Production

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Microeconomic Theory I
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Mas-Colell, A., Whinston, M. D. and Green, J. (1995). Microeconomic Theory

In the production theory, we deal with the supply side of the economy and study the process by which the goods and services consumed by individuals are produced. The supply side is composed of a number of production units or 'firms'.

A firm is an economic organization where a group of individuals coordinate their skills in order to produce goods and services.

The firm's technology is given by the production set. Consider an economy with L commodities. A *production vector* (also known as input-output or net output or production plan) is a vector

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_L \end{bmatrix} \in \mathbb{R}^L$$

that describes the (net) outputs of the L commodities from the production process.

We adopt the convention that positive numbers denote outputs and negative numbers denote inputs. Some elements of a production vector may be zero meaning that the process has no net output of the commodity. For example, $L = 5$ and

$$y = \begin{bmatrix} -5 \\ 2 \\ -6 \\ 3 \\ 0 \end{bmatrix} \in \mathbb{R}^5$$

means that 2 and 3 units of goods 2 and 4 respectively are produced while 5 and 6 units of goods 1 and 3 respectively are used as inputs. Good 5 is neither produced nor used as an input in this production vector.

Technological Possibility

The set of all production vectors that constitute feasible plans for the firm is known as the *production set* and is denoted by $Y \subset \mathbb{R}^L$. Any $y \in Y$ is possible and $y \notin Y$ is not.

The production set Y can be described using a function called the *transformation function*. The transformation function $g(\cdot)$ has the property that

$$Y = \{y \in \mathbb{R}^L : g(y) \leq 0\}$$

and $g(y) = 0$ if and only if y is a boundary point of Y. The set of boundary points of Y is known as the *transformation frontier*.

If g is differentiable and if the production vector $\bar{y}$ satisfies $g(\bar{y}) = 0$, then for any commodity pair ℓ and k,

$$MRT_{\ell k}(\bar{y}) = \left(\frac{\frac{\partial g(\bar{y})}{\partial y_\ell}}{\frac{\partial g(\bar{y})}{\partial y_k}} \right) \left(\text{assuming } \frac{\partial g(\bar{y})}{\partial y_k} \neq 0 \right)$$

is called the *marginal rate of transformation* (MRT) of good ℓ for good k at $\bar{y}$.

MRT is a measure of how much the (net) output of good k *can increase* if the firm decreases the (net) output of good ℓ by one marginal unit. Indeed, for any boundary point $\bar{y}$ (that is, for any point $\bar{y}$ such that $g(\bar{y}) = 0$), we get

$$\left(\frac{\partial g(\bar{y})}{\partial y_k}\right) dy_k + \left(\frac{\partial g(\bar{y})}{\partial y_\ell}\right) dy_\ell = 0.$$

and hence, the slope of the transformation frontier at $\bar{y}$ is precisely $-MRT_{\ell k}(\bar{y})$.

Technologies with distinct inputs and outputs

In many actual production processes, the set of goods that can be outputs is distinct from the set of goods that can be inputs, hence it is sometimes convenient to distinguish the firm's inputs and outputs.

For example, let $q = (q_1, \cdots, q_M) \geq 0$ denote the production levels of the firm's M outputs and $z = (z_1, \cdots, z_{L-M}) \geq 0$ denote the amounts of the firm's $L - M$ inputs with the convention that the amount of input z_ℓ used is measured in non-negative numbers.

A *single-output technology* is commonly described by means of a production function $f(z)$ that gives the maximum amount of output that can be produced using input amounts $z = (z_1, \cdots, z_{L-1}) \geq 0$.

For example, if the output is good L, then the production function $f(\cdot)$ gives rise to the production set

$$Y = \{(-z, q) : q - f(z) \leq 0 \text{ and } z \geq 0\},$$

where $z = (z_1, \cdots, z_{L-1})$. Holding the level of output fixed, we can define the *marginal rate of technical substitution* (*MRTS*) of input ℓ for input k at $\bar{z}$ as

$$MRTS_{\ell k}(\bar{z}) = \left(\frac{\frac{\partial f(\bar{z})}{\partial z_\ell}}{\frac{\partial f(\bar{z})}{\partial z_k}} \right).$$

The number $MRTS_{\ell k}(\bar{z})$ measures the additional amount of input k that can be used to keep output at level $\bar{q} = f(\bar{z})$ when the amount of input ℓ is decreased marginally.

It is the production theory analog to the consumer's marginal rate of substitution. In the consumer theory, we look at the trade-off between goods that keeps the utility constant, here; we examine the trade-off between inputs that keeps the amount of output constant.

Note that $MRTS_{\ell k}(\bar{z})$ is simply a renaming of the marginal rate of transformation of input ℓ for input k in the special case of a single-output many input technology.

Properties of the production set

Some commonly assumed properties of the production set are the following.

Y **is non-empty.** This assumption simply says that the firm has something it can plan to do. Otherwise, there is no need to study the behavior of the firm in question.

Y **is closed.** The set Y includes its boundary. Thus, the limit of a sequence of technologically feasible input-output vectors is also feasible. Formally, $y^n \to y$ and $y^n \in Y$ for all n imply that $y \in Y$. This assumption is primarily technical.

No free lunch. Suppose that $y \in Y \cap \mathbb{R}^L_+$, so that the vector y does not use any inputs. The no-free-lunch property is satisfied if this production vector cannot produce output either. That is, if $y \in Y \cap \mathbb{R}^L_+$ then $y = 0$. So, it is not possible to produce something out of nothing. Geometrically, $Y \cap \mathbb{R}^L_+ \subseteq \{0\}$.

Possibility of inaction. This property says that $0 \in Y$: *complete shutdown is possible*. You might be able to choose to do nothing (e.g. not to setup your own business now).

Free disposal. $Y - \mathbb{R}^L_+ \subset Y$. It is possible to use more inputs for a given amount of the outputs and it is also possible to produce less outputs with a given amount of inputs.

Irreversibility. If $y \in Y$ and $y \neq 0$ then $-y \notin Y$. It is impossible to reverse a technologically possible production vector to transform an amount of output into the same amount of input that was used to generate it. If the description of commodity includes the time of its availability, then irreversibility follows from the requirement that inputs be used before outputs emerge.

Non-increasing returns to scale. The production technology Y exhibits non-increasing returns to scale if for any $y \in Y$ and $\alpha \in [0, 1]$, we have $\alpha y \in Y$. *Any feasible input-output vector can be scaled down*. Note that this property imply that inaction is possible.

Non-decreasing returns to scale. The production technology Y exhibits non-decreasing returns to scale if for any $y \in Y$ and $\alpha \geq 1$, we have $\alpha y \in Y$. *Any feasible input-output vector can be scaled up*.

Constant returns to scale. The production technology Y exhibits constant returns to scale if for any $y \in Y$ and $\alpha \geq 0$, we have $\alpha y \in Y$. Geometrically, Y *is a cone*.

Exercise 1

Suppose that f is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if f is homogeneous of degree one.

Solution: Suppose that Y exhibits constant returns to scale. Let $z = (z_1, \cdots, z_{L-1}) \in \mathbb{R}_+^{L-1}$ and $\alpha > 0$. Since $(-z, f(z)) \in Y$ then by constant returns to scale, $(-\alpha z, \alpha f(z)) \in Y$ and hence $\alpha f(z) \leq f(\alpha z)$.

Again, since $(-\alpha z, f(\alpha z)) \in Y$, we have

$$\frac{1}{\alpha}(-\alpha z, f(\alpha z)) = \left(-z, \frac{1}{\alpha} f(\alpha z)\right) \in Y$$

which implies that

$$\frac{1}{\alpha} f(\alpha z) \leq f(z) \Rightarrow f(\alpha z) \leq \alpha f(z).$$

Therefore, $f(\alpha z) = \alpha f(z)$ for all $\alpha > 0$ which implies that f is homogeneous of degree one.

Conversely, suppose that f is homogeneous of degree one. Let $(-z, q) \in Y$ and $\alpha > 0$, then $q \leq f(z)$ and hence

$$\alpha q \leq \alpha f(z) = f(\alpha z).$$

Since $(-\alpha z, f(\alpha z)) \in Y$, we obtain that $(-\alpha z, \alpha q) \in Y$. For $\alpha = 0$ and $y \in Y$, we have $\alpha y = 0 \in Y$ since $0 \leq f(0)$. Thus, Y satisfies constant returns to scale.

Additivity (or free entry). If $y \in Y$ and $y' \in Y$ then $y + y' \in Y$. Additivity implies that $ky \in Y$ for any positive integer k. The economic interpretation of additivity is that if y and y' are both possible then one can set up two plants that do not interfere with each other and carry out production plans y and y' independently. The result is then a product vector $y + y'$.

Additivity is also related to the idea of entry. If $y \in Y$ is being produced by a firm and another firm enters and produces $y' \in Y$, then the net result is the vector $y + y'$. Hence, the aggregate production set (the production set describing feasible production plans for the economy as a whole) must satisfy additivity whenever unrestricted entry (or free entry) is possible.

Convexity. The production set is convex, that is, if $y, y' \in Y$ and $\alpha \in [0, 1]$, then $\alpha y + (1 - \alpha)y' \in Y$.

If inaction is possible (that is, if $0 \in Y$), then convexity implies that Y has non-increasing returns to scale. For any $\alpha \in [0, 1]$, we can write $\alpha y = \alpha y + (1 - \alpha) \cdot 0$. Therefore, if $y \in Y$ and $0 \in Y$ then convexity of Y implies that $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

If production plans y and y' produce exactly the same amount of output but uses different input combinations, then a production vector that uses a level of each input that is the average of the levels used in these two plans can do at least as well as either y or y'.

Exercise 2

Show that for a single-output technology, Y is convex if and only if the production function f is concave.

Solution: Suppose Y is convex. Let $z, z' \in \mathbb{R}_+^{L-1}$ and $\alpha \in [0,1]$, then $(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$. It follows from the convexity of Y that

$$(-(\alpha z + (1-\alpha)z'), \alpha f(z) + (1-\alpha) f(z')) \in Y.$$

Thus,

$$\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z').$$

So, f is concave.

Conversely, suppose that f is concave. Let $(-z, q) \in Y$,

$(-z', q') \in Y$ and $\alpha \in [0, 1]$. Then, $q \leq f(z)$ and $q' \leq f(z')$. Hence,

$$\alpha q + (1 - \alpha)q' \leq \alpha f(z) + (1 - \alpha)f(z') \leq f(\alpha z + (1 - \alpha)z').$$

The last inequality follows from concavity of f. Hence,

$$(-(\alpha z + (1 - \alpha)z'), \alpha q + (1 - \alpha)q') \in Y.$$

So, Y is convex.

Proposition

If Y is closed and convex and $-\mathbb{R}^L_+ \subset Y$, then free disposal holds.

Proof: Let $y \in Y$ and $v \in -\mathbb{R}^L_+$. Note that for every positive integer n, $nv \in -\mathbb{R}^L_+$. Since Y is convex,

$$\left(1-\frac{1}{n}\right)y+v=\left(1-\frac{1}{n}\right)y+\frac{1}{n}(nv)\in Y.$$

Since Y is closed,

$$y+v=\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)y+v\right)\in Y.$$

Convex Cone. Y is a convex cone, that is, for any $y, y' \in Y$ and $\alpha, \beta \in \mathbb{R}_+$, we have $\alpha y + \beta y' \in Y$.

The next result provides a justification for the convexity assumption in production. If feasible input-output combinations can always be scaled down and the simultaneous operation of several technologies without mutual interference is always possible then we obtain convexity.

Proposition

The production set is additive and satisfies the non-increasing returns to scale condition if and only if it is a convex cone.

Proof: The definition of convex cone directly implies non-increasing returns and additivity. By setting $\alpha \in [0, 1]$ and $\beta = 0$ in the definition of convex cone we get non-increasing returns to scale and by setting $\alpha = \beta = 1$ we get additivity.

Conversely, let the non-increasing returns and additivity assumption hold for the production set. Suppose that $y, y' \in Y$ and any $\alpha > 0$ and $\beta > 0$. We will show that $\alpha y + \beta y' \in Y$. Let k be an integer such that

$$k > \max\{\alpha, \beta\}.$$

By additivity, $ky \in Y$ and $ky' \in Y$. Since $\frac{\alpha}{k} < 1$ and $\alpha y = \frac{\alpha}{k} ky$, the non-increasing returns condition implies that $\alpha y \in Y$. Similarly, $\beta y' = \frac{\beta}{k} ky' \in Y$. Finally, by additivity, $\alpha y + \beta y' \in Y$.

Suppose that Y is a production set, interpreted now as the technology of a single production unit. Denote by Y^+ the *additive closure of* Y, that is, the smallest production set that is additive and contains Y. If the production set Y itself is additive then the additive closure of Y is Y itself, that is, $Y^+ = Y$.

Proposition

If Y is convex then $Y^+ = \bigcup\{nY : n \geq 1\}$ where for any positive integer n, $nY = \{ny \in \mathbb{R}^L : y \in Y\}$.

Proof: We first show that $\bigcup\{nY : n \geq 1\} \subseteq Y^+$. Let $y \in \bigcup\{nY : n \geq 1\}$. Then $y \in nY$ for some positive integer n.

So, $\frac{1}{n}y \in Y$. It follows from the definition of Y^+ that $\frac{1}{n}y \in Y^+$. Since Y^+ is additive, we have $y = n\frac{1}{n}y \in Y^+$. Thus,

$$\bigcup\{nY : n \geq 1\} \subseteq Y^+.$$

To show that $Y^+ \subseteq \bigcup\{nY : n \geq 1\}$, it is sufficient to show that $\bigcup\{nY : n \geq 1\}$ is additive. Let $y, y' \in \bigcup\{nY : n \geq 1\}$. Then there exists positive integers n and n' such that $y \in nY$ and $y' \in n'Y$. Thus, $\frac{1}{n}y, \frac{1}{n'}y' \in Y$. Consider

$$\alpha = \frac{n}{n+n'} \in (0,1).$$

By the convexity of Y,

$$\frac{1}{n+n'}(y+y') = \alpha\frac{1}{n}y + (1-\alpha)\frac{1}{n'}y' \in Y$$

and hence $y + y' \in (n+n')Y \subset \bigcup\{nY : n \geq 1\}$. Thus, $\bigcup\{nY : n \geq 1\}$ is an additive set that includes Y and hence $Y^+ \subseteq \bigcup\{nY : n \geq 1\}$.

So, $Y^+ = \bigcup\{nY : n \geq 1\}$.

The profit maximization problem (PMP)

We assume that there is a vector of prices quoted for the L goods, denoted by $p = (p_1, \cdots, p_L) >> 0$ and that these prices are independent of the production plans of the firm. The firm's objective is to maximize its profit. We assume that the production set satisfies non-emptiness, closedness and free disposal properties.

Given a price vector $p >> 0$ and a production vector $y \in \mathbb{R}^L$, the profit generated by implementing y is $p \cdot y = \sum_{\ell=1}^{L} p_\ell y_\ell$. By the sign convention, this is precisely the total revenue minus total cost.

Given the production set Y, the firm's profit maximization problem is then

$$\max\{p \cdot y : y \in Y\}$$

or equivalently,

$$\max\{p \cdot y : y \in \mathbb{R}^L,\ g(y) \leq 0\}.$$

The firm's profit function $\pi : \mathbb{R}^\ell_{++} \to \mathbb{R}$ is defined by

$$\pi(p) = \max\{p \cdot y : y \in Y\}.$$

Correspondingly, we define the firm's supply correspondence by

$$S(p) = \{y \in Y : p \cdot y = \pi(p)\}.$$

In general, $S(p)$ may be a set rather than a single vector. It is also possible that no profit maximizing production plan exists.

Exercise 3

Prove that if the production set Y exhibits non-decreasing returns to scale then either $\pi(p) \leq 0$ or $\pi(p) = \infty$.

Proof: Suppose not. That is, there exists $p >> 0$ such that $\pi(p) \in (0, \infty)$ which means $\{p \cdot y : y \in Y\}$ is bounded from above. Then there exists an $y^* \in Y$ such that $p \cdot y^* \in (0, \infty)$. Non-decreasing returns to scale implies that $ny^* \in Y$ for all $n \geq 1$. Clearly, $p \cdot ny^* \to \infty$ as $n \to \infty$, contradicting the fact that $\pi(p)$ is finite.

Proposition

Suppose that $\pi(\cdot)$ is the profit function of the production set Y and that $S(\cdot)$ is the associated supply correspondence. If Y is closed then

(a) $\pi(\cdot)$ is homogeneous of degree one.

(b) $\pi(\cdot)$ is convex.

(c) $S(\cdot)$ is homogeneous of degree zero.

(d) If Y is convex then $S(p)$ is a convex set for all p. Moreover, if Y is strictly convex then $S(p)$ is single valued (if $S(p)$ is non-empty).

Proof: (a) Let $\alpha > 0$ and $p \gg 0$. It follows from the definition of $\pi(\cdot)$ that $\alpha p \cdot y \leq \pi(\alpha p)$ for all $y \in Y$. Since $\pi(p) = \max\{p \cdot y : y \in Y\}$, we have $\alpha\pi(p) \leq \pi(\alpha p)$.

Similarly, for all $y \in Y$, $p \cdot y \leq \pi(p)$. So, $\alpha p \cdot y \leq \alpha\pi(p)$ for all $y \in Y$. Recall that $\pi(\alpha p) = \max\{\alpha p \cdot y : y \in Y\}$. So, $\pi(\alpha p) \leq \alpha\pi(p)$.

Thus, $\pi(\alpha p) = \alpha\pi(p)$ and so, $\pi(\cdot)$ is homogeneous of degree one.

(b) First, we prove that $\pi(p_1 + p_2) = \pi(p_1) + \pi(p_2)$ for all $p_1, p_2 \in \mathbb{R}^{\ell}_{++}$. Note that for any $x \in Y$, we have

$$p_1 \cdot x + p_2 \cdot x = (p_1 + p_2) \cdot x \leq \pi(p_1 + p_2).$$

Then $\pi(p_1) + \pi(p_2) \leq \pi(p_1 + p_2)$.

Again, for any $x \in Y$, we have

$$(p_1 + p_2) \cdot x = p_1 \cdot x + p_2 \cdot x \leq \pi(p_1) + \pi(p_2).$$

Then $\pi(p_1 + p_2) \leq \pi(p_1) + \pi(p_2)$. Thus, we have

$$\pi(p_1 + p_2) = \pi(p_1) + \pi(p_2).$$

From (a), for any $\lambda \in (0, 1)$, we get

$$\pi(\lambda p + (1-\lambda)p^*) = \pi(\lambda p) + \pi((1-\lambda)p^*) = \lambda\pi(p) + (1-\lambda)\pi(p^*).$$

(c) Let $\alpha > 0$. Note that

$$S(\alpha p) = \{y \in Y : \alpha p.y = \pi(\alpha p)\}$$

and

$$S(p) = \{y \in Y : p.y = \pi(p)\}.$$

By (a),

$$\alpha p.y = \pi(\alpha p) = \alpha \pi(p).$$

So,

$$S(\alpha p) = \{y \in Y : p.y = \pi(p)\} = S(p)$$

implying that S is homogeneous of degree zero.

(d) Let $p \gg 0$ and $\alpha \in (0,1)$. Pick $y, y' \in S(p)$. Then $p \cdot y = p \cdot y' = \pi(p)$. Since Y is convex, $\alpha y + (1-\alpha)y' \in Y$. Observe that

$$p \cdot (\alpha y + (1-\alpha)y') = \alpha p \cdot y + (1-\alpha) p \cdot y'$$

$$p = \alpha \pi(p) + (1-\alpha)\pi(p) = \pi(p)$$

implying $\alpha y + (1-\alpha)y' \in S(p)$. Hence, $S(p)$ is convex.

The production set Y is strictly convex if $y, y' \in Y$, $y \neq y'$ and $\alpha \in (0,1)$ implies that $\alpha y + (1-\alpha)y' \in int(Y)$.

To show that if Y is strictly convex then $S(p)$ is a singleton assume that $y, y' \in S(p)$ and $y \neq y'$.

Then

$$p \cdot y = p \cdot y' = \pi(p).$$

Let $\alpha \in (0,1)$ and $y'' = \alpha y + (1-\alpha)y'$. By the definition of strict convexity, $y'' \in int(Y)$. Note that $p \cdot y'' = \pi(p)$. However, since $y'' \in int(Y)$, there exists a $\bar{y} \in Y$ such that $\bar{y} > y''$ and hence $p.\bar{y} > \pi(p)$ contradicting our assumption that $y, y' \in S(p)$.

The Cost Minimization Problem (CMP)

We now analyze the combinations of inputs the firm selects in order to minimize its total cost of production, conditional on reaching a particular output level q.

For simplicity, we focus on the single output case. Let z denote the input vector and $f(\cdot)$ denote the production function. If $w \gg 0$ is the price vector for inputs then the *cost minimization problem* (CMP) can be stated as follows (we assume free disposal of output):

$$\min\{w \cdot z : z \geq 0\} \text{ subject to } f(z) \geq q.$$

In words, the firm selects the vector of input, z, that minimizes total cost, $w \cdot z$, subject to productive feasibility, that is, $f(z) \geq q$.

The cost minimization vector of inputs is denoted by $Z(w, q)$ and it is usually referred to as the *conditional factor demand correspondence* (or *function* if it is always single-valued). The term conditional arises because these factor demands are conditional on the requirement that the output level q be produced.

Suppose that $c(w, q)$ represents the lowest cost of producing output level q when input prices are w, and it is usually referred as the *cost function*.

Intuitively, $Z(w, q)$ reflects the optimal demand of inputs when input prices are w and the firm wants to reach a production level q. At an input combination $z \in Z(w, q)$, the firm cannot reduce its costs any further and still produce output level q. At this input combination, the firm's cost is $w \cdot z = c(w, q)$.

Proposition

Suppose that $c(\cdot,\cdot)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $Z(\cdot,\cdot)$ is the associated conditional factor demand correspondence. Assume that Y is closed and satisfies the free disposal property. Then

(a) $c(\cdot, q)$ is homogeneous of degree one and $c(w, \cdot)$ is non-decreasing.

(b) $c(\cdot, q)$ is a concave function.

(c) $z(\cdot, q)$ is homogeneous of degree zero.

(d) If the set $Y(q) = \{z \geq 0 : f(z) \geq q\}$ is convex, then $Z(w, q)$ is a convex set. Moreover, if $Y(q)$ is a strictly convex set, then $Z(w, q)$ is single valued.

(e) If f is homogeneous of degree one then $c(w, \cdot)$ and $Z(w, \cdot)$ are homogeneous of degree one.

(f) If f is concave, then $c(w, \cdot)$ is a convex function (in particular, marginal costs are non-decreasing in q).

Proof: (a) Let $\alpha > 0$ and $w \gg 0$. Note that $c(w, q) \leq w \cdot z$ for all $z \geq 0$ such that $f(z) \geq q$. Then $\alpha c(w, q) \leq \alpha w \cdot z$ for all $z \geq 0$ such that $f(z) \geq q$. Since

$$c(\alpha w, p) = \inf\{\alpha w \cdot z : z \geq 0,\ f(z) \geq q\},$$

we have

$$\alpha c(w, z) \leq c(\alpha w, z).$$

Now, $c(\alpha w, q) \leq \alpha w \cdot z$ for all $z \geq 0$ such that $f(z) \geq q$. Since

$$c(w, p) = \inf\{w \cdot z : z \geq 0,\ f(z) \geq q\},$$

we have

$$c(\alpha w, z) \leq \alpha c(w, z).$$

Hence, $c(\alpha w, z) = \alpha c(w, z)$.

Let $q' > q \geq 0$. Since $\{z \geq 0 : f(z) \geq q'\} \subseteq \{z \geq 0 : f(z) \geq q\}$, we have $c(w, q') \geq c(w, q)$. So, $c(w, \cdot)$ is non-decreasing.

(b) Let $w, w' \geq 0$. Consider any $\alpha \in [0, 1]$ and define $w'' = \alpha w + (1 - \alpha)w'$. For all $z \geq 0$ with $f(z) \geq q$, we have

$$w'' \cdot z = \alpha w \cdot z + (1 - \alpha)w' \cdot z \geq \alpha c(w, q) + (1 - \alpha)c(w', q).$$

So, $c(w'', q) \geq \alpha c(w, q) + (1 - \alpha)c(w', q)$.

(c) Observe that $Z(w, q) = \{z \geq 0 : f(z) \geq q, w \cdot z = c(w, q)\}$ and $Z(\alpha w, q) = \{z \geq 0 : f(z) \geq q, \alpha w \cdot z = c(\alpha w, q)\}$ for any $\alpha > 0$. Since $c(\alpha w, q) = \alpha c(w, q)$, we have

$$Z(\alpha w, q) = \{z \geq 0 : f(z) \geq q, w \cdot z = c(w, q)\} = Z(w, q).$$

(d) Let $z, z' \in Z(w,q)$, $\alpha \in [0,1]$. So, $w \cdot z = w \cdot z' = c(w,q)$ and $w \cdot (\alpha z + (1-\alpha)z') = \alpha c(w,q) + (1-\alpha)c(w,q) = c(w,q)$. Since $Y(q)$ is convex, $\alpha z + (1-\alpha)z' \in Y(q)$. Hence, $\alpha z + (1-\alpha)z' \in Z(w,q)$ and $Z(w,q)$ is convex.

Assume that $z, z' \in Z(w,q)$ and $z \neq z'$. Then $w \cdot z = w \cdot z' = c(w,q)$. Consider any $\alpha \in (0,1)$ and $z'' = \alpha z + (1-\alpha)z'$. By the definition of strict convexity, $z'' \in int(Y(q))$. Note that $w \cdot z'' = c(w,q)$. However, since $z'' \in int(Z(q))$, there exists a $\bar{z} \in Y(q)$ such that $\bar{z} < z''$ and can produce an output level of at least q. Hence, $w \cot \bar{z} < w \cdot z'' = c(w,q)$ contradicting our assumption that $z, z' \in Z(w,q)$.

Suppose there are J production units in the economy each specified by a production set $Y_1, \ldots, Y_J$ and we assume that each Y_j is non-empty, closed and satisfies the free disposal property.

If the supply correspondence of Y_j is S_j for $1 \leq j \leq J$, then the *aggregate supply correspondence* is defined as

$$S(p) = \sum_{j=1}^{J} S_j(p) = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \text{ for some } y_j \in S_j(p) \right\}.$$

Given $Y_1, \cdots, Y_J$, define the *aggregate production set* by

$$Y = \sum_{j=1}^{J} Y_j = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \text{ for some } y_j \in Y_j \right\}.$$

It describes the production vectors that are feasible in the aggregate if all the production sets are used together.

Let $\pi^* : \mathbb{R}^L_{++} \to \mathbb{R}$ and $S^* : \mathbb{R}^L_{++} \to Y$ be the profit function and the supply correspondence of the aggregate production set Y.

Proposition

For all $p >> 0$, we have

(i) $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$,

(ii) $S^*(p) = \sum_{j=1}^{J} S_j(p)$.

Proof: (i) For each $j = 1, \cdots, J$, we choose $y_j \in Y_j$. Then $\sum_{j=1}^{J} y_j \in Y$. Because $\pi^*(\cdot)$ is the profit function associated with Y,

$$\pi^*(p) \geq p \cdot \left(\sum_{j=1}^{J} y_j \right) = \sum_{j=1}^{J} p \cdot y_j.$$

Hence, $\pi^*(p) \geq \sum_{j=1}^{J} \pi_j(p)$.

For the other direction, consider any $y \in Y$. By the definition of Y, there is a $y_j \in Y_j$ for each $j = 1, \cdots, J$ such that $\sum_j y_j = y$. So,

$$p \cdot y = p \cdot \left(\sum_{j=1}^{J} y_j \right) = \sum_{j=1}^{J} p \cdot y_j \leq \sum_{j=1}^{J} \pi_j(p).$$

Thus, $\sum_{j=1}^{J} \pi_j(p) \geq \pi^*(p)$. So, $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$.

(ii) For $j = 1, \cdots, J$, let $y_j \in S_j(p)$. Then

$$p \cdot \left(\sum_{j=1}^{J} y_j \right) = \sum_{j=1}^{J} p \cdot y_j = \sum_{j=1}^{J} \pi_j(p) = \pi^*(p).$$

Hence, $\sum_{j=1}^{J} y_j \in S^*(p)$. Thus, if $\sum_{j=1}^{J} y_j \in \sum_{j=1}^{J} S_j(p)$ then $\sum_{j=1}^{J} y_j \in S^*(p)$. So,

$$\sum_{j=1}^{J} S_j(p) \subseteq S^*(p).$$

For the other direction, take any $y \in S^*(p)$. Then $y = \sum_{j=1}^{J} y_j$ for some $y_j \in Y_j$, $j = 1, \cdots, J$. So,

$$\sum_{j=1}^{J} \pi_j(p) = \pi^*(p) = p \cdot \sum_{j=1}^{J} y_j.$$

Note that $p \cdot y_j \leq \pi_j(p)$ for $j = 1, \cdots, J$. Thus, $p \cdot y_j = \pi_j(p)$ for all j implying that $y_j \in S_j(p)$ for all j. Hence, $y \in \sum_{j=1}^{J} S_j(p)$.

So, $S^*(p) \subseteq \sum_{j=1}^{J} S_j(p)$. Thus, $S^*(p) = \sum_{j=1}^{J} S_j(p)$.

Corollaries: (i) To find the solution of the aggregate profit maximization problem for given prices p, it is enough to add the solutions of the corresponding individual problems.

(ii) For the single-output case it means that given p and $w >> 0$, if $q = \sum_j q_j$ is the aggregate output produced by the firms, then the total cost of production is exactly equal to $c(w, q)$, the value of the *aggregate cost function* (the cost corresponding to the aggregate production set Y).

Proposition

If the production sets $Y_1, \cdots, Y_J$ are convex and satisfy the free disposal property and that $\sum_{j=1}^{J} Y_j$ is closed then

$$\sum_{j=1}^{J} Y_j = \left\{ y \in \mathbb{R}^L : p \cdot y \leq \sum_{j=1}^{J} \pi_j(p) \text{ for all } p >> 0 \right\}.$$

Proof: One can show that since Y_j is convex and satisfies free disposal property, $\sum_j Y_j$ is also convex and satisfies free disposal. Since $\sum_{j=1}^{J} Y_j$ is assumed to be closed, we get

$$\sum_{j=1}^{J} Y_j = \left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi^*(p) \text{ for all } p >> 0 \right\}.$$

Since $\pi^*(p) = \sum_{j=1}^{J} \pi_j(p)$, we get

$$\sum_{j=1}^{J} Y_j = \left\{ y \in \mathbb{R}^L : p \cdot y \leq \sum_{j=1}^{J} \pi_j(p) \text{ for all } p >> 0 \right\}.$$

We now focus on efficiency as it is useful to identify production plans that can be unambiguously regarded as non-wasteful. Here we define the concept of efficient production and study its relation to profit maximization. With some minor modifications, we see that profit-maximizing production plans are efficient and that when suitable convexity properties hold, the converse is also true. This constitutes a first look at the *fundamental theorems of welfare economics*.

A production vector $y \in Y$ is *efficient* if there is no $y' \in Y$ such that $y' \geq y$ and $y \neq y'$.

In words, a production vector y is efficient if there is no other feasible vector y' that generates as much output as y using no additional inputs and that actually produces more of some output or uses less of some input.

Every efficient production plan y must be on the boundary of Y but the converse is not necessarily true.

Proposition

If $y \in Y$ is profit maximizing for some $p >> 0$ then y is efficient.

Proof: Suppose that y is inefficient. Then there exists $y' \in Y$ such that $y' \neq y$ and $y' \geq y$. Since $p >> 0$, $p.y' > p.y$. This is a contradiction to the fact that y is profit maximizing. So, y is efficient.

Note that this result is valid even if the production set is not convex.

In the aggregate level, the above proposition states that *if a collection of firms each independently maximizes profits with* *respect to some $p >> 0$ then aggregate production is socially efficient*. That is, there is no other production plan for the economy as a whole that could produce more outputs using no additional input.

The restriction $p >> 0$ is necessary for the this proposition since one can find examples of $y \in Y$ that maximizes profit for some $p \geq 0$, $p \neq 0$ but is also inefficient.

The Separating Hyperplane Theorem

For any two disjoint non-empty convex sets $A, B \subset \mathbb{R}^L$ there is a $p \in \mathbb{R}^L$ with $p \neq 0$ and a value $c \in \mathbb{R}$ such that

$$p \cdot x \geq c \geq p \cdot y$$

for every $x \in A$ and every $y \in B$. That is, there is a hyperplane that separates A and B, leaving A and B on different sides of it.

Proposition

Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit maximizing production for some non-zero price vector $p \geq 0$.

Proof: Suppose that $y \in Y$ is efficient and define the set

$$S_y = \left\{ y' \in \mathbb{R}^L : y' \gg y \right\}.$$

Note that S_y is convex, and since y is efficient, $Y \cap S_y = \emptyset$. Hence, by Separating Hyperplane Theorem, there is some $p \neq 0$ such that for all $y' \in S_y$ and for all $y'' \in Y$,

$$p \cdot y' \geq p \cdot y''.$$

We claim that $p \geq 0$. Indeed, if $p_{\ell_0} < 0$ for some ℓ_0, then we choose an integer $n_0 > 0$ such that

$$\sum_{\ell \neq \ell_0} p_\ell + n_0 p_{\ell_0} < 0$$

Note that $y' = y + (1, \cdots, 1, n_0, 1, \cdots, 1) \in S_y$ and $y \in Y$. Clearly,

$$p \cdot y' < p \cdot y,$$

which is a contradiction. Now take any $y'' \in Y$. Since $y_n = y + (n, \cdots, n) \in S_y$,

$$p \cdot y_n \geq p \cdot y''.$$

By the limiting argument, we conclude that $p \cdot y \geq p \cdot y''$ for all $y'' \in Y$, that is, y is profit maximizing for p.

The restriction $p \geq 0$ in the last proposition cannot be strengthened to read $p >> 0$.