Francisco Moyet Vargas
HW5
3/11/2022

1. $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$, $x \in (0, L)$ with BC $u(0,t) = M > 0$
$u(L,t) = 0$

We know :
$u(x,t) = u_{ss}(x) + u_h(x,t)$

we know $u_{ss}(x) = T_1 + \frac{1}{L}(T_2 - T_1)x$,

where $u(0,t) = u_{ss}(0) = T_1 = M$
$u(L,t) = u_{ss}(L) = T_2 = 0$

then $u(x,t) = M - \frac{Mx}{L} + u_h(x,t)$

Initial Condition (i) $u(x,0) = f(x) = \begin{cases} M, & x < L/2 \\ 0, & x > L/2 \end{cases}$

then $u(x,0) = f(x) = u_{ss}(x) + u_h(x,0)$

$\Rightarrow u_h(x,0) = f(x) - u_{ss}(x)$ (Un homogeneous sol. of heat eq.)

then $f(x) - u_{ss}(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$

Using maple :

Finding coefficients for initial condition (i):

$$\text{phi} := \sin\left(\frac{n\cdot \text{Pi}\cdot x}{L}\right):$$

$$an := \frac{int\left(\frac{M\cdot x}{L}\cdot \text{phi}, x=0..\frac{L}{2}\right) + int\left(\left(0 - M + \frac{M\cdot x}{L}\right)\cdot \text{phi}, x=\frac{L}{2}..L\right)}{int\left(\phi^2, x=0..L\right)} \text{assuming}(M > 0, n, integer)$$

$$an := \frac{2\left(-\frac{L\left(n\,\pi\cos\left(\frac{n\,\pi}{2}\right) - 2\sin\left(\frac{n\,\pi}{2}\right)\right)M}{2\,\pi^2\,n^2} - \frac{L\left(n\,\pi\cos\left(\frac{n\,\pi}{2}\right) + 2\sin\left(\frac{n\,\pi}{2}\right)\right)M}{2\,\pi^2\,n^2}\right)}{L} \quad \textbf{(1)}$$

Coefficients are:

$$an := simplify(an)$$

$$an := -\frac{2\,M\cos\left(\frac{n\,\pi}{2}\right)}{n\,\pi} \quad \textbf{(2)}$$

(a) Solution u(x,t):

$$u := M - \frac{M\cdot x}{L} + Sum\left(an\cdot \text{phi}\cdot \exp\left(-\left(\frac{n\cdot \text{Pi}}{L}\right)^2\cdot \text{D}\cdot t\right), n=1..\text{infinity}\right)$$

$$u := M - \frac{M\,x}{L} + \sum_{n=1}^{\infty}\left(-\frac{2\,M\cos\left(\frac{n\,\pi}{2}\right)\sin\left(\frac{n\,\pi\,x}{L}\right)\mathrm{e}^{-\frac{n^2\,\pi^2\,\text{D}\,t}{L^2}}}{n\,\pi}\right) \quad \textbf{(3)}$$

$$plot\left(100 - \frac{100\cdot x}{10}, x=0..10\right):$$

(b) Partial sum, animation, plotting solution u(x,t) curves:

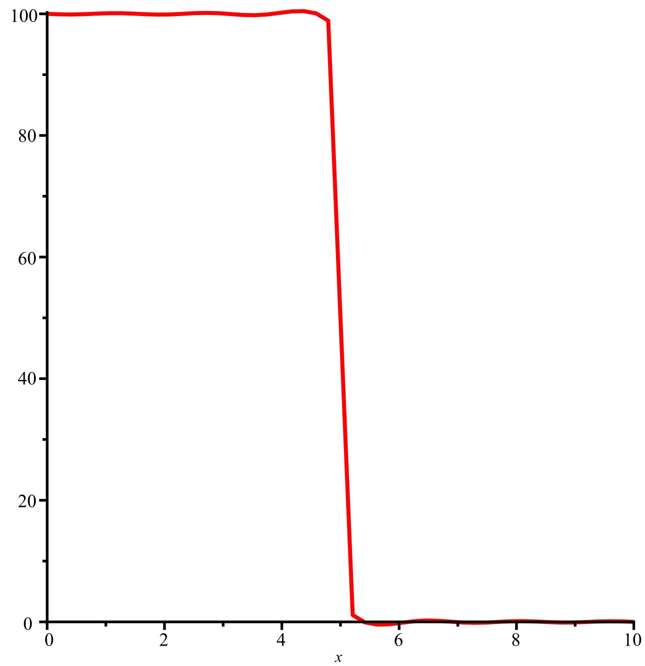
$$uss := subs\left(M=100, L=10, M - \frac{M\cdot x}{L}\right):$$

$$uh := subs\left(M=100, L=10, \text{D}=1, an\cdot \text{phi}\cdot \exp\left(-\left(\frac{n\cdot \text{Pi}}{L}\right)^2\cdot \text{D}\cdot t\right)\right):$$

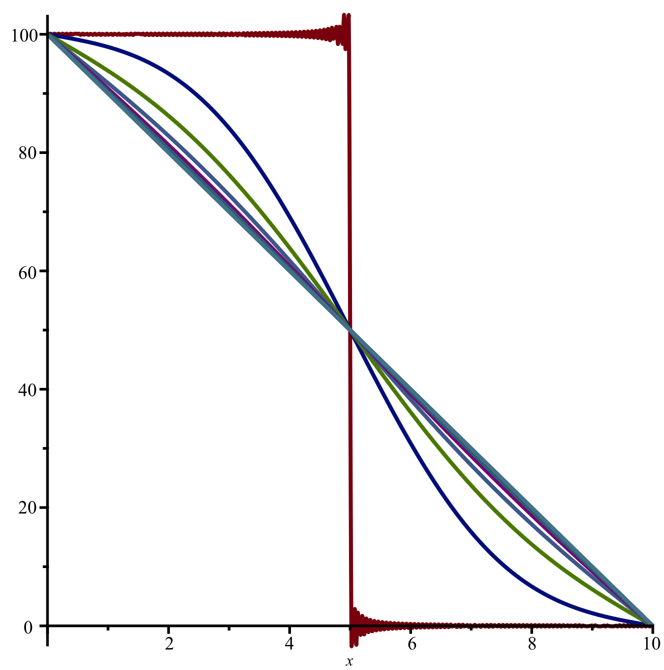
$$psum := uss + sum(uh, n=1..300):$$

$$with(plots):$$

$$animate(psum, x=0..10, t=0..40)$$



$curves := [seq(subs(t=2\cdot m, psum), m=0..10)]:$
$plot(curves, x=0..10)$



As t goes to infinity, the temperature u(x,t) approach the steady state.

(c) Steady state and asymptotic solution:

$$u(x, \text{infinity}) = M - \frac{M \cdot x}{L}$$

$u(x, t)$ is approximately equal to:

$$M - \frac{M \cdot x}{L} + subs\left(n=2, an \cdot \text{phi} \cdot \exp\left(-\left(\frac{n \cdot \text{Pi}}{L}\right)^2 \cdot \text{D} \cdot t\right)\right)$$

$$M - \frac{M x}{L} - \frac{M \cos(\pi) \sin\left(\frac{2 \pi x}{L}\right) \mathrm{e}^{-\frac{4 \pi^2 \mathrm{D} t}{L^2}}}{\pi} \tag{4}$$

1. (ii) IC : $u(x,0) = f(x) = \dfrac{Mx}{L}$

By 1.i we know $u(x,t) = u_{ss}(x) + u_h(x,t)$, where

$$u(x,t) = M - \frac{Mx}{L} + u_h(x,t)$$

and $u_h(x,t) = \displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$

Using maple :

For initial condition (ii) u(x,0) = M*x/L :

(a) Calculating our solution u(x,t)

$$an := simplify\left(\frac{int\left(\left(\frac{M\cdot x}{L} - M + \frac{M\cdot x}{L}\right)\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), x = 0..L\right)}{int\left(\sin^2\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right), x = 0..L\right)} \text{assuming}(M > 0, n, integer)\right)$$

$$an := -\frac{2M\left((-1)^n + 1\right)}{n\pi} \tag{1}$$

$$u := M - \frac{M\cdot x}{L} + Sum\left(an\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right)\cdot\exp\left(-\left(\frac{n\cdot \mathrm{Pi}}{L}\right)^2\cdot \mathrm{D}\cdot t\right), n = 1..\text{infinity}\right)$$

$$u := M - \frac{Mx}{L} + \sum_{n=1}^{\infty}\left(-\frac{2M\left((-1)^n + 1\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 \mathrm{D}\, t}{L^2}}}{n\pi}\right) \tag{2}$$

(b) Partial sum, animation, and plots:

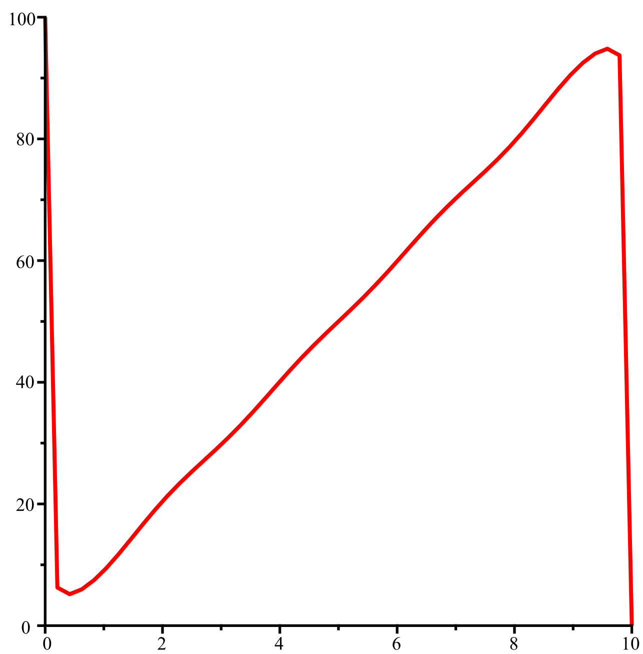
$$uss := subs\left(M = 100, L = 10, M - \frac{M\cdot x}{L}\right):$$

$$uh := subs\left(M = 100, L = 10, \mathrm{D} = 1, an\cdot\sin\left(\frac{n\cdot \mathrm{Pi}\cdot x}{L}\right)\cdot\exp\left(-\left(\frac{n\cdot \mathrm{Pi}}{L}\right)^2\cdot \mathrm{D}\cdot t\right)\right):$$

$psum := uss + sum(uh, n = 1..200):$

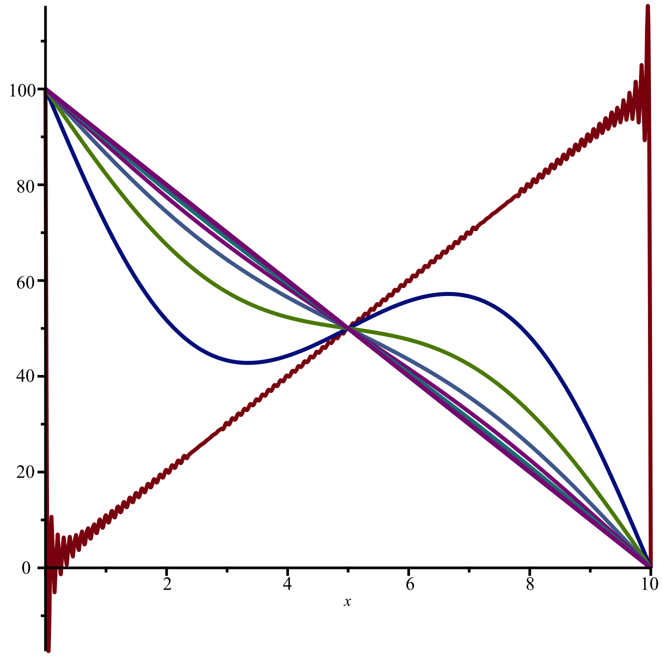
$with(plots):$

$animate(psum, x = 0..10, t = 0..20)$



$curves := [seq(subs(t = 2\cdot m, psum), m = 0..20)]:$

$plot(curves, x = 0..10)$



As t gets bigger, the temperature apporaches the linear steady state condition uss(x) = M-M*x/L.

(c)
Steady state is:

$uss := M - \frac{M \cdot x}{L}$

$$uss := M - \frac{M\,x}{L} \quad (3)$$

Then the solution u(x,t) is approximately equal to:

$an2 := subs(n = 2, an)$

$$an2 := -\frac{2\,M}{\pi} \quad (4)$$

$u := uss + subs\left(n = 2, \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right) \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)\right) \cdot an2$

$$u := M - \frac{M\,x}{L} - \frac{2 \sin\left(\frac{2\,\pi\,x}{L}\right) \mathrm{e}^{-\frac{4\,\pi^2\,\mathrm{D}\,t}{L^2}}\,M}{\pi} \quad (5)$$

2. $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, $x \in (0, L)$ IC: $u(x,0) = f(x)$

BC: $\frac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$

Let $u(x,t) = X(x)T(t)$, then, $T' \cdot X = D \cdot X'' \cdot T$

$\Rightarrow \frac{T'}{DT} = \frac{X''}{X} = K$, for K constant.

$\Rightarrow T' - KDT = 0$

$X'' - KX = 0$

Case 1 $K = 0$

$\Rightarrow X'' = 0$. Let $X(x) = ax+b$. BC: $\frac{\partial u}{\partial x}(0,t) = 0$

$\Rightarrow X'(0) = 0 = a \cdot 0 + b \Rightarrow \boxed{b = 0} \Rightarrow X(x) = ax$

BC: $u(L,t) = 0 \Rightarrow X(L) = 0 = X(L) = a \cdot L = 0$ — $a = 0$, $\cancel{L = 0}$

$\therefore a = 0$ $\therefore$ only trivial solution for $K = 0$.

Case $K > 0$ let $K = \nu^2$, then $X'' - KX = 0$

becomes $X'' - \nu^2 X = 0$. We know $X(x) = a\sinh(\nu x) + b\cosh(\nu x)$

is a solution to this ODE.

For BC $\frac{du}{dx}(0,t) = 0$:

$X'(x) = a\nu\cosh(\nu x) + b\nu\sinh(\nu x)$

$X'(0) = a\nu\cos(0) + \cancel{b\nu\sinh(0)}^{0} = a\nu \cdot 1 \Rightarrow \boxed{a = 0}$

for BC $u(L,t) = 0$: $(X(x) = b\cosh(\nu x))$

$X(L) = b\cosh(\nu L) = 0 \Rightarrow b = 0$ or $\cosh(\nu L) = 0$

$b = 0$ implies that we only have trivial solution.

$\cosh(\nu L) = 0 \iff \frac{e^{\nu L} + e^{-\nu L}}{2} = 0 \iff \underbrace{e^{\nu L} + e^{-\nu L} = 0}_{}$

no solution since $e^{\nu L} + e^{-\nu L} > 0$ always.

$\therefore$ for $K > 0$, only trivial solution.

case $K < 0$ $\Rightarrow$ let $K = -\nu^2$ then $X'' - KX = 0$

becomes $X'' + \nu^2 X = 0$. We know $X(x) = a\cos(\nu x) + b\sin(\nu x)$

is a solution to this ODE.

for BC $\frac{\partial u}{\partial x}(0,t) = 0$:

$$\Rightarrow X'(0) = -a\nu \cancel{\sin(0)} + b\nu\cos(0) = 0 = b\cdot\nu = 0$$

$$\boxed{\Rightarrow b = 0}$$

then $X(x) = a\cos(\nu x)$

for BC $u(L,t) = 0$ $\Rightarrow$ $X(L) = a\cos(\nu L) = 0$

$\Rightarrow$ $\cancel{a=0}$ trivial or $\nu L = \frac{(2n-1)\pi}{2}$ $\Rightarrow$ $\nu = \frac{\pi}{2L}\cdot(2n-1)$

$n = 1, 2, 3, \ldots$

$\therefore$ Our solution, for $X'' - KX = 0$ $\wedge$ $K < 0$ is

$$X(x) = a_n \cos\left((2n-1)\cdot\frac{\pi}{2L}\cdot x\right),$$ since $K = -\nu^2$,

$$K = -\left(\frac{\pi}{2L}(2n-1)\right)^2.$$

for $T' - \kappa D T = 0$ we have $\kappa = -\nu^2$ as before

$$\left(\kappa = -\left(\frac{\pi}{2L}\cdot(2n-1)\right)^2\right)$$

we know $T(t) = C \cdot e^{\lambda t}$ is a sol. for $T' + \nu^2 D T = 0$

$$\therefore \quad T' + \nu^2 D T = \cancel{C}\cdot \lambda \cancel{e^{\lambda t}} + \nu^2 D \cancel{C} \cancel{e^{\lambda t}} = 0$$

$$\Rightarrow C e^{\lambda t}\underbrace{\left(\lambda + \nu^2 D\right)}_{\text{eigenvalues}} = 0$$

$$\Rightarrow \lambda = -\nu^2 D = -\left(\frac{\pi}{2L}(2n-1)\right)^2 \cdot D$$

Our eigenfunction is :

$$T_n(t) = C_n e^{-\left(\frac{\pi}{2L}(2n-1)\right)^2 \cdot D t}$$

$$\therefore \quad u(x,t) = X(x)\cdot T(t)$$

$$\text{(a)} \quad u(x,t) = \cos\left(\frac{\pi}{2L}\cdot(2n-1)\cdot x\right)\cdot C_n \cdot e^{-\left(\frac{\pi}{L}(2n-1)\right)^2 \cdot D t}$$

2.b IC: $u(x,0)=f(x)$

Since $u_n(x,t)=\cos\left(\frac{(2n-1)x\cdot\pi}{2L}\right)\cdot C_n\cdot e^{-\left(\frac{\pi}{2\cdot L}(2n-1)\right)^2 D t}$

PDE & BC are linear & homogeneous.
General solution is:

$$u(x,t)=\sum_{n=1}^{\infty}\cos\left(\frac{(2n-1)x\cdot\pi}{2L}\right)\cdot C_n\cdot e^{-\left(\frac{\pi}{2L}(2n-1)\right)^2\cdot D\cdot t}$$

For $u(x,0)$ we have:

$$u(x,0)=\sum_{n=1}^{\infty}\cos\left(\frac{(2n-1)x\pi}{2L}\right)C_n=f(x)$$

2(c) Proving we have an orthogonal set:

$$phin := \cos\left(\frac{(2\cdot n-1)\cdot x\cdot \mathrm{Pi}}{2\cdot L}\right):$$

$$phim := \cos\left(\frac{(2\cdot m-1)\cdot x\cdot \mathrm{Pi}}{2\cdot L}\right):$$

$int(phin\cdot phim, x=0..L)\mathrm{assuming}(m \neq n, n, integer, m, integer)$

$$0 \tag{1}$$

Therefore, we have an orthogonal set. If m = n, we have:

$int\left(phin^2, x=0..L\right)\mathrm{assuming}(n, integer)$

$$\frac{L}{2} \tag{2}$$

(d) Finding the coefficients:

$$cn := \frac{int(f(x)\cdot phin, x=0..L)}{int\left(phin^2, x=0..L\right)}\mathrm{assuming}(n, integer, n>0)$$

$$cn := \frac{2\left(\int_0^L f(x)\cos\left(\frac{(2\,n-1)\,x\,\pi}{2\,L}\right)\mathrm{d}x\right)}{L} \tag{3}$$

(e) For initial condition f(x) = M, the solution is:

$$cn := \frac{int(M\cdot phin, x=0..L)}{int\left(phin^2, x=0..L\right)}\mathrm{assuming}(n, integer, n>0)$$

$$cn := -\frac{4\,(-1)^n M}{(2\,n-1)\,\pi} \tag{4}$$

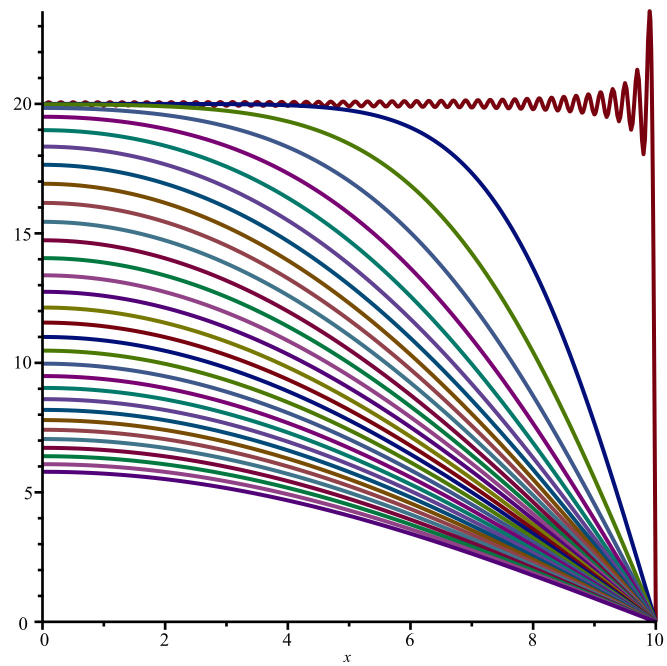
$$u := Sum\left(phin\cdot cn\cdot \exp\left(-\left(\frac{\mathrm{Pi}\cdot(2\cdot n-1)}{2\cdot L}\right)^2\cdot \mathrm{D}\cdot t\right), n=1..\mathrm{infinity}\right)$$

$$u := \sum_{n=1}^{\infty}\left(-\frac{4\cos\left(\frac{(2\,n-1)\,x\,\pi}{2\,L}\right)(-1)^n M\,\mathrm{e}^{-\frac{\pi^2(2\,n-1)^2\mathrm{D}\,t}{4\,L^2}}}{(2\,n-1)\,\pi}\right) \tag{5}$$

(f) Plotting:

$$uxt := subs\left(M=20, L=10, \mathrm{D}=1, phin\cdot cn\cdot \exp\left(-\left(\frac{\mathrm{Pi}\cdot(2\cdot n-1)}{2\cdot L}\right)^2\cdot \mathrm{D}\cdot t\right)\right):$$

$psum := sum(uxt, n=1..100):$
$with(plots):$
$animate(psum, x=0..10, t=0..20):$
$curves := [seq(subs(t=2\cdot m, psum), m=0..30)]:$
$plot(curves, x=0..10)$



0
5
10
15
20
0
2
4
6
8
10
x