That is, the law of supply can be expressed as $(p-p').(y-y') \geq 0$ for all p, p', $y \in y(p)$ and $y' \in y(p')$. Note that $(p-p').(y-y') = p.(y-y') + p'.(y'-y) = [\pi(p) - p.y'] + [\pi(p') - p'.y] \geq 0$.

3.5. **Aggregation.** Suppose there are J production units (firms or perhaps plants) in the economy each specified by a production set $Y_1, \dots, Y_J$ and we assume that each Y_j is non-empty, closed and satisfies the free disposal property. Denote the profit function and supply correspondences of Y_j by $\pi_j(p)$ and $y_j(p)$ respectively. The aggregate supply correspondence is the sum of the individual supply correspondences, that is $y(p) = \sum_{j=1}^{J} y_j(p) = \left\{y \in \Re^L \mid y = \sum_{j=1}^{J} y_j \text{ for some } y_j \in y_j(p),\ j = 1, \dots, J\right\}$. The law of supply holds in the aggregate: if a price increases, then so does the corresponding aggregate supply. This property holds for all price changes. We know that $(p-p').(y_j - y'_j) \geq 0$ for every $j = 1, \dots, J$ every p, every p', every $y_i \in y_j(p)$ and every $y'_j \in y_j(p')$. Therefore, adding over j, we get $(p-p').(y(p) - y(p')) \geq 0$.

The aggregate production set $Y = Y_1 + \ldots + Y_J = \{y \in \Re^L \mid y = \sum_j y_j$ for some $y_j \in Y_j$ for all $j = 1, \ldots, J\}$ describes the production vectors that are feasible in the aggregate if all the production sets are used together. Let $\pi^*(p)$ and $y^*(p)$ be the profit function and the supply correspondence that would arise if a single price-taking firm were to operate, under the same management so to speak, all the individual production sets. We have the following strong aggregation result for the supply side: The aggregate profit obtained by each production unit maximizing profit separately taking prices as given is the same as that which would be obtained if they were to coordinate their actions (that is, their y_js') in a joint profit maximizing decision. This is summarized in the next proposition. The proposition can be interpreted as a decentralization result: To find the solution to the aggregate profit maximization problem for given prices $p >> 0$, it is enough to add the solution of the corresponding individual problems.

Proposition 12. For all $p >> 0$, we have

(i) $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$ and we also have

(ii) $y^*(p) = \sum_{j=1}^{J} y_j(p)(= \{\sum_j y_j \mid y_j \in y_j(p)$ for every $j\})$.

Proof:

(i) For the first equality if we take a collection of production plans $y_j \in Y_j$, $j = 1, \ldots, J$, then $\sum_j y_j \in Y$. Because $\pi^*(.)$ is the profit function associated with Y, $\pi^*(p) \geq p.(\sum_j y_j) = \sum_j p.y_j$ and hence $\pi^*(p) \geq \sum_{j=1}^{J} \pi_j(p)$. In the other direction, consider any $y \in Y$. By definition of Y, there is $y_j \in Y_j$ for each $j = 1, \ldots, J$ such that $\sum_j y_j = y$. So $p.y = p.(\sum_j y_j) = \sum_j p.y_j \leq \sum_j \pi_j(p)$ for all $y \in Y$. Thus, $\sum_j \pi_j(p) \geq \pi^*(p)$.

(ii) Consider any set of individual production plans $y_j \in y_j(p)$, $j = 1, \ldots, J$. Then $p.(\sum_j y_j) = \sum_j p.y_j = \sum_j \pi_j(p) = \pi^*(p)$ and hence $\sum_j y_j \in y^*(p)$. Thus, if $\sum_j y_j \in \sum_j y_j(p)$, then $\sum_j y_j \in y^*(p)$ implying $\sum_j y_j(p) \subseteq y^*(p)$. In the other direction, take any $y \in y^*(p)$. Then $y = \sum_j y_j$ for some $y_j \in Y_j$, $j = 1, \ldots, J$. Since $\pi^*(p) = p. \sum_j y_j = \sum_j \pi_j(p)$ and since, for every $j = 1, \ldots, J$, we have $p.y_j \leq \pi_j(p)$, we get $p.y_j = \pi_j(p)$ for all j implying that $y_j \in y_j(p)$ for all j. Hence, $y \in \sum_j y_j(p)$. We have proved that if $\sum_j y_j \in y^*(p)$, then $\sum_j y_j \in \sum_j y_j(p)$ implying $y^*(p) \subseteq \sum_j y_j(p)$. □

3.6. **Efficient Production.** A significant portion of welfare economics focuses on efficiency as it is useful to identify production plans that can be unambiguously regarded as non-wasteful. Here we define the concept of efficient production and study its relation to profit maximization. With some minor qualifications, we see that profit-maximizing production plans are efficient and that when suitable convexity properties hold, the converse is also true. This constitutes a first look at the *fundamental theorems of welfare economics*.

Definition 9. A production vector $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y \neq y'$.

In words, a production vector y is efficient if there is no other feasible vector y' that generates as much output as y using no additional inputs and that actually produces more of some output or uses less of some input. Every efficient y must be on the boundary of Y but the converse is not necessarily true.

Proposition 13. If $y \in Y$ is profit maximizing for some $p >> 0$, then y is efficient.

Proof: Suppose not. Then there exists $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Because $p >> 0$, $p.y' > p.y$ contradicting our assumption that y is profit maximizing. □

Note that this result is valid even if the production set is not convex. In an aggregate level Proposition 13 states that if a collection of firms each independently maximizes profits with respect to some $p >> 0$, then aggregate production is socially efficient. The restriction $p >> 0$ is necessary for Proposition 13 since one can find examples of $y \in Y$ that maximizes profit for some $p \geq 0$, $p \neq 0$ but is also inefficient. For example, take a p such that one input price is zero. Before going to the next result we state a well known theorem that is relevant for the result.
The Separating Hyperplane Theorem: Suppose that there are two non-empty and convex sets $A, B \subset \Re^L$ that are disjoint (that is, $A \cap B = \emptyset$) then there is a $p \in \Re^L$ with $p \neq 0$ and a value $c \in \Re$ such that $p.x \geq c \geq p.y$ for every $x \in A$ and every $y \in B$. That is, there is a hyperplane that separates A and B, leaving A and B on different sides of it.

Proposition 14. Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit maximizing production for some non-zero price vector $p \geq 0$.

Proof: Suppose that $y \in Y$ is efficient and define the set $S_y = \{y' \in \Re^L \mid y' >> y\}$. S_y is convex and, because y is efficient, $Y \cap S_y = \emptyset$. Hence by Separating Hyperplane Theorem there is some $p \neq 0$ such that $p.y' \geq p.y''$ for all $y' \in S_y$ and for all $y'' \in Y$. We must have $p \geq 0$ because if $p_l < 0$ for some l, then we would get $p.y' < p.y$ for some $y' >> y$ with $y'_l - y_l$ sufficiently large. Now take any $y'' \in Y$. Then $p.y' \geq p.y''$ for all $y' \in S_y$. Because y' can be chosen arbitrary close to y, we conclude that $p.y \geq p.y''$ for all $y'' \in Y$, that is y is profit maximizing for p. □

4. General equilibrium: the basic model and definitions

Consider an economy consisting of I consumers (indexed $i = 1, \ldots, I$), J firms (indexed $j = 1, \ldots, J$) and L commodities (indexed $l = 1, \ldots, L$). Each consumer i is characterized by a consumption set $X_i \subset \Re^L$ and a *rational* preference relation R_i defined on X_i. Each firm j has available to it the production possibilities summarized by the production set $Y_j \subset \Re^L$. We assume that Y_j is non-empty and closed. The initial resources of commodities in the economy, that is, the economy's endowments are given by a vector $\hat{\omega} = (\hat{\omega}_1, \ldots, \hat{\omega}_L) \in \Re^L$. Thus, the basic data on preferences, technology, and resources for this economy are summarized by $(\{(X_i; R_i)\}_{i \in I}, \{Y_j\}_{j \in J}, \hat{\omega})$. An *allocation* $(x, y) = (x_1, \ldots, x_I; y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X_i$ for each consumer $i = 1, \ldots, I$ and a production vector $y_j \in Y_j$ for each firm $j = 1, \ldots, J$.

Definition 10. An allocation (x, y) is *feasible* if $\sum_{i \in I} x_{li} = \hat{\omega}_l + \sum_{j \in J} y_{lj}$ for every commodity l, that is, if $\sum_{i \in I} x_i = \hat{\omega} + \sum_{j \in J} y_j$.

An allocation is feasible if the total amount of each commodity consumed does not exceed the total amount available from both the initial endowment and production. We denote the set of allocations by

$$A = \left\{ (x, y) \in X_1 \times \ldots \times X_I \times Y_1 \times \ldots \times Y_J \mid \sum_{i \in I} x_i = \hat{\omega} + \sum_{j \in J} y_j \right\} \subset \Re^{L(I+J)}.$$

Definition 11. A feasible allocation (x, y) is said to be *Pareto optimal* (or *Pareto efficient*) if there is no other feasible allocation $(x', y') \in A$ that Pareto dominates it, that is, if there is no feasible allocation (x', y') such that $x'_i R_i x_i$ for all i and $x'_i P_i x_i$ for some i.

An allocation is Pareto optimal if there is no waste: It is impossible to make any consumer strictly better off without making some other consumer worse off.

Definition 12. A feasible allocation (x, y) is said to be *weak Pareto optimal* (or *weak Pareto efficient*) if there is no other feasible allocation (x', y') such that $x'_i P_i x_i$ for all $i = 1, \ldots, I$.

Proposition 15. Prove that Pareto efficiency implies weak Pareto efficiency. Also show that if $X_i = \Re^L_+$ for all $i = 1, \ldots, I$, and all consumers' preferences are continuous and strongly monotonic, then weak Pareto efficiency implies Pareto efficiency.

Proof: Suppose that a feasible allocation (x, y) is Pareto efficient but is not weak Pareto efficient. Then there exists a feasible allocation (x', y') for which $x'_i P_i x_i$ for all $i = 1, \ldots, I$. But by Pareto optimality of (x, y) it follows that the allocation (x', y') is not feasible. Hence (x, y) must also be weak Pareto efficient.

Suppose that a feasible allocation (x, y) is weak Pareto efficient but is not Pareto efficient. Hence, there exists a feasible allocation (x', y') for which $U_i(x'_i) \geq U_i(x_i)$ for all i and $U_k(x'_k) > U_k(x_k)$ for some k. Since $X_i = \Re^L_+$ for all $i = 1, \ldots, I$ and preferences are strongly monotonic, we must have $U_k(x'_k) > U_k(x_k) \geq U_k(\underline{0})$ and therefore, $x'_k \geq 0$ and $x'_k \neq 0$. Therefore, we must have at least one commodity s such that $x'_{sk} > 0$. Consider the new allocation (x'', y') such that (i) $x''_{li} = x'_{li}$ for all i and all $l \neq s$, (ii) $x''_{si} = x'_{si} + \left(\frac{1}{I-1}\right) \epsilon$ for all $i \neq k$ and (iii) $x''_{sk} = x'_{sk} - \epsilon$. Observe that as long as $x'_{sk} > \epsilon > 0$, the allocation (x'', y') is feasible. By strong monotonicity of preferences $U_i(x''_i) > U_i(x'_i) \geq U_i(x_i)$ for all $i \neq k$, for any $\epsilon \in (0, x'_{sk})$. By continuity of $U_k(.)$ we have $U_k(x''_k) > U_k(x_k)$ for small enough ϵ. Therefore, we can find an ϵ small enough such that the corresponding allocation $(x''; y')$ is feasible and makes every consumer strictly better off in comparison to the original bundle (x, y). Hence (x, y) is not weak Pareto optimal which contradicts our assumption. □

The assumption of strong monotonicity is crucial for the second part of Proposition 15. For example, consider $I = 1, 2$, $L = 1$, $X_1 = X_2 = \Re_+$, $U_1(x_1) = 0$ for all $x_1 \in X_1$ and $U_2(x_2) = x_2$ for all $x_2 \in X_2$, $w > 0$ and $Y = -\Re_+$ (no production). Then $(x^*_1, x^*_2) = (\frac{w}{2}, \frac{w}{2})$ is a weakly Pareto efficient allocation but it is not Pareto efficient. Note that the preference of individual 1 fails to satisfy strong monotonicity.

4.1. **Private ownership economies.** We study the properties of competitive private ownership economies. In such economies, every good is traded in a market at publicly known prices that consumers and firms take as unaffected by their own actions. Consumers trade in the market place to maximize their well-being, and firms produce and trade to maximize profits. The wealth of consumers is derived from initial endowments of commodities and from ownership claims (shares) to the profits of the firms. Therefore, the society's initial endowments and technological possibilities (that is, the firms) are owned by consumers. For each commodity l, we suppose that consumer i initially owns ω_{li} of commodity l where $\sum_{i \in I} \omega_{li} = \hat{\omega}_l$. We denote consumer i's vector of endowments by $\omega_i = (\omega_{1i}, \ldots, \omega_{Li})$. Note that $\hat{\omega} = \sum_{i \in I} \omega_i$. In addition consumer i owns a share $\theta_{ij} \in [0, 1]$ of firm j giving the consumer a claim to fraction θ_{ij} of firm j's profit. Note that for each $j = 1, \ldots, J$, $\sum_{i \in I} \theta_{ij} = 1$. For each $i \in I$, define $\theta_i = (\theta_{i1}, \ldots, \theta_{iJ}) \in [0, 1]^J$. Thus, the basic preferences, technological resource, and ownership data of a private ownership economy are summarized by $(\{(X_i; R_i)\}_{i \in I}, \{Y_j\}_{j \in J}, \{(\omega_i, \theta_i)\}_{i \in I})$. In a competitive private ownership economy, a market exists for each of the L goods and all consumers and producers act as price takers. The idea behind price-taking assumption is that if consumers and producers are small relative to the size of the market, they will regard market prices as unaffected by their own actions. We denote the vector of prices by $p = (p_1, \ldots, p_L)$.

Definition 13. Given a $(\{(X_i; R_i)\}_{i\in I}, \{Y_j\}_{j\in J}, \{(\omega_i, \theta_i)\}_{i\in I})$, an allocation (x^*, y^*) and a price vector p^* constitute a *Walrasian* (or *competitive*) *equilibrium* if the following conditions are satisfied:

(W1) *Profit maximization*: For every firm j, y_j^* maximizes profits in Y_j, that is, $p^*.y_j \leq p^*.y_j^*$ for all $y_j \in Y_j$.

(W2) *Preference maximization*: For every consumer i, x_i^* is maximal for R_i in the budget set $\left\{ x_i \in X_i \mid p^*.x_i \leq p^*.\omega_i + \sum_{j\in J} \theta_{ij}(p^*.y_j^*) \right\}$.

(W3) *Market clearing*: For every commodity l, $\sum_{i\in I} x_{li}^* = \hat{\omega}_l + \sum_{j\in J}^{J} y_{lj}^*$, that is, $\sum_{i\in I}^{I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^*$.

Definition 13 delineates three sorts of conditions that must be met for a competitive private ownership economy to be in Walrasian equilibrium. Condition (W1) says that firms are maximizing their profits given equilibrium prices p^*. Condition (W2) says that consumers are maximizing their well-being given, first, the equilibrium prices p^* and, second, the wealth derived from their holdings of commodities and from their shares of profits. Note that wealth is a function of prices. This dependence is due to the following two reasons: (1) Prices determine the value of the consumer's initial endowments. (2) The equilibrium price affect firms' profits and hence the value of the consumer's shareholdings. Condition (W3) requires that, at equilibrium prices, the desired consumption and production levels identified in conditions (W1) and (W2) are in fact compatible; that is, the aggregate supply of each commodity (its total endowment plus its net production) equals the aggregate demand for it. Putting it differently, markets must clear at an equilibrium, that is, all consumers and firms must be able to achieve their trades at the going market prices p^*.

From Definition 13 it follows that if an allocation (x^*, y^*) and price vector p^* (with $p_l^* > 0$ for some $l = 1, \ldots, L$) constitute a competitive equilibrium, then so do the allocation (x^*, y^*) and price vector αp^* for any scalar $\alpha > 0$. First consider condition (W1) of Definition 13, that is, profit maximization. If y_j^* solves $\max_{y_j\in Y_j} p^*.y_j$, then y_j^* solves $\max_{y_j\in Y_j} \alpha p^*.y_j = \alpha \max_{y_j\in Y_j} p^*.y_j$. Consider condition (W2) of Definition 13, that is preference maximization. Note that the consumer's new budget constraint is $\alpha p^*.x_i \leq \alpha p^*.\omega_i + \sum_{j=1}^{J} \theta_{ij}(\alpha p^*.y_j^*) \Leftrightarrow p^*.x_i \leq p^*.\omega_i + \sum_{j=1}^{J} \theta_{ij}(p^*.y_j^*)$ which is the old budget constraint. Therefore, if x_i^* maximizes preference with the old budget set, then x_i^* also maximizes preference with the new budget constraint. Finally, condition (W3) that is, market clearing does not depend on prices. Hence, the result follows implying that we can normalize prices without loss of generality, that is, we can always set the price of one good (say good l) at unity (by selecting $\alpha = \frac{1}{p_l} > 0$).

Lemma 1. If an allocation $(x; y)$ and price vector $p \neq 0$ satisfy the market clearing condition (that is, condition (W3) of Definition 13) for all $l \neq k$, $p_k \neq 0$ and if every consumer's budget constraint is satisfied with equality, that is, for all $i = 1, \ldots, I$, $p.x_i = p.\omega_i + \sum_{j=1}^{J} \theta_{ij}(p.y_j)$, then the market for good k also clears.

Proof: Adding up the consumers' budget constraints over the I consumers we get

$$\sum_{i=1}^{I} p.x_i = \sum_{i=1}^{I} p.\omega_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \theta_{ij}(p.y_j)$$
$$\Rightarrow \sum_{i=1}^{I}\left\{\sum_{l=1}^{L} p_l x_{li}\right\} = \sum_{i=1}^{I}\left\{\sum_{l=1}^{L} p_l \omega_{li}\right\} + \sum_{j=1}^{J}\left\{(p.y_j)\left(\sum_{i=1}^{I}\theta_{ij}\right)\right\}$$
$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I}\omega_{li}\right\} + \sum_{j=1}^{J}\left\{(p.y_j)(1)\right\}$$
$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\hat{\omega}_l + \sum_{j=1}^{J}\left\{\sum_{l=1}^{L} p_l y_{lj}\right\}$$
$$\Rightarrow \sum_{l=1}^{L} p_l\left\{\sum_{i=1}^{I} x_{li}\right\} = \sum_{l=1}^{L} p_l\hat{\omega}_l + \sum_{l=1}^{L} p_l\left\{\sum_{j=1}^{J} y_{lj}\right\}$$
$$\Rightarrow \sum_{l=1}^{L} p_l\left(\sum_{i=1}^{I} x_{li} - \hat{\omega}_l - \sum_{j=1}^{J} y_{lj}\right) = 0$$
$$\Rightarrow \sum_{l\neq k} p_l\left(\sum_{i=1}^{I} x_{li} - \hat{\omega}_l - \sum_{j=1}^{J} y_{lj}\right) = -p_k\left(\sum_{i=1}^{I} x_{ki} - \hat{\omega}_k - \sum_{j=1}^{J} y_{kj}\right).$$

By market clearing in goods $l \neq k$, the left hand side of the last equation is equal to zero. Thus, the right hand side must also be zero. Since $p_k \neq 0$, $\sum_{i=1}^{I} x_{ki} - \hat{\omega}_k - \sum_{j=1}^{J} y_{kj} = 0 \Leftrightarrow \sum_{i=1}^{I} x_{ki} = \hat{\omega}_k + \sum_{j=1}^{J} y_{kj}$ implying market clearance for good k. $\square$

If consumers' budget constraints hold with equality then the rupee value of each consumer's planned purchases equals the rupee value of what he plans to sell plus the rupee value of his share (θ_{ij}) of the firms' (net) supply and so the total value of planned purchase in the economy must equal the total value of planned sales. If these values are equal to each other in all markets but one then equality must hold in the remaining market as well.

4.2. **Price equilibria with transfers.** We want to show how Pareto optimal allocations can be supported by means of price taking behavior. Therefore, it is useful to introduce a notion of equilibrium that allows for a more general determination of consumers' wealth levels than that in a private ownership economy. We can imagine a situation where a social planner is able to carry out (lump-sum) redistribution of wealth, and where society's aggregate wealth can therefore be redistributed among consumers in any desired manner.

Definition 14. Given a $(\{(X_i; R_i)\}_{i\in I}, \{Y_j\}_{j\in J}, \hat{\omega})$, an allocation (x^*, y^*) and a price vector p^* constitute a *price equilibrium with transfers* if there is an assignment of wealth levels $w = (w_1, \ldots, w_I)$ with $\sum_{i\in I} w_i = p^*.\hat{\omega} + \sum_{j\in J} p^*.y_j^*$ such that the following conditions are satisfied:

(T1) *Profit maximization*: For every firm j, y_j^* maximizes profits in Y_j, that is, $p^*.y_j \leq p^*.y_j^*$ for all $y_j \in Y_j$.

(T2) *Preference maximization*: For every consumer i, x_i^* is maximal for R_i in the budget set $\{x_i \in X_i \mid p^*.x_i \leq w_i\}$.

(T3) *Market clearing*: $\sum_{i\in I}^{I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^*$.

The concept of a price equilibrium with transfers requires only that there be some wealth distribution such that allocation (x^*, y^*) and price vector p^* constitute an equilibrium. It captures the idea of price-taking market behavior without any supposition about the determination of consumers' wealth levels. A Walrasian equilibrium is a special case of a price equilibrium with transfers in which every i's wealth level is determined by the initial endowment vectors ω_i and by the profit shares θ_i without any further wealth transfers, that is, where $w_i = p^*.\omega_i + \sum_{j\in J} \theta_{ij} p^*.y_j^*$ for all $i \in I$.

5. THE FIRST FUNDAMENTAL THEOREM OF WELFARE ECONOMICS

The first fundamental theorem of welfare economics states conditions under which any price equilibrium with transfers, and, in particular, any Walrasian equilibrium, is Pareto optimum. For competitive market economies, it provides a formal and very general confirmation of Adam Smith's asserted "invisible hand" property of the market.

Exercise 5. Show that if a consumption set $X_i \subset \Re_+^L$ is non-empty, closed and bounded and the preference relation R_i on X_i is rational and continuous, then R_i cannot be locally non-satiated.

Proof: Given that R_i on X_i is rational and continuous, there exists a continuous utility function $U_i : X_i \to \Re$ that represents this preference R_i on X_i. Since X_i is closed and bounded, it is compact. Therefore, $U_i : X_i \to \Re$ is a continuous function on a compact domain and hence (by Bolzano-Weierstrass theorem) it has a maximizer (say x_i^*). Therefore, $U_i(x_i^*) \geq U_i(x_i)$ for all $x_i \in X_i$. Thus, R_i on X_i is globally (and hence locally) satiated at x_i^*. □

From Exercise 5 we have: If R_i on X_i is continuous and locally non-satiated, then any closed consumption set X_i must be unbounded.

Exercise 6. Suppose that the rational preference relation R_i is locally non-satiated and that x_i^* is maximal for R_i in the set $\{x_i \in X_i \mid p.x_i \leq w_i\}$. Prove the following statement: "If $x_i R_i x_i^*$, then $p.x_i \geq w_i$".

Proof: Suppose that x_i^* is maximal for R_i in the set $\{x_i \in X_i \mid p.x_i \leq w_i\}$ and there exists $x_i \in X_i$ such that $x_i R_i x_i^*$ and $p.x_i < w_i$. Then, by local non-satiation, for every $\epsilon > 0$ there exists $x_i'(\epsilon) \in X$ such that $||x_i'(\epsilon) - x_i|| \leq \epsilon$ and $x_i'(\epsilon) P_i x_i$. Moreover, for small enough $\epsilon > 0$, say ϵ^*, it is also true that $p.x_i'(\epsilon^*) \leq w_i$. Given $x_i'(\epsilon^*) P_i x_i R_i x_i^*$ and by transitivity of preference we get $x_i'(\epsilon^*) P_i x_i^*$. But this contradicts the fact that x_i^* is maximal for R_i in the set

$\{x_i \in X_i \mid p.x_i \leq w_i\}$ since $x_i'(\epsilon^*)P_i x_i^*$ and $x_i'(\epsilon^*) \in \{x_i \in X_i \mid p.x_i \leq w_i\}$. Hence, the result follows. □

Theorem 1. (The First Fundamental Theorem of Welfare Economics) If all agents have rational and locally non-satiated preferences, and if (x^*, y^*) and p^* is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium is Pareto optimal.

Proof : Suppose that (x^*, y^*) and p^* is a price equilibrium with transfers and that the associated wealth levels are $w = (w_1, \ldots, w_I)$ and hence $\sum_{i \in I} w_i = p^*.\hat{\omega} + \sum_{j \in J} p^*.y_j^*$. The preference maximization part (T2) of the definition of a price equilibrium with transfers imply that (A) if $x_i P_i x_i^*$, then $p^* x_i > w_i$, that is, anything that is strictly preferred by consumer i to x_i^* must be unaffordable. Non-satiation also implies that (B) if $x_i R_i x_i^*$, then $p^*.x_i \geq w_i$, that is, anything that is at least as good as x_i^* is at best affordable (see Exercise 6).

Suppose that the result is not true, that is, suppose that an allocation (x, y) Pareto dominates (x^*, y^*). That is, $x_i R_i x_i^*$ for all i and $x_i P_i x_i^*$ for some i. From (B) we have $p^*.x_i \geq w_i$ for all i and from (A) we have $p^*.x_i > w_i$ for some i. Hence, $\sum_{i \in I} p^*.x_i > \sum_{i \in I} w_i = p^*.\hat{\omega} + \sum_{j \in J}(p^*.y_j^*)$. Moreover, because y_j^* is profit maximizing for each j, we have $p^*.\hat{\omega} + \sum_{j \in J}(p^*.y_j^*) \geq p^*.\hat{\omega} + \sum_{j \in J}(p^*.y_j)$. Thus, for the allocation (x, y), we have (C) $\sum_{i \in I} p^*.x_i > p^*.\hat{\omega} + \sum_{j \in J}(p^*.y_j)$. But then (x, y) cannot be feasible since feasibility requires $\sum_{i \in I} x_i = \hat{\omega} + \sum_{j \in J} y_j$ and it implies (D) $\sum_{i \in I} p^*.x_i = p^*.\hat{\omega} + \sum_{j \in J}(p^*.y_j)$. But (D) contradicts (C). □

At any feasible (x, y), the total cost of $(x_1, \ldots, x_I)$ with price p^*, must be equal $p^*.\hat{\omega} + \sum_{j \in J} p^*.y_j$. Moreover, because preferences are locally non-satiated, if (x, y) Pareto dominates (x^*, y^*), then the total cost of the consumption bundle x at prices p^*, and therefore the social wealth at those prices, must exceed the total cost of the equilibrium consumption allocation $p^*.(\sum_{i \in I} x_i^*) = p^*.\hat{\omega} + \sum_{j \in J} p^*.y_j^*$. But by (T1) of Definition 14, there is no technologically feasible production levels that attain a value of social wealth at price p^* in excess of $p^*.\hat{\omega} + \sum_{j \in J} p^*.y_j^*$.

6. THE SECOND FUNDAMENTAL THEOREM OF WELFARE ECONOMICS

The second fundamental theorem of welfare economics gives conditions under which a Pareto optimal allocation can be supported as a price equilibrium with transfers. It is a converse of the first fundamental theorem of welfare economics in the sense that it tells us that, under its assumption, we can achieve any desired Pareto optimal allocation as a market-based equilibrium using appropriate lump-sum wealth distribution schemes. We first define quasi-equilibrium with transfers which is similar to the definition of price equilibrium with transfers (Definition 14) except that the preference maximization condition is replaced by the weaker requirement that anything preferred to x_i^* cannot cost less than w_i.

Definition 15. Given a $(\{(X_i; R_i)\}_{i\in I}, \{Y_j\}_{j\in J}, \hat{\omega})$, an allocation (x^*, y^*) and a price vector $p \neq 0$ constitute a *price quasi-equilibrium with transfers* if there is an assignment of wealth levels $w = (w_1, \ldots, w_I)$ with $\sum_{i\in I} w_i = p.\hat{\omega} + \sum_{j\in J} p.y_j^*$ such that the following conditions are satisfied:

(Q1) *Profit maximization*: For every firm j, y_j^* maximizes profits in Y_j, that is, $p.y_j \leq p.y_j^*$ for all $y_j \in Y_j$.
(Q2) *Weak preference restriction:* For every consumer i, if $x_i P_i x_i^*$, then $p.x_i \geq w_i$.
(Q3) *Market clearing*: $\sum_{i\in I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^*$.

Part (Q2) of Definition 15 is implied by the preference maximization condition (T2) of Definition 14. If x_i^* is preference maximizing in the set $\{x_i \in X_i \mid p.x_i \leq w_i\}$, then no $x_i P_i x_i^*$ with $p.x_i < w_i$ can exist. Hence any price equilibrium with transfers is a price quasi-equilibrium with transfers.

When consumers' preferences are locally non-satiated, part (Q2) of Definition 15 implies $p.x_i^* \geq w_i$ for every i. To see this note that if $p.x^* < w_i$, then (by local non-satiation) there exists an x_i close to x_i^* such that $x_i P_i x_i^*$ and $p.x_i < w_i$ contradicting the fact that if $x_i P_i x_i^*$, then $p.x_i \geq w_i$. In addition, from (Q3), we get $\sum_{i\in I} p.x_i^* = p.\hat{\omega} + \sum_{j\in J} p.y_j^* = \sum_{i\in I} w_i$ and hence we have $p.x_i^* = w_i$ for every i. This means that we could just as well replace w_i in (Q2) of Definition 15 by

(Q2a) If $x_i P_i x_i^*$, then $p.x_i \geq p.x_i^*$.

Therefore it follows that an allocation (x^*, y^*) and price vector $p \neq 0$ constitutes a price quasi-equilibrium with transfers if and only if (Q1), (Q2a) and (Q3) holds.

Exercise 7. Prove that if preference relations are locally non-satiated, then condition (Q2a): "If $x_i P_i x_i^*$, then $p.x_i \geq p.x_i^*$" is equivalent to the condition: "x_i^* is expenditure minimizing in the set $\{x_i \in X_i \mid x_i R_i x_i^*\}$ given the price vector $p \neq 0$".

Proof: Given p, if x_i^* is expenditure minimizing in the set $\{x_i \in X_i \mid x_i R_i x_i^*\}$, then $p.x_i' \geq p.x_i^*$ for all $x_i' \in \{x_i \in X_i \mid x_i R_i x_i^*\}$. If $x_i'' P_i x_i^*$, then clearly $x_i'' \in \{x_i \in X_i \mid x_i R_i x_i^*\}$ and hence $p.x_i'' \geq p.x_i^*$ and we have (Q2a).

For the converse suppose that (Q2a) is true and x_i^* is not expenditure minimizing in $\{x_i \in X_i \mid x_i R_i x_i^*\}$. Then there exists $x_i' \in \{x_i \in X_i \mid x_i R_i x_i^*\}$ such that $p.x_i' < p.x_i^*$. Then, by local non-satiation, for every $\epsilon > 0$ there exists $x_i'(\epsilon) \in X$ such that $||x_i'(\epsilon) - x_i'|| \leq \epsilon$ and $x_i'(\epsilon) P_i x_i'$. Moreover, for small enough $\epsilon > 0$, say ϵ^*, it is also true that $p.x_i'(\epsilon^*) < p.x_i^*$.

Given $x_i'(\epsilon^*)P_ix_i$ and $x_iR_ix_i^*$ and given transitivity of preference, $x_i'(\epsilon^*)P_ix_i^*$. But $x_i'(\epsilon^*)P_ix_i^*$ and $p.x_i'(\epsilon^*) < p.x_i^*$ is a violation of (Q2a). $\square$

From Exercise 7 it follows that (Q2a) is equivalent to saying that "x_i^* is expenditure minimizing in the set $\{x_i \in X_i \mid x_iR_ix_i^*\}$. Therefore, assuming locally non-satiated preferences, when we will try to identify conditions under which the price quasi-equilibrium with transfers implies the price equilibrium with transfers, it will be like providing conditions under which expenditure minimization on $\{x_i \in X_i \mid x_iR_ix_i^*\}$ implies preference maximization on $\{x_i \in X_i \mid p.x_i \leq p_i.x_i^*\} = \{x_i \in X_i \mid p.x_i \leq w_i\}$.

Theorem 2. (The Second Fundamental Theorem of Welfare Economics) Consider an economy specified by $(\{(X_i; R_i)\}_{i\in I}, \{Y_j\}_{j\in J}, \hat{\omega})$ and suppose that every Y_j is convex and every R_i on X_i is rational, convex (so that for every $x_i \in X_i$, $\{x_i \in X_i \mid x_iR_ix_i^*\}$ is a convex set) and locally non-satiated. Then for every Pareto optimal allocation (x^*, y^*), there is a price vector $p \neq 0$ such that (x^*, y^*) and p constitutes a price quasi-equilibrium with transfers.

Proof: For every i, define the set $V_i = \{x_i \in X_i \mid x_iP_ix_i^*\} \subset \Re^L$. Define $V = \sum_{i\in I} V_i = \{\sum_{i\in I} x_i \in \Re^L \mid x_1 \in V_1, \ldots, x_I \in V_I\}$ and define $Y = \sum_{j\in J} Y_j = \{\sum_{j\in J} y_j \in \Re^L \mid y_1 \in Y_1, \ldots, y_J \in Y_J\}$. Thus, V is the set of aggregate consumption bundles that could be split into I individual consumptions, each preferred by its corresponding consumer to x_i^*. The set Y is simply the aggregate production set. Note that the set $Y + \{\hat{\omega}\}$, which is the aggregate production set with its origin shifted to $\hat{\omega}$, is the set of aggregate bundles producible with the given technology and endowments and usable, in principle, for consumption.

Step 1: *Every set V_i is convex.* Suppose that $x_iP_ix_i^*$, $x_i'P_ix_i^*$ and, without loss of generality, assume $x_iR_ix_i'$. By convexity of preference, for any $\alpha \in [0, 1]$, $\alpha x_i + (1-\alpha)x_i'R_ix_i'$ and given $x_i'P_ix_i^*$ and transitivity of preference we get $\alpha x_i + (1-\alpha)x_i'P_ix_i^*$.
Step 2: *The sets V and $Y + \{\hat{\omega}\}$ are convex.* Follows from the fact that the sum of any two (and therefore any number of) convex sets is convex.
Step 3: $V \cap (Y + \{\hat{\omega}\}) = \emptyset$. This is a consequence of the Pareto optimality of (x^*, y^*). If there were a vector z included in V as well as in $Y + \{\hat{\omega}\}$, then this would mean that with the given endowments and technologies it would be possible to produce an aggregate vector that could be used to give every consumer i a consumption bundle that is preferred to x_i^*.
Step 4: *There is a vector $p = (p_1, \ldots, p_L) \neq 0$ and a number r such that $p.z \geq r$ for every $z \in V$ and $p.z \leq r$ for every $z \in Y + \{\hat{\omega}\}$.* This step follows from the Separating Hyperplane Theorem.
Step 5: *If $x_iR_ix_i^*$ for all i, then $p.(\sum_{i\in I} x_i) \geq r$.* Suppose $x_iR_ix_i^*$ for every i. By local non-satiation, for each i there is a consumption bundle $\hat{x}_i$ arbitrarily close to x_i such that $\hat{x}_iP_ix_i$. Therefore, $\sum_{i\in I} \hat{x}_i \in V$ and so $p.(\sum_{i\in I} \hat{x}_i) \geq r$. By taking the limit as $\hat{x}_i \to x_i$ gives $p.(\sum_{i\in I} x_i) \geq$

r. Observe that the limit argument follows from the fact that the upper half space, that is, the set $\{z \in \Re^L \mid p.z \geq r\}$ is a closed set. Geometrically, what we have shown here is that the set $\sum_{i\in I}\{x_i \in X_i \mid x_i R_i x_i^*\}$ is contained in the closure of V.

Step 6: $p.(\sum_{i\in I} x_i^*) = p.(\hat{\omega} + \sum_{j\in J} y_j^*) = r$. From Step 5 we have $p.(\sum_{i\in I} x_i^*) \geq r$. On the other hand, Pareto optimality of (x^*, y^*) implies that $\sum_{i\in I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^* \in Y + \{\hat{\omega}\}$, and therefore, $p.(\sum_{i\in I} x_i^*) \leq r$. Thus, $p.(\sum_{i\in I} x_i^*) = r$. Since Pareto optimality means $\sum_{i\in I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^*$, we also have $p.(\hat{\omega} + \sum_{j\in J} y_j^*) = r$.

Step 7: *For every j, we have $p.y_j \leq p.y_j^*$ for all $y_j \in Y_j$.* For any firm j and any $y_j \in Y_j$ we have $y_j + \sum_{h\neq j} y_h^* \in Y$. Therefore, $p.(\hat{\omega} + y_j + \sum_{h\neq j} y_h^*) \leq r = p.(\hat{\omega} + y_j^* + \sum_{h\neq j} y_h^*)$. Hence $p.y_j \leq p.y_j^*$.

Step 8: *For every i, if $x_i P_i x_i^*$, then $p.x_i \geq p.x_i^*$.* Consider any x_i such that $x_i P_i x_i^*$. From Step 5 and Step 6 we have $p.(x_i + \sum_{k\neq i} x_k^*) \geq r = p.(x_i^* + \sum_{k\neq i} x_k^*)$. Hence $p.x_i \geq p.x_i^*$.

Step 9: *The wealth levels $w_i = p.x_i^*$ for all i supports (x^*, y^*) and p as a price quasi-equilibrium with transfers.* Given Step 8, using local non-satiation and using feasibility of Pareto optimal allocation (x^*, y^*) we get $p.x_i^* = w_i$ for all $i \in N$.

Condition (Q1) of Definition 15 follows from Step 7, condition (Q2) follows from Step 8 and Step 9 and, finally, condition (Q3) follows from the feasibility of the Pareto optimal allocation (x^*, y^*). □

When is price quasi-equilibrium with transfers also a price equilibrium with transfers?

Proposition 16. Assume that X_i is convex and R_i is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p, and the wealth level w_i are such that $x_i P_i x_i^*$ implies $p.x_i \geq w_i$. Then, if there is a consumption vector $x_i' \in X_i$ such that $p.x_i' < w_i$, it follows that $x_i P_i x_i^*$ implies $p.x_i > w_i$.

Proof: Contrary to the assertion suppose there is $x_i \in X_i$ such that $x_i P_i x_i^*$ and $p.x_i = w_i$. By the cheaper consumption assumption, there exists an $x_i' \in X_i$ such that $p.x_i' < w_i$. Then for all $\alpha \in [0, 1)$, we have $\alpha x_i + (1-\alpha)x_i' \in X_i$ and $p.(\alpha x_i + (1-\alpha)x_i') < w_i$. But if α is sufficiently close to 1, then by continuity of R_i, $(\alpha x_i + (1-\alpha)x_i')P_i x_i^*$ which is a contradiction because we have found a consumption bundle that is preferred to x_i^* and costs less than w_i. □

An immediate consequence of Proposition 16 is Proposition 17.

Proposition 17. Suppose that for every i, X_i is convex, $\underline{0} \in X_i$, and R_i is continuous. Then any price quasi-equilibrium with transfers that has $(w_1, \ldots, w_I) >> 0$ is a price equilibrium with transfers.

Proof: Let (x^*, y^*) and $p \neq 0$ constitute a price quasi-equilibrium with transfers. Firstly, for any $j \in J$, $p.y_j^* \geq p.y_j$ for all $y_j \in Y_j$. Hence, condition (T1) of price equilibrium with transfers hold. Secondly, from the definition of price quasi-equilibrium with transfers, market clearing (that is, $\sum_{i\in I} x_i^* = \hat{\omega} + \sum_{j\in J} y_j^*$) holds implying that condition (T3) of price equilibrium with transfers is also satisfied. Finally, given $\underline{0} \in X_i$ and $w_i > 0$ it follows that for the price vector $p \neq 0$, $p.\underline{0} = 0 < w_i$. Using Proposition 16 it follows that if $x_i P_i x_i^*$, then $p.x_i > w_i$. Thus, for any $i \in I$ and any $x_i' \in \{x_i \in X_i \mid p.x_i \leq w_i\}$, we must have $\sim (x_i' P_i x_i^*)$ implying $x_i^* R_i x_i'$. Hence, for each $i \in I$, x_i^* is a maximal element in the budget set $\{x_i \in X_i \mid p.x_i \leq w_i\}$ implying that condition (T2) of the price equilibrium with transfers also holds. □

(1) *When can we get $p \neq 0$ and $p \geq 0$?* Consider the implication of Proposition 17 for an economy in which $\hat{\omega} >> 0$ and every consumer has $X_i = \Re^L_+$ (which is a convex set) and continuous, locally non-satiated preferences. In such an economy, if free disposal holds, then profit maximization ensures that $p \geq 0$ and $p \neq 0$ for any price quasi-equilibrium with transfers.[5] Thus, under these assumptions, any price quasi-equilibrium with transfers in which $x_i^* >> 0$ for every i is also a price equilibrium with transfers (since then $w_i = p.x_i^* > 0$ for all i).

(2) *When can we get $p >> 0$?* Suppose that, in addition, preferences are strong monotone (so that we replace local non-satiation by strong monotone). Then we must have $p >> 0$ in any price quasi-equilibrium with transfers. To see this note that $p \geq 0$ and $p \neq 0$ and $\hat{\omega} >> 0$ imply $\sum_{i\in I} w_i = p.\hat{\omega} + p.(\sum_{j\in J} y_j) > 0$. Therefore, $w_i > 0$ for some i. But by Proposition 16, this consumer must then be maximizing preferences in the budget set $\{x_i \in \Re^L_+ \mid p.x_i \leq w_i\}$, which by strong monotonicity cannot occur if prices are not strictly positive. Then we can conclude that any price quasi-equilibrium with transfers is also a price equilibrium with transfers: If consumer i's allocation satisfies $x_i^* \neq \underline{0}$, then $p.x_i^* > \underline{0}$ and Proposition 17 applies. On the other hand if $x_i^* = \underline{0}$, then $w_i = 0$ and the result follows from the fact that x_i^* is the only vector in the set $\{x_i \in \Re^L_+ \mid p.x_i \leq 0\}$.

7. PARETO EFFICIENCY AND SOCIAL WELFARE OPTIMA

We discuss the relationship between the Pareto efficiency and the maximization of a social welfare function. Given a family $u_i(.)$ of continuous utility functions representing the preferences R_i of the I consumers, we capture the attainable vectors of utility levels for $(\{(X_i; R_i)\}_{i\in I}, \{Y_j\}_{j\in J}, \hat{\omega})$ by the *utility possibility set*: $U = \{u \in \Re^I \mid$ there is an allocation $(x, y) \in A$ such that $u_i \leq u_i(x_i)\ \forall\, i \in I\}$.

[5] If $p_l < 0$, then unboundedly large profit can be generated through disposing off commodity l.

(1) Suppose that the set of feasible allocations A is non-empty, closed and bounded and the utility functions $u_i(.)$ are continuous. Define $U' = \{(u_1(x_1), \dots, u_I(x_I)) \mid (x, y) \in A\} \subset \Re^I$. Thus, U' is the image of a compact set A under continuous utility functions and is therefore itself a compact set. Define $U' - \Re^I_+ = \{v \in \Re^I \mid v = u - w$ for some $u \in U'$ and some $w \in \Re^I_+\}$. Clearly, the utility possibility set $U = U' - \Re^L_+$. Hence, *given that A is non-empty, closed and bounded and the utility functions $u_i(.)$ are continuous, the utility possibility set U is closed and bounded from above.*

(2) *If every X_i and every Y_j is convex and if the utility functions $u_i(.)$ are concave, then the utility possibility set U is convex.* To see this suppose that $u, u' \in U$. Then there exists $(x, y) \in A$ and $(x', y') \in A$ such that $u_i \leq u_i(x_i)$ and $u'_i \leq u_i(x'_i)$ for all $i \in I$. Consider any $\alpha \in [0, 1]$. Since $(\alpha x + (1 - \alpha)x', \alpha y + (1 - \alpha)y') \in A$, and since by concavity of the utility functions, $u_i(\alpha x_i + (1 - \alpha)x'_i) \geq \alpha u_i(x_i) + (1 - \alpha)u_i(x'_i) \geq \alpha u_i + (1 - \alpha)u'_i$ for all $i \in I$, we have $\alpha u + (1 - \alpha)u' \in U$.

By the definition of Pareto efficiency, the utility values of a Pareto efficient allocation must belong to the boundary of the utility possibility set. More precisely, we define the *Pareto frontier* $UP = \{u \in U \mid \nexists u' \in U$ such that $u'_i \geq u_i \; \forall \, i \in I$ and $u'_i > u_i$ for some $i\}$.

Proposition 18. A feasible allocation (x, y) is a Pareto efficient if and only if $(u_1(x_1), \dots, u_I(x_I)) \in UP$.

Proof: If (x, y) is Pareto efficient and $(u_1(x_1), \dots, u_I(x_I)) \notin UP$, then there is $u' \in U$ such that $u'_i \geq u_i(x_i)$ for all i and $u'_i > u_i(x_i)$ for some i. But $u' \in U$ only if there is a feasible allocation (x', y') such that $u_i(x'_i) \geq u'_i$ for all i. It follows that (x', y') Pareto dominates (x, y). To prove the converse we prove the contra-positive, that is, we prove that if (x, y) is not Pareto efficient, then $(u_1(x_1), \dots, u_I(x_I)) \notin UP$. If (x, y) is not Pareto efficient, then it is Pareto dominated by some feasible (x', y'), which means that $u_i(x'_i) \geq u_i(x_i)$ for all i and $u_i(x'_i) > u_i(x_i)$ for some i. Hence, $(u_1(x_1), \dots, u_I(x_I)) \notin UP$. □

Definition 16. A (Bergson-Samuelson) *social welfare function* is a function $W : \Re^I \to \Re$ that assigns a utility value to each possible vector $(u_1, \dots, u_I) \in \Re^I$ of utility levels for the I consumer.

The idea behind a social welfare function $W(u_1, \dots, u_I)$ is that it accurately expresses society's judgement on how individual utilities have to be compared to produce an ordering of possible social outcomes. We concentrate here on a particularly simple class of social welfare functions: $W(u) = \sum_{i \in I} \lambda_i u_i$ for some constants $\lambda = (\lambda_1, \dots, \lambda_I)$. Hence, $W(u) = \lambda.u$. Because $W(u)$ should be non-decreasing in the consumer's utility levels, $\lambda \geq 0$. Since society cannot be unconcerned we also assume $\lambda \neq 0$. We can select points in the utility possibility set U that maximize our measure of social welfare by solving the optimization problem (MSW): $\max_{u \in U} \lambda.u$.

Proposition 19. If u^* is a solution to (MSW) with $\lambda >> 0$, then $u^* \in UP$, that is, u^* is the utility vector of a Pareto efficient allocation. Moreover, if the utility possibility set U is convex, then for any $\bar{u} \in UP$, there is a vector $\lambda \geq 0$ with $\lambda \neq 0$, such that $\lambda.\bar{u} \geq \lambda.u$ for all $u \in U$, that is, $\bar{u}$ is a solution to (MSW).

Proof: If u^* is a solution to (MSW) and is not Pareto efficient, then there exists $u \in U$ such that $u \geq u^*$ and $u \neq u^*$. Given $\lambda >> 0$, we get $\lambda.u > \lambda.u^*$ which is a contradiction to u^* being a solution to (MSW). For the other part, if $\bar{u} \in UP$, then $\bar{u}$ is on the boundary of U. By Supporting Hyperplane Theorem, there exists $\lambda \neq 0$ such that $\lambda.\bar{u} \geq \lambda.u$ for all $u \in U$. Moreover, the set U has been constructed so that $U - \Re^L_+ \subset U$, we must have $\lambda \geq 0$. The reason being if $\lambda_i < 0$, then by choosing a $u \in U$ with $u_i < 0$ large enough in absolute terms, we would have $\lambda.u > \lambda.\bar{u}$. □

By using the social welfare weights associated with a particular Pareto optimal allocation (can be a Walrasian equilibrium), we can view the latter as the welfare optimum in a certain single-consumer and single-firm economy. To see this, let (x^*, y^*) be a Pareto optimal allocation and suppose that $\lambda = (\lambda_1, \ldots, \lambda_I) >> 0$ is a vector of welfare weights supporting U at $(u_1(x_1^*), \ldots, u_I(x_I^*))$. Define the utility function $u_\lambda(\bar{x})$ on the aggregate consumption vectors in $X = \sum_{i \in I} X_i \subset \Re^L$ by $u_\lambda(\bar{x}) = \max_{(x_1,\ldots,x_I)} \sum_{i \in I} \lambda_i u_i(x_i)$ such that $x_i \in X_i$ for all i and $\sum_{i \in I} x_i = \bar{x}$. The utility function $u_\lambda(.)$ is the direct utility function of a *normative representative consumer*. Let $Y = \sum_{j \in J} Y_j$ be the aggregate production set. The pair $(\sum_{i \in I} x_i^*, \sum_{j \in J} y_j^*)$ is then a solution to the following problem: $\max_{(\bar{x},\bar{y})} u_\lambda(\bar{x})$ subject to $\bar{x} = \hat{\omega} + \bar{y}$, $\bar{x} \in X$, $\bar{y} \in Y$. The particular utility function chosen for the representative consumer depends on the weights $(\lambda_1, \ldots, \lambda_I)$ and therefore on the Pareto optimal allocation under consideration.

INDIAN STATISTICAL INSTITUTE, 203, B. T. ROAD, KOLKATA-700 108, INDIA.
Email address: mmitra@isical.ac.in