1. $\frac{d^2u}{dx^2} + u = 0$. $u'' + u = 0$ best guess: $u(x) = ce^{\lambda x}$

$u' = c\lambda e^{\lambda x}$
$u'' = c\lambda^2 e^{\lambda x}$

$u'' + u = c\lambda^2 e^{\lambda x} + ce^{\lambda x} = ce^{\lambda x}(\lambda^2 + 1) = 0 \Rightarrow$ $c = 0$ or
$\lambda^2 + 1 = 0$
$\lambda^2 = -1$
$\lambda = i, -i$

Eigenfunctions: $u_1(x) = c_1 e^{ix} = c_1 \cos(x) + c_1 i \sin(x)$
$u_2(x) = c_2 e^{-ix} = c_2 \cos(x) - i c_2 \sin(x)$

General solution: $u_1 + u_2 = \cos(x)(c_1 + c_2) + i\sin(x)(c_1 - c_2) = u(x)$

Initial conditions: $u(0) = 0$, $u(\pi) = 0$

$u(0) = \cos(0)(c_1 + c_2) + \cancel{i \sin(0)(c_1 - c_2)} = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$u(\pi) = 0 = \cos(\pi)(c_1 + c_2) + \cancel{i \sin(\pi)(c_1 - c_2)}^{\,0} = -(c_1 + c_2) = 0$

$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

then, replacing c_1 by $-c_2$:
$u(x) = \cos(x)\cancel{(-c_2 + c_2)}^{\,0} + i\sin(x)(-c_2 - c_2)$

$u(x) = i\sin(x)(-2c_2) = -2c_2 i \sin(x)$

$\therefore u(x) = -2c_2 i \sin(x)$ infinitely many

b) $\frac{d^2u}{dx^2} + u = 1 \Rightarrow u'' + u = 1$. u_h for $u'' + u = 0$

let $u_1(x) = c_1\cos(x)$ $u_1' = -c_1\sin(x)$, $u_1''(x) = -c_1\cos(x)$

and $u_1'' + u_1 = -c_1\cos(x) + c_1\cos(x) = 0$ ✓

Similarly, $u_2(x) = c_2\sin(x)$, $u_2' = c_2\cos(x)$, $u_2''(x) = -c_2\sin(x)$

$\Rightarrow u_2'' + u_2 = -c_2\sin(x) + c_2\sin(x) = 0$ ✓

$\therefore u_h = c_1\cos(x) + c_2\sin(x)$ Let $u_p = 1$, $u_p'' + u_p = 0+1 = 1$

~~Let $u_p =$~~ then $u(x) = c_1\cos(x) + c_2\sin(x) + 1$

Initial conditions: $u(0) = 0$, $u(1) = 0$

$u(0) = c_1\cos(0) + c_2\sin(0) + 1 \Rightarrow \boxed{c_1 + 1 = 0} \Rightarrow c_1 = -1$

$u(1) = 0 = -\cos(1) + c_2\sin(1) + 1 = 0$

$\Rightarrow c_2\sin(1) = \cos(1) - 1 \Rightarrow c_2 = \frac{\cos(1) - 1}{\sin(1)}$

then, $\frac{d^2u}{dx^2} + u = 1$ with $u(0) = 0$, $u(1) = 0$ have just one solution.

1.c) $\frac{d^2u}{dx^2} + u = 0$, $u(0) = 0$, $u(\pi) = 1$

$u'' + u = 0$ let $u_1(x) = c_1\cos(x)$, $u_2(x) = c_2\sin(x)$

$u_1' = -c_1\sin(x)$, $u_1'' = -c_1\cos(x)$

$u_1'' + u = -c_1\cos(x) + c_1\cos(x) = 0$ ✓

$u_2' = c_2\cos(x)$, $u_2'' = -c_2\sin(x)$

$u_2'' + u_2 = -c_2\sin(x) + c_2\sin(x) = 0$. General solution is :

$u_1 + u_2 = u = c_1\cos(x) + c_2\sin(x)$

$u(0) = 0 = c_1\cos(0) + \cancel{c_2\sin(0)}^{0} = c_1 \Rightarrow \boxed{c_1 = 0}$

$$\left.\begin{aligned} u(\pi) &= 0\cdot\cos(\pi) + c_2\sin(\pi) = 1 \\ &c_2 \cdot 0 = 1 \end{aligned}\right\} \text{not true}$$

∴ not solution

2.(a)

sin (x)

By Eulers formula : $e^{ix} = \cos(x) + i\sin(x)$

$e^{-ix} = \cos(x) - i\sin(x)$

$$\Rightarrow \quad e^{ix} - e^{-ix} = \cancel{\cos(x)} + i\sin(x) - \cancel{\cos(x)} + i\sin(x)$$

$$e^{ix} - e^{-ix} = 2 \cdot i \cdot \sin(x)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2 \cdot i}$$

Since $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$

$$\therefore \quad \sin(x) = -i \cdot \left(\frac{e^{ix} - e^{-ix}}{2} \right)$$

2.b

cos(x)

$$e^{ix} + e^{-ix} = \cos(x) + i\cancel{\sin(x)} + \cos(x) - i\cancel{\sin(x)}$$

$$= 2\cos(x)$$

$$\therefore \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

3. a $\quad \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

by exercise 2: $\quad \sin(x) = \left(\dfrac{e^{ix} - e^{-ix}}{2}\right) \cdot -i$

$$\Rightarrow \sin(a+b) = \left(\frac{e^{i(a+b)} - e^{-i(a+b)}}{2}\right) \cdot -i$$

Aside:

For $e^{i(a+b)} = e^{ia} \cdot e^{ib}$

$$= (\cos(a) + i\sin(a)) \cdot (\cos(b) + i\sin(b)) = \cancel{\cos(a) \cdot \cos(b)} + \cancel{\cos(a) \cdot i \cdot \sin(b)} + \cancel{i\sin(a) \cdot \cos(b)} - \cancel{\sin(a)\sin(b)}$$

For $e^{-i(a+b)} = e^{i(-a-b)} = e^{-ia} \cdot e^{-ib}$

$$= (\cos(a) - i\sin(a)) \cdot (\cos(b) - i\sin(b))$$

$$= \cancel{\cos(a) \cdot \cos(b)} - \cancel{\cos(a) \cdot i \sin(b)} - \cancel{i\sin(a)\cos(b)} - \cancel{\sin(a) \cdot \sin(b)}$$

then $e^{i(a+b)} - e^{-i(a+b)}$

$$= i\cos(a)\sin(b) + i\cos(a)\sin(b) + i\sin(a)\cos(b) + i\sin(a)\cos(b)$$

$$= 2 \cdot i \cos(a)\sin(b) + 2 \cdot i \sin(a) \cdot \cos(b)$$

$$\therefore e^{i(a+b)} - e^{-i(a+b)} = 2 \cdot i(\cos(a)\sin(b) + \sin(a)\cos(b))$$

then

$$\left(e^{i(a+b)} - e^{-i(a+b)}\right)\cdot\frac{-i}{2} = \frac{-i}{\cancel{4}}\cdot\cancel{2}i\cdot\left(\cos(a)\sin(b) + \sin(a)\cos(b)\right)$$

$$= \cos(a)\cdot\sin(b) + \sin(a)\cos(b) = \sin(a+b)_{//}$$

3.b show

$$\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$$

$$\sinh(a+b) = \frac{e^{a+b} - e^{-a-b}}{2} = \frac{2(e^{a+b} - e^{-a-b})}{4}$$

$$= \frac{(e^{a+b} - e^{-a-b}) + (e^{a+b} - e^{-a-b}) + 0 + 0}{4}$$

$$= \frac{(e^{a+b} - e^{-a-b}) + (e^{a+b} - e^{-a-b}) + e^{a-b} - e^{a-b} + e^{b-a} - e^{b-a}}{4}$$

$$= \frac{\overbrace{\left(e^{a+b} + e^{a-b} - e^{b-a} - e^{-a-b}\right)}^{= (e^{a}-e^{-a})(e^{b}+e^{-b})} + \underbrace{\left(e^{a+b} - e^{a-b} + e^{b-a} - e^{-a-b}\right)}_{(e^{a}+e^{-a})(e^{b}-e^{-b})}}{4}$$

$$= \underbrace{\frac{(e^{a} - e^{-a})}{2}}_{\sinh(a)} \cdot \underbrace{\frac{(e^{b} + e^{-b})}{2}}_{\cosh(b)} + \underbrace{\frac{(e^{a} + e^{-a})}{2}}_{\cosh(a)} \cdot \underbrace{\frac{(e^{b} - e^{-b})}{2}}_{\sinh(b)}$$

$$\Rightarrow \sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b) \,_{//}$$

4. By Eulers formula:

$$i = \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} + \underbrace{i\sin(\pi/2)}_{=i\cdot 1} = e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad i = e^{\frac{i\pi}{2}}, \quad i^{i} = \left(e^{\frac{i\pi}{2}}\right)^{i} = e^{-\pi/2}$$

then $i^i = e^{-\pi/2}$ is real! (crazy!) $\quad i^i = \dfrac{1}{e^{\pi/2}}$

5. a) $y'' + 5y' - 2y = 0$, $y(x) = Ce^{\lambda x}$

$$y' = C\lambda e^{\lambda x}$$
$$y'' = C\lambda^2 e^{\lambda x}$$

$$y'' + 5y' - 2y = C\lambda^2 e^{\lambda x} + 5C\lambda e^{\lambda x} - 2Ce^{\lambda x}$$
$$= Ce^{\lambda x}(\lambda^2 + 5\lambda - 2) = 0 \Rightarrow \lambda = \frac{-5 \pm \sqrt{25 - 4(-2)}}{2}$$
$$\Rightarrow \lambda = \frac{-5 \pm \sqrt{33}}{2} \Rightarrow \lambda = \frac{-5 + \sqrt{33}}{2}, \frac{-5 - \sqrt{33}}{2}$$

Eigenfunctions: $u_1(x) = c_1 e^{\left(\frac{-5+\sqrt{33}}{2}\right)x}$

$$u_2(x) = c_2 e^{\left(\frac{-5-\sqrt{33}}{2}\right)x}$$

General function: $u_1 + u_2 = u(x) = c_1 e^{\left(\frac{-5+\sqrt{33}}{2}\right)x} + c_2 e^{\left(\frac{-5-\sqrt{33}}{2}\right)x}$

Let $a = \frac{-5+\sqrt{33}}{2}$, $b = \frac{-5-\sqrt{33}}{2}$. $y'' - 2y' + 5y$:

$$\cancel{5c_1e^{ax}} + \cancel{5c_2e^{bx}} + \cancel{a^2c_1e^{ax}} + \cancel{b^2c_2e^{bx}} - \cancel{2ac_1e^{ax}} - \cancel{2bc_2e^{bx}} = 0$$

$$= c_1e^{ax}(5 + a^2 - 2a) + c_2e^{bx}(5 + b^2 - 2b) = 0 \quad (*)$$

$$e^{ax} = e^{\frac{-5x+\sqrt{33}x}{2}} = e^{-\frac{5x}{2}} \cdot e^{\frac{\sqrt{33}x}{2}}$$

$$e^{bx} = e^{\frac{-5x-\sqrt{33}x}{2}} = e^{-\frac{5x}{2}} \cdot e^{-\frac{\sqrt{33}x}{2}}$$

since $e^{-\frac{5x}{2}} \neq 0$ divide both sides of (*) by $e^{-\frac{5x}{2}}$

$$c_1e^{\frac{\sqrt{33}x}{2}}(5 + a^2 - 2a) + c_2e^{-\frac{\sqrt{33}x}{2}}(5 + b^2 - 2b) = 0$$

$$5c_1 + a^2 - 2a = 5 + \frac{25 - 10\sqrt{33} + 33}{4} + \frac{10 - 2\sqrt{33}}{2}$$

$$= 5 + \frac{\cancel{25}\cancel{+33} - 10\sqrt{33} + \cancel{20} - 4\sqrt{33}}{4} = 5c_1 + \frac{45 + 33 - 14\sqrt{33}}{4}$$

$$= 5 + \frac{78 - 14\sqrt{33}}{4} \approx 4.3940$$

$$\Rightarrow c_1e^{\frac{\sqrt{33}}{2}x}(4.3940) + c_2e^{-\frac{\sqrt{33}}{2}x}(5 + b^2 - 2b) = 0$$

$$5 + b^2 - 2b = 5 + \frac{\cancel{25} + 10\sqrt{33} + \cancel{33}}{4} + \frac{10 + 2\sqrt{33}}{2}$$

$$= 5 + \frac{58 + 10\sqrt{33}}{4} + \frac{10 + 2\sqrt{33}}{2} = 5 + \frac{58 + 10\sqrt{33} + \cancel{20} + 4\sqrt{33}}{4}$$

$$5 + \frac{78 + 14\sqrt{33}}{4} \approx 44.6060$$

$$\Rightarrow c_1 e^{\frac{\sqrt{33}}{2}x}(4.3940) + c_2 e^{-\frac{\sqrt{33}}{2}x}(44.6060) = 0$$

$$\Rightarrow c_1 e^{\frac{\sqrt{33}}{2}x} \cdot e^{\frac{\sqrt{33}}{2}x}(4.3940) + c_2 \cancel{\frac{e^{\frac{\sqrt{33}}{2}x}}{e^{\frac{\sqrt{33}}{2}x}}}(44.6060) = 0$$

$$\Rightarrow c_1 e^{\frac{33}{4}x^2}(4.3940) + c_2(44.6060) = 0$$

$$\Rightarrow c_1 e^{33/4 x^2} = -c_2 \frac{(44.6060)}{(4.3940)}$$

$$\Rightarrow e^{33/4 \cdot x^2} = \left(\frac{-c_2(44.6060)}{c_1\ 4.3940}\right)$$

$$\Rightarrow \frac{33}{4} \cdot x^2 = \ln\left(-\frac{c_2(44.6060)}{c_1(4.3940)}\right)$$

then $y'' - 2y' + 5y = 0 \iff \frac{33}{4} \cdot x^2 = \ln\left(\frac{-c_2(44.6060)}{c_1(4.3940)}\right)$

for some constant c_1 and c_2 i.e. $\frac{33}{4} \cdot x^2$ is constant

which is not true.

check (c) $y'' - 2y' + 5y = 2x$, by our previous analysis :

$$\Rightarrow u'' - 2u' + 5u = \frac{33}{4} x^2 - \ln\left(\frac{-c_2 (44.6060)}{c_1 (4.3940)}\right) = 2x$$

if $$\frac{33}{4} x^2 = 2x + \ln\left(\frac{-c_2}{c_1} \cdot \left(\frac{44.6060}{4.3940}\right)\right)$$

$\Rightarrow \frac{33}{4} x^2$ is equal to a linear function $\Rightarrow\Leftarrow$

$\therefore u(x) = c_1 e^{\left(\frac{-5+\sqrt{33}}{2}\right)x} + c_2 e^{\left(\frac{-5-\sqrt{33}}{2}\right)x}$ is not solution

for $y'' - 2y' + 5y = 2x$ //

5.b) $y''-2y'+5y=0$ = ODE, linear, const. coeff., ODE,

$u(x)=ce^{\lambda x}$, $u'(x)=c\lambda e^{\lambda x}$, $u''(x)=c\lambda^2 e^{\lambda x}$

$\Rightarrow u''-2u'+5u = c\lambda^2 e^{\lambda x}-2c\lambda e^{\lambda x}+5ce^{\lambda x}=0$

$\Rightarrow ce^{\lambda x}(\lambda^2-2\lambda+5)=0 \Rightarrow c=0$ (trivial) or

$\lambda^2-2\lambda+5=0 \Rightarrow \lambda = \dfrac{2\pm\sqrt{4-4(5)}}{2(1)}$

$\Rightarrow \lambda = \dfrac{2\pm\sqrt{4-20}}{2} = \dfrac{2\pm\sqrt{-16}}{2} = \dfrac{2\pm i\sqrt{16}}{2} = \dfrac{2\pm i4}{2}$

$\Rightarrow \lambda = 1\pm 2i \Rightarrow \lambda = 1+2i,\ 1-2i$

Eigen function: $u_1(x)=c_1e^{(1+2i)x}$

$u_2(x)=c_2e^{(1-2i)x}$

and $u_1+u_2=u=c_1e^{(1+2i)x}+c_2e^{(1-2i)x}$ is the general solution.

$$u_1 = c_1 e^{(1+2i)x} = c_1 e^{x} e^{2ix} = c_1 e^{x}\left(\cos(2x) + i\sin(2x)\right)$$

$$u_2 = c_2 e^{(1-2i)x} = c_2 e^{x} e^{-2ix}\left(\cos(2x) - i\sin(2x)\right)$$

$$\Rightarrow u(x) = e^{x}\cos(2x)\left(c_1 + c_2\right) + i\sin(2x)\left(c_1 - c_2\right)$$

$$u'' - 2u' + 5u = i\sin(2x)\left(c_1 - c_2\right) - 4i\cos(2x)\left(c_1 - c_2\right)$$

$$= i\underbrace{\left(c_1 - c_2\right)}\left(\sin(2x) - 4\cos(2x)\right) = 0$$

$\boxed{c_1 = c_2}$ $\therefore$ u is a solution

Checking u is not a solution for (a) $u'' + 5u' - 2u = 0$

$$u'' + 5u' - 2u =$$

$$\cancel{10i\cos(2x)(c_1 - c_2)} - 6i\sin(2x)(c_1 - c_2) - 14\sin(2x)e^{x}(c_1 + c_2)$$

$$= i(c_1 - c_2)\left(10\cos(2x) - 6\sin(2x)\right) - 14\sin(2x)e^{x}(c_1 + c_2) = 0$$

$$= i(c_1 - c_2)\left(10\,\frac{e^{i2x} + e^{-i2x}}{2} - 6\,\frac{e^{i2x} - e^{-i2x}}{2i}\right) - 14\,\frac{e^{2ix} - e^{-2ix}}{2i}\cdot e^{x}(c_1 + c_2) = 0$$

Case 2 :

if $c_1+c_2=0 \Rightarrow c_1=-c_2$ and

$; (c_1-c_2)\left(8e^{2ix}+2e^{-2ix}\right)=0 \Rightarrow 8e^{2ix}+2e^{-2ix}=0$

$\Rightarrow 4e^{2ix}+e^{-2ix}=0=4\left(\cancel{\cos(2x)}+i\sin(2x)\right)+\cancel{\cos(2x)}-i\sin(2x)=0$

$=5\cos(2x)+3i\sin(2x)=0$ not true,

Case 3 $c_1-c_2\neq 0$ and $c_1+c_2\neq 0$ then let $c_1-c_2=\alpha$

and $c_1+c_2=\phi$

$\Rightarrow i\alpha\left(8e^{2ix}+2e^{-2ix}\right)+e^{x}\phi\left(7e^{2ix}-7e^{-2ix}\right)=0$

$\Rightarrow i\alpha\left(8\cancel{\cos(2x)}+i8\cancel{\sin(2x)}+2\cancel{\cos(2x)}-\cancel{i2\sin(2x)}\right)$
$+e^{x}\phi\left(7\cancel{\cos(2x)}+\cancel{7i\sin(2x)}-\cancel{7\cos(2x)}+7\,\cancel{i\sin(2x)}\right)=0$

$\Rightarrow i\alpha\left(10\cos(2x)+6i\sin(2x)\right)+e^{x}\phi\left(14i\sin(2x)\right)=0$

$=\underbrace{i\alpha 10\cos(2x)-6\alpha\sin(2x)+e^{x}\phi 14i\sin(2x)}_{\text{never } 0}=0$

$\therefore u(x)=c_1e^{(1+2i)x}+c_2e^{(1-2i)x}$ is not

a solution for (a)

$$\Rightarrow i(c_1-c_2)\left(5e^{i2x}+5e^{-2ix}-3(-i)e^{i2x}+3(-i)e^{-2ix}\right)$$

$$+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow i(c_1-c_2)\left(5e^{i2x}\cancel{+5e^{-2ix}}\cancel{+3ie^{2ix}}\cancel{-3ie^{-2ix}}\right)+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow i(c_1-c_2)\left(8e^{2ix}+2e^{-2ix}\right)+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

case 1 : $c_1-c_2=0 \Rightarrow c_1=c_2$ then

$$\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow 7ie^{2ix}-7ie^{-2ix}=0 \Rightarrow \cancel{7i}e^{2ix}=7ie^{-2ix} \quad \forall x$$

$$\Rightarrow e^{2ix}=e^{-2ix} \Rightarrow \cancel{\cos(2x)}+i\sin(2x)=\cancel{\cos(2x)}-i\sin(2x)$$

$$\Rightarrow i\sin(2x)=-i\sin(2x) \Rightarrow 2i\sin(2x)=0 \quad \forall x$$

which is not true, since $\sin(2x)\neq 0 \ \forall x$

checking if $u(x) = Ke^{(1+2i)x} + Ke^{(1-2i)x}$

is a solution for $u'' - 2u' + 5u = 2x$

Since $u(x) = Ke^{(1+2i)x} + Ke^{(1-2i)x}$ is a solution for

$u'' - 2u' + 5u = 0$ i.e.

$u'' - 2u' + 5u = i(K-K)(\sin(2x) - 4\cos(2x)) = 0$ $\forall x$ in

the domain of u,

$u'' - 2u' + 5u \neq 2x$. $\therefore u(x)$ is not a solution for 5.c

$u'' - 2u' + 5u = 2x$.