Francisco Moyet Varges HW 3

1.a) Laplace equation

(a) $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ PDE is: linear homogeneous

Boundary conditions: second order in x and y , H B.C.

Separation of variables: guess $w(x,y) = X(x) \cdot Y(y)$

$$\frac{\partial w}{\partial x} = \frac{\partial (X(x) \cdot Y(y))}{\partial x} = Y(y) \cdot X'(x)$$

$$\boxed{\frac{\partial^2 w}{\partial x^2} = Y(y) \cdot X''(x)} \quad \text{and} \quad \boxed{\frac{\partial^2 w}{\partial y^2} = Y''(y) \cdot X(x)}$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = Y(y) \cdot X''(x) + Y''(y) \cdot X(x) = 0$$

$$\Rightarrow Y(y) \cdot X''(x) = -Y''(y) \cdot X(x)$$

$$\boxed{\Rightarrow \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)}}$$

yes ✓ it is separable

then $\dfrac{X''(x)}{X(x)} = \dfrac{-Y''(y)}{Y(y)} = k$

$\Rightarrow X''(x) - kX(x) = 0$

and

$-Y''(y) - kY(y) = 0 = Y''(y) + kY(y) = 0$

$$\boxed{\begin{aligned} \therefore\ & X''(x) - kX(x) = 0 \\ & Y''(y) + kY(y) = 0 \end{aligned}}$$

1.b) $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = A$ Poisson eq.

not homogeneous

linear

second order in $x \Rightarrow 2$ Boundary Conditions

second order in $y \Rightarrow 2$ Boundary Conditions

$\therefore 4$ B.C.

Let $w(x,y) = X(x)\,Y(y)$

$$\frac{\partial^2 w}{\partial x^2} = X''(x) \cdot Y(y)$$

$$\frac{\partial^2 w}{\partial y^2} = Y''(x) \cdot X(x)$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = A$$

$$X''(x)\cdot Y(y) + Y''(y)\cdot X(x) = A$$

$$\Rightarrow Y''(y)\cdot X(x) = A - X''(x)\cdot Y(y)$$

$$\Rightarrow \frac{Y''(y)\cdot}{Y(y)} = \frac{A}{X(x)\cdot Y(y)} - \frac{X''(x)}{X(x)}$$

$\therefore \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = A$ is not separable

1.c) $\frac{du}{dt} + c(1+au)\frac{du}{dx} = D \cdot \frac{d^2u}{dx^2}$

$$\frac{du}{dt} + c(1+au)\frac{du}{dx} - D \cdot \frac{d^2u}{dx^2} = 0$$

a, c, D are parameters → nonlinear because the coefficient of derivative terms depends on the unknown function.

Boundary Conditions : 2 (2nd order in x)

Initial Conditions : 1 (order 1 in t)

ii) Is it separable?

let $u(x,t) = X(x) \cdot T(t)$

$$\frac{du}{dt} = X(x) \cdot T'(t) \qquad \frac{d^2u}{dx^2} = X''(x) \cdot T(t)$$

$$\frac{du}{dx} = X'(x) \cdot T(t)$$

Now we have:

$$T'(t)\cdot X(x) + c(1+au)\,X'(x)\cdot T(t) - D\,X''(x)\cdot T(t) = 0$$

$$\Rightarrow \frac{T'(t)\cdot \cancel{X(x)}}{T(t)\cdot \cancel{X(x)}} = \frac{-c(1+au)\,X'(x)\cdot T(t) + D\,X''(x)\cdot T(t)}{T(t)\cdot X(x)}$$

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{-c(1+au)\,X'(x)}{X(x)} \qquad (*)$$

but $u(x,t) = X(x)\cdot T(t)$

$$\Rightarrow \frac{-c(1+au)\,X'(x)}{X(x)} = -\frac{cX'(x)}{X(x)} - \frac{ca\,\cancel{X(x)}\cdot T(t)\cdot X'(x)}{\cancel{X(x)}}$$

$$= -\frac{cX'(x)}{X(x)} - ca\,T(t)\cdot X'(x)$$

Then $(*)$ becomes:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} - \frac{cX'(x)}{X(x)} - \underbrace{ca\,T(t)\cdot X'(x)}_{\text{separation fails}}$$

$\therefore$ is not separable.

1. d) $\frac{d^2u}{dx^2} + \frac{d^2u}{dx\,dy} + \frac{d^2u}{dy^2} = 0$

PDE
constant coefficiats
linear

homogeneous
(since $u = 0$ is sol.)

Boundary Condition:

Since is 2nd order in x and y, 4 B.C.

(i) Is it separable?

Let $u(x,y) = X(x) \cdot Y(y)$

$$\frac{d^2u}{dx^2} = X''(x) \cdot Y(y)$$

$$\frac{d^2u}{dy^2} = X(x) \cdot Y''(y)$$

$$\frac{d^2u}{dx\,dy} = \frac{d}{dx}\left(\frac{du}{dy}\right) = \frac{d}{dx}\left(Y'(y) \cdot X(x)\right) = X'(x) \cdot Y'(y)$$

Now, we have:

$$\left(X''(x)\cdot Y(y) + X'(x)\cdot Y'(y) + Y''(y)\cdot X(x) = 0\right)\frac{1}{Y(y)\cdot X(y)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \underbrace{\frac{X'(x)\cdot Y'(y)}{X(x)\cdot Y(y)}}_{\text{separation fails for this value}} + \frac{Y''(y)}{Y(y)} = 0$$

i.e. $$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} - \underbrace{\frac{X'(x)\,Y'(y)}{X(x)\,Y(y)}}_{\text{separation fails for this value.}}$$

$\therefore$ Separation fails

$u(x,y)$

1\. e) $(\sin(x)) \cdot \dfrac{d^2u}{dx^2} + (\cos(y)) \cdot \dfrac{d^2u}{dy^2} = 0$

function of independent variable

linear PDE, homogeneous

Since second order in x and second order in y,

4 Boundary Conditions

ii) Is it separable? Let $u(x,y) = X(x) \cdot Y(y)$

$$\Rightarrow \sin(x) \cdot \frac{d^2u}{dx^2} + \cos(y) \cdot \frac{d^2u}{dy^2} = \sin(x) \cdot X''(x) \cdot Y(y) + \cos(y) \cdot Y''(y) \cdot X(x) = 0$$

$$\Rightarrow \cos(y) \cdot Y''(y) \cdot X(x) = -\sin(x) \cdot X''(x) \cdot Y(y)$$

$$\Rightarrow \cos(y) \cdot \frac{Y''(y)}{Y(y)} = -\sin(x) \cdot \frac{X''(x)}{X(x)} = k$$

<u>yes</u> it is separable.

ODE's :

$$\cos(y) \cdot Y'' - kY = 0$$

$$-\sin(x)\,X'' - kX = \sin(x)X'' + kX = 0$$

2. For the Wave Equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(a) $u_1(x,t) = f(x-ct)$

$\frac{\partial u_1}{\partial t} = -c \cdot f'(x-ct)$

$\frac{\partial^2 u_1}{\partial t^2} = c^2 \cdot f''(x-ct)$

$\frac{\partial u_1}{\partial x} = f'(x-ct)$

$\frac{\partial^2 u_1}{\partial x^2} = f''(x-ct)$

$\therefore$ since $\frac{\partial^2 u_1}{\partial x^2} = f''(x-ct)$

$\frac{\partial^2 u_1}{\partial t^2} = c^2 f''(x-ct)$

$\Rightarrow \frac{\partial^2 u_1}{\partial t^2} = c^2 \cdot \frac{\partial^2 u_1}{\partial x^2}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ wave equation

(b). $u_2(x,t) = f(x+ct)$

$$\frac{\partial u_2}{\partial t} = c \cdot f'(x+ct)$$

$$\frac{\partial^2 u_2}{\partial t^2} = c^2 \cdot f''(x+ct)$$

and $$\frac{\partial u_2}{\partial x} = f'(x+ct), \quad \frac{\partial^2 u_2}{\partial x^2} = f''(x+ct)$$

$\therefore$ Since $\frac{\partial^2 u_2}{\partial t^2} = c^2 \cdot f''(x+ct)$ and

$$f''(x+ct) = \frac{\partial^2 u_2}{\partial x^2} \Rightarrow \frac{\partial^2 u_2}{\partial t^2} = c^2 \cdot \frac{\partial^2 u_2}{\partial x^2}$$

$\therefore$ $u_2(x,t)$ is a solution.

(c) $u_3(x,t) = f(x-ct) + f(x+ct)$

Note

$\Rightarrow u_3 = u_1 + u_2$

$$\frac{\partial u_3}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} = -c\, f'(x-ct) + c \cdot f'(x+ct)$$

$$\frac{\partial^2 u_3}{\partial t^2} = c^2 f''(x-ct) + c^2 \cdot f''(x+ct) = c^2\left(f''(x-ct) + f''(x+ct)\right)$$

$$\boxed{\Rightarrow \frac{\partial^2 u_3}{\partial t^2} = c^2\left(f''(x-ct) + f''(x+ct)\right)}$$

$$\frac{\partial u_3}{\partial x} = f'(x-ct) + f'(x+ct)$$

$$\boxed{\frac{\partial^2 u_3}{\partial x^2} = f''(x-ct) + f''(x+ct)}$$

$$\therefore \frac{\partial^2 u_3}{\partial t^2} = c^2 \cdot \frac{\partial^2 u_3}{\partial x^2}$$

$\therefore$ c is solution.