Microeconomic Theory: problems and solutions

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Contents

Chapter 1

Preferences and choices

Proposition 1 *Suppose that X is countable and $\succeq$ is a rational preference over X. Then there exists a utility function $U : X \to \mathbb{R}$ that represents $\succeq$.*

Proof: Let $X = \{x_1, x_2, \cdots\}$. Define

$$\delta_{ij} = \begin{cases} 1, & \text{if } x_i, x_j \in X \text{ and } x_i \succ x_j; \\ 0, & \text{otherwise.} \end{cases}$$

For each $x_i \in X$, define

$$U(x_i) = \sum_{j \geq 1} \frac{1}{2^j} \delta_{ij}.$$

Since $\sum_{j \geq 1} \frac{1}{2^j} < \infty$, U is well defined. We show that U is a utility function representing $\succeq$, that is, for any $x_m, x_k \in X$, $x_m \succeq x_k \Leftrightarrow U(x_m) \geq U(x_k)$. To see the implication "$\Rightarrow$", let $x_m, x_k \in X$ and $x_m \succeq x_k$. Define $A = \{j : x_k \succ x_j\}$. The rest of the proof of "$\Rightarrow$" is completed by considering the following two cases.

Case 1. $A = \emptyset$. Then $\delta_{kj} = 0$ for all $j \geq 1$ and so, $U(x_k) = 0$. Thus, $U(x_m) \geq U(x_k)$.

Case 2. $A \neq \emptyset$. Then

$$\delta_{kj} = \begin{cases} 1, & \text{if } j \in A; \\ 0, & \text{if } j \notin A. \end{cases}$$

So, $U(x_k) = \sum_{j \in A} \frac{1}{2^j}$. Since $x_m \succeq x_k$ and $x_k \succ x_j$ for all $j \in A$, applying the transitivity of $\succeq$, we can show that $x_m \succ x_j$ for all $j \in A$. So, $\delta_{mj} = 1$ for all $j \in A$. Hence,

$$U(x_m) \geq \sum_{j \in A} \frac{1}{2^j} = U(x_k).$$

To see the implication "$\Leftarrow$", let $x_m, x_k \in X$ and $U(x_m) \geq U(x_k)$. If $x_k \succ x_m$, then $\delta_{km} = 1$. So, applying an argument similar to that in the case of "$\Rightarrow$", we can show that

$$U(x_k) \geq \frac{1}{2^m} + U(x_m) > U(x_m),$$

which is a contradiction. By the completeness of $\succeq$, we have $x_m \succeq x_k$.

Exercise 1 *Show that a choice structure $(\mathscr{B}, C(.))$ for which a rationalizing preference relation exists satisfies the path-invariance property: For every pair $B_1, B_2 \in \mathscr{B}$ such that $B_1 \cup B_2 \in \mathscr{B}$ and $C(B_1) \cup C(B_2) \in \mathscr{B}$, we have $C(B_1 \cup B_2) = C(C(B_1) \cup C(B_2))$, that is, the decision problem can be safely sub-divided.*

Solution: Let $\succeq$ be a rational preference relation that rationalize $C(.)$ relative to $\mathscr{B}$ for the choice structure $(\mathscr{B}, C(.))$. Suppose that $x \in C(B_1 \cup B_2) = C^*(B_1 \cup B_2; \succeq)$. This implies that for all $y \in B_1 \cup B_2$, $x \succeq y$. Since $C(B_1) \subseteq B_1$ and $C(B_2) \subseteq B_2$, $C(B_1) \cup C(B_2) \subseteq B_1 \cup B_2$. So, $x \succeq y'$ for all $y' \in C(B_1) \cup C(B_2)$ and we have $x \in C^*(C(B_1) \cup C(B_2); \succeq) = C(C(B_1) \cup C(B_2))$. Thus, we have proved that $C(B_1 \cup B_2) \subseteq C(C(B_1) \cup C(B_2))$.

Consider any $x \in C(C(B_1) \cup C(B_2)) = C^*(C(B_1) \cup C(B_2); \succeq)$. This implies that for all $y \in C(B_1) \cup C(B_2)$, $x \succeq y$. Take any $y_1 \in C(B_1) = C^*(B_1; \succeq)$. Since $y_1 \in C^*(B_1; \succeq)$, $y_1 \succeq z_1$ for all $z_1 \in B_1$. Therefore, we get $x \succeq y_1 \succeq z_1$ for all $z_1 \in B_1$ and using transitivity of $\succeq$, it follows that $x \succeq z_1$ for all $z_1 \in B_1$. Similarly, take any $y_2 \in C(B_2) = C^*(B_2; \succeq)$. Since $y_2 \in C^*(B_2; \succeq)$, $y_2 \succeq z_2$ for all $z_2 \in B_2$. Therefore, we get $x \succeq y_2 \succeq z_2$ for all $z_2 \in B_2$ and using transitivity, we get $x \succeq z_2$ for all $z_2 \in B_2$. It follows that $x \succeq z$ for all $z \in B_1 \cup B_2$ implying that $x \in C^*(B_1 \cup B_2; \succeq) = C(B_1 \cup B_2)$. Hence, we have proved that $C(C(B_1) \cup C(B_2)) \subseteq C(B_1 \cup B_2)$.

Exercise 2 *Let $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{z, x\}\}$. Suppose that the choice structure is stochastic, that is, for every $B \in \mathscr{B}$, $C(B)$ is a frequency distribution over alternatives in B. For example, if $B = \{x, y\}$ then $C(B) = (C_x(B), C_y(B))$ is such that $C_x(B) \geq 0$,*

$C_y(B) \geq 0$ and $C_x(B) + C_y(B) = 1$. We say that the stochastic choice function can be **rationalized by preferences** *if we can find a probability distribution $Pr(\cdot)$ over the six possible strict preference relations on X such that for all $B \in \mathscr{B}$, $C(B)$ is precisely the frequency of choices induced by $Pr(\cdot)$. For example, if $B = \{x, y\}$, then $C_x(B) = Pr(\{\succ : x \succ y\})$.*

(A) *Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{2}, \frac{1}{2})$ can be rationalizes by preferences.*

(B) *Show that the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\frac{1}{4}, \frac{3}{4})$ cannot be rationalizes by preferences.*

(C) *Determine the $\alpha \in (0, 1)$ at which the stochastic choice function $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ switches from rationalizability to non-rationalizability.*

Solution: Let $\mathscr{P} = \{\succ^1, \cdots, \succ^6\}$ be the set of all possible strict preference orderings with the set of alternatives $X = \{x, y, z\}$. In particular, let

$$\succ^1 : x \succ y \succ z,\ \succ^2 : x \succ z \succ y,\ \succ^3 : y \succ x \succ z,$$

$$\succ^4 : y \succ z \succ x,\ \succ^5 : z \succ x \succ y \text{ and } \succ^6 : z \succ y \succ x.$$

Also let $Pr(\succ^k) = p_k$ for all $k = \{1, \cdots, 6\}$. Therefore, a probability distribution on $\mathscr{P}$ is a vector $p = (p_1, \cdots, p_6) \in [0, 1]^6$ such that $\sum_{k=1}^{6} \succ_k = 1$. To rationalize a stochastic choice structure, we need to find a probability distribution on $\mathscr{P}$ that rationalizes it. We start from part (C) of this question, that is we try to rationalize the stochastic choice structure $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$.

1. $C_x(\{x, y\}) = Pr(\succ : x \succ y) = p_1 + p_2 + p_5 = \alpha$,

2. $C_y(\{x, y\}) = Pr(\succ : y \succ x) = p_3 + p_4 + p_6 = 1 - \alpha$,

3. $C_y(\{y, z\}) = Pr(\succ : y \succ z) = p_1 + p_3 + p_4 = \alpha$,

4. $C_z(\{y, z\}) = Pr(\succ : z \succ y) = p_2 + p_5 + p_6 = 1 - \alpha$,

5. $C_z(\{z, x\}) = Pr(\succ : z \succ x) = p_4 + p_5 + p_6 = \alpha$, and

6. $C_x(\{z, x\}) = Pr(\succ : x \succ z) = p_1 + p_2 + p_3 = 1 - \alpha$.

From (1) and (6), (2) and (3), and (4) and (5) we get

$$p_5 - p_3 = p_1 - p_6 = p_4 - p_2 = 2\alpha - 1. \tag{1.1}$$

From (1.1), we get $[p_1 + p_4 + p_5] - [p_2 + p_3 + p_6] = 6\alpha - 3 \Rightarrow 2[p_1 + p_4 + p_5] = 6\alpha - 2 \Rightarrow p_1 + p_4 + p_5 = 3\alpha - 1$ and since $p_1 + p_4 + p_5 \in [0, 1]$ we get $3\alpha - 1 \in [0, 1] \Rightarrow \alpha \in [1/3, 2/3]$. Thus the stochastic choice structure $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ is rationalizable only if α lies in the interval $[1/3, 2/3]$.

Part (A) of this question is a special case of the stochastic choice structure $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ where $\alpha = 1/2 \in [1/3, 2/3]$ and is hence rationalizable. Finally part (B) of this question is a special case of the stochastic choice structure $C(\{x, y\}) = C(\{y, z\}) = C(\{z, x\}) = (\alpha, 1 - \alpha)$ where $\alpha = 1/4 \notin [1/3, 2/3]$ and is hence non-rationalizable.

Chapter 2

Production

Proposition 2 *Suppose that $c(\cdot,\cdot)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $Z(\cdot,\cdot)$ is the associated conditional factor demand correspondence.*

(i) If f is homogeneous of degree one then $c(w,\cdot)$ and $Z(w,\cdot)$ are homogeneous of degree one.

(ii) If f is concave, then $c(w,\cdot)$ is a convex function (in particular, marginal costs are non-decreasing in q).

Proof: (i) Let $\alpha > 0$ and $q \geq 0$. Define $Y(q) = \left\{z \in \mathbb{R}^L_+ : f(z) \geq q\right\}$. We first show that

$$\alpha Z(w,q) \subseteq Z(w,\alpha q).$$

Pick an element $z \in Z(w,q)$. Then $f(\alpha z) = \alpha f(z) \geq \alpha q$. For all $z' \in Y(\alpha q)$,

$$f(z') \geq \alpha q \Rightarrow f\left(\frac{1}{\alpha}z'\right) = \frac{1}{\alpha}f(z') \geq q.$$

Since $z \in Z(w,q)$ and $\frac{1}{\alpha}z' \in Y(q)$ for all $z' \in Y(\alpha q)$,

$$w \cdot \left(\frac{1}{\alpha}z'\right) \geq w \cdot z.$$

Hence, for all $z' \in Y(\alpha q)$, $w \cdot z' \geq w \cdot (\alpha z)$ implying that

$$\alpha z \in Z(w, \alpha q).$$

Next, we show that

$$Z(w, \alpha q) \subseteq \alpha Z(w, q).$$

Let $z \in Z(w, \alpha q)$. Then $f(z) \geq \alpha q$ and for all $z' \in Y(q)$, $f(z') \geq q$ implying that $f(\alpha z') \geq \alpha q$. Since $z \in Z(w, \alpha q)$, $w \cdot z \leq \alpha w \cdot z'$ for all $z' \in Y(q)$ implying $z \in \alpha Z(w, q)$. Thus, we conclude that $Z(w, \alpha q) = \alpha Z(w, q)$.

Note that $c(w, \alpha q) = w \cdot z$ for all $z \in Z(w, \alpha q)$. Take an element $z \in Z(w, \alpha q)$. Since $Z(w, \alpha q) = \alpha Z(w, q)$, $z = \alpha z'$ for some $z' \in Z(w, q)$. Thus,

$$c(w, \alpha q) = \alpha w \cdot z' = \alpha c(w, q).$$

(i) Let $q \geq 0$, $q' \geq 0$ and $\alpha \in [0, 1]$. Define $Y(q) = \left\{z \in \mathbb{R}^L_+ : f(z) \geq q\right\}$. Choose $z \in Y(q)$ and $z' \in Y(q')$. Since f is concave,

$$f(\alpha z + (1 - \alpha) z') \geq \alpha f(z) + (1 - \alpha) f(z') \geq \alpha q + (1 - \alpha) q'.$$

So, $\alpha z + (1 - \alpha) z' \in Y(\alpha q + (1 - \alpha) q')$. Hence,

$$c(w, \alpha q + (1 - \alpha) q') \leq w \cdot (\alpha z + (1 - \alpha) z') = \alpha w \cdot z + (1 - \alpha) w \cdot z',$$

which further implies

$$c(w, \alpha q + (1 - \alpha) q') \leq \alpha c(w, q) + (1 - \alpha) c(w, q').$$

In the proof of next result, we use the following fact: If A is a closed convex subset of $\mathbb{R}^L$ and $x \notin A$, then there exists some non-zero element $p \in \mathbb{R}^L$ such that $p \cdot x > \sup\{p \cdot y : y \in A\}$. This result is a consequence of the *separating hyperplane theorem*.

Proposition 3 Suppose that $\pi(\cdot)$ is the profit function of the production set Y. Assume that Y is closed, convex, and satisfies the free disposal property. Then

$$Y = \left\{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p >> 0\right\}.$$

Proof: From the definition of $\pi(\cdot)$, $y \in Y$ implies $p \cdot y \leq \pi(p)$ for all $p >> 0$. Thus,

$$Y \subseteq \left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\}.$$

Pick an element $y \in \mathbb{R}^L$ such that $p \cdot y \leq \pi(p)$ for all $p \gg 0$. We claim that $y \in Y$. Then there exists a non-zero element $p \in \mathbb{R}^L$ such that $p \cdot y > \pi(p)$. First, we show that $p \geq 0$. Assume that $p_\ell < 0$ for some $1 \leq \ell \leq L$. Since Y satisfies the free disposal condition,

$$Y - (0, \cdots, 0, n, 0, \cdots, 0) \subset Y$$

for all $n \geq 1$, where n is corresponding to the ℓ^{th}-coordinate of $(0, \cdots, 0, n, 0, \cdots, 0)$. Let $z \in Y$ and define $z_n = z - (0, \cdots, 0, n, 0, \cdots, 0)$ for all $n \geq 1$. Then the sequence $\{p \cdot z_n : n \geq 1\}$ is unbounded from above which is a contradiction with the fact that $\pi(p) < p \cdot y$. So, we conclude that $p \geq 0$. Let $\varepsilon = p \cdot y - \pi(p)$. Consider an element $p_0 \gg 0$ and then choose an $\frac{1}{2} < \alpha < 1$ such that

$$|\pi(p_0) - p_0 \cdot y| < \frac{\varepsilon}{2(1-\alpha)}.$$

Since $\pi(\cdot)$ is convex,

$$\begin{aligned}
\pi(\alpha p + (1-\alpha)p_0) &\leq \alpha\pi(p) + (1-\alpha)\pi(p_0) \\
&< \alpha(p \cdot y - \varepsilon) + (1-\alpha)\left(p_0 \cdot y + \frac{\varepsilon}{2(1-\alpha)}\right) \\
&= (\alpha p + (1-\alpha)p_0) \cdot y + \left(\frac{1}{2} - \alpha\right)\varepsilon \\
&< (\alpha p + (1-\alpha)p_0) \cdot y
\end{aligned}$$

Since $\alpha p + (1-\alpha)p_0 \gg 0$, we arrived at a contradiction. Thus, $y \in Y$ and hence,

$$\left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\} \subseteq Y.$$

So, we conclude that

$$Y = \left\{ y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0 \right\}.$$