MA 501 SP22 HW1 solutions   Pardon the numbering, which corresponds to a previous semester.

2. 1. Just work from Euler's formula:
exp(ix) = cos(x) + i sin(x)
exp(-ix) = cos(x) - i sin(x)
adding: cos(x) = (exp(ix) + exp(-ix))/2
subtracting: sin(x) = (exp(ix) - exp(-ix))/2i

3a 2. Similar to #3.

3b 3.
sinh(a+b) = (exp(a+b) - exp(-a-b))/2 from the definition.
sinh(a)*cosh(b) + cosh(a)*sinh(b) = (exp(a) - exp(-a))*(exp(b) + exp(-b))/4 + (exp(a) + exp(-a))*(exp(b) - exp(-b))/4
= (exp(a+b) - exp(b-a) + exp(a-b) - exp(-a-b) + exp(a+b) + exp(b-a) - exp(a-b) - exp(-a-b))/4
= (2*exp(a+b) - 2*exp(-a-b))/4
= (exp(a+b) - exp(-a-b))/2
= sinh(a+b) from the first line above.

4. Using Euler's formula,
i = exp(i*theta) where theta = pi/2
so i^i = exp(i*pi/2)^i = exp(i*i*pi/2) = exp(-pi/2) which is real and ~ 0.21

Actually, there are an infinite number of solutions! Why? You can use different angles (add 2Pi etc).

5. Maple is so helpful here:

> $ode1 := diff(y(x), x, x) + 5 \cdot diff(y(x), x) - 2 \cdot y(x) = 0$

$$ode1 := \frac{d^2}{dx^2}\, y(x) + 5 \left(\frac{d}{dx}\, y(x)\right) - 2\, y(x) = 0 \quad \textbf{(1)}$$

> $sol1 := dsolve(ode1)$

$$sol1 := y(x) = \_C1\, e^{\frac{(-5+\sqrt{33})x}{2}} + \_C2\, e^{-\frac{(5+\sqrt{33})x}{2}} \quad \textbf{(2)}$$

> $simplify(eval(subs(sol1, ode1)))$

$$0 = 0 \quad \textbf{(3)}$$

> $ode2 := diff(y(x), x, x) - 2 \cdot diff(y(x), x) + 5 \cdot y(x) = 0$

$$ode2 := \frac{d^2}{dx^2}\, y(x) - 2 \left(\frac{d}{dx}\, y(x)\right) + 5\, y(x) = 0 \quad \textbf{(4)}$$

> $sol2 := dsolve(ode2)$

$$sol2 := y(x) = \_C1\, e^{x} \sin(2\,x) + \_C2\, e^{x} \cos(2\,x) \quad \textbf{(5)}$$

> $simplify(eval(subs(sol2, ode2)))$

$$0 = 0 \quad \textbf{(6)}$$

> $ode3 := diff(y(x), x, x) - 2 \cdot diff(y(x), x) + 5 \cdot y(x) = 2 \cdot x$

$$ode3 := \frac{d^2}{dx^2}\, y(x) - 2 \left(\frac{d}{dx}\, y(x)\right) + 5\, y(x) = 2\, x \quad \textbf{(7)}$$

> $sol3 := dsolve(ode3)$

$$sol3 := y(x) = \_C2\, e^{x} \sin(2x) + \_C1\, e^{x} \cos(2x) + \frac{2x}{5} + \frac{4}{25} \quad \textbf{(8)}$$

> simplify(eval(subs(sol3, ode3)))

$$2x = 2x \quad \textbf{(9)}$$

> simplify(eval(subs(sol2, ode1)))

$$14\, e^{x} (\cos(2x)\, \_C1 - \sin(2x)\, \_C2) = 0 \quad \textbf{(10)}$$

> simplify(eval(subs(sol1, ode2)))

$$-\frac{7\, \_C1\, (\sqrt{33} - 7)\, e^{\frac{(-5+\sqrt{33})x}{2}}}{2} + \frac{7\, e^{-\frac{(5+\sqrt{33})x}{2}}\, \_C2\, (\sqrt{33} + 7)}{2} = 0 \quad \textbf{(11)}$$

> simplify(eval(subs(sol3, ode2)))

$$2x = 0 \quad \textbf{(12)}$$

> simplify(eval(subs(sol3, ode1)))

$$14\, \_C2\, e^{x} \cos(2x) - 14\, \_C1\, e^{x} \sin(2x) + \frac{42}{25} - \frac{4x}{5} = 0 \quad \textbf{(13)}$$

1.
For (a), (b), and the homogeneous version of (b), we have eigenvalues i and -i, giving eigenfunctions sin(x) and cos(x).

(a) General solution is u(x) = A sin(x) +B cos(x). The BC mandate B = 0, but A can be anything. This BVP has an *infinite number of solutions*.

(b) Particular solution is u_p = 1. General solution u(x) = A sin(x) +B cos(x) + 1. The x=0 BC gives B = -1. The BC at x=1 gives A = (cos(1)-1)/sin(1). Therefore this BVP has a *unique solution*.

(c) General solution is u(x) = A sin(x) +B cos(x). The x=0 BC mandates B = 0. But the Pi BC is incompatible with u(x) = A sin(x) so this has *no solution*.

This problem shows why many mathematicians are concerned with "existence and uniqueness."