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Project 3

Laplace Eq. in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

where $x = r\cos(\theta)$
$y = r\sin(\theta)$
r, θ functions of x

Domain: $L \leq r$
$0 \leq \theta < 2\pi$

Periodic bc: $u(r,\theta) = u(r, \theta + 2\pi)$

BC in r: $u(L,\theta) = 100$ where $0 < \theta < \frac{\pi}{2}$

$u(L,\theta) = 0$ where $\frac{\pi}{2} < \theta < 2\pi$

Let $u(r,\theta) = R(r) \cdot \Theta(\theta)$

Separations of variables:

$$R'' \cdot \Theta + \frac{1}{r} \cdot R' \Theta + \frac{1}{r^2} \cdot R \cdot \Theta'' = 0$$

$$\Rightarrow r^2 \cdot \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$\Rightarrow \quad \Theta'' + \lambda \Theta = 0 \qquad \wedge \qquad r^2 R'' + r R' - \lambda R = 0$

Periodic BC: $\Theta(\theta) = \Theta(\theta + 2\pi)$ and $\frac{\partial \Theta}{\partial \theta}(\theta) = \frac{\partial \Theta}{\partial \theta}(\theta + 2\pi)$

We know the solution is $\Theta(\theta) = \text{const} \cdot \sin(n\theta) + \text{const} \cdot \cos(n\theta)$,

$n = 0, 1, 2, \ldots$ and $\lambda_n = n^2$. Now, we have $r^2 R'' + r R' - n^2 R = 0$

Let $R(r) = c r^K \Rightarrow R' = cK r^{K-1} \wedge R'' = cK(K-1) r^{K-2}$

plugging in: $r^2 R'' + r R' - n^2 R = r^2 \cdot cK(K-1) r^{K-2} + r \cdot cK r^{K-1} - n^2 c r^K = 0$

$$\Rightarrow cK(K-1) r^K + cK r^K - n^2 c r^K = 0$$

$$c r^K (K(K-1)) + K - n^2) = 0$$

$$\Rightarrow c = 0 \text{ (trivial)} \quad \vee \quad K^2 - \cancel{K} + \cancel{K} - n^2 = 0$$

$$\Rightarrow K^2 - n^2 = 0$$

$$K^2 = n^2$$

$$K = \pm \sqrt{n^2} = \pm n$$

Then $R_n(r) = a_n r^{K_1} + b_n r^{K_2} = a_n r^n + b_n r^{-n}$

$\therefore R_n(r) = a_n r^n + b_n r^{-n}$ for $n > 0$

$n = 1, 2, 3, \ldots$

for case $n=0$: $r^2 R'' + r R' - n^2 R = 0$

$$\Rightarrow r^2 R'' + r R' = 0$$

$$\Rightarrow r R'' + R' = 0$$

Let $R' = Q$ and $R'' = Q'$, then $rR'' + R' = rQ' + Q = 0$

$$\Rightarrow r\frac{\partial Q}{\partial r} + Q = 0 \Rightarrow r\frac{\partial Q}{\partial r} = -Q \Rightarrow \frac{1}{Q}\partial Q = -\frac{1}{r}dr$$

$$\Rightarrow \int \frac{1}{Q}\partial Q = \int -\frac{1}{r}dr \Rightarrow \ln|Q| = -\ln|r| + \text{const.}$$

$$\Rightarrow e^{\ln|Q|} = e^{-\ln|r|} \cdot e^{\text{const}}, \text{ let } e^{\text{const.}} = K_0$$

$$\Rightarrow |Q| = \frac{1}{|r|} \cdot K_0 \Rightarrow Q = \frac{1}{r} \cdot K_0$$

(Since sign of K_0 is not important)

then; since $R' = Q$, $R' = \frac{1}{r} K_0 \Rightarrow \int \partial R = \int \frac{K_0}{r} dr$

$$\boxed{\Rightarrow R(r) = K_0 \ln|r| + d}$$

2 linearly independent ODE

Now, note that $\lim_{r\to\infty} R_0(r) = k_0 \ln(r) + d = \infty$

but we need $u(\infty, \theta) = 25 \neq \infty$ $\therefore$, let $R_0 = d$. Moreover, let $R_0(r) = 25$ to satisfy the BC.

for $R_n(r) = a_n r^n + \frac{b_n}{r^n}$, $n > 1$, note that $\lim_{r\to\infty} R_n = \infty$ which can't happen since $u(\infty, \theta) = 25$. Then, let $a_n = 0$ $\forall n$.

$$\Rightarrow R_n(r) = \frac{b_n}{r^n}$$

Our Solution

$$u(r, \theta) = 25 + \sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n \left(A_n \sin(n\theta) + B_n \cos(n\theta)\right)$$

and our BC at $r = L$ is $u(L, \theta) = f(\theta) = \begin{cases} 100, & 0 \leq \theta \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta < 2\pi \end{cases}$

$$\Rightarrow f(\theta) = 25 + \sum_{n=1}^{\infty} \left(\frac{1}{L}\right)^n \left(A_n \sin(n\theta) + B_n \cos(n\theta)\right)$$

MA 501 FA21 Lubkin, Project 3, due 4/15/22

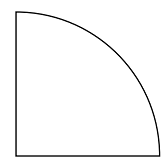
Objectives:

- Practice solving the Laplace equation in polar coordinates
- Experience with knowing when to re-derive something
- Experience visualizing and displaying functions of 2 spatial variables in polar coordinates

Remember to check in your plots whether you are satisfying all the BC. Does your plot look reasonable? Please use enough terms in your partial sum so that it looks nice.

1. Solve the Laplace equation in polar coordinates on the domain *outside* a disk of radius *L.* Let the temperature at r=L be 100 on 1/4 of the circumference, and 0 on the other 3/4. Where is the other BC in *r* if we don't have *r*=0? It's at $r = \infty$, where you should require finite temperature (best choice: 25, why?). Plot your solution as a filled contourplot, including the hole and some of the domain around it (at least one more diameter away from the hole). Use constrained scaling.

2. Solve the Laplace equation $\nabla^2 u(r,\theta) = 0$ on the quarter-disk (that is, 1/4 of a pizza) of radius L, with the BC provided, and plot partial sums as filled `contourplot` as we have done previously. Use constrained scaling. For the 4th BC, which is at r=0, you can use boundedness, or whatever makes your solution seem reasonable (and still satisfying the other BC). There can be contradictions at the corners where the BC meet.



(a) $u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(L,\theta) = 100$

(b) $u(r,0) = 0$, $\frac{\partial u}{\partial \theta}(r,\frac{\pi}{2}) = 0$, $u(L,\theta) = 100$

(c) $u(r,0) = 0$, $u(r,\frac{\pi}{2}) = 100$, $\frac{\partial u}{\partial r}(L,\theta) = 0$