MA 501 SP22 Lubkin HW2 solutions

1. exp(x) = cosh(x) + sinh(x).

2. Let f + g = h, where f and g are periodic with period T. Then h(t+T) = f(t+T) + g(t+T) = f(t) + g(t) = h(t) so h is periodic with period T too.

3. We know that for x=0, this function =0. We also know that sin(x) = 0 at x=n*Pi where n is any integer. We also know that sin(Pi*x) = 0 at x = m where m is any integer. There are no other zeros of either sin function. However, since Pi is irrational, there is no pair of m and n except for 0 and 0 which will yield m = n*Pi. So this is not a periodic function. It resembles a periodic function, so we can call it "quasiperiodic" but it is not actually periodic.

4. 1. Note how I am being careful not to divide by zero! If you just plow through carelessly, you might miss the exceptions when m=n which need to be proven separately!

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot m\cdot t}{T}\right), t=0..T\right)$

$$\frac{1}{\pi\,(m^2-n^2)}\left(T\left(2\,m\sin(\pi\,m)\cos(\pi\,m)\sin(\pi\,n)\cos(\pi\,n) + 2\,n\cos(\pi\,m)^2\cos(\pi\,n)^2 - n\cos(\pi\,m)^2 - n\cos(\pi\,n)^2\right)\right) \quad (1)$$

This will have a div by 0 if m=n, so we need to do that separately (see a few lines down)

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot m\cdot t}{T}\right), t=0..T\right)$assuming$(n, integer)$

$$\frac{n\,T\left(\cos(\pi\,m)^2-1\right)}{\pi\,(m^2-n^2)} \quad (2)$$

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot m\cdot t}{T}\right), t=0..T\right)$assuming$(m, integer)$

$$\frac{n\,T\left(\cos(\pi\,n)^2-1\right)}{\pi\,(m^2-n^2)} \quad (3)$$

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot m\cdot t}{T}\right), t=0..T\right)$assuming$(n, integer, m, integer)$

$$0 \quad (4)$$

Thus we show that the functions are orthogonal except for the case of m=n, which we have to do separately!

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right), t=0..T\right)$

$$-\frac{T\cos(\pi n)^2\left(\cos(\pi n)^2-1\right)}{\pi n} \qquad \mathbf{(5)}$$

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right), t=0..T\right)$ assuming$(n, integer)$

$$0 \qquad \mathbf{(6)}$$

And of course we need to separately consider m=0:

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot 0\cdot t}{T}\right), t=0..T\right)$

$$-\frac{T\left(\cos(\pi n)^2-1\right)}{\pi n} \qquad \mathbf{(7)}$$

> $int\left(\sin\left(\frac{2\cdot\pi\cdot n\cdot t}{T}\right)\cdot\cos\left(\frac{2\cdot\pi\cdot 0\cdot t}{T}\right), t=0..T\right)$ assuming$(n, integer)$

$$0 \qquad \mathbf{(8)}$$

5. Note that I am putting > 1 Maple command per line, by ending each command with a semicolon

> $f1 := 1; f2 := x; f3 := \frac{(3\cdot x^2-1)}{2}$

$$f1 := 1$$
$$f2 := x$$
$$f3 := \frac{3x^2}{2}-\frac{1}{2} \qquad \mathbf{(1)}$$

> $int(f1\cdot f2, x=0..1); int(f2\cdot f3, x=0..1); int(f1\cdot f3, x=0..1)$

$$\frac{1}{2}$$
$$\frac{1}{8}$$
$$0 \qquad \mathbf{(2)}$$

> $int(f1\cdot f2, x=-1..1); int(f2\cdot f3, x=-1..1); int(f1\cdot f3, x=-1..1)$

$$0$$
$$0$$
$$0 \qquad \mathbf{(3)}$$

We conclude that these functions do form an orthogonal set on (-1, 1) but not on (0,1).

6.

7. a. Period = Pi

> $phin := \sin(2 \cdot n \cdot t)$

$$phin := \sin(2\, n\, t) \tag{1}$$

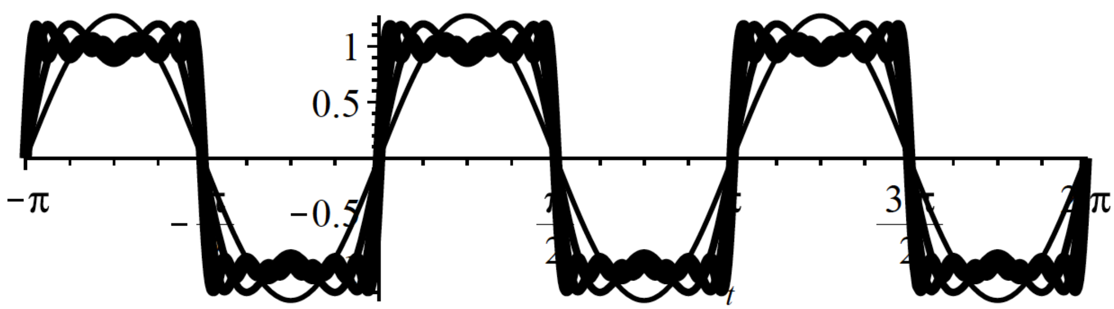
> $An := \dfrac{\left(int\left(-phin,\, t = -\frac{\pi}{2} ..0\right) + int\left(phin,\, t = 0 .. \frac{\pi}{2}\right)\right)}{int\left(phin^2,\, t = -\frac{\pi}{2} .. \frac{\pi}{2}\right)}$ assuming$(n,\, integer,\, n \neq 0)$

$$An := -\frac{2\left((-1)^n - 1\right)}{n\,\pi} \tag{2}$$

> $psum := Sum(An \cdot phin,\, n = 1 ..N);$

$$psum := \sum_{n=1}^{N} \left(-\frac{2\left((-1)^n - 1\right) \sin(2\, n\, t)}{n\,\pi}\right) \tag{3}$$

> $plot([subs(N = 1, psum), subs(N = 2, psum), subs(N = 4, psum), subs(N = 8, psum), subs(N = 16, psum)], t = -\pi ..2 \cdot \pi, color = black, thickness = [1, 2, 3, 4, 5], scaling = constrained)$




>

I should have had more terms in my plot, since I did ask you to plot up to 16 *nonzero* terms.

The square wave and the sawtooth Fourier series have a Gibbs phenomenon in their partial sums, but otherwise the convergence is OK. The An converge as 1/n which is slow, but at least each term is smaller than the previous term.

1. sawtooth

6 b. Period = 2

> $phin := \sin\left(\frac{2\cdot\pi\cdot n\cdot t}{2}\right)$

$$phin := \sin(\pi\, n\, t) \tag{11}$$

> $An := \frac{int(-t\cdot phin,\, t=-1..1)}{int(phin^2,\, t=-1..1)}$

$$An := -\frac{2\,(\pi\, n\cos(\pi\, n) - \sin(\pi\, n))}{\pi\, n\,(\sin(\pi\, n)\cos(\pi\, n) - \pi\, n)} \tag{12}$$

> $An := \frac{int(-t\cdot phin,\, t=-1..1)}{int(phin^2,\, t=-1..1)}$ assuming$(n, integer)$

$$An := \frac{2\,(-1)^n}{\pi\, n} \tag{13}$$

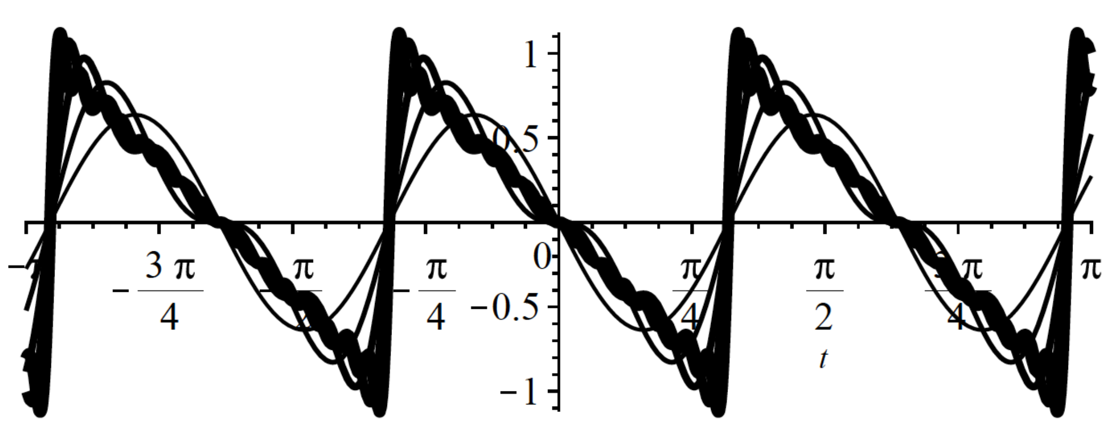
> $psum1 := eval(subs(n=1, An\cdot phin));\, psum2 := sum(An\cdot phin, n=1..2);\, psum4 := sum(An\cdot phin, n=1..4);\, psum8 := sum(An\cdot phin, n=1..8) : psum16 := sum(An\cdot phin, n=1..16) :$

$$psum1 := -\frac{2\sin(\pi\, t)}{\pi}$$

$$psum2 := -\frac{2\sin(\pi\, t)}{\pi} + \frac{\sin(2\,\pi\, t)}{\pi}$$

$$psum4 := -\frac{2\sin(\pi\, t)}{\pi} + \frac{\sin(2\,\pi\, t)}{\pi} - \frac{2}{3}\,\frac{\sin(3\,\pi\, t)}{\pi} + \frac{1}{2}\,\frac{\sin(4\,\pi\, t)}{\pi} \tag{14}$$

> $plot([psum1, psum2, psum4, psum8, psum16], t=-\pi..\pi, color = black, thickness = [1, 2, 3, 4, 5], scaling = constrained)$




2.

7. 3. $x^3 + 1$ is its own Taylor series! No calculations need to be done.

8. 4. $1 + \sin(t)$ is its own Fourier series! No calculations need to be done.

9. 5. $\sin(t)*\cos(t) = 1/2 \sin(2t)$ which is its own Fourier series. No further calculations need to be done.

In both of these (4-5), if you tried to calculate the coefficients using dot products and integrals, very likely Maple would have misled you into thinking all of these coefficients were zero. Clearly the coefficients are not all zero, since neither function is the zero function.