1. $\sinh(x) = \frac{e^{x} - e^{-x}}{2}$, $\cosh(x) = \frac{e^{x} + e^{-x}}{2}$

$$\Rightarrow \sinh(x) + \cosh(x) = \frac{e^{x} - \cancel{e^{-x}} + e^{x} + \cancel{e^{-x}}}{2} = \frac{\cancel{2}e^{x}}{\cancel{2}} = e^{x}$$

$\Rightarrow$ $\boxed{\sinh(x) + \cosh(x) = e^{x}}$ where $\sinh(x)$ is odd

and $\cosh(x)$ is even

---

2. $f(x)$ is periodic if $f(x) = f(x+p)$ $\forall x$.
with period p.

Let $f(x)$ and $g(x)$ be periodic functions with period T.

Then $f(x+T) = f(x)$ and $g(x+T) = g(x)$

Let $h(x) = f(x) + g(x)$

Claim $h(x) = h(x+T)$ always.

Proof:

Since $h(x) = f(x) + g(x)$, $h(x+T) = f(x+T) + g(x+T)$

note that $f(x+T) = f(x)$ and $g(x+T) = g(x)$,

f, g are periodic $\Rightarrow h(x+T) = g(x) + f(x) = h(x)$

$\therefore h(x) = h(x+T)$ $\therefore$ $h(x)$ is periodic.

3. $\sin(x) + \sin(\pi x)$. The period of $\sin(x)$ is 2π.

The period of $\sin(\pi x)$ is $\frac{2\pi}{\pi} = 2$

at the end

Let $h(x) = \sin(x) + \sin(\pi x)$.

if $h(x) = h(x+p)$ for some period p

$\Rightarrow \sin(x) + \sin(\pi x) = \sin(x+p) + \sin(\pi(x+p))$

if $\sin(x) = \sin(x+p) \Rightarrow p = 2\pi$

if $\sin(\pi x) = \sin(\pi(x+p)) \Rightarrow p = 2$

$\Big\} \Rightarrow \Leftarrow$

therefore $\sin(x) + \sin(\pi x)$ can't be periodic.

4. Show $\sin\left(\frac{2\pi n t}{T}\right)$ and $\cos\left(\frac{2\pi m t}{T}\right)$ are orthogonal

if $m=n \Rightarrow \sin\left(\frac{2\pi n t}{T}\right)$ and $\cos\left(\frac{2\pi n t}{T}\right)$. $\{n=m\neq 0\}$

by the identity $\cos(x)\sin(x) = \frac{\sin(2x)}{2}$

$$\Rightarrow \int_0^T \sin\left(\frac{2\pi n t}{T}\right)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{1}{2}\int_0^T \sin\left(\frac{4\pi n t}{T}\right)dt$$

$w = \frac{4\pi n t}{T}$, $dw = \frac{4\pi n}{T}dt \Rightarrow \frac{T}{4\pi n}dw = dt$ $\quad n\neq 0$

then $\frac{1}{2}\int_0^T \sin\left(\frac{4\pi n t}{T}\right)dt = \frac{1}{2}\int_0^T \sin(w)\cdot\frac{T}{4\pi n}dw$

$$= \frac{1}{2}\cdot\frac{T}{4\pi n}\cdot -\cos\left(\frac{4\pi n t}{T}\right)\Bigg|_0^T$$

$$= \frac{T}{8\pi n}\left(-\cos(4\pi n) + \cos(0)\right) = \frac{T}{8\pi n}\left(-\cos(4\pi n)+1\right)$$

sin $\cos(x)$ is even, $\forall n, n\neq 0$ $\frac{T}{8\pi n}\left(-\cos(4\pi n)+1\right) = 0$ ✓

Case 2 $m \neq n$

by the trigonometric identity $\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$

we get $\sin\left(\frac{2\pi n t}{T}\right)\cos\left(\frac{2\pi m t}{T}\right) = \left(\sin\left(\frac{2\pi n t}{T} + \frac{2\pi m t}{T}\right)\right.$

$$\left. - \sin\left(\frac{2\pi n t}{T} - \frac{2\pi m t}{T}\right)\right)\frac{1}{2}$$

Aside: $\frac{2\pi n t}{T} + \frac{2\pi m t}{T} = \frac{2\pi n t + 2\pi m t}{T} = \frac{2\pi t(n+m)}{T}$

$$\frac{2\pi n t}{T} - \frac{2\pi m t}{T} = \frac{2\pi t(n-m)}{T}$$

$\Rightarrow \sin\left(\frac{2\pi n t}{T}\right)\cos\left(\frac{2\pi m t}{T}\right) = \frac{1}{2}\left(\sin\left(\frac{2\pi t(n+m)}{T}\right) - \sin\left(\frac{2\pi t(n-m)}{T}\right)\right)$

then, $\left(\sin\left(\frac{2\pi n t}{T}\right), \cos\left(\frac{2\pi m t}{T}\right)\right) = \frac{1}{2}\int_0^T \sin\left(\frac{2\pi t(n+m)}{T}\right)dt$

$$-\frac{1}{2}\int_0^T \sin\left(\frac{2\pi t(n-m)}{T}\right)dt$$

$$u = \frac{2\pi t(n+m)}{T} \qquad\qquad W = \frac{2\pi t(n-m)}{T}$$

$$du = \frac{2\pi(n+m)}{T}\,dt \qquad\qquad dw = \frac{2\pi(n-m)}{T}\,dt$$

$$\frac{T}{2\pi(n+m)}\,du = dt \qquad\qquad \frac{T}{2\pi(n-m)}\,dw = dt$$

Now, :

$$\frac{1}{2}\int_0^T \frac{\sin(u)\cdot T}{2\pi(n+m)}\,du - \frac{1}{2}\int_0^T \frac{\sin(w)\cdot T}{2\pi(n-m)}\,dw$$

$$= \frac{T}{4\pi(n+m)}\cdot -\cos\left(\frac{2\pi t(n+m)}{T}\right)\Bigg|_0^T + \frac{T}{4\pi(n-m)}\cos\left(\frac{2\pi t(n-m)}{T}\right)\Bigg|_0^T$$

$$= \frac{T}{4\pi(n+m)}\left(-\cos\left(2\pi(n+m)\right) + \cos(0)\right) + \frac{T}{4\pi(n-m)}\left(\cos\left(2\pi(n-m)\right) - \cos(0)\right)$$

Since $\cos(x) = \cos(-x)$, $\cos(2\pi(n+m)) = \cos(-2\pi(n+m)) = 1$

$$= \frac{T}{4\pi(n+m)}\left(-1+1\right) + \frac{T}{4\pi(n-m)}\left(1-1\right) = 0 \checkmark$$

If $m=0$ always $\Rightarrow$ check $\sin\left(\frac{2\pi n t}{T}\right)$, $\cos(0)=1$

is $\sin\left(\frac{2\pi n t}{T}\right) \perp 1$?

$$\int_0^T \sin\left(\frac{2\pi n t}{T}\right) dt, \quad w = \frac{2\pi n t}{T}, \quad dw = \frac{2\pi n}{T} dt \quad \Bigg\} \quad \frac{T\,dw}{2\pi n} = dt$$

$$\Rightarrow \int_0^T \frac{\sin(w)\cdot T}{2\pi n}\, dw = \left. \frac{-\cos\left(\frac{2\pi n t}{T}\right)\cdot T}{2\pi n} \right|_0^T$$

$$= -\frac{\cos(2\pi n)\cdot T}{2\pi n} + \frac{T}{2\pi n} = 0 \checkmark$$

$$\therefore \left(\sin\left(\frac{2\pi n t}{T}\right), \cos\left(\frac{2\pi m t}{T}\right)\right) = 0 \quad \text{for } \begin{array}{l} m=n,\ m\neq 0 \\ m=0,\ n\neq 0 \\ m\neq n \end{array}$$

$\therefore \sin\left(\frac{2\pi n t}{T}\right)$, $\cos\left(\frac{2\pi m t}{T}\right)$ are orthogonal.

5. $\{f_1, f_2, f_3\}$ with $\left.\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= x \\ f_3(x) &= \frac{3x^2-1}{2} \end{aligned}\right\}$ is an orthogonal set if $(f_i, f_j) = 0$ $\forall i,j$, $i \neq j$

$$\int_0^1 f_1 \cdot f_2 \, dx = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 f_1 \cdot f_3 \, dx = \int_0^1 \frac{3x^2-1}{2} \, dx = \frac{1}{2}\int 3x^2 \, dx - \frac{1}{2}\int_0^1 1 \, dx$$

$$= \frac{1}{2} \cdot \left(\frac{\cancel{3} \cdot x^3}{\cancel{3}} - \frac{1}{2} \cdot x\right)\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{2}\left(\frac{2-1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}$$

$$\int_0^1 f_2 \cdot f_3 \, dx = \int_0^1 x \cdot \left(\frac{3x^2-1}{2}\right) dx = \int_0^1 \frac{3x^3 - x}{2} \, dx = \frac{1}{2}\int_0^1 (3x^3 - x) \, dx$$

$$= \frac{1}{2}\left(\frac{3x^4}{4} - \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$$

$\{f_1, f_2, f_3\}$ is not an orthogonal set on the interval $(0,1)$.

on the interval $(-1,1)$: $\int_{-1}^{1} f_1 f_2 \, dx = \int_{-1}^{1} x \cdot 1 \, dx = \frac{x^2}{2} \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$ ✓

$$\int_{-1}^{1} f_1 \cdot f_3 \, dx = \int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2-1) \, dx = \frac{1}{2}\left(\frac{3x^3}{3} - x\right)\Big|_{-1}^{1} = 0$$ ✓

$$= 0.5(1-1) - 0.5(1-1) = 0$$

$$\int_{-1}^{1} f_2 \cdot f_3 \, dx = \int_{-1}^{1} x \cdot \frac{1}{2}(3x^2-1) \, dx = \int_{-1}^{1} \frac{3x^3 - x}{2} \, dx = \frac{1}{2}\left(\frac{3x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{1}$$

$$= 0.5\left(\frac{3}{4} - \frac{1}{2}\right) - 0.5\left(\frac{3}{4} - \frac{1}{2}\right) = 0$$ ✓

$\therefore$, $\{f_1, f_2, f_3\}$ is an $\perp$ set on the interval $(-1,1)$