5.b) $y''-2y'+5y=0$ ODE, linear, const. coeff., ODE,

$u(x)=ce^{\lambda x}$, $u'(x)=c\lambda e^{\lambda x}$, $u''(x)=c\lambda^2 e^{\lambda x}$

$\Rightarrow u''-2u'+5u=c\lambda^2 e^{\lambda x}-2c\lambda e^{\lambda x}+5ce^{\lambda x}=0$

$\Rightarrow ce^{\lambda x}(\lambda^2-2\lambda+5)=0 \Rightarrow c=0$ (trivial) or

$\lambda^2-2\lambda+5=0 \Rightarrow \lambda=\dfrac{2\pm\sqrt{4-4(5)}}{2(1)}$

$\Rightarrow \lambda=\dfrac{2\pm\sqrt{4-20}}{2}=\dfrac{2\pm\sqrt{-16}}{2}=\dfrac{2\pm i\sqrt{16}}{2}=\dfrac{2\pm i4}{2}$

$\Rightarrow \lambda=1\pm 2i \Rightarrow \lambda=1+2i, 1-2i$

Eigen function: $u_1(x)=c_1e^{(1+2i)x}$
$u_2(x)=c_2e^{(1-2i)x}$

and $u_1+u_2=u=c_1e^{(1+2i)x}+c_2e^{(1-2i)x}$ is the general solution.

$$u_1 = c_1 e^{(1+2i)x} = c_1 e^{x} e^{2ix} = c_1 e^{x}\left(\cos(2x) + i\sin(2x)\right)$$

$$u_2 = c_2 e^{(1-2i)x} = c_2 e^{x} e^{-2ix}\left(\cos(2x) - i\sin(2x)\right)$$

$$\Rightarrow u(x) = e^{x}\cos(2x)\left(c_1 + c_2\right) + i\sin(2x)\left(c_1 - c_2\right)$$

$$u'' - 2u' + 5u = i\sin(2x)\,(c_1 - c_2) - 4i\cos(2x)\,(c_1 - c_2)$$

$$= i\,(c_1 - c_2)\left(\sin(2x) - 4\cos(2x)\right) = 0$$

$\boxed{c_1 = c_2}$ $\therefore$ u is a solution

Checking u is not a solution for (a) $u'' + 5u' - 2u = 0$

$$u'' + 5u' - 2u = 10i\cos(2x)(c_1 - c_2) - 6i\sin(2x)(c_1 - c_2) - 14\sin(2x)e^{x}(c_1 + c_2)$$

$$= i(c_1 - c_2)\left(10\cos(2x) - 6\sin(2x)\right) - 14\sin(2x)\,e^{x}(c_1 + c_2) = 0$$

$$= i(c_1 - c_2)\left(10\,\frac{e^{i2x} + e^{-i2x}}{2} - 6\,\frac{e^{i2x} - e^{-i2x}}{2i}\right) - 14\,\frac{e^{2ix} - e^{-2ix}}{2i}\cdot e^{x}(c_1 + c_2) = 0$$

Case 2 :

if $c_1+c_2=0 \Rightarrow c_1=-c_2$ and

$$;(c_1-c_2)\left(8e^{2ix}+2e^{-2ix}\right)=0 \Rightarrow 8e^{2ix}+2e^{-2ix}=0$$

$$\Rightarrow 4e^{2ix}+e^{-2ix}=0=4\left(\cancel{\cos(2x)}+i\sin(2x)\right)+\cancel{\cos(2x)}-i\sin(2x)=0$$

$$=5\cos(2x)+3i\sin(2x)=0$$ not true,

Case 3 $c_1-c_2\neq 0$ and $c_1+c_2\neq 0$ then let $c_1-c_2=\alpha$

and $c_1+c_2=\phi$

$$\Rightarrow i\alpha\left(8e^{2ix}+2e^{-2ix}\right)+e^{x}\phi\left(7e^{2ix}-7e^{-2ix}\right)=0$$

$$\Rightarrow i\alpha\left(8\cancel{\cos(2x)}+i8\cancel{\sin(2x)}+2\cancel{\cos(2x)}-\cancel{i2\sin(2x)}\right)$$
$$+e^{x}\phi\left(7\cancel{\cos(2x)}+\cancel{7i\sin(2x)}-\cancel{7\cos(2x)}+7\cancel{i\sin(2x)}\right)=0$$

$$\Rightarrow i\alpha\left(10\cos(2x)+6i\sin(2x)\right)+e^{x}\phi\left(14i\sin(2x)\right)=0$$

$$=\underbrace{i\alpha 10\cos(2x)-6\alpha\sin(2x)+e^{x}\phi 14i\sin(2x)}_{\text{never } 0}=0$$

$\therefore u(x)=c_1e^{(1+2i)x}+c_2e^{(1-2i)x}$ is not a solution for (a)

$$\Rightarrow i(c_1-c_2)\left(5e^{i2x}+5e^{-2ix}-3(-i)e^{i2x}+3(-i)e^{-2ix}\right)$$

$$+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow i(c_1-c_2)\left(\cancel{5e^{i2x}}+\cancel{5e^{-2ix}}+\cancel{3ie^{2ix}}-\cancel{3ie^{-2ix}}\right)+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow i(c_1-c_2)\left(8e^{2ix}+2e^{-2ix}\right)+\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

Case 1: $c_1-c_2=0 \Rightarrow c_1=c_2$ then

$$\left(7ie^{2ix}-7ie^{-2ix}\right)e^{x}(c_1+c_2)=0$$

$$\Rightarrow 7ie^{2ix}-7ie^{-2ix}=0 \Rightarrow \cancel{7i}e^{2ix}=\cancel{7i}e^{-2ix} \quad \forall x$$

$$\Rightarrow e^{2ix}=e^{-2ix} \Rightarrow \cancel{\cos(2x)}+i\sin(2x)=\cancel{\cos(2x)}-i\sin(2x)$$

$$\Rightarrow i\sin(2x)=-i\sin(2x) \Rightarrow 2i\sin(2x)=0 \quad \forall x$$

which is not true, since $\sin(2x)\neq 0 \ \forall x$

checking if $u(x) = k e^{(1+2i)x} + k e^{(1-2i)x}$

is a solution for $u'' - 2u' + 5u = 2x$

Since $u(x) = k e^{(1+2i)x} + k e^{(1-2i)x}$ is a solution for

$u'' - 2u' + 5u = 0$ i.e.

$u'' - 2u' + 5u = i(k-k)(\sin(2x) - 4\cos(2x)) = 0$ $\forall x$ in

the domain of u,

$u'' - 2u' + 5u \neq 2x$. $\therefore u(x)$ is not a solution for 5.c

$u'' - 2u' + 5u = 2x$.