(iii)

(iii)

$u(x,0) = g(x)$, $\dfrac{\partial u}{\partial t}(x,0) = 0$

$$g(x) = \begin{cases} M, & \frac{L}{4} < x < \frac{L}{2} \\ 0, & \text{else} \end{cases}$$

We have $X(x) = a \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$

$$T(t) = a \sin(c\nu t) + b \cos(c\nu t)$$

$$\nu = \frac{n \cdot \pi}{L} \Rightarrow T(t) = a \sin\left(\frac{n \cdot \pi}{L} \cdot ct\right) + b \cos\left(\frac{n \cdot \pi \cdot c}{L} \cdot t\right)$$

$$\therefore u(x,t) = X \cdot T = \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \left(a \sin\left(\frac{n\pi c}{L} \cdot t\right) + b \cos\left(\frac{n\pi c}{L} t\right)\right)$$

Since $\dfrac{\partial u}{\partial t}(x,0) = 0$

$$\Rightarrow \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)\left(a \cos(0) - b \overset{0}{\cancel{\sin(0)}}\right) = 0$$

$$\Rightarrow \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot a = 0 \Rightarrow a = 0$$

then $u(x,t) = \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot b \cos\left(\frac{n \cdot \pi \cdot c}{L} \cdot t\right)$

$$u_n(x,t) = \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot b \cos\left(\frac{n\pi c}{L} \cdot t\right)$$

for $u_n(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot b$

note $g = \begin{cases} M, & \frac{L}{4} < x < \frac{L}{2} \\ 0, & \text{else.} \end{cases}$

finding coefficients:

$$b_n = \frac{\int_{\frac{L}{4}}^{\frac{L}{2}} M \cdot \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

$$b_n = \frac{-2M \cdot \left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)}{n\pi}$$

$$\therefore u_n(x,t) = \sum_{n=1}^{\infty} \frac{-2M\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right)}{n\pi} \cdot \sin\left(\frac{n\pi}{L} \cdot x\right) \cdot \cos\left(\frac{n\pi \cdot ct}{L}\right)$$

with period in $t = \frac{2\cancel{\pi}}{n\cancel{\pi}} \cdot L = \frac{2L}{n}$