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Project 3

Laplace Eq. in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

where $x = r\cos(\theta)$
$y = r\sin(\theta)$
r, θ functions of x

Domain: $L \leq r$
$0 \leq \theta < 2\pi$

Periodic bc: $u(r,\theta) = u(r, \theta + 2\pi)$

BC in r: $u(L,\theta) = 100$ where $0 < \theta < \frac{\pi}{2}$

$u(L, \theta) = 0$ where $\frac{\pi}{2} < \theta < 2\pi$

Let $u(r,\theta) = R(r)\cdot\Theta(\theta)$

Separations of variables:

$$R''\cdot\Theta + \frac{1}{r}\cdot R'\Theta + \frac{1}{r^2}\cdot R\cdot\Theta'' = 0$$

$$\Rightarrow r^2\cdot\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$\Rightarrow \quad \Theta'' + \lambda \Theta = 0 \quad \wedge \quad r^2 R'' + r R' - \lambda R = 0$

Periodic BC: $\Theta(\theta) = \Theta(\theta + 2\pi)$ and $\dfrac{\partial \Theta}{\partial \theta}(\theta) = \dfrac{\partial \Theta}{\partial \theta}(\theta + 2\pi)$

We know the solution is $\Theta(\theta) = \text{const} \cdot \sin(n\theta) + \text{const} \cdot \cos(n\theta)$,

$n = 0, 1, 2, \ldots$ and $\lambda_n = n^2$. Now, we have $r^2 R'' + r R' - n^2 R = 0$

Let $R(r) = c r^k \Rightarrow R' = c k r^{k-1} \wedge R'' = c k(k-1) r^{k-2}$

plugging in: $r^2 R'' + r R' - n^2 R = r^2 \cdot c k(k-1) r^{k-2} + r \cdot c k r^{k-1} - n^2 c r^k = 0$

$$\Rightarrow c k(k-1) r^k + c k r^k - n^2 c r^k = 0$$

$$c r^k (k(k-1) + k - n^2) = 0$$

$$\Rightarrow c = 0 \text{ (trivial)} \quad \vee \quad k^2 - \cancel{k} + \cancel{k} - n^2 = 0$$

$$\Rightarrow k^2 - n^2 = 0$$

$$k^2 = n^2$$

$$k = \pm\sqrt{n^2} = \pm n$$

Then $R_n(r) = a_n r^{k_1} + b_n r^{k_2} = a_n r^n + b_n r^{-n}$

$\therefore R_n(r) = a_n r^n + b_n r^{-n}$ for $n > 0$

$n = 1, 2, 3 \ldots$

for case $n=0$: $r^2 R'' + rR' - n^2 R = 0$

$$\Rightarrow r^2 R'' + rR' = 0$$

$$\Rightarrow rR'' + R' = 0$$

Let $R' = Q$ and $R'' = Q'$, then $rR'' + R' = rQ' + Q = 0$

$$\Rightarrow r\frac{\partial Q}{\partial r} + Q = 0 \Rightarrow r\frac{\partial Q}{\partial r} = -Q \Rightarrow \frac{1}{Q}\partial Q = -\frac{1}{r}dr$$

$$\Rightarrow \int \frac{1}{Q}\partial Q = \int -\frac{1}{r}dr \Rightarrow \ln|Q| = -\ln|r| + \text{const.}$$

$$\Rightarrow e^{\ln|Q|} = e^{-\ln|r|} \cdot e^{\text{const}}, \quad \text{let } e^{\text{const.}} = K_0$$

$$\Rightarrow |Q| = \frac{1}{|r|} \cdot K_0 \Rightarrow Q = \frac{1}{r} \cdot K_0$$

(Since sign of K_0 is not important)

then, since $R' = Q$, $R' = \frac{1}{r} K_0 \Rightarrow \int \partial R = \int \frac{K_0}{r} dr$

$$\boxed{\Rightarrow R(r) = K_0 \ln|r| + d}$$

2 linearly independent ODE

Now, note that $\lim_{r\to\infty} R_0(r) = K_0 \ln(r) + d = \infty$

but we need $u(\infty, \theta) = 25 \neq \infty$ $\therefore$, let $R_0 = d$. Moreover,

let $R_0(r) = 25$ to satisfy the BC.

for $R_n(r) = a_n r^n + \frac{b_n}{r^n}$, $n > 1$, note that $\lim_{r\to\infty} R_n = \infty$

which can't happen since $u(\infty, \theta) = 25$. Then, let $a_n = 0$ $\forall n$.

$\Rightarrow R_n(r) = \frac{b_n}{r^n}$

Our Solution

$$u(r, \theta) = 25 + \sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n \left(A_n \sin(n\theta) + B_n \cos(n\theta)\right)$$

and our BC at $r = L$ is $u(L, \theta) = f(\theta) = \begin{cases} 100, & 0 \leq \theta \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta < 2\pi \end{cases}$

$$\Rightarrow f(\theta) = 25 + \sum_{n=1}^{\infty} \left(\frac{1}{L}\right)^n \left(A_n \sin(n\theta) + B_n \cos(n\theta)\right)$$