1. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ on a ring $x \in (0, 2\pi)$

Separation of variables:

Let $u(x,t) = X(x) \cdot T(t) \Rightarrow X \cdot T'' = c^2 \cdot X'' \cdot T$

$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = K$, K cont.

Then, $\left.\begin{array}{l} T'' = K c^2 \cdot T \\ X'' = KX \end{array}\right\} \Rightarrow \left.\begin{array}{l} T'' - K c^2 \cdot T = 0 \\ X'' - KX = 0 \end{array}\right\}$ ODE system

We know the solution is for $K < 0$, so let $\boxed{K = -\nu^2}$.

Then our ODE system becomes:

$T'' - (-\nu^2) \cdot c^2 T = T'' + (\nu c)^2 \cdot T = 0$

$\wedge$

$X'' - (-\nu^2) X = X'' + \nu^2 X = 0$

For $X'' + \nu^2 X = 0$, we know solution is $X(x) = a\sin(\nu x) + b\cos(\nu x)$

since $X(x) = X(x+2\pi)$ (periodicity in boundary conditions)

$\Rightarrow \nu = 1, 2, 3, \dots = n$ i.e. ν is an integer > 0

$\therefore \nu = n \quad n = 1, 2, \dots$

$\Rightarrow k = -\nu^2 = -(n^2)$

For IC. :

$T(t) = b_1 \sin(nct) + b_2 \cos(nct)$ is a solution for

we know

$T'' + (nc)^2 T = 0$

homogeneous IC is $T'(0) = 0$,

$\Rightarrow T'(0) = ncb_1 \cos(0) - ncb_2 \sin(0) = 0$

$T'(0) = ncb_1 = 0 \quad \Rightarrow b_1 = 0$

$\Rightarrow T(t) = \cos(nct)$

Now, we have a family of solutions

$$u_n(x,t) = X \cdot T = (a_n \sin(nx) + b_n \cos(nx)) \cdot \cos(nct)$$

Since we have periodic BC $\wedge$ homogeneous IC, we can superimpose.

Then, General solution is:

$$u(x,t) = \sum_{n=0}^{\infty} (a_n \sin(nx) + b_n \cos(nx)) \cdot \cos(nct)$$

PDE ✓

BC ✓✓

IC ✓

For IC $u(x,0) = f(x) = \sum_{n=0}^{\infty} (a_n \sin(nx) + b_n \cos(nx)) \cdot 1$

$$= \sum_{n=0}^{\infty} a_n \sin(nx) + \sum_{n=0}^{\infty} b_n \cos(nx)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx) + b_0 + \sum_{n=1}^{\infty} b_n \cos(nx)$$

$$= b_0 + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$$

$$f(x) = \begin{cases} M, & \frac{\pi}{2} < x < \pi \\ 0, & \text{else} \end{cases}$$

Maple ↓

$phisin := \sin(n \cdot x)$:
$phicos := \cos(n \cdot x)$:

$$An := \frac{int\left(M \cdot phisin, x = \frac{\text{Pi}}{2} .. \text{Pi}\right)}{int\left(phisin^2, x = 0 .. 2 \cdot \text{Pi}\right)} \text{assuming}(n > 0, n, integer)$$

$$An := -\frac{M\left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right)}{n\pi} \quad (1)$$

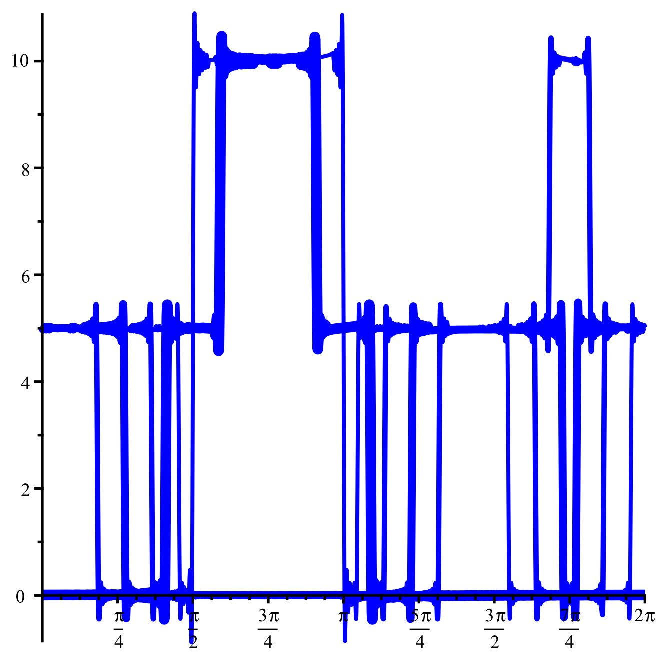
$$B0 := \frac{int\left(M, x = \frac{\text{Pi}}{2} .. \text{Pi}\right)}{int(1, x = 0 .. 2 \cdot \text{Pi})}$$

$$B0 := \frac{M}{4} \quad (2)$$

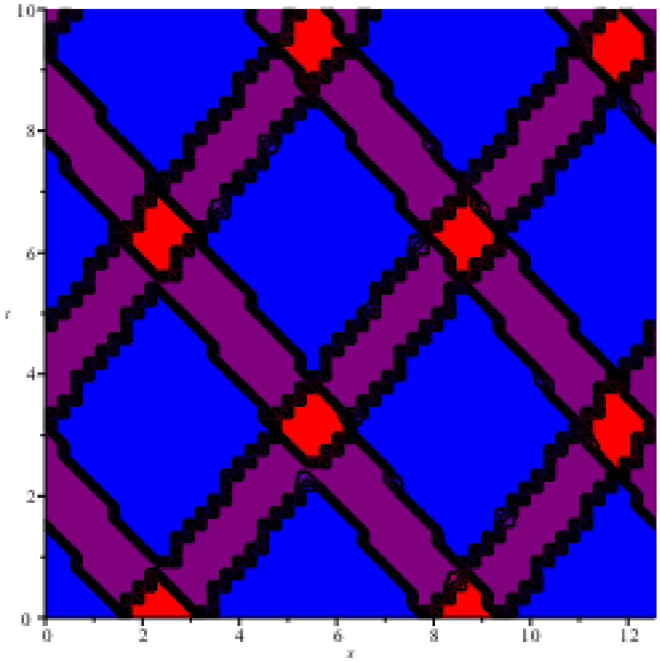
$$Bn := \frac{int\left(M \cdot phicos, x = \frac{\text{Pi}}{2} .. \text{Pi}\right)}{int\left(phicos^2, x = 0 .. 2 \cdot \text{Pi}\right)} \text{assuming}(n > 0, n, integer)$$

$$Bn := -\frac{\sin\left(\frac{n\pi}{2}\right) M}{n\pi} \quad (3)$$

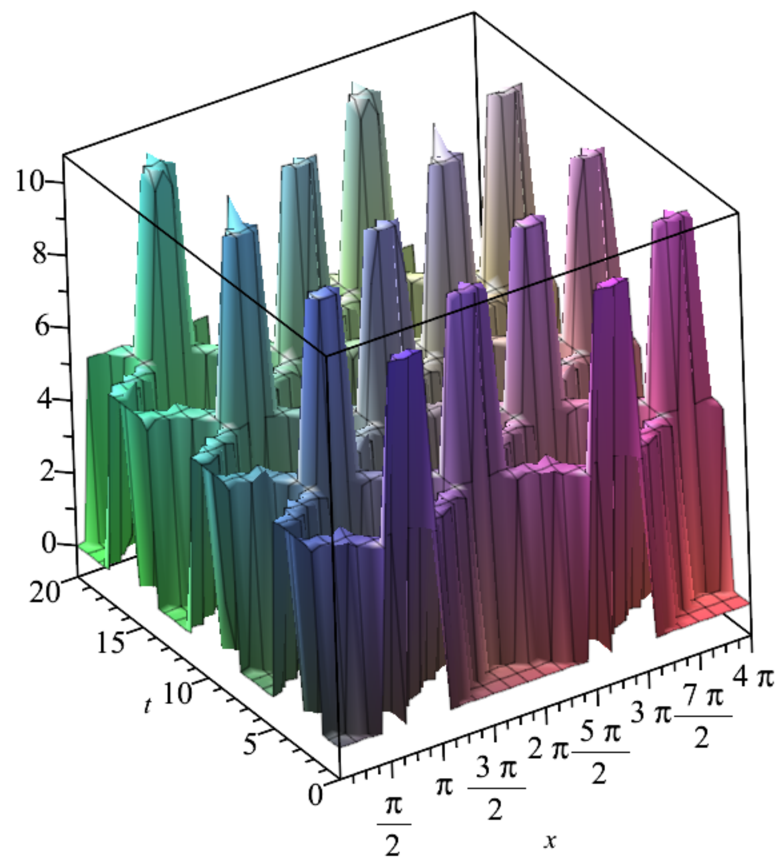
$ux := B0 + sum((An \cdot phisin + Bn \cdot phicos) \cdot \cos(n \cdot c \cdot t), n = 1 .. 200)$:
$with(plots)$:
$uxsubs := subs(M = 10, c = 1, ux)$:
$curves := [seq(subs(t = 2 \cdot m, uxsubs), m = 0 .. 5)]$:
$plot(curves, x = 0 .. 2 \cdot \text{Pi}, thickness = [1, 2, 3, 4], color = blue)$



$contourplot(uxsubs, x = 0 .. 4 \cdot \text{Pi}, t = 0 .. 10, coloring = [blue, red], filledregions = true)$



$plot3d(uxsubs, x = 0 ..4 \cdot \text{Pi}, t = 0 ..20)$



What do you notice?

We have displacement every 2*Pi distances starting at Pi/2 when t=0. The pattern of the waves at the boundaries is always the same, but "inside" the rectangle we do not have the same wave patterns. Inside the rectangle there is a "grid-like" shape pattern.

Why "scaling constrained" is not emphasized?

Maple automatically creates the countour plot or 3dplot in a square domain and not in the domain we specified. The "scaling constrained" command tells Maple to plot the graph in the domained we specified. "Scaling constrained" is not emphasized in this case because we have periodic boundary conditions. The displacement at the boundaries will repeat itself every 2*Pi distances. We previously had to use the "scaling constrained" command because the boundary conditions were not periodic and we were working on the boundaries of a rectangle i.e. our function did not exists beyond this rectangle.