Preferences and Choices

Anuj Bhowmik

Economic Research Unit
Indian Statistical Institute
203 Barackpore Trunk Road
Kolkata 700108
India

Email: anuj.bhowmik@isical.ac.in,
anujbhowmik09@gmail.com

Homepage: http://www.isical.ac.in/∼anuj.bhowmik/

Microeconomic Theory I
Semester I, 2013

Outline

Outline

Outline

Outline

Outline

Mas-Colell, A., Whinston, M. D. and Green, J. (1995). Microeconomic Theory

Introduction

In consumer behavior we deal with the individual's decision problem of making choices from a set of mutually exclusive alternatives X.

Example

[1] The set X can be something like a student's career decision problem, that is, $X = \{$go to a law school, go to a graduate school and study economics, go to the army,$\cdots$, be a pop star$\}$.

[2] The set X can also be a combo pack of fruits, that is, $X = \{$one orange and one banana, one orange and one apple,$\cdots$, two bananas, $\cdots\}$.

[3] $X = \{(\text{apple}, \text{money}) : \text{apple} \in \{1, 2, \cdots, 20\}, \text{money} \in [0, 5]\}$.

In consumer behavior we deal with the individual's decision problem of making choices from a set of mutually exclusive alternatives X.

Example

[1] The set X can be something like a student's career decision problem, that is, $X = \{$go to a law school, go to a graduate school and study economics, go to the army,$\cdots$, be a pop star$\}$.

[2] The set X can also be a combo pack of fruits, that is, $X = \{$one orange and one banana, one orange and one apple,$\cdots$, two bananas, $\cdots\}$.

[3] $X = \{(\text{apple}, \text{money}) : \text{apple} \in \{1, 2, \cdots, 20\}, \text{money} \in [0, 5]\}$.

In general, the set X can be many things and therefore we can write $X = \{x, y, \cdots\}$ where x, y etc. are elements in the set that can be scalar or vector and the set X can be both finite or infinite.

Given the set X, there are two approaches to modelling individual choice behavior.

1. The preference based approach
2. The choice based approach

In general, the set X can be many things and therefore we can write $X = \{x, y, \cdots\}$ where x, y etc. are elements in the set that can be scalar or vector and the set X can be both finite or infinite.

Given the set X, there are two approaches to modelling individual choice behavior.

1. The preference based approach
2. The choice based approach

In general, the set X can be many things and therefore we can write $X = \{x, y, \cdots\}$ where x, y etc. are elements in the set that can be scalar or vector and the set X can be both finite or infinite.

Given the set X, there are two approaches to modelling individual choice behavior.

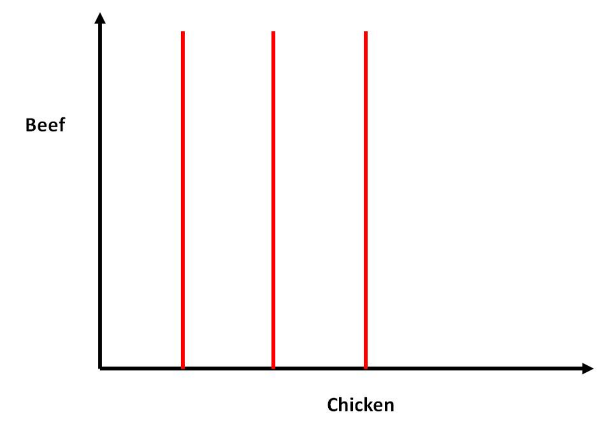
1. The preference based approach
2. The choice based approach

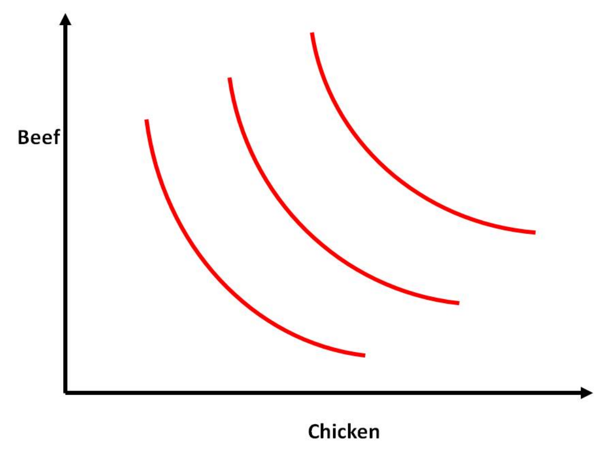
The preference based approach

In the preference based approach, the objective of the agent (individual or decision maker) are summarized by a *preference relation*, which is denoted by $\succeq$.

- Let $X = \{x, y, z, \cdots\}$ be the set of possible alternatives that are available to the agent.
- $\succeq$ is a binary relation on X allowing the comparison of pairs of alternatives $x, y \in X$.

Here $x \succeq y$ means that "x is at least as good as y".


Beef
Chicken


Beef
Chicken

Given $\succeq$, we can derive two other important binary relations on X:

1. The *strict preference* relation $\succ$ is defined as

$$x \succ y \Leftrightarrow x \succeq y \text{ and } y \not\succeq x.$$

   This is read as "x is preferred to y".

2. The *indifference* relation $\sim$ is defined as

$$x \sim y \Leftrightarrow x \succeq y \text{ and } y \succeq x.$$

   This is read as "x is indifferent to y".

Given $\succeq$, we can derive two other important binary relations on X:

1. The *strict preference* relation $\succ$ is defined as

$$x \succ y \Leftrightarrow x \succeq y \text{ and } y \not\succeq x.$$

This is read as "x is preferred to y".

2. The *indifference* relation $\sim$ is defined as

$$x \sim y \Leftrightarrow x \succeq y \text{ and } y \succeq x.$$

This is read as "x is indifferent to y".

Given $\succeq$, we can derive two other important binary relations on X:

1. The *strict preference* relation $\succ$ is defined as

$$x \succ y \Leftrightarrow x \succeq y \text{ and } y \not\succeq x.$$

This is read as "x is preferred to y".

2. The *indifference* relation $\sim$ is defined as

$$x \sim y \Leftrightarrow x \succeq y \text{ and } y \succeq x.$$

This is read as "x is indifferent to y".

Completeness

For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).

1. Completeness says that the agent has a well-defined preference between any pair of alternatives.
2. This might be a relatively strong assumption if we think about goods that we have not consumed in the past or goods that we have not even seen before.

The above completeness assumption is equivalent to “for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Completeness

For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).

1. Completeness says that the agent has a well-defined preference between any pair of alternatives.

2. This might be a relatively strong assumption if we think about goods that we have not consumed in the past or goods that we have not even seen before.

The above completeness assumption is equivalent to "for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Completeness

For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).

1. Completeness says that the agent has a well-defined preference between any pair of alternatives.
2. This might be a relatively strong assumption if we think about goods that we have not consumed in the past or goods that we have not even seen before.

The above completeness assumption is equivalent to "for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Completeness

For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).

1. Completeness says that the agent has a well-defined preference between any pair of alternatives.
2. This might be a relatively strong assumption if we think about goods that we have not consumed in the past or goods that we have not even seen before.

The above completeness assumption is equivalent to "for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Completeness

For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$ (or both).

1. Completeness says that the agent has a well-defined preference between any pair of alternatives.
2. This might be a relatively strong assumption if we think about goods that we have not consumed in the past or goods that we have not even seen before.

The above completeness assumption is equivalent to “for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Incomplete binary relations

[1] Define $\succeq \subseteq \mathbb{R}^2_+ \times \mathbb{R}^2_+$ by

$$(x, y) \succeq (x', y') \Leftrightarrow x \geq x' \text{ and } y \geq y'.$$

[2] The binary relation "is the brother of" does not satisfy completeness for all elements (persons) in the set of available alternatives (set X in this case could be a given group of people).

Transitivity

For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

1. Transitivity says that if x is at least as good as y and y is at least as good as z then x is at least as good as z.

2. The assumption of transitivity is often understood as that individual preferences should not cycle.

Suppose that Rahul prefers a mango to an orange, and he prefers an orange to a banana. However, he prefers a banana to a mango. Formally,

$$\text{mango} \succeq \text{orange and orange} \succeq \text{banana, but banana} \succeq \text{mango.}$$

Assume Rahul owns a banana and 10 rupees. Consider the the following deals.

1. Vivek offer him an orange for a banana and 1 rupee;
2. Vivek offer him a mango for an orange and 1 rupee; and
3. Vivek offer him a banana for a mango and 1 rupee.

Some consequences of transitivity

Let $x, y, z \in X$.

1. If $x \sim y$ and $y \sim z$, then $x \sim z$.
2. If $x \succeq y$ and $y \succ z$, then $x \succ z$.
3. If $x \succ y$ and $y \succeq z$, then $x \succ z$.

Some consequences of transitivity

Let $x, y, z \in X$.

1. If $x \sim y$ and $y \sim z$, then $x \sim z$.
2. If $x \succeq y$ and $y \succ z$, then $x \succ z$.
3. If $x \succ y$ and $y \succeq z$, then $x \succ z$.

Some consequences of transitivity

Let $x, y, z \in X$.

1. If $x \sim y$ and $y \sim z$, then $x \sim z$.
2. If $x \succeq y$ and $y \succ z$, then $x \succ z$.
3. If $x \succ y$ and $y \succeq z$, then $x \succ z$.

Intransitive preferences

Comparing elements that are too close to be distinguishable. When two alternatives are extremely similar we are often unable to state which of them we prefer.

You might not be able to distinguish 3 milligrams more of sugar in your coffee, but you can probably detect when 5 milligrams more of sugar in your coffee.

(i) A cup of coffee with amount of sugar is 6 milligrams.

(ii) A cup of coffee with amount of sugar is 3 milligrams.

(iii) A cup of coffee with amount of sugar is 1 milligrams.

Note that (i)$\sim$(ii) and (ii)$\sim$(iii), but (i)$\succ$(iii).

Intransitive preferences (continued)

Consider the situation where several individual preferences must be aggregated into only one.

For instance, Amit, Raju, and Sahil want to study M.S. in Economics in any one of the following departments: ERU, PU, and DSE. Let $\succeq_1$, $\succeq_2$, and $\succeq_3$ be the preference relations of Amit, Raju, and Sahil respectively.

(i) ERU$\succeq_1$PU$\succeq_1$DSE;

(ii) PU$\succeq_2$DSE$\succeq_2$ERU;

(iii) DSE$\succeq_3$ERU$\succeq_3$PU.

If they want to study together, then we must use the majority rule. Note that

$$\text{ERU} \succeq_{1,3} \text{PU} \succeq_{1,2} \text{DSE} \succeq_{2,3} \text{ERU}.$$

Thus, the aggregate preference relation is a cycle and so it violates transitivity.

Rational preference relations

A preference relation $\succeq$ is called *rational* if it is complete and transitive.

Suppose that $\succeq$ is rational. Then

(1)$\succeq$ is reflexive.

(2) $\succ$ is irreflexive, transitive, and not symmetric.

(3) $\sim$ is reflexive, symmetric, and transitive.

Proofs of (1)-(3)

(1) It is trivial.

(2) $\succ$ is irreflexive: For any $x \in X$, $x \succ x \Leftrightarrow [x \succeq x$ and $x \not\succeq x]$. Since $x \succeq x$ and $x \not\succeq x$ cannot hold simultaneously, $x \not\succ x$ for any $x \in X$.

$\succ$ is transitive: Suppose not. Then there exist $x, y, z \in X$ such that $x \succ y$, $y \succ z$ and $x \not\succ z$.

1. Since $\succeq$ is complete, $x \not\succ z$ is equivalent to $z \succeq x$.
2. $x \succ y \Leftrightarrow [x \succeq y$ and $y \not\succeq x]$
3. From the transitivity of $\succeq$, we get $z \succeq y$ which contradicts our assumption $y \succ z$ since $y \succ z \Rightarrow z \not\succeq y$.

$\succ$ is not symmetric: For any $x, y \in X$,

$$x \succ y \Leftrightarrow [x \succeq y \text{ and } y \not\succeq x] \text{ and } y \succ x \Leftrightarrow [y \succeq x \text{ and } x \not\succeq y].$$

So,

$$x \succ y \Rightarrow y \not\succeq x \Rightarrow y \not\succ x.$$

(3) It is trivial.

Utility functions

Preference relations of the individual is represented by a *utility function* (whenever possible).

A utility function $U : X \to \mathbb{R}$ assigns a numerical value to each element in X, ranking the elements of X in accordance with the individual's preferences.

Definition

A function $U : X \to \mathbb{R}$ is a utility function representing preference relation $\succeq$ on X if for all $x, y \in X$,

$$x \succeq y \Leftrightarrow U(x) \geq U(y).$$

Proposition

A preference relation $\succeq$ on X can be represented by a utility function U only if it is rational.

Proof: (i) Since U is a real-valued function defined on X, it must be that for any $x, y \in X$, either

$$U(x) \geq U(y) \text{ or, } U(y) \geq U(x).$$

Since U is a utility function representing $\succeq$, this implies either $x \succeq y$ or, $y \succeq x$. Hence, $\succeq$ is complete.

(ii) Suppose that $x \succeq y$ and $y \succeq z$. Since U represents $\succeq$, we have $U(x) \geq U(y)$ and $U(y) \geq U(z)$. So, $U(x) \geq U(z)$, which means $x \succeq z$. Thus, we have transitivity.

Proposition

Suppose that X is countable and $\succeq$ is a rational preference over X. Then there exists a utility function $U : X \to \mathbb{R}$ that represents $\succeq$.

Proof: Let $X = \{x_1, x_2, \cdots\}$. Define

$$\delta_{ij} = \begin{cases} 1, & \text{if } x_i, x_j \in X \text{ and } x_i \succ x_j; \\ 0, & \text{otherwise.} \end{cases}$$

For each $x_i \in X$, define

$$U(x_i) = \sum_{j \geq 1} \frac{1}{2^j} \delta_{ij}.$$

Since $\sum_{j\geq 1} \frac{1}{2^j} < \infty$, U is well defined. It is easy to see that U represents $\succeq$.

Theorem

Suppose that $\succeq$ is rational. The following are equivalent:

(1) U is a utility function representing $\succeq$.

(2) For all $x, y \in X$, $x \succ y \Leftrightarrow U(x) > U(y)$.

Proof: (1) $\Rightarrow$ (2): Let $x, y \in X$ and $x \succ y$. Now,

$$x \succ y \Leftrightarrow [x \succeq y \text{ and } y \not\succeq x],$$

$$x \succeq y \Leftrightarrow U(x) \geq U(y), \text{ and}$$

$$y \not\succeq x \Leftrightarrow U(y) < U(x).$$

So,

$$x \succ y \Rightarrow y \not\succeq x \Rightarrow U(x) > U(y),$$

and

$$U(x) > U(y) \Rightarrow [x \succeq y \text{ and } y \not\succeq x] \Rightarrow x \succ y.$$

Thus, $x \succ y \Leftrightarrow U(x) > U(y)$.

(2) $\Rightarrow$ (1): Let $x, y \in X$. Then $x \succeq y \Rightarrow y \not\succ x \Rightarrow U(x) \geq U(y)$. Conversely, let $U(x) \geq U(y)$.

(i) $U(x) > U(y) \Rightarrow x \succ y \Rightarrow x \succeq y$.

(ii) $U(x) = U(y) \Rightarrow [U(x) \not> U(y) \text{ and } U(y) \not> U(x)] \Rightarrow [x \not\succ y \text{ and } y \not\succ x] \Rightarrow x \sim y \Rightarrow x \succeq y$.

So, $x \succeq y \Leftrightarrow U(x) \geq U(y)$.

If U is a utility function representing $\succeq$, then for all $x, y \in X$,

$$x \sim y \Leftrightarrow U(x) = U(y).$$

Exercise

Consider a rational preference relation $\succeq$ over X. Show that if $U(x) = U(y) \Rightarrow x \sim y$ and $U(x) > U(y) \Rightarrow x \succ y$, then U is a utility function representing $\succeq$.

Solution: (i) $U(x) \geq U(y) \Rightarrow [x \sim y$ or, $x \succ y] \Rightarrow x \succeq y$.

(ii) It remains to show that $x \succeq y \Rightarrow U(x) \geq U(y)$. Assume not. Then there exists $x, y \in X$ such that

$$x \succeq y \text{ and } U(x) < U(y).$$

By definition, $U(x) < U(y) \Rightarrow y \succ x$, which is a contradiction since $y \succ x \Rightarrow x \not\succeq y$. Hence $x \succeq y \Rightarrow U(x) \geq U(y)$.

A utility function U that represents a preference relation $\succeq$ over X is not unique. Hence, it is only the ranking that matters.

Exercise

Show that if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $U : X \to \mathbb{R}$ is a utility function representing the preference relation $\succeq$ over X, then $V : X \to \mathbb{R}$ defined by $V(x) = f(U(x))$ for all $x \in X$ is also a utility function representing the same $\succeq$ over X.

Solution: Since U is a utility function representing $\succeq$ over X then for all $x, y \in X$,

$$x \succeq y \Leftrightarrow U(x) \geq U(y).$$

Since f is a strictly increasing function and $V(x) = f(U(x))$, one has for all $x, y \in X$,

$$U(x) \geq U(y) \Leftrightarrow f(U(x)) \geq f(U(y)).$$

Thus,

$$x \succeq y \Leftrightarrow V(x) \geq V(y),$$

which means that $V(\cdot)$ is also a valid utility function representing the same $\succeq$ over X.

Question

Does a similar result true if one take a monotonically increasing function f?

Properties of utility functions that are invariant for any strictly increasing transformation are called *ordinal* properties.

Cardinal properties are those not preserved under all such transformations.

Thus, the preference relation $\succeq$ associated with a utility function is an ordinal property. On the other hand, the numerical values associated with the alternatives in X and hence the magnitude of any differences in the utility measure between alternatives, are cardinal properties.

The choice based approach

In the choice based approach, we focus on the actual choices made by the individual, rather than on the process of introspection by which the individual discovers his own preferences by systematically comparing different alternatives. In the choice based approach, we use the so-called "choice structure".

The *choice structure* is a pair $(\mathscr{B}, C(\cdot))$ where $\mathscr{B}$ is a family of subsets of X and $C(\cdot)$ is a choice rule , that is, $C(B)$ is a non-empty subset of B for all $B \in \mathscr{B}$.

Example

[1] In consumer theory, each element $B \in \mathscr{B}$ can be understood as a particular set of all the affordable bundles for a consumer, given his wealth and the market prices. In this case, B is known as the consumer's *budget set*.

$c(B)$ would be the bundle/s that the individual chooses to buy among in the budget set B.

[2] B as a particular list of all the schools and colleges where you were admitted, among all schools and colleges in the scope of your imagination X.

$c(B)$ would contain the schools and colleges that you choose to attend.

Example

Consider $X = \{x, y, z\}$ and $\mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$. Choice structures can be of the following type.

(c_1) Choice structure is $(\mathscr{B}, C_1(\cdot))$, where $C_1(\{x, y\}) = \{x\}$, $C_1(\{x, y, z\}) = \{x\}$.

(c_2) Choice structure is $(\mathscr{B}, C_2(\cdot))$, where $C_2(\{x, y\}) = \{x\}$, $C_2(\{x, y, z\}) = \{x, y\}$.

The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the *weak axiom of revealed preference* (or, WARP) if the following property holds:

If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$ we must have $x \in C(B')$.

Revealed preference relation

Given a choice structure $(\mathscr{B}, C(.))$, the *revealed preference relation* $\succeq^*$ is defined by

$$x \succeq^* y \Leftrightarrow \exists\ B \in \mathscr{B} \text{ such that } x, y \in B \text{ and } x \in C(B).$$

We read $x \succeq^* y$ as "*x is revealed at least as good as y*".

Note that $\succeq^*$ need not be complete or transitive. In particular, for any $x, y \in X$ to be comparable it is necessary that for some $B \in \mathscr{B}$ we have $x, y \in B$ and either $x \in C(B)$ or, $y \in C(B)$ or, both.

In particular, $C_1(\{x,y\}) = \{x\}$ and $C_1(\{x,y,z\}) = \{x\}$ means that $x \succeq^* y$ and $x \succeq^* z$. No revealed preference relation can be inferred between y and z.

Definition

Given a choice structure $(\mathscr{B}, C(\cdot))$, the *strict revealed preference relation* $\succ^*$ is defined by

$$x \succ^* y \Leftrightarrow \exists\, B \in \mathscr{B} \text{ such that } x, y \in B, x \in C(B) \text{ and } y \notin C(B).$$

We read $x \succ^* y$ as "*x is revealed preferred to y*".

$\succ^*$ is not necessarily transitive. Indeed, consider $X = \{x,y,z\}$ and the choice structure $(\mathscr{B}, C(\cdot))$ such that

$\mathscr{B} = \{\{x, y\}, \{y, z\}\}, C(\{x, y\}) = \{x\}$, and $C(\{y, z\}) = \{y\}$.

Then $x \succ^* y$ and $y \succ^* z$. However, no revealed preference relation can be established between x and z since there does not exist $B \in \mathscr{B}$ such that $x, z \in B$.

Proposition

The following are equivalent:

1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies WARP.
2. If $x \succeq^* y$ then $y \not\succ^* x$ for $x, y \in X$.
3. For $B, B' \in \mathscr{B}$, $x, y \in B$ and $x, y \in B'$, if $x \in C(B)$ and $y \in C(B')$, we have $\{x, y\} \subseteq C(B)$ and $\{x, y\} \subseteq C(B')$.

Exercise

If $(\mathscr{B}, C(\cdot))$ satisfies WARP and $\mathscr{B}$ includes all three element subsets of X, then $\succ^*$ is transitive.

Solution: Assume $x \succ^* y$ and $y \succ^* z$. Let $B = \{x, y, z\} \in \mathscr{B}$. Note that $y \notin C(B)$ (if $y \in C(B)$ then $y \succeq^* x$ and this together with $x \succ^* y$ violate WARP). Similarly, $z \notin C(B)$. Since $C(B) \neq \emptyset$, we have $C(B) = \{x\}$. Thus, $x, z \in B$, $x \in C(B)$ and $z \notin C(B)$ implying that $x \succ^* z$.

Example

(c_1) The choice structure $(\mathscr{B}, C_1(\cdot))$ satisfies WARP.

(c_2) The choice structure $(\mathscr{B}, C_2(\cdot))$ fails to satisfy WARP.

Exercise

Let $\mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$ and $C(\{x, y\}) = \{x\}$. Show that if $(\mathscr{B}, C(\cdot))$ satisfies WARP then $C(\{x, y, z\}) = \{x\}$ or, $= \{z\}$ or, $= \{x, z\}$.

Solution: To complete the result, it is enough to show that $y \notin C(\{x, y, z\})$. Since $x \in C(\{x, y\})$ and $y \notin C(\{x, y\})$, we get $x \succ^* y$. If $y \in C(\{x, y, z\})$, then $y \succeq^* x$ and thus, WARP not holds, a contradiction. Hence, $y \notin C(\{x, y, z\})$.

Exercise

Suppose that $(\mathscr{B}, C(\cdot))$ satisfies WARP. Show that $\succ^*$ and $\succ^{**}$ are identical, that is, for any $x, y \in X$, $x \succ^* y \Leftrightarrow x \succ^{**} y$, where $x \succ^{**} y \Leftrightarrow [x \succeq^* y \text{ and } y \not\succeq^* x]$.

Solution: We first prove that $x \succ^{**} y \Rightarrow x \succ^{*} y$. Note that $x \succ^{**} y \Leftrightarrow [x \succeq^{*} y \text{ and } y \not\succeq^{*} x]$.

(**i**) $x \succeq^{*} y \Rightarrow \exists\, B \in \mathscr{B}$ such that $x, y \in B$ and $x \in C(B)$.

(**ii**) $y \not\succeq^{*} x \Rightarrow$ for all $B' \in \mathscr{B}$ such that $x, y \in B'$, we have $y \notin C(B')$.

(**i**) and (**ii**)$\Rightarrow \exists\, B \in \mathscr{B}$ such that $x, y \in B$, $x \in C(B)$, and $y \notin C(B)$. Hence, $x \succ^{*} y$.

We now prove that $x \succ^{*} y \Rightarrow x \succ^{**} y$. Obviously, $x \succ^{*} y \Rightarrow x \succeq^{*} y$. Suppose that $y \succeq^{*} x$. Since $(\mathscr{B}, C(\cdot))$ satisfies WARP, $x \not\succ^{*} y$ which is a contradiction. Thus, $y \not\succeq^{*} x$. Hence, $[x \succeq^{*} y \text{ and } y \not\succeq^{*} x] \Rightarrow x \succ^{**} y$.

In this section, we try to answer the following questions.

(Q1) If $\succeq$ on X is rational, is it true that the choice structure $(\mathscr{B}, C(\cdot))$ satisfies WARP?

(Q2) If the choice structure $(\mathscr{B}, C(\cdot))$ satisfies WARP, is it true that there exists a rational preference relation that is consistent with these choices?

For each $B \subseteq X$, we define the most preferred set of the agent as

$$C^*(B; \succeq) = \{x \in B : x \succeq y \text{ for all } y \in B\}.$$

In principle, we could have $C^*(B; \succeq) = \emptyset$ for some B.

Exercise

If X is finite, then any rational preference relation $\succeq$ on X generates a non-empty choice rule, that is, $C^*(B; \succeq) \neq \emptyset$ for all non-empty $B \subseteq X$.

Solution: Consider any $B = \{x_1, \cdots, x_n\} \subseteq X$.

Case 1. $x_1 \sim \cdots \sim x_m$.

So, $C^*(B; \succeq) = B$ and we are done.

Case 2. There are x_i, x_j such that $x_i \succ x_j$.

(i) If $x_1 \in C^*(B; \succeq)$, then we are done.

(ii) If $x_1 \notin C^*(B; \succeq)$, then there exists some element in B that is strictly preferred to x_1. Without loss of generality, assume that element to be x_2. Since $\succ$ is irreflexive, we have $x_1 \neq x_2$. If $x_2 \in C^*(B;\succeq)$, then again we are done.

(iii) If $x_2 \notin C^*(B;\succeq)$, then $\exists\ x_3$ (wlog) such that $x_3 \succ x_2$. By transitivity of $\succ$, $x_3 \succ x_1$. Moreover, $x_3 \neq x_1, x_2$. If $x_3 \in C^*(B;\succeq)$ then we are done.

Thus, in each stage of dominance either we are done (if $x_k \in C^*(B;\succeq)$) or, we go to the next stage (if $x_k \notin C^*(B;\succeq)$). If somehow we reach the stage where $x_{n-1} \notin C^*(B;\succeq)$ (given $x_k \notin C^*(B;\succeq)$ for all $k \in \{1, \cdots, n-2\}$) then we already have

$$x_n \succ x_{n-1} \succ \cdots \succ x_2 \succ x_1$$

implying $x_n \in C^*(B;\succeq)$.

Proposition

Suppose $\succeq$ over X is rational. Then the choice structure generated by $\succeq$, that is, $(\mathscr{B}, C^*(., \succeq))$, satisfies WARP.

Solution. Suppose that for some $B \in \mathscr{B}$, we have $x, y \in B$ and $x \in C^*(B; \succeq)$. By definition of $C^*(B; \succeq)$, this implies that $x \succeq y$. To check WARP, suppose that for some $B' \in \mathscr{B}$ with $x, y \in B'$, we have $y \in C^*(B'; \succeq)$. So, $y \succeq z$ for all $z \in B'$. But we already know that $x \succeq y$ and hence by transitivity, we have $x \succeq z$ for all $z \in B'$ implying that $x \in C^*(B'; \succeq)$. Hence, we have WARP.

From the above proposition, it follows that the answer to the question (Q1) is *yes*.

Definition

Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference $\succeq$ *rationalizes* $C(\cdot)$ relative to $\mathscr{B}$ if $C(B) = C^*(B; \succeq)$ for all $B \in \mathscr{B}$, that is, $\succeq$ generates the choice structure $(\mathscr{B}, C^*(.; .))$.

WARP is not sufficient to ensure the existence of a rationalizing preference relation. This is explained in the next example.

Example

Consider $X = \{x, y, z\}$ and let $(\mathscr{B}, C_3(\cdot))$ be such that

1. $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$;
2. $C_3(\{x, y\}) = \{x\}$, $C_3(\{y, z\}) = \{y\}$ and $C_3(\{x, z\}) = \{z\}$.

Example

Here $(\mathscr{B}, C_3(\cdot))$ satisfies WARP. Note that

$$C^*(\{x, y\}; \succeq) = \{x\} \Rightarrow x \succeq y \text{ and } y \not\succeq x \Rightarrow x \succ y,$$

since $y \succeq x$ together with $y \succeq y$ imply $y \in C^*(\{x, y\}; \succeq)$ which is a contradiction.

Similarly,

$$C^*(\{y, z\}; R) = \{y\} \Rightarrow y \succ z$$

and

$$C^*(\{x, z\}; \succeq) = \{z\} \Rightarrow z \succ x.$$

By the transitivity of $\succ$, $x \succ y$ and $y \succ z \Rightarrow x \succ z$ which is a contradiction.

Proposition

If $(\mathscr{B}, C(\cdot))$ is a choice structure such that

(i) WARP is satisfied and

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements,

then there is a rational preference relation $\succeq$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$.

Furthermore, this rational preference relation is the only preference relation that does so.

Proof: We show that the revealed at least as good as relation $\succeq^*$ rationalizes $C(\cdot)$.

To see that $\succeq^*$ is complete, let $x, y \in X$. By assumption (ii), $\{x, y\} \in \mathscr{B}$. Since either x or y must belong to $C(\{x, y\})$, either $x \succeq^* y$ or $y \succeq^* x$ or both implying that $\succeq^*$ is complete.

To see that $\succeq^*$ is transitive, let $x, y, z \in X$, $x \succeq^* y$ and $y \succeq^* z$. Then $\{x, y, z\} \in \mathscr{B}$. We show that $x \in C(\{x, y, z\})$.

(a) If $y \in C(\{x, y, z\})$ then given $x \succeq^* y$, WARP implies that $x \in C(\{x, y, z\})$.

(b) If $z \in C(\{x, y, z\})$ then given $y \succeq^* z$, WARP implies that $y \in C(\{x, y, z\})$ and then by (a), $x \in C(\{x, y, z\})$.

(c) Finally if $y \notin C(\{x, y, z\})$ and $z \notin C(\{x, y, z\})$ then $x \in C(\{x, y, z\})$ since $C(\{x, y, z\}) \neq \emptyset$.

Now, we show that $C(B) = C^*(B; \succeq^*)$ for all $B \in \mathscr{B}$.

To see that $C(B) \subseteq C^*(B; \succeq^*)$ for all $B \in \mathscr{B}$, consider any $B \in \mathscr{B}$ and $x \in C(B)$. Then $x \succeq^* y$ for all $y \in B$ which implies that $x \in C^*(B; \succeq^*)$.

To see that $C^*(B; \succeq^*) \subseteq C(B)$ for all $B \in \mathscr{B}$. If $x \in C^*(B; \succeq)$ then $x \succeq^* y$ for all $y \in B$. Thus, for each $y \in B$, there exists $B_y \in \mathcal{B}$ such that $x, y \in B_y$ and $x \in C(B_y)$. Since $C(B) \neq \emptyset$, by WARP, $x \in C(B)$.

Finally, to establish uniqueness simply note that because $\mathscr{B}$ includes all two element subsets of X, the choice behavior in $C(\cdot)$ completely determines the pairwise preference relations over X of any rationalizing preference.

Mathematical preliminaries

A *binary relation* or simply a *relation* R from a non-empty set A into a non-empty set B is a non-empty subset of $A \times B$. If $A = B$, then R is termed as a binary relation on A.

Example

[1] Let $A = \{1, 3, 7\}$ and $B = \{3, 5\}$. Define $R_1 \subseteq A \times B$ by

$$R_1 = \{(a, b) \in A \times B : a < b\} = \{(1, 3), (1, 5), (3, 5)\}.$$

[2] Let A be the set of all straight lines. Consider $R_2 \subseteq A \times A$ defined by

$$R_2 = \{(\ell_1, \ell_2) : \ell_1 \text{ and } \ell_2 \text{ are parallel}\}.$$

Mathematical preliminaries

A *binary relation* or simply a *relation* R from a non-empty set A into a non-empty set B is a non-empty subset of $A \times B$. If $A = B$, then R is termed as a binary relation on A.

Example

[1] Let $A = \{1, 3, 7\}$ and $B = \{3, 5\}$. Define $R_1 \subseteq A \times B$ by

$$R_1 = \{(a, b) \in A \times B : a < b\} = \{(1, 3), (1, 5), (3, 5)\}.$$

[2] Let A be the set of all straight lines. Consider $R_2 \subseteq A \times A$ defined by

$$R_2 = \{(\ell_1, \ell_2) : \ell_1 \text{ and } \ell_2 \text{ are parallel}\}.$$

Let R be a binary relation on a set A. If $(a, b) \in R$, we write aRb and we say that a is related to b with respect to R.

Definition

Let R be a binary relation on a set A. The R is said to be

Let R be a binary relation on a set A. If $(a, b) \in R$, we write aRb and we say that a is related to b with respect to R.

Definition

Let R be a binary relation on a set A. The R is said to be

1. reflexive if for all $a \in A$, aRa,
2. symmetric if for all $a, b \in A$, aRb implies bRa,
3. transitive if for all $a, b, c \in A$, aRb and bRc imply aRc.

Let R be a binary relation on a set A. If $(a, b) \in R$, we write aRb and we say that a is related to b with respect to R.

Definition

Let R be a binary relation on a set A. The R is said to be

1. reflexive if for all $a \in A$, aRa,
2. symmetric if for all $a, b \in A$, aRb implies bRa,
3. transitive if for all $a, b, c \in A$, aRb and bRc imply aRc.

Let R be a binary relation on a set A. If $(a, b) \in R$, we write aRb and we say that a is related to b with respect to R.

Definition

Let R be a binary relation on a set A. The R is said to be

1. reflexive if for all $a \in A$, aRa,
2. symmetric if for all $a, b \in A$, aRb implies bRa,
3. transitive if for all $a, b, c \in A$, aRb and bRc imply aRc.

Let R be a binary relation on a set A. If $(a, b) \in R$, we write aRb and we say that a is related to b with respect to R.

Definition

Let R be a binary relation on a set A. The R is said to be

1. reflexive if for all $a \in A$, aRa,
2. symmetric if for all $a, b \in A$, aRb implies bRa,
3. transitive if for all $a, b, c \in A$, aRb and bRc imply aRc.

Comparison test

Suppose that $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ are two series of positive real numbers and there is a natural number N such that $u_n \leq kv_n$ for all $n \geq N$, k being a fixed positive real number. Then

1. $\sum_{n\geq 1} u_n$ is convergent if $\sum_{n\geq 1} v_n$ is convergent;
2. $\sum_{n\geq 1} v_n$ is divergent if $\sum_{n\geq 1} u_n$ is divergent.

A function $f : X(\subseteq \mathbb{R}) \to \mathbb{R}$ is called *monotonically increasing* if for any $x, y \in X$ with $y > x$, one has $f(y) \geq f(x)$.

A function $f : X(\subseteq \mathbb{R}) \to \mathbb{R}$ is called *strictly increasing* if for any $x, y \in X$ with $y > x$, one has $f(y) > f(x)$.