1. $\frac{d^2u}{dx^2} + u = 0$. $u'' + u = 0$ best guess: $u(x) = ce^{\lambda x}$

$u' = c\lambda e^{\lambda x}$
$u'' = c\lambda^2 e^{\lambda x}$

$u'' + u = c\lambda^2 e^{\lambda x} + ce^{\lambda x} = ce^{\lambda x}(\lambda^2 + 1) = 0 \Rightarrow c = 0$ or
$\lambda^2 + 1 = 0$
$\lambda^2 = -1$
$\lambda = i, -i$

Eigenfunctions: $u_1(x) = c_1 e^{ix} = c_1 \cos(x) + c_1 i \sin(x)$
$u_2(x) = c_2 e^{-ix} = c_2 \cos(x) - i c_2 \sin(x)$

General solution: $u_1 + u_2 = \cos(x)(c_1 + c_2) + i\sin(x)(c_1 - c_2) = u(x)$

Initial conditions: $u(0) = 0$, $u(\pi) = 0$:
$u(0) = \cos(0)(c_1 + c_2) + \cancel{i\sin(0)(c_1 - c_2)} = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$u(\pi) = 0 = \cos(\pi)(c_1 + c_2) + \cancel{i\sin(\pi)(c_1 - c_2)}^{0} = -(c_1 + c_2) = 0$

$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

then, replacing c_1 by $-c_2$:
$u(x) = \cos(x)\cancel{(-c_2 + c_2)}^{0} + i\sin(x)(-c_2 - c_2)$

$u(x) = i\sin(x)(-2c_2) = -2c_2 i \sin(x)$

$\therefore u(x) = -2c_2 i \sin(x)$ infinitely many

b) $\frac{d^2u}{dx^2} + u = 1 \Rightarrow u'' + u = 1$. u_h for $u'' + u = 0$

let $u_1(x) = c_1\cos(x)$ $u_1' = -c_1\sin(x)$, $u_1''(x) = -c_1\cos(x)$

and $u_1'' + u_1 = -c_1\cos(x) + c_1\cos(x) = 0$ ✓

Similarly, $u_2(x) = c_2\sin(x)$, $u_2' = c_2\cos(x)$, $u_2''(x) = -c_2\sin(x)$

$\Rightarrow u_2'' + u_2 = -c_2\sin(x) + c_2\sin(x) = 0$ ✓

$\therefore u_h = c_1\cos(x) + c_2\sin(x)$ Let $u_p = 1$, $u_p'' + u_p = 0+1 = 1$

~~Let $u_p =$~~ then $u(x) = c_1\cos(x) + c_2\sin(x) + 1$

Initial conditions: $u(0) = 0$, $u(1) = 0$

$u(0) = c_1\cos(0) + c_2\sin(0) + 1 \Rightarrow \boxed{c_1 + 1 = 0} \Rightarrow c_1 = -1$

$u(1) = 0 = -\cos(1) + c_2\sin(1) + 1 = 0$

$\Rightarrow c_2\sin(1) = \cos(1) - 1 \Rightarrow c_2 = \frac{\cos(1) - 1}{\sin(1)}$

then, $\frac{d^2u}{dx^2} + u = 1$ with $u(0) = 0$, $u(1) = 0$ have just one solution.

1.c) $\frac{d^2u}{dx^2} + u = 0$, $u(0) = 0$, $u(\pi) = 1$

$u'' + u = 0$ let $u_1(x) = c_1 \cos(x)$, $u_2(x) = c_2 \sin(x)$

$u_1' = -c_1 \sin(x)$, $u_1'' = -c_1 \cos(x)$

$u_1'' + u = -c_1 \cos(x) + c_1 \cos(x) = 0$ ✓

$u_2' = c_2 \cos(x)$, $u_2'' = -c_2 \sin(x)$

$u_2'' + u_2 = -c_2 \sin(x) + c_2 \sin(x) = 0$. General solution is :

$u_1 + u_2 = u = c_1 \cos(x) + c_2 \sin(x)$

$u(0) = 0 = c_1 \cos(0) + c_2 \cancel{\sin(0)}^{0} = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\left.\begin{aligned} u(\pi) = 0 \cdot \cos(\pi) + c_2 \sin(\pi) &= 1 \\ c_2 \cdot 0 &= 1 \end{aligned}\right\} \text{not true}$$

∴ not solution