1. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ on a ring $x \in (0, 2\pi)$

Separation of variables:

Let $u(x,t) = X(x) \cdot T(t) \Rightarrow X \cdot T'' = c^2 \cdot X'' \cdot T$

$\Rightarrow \dfrac{T''}{c^2 T} = \dfrac{X''}{X} = K$, K const.

Then, $\left.\begin{array}{l} T'' = K c^2 \cdot T \\ X'' = K X \end{array}\right\} \Rightarrow \left.\begin{array}{l} T'' - K c^2 \cdot T = 0 \\ X'' - K X = 0 \end{array}\right\}$ ODE system

We know the solution is for $K < 0$, so let $\boxed{K = -\nu^2}$.

Then our ODE system becomes:

$T'' - (-\nu^2) \cdot c^2 T = T'' + (\nu c)^2 \cdot T = 0$

$\wedge$

$X'' - (-\nu^2) X = X'' + \nu^2 X = 0$

For $X'' + \nu^2 X = 0$, we know solution is $X(x) = a \sin(\nu x) + b \cos(\nu x)$

since $X(x) = X(x+2\pi)$ (periodicity in boundary conditions)

$\Rightarrow \nu = 1, 2, 3, \dots = n$ i.e. ν is an integer > 0

$\therefore \nu = n \quad n = 1, 2, \dots$

$\Rightarrow k = -\nu^2 = -(n^2)$

For IC. :

we know $T(t) = b_1 \sin(nct) + b_2 \cos(nct)$ is a solution for

$T'' + (nc)^2 T = 0$

homogeneous IC is $T'(0) = 0$,

$\Rightarrow T'(0) = ncb_1 \cos(0) - ncb_2 \sin(0) = 0$

$T'(0) = ncb_1 = 0 \Rightarrow b_1 = 0$

$\Rightarrow T(t) = \cos(nct)$

Now, we have a family of solutions

$$u_n(x,t) = X \cdot T = (a_n \sin(nx) + b_n \cos(nx)) \cdot \cos(nct)$$

Since we have periodic BC $\wedge$ homogeneous IC, we can superimpose.

Then, General solution is:

$$u(x,t) = \sum_{n=0}^{\infty} (a_n \sin(nx) + b_n \cos(nx)) \cdot \cos(nct)$$

PDE ✓

BC ✓✓

IC ✓

For IC $u(x,0) = f(x) = \sum_{n=0}^{\infty} (a_n \sin(nx) + b_n \cos(nx)) \cdot 1$

$$= \sum_{n=0}^{\infty} a_n \sin(nx) + \sum_{n=0}^{\infty} b_n \cos(nx)$$

$$= \sum_{n=1}^{\infty} a_n \sin(nx) + b_0 + \sum_{n=1}^{\infty} b_n \cos(nx)$$

$$= b_0 + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$$

$$f(x) = \begin{cases} M, & \frac{\pi}{2} < x < \pi \\ 0, & \text{else} \end{cases}$$

Maple ↓