MA 501 SP22 Lubkin
HW5 due Fri 3/11/22

Objectives:

- Practice solving the heat equation
- Consolidating understanding of behavior of the heat equation and the significance of BC through observation
- Experience with inhomogeneous BC
- Practice thinking about asymptotic behavior and averages
- More independence deriving solutions
- Deepen understanding of the role of orthogonality in representing a function as a series

1. Solve the heat equation $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, $x \in (0,L)$, with one inhomogeneous BC $u(0,t) = M > 0$, $u(L,t) = 0$ for the initial conditions given.

(i) $u(x,0) = \begin{cases} M, x < L/2 \\ 0, x > L/2 \end{cases}$

(ii) $u(x,0) = Mx/L$

(a) Write the solution $u(x,t)$.

(b) Create a partial sum of satisfactory precision (in your opinion), look at an animation to see how it behaves, and plot solution curves at several times to illustrate the motion. If you can't indicate time by curve color or thickness, then hand-annotate the plot to indicate times. Make sure you run it long enough for us to see both short-term and long-term behavior. I would like to see multiple curves on the same axes but at different times.

(c) State both the solution at steady state and also the asymptotic solution taking the steady-state solution plus the slowest-decaying mode. That is, state

$u(x,\infty) =$ _______

$u(x,t) \approx u(x,\infty) +$ _____ as $t \to \infty$.

2. For the heat equation $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, $x \in (0,L)$, $u(x,0) = f(x)$ with *different* BC, $\frac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$

(a) Separate variables, examine cases 1-2-3 for the separation constant, and find the solutions $u_n(x,t)$ which satisfy the PDE and its BC. Verify to yourself that they do satisfy the PDE and BC.

(b) Superimpose the solutions from (a) to form the general solution, and attempt to satisfy the IC. You will have something which is *not* exactly a Fourier series, because it doesn't have the specific set of periods that a Fourier series should have. Do not attempt to find the coefficients until you know that you have an orthogonal set of functions!

(c) Prove that you do have an orthogonal set of functions $\varphi_n(x) = u_n(x,0)$. Use the same domain for the inner product as for the PDE. You can now be confident that although $u(x,0) = \sum A_n\varphi_n(x)$ isn't a true Fourier series, you do have an orthogonal set of functions on that domain.

(d) Find the coefficients that will satisfy the general IC, by our usual projection method, using the inner product that you used in part (c).

(e) For the specific IC $f(x) = M$, where M is a constant, write the solution.

(f) Plot the time series for u from (e) as you have done before, with multiple curves on the same axes, each at a set time, including $t = 0$ and infinity. Your solution curves should satisfy the IC and BC. (If they don't look right, you either have the wrong coefficients, the wrong form of the solution, or you are not counting all the terms correctly.)