Exercise (2) for condition (ii):

$$f := piecewise\left(x < \frac{L}{2}, M, x > \frac{L}{2}, 0\right)$$

$$f := \begin{cases} M & x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \end{cases} \tag{1}$$

(a) Calculating the solution u(x,t):

Family of solutions:

$$u := \cos\left(\frac{n \cdot \mathrm{Pi}}{L} \cdot x\right) \cdot an \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := \frac{2 \cos\left(\frac{n \pi x}{L}\right) \sin\left(\frac{\pi n}{2}\right) M \mathrm{e}^{-\frac{n^2 \pi^2 \mathrm{D}\, t}{L^2}}}{n \pi} \tag{2}$$

$$a0 := \frac{int\left(M \cdot \cos\left(\frac{\mathrm{Pi} \cdot x}{L}\right), x = 0 .. \frac{L}{2}\right)}{int\left(\cos^2\left(\frac{\mathrm{Pi} \cdot x}{L}\right), x = 0 .. L\right)}$$

$$a0 := \frac{2 M}{\pi} \tag{3}$$

$$an := \frac{int\left(M \cdot \cos\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0 .. \frac{L}{2}\right)}{int\left(\cos^2\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0 .. L\right)} \text{assuming}(L > 0, n, integer)$$

$$an := \frac{2 \sin\left(\frac{\pi n}{2}\right) M}{n \pi} \tag{4}$$

Then, our family of solutions u(x,t) is:

$$u := a0 + \cos\left(\frac{n \cdot \mathrm{Pi}}{L} \cdot x\right) \cdot an \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := \frac{2 M}{\pi} + \frac{2 \cos\left(\frac{n \pi x}{L}\right) \sin\left(\frac{\pi n}{2}\right) M \mathrm{e}^{-\frac{n^2 \pi^2 \mathrm{D}\, t}{L^2}}}{n \pi} \tag{5}$$

(b) Average temperature at time t=0:

$$avg\_t0 := \frac{1}{L} int(u, x = 0 .. L)$$

$$avg\_t0 := \frac{2\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^{-\frac{n^2\pi^2 D t}{L^2}}\right)M}{n^2\pi^2} \tag{6}$$

$avg\_t0 := subs(t = 0, L = 10, M = 100, avg\_t0)$

$$avg\_t0 := \frac{200\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^0\right)}{n^2\pi^2} \tag{7}$$

$avg\_t0 := evalf(sum(avg\_t0, n = 1 ..900))$

$$avg\_t0 := 57295.77950 \tag{8}$$

The average temperature at t=0 is 57295.77950 degrees for M=100,L=10.

(c) Expression for average temperature at time t:

$avg\_t := \left(\frac{1}{L}\right) \cdot int(u, x = 0 ..L) :$

$avg\_t := subs(L = 10, M = 100, D = 1, avg\_t)$

$$avg\_t := \frac{200\left(n^2\pi + \sin(\pi n)\sin\left(\frac{\pi n}{2}\right)e^{-\frac{n^2\pi^2 t}{100}}\right)}{n^2\pi^2} \tag{9}$$

This is the expression for average temperature at time t.

(d) Since we are not losing heat, the temperature at t0 will not decrease to 10% of its value.Since our initial temperature is given by our initial condition, we know the following:

f

$$\begin{cases} M & x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \end{cases} \tag{10}$$

Therefore, we can expect for the heat to come to an equlibrium temperature on the whole domain (0,L), but it will not go as low as the 10% of the initial average temperature.

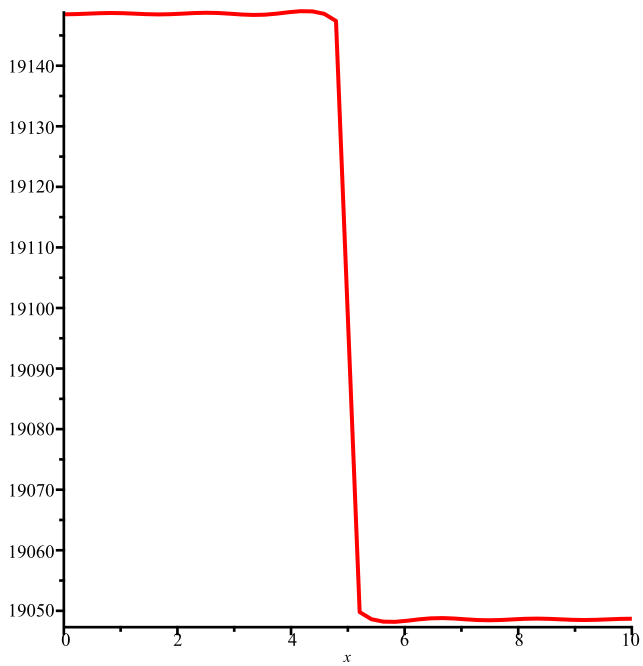
(e) Plots:

$with(plots) :$

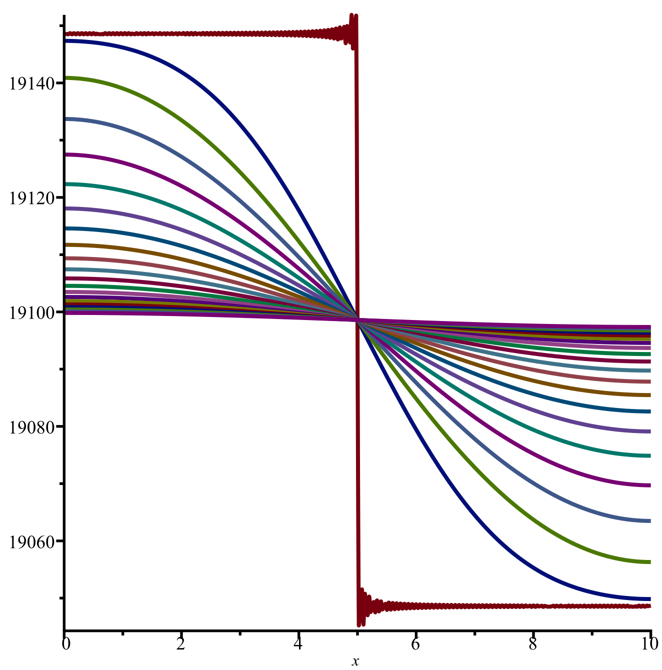
$u\_sol := subs(M = 100, L = 10, D = 1, u) :$

$psum := sum(u\_sol, n = 1 ..300) :$

$animate(psum, x = 0 ..10, t = 0 ..20)$



$curves := [seq(subs(t=2\cdot m, psum), m=0..20)]:$
$plot(curves, x=0..10)$



We can see that as time passes, we arrive to an equlibrium temperature on the whole domain (0,10) of approximately 19100 degrees.