2. $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$, $x \in (0, L)$ IC: $u(x,0) = f(x)$

BC: $\dfrac{\partial u}{\partial x}(0,t) = 0$, $u(L,t) = 0$

Let $u(x,t) = X(x)T(t)$, then, $T' \cdot X = D \cdot X'' \cdot T$

$\Rightarrow \dfrac{T'}{DT} = \dfrac{X''}{X} = K$, for K constant.

$\Rightarrow T' - KDT = 0$

$X'' - KX = 0$

case 1 $K = 0$

$\Rightarrow X'' = 0$, Let $X(x) = ax + b$. BC: $\dfrac{\partial u}{\partial x}(0,t) = 0$

$\Rightarrow X'(0) = 0 = a \cdot 0 + b \Rightarrow \boxed{b = 0} \Rightarrow X(x) = ax$

BC: $u(L,t) = 0 \Rightarrow X(L) = 0 = X(L) = a \cdot L = 0$ $a = 0$ $\cancel{L = 0}$

$\therefore a = 0$ $\therefore$ only trivial solution for $K = 0$.

case $K > 0$ let $K = \nu^2$, then $X'' - KX = 0$

becomes $X'' - \nu^2 X = 0$. We know $X(x) = a\sinh(\nu x) + b\cosh(\nu x)$

is a solution to this ODE.

For BC $\frac{du}{dx}(0,t) = 0$:

$$X'(x) = a\nu\cosh(\nu x) + b\nu\sinh(\nu x)$$

$$X'(0) = a\nu\cos(0) + b\nu\cancel{\sinh(0)}^{0} = a\nu \cdot 1 \Rightarrow \boxed{a = 0}$$

for BC $u(L,t) = 0$: $(X(x) = b\cosh(\nu x))$

$$X(L) = b\cosh(\nu L) = 0 \Rightarrow b = 0 \text{ or } \cosh(\nu L) = 0$$

$b = 0$ implies that we only have trivial solution.

$$\cosh(\nu L) = 0 \iff \frac{e^{\nu L} + e^{-\nu L}}{2} = 0 \iff \underbrace{e^{\nu L} + e^{-\nu L}}_{} = 0$$

no solution since $e^{\nu L} + e^{-\nu L} > 0$ always.

$\therefore$ for $K > 0$, only trivial solution.

case $K < 0$ $\Rightarrow$ let $K = -\nu^2$ then $X'' - KX = 0$

becomes $X'' + \nu^2 X = 0$. We know $X(x) = a\cos(\nu x) + b\sin(\nu x)$

is a solution to this ODE.

for BC $\frac{\partial u}{\partial x}(0,t) = 0$:

$$\Rightarrow X'(0) = \cancel{-a\nu\sin(0)} + b\nu\cos(0) = 0 = b\cdot\nu = 0$$

$$\boxed{\Rightarrow b = 0}$$

then $X(x) = a\cos(\nu x)$

for BC $u(L,t) = 0$ $\Rightarrow$ $X(L) = a\cos(\nu L) = 0$

$\Rightarrow$ $\cancel{a = 0}$ trivial or $\nu L = \frac{(2n-1)\pi}{2}$ $\Rightarrow$ $\nu = \frac{\pi}{2L}\cdot(2n-1)$

$n = 1, 2, 3, \ldots$

$\therefore$ Our solution, for $X'' - KX = 0$ $\wedge$ $K < 0$ is

$X(x) = a_n \cos\left((2n-1)\cdot\frac{\pi}{2L}\cdot x\right)$, since $K = -\nu^2$,

$$K = -\left(\frac{\pi}{2L}(2n-1)\right)^2.$$

for $T' - K D T = 0$ we have $K = -\nu^2$ as before
$\left(K = -\left(\frac{\pi}{2L} \cdot (2n-1)\right)^2 \right)$

we know $T(t) = C \cdot e^{\lambda t}$ is a sol. for $T' + \nu^2 D T = 0$

$$\therefore \quad T' + \nu^2 D T = \cancel{C} \cdot \lambda \cancel{e^{\lambda t}} + \nu^2 D \cancel{C} \cancel{e^{\lambda t}} = 0$$

$$\Rightarrow C e^{\lambda t} \underbrace{\left(\lambda + \nu^2 D\right)}_{\text{eigenvalues}} = 0$$

$$\Rightarrow \lambda = -\nu^2 D = -\left(\frac{\pi}{2L}(2n-1)\right)^2 \cdot D$$

Our eigenfunction is :

$$T_n(t) = C_n e^{-\left(\frac{\pi}{2L}(2n-1)\right)^2 \cdot D t}$$

$$\therefore \quad u(x,t) = X(x) \cdot T(t)$$

$$(a) \quad u(x,t) = \cos\left(\frac{\pi}{2L} \cdot (2n-1) \cdot x\right) \cdot C_n \cdot e^{-\left(\frac{\pi}{L}(2n-1)\right)^2 \cdot D \cdot t}$$