Neutrinoless double beta decay and search for sterile neutrinos

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Abstract

This is just a technical note that will explain step by step and with certain detail the process of creation of a "lobster plot" (A plot of the effective majorana mass $\langle m_{ee} \rangle$ in function of the lightest neutrino m_{light}) with updated data and also expanding the original lobster plot with the inclusion of the possible existence of 1 sterile neutrino and also the case of the existence of 2 sterile neutrinos with all the possible mass hierarchies, and in addition to that, some animations of how the angles for sterile neutrinos change with respect of time.

1 Introduction

In 1930, Wolfgang Pauli proposed the existence of the neutrino as a way to balance the energy and momentum of beta decay process, it was a theoretical particle until 1956 when Frederick Reines and Clyde Cowan with the project Poltergeist, successfully were able to detect a neutrino due to a beta decay process.

In the late 1960s Homestake experiment headed by Raymond Davis Jr and John Bahcall has the purpose of detecting neutrinos that come from the Sun to prove that the Standard Solar model was correct, however, after a period of 25 years of collecting data from the solar neutrinos, the experiment was able to detect 1 neutrino each 2 day, however, the expected quantity of neutrinos according to the Standard Solar model was 3 neutrinos each 2 days, this discrepancy was named as the Solar Neutrino Problem.

Several hypothesis were made around this, some of them said that Standard Solar model was not correct and some others that the Sun ran out of hydrogen, but Bruno Pontecorvo proposed the hypothesis that neutrinos actually have the property to oscillate into different flavors, this theory contradicted the standard model of particle physics because the model theorized that neutrinos were massless particles, but for a neutrino to oscillate into different flavors must have a mass, and in 2001, the Sudbury Neutrino Observatory was able to detect the three neutrino flavors, all of them comming from the Sun, but the Standard Solar Model says that the Sun only produces electron neutrinos, so the conclusion was that neutrino oscillates and actually has a mass value.

2 PMNS Matrix

Since the neutrino oscillates and has a mass value, to measure the neutrino oscillations it is used something called the Pontecorvo Maki Nakagawa Sakata Matrix (also known as PMNS matrix, lepton mixing matrix, or neutrino mixing matrix)(eq 1 and 2), where the left part of equation is the expresion of a neutrino type, and the right part is the product of the lepton mixing matrix, which U components are amplitude of the mass eigenstates, and the mass eigenstates of the neutrino, if we apply matrix multiplication we can get a neutrino flavor in terms of the mass eigenstates.

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1}\nu_1 + U_{e2}\nu_2 + U_{e3}\nu_3 \\ U_{\mu 1}\nu_1 + U_{\mu 2}\nu_2 + U_{\mu 3}\nu_3 \\ U_{\tau 1}\nu_1 + U_{\tau 2}\nu_2 + U_{\tau 3}\nu_3 \end{bmatrix} \tag{2}$$

$$\nu_e = \sum_{i=1}^{3} U_{ei}\nu_i \tag{3}$$

3 Neutrinoless double beta decay

Neutrinoless double beta decay is a theoretical type of beta decay that, unlike the standard beta decay ${}^{A}_{Z}X \longrightarrow {}^{A}_{Z+1}X + e^- + \nu_e$ and the not so common double beta decay ${}^{A}_{Z}X \longrightarrow {}^{A}_{Z+2}X + 2e^- + 2\nu_e$, the neutrinoless double beta decay is characterized for being a beta decay process where the trajectory of the ejected neutrinos collide and annihilate each other (as the name suggests) ${}^{A}_{Z}X \longrightarrow {}^{A}_{Z+2}X + 2e^-$. Even thought the hypothesis is attractive, for this to be possible, neutrino has to be their own antiparticle so it can annihilate with another one, but for technical reasons, instead of theorizing the existence and probability to find a neutrinoless double beta decay, this document will assume it exists and based on that, use the effective mass $\langle m_{ee} \rangle$ that is derived from the PMNS matrix, and m_{light} relationship to find possible mass values for the neutrino including the possibility of the existance of 1 and 2 sterile neutrinos

4 Lobster plot

Methodology followed to recreate the original lobster of the 3 neutrinos masses was confirmed that exist (m_1, m_2, m_3) was using equation 1 and expand it until we get 3 different mass terms (showed in equation 4, 5, and 6) that are related to the effective mass.

$$\text{Effective mass} \ \ \langle m_{ee} \rangle = \left| \sum U_{ei}^2 m_i \right| \tag{4}$$

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^{3} U_{ei}^2 m_i \right| = \left| \left| m_{ee}^{(1)} \right| + \left| m_{ee}^{(2)} \right| e^{2i\alpha} + \left| m_{ee}^{(3)} \right| e^{2i\beta} \right| \tag{5}$$

$$\begin{gathered} \langle m_{ee} \rangle = \left| c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 m_2 e^{i\phi_{12}} + s_{13}^2 m_3 e^{i\phi_{13}} \right| \\ s_{ij}^2 = \sin^2 \theta_{ij} \\ c_{ij}^2 = \cos^2 \theta_{ij} \end{gathered} \tag{6}$$

At the end, we got an equation with three terms that each of them is related to an specific mass term, however,the problem here is that we do not know the mass values, but we know the absolute value of the square difference of two masses ($\left|\Delta m_{ij}^2\right|$) and the value of s_{12}^2 and s_{13}^2, so based on that it is possible to put the mass terms in terms of the lightest mass, but since we know $\left|\Delta m_{ij}^2\right|$, we have to consider the case were $\Delta m_{ij}^2 > 0$ and $\Delta m_{ij}^2 < 0$ and for doing that, we have to make a plot considering different scenarios of mass hierarchy, and those are scenarios are represented as normal hierarchy NH, and inverted hierarchy IH, since experimental results showed that $\Delta m_{21}^2 > 0$ (also known as solar mass Δm_{solar}^2) we know that the lightest mass can only be m_1 or m_3, and Δm_{31}^2 (also known as atmospheric mass Δm_A^2) is the one that defines the hierarchy.

Hierarchy name	Hierarchy label	Mass ordering
Normal hierarchy	NH	$m_3 > m_2 > m_1$
Inverted hierarchy	IH	$m_2 > m_1 > m_3$

$$\text{NH:}\;\; m_2 = \sqrt{m_1^2 + \Delta m_{solar}^2} \quad m_3 = \sqrt{m_1^2 + \Delta m_A^2}$$

$$\text{IH:}\;\; m_2 = \sqrt{m_3^2 + \Delta m_{solar}^2 + \Delta m_A^2} \quad m_1 = \sqrt{m_3^2 + \Delta m_A^2} \tag{7}$$

$$\text{where } \Delta m_{solar}^2 = \Delta m_{21}^2 \text{ and } \; \Delta m_A^2 = \Delta m_{31}^2$$

$$(\Delta m_{31}^2 > 0 \text{ for NH and } \Delta m_{31}^2 < 0 \text{ for IH})$$

5 Data used for 3 masses

Parameter	Best Fit	3σ
Δm_{21}^2 (10^{-5} eV^2)	7.53	8.07-6.99
$\lvert\Delta m_{31}^2\rvert$ (10^{-3} eV^2)	2.5283	2.5631-2.4935
s_{12}^2	0.307	0.346-0.268
s_{13}^2	0.022	0.0241-0.0199

6 Step by step to make it work for 3 neutrinos

So, after having the required materials to mathematically build the plot, next step is creating the plot using Jupyter Notebook (Details of how the code was created and how does it works are located here or go to the next url https://github.com/FranciscoTapia61199/Sterile-neutrinos)

It is tricky to get this plot, because Python reads it as a 1-dimensional variable (as it is required to do) but there is also the free parameters of ϕ_{12} and ϕ_{13}, and with each different combination, the code will create another curve, and since there are infinite many combinations of angles, it would be impossible for the code to run it without burning the PC, so, another strategy is to find the angles where the boundaries of this function are located and fill each the space between them to make sure all possible angle combinations are included without making the PC consume a lot of time and a lot of resources, to calculate the angles that made the boundaries, i used an optimization method using derivatives to find the max and min curves of the function.

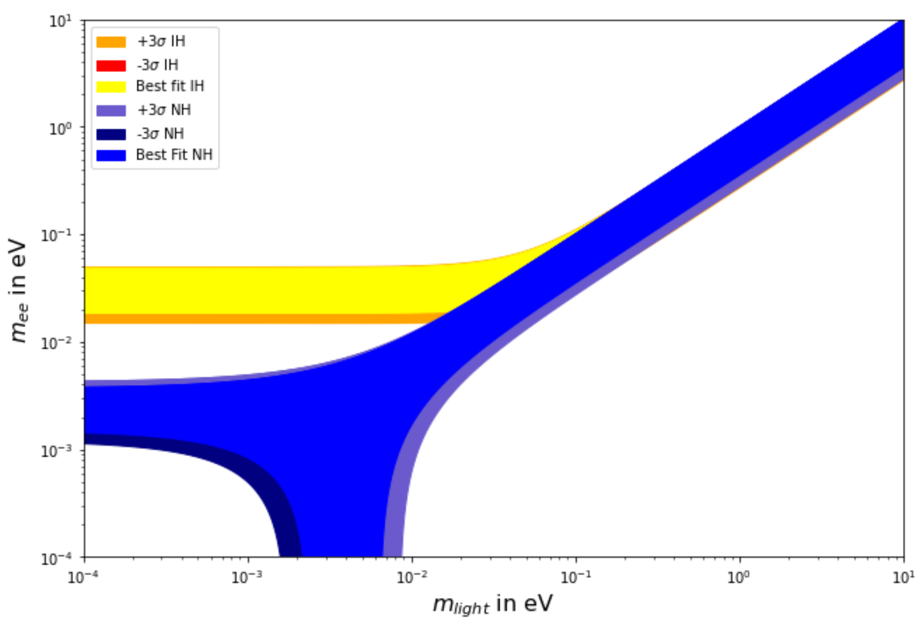
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{12})} = 0$ if $\phi_{12} = (0, \pi)$

$\frac{\partial(\langle m_{ee} \rangle)}{\partial(\phi_{13})} = 0$ if $\phi_{13} = (0, \pi)$

upper bound at ϕ_{12}=0 and ϕ_{13}=0

lower bound at $\phi_{12} = \pi$ and $\phi_{13} = \pi$

lower inferior bound (only for NH case) at $\phi_{12} = \pi$ and $\phi_{13} = 0$



7 Extending the definition for sterile neutrinos

Liquid Scintillator Neutrino Detector (LSND) was a neutrino detector which main purpose was to look for evidence for the neutrino oscillation, and that resulted on the detection of an anomaly where the recorded value $\Delta m^2_{41} = 1.78eV^2$ but the cosmological limit for neutrino mass is 1 eV, so this anomaly must be something apart for the solar and atmospheric masses.

The formulas and techniques used before were defined just for the existence of 3 neutrinos, however, we can actually extend the formulas to include the existence of 4, 5.....n neutrinos (but for the sake of time, coherence and computational resources, here are covered only the cases for 1 and 2 sterile neutrinos, but as is mentioned, in theory, this is can be applied for n neutrinos) and the formula used would be equations 8 and 9.

$$\langle m_{ee} \rangle = \left| \sum_{i=1}^{4} U_{ei}^2 m_i \right| = \left| \left| m_{ee}^{(1)} \right| + \left| m_{ee}^{(2)} \right| e^{2i\alpha} + \left| m_{ee}^{(3)} \right| e^{2i\beta} + \left| m_{ee}^{(4)} \right| e^{2i\gamma} \right| \tag{8}$$

$$\langle m_{ee} \rangle = \left| c_{12}^2 c_{13}^2 c_{14}^2 m_1 + s_{12}^2 c_{13}^2 c_{14}^2 m_2 e^{i\phi_{12}} + s_{13}^2 c_{14}^2 m_3 e^{i\phi_{13}} + s_{14}^2 m_4 e^{i\phi_{14}} \right| \tag{9}$$

As we can see in the formula, including another mass term to the equation, is not just adding some m_4 times some $s_{ij}^2 e^{i\phi_{ij}}$ term, it is necessary to multiply each mass term by some c_{ij}^2 (in this case, c_{14}^2) and that makes sense, because, if $\theta_{14} = 0$ we get back the equation with 3 mass terms, so, following that pattern, is possible to expand it for n neutrinos.

Besides including the sterile mass, it is necessary to also include a m_{light} dependence to m_4, so, for that, we just repeat the process used previously of the experimental measurement $\left| \Delta m_{ij}^2 \right|$ to add the m_{light} dependence, that in this case we use Δm_{41}^2 (also known as Experiment LSND mass $\Delta m_{\mathrm{LSND}}^2$), but since the experimental result is an absolute value, besides considering the previous hierarchies, we have to consider the cases were m_4 is the heaviest mass, and were m_4 is the lightest mass, and with all of this in consideration, we can start making the formulas for the plot.

Hierarchy name	Hierarchy label	Mass ordering
Heavy sterile neutrino (Normal hierarchy)	SNH	$m_4 > m_3 > m_2 > m_1$
Heavy sterile neutrino (Inverted hierarchy)	SIH	$m_4 > m_2 > m_1 > m_3$
Light sterile neutrino (Normal hierarchy)	NHS	$m_3 > m_2 > m_1 > m_4$
Light sterile neutrino (Inverted hierarchy)	IHS	$m_2 > m_1 > m_3 > m_4$

Heavy sterile neutrino ($\Delta m_{41}^2 > 0$)

$$\text{NH:} \;\; m_2 = \sqrt{m_1^2 + \Delta m_{solar}^2} \quad m_3 = \sqrt{m_1^2 + \Delta m_A^2} \quad m_4 = \sqrt{m_1^2 + \Delta m_{\mathrm{LSND}}^2}$$

$$\text{IH:} \;\; m_2 = \sqrt{m_3^2 + \Delta m_{solar}^2 + \Delta m_A^2} \quad m_1 = \sqrt{m_3^2 + \Delta m_A^2} \quad m_4 = \sqrt{m_3^2 + \Delta m_{\mathrm{LSND}}^2 + \Delta m_A^2}$$

$$\text{where } \Delta m_{solar}^2 = \Delta m_{21}^2 \;,\; \Delta m_A^2 = \Delta m_{31}^2 \text{ and, } \Delta m_{\mathrm{LSND}}^2 = \Delta m_{41}^2$$

$$\Delta m_{31}^2 > 0 \text{ for NH and } \Delta m_{31}^2 < 0 \text{ for IH} \tag{10}$$

8 Data used for 4 masses

Parameter	Best Fit	3σ
Δm^2_{21} (10^{-5} eV^2)	7.53	8.07-6.99
$\left\|\Delta m^2_{31}\right\|$ (10^{-3} eV^2)	2.5283	2.5631-2.4935
$\left\|\Delta m^2_{41}\right\|$ (eV^2)	1.78	2.01-1.61
s^2_{12}	0.307	0.346-0.268
s^2_{13}	0.022	0.0241-0.0199
s^2_{14}	0.023	0.04-0.006

9 Step by step to make it work for 3+1 neutrinos

To make this plot, we follow exactly the same process as for 3 neutrinos, but since we are including a new ϕ_{14} dependence, then, the original boundaries for the 3 neutrino case must be different ones (at least for the NH case), so, following the same optimization method with derivatives, we found that critical angles are 0 and π (in fact, even with n neutrinos, the critical values will always be a mix of 0 and π angles) and based on that, we can construct the plots.

$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{12})} = 0$ if $\phi_{12} = (0, \pi)$

$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{13})} = 0$ if $\phi_{13} = (0, \pi)$

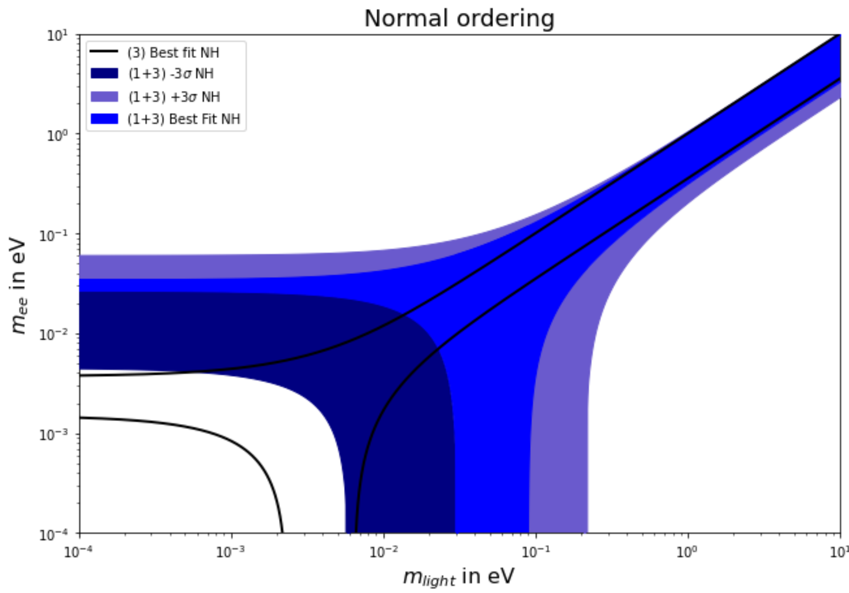
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{14})} = 0$ if $\phi_{14} = (0, \pi)$

upper bound at ϕ_{12}=0, ϕ_{13}=0, and ϕ_{14}=0

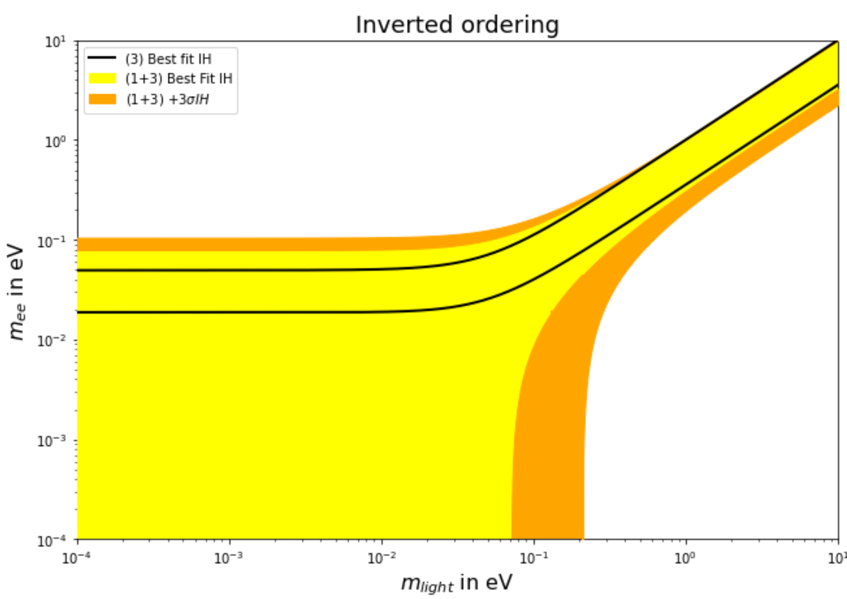
lower bound at $\phi_{12} = \pi$, $\phi_{13} = \pi$, and $\phi_{14} = \pi$

lower inferior bound (only for SNH case) at $\phi_{12} = 0$, $\phi_{13} = 0$, and $\phi_{14} = \pi$

9.1 Heavy sterile neutrino (SNH)



9.2 Heavy sterile neutrino (SIH)



We also have to include the case were m_4 is the lightest mass, however, according to the previous hierarchies, m_{light} in case for the active neutrinos (the ones we know that actually exist) is m_1 or m_3, but neither of them are the lightest mass anymore, so, for experimental purposes, we are going to include 2 types of plots, one where we put everything in terms of the lightest mass, regardless if is active or sterile, and one where we put everything in terms of the lightest active mass, as mentioned before, we include one of the active neutrino mass separated to be able to see the possibilities to find the sterile neutrino via experimental measurements. (Details of how the plots were made in https://github.com/FranciscoTapia61199/Sterile-neutrinos).

Light sterile neutrino ($\Delta m_{41}^2 < 0$)

$$\text{NH:}\;\; m_2 = \sqrt{m_4^2 + \Delta m_{solar}^2 + \Delta m_{\text{LSND}}^2} \quad m_3 = \sqrt{m_4^2 + \Delta m_A^2 + \Delta m_{\text{LSND}}^2} \quad m_1 = \sqrt{m_4^2 + \Delta m_{\text{LSND}}^2}$$

$$\text{IH:}\;\; m_2 = \sqrt{m_4^2 + \Delta m_{solar}^2 + \Delta m_{\text{LSND}}^2} \quad m_3 = \sqrt{m_4^2 + \Delta m_A^2 - \Delta m_{\text{LSND}}^2} \quad m_1 = \sqrt{m_4^2 + \Delta m_{\text{LSND}}^2}$$

$$\text{where } \Delta m_{solar}^2 = \Delta m_{21}^2 \;,\; \Delta m_A^2 = \Delta m_{31}^2 \text{ and, } \Delta m_{\text{LSND}}^2 = \Delta m_{41}^2$$

$$\Delta m_{31}^2 > 0 \text{ for NH and } \Delta m_{31}^2 < 0 \text{ for IH} \tag{11}$$

(active) Light sterile neutrino ($\Delta m_{41}^2 < 0$)

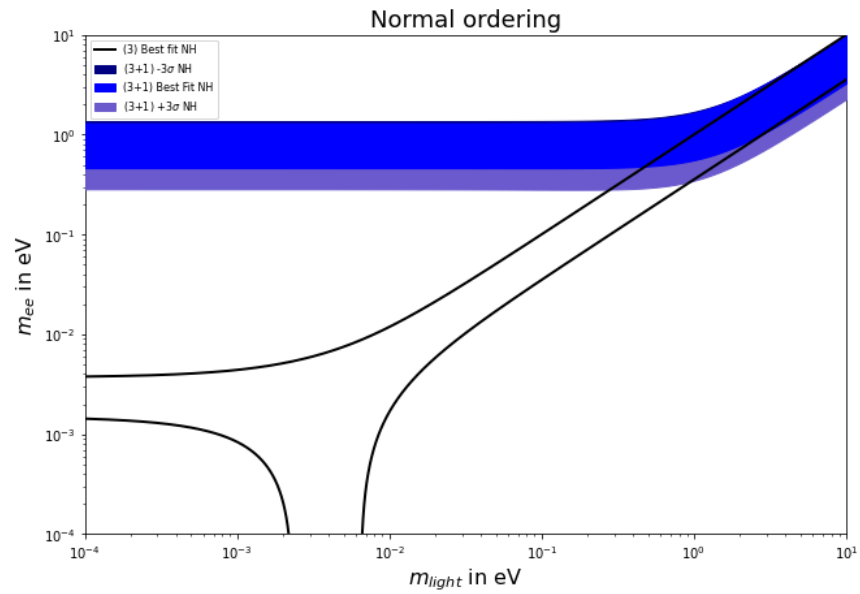
$$\text{NH:}\;\; m_2 = \sqrt{m_1^2 + \Delta m_{solar}^2} \quad m_3 = \sqrt{m_1^2 + \Delta m_A^2} \quad m_4 = \sqrt{m_1^2 - \Delta m_{\text{LSND}}^2}$$

$$\text{IH:}\;\; m_2 = \sqrt{m_3^2 + \Delta m_{solar}^2 + \Delta m_{\text{A}}^2} \quad m_1 = \sqrt{m_3^2 + \Delta m_A^2} \quad m_4 = \sqrt{m_3^2 + \Delta m_A^2 - \Delta m_{\text{LSND}}^2}$$

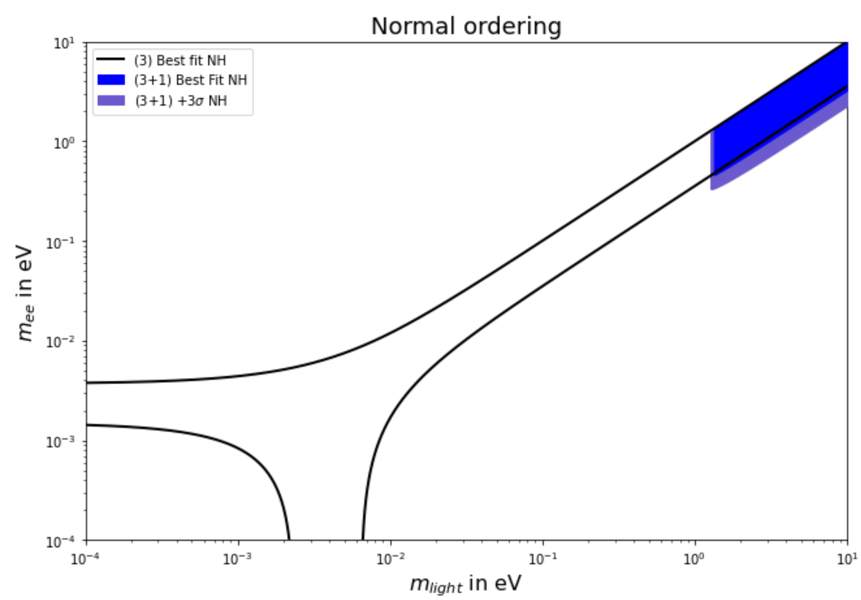
$$\text{where } \Delta m_{solar}^2 = \Delta m_{21}^2 \;,\; \Delta m_A^2 = \Delta m_{31}^2 \text{ and, } \Delta m_{\text{LSND}}^2 = \Delta m_{41}^2$$

$$\Delta m_{31}^2 > 0 \text{ for NH and } \Delta m_{31}^2 < 0 \text{ for IH} \tag{12}$$

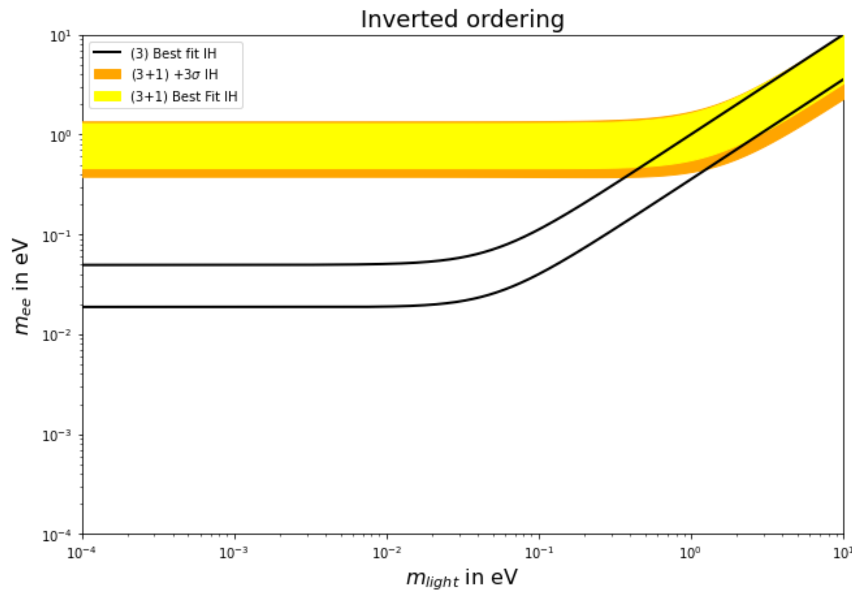
9.3 Light sterile neutrino (NHS)



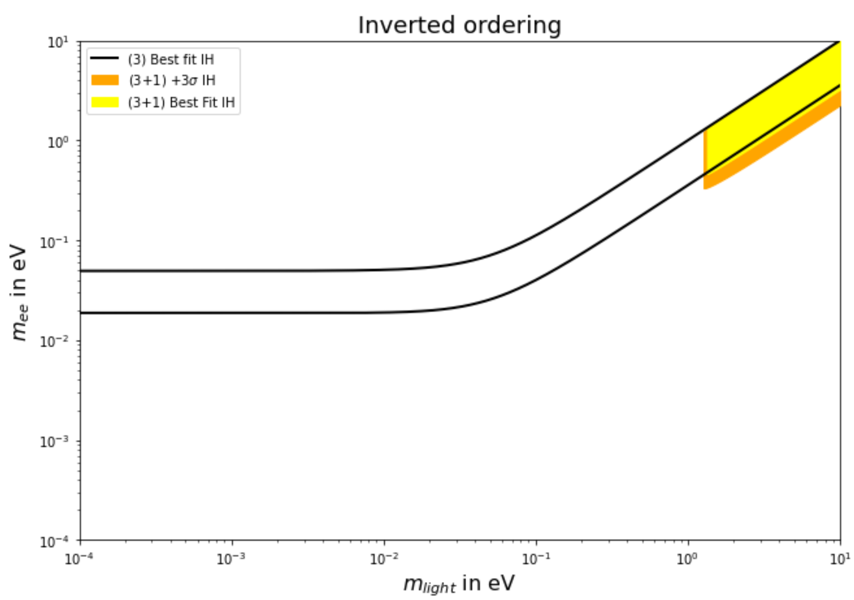
9.4 (active) Light sterile neutrino (NHS)



9.5 Light sterile neutrino (IHS)



9.6 (active) Light sterile neutrino (IHS)



10 2 sterile neutrinos

Again, following the same pattern explained for 1 sterile neutrino case, we can expand $\langle m_{ee}\rangle$ for 2 sterile neutrinos, again, including the s_{15}^2 and m_5 terms.

$$\langle m_{ee}\rangle = \left|\sum_{i=1}^{5} U_{ei}^2 m_i\right| = \left|\left|m_{ee}^{(1)}\right| + \left|m_{ee}^{(2)}\right|e^{2i\alpha} + \left|m_{ee}^{(3)}\right|e^{2i\beta} + \left|m_{ee}^{(4)}\right|e^{2i\gamma} + \left|m_{ee}^{(5)}\right|e^{2i\delta}\right| \tag{13}$$

$$\langle m_{ee}\rangle = \left|c_{12}^2c_{13}^2c_{14}^2c_{15}^2m_1 + s_{12}^2c_{13}^2c_{14}^2c_{15}^2m_2e^{i\phi_{12}} + s_{13}^2c_{14}^2c_{15}^2m_3e^{i\phi_{13}} + s_{14}^2c_{15}^2m_4e^{i\phi_{14}} + s_{15}^2m_5e^{i\phi_{15}}\right| \tag{14}$$

And again, similar to the equation of 4 neutrinos, if we set $\theta_{15} = 0$ all s_{15}^2 become 0, and all c_{15}^2 become 1 and we get the 4 neutrinos equation back.

Since we are including a new mass m_5, we have to put it in terms of m_{light}, for that, we use the experimental value $\left|\Delta m_{51}^2\right|$ (named Δm_{New}^2 on this note.), and since is an absolute value, we have to consider again the different hierarchies SSN, SSI, SNSa, SISa, SNSb, SISb, NSS, and ISS.

Hierarchy name	Hierarchy label	Mass ordering
2 Heavy steriles (Normal hierarchy)	SSN	$m_5 > m_4 > m_3 > m_2 > m_1$
2 Heavy steriles (Inverted hierarchy)	SSI	$m_5 > m_4 > m_2 > m_1 > m_3$
Heavy m_5 light m_4 (Normal hierarchy)	SNS[m5]	$m_5 > m_3 > m_2 > m_1 > m_4$
Heavy m_5 light m_4 (Inverted hierarchy)	SIS[m5]	$m_5 > m_2 > m_1 > m_3 > m_4$
Heavy m_4 light m_5 (Normal hierarchy)	SNS[m4]	$m_4 > m_3 > m_2 > m_1 > m_5$
Heavy m_4 light m_5 (Inverted hierarchy)	SIS[m4]	$m_4 > m_2 > m_1 > m_3 > m_5$
2 Light steriles (Normal hierarchy)	NSS	$m_3 > m_2 > m_1 > m_5 > m_4$
2 Light steriles (Inverted hierarchy)	ISS	$m_2 > m_1 > m_3 > m_5 > m_4$

2 Heavy sterile neutrinos ($\Delta m^2_{41} > 0,$ and $\Delta m^2_{51} > 0$)

$$
\begin{gathered}
\text{NH:}\;\; m_2 = \sqrt{m_1^2 + \Delta m^2_{solar}} \quad m_3 = \sqrt{m_1^2 + \Delta m^2_A} \\
m_4 = \sqrt{m_1^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_5 = \sqrt{m_1^2 + \Delta m^2_{New}} \\
\\
\text{IH:}\;\; m_2 = \sqrt{m_3^2 + \Delta m^2_{solar} + \Delta m^2_A} \quad m_1 = \sqrt{m_3^2 + \Delta m^2_A} \\
m_4 = \sqrt{m_3^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_A} \;\; m_5 = \sqrt{m_3^2 + \Delta m^2_{New} + \Delta m^2_A}
\end{gathered}
\tag{15}
$$

where $\Delta m^2_{solar} = \Delta m^2_{21}$, $\Delta m^2_A = \Delta m^2_{31}$ and, $\Delta m^2_{\mathrm{LSND}} = \Delta m^2_{41}$

$\Delta m^2_{31} > 0$ for NH and $\Delta m^2_{31} < 0$ for IH

m_5 Heavy m_4 Light neutrinos ($\Delta m^2_{41} < 0,$ and $\Delta m^2_{51} > 0$)

$$
\begin{gathered}
\text{NH:}\;\; m_1 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_2 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{solar}} \\
m_3 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_A} \quad m_5 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{New}} \\
\\
\text{IH:}\;\; m_1 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_2 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{solar}} \\
m_3 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} - \Delta m^2_A} \quad m_5 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{New}}
\end{gathered}
\tag{16}
$$

where $\Delta m^2_{solar} = \Delta m^2_{21}$, $\Delta m^2_A = \Delta m^2_{31}$ and, $\Delta m^2_{\mathrm{LSND}} = \Delta m^2_{41}$

$\Delta m^2_{31} > 0$ for NH and $\Delta m^2_{31} < 0$ for IH (active) m_5 Heavy m_4 Light neutrinos $(\Delta m^2_{41} < 0,\ \text{and}\ \ \Delta m^2_{51} > 0)$

$$\begin{aligned}
&\text{NH:}\ \ m_2 = \sqrt{m_1^2 + \Delta m^2_{solar}} \quad m_3 = \sqrt{m_1^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_1^2 - \Delta m^2_{\text{LSND}}} \quad m_5 = \sqrt{m_1^2 + \Delta m^2_{New}} \\
&\\
&\text{IH:}\ \ m_2 = \sqrt{m_3^2 + \Delta m^2_{solar} + \Delta m^2_A} \quad m_1 = \sqrt{m_3^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_3^2 + \Delta m^2_{\text{LSND}} + \Delta m^2_A} \ \ m_5 = \sqrt{m_3^2 + \Delta m^2_{New} + \Delta m^2_A} \\
&\\
&\text{where } \Delta m^2_{solar} = \Delta m^2_{21}\ ,\ \Delta m^2_A = \Delta m^2_{31} \text{ and, } \Delta m^2_{\text{LSND}} = \Delta m^2_{41} \\
&\\
&\Delta m^2_{31} > 0 \text{ for NH and } \Delta m^2_{31} < 0 \text{ for IH}
\end{aligned} \tag{17}$$

m_4 Heavy m_5 Light neutrinos $(\Delta m^2_{41} > 0,\ \text{and}\ \ \Delta m^2_{51} < 0)$

$$\begin{aligned}
&\text{NH:}\ \ m_1 = \sqrt{m_5^2 + \Delta m^2_{New}} \quad m_2 = \sqrt{m_5^2 + \Delta m^2_{New} + \Delta m^2_{solar}} \\
&m_3 = \sqrt{m_5^2 + \Delta m^2_{New} + \Delta m^2_A} \quad m_4 = \sqrt{m_5^2 + \Delta m^2_{\text{LSND}} + \Delta m^2_{New}} \\
&\\
&\text{IH:}\ \ m_1 = \sqrt{m_5^2 + \Delta m^2_{New}} \quad m_2 = \sqrt{m_5^2 + \Delta m^2_{New} + \Delta m^2_{solar}} \\
&m_3 = \sqrt{m_5^2 + \Delta m^2_{New} - \Delta m^2_A} \quad m_4 = \sqrt{m_5^2 + \Delta m^2_{\text{LSND}} + \Delta m^2_{New}} \\
&\\
&\text{where } \Delta m^2_{solar} = \Delta m^2_{21}\ ,\ \Delta m^2_A = \Delta m^2_{31} \text{ and, } \Delta m^2_{\text{LSND}} = \Delta m^2_{41} \\
&\\
&\Delta m^2_{31} > 0 \text{ for NH and } \Delta m^2_{31} < 0 \text{ for IH}
\end{aligned} \tag{18}$$

(active) m_4 Heavy m_5 Light neutrinos ($\Delta m^2_{41} > 0,$ and $\Delta m^2_{51} < 0$)

$$\begin{aligned}
&\text{NH:}\;\; m_2 = \sqrt{m_1^2 + \Delta m^2_{solar}} \quad m_3 = \sqrt{m_1^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_1^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_5 = \sqrt{m_1^2 - \Delta m^2_{New}} \\
&\text{IH:}\;\; m_2 = \sqrt{m_3^2 + \Delta m^2_{solar} + \Delta m^2_A} \quad m_1 = \sqrt{m_3^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_3^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_A} \quad m_5 = \sqrt{m_3^2 - \Delta m^2_{New} + \Delta m^2_A} \\
&\text{where } \Delta m^2_{solar} = \Delta m^2_{21}\;,\; \Delta m^2_A = \Delta m^2_{31} \text{ and, } \Delta m^2_{\mathrm{LSND}} = \Delta m^2_{41} \\
&\Delta m^2_{31} > 0 \text{ for NH and } \Delta m^2_{31} < 0 \text{ for IH}
\end{aligned} \tag{19}$$

2 Light sterile neutrinos ($\Delta m^2_{41} < 0,$ and $\Delta m^2_{51} < 0$)

$$\begin{aligned}
&\text{NH:}\;\; m_1 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_2 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{solar}} \\
&m_3 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_A} \quad m_5 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} - \Delta m^2_{New}} \\
&\text{IH:}\;\; m_1 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}}} \quad m_2 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} + \Delta m^2_{solar}} \\
&m_3 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} - \Delta m^2_A} \quad m_5 = \sqrt{m_4^2 + \Delta m^2_{\mathrm{LSND}} - \Delta m^2_{New}} \\
&\text{where } \Delta m^2_{solar} = \Delta m^2_{21}\;,\; \Delta m^2_A = \Delta m^2_{31} \text{ and, } \Delta m^2_{\mathrm{LSND}} = \Delta m^2_{41} \\
&\Delta m^2_{31} > 0 \text{ for NH and } \Delta m^2_{31} < 0 \text{ for IH}
\end{aligned} \tag{20}$$

(active) 2 Light sterile neutrinos ($\Delta m^2_{41} < 0,$ and $\Delta m^2_{51} < 0$)

$$
\begin{aligned}
&\text{NH:}\;\; m_2 = \sqrt{m_1^2 + \Delta m^2_{solar}} \quad m_3 = \sqrt{m_1^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_1^2 - \Delta m^2_{\text{LSND}}} \quad m_5 = \sqrt{m_1^2 - \Delta m^2_{New}} \\
&\text{IH:}\;\; m_2 = \sqrt{m_3^2 + \Delta m^2_{solar} + \Delta m^2_A} \quad m_1 = \sqrt{m_3^2 + \Delta m^2_A} \\
&m_4 = \sqrt{m_3^2 - \Delta m^2_{\text{LSND}} + \Delta m^2_A} \;\; m_5 = \sqrt{m_3^2 - \Delta m^2_{New} + \Delta m^2_A}
\end{aligned}
\tag{21}
$$

where $\Delta m^2_{solar} = \Delta m^2_{21}$, $\Delta m^2_A = \Delta m^2_{31}$ and, $\Delta m^2_{\text{LSND}} = \Delta m^2_{41}$

$\Delta m^2_{31} > 0$ for NH and $\Delta m^2_{31} < 0$ for IH

11 Data used for 5 masses

Parameter	Best Fit	3σ
Δm^2_{21} $(10^{-5}\ eV^2)$	7.53	8.07-6.99
$\left\|\Delta m^2_{31}\right\|$ $(10^{-3}\ eV^2)$	2.5283	2.5631-2.4935
$\left\|\Delta m^2_{41}\right\|$ (eV^2)	1.78	2.01-1.61
$\left\|\Delta m^2_{51}\right\|$ (eV^2)	0.87	0.97-0.77
s^2_{12}	0.307	0.346-0.268
s^2_{13}	0.022	0.0241-0.0199
s^2_{14}	0.023	0.04-0.006
s^2_{15}	0.02	0.035-0.005

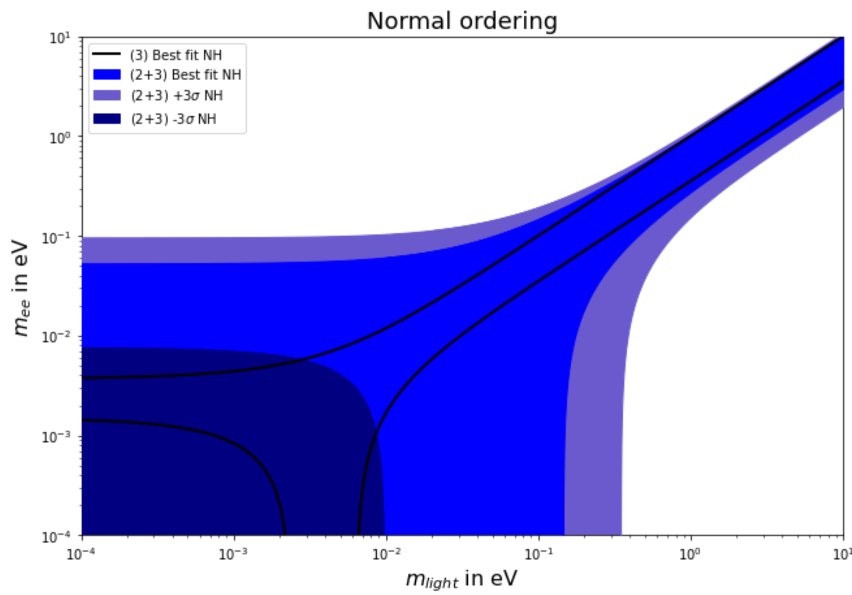
12 Step by step to make it work for 3+2 neutrinos

Basically to plot the graph for 2 sterile neutrinos we use the same approach as the previous plots, however, as we have new mass terms, then, the boundaries of the plot change again, fortunately, we just have to use the same optimization technique but including a ϕ_{15} angle.

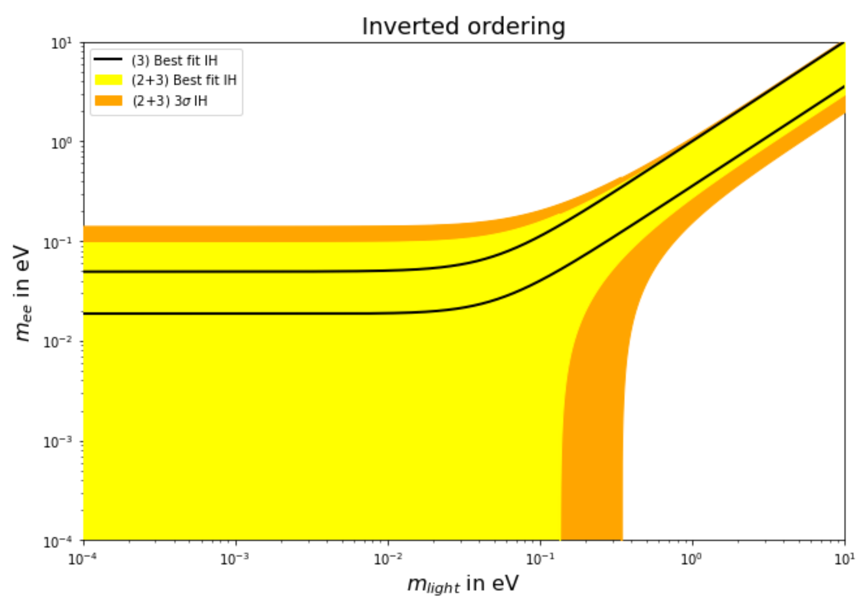
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{12})} = 0$ if $\phi_{12} = (0, \pi)$
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{13})} = 0$ if $\phi_{13} = (0, \pi)$
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{14})} = 0$ if $\phi_{14} = (0, \pi)$
$\frac{\partial(\langle m_{ee}\rangle)}{\partial(\phi_{14})} = 0$ if $\phi_{15} = (0, \pi)$

upper bound at ϕ_{12}=0, ϕ_{13}=0, ϕ_{14}=0, and ϕ_{15}=0
lower bound at $\phi_{12} = \pi$, $\phi_{13} = \pi$, $\phi_{14} = \pi$, and $\phi_{15} = \pi$
lower inferior bound (only for SSN case) at $\phi_{12} = 0$, $\phi_{13} = 0$, $\phi_{14} = \pi$, and $\phi_{15} = 0$

12.1 2 Heavy steriles (SSN)

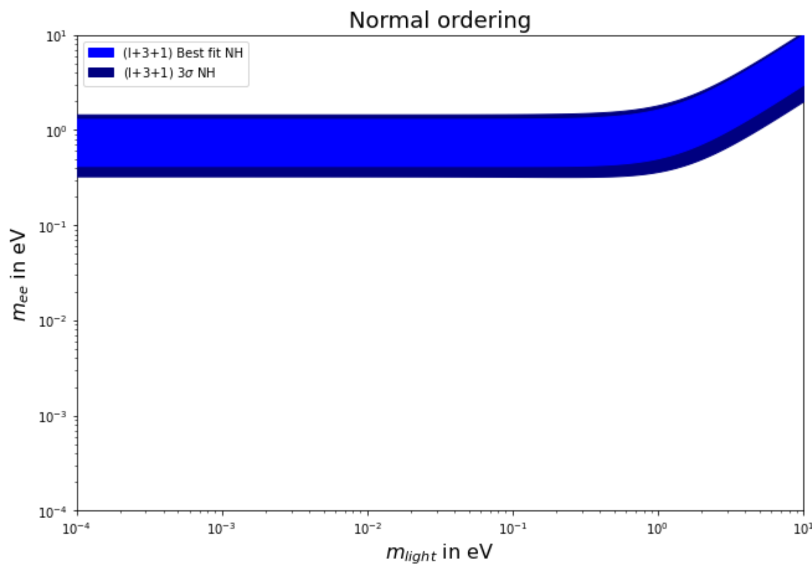


12.2 2 Heavy steriles (SSI)

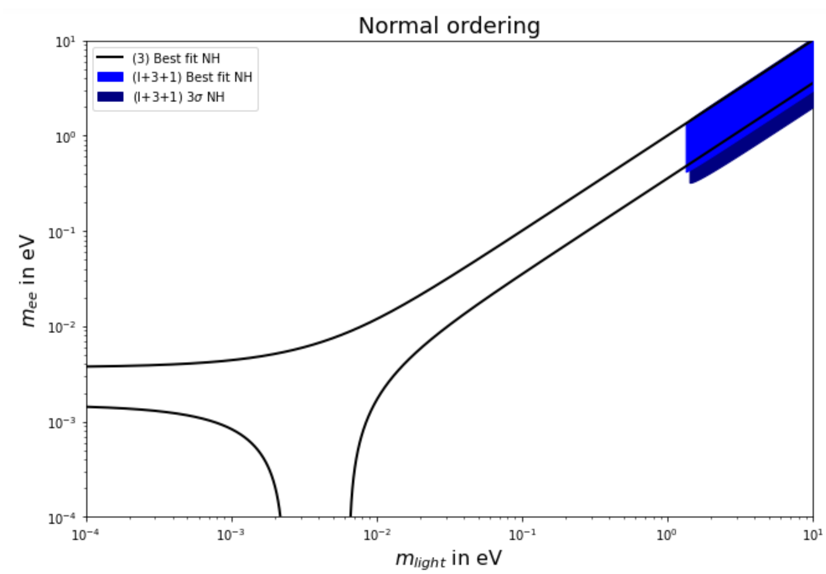


And like the 1 sterile neutrino case, even when the lightest mass for these 3 hierarchies are m_4 and m_5, for experimental purposes, we also include separated plots in terms of m_1 for NH and m_3 for IH (Details of how the plots were made in https://github.com/FranciscoTapia61199/Sterile-neutrinos and go to "2 steriles" folder).

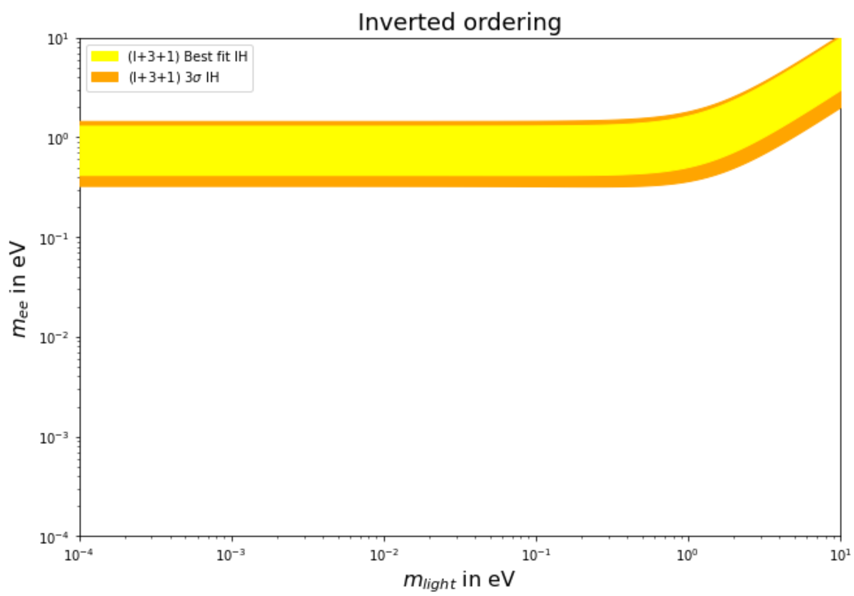
12.3 Heavy m_5 light m_4 (SNSa)



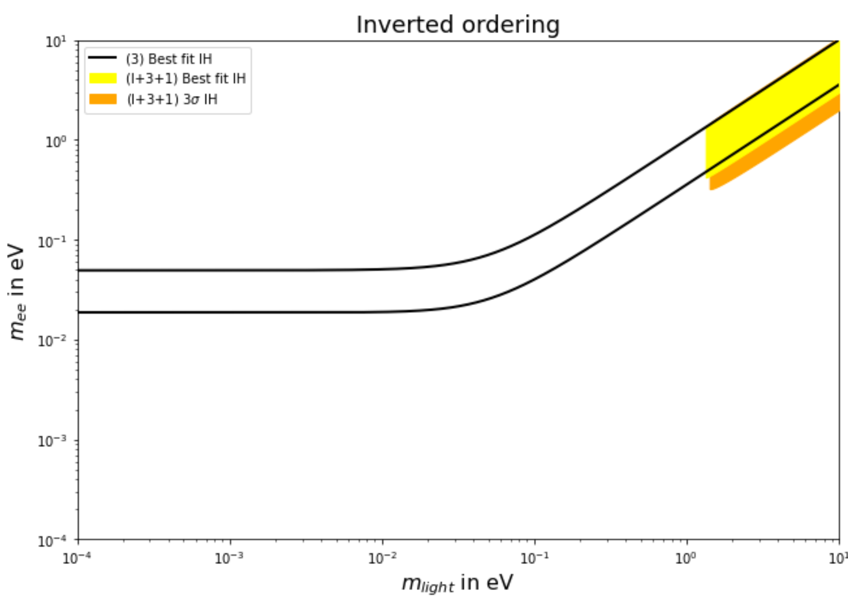
12.4 (active) Heavy m_5 light m_4 (SNSa)



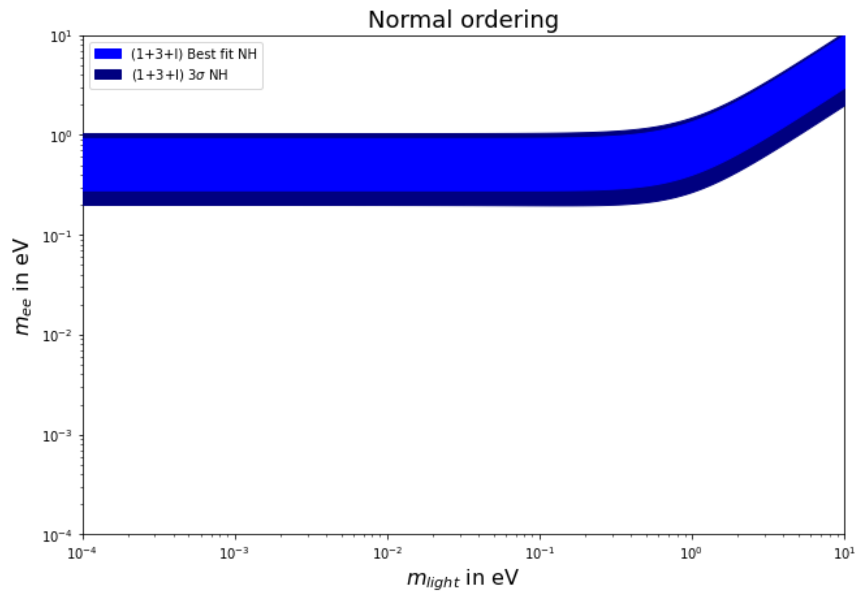
12.5 Heavy m_5 light m_4 (SISa)



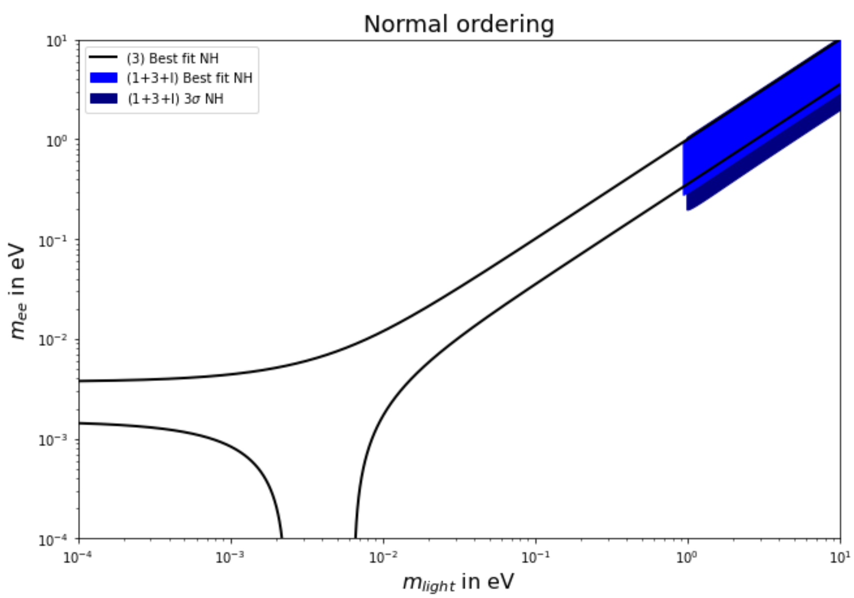
12.6 (active) Heavy m_5 light m_4 (SISa)



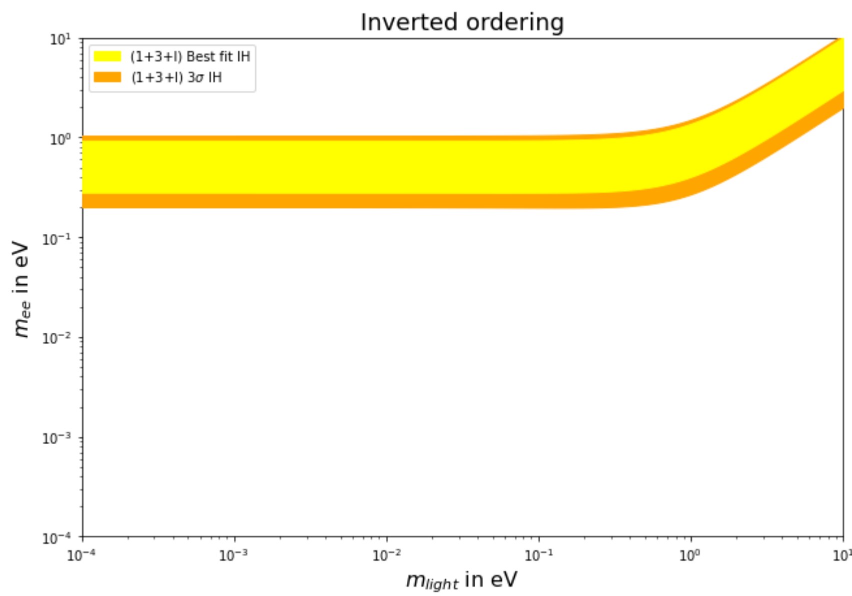
12.7 Heavy m_4 light m_5 (SNSb)



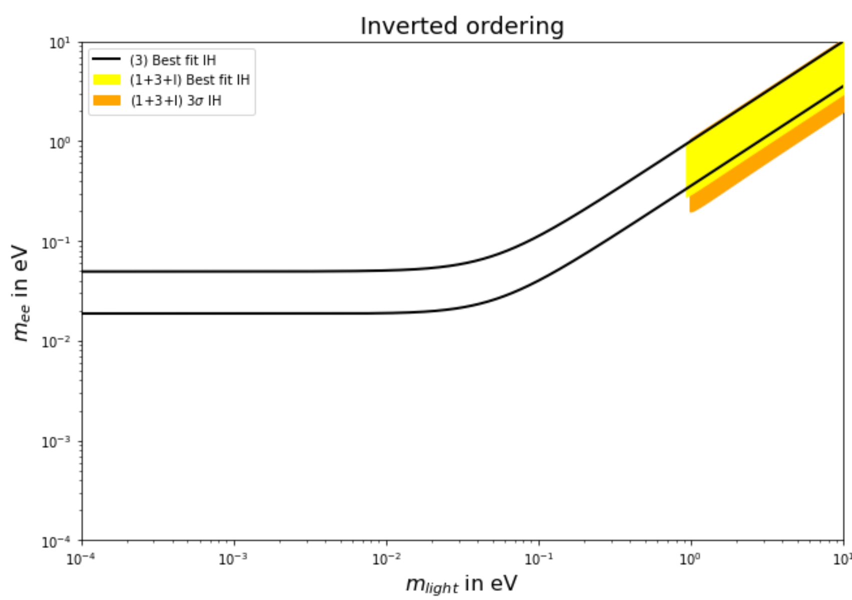
12.8 (active) Heavy m_4 light m_5 (SNSb)



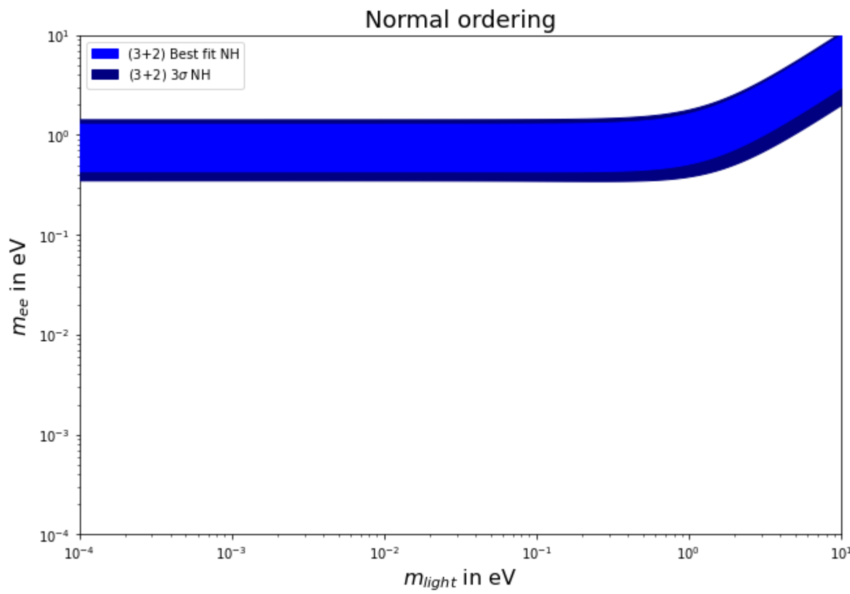
12.9 Heavy m_4 light m_5 (SISb)



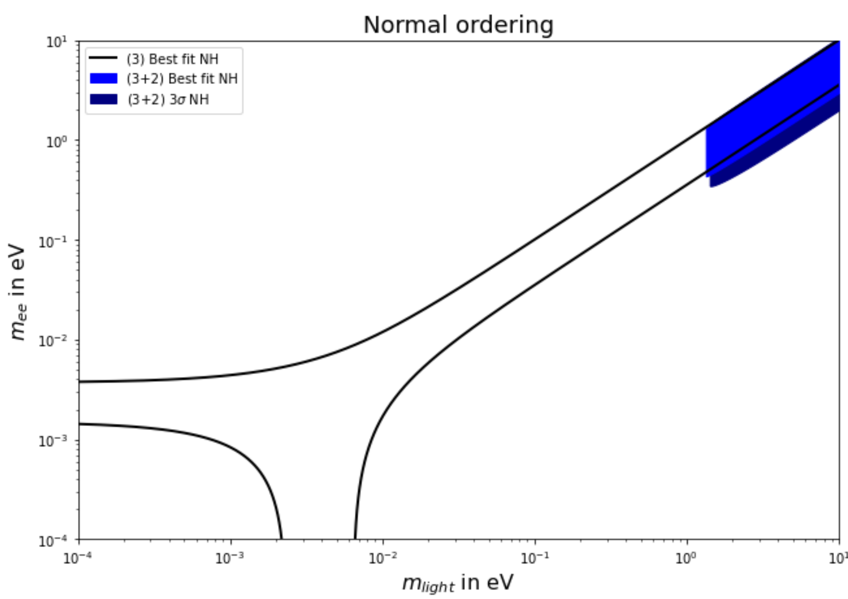
12.10 (active) Heavy m_4 light m_5 (SISb)



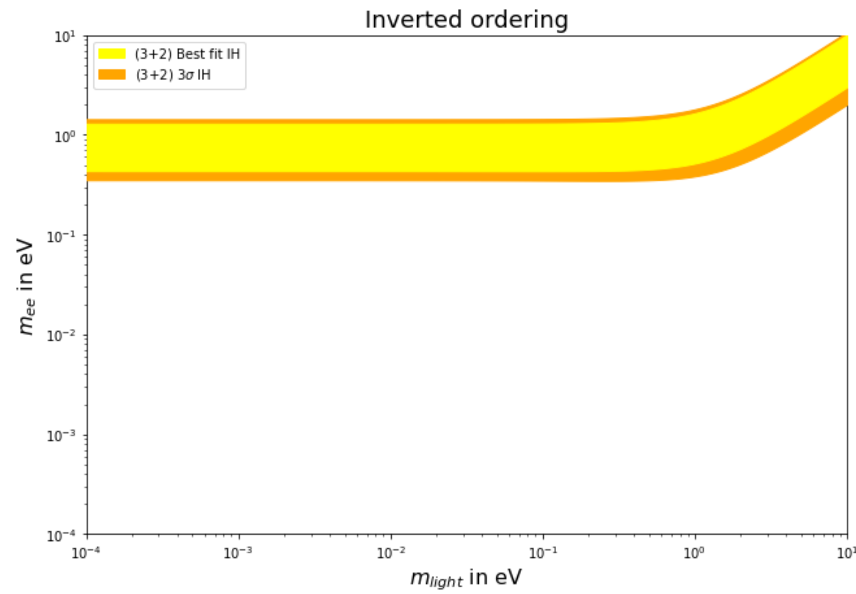
12.11 2 Light steriles (NSS)



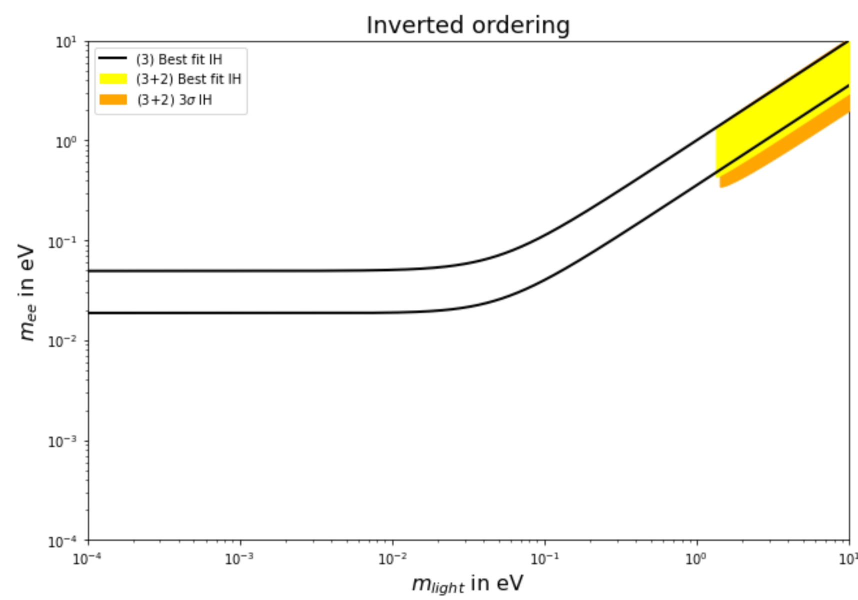
12.12 (active) 2 Light steriles (NSS)



12.13 2 Light steriles (ISS)



12.14 (active) 2 Light steriles (ISS)



13 Dynamic plots and how to make them

In addition to the previously shown plots, we include also animations that shows how the plots change with respect to θ_{14} for the case of 1 sterile neutrino and θ_{15} for the case of 2 sterile neutrinos, (For simplicity reasons, the animations made are first, the ones that transforms the 3 neutrinos model to the 3+1 neutrino model, and then the ones that transforms the 3+1 neutrino models to 3+2 neutrino models.) the purpose of why making the dynamic plots in function of those angles is because that, if those angles become zero, we get back the previous plot (for example, if $\theta_{14} = 0$ we get the 3 neutrino plot, and if $\theta_{15} = 0$ we get back the 3+1 neutrino plot.)

$$\langle m_{ee} \rangle = \left| c_{12}^2 c_{13}^2 c_{14}^2 c_{15}^2 m_1 + s_{12}^2 c_{13}^2 c_{14}^2 c_{15}^2 m_2 e^{i\phi_{12}} + s_{13}^2 c_{14}^2 c_{15}^2 m_3 e^{i\phi_{13}} + s_{14}^2 c_{15}^2 m_4 e^{i\phi_{14}} + s_{15}^2 m_5 e^{i\phi_{15}} \right| \tag{22}$$

if $\theta_{15} = 0$

$$\langle m_{ee} \rangle = \left| c_{12}^2 c_{13}^2 c_{14}^2 m_1 + s_{12}^2 c_{13}^2 c_{14}^2 m_2 e^{i\phi_{12}} + s_{13}^2 c_{14}^2 m_3 e^{i\phi_{13}} + s_{14}^2 m_4 e^{i\phi_{14}} \right| \tag{23}$$

if $\theta_{14} = 0$

$$\langle m_{ee} \rangle = \left| c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 m_2 e^{i\phi_{12}} + s_{13}^2 m_3 e^{i\phi_{13}} \right| \tag{24}$$

The first step to make an animation is choose a "time dependent" value (more than time dependent, in this case would be angle dependent, but for technical reasons we assume that those are angles that change with time.) and as showed in the previous equations 19-21, a good choice are θ_{14} and θ_{15}.

Next, we have to see how those angles will change with respect of time; θ_{14} and θ_{15} have fix values, but we are also interested in see how the animation change with respect of those angles, so, a clever way to solve this could be to make each frame of the animation a fractional part of those angles (or in this case, the s_{ij}^2 and c_{ij}^2), we can represent that like this:

$$s_{ijframe}^2 = \left(\frac{currentframe}{totalframes - 1}\right) s_{ij}^2 \tag{25}$$

$$c_{ijframe}^2 = 1 - s_{ijframe}^2 \tag{26}$$

For example, for these animations, the total frames used are 100, and since we want to start on a point where we do not have the θ_{ij}, the first frame would be zero, (that is why in the equation 25 we substract 1 from the total frames, to include the zero frame on the top).

Now that we know how the animation will change, another important step is to preserve the boundaries of each graph to include all possible angles, but at the same time keep it consistent with how the animation change frame by frame, considering the fact that the boundaries of the plots could change dramatically with the change of θ_{ij}, it is challenging to achieve that. Surprisingly in the case of all IH plots and NH that are not SNH and SSN, the boundaries actually preserved and if the angles of θ_{ij} change, the boundaries were still able to contain and include all possible combinations (Note, in fact the inferior boundaries actually changed a little bit, however since the boundaries were beyond the graph y-scale, it was not necessary to calculate them frame by frame.)

However, the most challenging ones to actually be able to do where the SNH model, due that the inferior left boundary changes, and the transition from one model to the new model was everything but smooth and precise. Fortunately, there was a method that works not only for 4, 5, but potentially for n neutrinos.

14 Boundary generator

To locate and calculate the boudaries for the SNH model, the methodology used was the following, first was use the equations 19-21 (Depending on which model was required to transform) and separate each mass term, but instead of making them dependent to the lightest mass, those were made dependent to the current frame in where there angles are, and for the lightest mass, we pick the smallest value possible for the plot, in this case 0.0001, as it shows here:

14.1 3 neutrinos to 3+1 neutrinos

$$s^2_{14frame} = (\frac{currentframe}{totalframes - 1})s^2_{14} \tag{27}$$

$$c^2_{14frame} = 1 - s^2_{14frame} \tag{28}$$

$m_1 = 0.0001$
$m_2 = \sqrt{m_1^2 + \Delta m^2_{solar}}$
$m_3 = \sqrt{m_1^2 + \Delta m^2_{A}}$
$m_4 = \sqrt{m_1^2 + \Delta m^2_{\mathrm{LSND}}}$

$m^{(1)}_{ee} = c^2_{12}c^2_{13}c^2_{14}m_1$
$m^{(2)}_{ee} = s^2_{12}c^2_{13}c^2_{14}m_2$
$m^{(3)}_{ee} = s^2_{13}c^2_{14}m_3$
$m^{(4)}_{ee} = s^2_{14}m_4$

(Note, we are not including the $e^{i\phi_{ij}}$ because we know from previous pages that the critical values are 0 and π, but that is the equivalent to just add or subtract the $m^{(i)}_{ee}$ in different combinations that we will show now.)

Since we fix the lightest mass to be 0.001, that means those mass terms are in terms of the current frame, but we are not interested in the mass terms by themselves, but interested in how those mass terms add and subtract on this way.

$$m0 = \left|m_{ee}^{(1)} + m_{ee}^{(2)} + m_{ee}^{(3)} + m_{ee}^{(4)}\right|$$
$$m1 = \left|m_{ee}^{(1)} + m_{ee}^{(2)} + m_{ee}^{(3)} - m_{ee}^{(4)}\right|$$
$$m2 = \left|m_{ee}^{(1)} + m_{ee}^{(2)} - m_{ee}^{(3)} + m_{ee}^{(4)}\right|$$
$$m3 = \left|m_{ee}^{(1)} + m_{ee}^{(2)} - m_{ee}^{(3)} - m_{ee}^{(4)}\right|$$
$$m4 = \left|m_{ee}^{(1)} - m_{ee}^{(2)} + m_{ee}^{(3)} + m_{ee}^{(4)}\right|$$
$$m5 = \left|m_{ee}^{(1)} - m_{ee}^{(2)} + m_{ee}^{(3)} - m_{ee}^{(4)}\right|$$
$$m6 = \left|m_{ee}^{(1)} - m_{ee}^{(2)} - m_{ee}^{(3)} + m_{ee}^{(4)}\right|$$
$$m7 = \left|m_{ee}^{(1)} - m_{ee}^{(2)} - m_{ee}^{(3)} - m_{ee}^{(4)}\right|$$

Now that we have this, what we actually want to know is which combination of masses will give the minimum value for each frame, and by looking at which combination of masses gives us the minimum value for certain specific frame, we can then assign specific boundaries for specific frames, making sure the animation is smooth and contains all possible angle combinations (For more details on how this work, here is attached a code named "Boundary generator" at https://github.com/FranciscoTapia61199/Sterile-neutrinos that explains how it works.)

These method was used exclusively to calculate the lower left boundary and how it changes with respect to the frame for the SNH model, however, it can be applied for all models and potentially for n neutrinos, but due that those other models keep the same boundaries even with the change of frames, this was not necessary.

And in this repository https://github.com/FranciscoTapia61199/Sterile-neutrinos at the animation section are the results of how the plots change with respect of angle (Note, all plots regardless of what is the actual lightest mass are for NH in terms of m_1 and for IH in terms of m_3 for experimental purposes, and all plots runs at 15 fps.)

15 Remarkable Observations

The evolution of the plots in comparison to the original 3 neutrino plots is...interesting, the first remarkable observation to made is that the plot, depending on which hierarchy is used, is extremely sensitive, small changes to the mixing angles and mass values would result into changes in shape of the whole graph that are extremely visible for the transition from the 3 neutrino model to the heavy sterile models, it could be with 1 or 2 steriles, the changes are visible for NH and IH cases, each time we are including a heavy mass term, it is hard to distinguish if a neutrino mass is from a NH or IH, but even that is almost indistinguishable, due that that region is located on a low energy area (10^{-2} eV - 10^{-5} eV) is extremely hard to detect with actual equipment.

Thinks are also not looking well for the case where the 1 sterile model transforms to 2 sterile model, if a big change is visible from 3 neutrinos to 1 sterile neutrino model, the inclusion of another sterile makes just small changes to the plot, basically detection regions are the same for NH and IH.

For the case where 1 sterile neutrino was heavy and the other one light, we see little to no visible change in the graph, and also the results are very similar with the case of 2 light sterile neutrinos, regardless of the hierarchy used.

The main observation of this project is that, the most accurate model to predict the mass of the neutrino is the 3 neutrino model due that NH and IH are enough separated to be distinguished and the fact that the hierarchy that is more likely to detect is IH, the moment we are sure of the actual mass value of the active neutrinos, then, it is expected to formulate better techniques to detect the sterile neutrino.