Choice under Uncertainty

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Microeconomic Theory II
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Outline

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Maschler, M., Solan, E. and Zamir, S. (2013), Game Theory

Mas-Colell, A., Whinston, M. D. and Green, J. (1995), Microeconomic Theory

Certain outcomes

Let $\mathcal{O}$ be a finite set of given certain alternatives/outcomes. To analyze the behaviour agents/players, we need to know the preferences of each agent/player over the set of alternatives/ outcomes.

This means that for every pair of outcomes x and y, we need to know for each player whether he prefers x to y, whether he prefers y to x, or whether he is indifferent between them. The preferences of each player is captured by a binary relation, denoted by $\succeq$, is termed *preference relation*.

Given $x, y \in \mathcal{O}$, $x \succeq y$ means that "x is at least as good as y".

Given $\succeq$, we can derive two other important binary relations on $\mathcal{O}$:

1. The *strict preference* relation $\succ$ is defined as

$$x \succ y \Leftrightarrow x \succeq y \text{ and } y \not\succeq x.$$

   This is read as "x is preferred to y".

2. The *indifference* relation $\sim$ is defined as

$$x \sim y \Leftrightarrow x \succeq y \text{ and } y \succeq x.$$

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Completeness

For all $x, y \in \mathcal{O}$, either $x \succeq y$ or $y \succeq x$ (or both).

Completeness says that the agent has a well-defined preference between any pair of alternatives.

The above completeness assumption is equivalent to "for any $x, y \in X$, one and only one of the following hold: $x \succ y$, $y \succ x$ and $x \sim y$.

Transitivity

For all $x, y, z \in \mathcal{O}$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

1. Transitivity says that if x is at least as good as y and y is at least as good as z then x is at least as good as z.
2. The assumption of transitivity is often understood as that individual preferences should not cycle.

Some consequences of transitivity

Let $x, y, z \in X$.

1. If $x \sim y$ and $y \sim z$, then $x \sim z$.
2. If $x \succeq y$ and $y \succ z$, then $x \succ z$.
3. If $x \succ y$ and $y \succeq z$, then $x \succ z$.

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Rational preference relations

A preference relation $\succeq$ is called *rational* if it is complete and transitive.

Suppose that $\succeq$ is rational. Then

(1)$\succeq$ is reflexive.

(2) $\succ$ is irreflexive, transitive, and not symmetric.

(3) $\sim$ is reflexive, symmetric, and transitive.

Utility functions

A utility function $U : \mathcal{O} \to \mathbb{R}$ assigns a numerical value to each element in X, ranking the elements of X in accordance with the individual's preferences.

Definition

A function $U : X \to \mathbb{R}$ is a utility function representing preference relation $\succeq$ on X if for all $x, y \in X$,

$$x \succeq y \Leftrightarrow U(x) \geq U(y).$$

A preference relation $\succeq$ over $\mathcal{O}$ can be represented by a utility function U if and only if it is rational.

Uncertain outcomes

In reality, many important economic decisions involve an element of risk. Let us imagine that an agent faces a choice among a number of risky alternatives. Each risky alternative may result in one of a number of possible outcomes, but which outcome will actually occur is uncertain to the agent at the time of making a choice.

Example

Consider the situation involving one player who has two possible moves, A and B. The outcome is the amount of INR that player receives.

1. If she chooses A, she receives INR $7,000$.
2. If she chooses B, she receives a result of a lottery that grants a payoff of INR 0 or INR 20, 000 with equal probabilities, denoted by

$$\left[\frac{1}{2}(0), \frac{1}{2}(20,000)\right].$$

Which moves can we expect a player receives? The answer depend on the player's attitude to risk.

Lotteries

Let $\mathcal{O} = \{A_1, \cdots, A_K\}$ be the set of outcomes. A *lottery* L in which the outcome A_k has probability p_k (where $p_1, \cdots, p_K$ are non-negative real numbers summing to 1) is denoted by

$$L = [p_1(A_1), \cdots, p_K(A_K)].$$

The set of lotteries over $\mathcal{O}$ is denoted by $\mathscr{L}(\mathcal{O})$ or simply, $\mathscr{L}$ when the set of outcomes $\mathcal{O}$ is understood. In the previous example, $A_1 = \mathrm{INR}0$, $A_2 = \mathrm{INR}7,000$, and $A_3 = \mathrm{INR}20,000$.

Since each A_k is equivalence to the lottery

$$L = [0(A_1), \cdots, 1(A_k), \cdots 0(A_K)],$$

we can write $\mathcal{O} \subseteq \mathscr{L}$.

Preferences over lotteries

Let $\succeq$ be a preference relation over $\mathscr{L}$. A real-valued function $U : L \to \mathbb{R}$ is called a *utility function representing* $\succeq$ if

$$L_1 \succeq L_2 \Leftrightarrow U(L_1) \geq U(L_2) \text{ for all } L_1, L_2 \in \mathscr{L}.$$

A utility function $U : L \to \mathbb{R}$ is called *linear* if for every lottery $L = [p_1(A_1), \cdots, p_K(A_K)]$, it satisfies

$$U(L) = p_1 U(A_1) + \cdots + p_K U(A_K).$$

A linear utility function is called a *von Neumann-Morgenstern utility function*.

Utility representation

Which preference relation can be represented by a linear utility function?

First of all, since $\geq$ is rational, it cannot possibly represent a preference relation $\succeq$ which is not rational.

However, rationality is not sufficient for the existence of a linear utility function. For example, take

$$\mathcal{O} = \{(x, y) : x, y \in Z_{+}, x, y \leq 3\}$$

and the lexicographic preference relation over $\mathscr{L}$.

Compound lottery

A *compound lottery* is a lottery of lotteries. It can be expressed as

$$\hat{L} = [q_1(L_1), \cdots, q_J(L_J)],$$

where $L_j \in \mathscr{L}$ for all $1 \leq j \leq J$ and $q_1, \cdots, q_J$ are non-negative numbers such that $\sum_{j=1}^{J} q_j = 1$.

Compound lotteries naturally arise in many situations.

For example, consider an individual who chooses his route to work based on the weather: on rainy days he travels by Route 1, and on sunny days he travels by Route 2.

Travel time along each route is inconstant, because it depends on many factors (beyond the weather).

We are dealing with a “travel time to work” random variable, whose value depends on a lottery of lotteries.

(**i**) There is some probability that tomorrow morning will be rainy, in which case travel time will be determined by a probability distribution depending on the factors affecting travel along Route 1.

(**ii**) There is a complementary probability that tomorrow morning will be sunny, so that travel time will be determined by a probability distribution depending on the factors affecting travel along Route 2.

We will show that under standard assumptions there is no need to consider lotteries that are more compound than compound lotteries, namely, lotteries of compound lotteries. All our our analysis can be conducted by limiting consideration to only one level of compounding.

To distinguish between two types of lotteries, we call the lotteries in $\mathscr{L}$ *simple lotteries*. The set of compound lotteries is denoted by $\hat{\mathscr{L}}(\mathcal{O})$ or simply, $\hat{\mathscr{L}}$.

Every simple lottery L can be identified with the compound lottery $\hat{L}$ that yields the simple lottery L with probability 1, that is, $\hat{L} = [1(L)]$.

Each outcome A_k can be identified with the simple lottery

$$L = [0(A_1), \cdots, 0(A_{k-1}), 1(A_k), 0(A_{k+1}), \cdots, 0(A_K)]$$

and thus, it also can be identified with the compound lottery $[1(L)]$, in which L is the simple lottery given above.

We now assume that the preference relation $\succeq$ is defined over $\hat{\mathscr{L}}$. A function $U : \hat{\mathscr{L}} \to \mathbb{R}$ is a *utility function representing* $\succeq$ if

$$\hat{L}_1 \succeq \hat{L}_2 \Leftrightarrow U(\hat{L}_1) \geq U(\hat{L}_2) \text{ for all } \hat{L}_1, \hat{L}_2 \in \hat{\mathscr{L}}.$$

Given the identification of outcomes with simple lotteries, $U(A_k)$ and $U(L)$ denote the utility of compound lotteries corresponding to the outcome A_k and the simple lottery L, respectively.

Continuity

Every player will prefer receiving $\mathrm{INR}300$ to $\mathrm{INR}100$, and prefer receiving $\mathrm{INR}100$ to $\mathrm{INR}0$, that is,

$$\mathrm{INR}300 \succ \mathrm{INR}100 \succ \mathrm{INR}0.$$

It is also a reasonable assumption that a player will prefer receiving $\mathrm{INR}300$ with probability 0.9999 (and $\mathrm{INR}0$ with probability 0.0001) to receiving $\mathrm{INR}100$ with probability 1.

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Continuity cont.

Formally,

$$[0.9999(300), 0.0001(0)] \succ 100 \succ [0.0001(300), 0.9999(0)].$$

The higher the probability of receiving $\mathrm{INR}300$ (that is, the lower the probability of receiving $\mathrm{INR}0$), the more the lottery will be preferred.

By continuity, it is reasonable to suppose that there will be a particular probability p at which the player will be indifferent between receiving $\mathrm{INR}100$ and a lottery granting $\mathrm{INR}300$ with probability p and $\mathrm{INR}0$ with probability $1 - p$:

$$100 \sim [p(300), (1 - p)(0)].$$

Continuity cont.

The value of p will vary depending on the player. Next, we define formally the continuity axiom.

($\mathbf{A}_1$) For every triple of outcomes $A \succeq B \succeq C$, there exists a number $\theta \in [0, 1]$ such that

$$B \sim [\theta(A), (1 - \theta)(C)].$$

Consider a game with three possible outcomes:$c_1 =$ win, $c_2 =$ draw and $c_3 =$ lose.

Thus the consequence vector of this game is $c = (c_1, c_2, c_3)$. Different strategies of playing the game leads to different probability distribution over these outcomes. So strategies can be treated as simple lotteries $L = (p_1, p_2, p_3)$ such that $p_n \geq 0$ for all $n \in \{1, 2, 3\}$ and $p_1 + p_2 + p_3 = 1$.

Suppose that the preference relation over these lotteries is as follows:

$$L \succeq L' \text{ if either } [p_3 < p_3'] \text{ or } [p_3 = p_3' \text{ and } p_2 \leq p_2'].$$

Verify that this preference relation violates continuity.

Monotonicity

Every reasonable player will prefer to increase his probability of receiving a more-preferred outcome and lower the probability of receiving a less-preferred outcome. This natural property is captured in the next axiom.

(**A**$_2$) Let α, β be real numbers in $[0, 1]$, and suppose that $A \succ B$. Then

$$[\alpha(A), (1-\alpha)(B)] \succeq [\beta(A), (1-\beta)(B)]$$

if and only if $\alpha \geq \beta$.

Consequences of monotonicity

Theorem

If a preference relation $\succeq$ satisfies ($\mathbf{A}_1$)-($\mathbf{A}_2$), and if $A \succeq B \succeq C$, and $A \succ C$, then the value of θ defined in ($\mathbf{A}_1$) is unique.

Corollary

Suppose that $A_K \succeq \cdots \succeq A_1$. If a preference relation $\succeq$ over $\hat{\mathscr{L}}$ satisfies ($\mathbf{A}_1$)-($\mathbf{A}_2$), and if $A_K \succ A_1$, then for each $1 \leq k \leq K$, there exists a unique θ_k such that

$$A_k \sim [\theta_k(A_K), (1-\theta_k)(A_1)].$$

Simplification of lotteries

The following axiom states that the only considerations that determine the preference between lotteries are the probabilities attached to each outcome, and not the way that the lottery is conducted.

(**A**$_3$) For each $j \in \{1, \cdots, J\}$, let L_j be the following simple lottery:

$$L_j = \left[p_1^j(A_1), \cdots, p_K^j(A_K)\right],$$

and let $\hat{L}$ be the following compound lottery:

$$\hat{L} = [q_1(L_1), \cdots, q_K(A_K)].$$

Simplification of lotteries cont.

For each $k \in \{1, \cdots, K\}$, define

$$r_k = q_1 p_k^1 + \cdots + q_J p_k^J;$$

this is the overall probability that the outcome of the compound lottery $\hat{L}$ will be A_k.

Consider the simple lottery

$$L = [r_1(A_1), \cdots, r_K(A_K)].$$

Then we have

$$\hat{L} \sim L.$$

Simplification of lotteries cont.

The motivation of ($\mathbf{A}_3$) is that it should not matter whether a lottery is conducted in a single stage or in several stages, provided the probability of receiving various outcomes is identical in the two stages.

Independence

Suppose that we create a new compound lottery out of a given compound lottery by replacing one of the simple lotteries involve in the compound lottery with a different simple lottery.

The axiom then requires a player who is indifferent between the original simple lottery and its replacement to be indifferent between the two corresponding compound lotteries.

(**A**$_4$) Let $\hat{L} = [q_1(L_1), \cdots, q_J(L_J)]$ be a compound lottery, and let M be a simple lottery. If $L_j \sim M$ then

$$\hat{L} \sim [q_1(L_1), \cdots, q_{j-1}(A_{j-1}), q_j(M), q_{j+1}(A_{j+1}), \cdots, q_J(L_J)].$$

One can extend Axioms of Simplification and Independence to compound lotteries of any order (that is, lotteries over lotteries over lotteries $\cdots$ over lotteries over outcomes) in a natural way.

By induction over the levels of compounding, it follows that preference relation over all compound lotteries (of any order) is determined by preference relation over simple lotteries (why?).

The characterization theorem

Theorem

If a preference relation $\succeq$ over $\hat{\mathscr{L}}$ is rational, and satisfies ($\mathbf{A}_1$)-($\mathbf{A}_4$) then the preference relation can be represented by a linear utility function.

The next example shows a player whose preference relation is rational, and satisfies ($\mathbf{A}_1$)-($\mathbf{A}_4$) compares two lotteries based on his utility from the outcomes of the lottery.

Example

Suppose that Joshua is choosing which of the following two lotteries he prefers: $\left[\frac{1}{2}(\text{Car}), \frac{1}{2}(\text{Computer})\right]$ and $\left[\frac{1}{3}(\text{Motorcycle}), \frac{2}{3}(\text{Trip around the world})\right]$.

Suppose that Joshua's preference relation over the lotteries satisfies ($\mathbf{A}_1$)-($\mathbf{A}_4$). By the characterization theorem, there is a linear utility function U representing his preference relation. Assume that

$$U(\text{Car}) = 25, U(\text{Trip around the world}) = 14,$$

$$U(\text{Motorcycle}) = 3, and U(\text{Computer}) = 1.$$

Then Joshua's utility from the first lottery is

$$\left[\frac{1}{2}(\text{Car}), \frac{1}{2}(\text{Computer})\right] = \frac{1}{2} \times 25 + \frac{1}{2} \times 1 = 13$$

and his utility from the second lottery is

$$\left[\frac{1}{3}(\text{Motorcycle}), \frac{2}{3}(\text{Trip around the world})\right] = \frac{1}{3} \times 3 + \frac{2}{3} \times 14 = 10\frac{1}{3}.$$

Proof of the characterization theorem

Let $A_K \succeq \cdots \succeq A_1$. Rest of the proof is decomposed into two cases.

Case 1. $A_K \succ A_1$.

Step 1: Definition of a function U over the set of lotteries.

By Corollary 1, for each $1 \leq k \leq K$ there exists a unique real number $0 \leq \theta_i^k \leq 1$ satisfying

$$A_k \sim [\theta_k(A_K), (1-\theta_k)(A_1)].$$

Suppose that $\hat{L} = [q_1(L_1), \cdots, q_J(A_J)] \in \hat{\mathscr{L}}$, where $q_1, \cdots, q_J$ are non-negative numbers summing to 1 and $L_1, \cdots, L_J$ are simple lotteries given by

$$L_j = \left[p_1^j(A_1), \cdots, p_K^j(A_K)\right].$$

For each $1 \leq k \leq K$, define

$$r_k = q_1 p_k^1 + \cdots + q_J p_k^J.$$

This is the probability that the outcome of the lottery is A_k. Define $U : \hat{\mathscr{L}} \to \mathbb{R}$ by letting

$$U(\hat{L}) = \sum_{k=1}^{K} r_k \theta_k.$$

In particular, for every simple lottery $L = [p_1(A_1), \cdots, p_K(A_K)]$,

$$U(L) = \sum_{k=1}^{K} p_k \theta_k.$$

Step 2: *$U(A_k) = \theta_k$ for all $1 \leq k \leq K$*.

Since the outcome A_k is equivalence to the lottery $L = [1(A_k)]$, which in turn is equivalent to the compound lottery $\hat{L} = [1(L)]$.

The outcome of this lottery is A_k with probability 1. Thus,

$$r_k = \begin{cases} 1, & \text{if } l = k; \\ 0, & \text{otherwise.} \end{cases}$$

So, $U(A_k) = \theta_k$ for all $1 \leq k \leq K$.

Step 3: *U is linear*.

By Step 3, for any simple lottery $L = [p_1(A_1), \cdots, p_K(A_K)]$,

$$U(L) = \sum_{k=1}^{K} p_k \theta_k = \sum_{k=1}^{K} p_k U(A_k).$$

Step 4: $\hat{L} = [U(\hat{L}), (1 - U(\hat{L}))(A_1)]$ *for every compound lottery* $\hat{L}$.

Let $\hat{L} = [q_1(L_1), \cdots, q_J(A_J)] \in \hat{\mathscr{L}}$, where

$$L_j = \left[p_1^j(A_1), \cdots, p_K^j(A_K)\right]$$

for all $1 \leq j \leq J$.

Recall that $r_k = \sum_{j=1}^{J} q_j p_j^k$ for all $1 \leq k \leq K$.

By $(\mathbf{A}_3)$,

$$\hat{L} \sim [r_1(A_1), \cdots, r_K(A_K)].$$

Let

$$M_k = [\theta_k(A_K), (1 - \theta_k)(A_1)]$$

for all $1 \leq k \leq K$.

By definition, $A_k \sim M_k$ for all $1 \leq k \leq K$. So, K applications of the Independent Axiom yield

$$\hat{L} \sim [r_1(M_1), \cdots, r_K(M_K)].$$

S5nce each lottery M_k is a lottery over outcomes A_1 and A_K, the lottery on the right-hand side of the above equation ia a lottery over these two outcomes.

So, if we denote by r_* the total probability of A_K in the lottery on the right-hand of the above equation, then

$$R_* = \sum_{k=1}^{K} r_k \theta_k = U(\hat{L}),$$

and the Simplification Axiom implies that

$$\hat{L} \sim [r_*(A_K), (1 - r_*)(A_1)] = \Big[U(\hat{L})(A_K), (1 - U(\hat{L}))(A_1)\Big].$$

Step 5: U is a utility function.

By Step 4 and the Monotonicity Axiom,

$$\hat{L} \succeq \hat{L}' \Leftrightarrow U(\hat{L}) \geq U(\hat{L}') \text{ for all } \hat{L}, \hat{L}' \in \hat{\mathscr{L}}.$$

So, U is a utility function.

Case 2. $A_K \sim A_1$.

So, $A_1 \sim \cdots \sim A_K$. Consider a simple lottery

$$L = [p_1(A_1), \cdots, p_K(A_K)].$$

By repeated use of the Axiom of Independence,

$$L \sim [p_1(A_1), \cdots, p_1(A_1)].$$

The Axiom of Simplification implies that $L \sim [1(A_1)]$. So every compound lottery $\hat{L}$ satisfies $\hat{L} \sim [1(A_1)]$.

Thus, the player is indifferent between any two compound lotteries. Hence, any constant function U represents his preference relation.