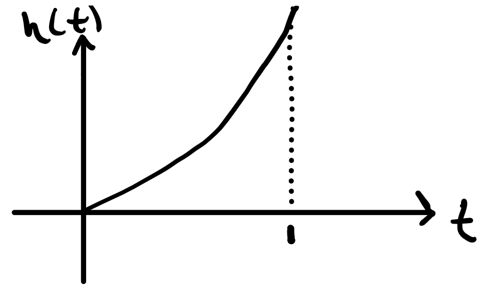
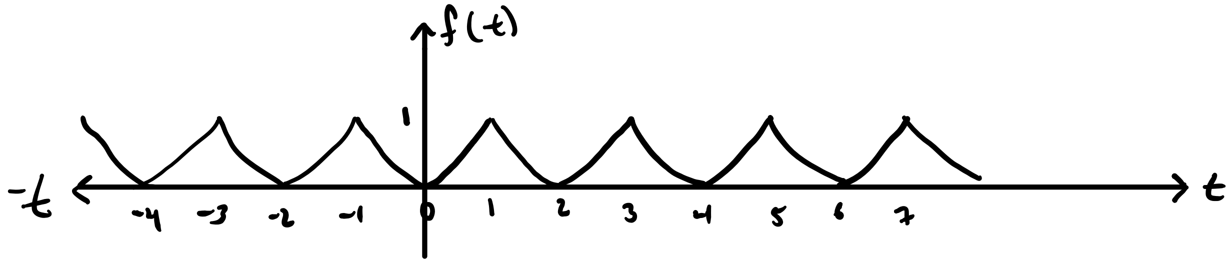
1.) t^2 , $t \in [0,1]$

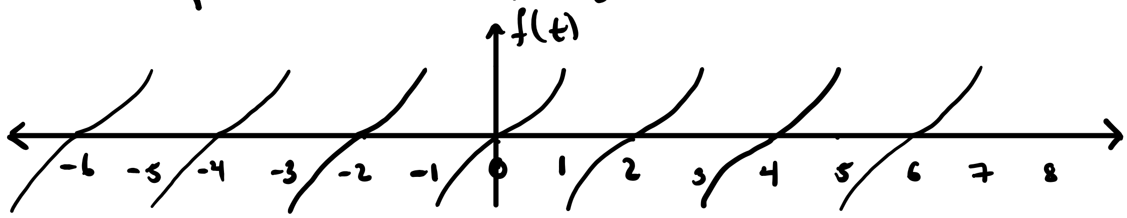


even periodic extension:



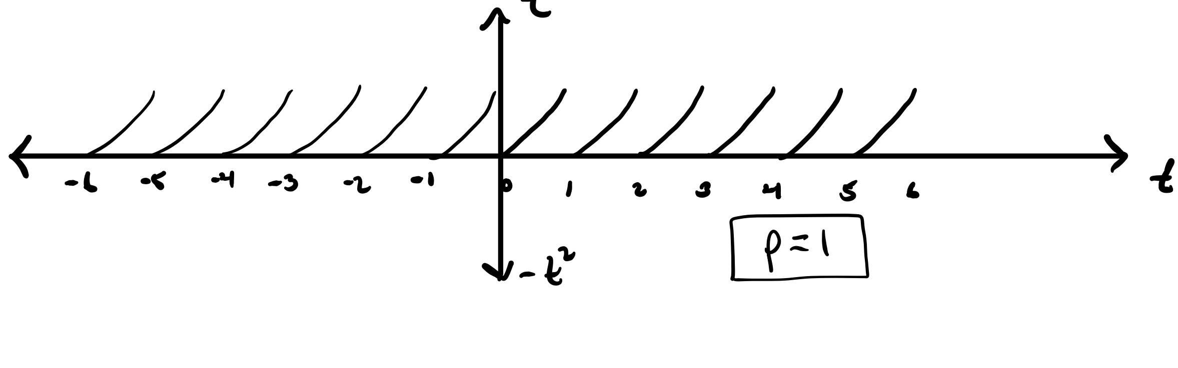
period = 2

odd periodic extension:



period $p = 2$

Periodic Extension:



$\boxed{p = 1}$

For the even periodic extension:

$$t^2 \sim \sum_{n=0}^{\infty} A_n \phi_n \quad , \quad \phi_n(t) = \left\{ \cos\left(\frac{\cancel{2}\pi n t}{\cancel{2}} \right) \right\}$$

$$t^2 \sim \sum_{n=0}^{\infty} A_n \cos(\pi n x) \quad . \quad f(t) = t^2$$

$$A_0 = \frac{\int_{-1}^{1} t^2 \cdot 1 \, dt}{\int_{-1}^{1} 1 \, dt} = \frac{\left. \frac{t^3}{3} \right|_{-1}^{1}}{\left. t \right|_{-1}^{1}} = \frac{\frac{1}{3} + \frac{1}{3}}{1+1} = \frac{\frac{2}{3}}{2}$$

$$= \frac{1}{3} = A_0$$

Our even Fourier Series is:

$$f(t) \sim \frac{1}{3} + \sum_{n=1}^{\infty} A_n \cos(n\pi t)$$

Using Maple to find A_n:

$$A_n = \frac{\int_{-1}^{1} \overbrace{t^2 \cdot \cos(n\pi t)}^{\text{even}} \, dt}{\int_{-1}^{1} \underbrace{\cos^2(n\pi t)}_{\text{even}} \, dt} = \frac{\cancel{2} \int_0^1 t^2 \cos(n\pi t) \, dt}{\cancel{2} \int_0^1 \cos^2(n\pi t) \, dt}$$

$$A_n = \frac{4(-1)^n}{(n\cdot\pi)^2} \qquad n \geq 1.$$

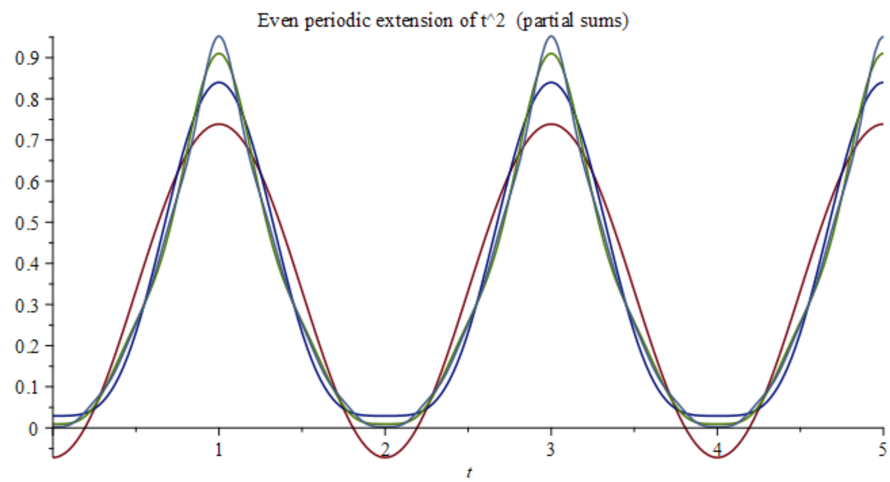
Answer for even extension:

$\therefore$ Fourier extension is $f(t) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{(n\cdot\pi)^2} \cdot \cos(\pi n t)$

(a) period $T = \frac{2\cancel{\pi}}{\cancel{\pi} n} = \frac{2}{n}$, common period $= 2$

$$\phi_n(t) = \cos(\pi n t), \quad A_0 = \frac{1}{3}$$

$$A_n = \frac{4(-1)^n}{(n\cdot\pi)^2}, \quad n > 0$$



For the odd periodic extension:

$$f(t) = t^2 \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n t}{p}\right) \qquad p = 2$$

Fourier terms:

$$\phi_n(t) = \sin\left(\frac{\cancel{2}\pi n t}{\cancel{2}}\right) = \sin(\pi n t)$$

The function is : $f(t) = \begin{cases} t^2 , & 0 < t < 1 \\ -t^2 , & -1 < t < 0 \end{cases}$

$$B_n = \frac{\int_{-1}^{1} f \cdot \sin(\pi n t)\, dt}{\int_{-1}^{1} \underbrace{\sin^2(\pi n t)}_{\text{even}}\, dt} \quad = 2\int_0^1 \sin^2(\pi n t)\, dt$$

$$\Rightarrow B_n = \frac{\int_{-1}^{0} -t^2 \sin(n\pi t)\, dt + \int_0^1 t^2 \sin(n\pi t)\, dt}{2 \cdot \int_0^1 \sin^2(n\pi t)\, dt}$$

Using Maple, we arrive to :

$$B_n = \frac{-2\pi^2 n^2 (-1)^n + 4(-1)^n - 4}{n^3 \cdot \pi^3} \qquad n \geq 1, \; n \text{ integer}$$

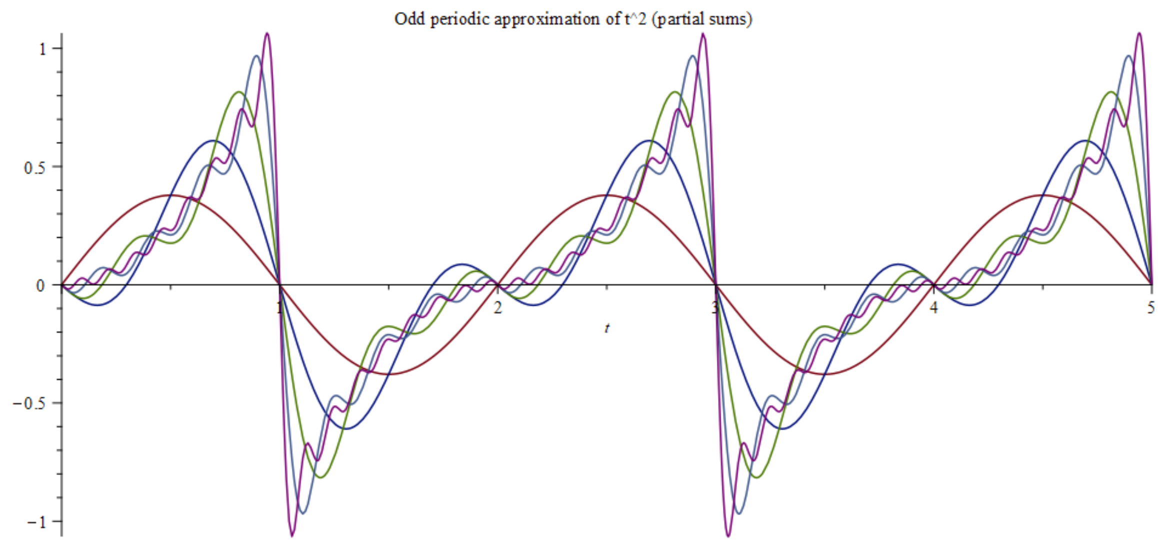
Odd :

(b) Extension period $T = \frac{2\cancel{\pi}}{\cancel{\pi} n} = \frac{2}{n}$

Fourier terms $\phi_n(x) = \sin(n \cdot \pi t)$

(c) Fourier Coefficients : (using Maple)

$$B_n = \frac{-2\pi^2 n^2 (-1)^n + 4(-1)^n - 4}{n^3 \pi^3} \qquad n > 0$$



Observation: the quality (i.e. looks more like the sketch) is better if n is big.

Periodic Extension:

period $p=1$

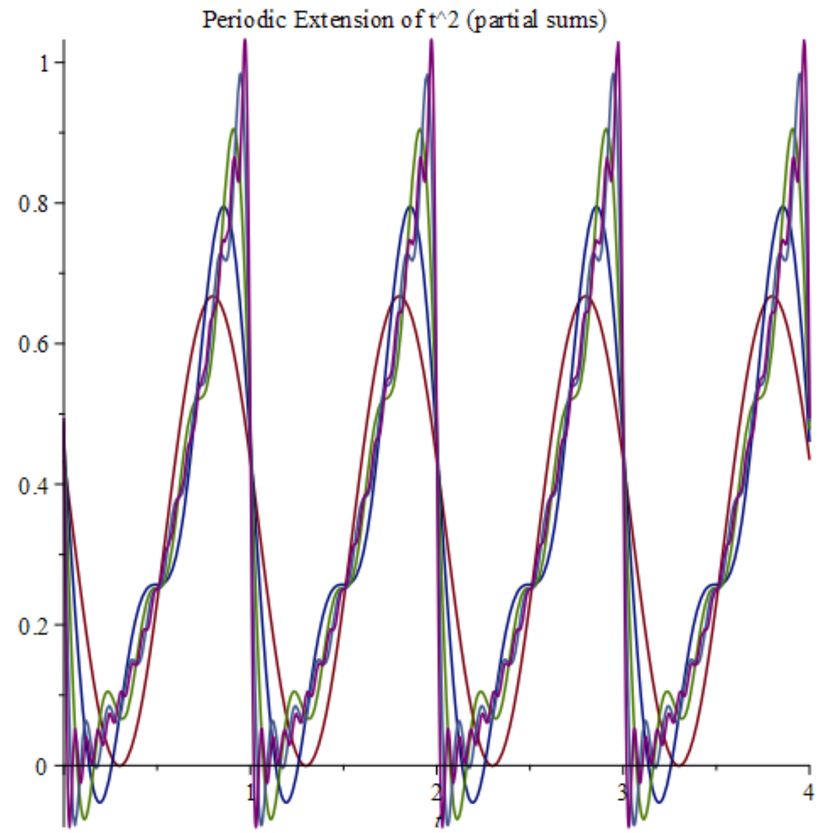
$$\phi_n(x) = \{\cos(2\pi n x), \sin(2\pi n x)\}$$

F.S. is given by

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n x) + B_n \sin(2\pi n x)$$

where $A_0 = \frac{1}{3}$, $A_n = \frac{1}{(\pi \cdot n)^2}$

$B_n = \frac{-1}{\pi n}$.



2. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ Let $u(x,t) = X(x)T(t)$

BC: $u(0,t) = u(L,t) = 0$

$\Rightarrow$ wave equation becomes $T''(t) \cdot X(x) = c^2 X''(x) \cdot T(t)$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = k \quad \Rightarrow \quad \left.\begin{aligned} X'' - kX &= 0 \\ T'' - kc^2 T &= 0 \end{aligned}\right\} \text{two ODE's}$$

We know that the solution for $X'' - kX = 0$ occurs for $k < 0$

Then, let $k = -\nu^2$. $\nu^2 > 0$

$$X'' - kX = X'' - (-\nu^2)X = X'' + \nu^2 X = 0$$

Let $X(x) = a \sin(\nu x) + b \cos(\nu x)$

$X'(x) = a\nu \cos(\nu x) - b\nu \sin(\nu x)$

$X''(x) = -a\nu^2 \sin(\nu x) - b\nu^2 \cos(\nu x)$

$X'' = -\nu^2 (a \sin(\nu x) + b \cos(\nu x)) = -\nu^2 X(x)$

then $X'' + \nu^2 X = -\nu^2 X + \nu^2 X = 0$ ✓ $\therefore X(x)$ is solution.

$\therefore X(x) = a \sin(\nu x) + b \cos(\nu x)$ is solution.

Checking boundary Conditions :

$$u(0,t) = X(0) = 0 = a \cancelto{0}{\sin(0)} + b\cos(0) = 0 = b \cdot 1 = 0 = b$$

$\therefore b = 0$

$$\Rightarrow X(x) = a \sin(\nu x)$$

$$u(L,t) = X(L) = a \sin(\nu L) = 0 \begin{cases} \cancel{a = 0} \text{ trivial} \\ \sin(\nu L) = 0 \end{cases}$$

$$\sin(\nu L) = 0 \iff \nu L = n \cdot \pi \iff \nu = \frac{n \cdot \pi}{L}$$

$$n = 1, 2, 3, \ldots$$

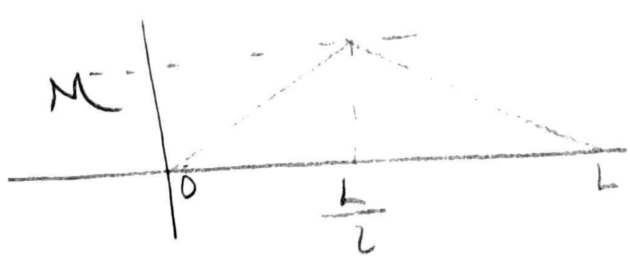
since $k = -\nu^2 \Rightarrow k = -\left(\frac{n \cdot \pi}{L}\right)^2$

$$\Rightarrow X(x) = a \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$$

For the initial condition (i)

(i) $u(x,0) = f(x)$

$$\frac{\partial u(x,0)}{\partial t} = 0$$



$$f(x) = \begin{cases} \frac{2 \cdot M}{L} \cdot x, & 0 < x < \frac{L}{2} \\ \frac{2M}{L}(L - x), & \frac{L}{2} < x < L \end{cases}$$

$$T'' - c^2 k T = 0 = T'' - c^2 \cdot (-\nu^2) T = T'' + c^2 \nu^2 T = 0$$

where $\nu^2 = \left(\frac{n \cdot \pi}{L}\right)^2$

Let $T(t) = a_2 \sin(c\nu t) + b_2 \cos(c\nu t)$

$$T' = a_2 c\nu \cos(c\nu t) - b_2 c\nu \sin(c\nu t)$$

$$T'' = -a_2 (c\nu)^2 \sin(c\nu t) - b_2 (c\nu)^2 \cos(c\nu t)$$

$$\Rightarrow T'' = -(c\nu^2)\left(a_2 \sin(c\nu t) + b_2 \cos(c\nu t)\right) = -(c\nu)^2 T$$

$$\therefore T'' + (c\nu)^2 T = -(c\nu)^2 T + (c\nu)^2 T = 0 \checkmark$$

I.C.

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \left(a_2 c\nu \cos(0) - \cancel{b_2 c\nu \sin(0)}^{\,0}\right) = 0$$

$$= \sin\left(\left(\frac{n \cdot \pi}{L}\right) \cdot x\right) \cdot a_2 c\nu \cdot 1 = 0 \qquad \forall x$$

$$\Rightarrow a_2 = 0 \checkmark$$

$$\Rightarrow T_n(t) = b_n \cos(c\omega t) = b_n \cos\left(\frac{n \cdot \pi}{L} \cdot t\right)$$

(a)

By Principle of superposition, we have the family of solutions:

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot b_n \cos\left(\frac{n \cdot \pi}{L} \cdot t\right)$$

with period on t $\quad p = \frac{2\pi}{n\pi} \cdot L = \frac{2L}{n}$ and

common period $2L$.

$u_n(x,t)$ satisfies BC ✓✓

IC ✓ $\left(\frac{\partial u}{\partial t}(x,0) = 0\right)$

For IC $f(x) = u_n(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$

$$f(x) = \begin{cases} \frac{2M}{L} \cdot x, & 0 < x < L/2 \\ \frac{2M}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

$$b_n = \frac{\int_0^{L/2} \frac{2M \cdot x}{L} \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) dx + \int_{L/2}^{L} \frac{2M}{L}(L-x) \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) dx}{\int_0^L \sin^2\left(\frac{n \cdot \pi}{L} \cdot x\right) dx}$$

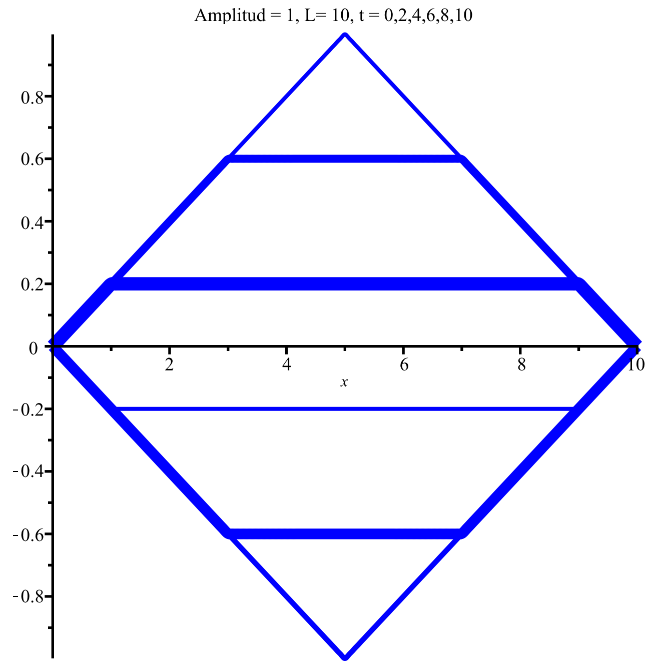
by using Maple, we arrive to:

$$b_n = \frac{8 \cdot M \sin\left(\frac{n\pi}{2}\right)}{(\pi n)^2}$$ assuming $L > 0$, n, integer

$\therefore$ Our general solution is:

$$u_n(x,t) = \sum_{n=1}^{\infty} \frac{8 \cdot M \cdot \sin\left(\frac{n \cdot \pi}{2}\right)}{(\pi n)^2} \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \cos\left(\frac{n \cdot \pi}{L} \cdot t\right)$$

of period in t, $p = \frac{2\cancel{\pi}}{n \cdot \cancel{\pi}} \cdot L = \frac{2L}{n}$



Amplitud = 1, L= 10, t = 0,2,4,6,8,10
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
2
4
6
8
10
x

(ii) $u(x,0)=0$

$$\frac{\partial u}{\partial t}(x,0)=f(x)$$

where $f(x)=\begin{cases} \frac{2M}{L}\cdot x, & 0<x<\frac{L}{2} \\ \frac{2M(L-x)}{L}, & \frac{L}{2}<x<L \end{cases}$

We know:

$X(x)= \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)$

$T(t)= a\sin(c\nu t)+b\cos(c\nu t)$, $\nu=\frac{n\cdot\pi}{L}$

since $u(x,t)=X(x)T(t)$

$u(x,0)=X(x)\cdot T(0)=0$

$\Rightarrow a\cancel{\sin(0)}+b\cos(0)=0 \;=b \quad \Rightarrow b=0$

$\Rightarrow T(t)$ becomes: $T(t)=a\sin(c\nu t)$

and $T'(t)=a\frac{c\,n\cdot\pi}{L}\cdot\cos\left(\frac{n\cdot\pi c}{L}\cdot x\right)$

since $\frac{\partial u}{\partial t}(x,0) = f(x)$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{a_n \cdot c \cdot n \cdot \pi}{L} \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) = g(x)$$

let $K_n = \frac{a_n c n \cdot \pi}{L} \Rightarrow a_n = \frac{K_n}{c \cdot n \cdot \pi} \cdot L$

then $\sum_{n=1}^{\infty} \frac{a_n c \cdot n \cdot \pi}{L} \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$

and $u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot a_n \cdot \sin\left(\frac{c \cdot n \cdot \pi}{L} \cdot t\right)$

with period (in t) $\frac{2\cancel{\pi}}{cn\cancel{\pi}} \cdot L = \frac{2L}{cn}$

$$> kn := \frac{int\left(\frac{2\cdot M\cdot x}{L}\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), x=0..\frac{L}{2}\right) + int\left(\frac{2\cdot M\cdot(L-x)}{L}\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), x=\frac{L}{2}..L\right)}{int\left(\sin^2\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), x=-L..L\right)} :$$

$> simplify(kn) :$

$> kn := simplify(kn) :$

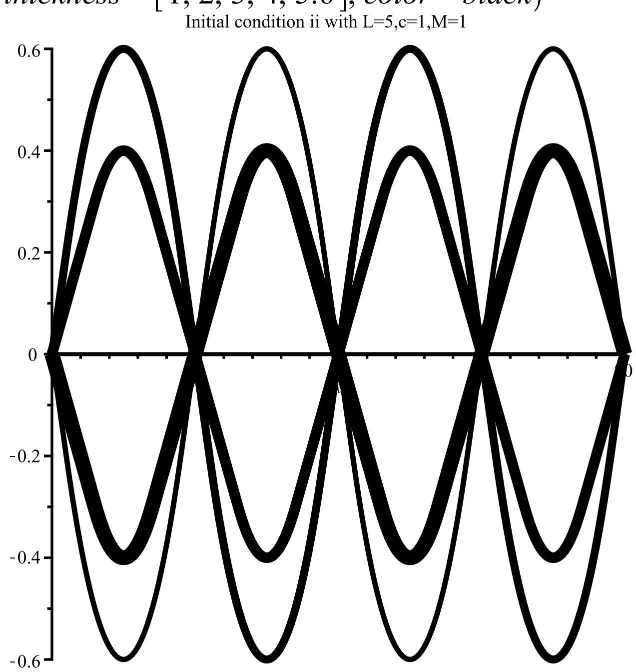
$$> an := \frac{kn\cdot L}{c\cdot n\cdot \mathrm{Pi}} :$$

$> with(plots) :$

$$> psum := subs\left(M=1, L=5, c=1, sum\left(an\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right)\cdot\sin\left(\frac{c\cdot n\cdot \mathrm{Pi}\cdot t}{L}\right), n=1..100\right)\right) :$$

$> curves := \{seq(subs(t=2\cdot m, psum), m=0..10)\} :$

$> plot(curves, x=0..20, thickness=[1, 2, 3, 4, 5.6], color=black)$



$> animate(psum, x=0..20, t=0..10) :$

>

(iii)

We have $u(x,0) = 0$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{where}$$

$$g(x) = \begin{cases} M, & \frac{L}{4} < x < \frac{L}{2} \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow T(0) = \cancel{a \sin(0)}^{0} + b \cos(0) = 0 = b$$

$\therefore b = 0$

$$\Rightarrow T(t) = a \sin\left(\frac{c \cdot n \cdot \pi}{L} \cdot t\right)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{and} \quad T'(0) = \frac{a c n \cdot \pi}{L} \cdot \cos(0)$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \underbrace{\frac{a_n c \cdot n \cdot \pi}{L}}_{= K_n}$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot K_n$$

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot b_n \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right).$$ for IC

$$u_n(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot b_n$$ where

$$g(x) = \begin{cases} M, & \frac{L}{4} < x < \frac{L}{2} \\ 0, & \text{else} \end{cases}$$

By Maple, we end up with

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-M\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)}{n\pi} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot \cos\left(\frac{c \cdot n\pi}{L} \cdot t\right)$$

period in t : $\frac{2\pi}{cn\pi} L = \frac{2L}{cn}$

Maple :

> $bn := \dfrac{int\left(M\cdot\sin\left(\dfrac{n\cdot\mathrm{Pi}\cdot x}{L}\right), x=\dfrac{L}{4}..\dfrac{L}{2}\right)}{int\left(\sin^2\left(\dfrac{n\cdot\mathrm{Pi}\cdot x}{L}\right), x=-L..L\right)}$ assuming$(L > 0, n, integer, n > 0)$

$$bn := -\frac{M\left(\cos\left(\frac{n\,\pi}{2}\right)-\cos\left(\frac{n\,\pi}{4}\right)\right)}{n\,\pi} \quad \textbf{(1)}$$

> bn

$$-\frac{M\left(\cos\left(\frac{n\,\pi}{2}\right)-\cos\left(\frac{n\,\pi}{4}\right)\right)}{n\,\pi} \quad \textbf{(2)}$$

> $phin := \sin\left(\dfrac{n\cdot\mathrm{Pi}\cdot x}{L}\right)$

$$phin := \sin\left(\frac{n\,\pi\,x}{L}\right) \quad \textbf{(3)}$$

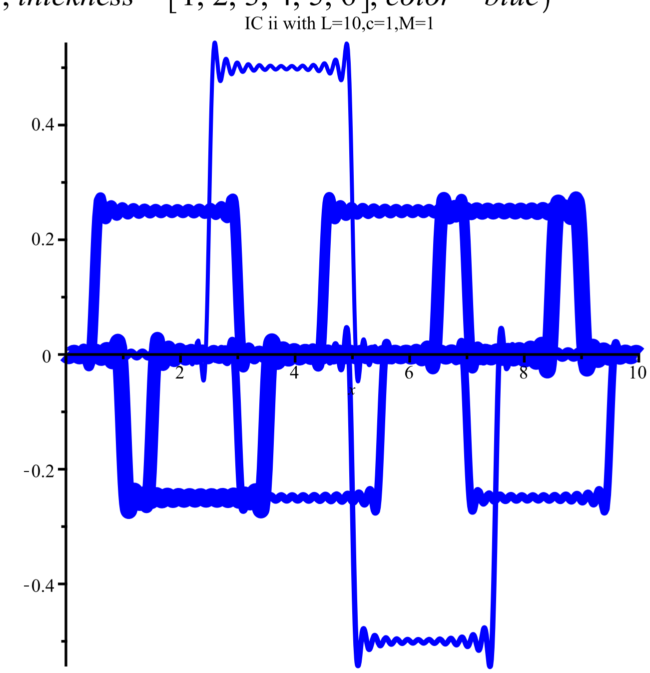
> $fs := subs\left(M=1, L=10, c=1, sum\left(bn\cdot phin\cdot\cos\left(\dfrac{n\cdot\mathrm{Pi}\cdot c\cdot t}{L}\right), n=1..100\right)\right):$

> $with(plots):$

> $animate(fs, x=0..10, t=0..20):$

> $curves := \{seq(subs(t=2\cdot m, fs), m=0..10)\}:$

> $plot(curves, x=0..10, thickness=[1,2,3,4,5,6], color=blue)$



>

(iv)

$u(x,0)=0$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ where

$$g(x) = \begin{cases} M, & \frac{L}{4} < x < \frac{L}{2} \\ 0, & \text{else} \end{cases}$$

we know $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)\cdot T_n(t)$ where

$$T_n(t) = a_n \sin\left(\frac{cn\pi}{L}\cdot t\right) + b_n \cos\left(\frac{cn\pi}{L}\cdot t\right)$$

$$u_n(x,0) \Rightarrow T_n(0) = a_n \cancel{\sin(0)}^{0} + b_n \cdot 1 = 0$$

$$\boxed{b_n = 0}$$

$$\Rightarrow T_n(t) = a_n \sin\left(\frac{cn\pi}{L}\cdot t\right)$$

then

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)\cdot a_n \cdot \frac{c\cdot n\cdot\pi}{L}$$

where g is piecewise. Let $K_n = \frac{a_n\cdot c\cdot n\cdot\pi}{L}$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)\cdot K_n$$

by Maple, we end up with:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-M\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)L}{n^2 \pi^2 c} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot \sin\left(\frac{cn\pi}{L} t\right)$$

with period (in t) $\frac{2\cancel{\pi}}{cn\cancel{\pi}} \cdot L = \frac{2L}{cn}$ and common period $2L$.

Maple:

> $kn := \dfrac{int\left(M\cdot\sin\left(\frac{n\cdot\mathrm{Pi}\cdot x}{L}\right), x=\frac{L}{4}..\frac{L}{2}\right)}{int\left(\sin^2\left(\frac{n\cdot\mathrm{Pi}\cdot x}{L}\right), x=-L..L\right)}$ assuming$(L > 0, n > 0, n, integer)$

$$kn := -\frac{M\left(\cos\left(\frac{n\,\pi}{2}\right)-\cos\left(\frac{n\,\pi}{4}\right)\right)}{n\,\pi} \qquad \textbf{(1)}$$

> $an := \dfrac{kn\cdot L}{c\cdot n\cdot \mathrm{Pi}}$

$$an := -\frac{M\left(\cos\left(\frac{n\,\pi}{2}\right)-\cos\left(\frac{n\,\pi}{4}\right)\right)L}{n^2\,\pi^2\,c} \qquad \textbf{(2)}$$

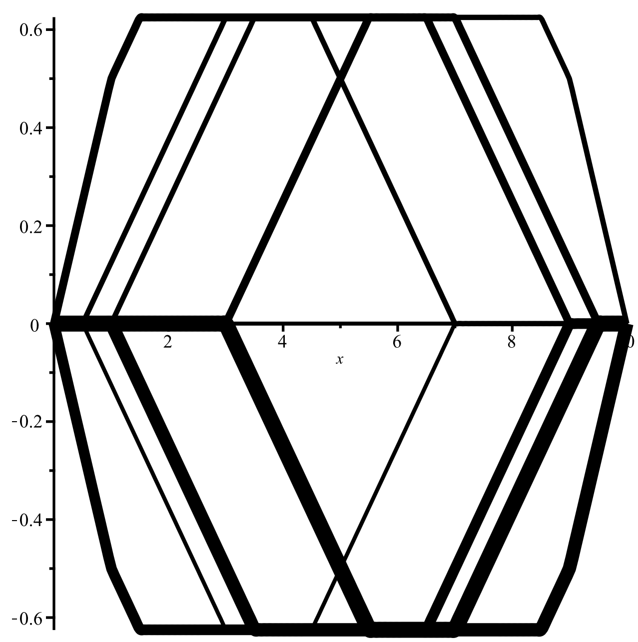
> $with(plots):$

> $psum := subs\left(M=1, L=10, c=1, sum\left(an\cdot\sin\left(\frac{n\cdot\mathrm{Pi}\cdot x}{L}\right)\cdot\sin\left(\frac{c\cdot n\cdot\mathrm{Pi}\cdot t}{L}\right), n=1..200\right)\right):$

> $animate(psum, x=0..50, t=0..20):$

> $curve := \{seq(subs(t=2\cdot m, psum), m=0..10)\}:$

> $plot(curve, x=0..10, thickness=[1, 2, 3, 4, 5, 6], color=black)$



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