For the initial condition (i) u(x,0) = M:
(a) Write the solution u(x,t):
For condition (i) we have the family of solutions:

$$u := \cos\left(\frac{n \cdot \mathrm{Pi}}{L} \cdot x\right) \cdot an \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right):$$

Finding coefficients an:

$$a0 := \frac{int\left(M \cdot \cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0..L\right)}{int\left(\cos^2\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0..L\right)} \text{assuming}(n = 0, M > 0, L > 0)$$

$$a0 := M \tag{1}$$

$$an := \frac{int\left(M \cdot \cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0..L\right)}{int\left(\cos^2\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0..L\right)} \text{assuming}(L > 0, n = 1)$$

$$an := 0 \tag{2}$$

Then, u(x,t) becomes:
$u := a0$

$$u := M \tag{3}$$

(the temperature is constant for all times)

(b) Average temperature at t = 0:

$$avg\_t0 := \left(\frac{1}{L}\right) \cdot int(u, x = 0..L)$$

$$avg\_t0 := M \tag{4}$$

Average temperature is constant. If we let M=100, then
$avg\_t0 := 100$

$$avg\_t0 := 100 \tag{5}$$

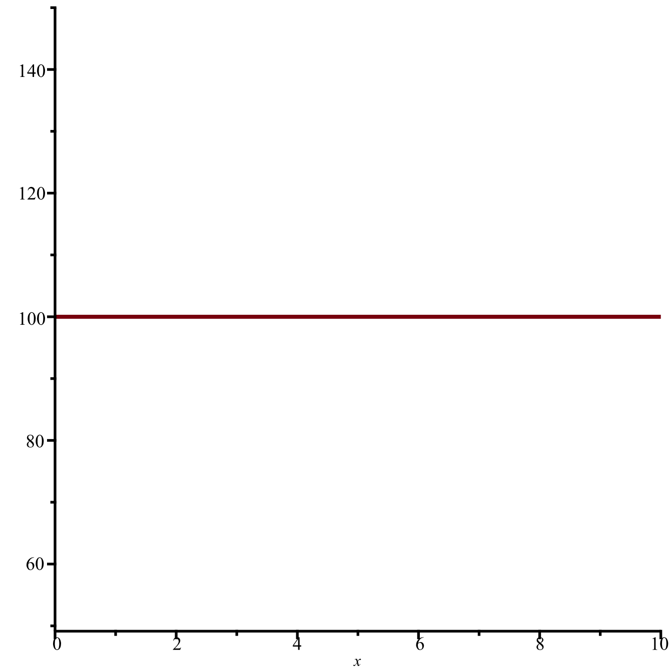
(c) Average temperature expression for any t (and M=100):
$avg\_t := 100$

$$avg\_t := 100 \tag{6}$$

(d) Since the temperature u(x,t) is always constant, it will never decrease

(e)Plotting temperature function u(x,t):

$psum := 100 :$
$with(plots) :$
$plot(psum, x = 0..10)$


140
120
100
80
60
0
2
4
6
8
10
x

Exercise (2) for condition (ii):

$$f := piecewise\left(x < \frac{L}{2}, M, x > \frac{L}{2}, 0\right)$$

$$f := \begin{cases} M & x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \end{cases} \quad (1)$$

(a) Calculating the solution u(x,t):
Family of solutions:

$$u := \cos\left(\frac{n \cdot \mathrm{Pi}}{L} \cdot x\right) \cdot an \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := \frac{2 \cos\left(\frac{n \pi x}{L}\right) \sin\left(\frac{\pi n}{2}\right) M \mathrm{e}^{-\frac{n^2 \pi^2 \mathrm{D} t}{L^2}}}{n \pi} \quad (2)$$

$$a0 := \frac{int\left(M \cdot \cos\left(\frac{\mathrm{Pi} \cdot x}{L}\right), x = 0 .. \frac{L}{2}\right)}{int\left(\cos^2\left(\frac{\mathrm{Pi} \cdot x}{L}\right), x = 0 .. L\right)}$$

$$a0 := \frac{2 M}{\pi} \quad (3)$$

$$an := \frac{int\left(M \cdot \cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0 .. \frac{L}{2}\right)}{int\left(\cos^2\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0 .. L\right)} \mathrm{assuming}(L > 0, n, integer)$$

$$an := \frac{2 \sin\left(\frac{\pi n}{2}\right) M}{n \pi} \quad (4)$$

Then, our family of solutions u(x,t) is:

$$u := a0 + \cos\left(\frac{n \cdot \mathrm{Pi}}{L} \cdot x\right) \cdot an \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := \frac{2 M}{\pi} + \frac{2 \cos\left(\frac{n \pi x}{L}\right) \sin\left(\frac{\pi n}{2}\right) M \mathrm{e}^{-\frac{n^2 \pi^2 \mathrm{D} t}{L^2}}}{n \pi} \quad (5)$$

(b) Average temperature at time t=0:

$$avg\_t0 := \frac{1}{L} int(u, x = 0 .. L)$$

$$avg\_t0 := \frac{2\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^{-\frac{n^2\pi^2 \mathrm{D}\, t}{L^2}}\right)M}{n^2\pi^2} \tag{6}$$

$avg\_t0 := subs(t=0, L=10, M=100, avg\_t0)$

$$avg\_t0 := \frac{200\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^{0}\right)}{n^2\pi^2} \tag{7}$$

$avg\_t0 := evalf(sum(avg\_t0, n=1..900))$

$$avg\_t0 := 57295.77950 \tag{8}$$

The average temperature at t=0 is 57295.77950 degrees for M=100,L=10.

(c) Expression for average temperature at time t:

$avg\_t := \left(\frac{1}{L}\right)\cdot int(u, x=0..L):$

$avg\_t := subs(L=10, M=100, \mathrm{D}=1, avg\_t)$

$$avg\_t := \frac{200\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^{-\frac{n^2\pi^2 t}{100}}\right)}{n^2\pi^2} \tag{9}$$

This is the expression for average temperature at time t.

(d) Since we are not losing heat, the temperature at t0 will not decrease to 10% of its value.Since our initial temperature is given by our initial condition, we know the following:

f

$$\begin{cases} M & x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \end{cases} \tag{10}$$

Therefore, we can expect for the heat to come to an equlibrium temperature on the whole domain (0,L), but it will not go as low as the 10% of the initial average temperature.

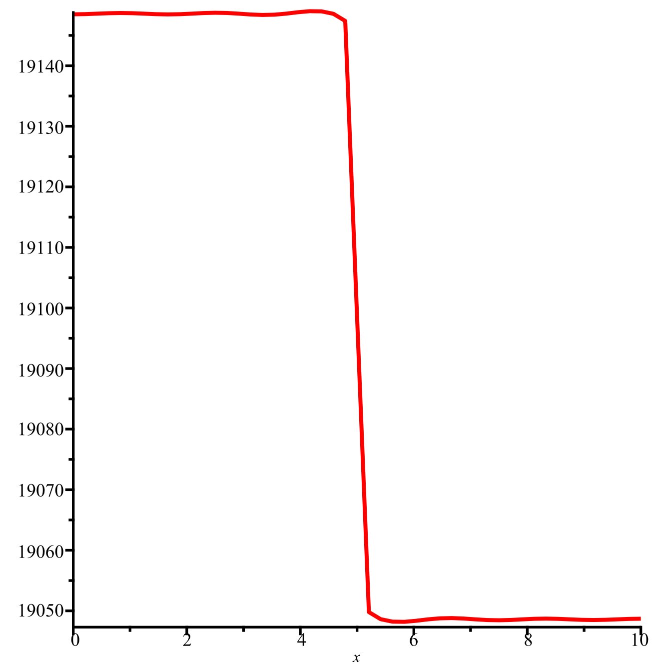
(e) Plots:

$with(plots):$

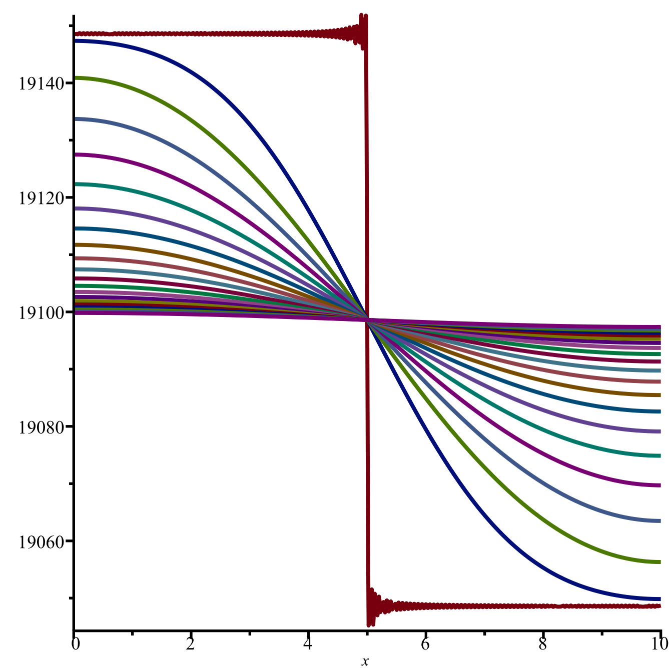
$u\_sol := subs(M=100, L=10, \mathrm{D}=1, u):$

$psum := sum(u\_sol, n=1..300):$

$animate(psum, x=0..10, t=0..20)$



$curves := [seq(subs(t=2\cdot m, psum), m=0..20)]:$
$plot(curves, x=0..10)$



We can see that as time passes, we arrive to an equlibrium temperature on the whole domain (0,10) of approximately 19100 degrees.