2. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ Let $u(x,t) = X(x)T(t)$

BC: $u(0,t) = u(L,t) = 0$

$\Rightarrow$ wave equation becomes $T''(t) \cdot X(x) = c^2 X''(x) \cdot T(t)$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = k \Rightarrow \left.\begin{array}{l} X'' - kX = 0 \\ T'' - kc^2T = 0 \end{array}\right\} \text{two ODE's}$$

We know that the solution for $X'' - kX = 0$ occurs for $k < 0$

Then, let $k = -\nu^2$. $\nu^2 > 0$

$$X'' - kX = X'' - (-\nu^2)X = X'' + \nu^2 X = 0$$

Let $X(x) = a\sin(\nu x) + b\cos(\nu x)$

$X'(x) = a\nu\cos(\nu x) - b\nu\sin(\nu x)$

$X''(x) = -a\nu^2\sin(\nu x) - b\nu^2\cos(\nu x)$

$X'' = -\nu^2(a\sin(\nu x) + b\cos(\nu x)) = -\nu^2 X(x)$

then $X'' + \nu^2 X = -\nu^2 X + \nu^2 X = 0$ ✓ $\therefore X(x)$ is solution.

$\therefore X(x) = a\sin(\nu x) + b\cos(\nu x)$ is solution.

Checking boundary Conditions :

$$u(0,t) = X(0) = 0 = a \cancel{\sin(0)}^{0} + b\cos(0) = 0 = b \cdot 1 = 0 = b$$

$\therefore b = 0$

$\Rightarrow X(x) = a \sin(\nu x)$

$$u(L,t) = X(L) = a\sin(\nu L) = 0 \begin{cases} \cancel{a = 0} \text{ trivial} \\ \sin(\nu L) = 0 \end{cases}$$

$$\sin(\nu L) = 0 \iff \nu L = n \cdot \pi \iff \nu = \frac{n \cdot \pi}{L}$$

$n = 1, 2, 3, \ldots$

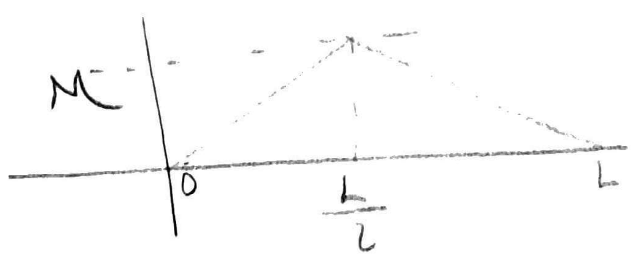
Since $k = -\nu^2 \Rightarrow k = -\left(\frac{n \cdot \pi}{L}\right)^2$

$$\Rightarrow X(x) = a \sin\left(\frac{n \cdot \pi}{L} \cdot x\right)$$

For the initial condition (i)

(i) $u(x,0) = f(x)$

$$\frac{\partial u(x,0)}{\partial t} = 0$$



$$f(x) = \begin{cases} \frac{2 \cdot M}{L} \cdot x, & 0 < x < \frac{L}{2} \\ \frac{2M}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

$$T'' - c^2 k T = 0 = T'' - c^2 \cdot (-\nu^2) T = T'' + c^2 \nu^2 T = 0$$

where $\nu^2 = \left(\frac{n \cdot \pi}{L}\right)^2$

Let $T(t) = a_2 \sin(c\nu t) + b_2 \cos(c\nu t)$

$$T' = a_2 c\nu \cos(c\nu t) - b_2 c\nu \sin(c\nu t)$$

$$T'' = -a_2 (c\nu)^2 \sin(c\nu t) - b_2 (c\nu)^2 \cos(c\nu t)$$

$$\Rightarrow T'' = -(c\nu^2)\left(a_2 \sin(c\nu t) + b_2 \cos(c\nu t)\right) = -(c\nu)^2 T$$

$$\therefore T'' + (c\nu)^2 T = -(c\nu)^2 T + (c\nu)^2 T = 0 \checkmark$$

I.C.

$u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{\partial u}{\partial t}(x,0) = \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \left(a_2 c\nu \cos(0) - \cancel{b_2 c\nu \sin(0)}^{\,0}\right) = 0$$

$$= \sin\left(\left(\frac{n \cdot \pi}{L}\right) \cdot x\right) \cdot a_2 c\nu \cdot 1 = 0 \qquad \forall x$$

$$\Rightarrow a_2 = 0 \checkmark$$

$$\Rightarrow T_n(t) = b_n \cos(c\omega t) = b_n \cos\left(\frac{n\cdot\pi}{L}\cdot t\right)$$

(a)

By Principle of superposition, we have the family of solutions:

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)\cdot b_n \cos\left(\frac{n\cdot\pi}{L}\cdot t\right)$$

with period on t $\quad p = \frac{2\pi}{n\pi}\cdot L = \frac{2L}{n}$ and

common period $2L$.

$u_n(x,t)$ satisfies BC ✓✓

IC ✓ $\left(\frac{\partial u}{\partial t}(x,0) = 0\right)$

For IC $f(x) = u_n(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\cdot\pi}{L}\cdot x\right)$

$$f(x) = \begin{cases} \frac{2M}{L}\cdot x, & 0 < x < L/2 \\ \frac{2M}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

$$b_n = \frac{\int_0^{L/2} \frac{2M\cdot x}{L}\cdot \sin\left(\frac{n\cdot\pi}{L}\cdot x\right) dx + \int_{L/2}^{L} \frac{2M}{L}(L-x)\sin\left(\frac{n\cdot\pi}{L}\cdot x\right) dx}{\int_0^L \sin^2\left(\frac{n\cdot\pi}{L}\cdot x\right) dx}$$

by using Maple, we arrive to :

$$b_n = \frac{8 \cdot M \sin\left(\frac{n\pi}{2}\right)}{(\pi n)^2}$$ assuming $L > 0$, n, integer

$\therefore$ Our general solution is :

$$u_n(x,t) = \sum_{n=1}^{\infty} \frac{8 \cdot M \cdot \sin\left(\frac{n \cdot \pi}{2}\right)}{(\pi n)^2} \cdot \sin\left(\frac{n \cdot \pi}{L} \cdot x\right) \cdot \cos\left(\frac{n \cdot \pi}{L} \cdot t\right)$$

of period in t, $p = \frac{2\cancel{\pi}}{n \cdot \cancel{\pi}} \cdot L = \frac{2L}{n}$