IC (i):
Calculating coefficients bn:

$$bn := \frac{int\left(M\cdot\sin\left(\frac{n\cdot\text{Pi}\cdot x}{L}\right), x=0..L\right)}{int\left(\sin^2\left(\frac{n\cdot\text{Pi}\cdot x}{L}\right), x=0..L\right)} \text{assuming}(M>0, n, integer)$$

$$bn := -\frac{2M\left((-1)^n-1\right)}{\pi n} \tag{1}$$

(a) u(x,t) is equalt to:

$$u := bn\cdot\sin\left(\frac{n\cdot\text{Pi}\cdot x}{L}\right)\cdot\exp\left(-\left(\frac{n\cdot\text{Pi}}{L}\right)^2\cdot\text{D}\cdot t\right)$$

$$u := -\frac{2M\left((-1)^n-1\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{e}^{-\frac{n^2\pi^2\text{D}\,t}{L^2}}}{\pi n} \tag{2}$$

(b) Then, calculating the average temperature at t=0:

$$avg := \left(\frac{1}{L}\right)\cdot int\left(bn\cdot\sin\left(\frac{n\cdot\text{Pi}}{L}\cdot x\right)\cdot\exp\left(-\left(\frac{n\cdot\text{Pi}}{L}\right)^2\cdot\text{D}\cdot t\right), x=0..L\right)$$

$$avg := \frac{2M\left((-1)^n-1\right)\mathrm{e}^{-\frac{n^2\pi^2\text{D}\,t}{L^2}}\left(\cos(n\pi)-1\right)}{n^2\pi^2} \tag{3}$$

$$avg := subs(t=0, M=100, L=10, \text{D}=1, avg)$$

$$avg := \frac{200\left((-1)^n-1\right)\mathrm{e}^0\left(\cos(n\pi)-1\right)}{n^2\pi^2} \tag{4}$$

$psum := sum(avg, n=1..1000):$
$evalf(psum)$

$$99.95947149 \tag{5}$$

(c) Expression for calculating the average temperature at time t:

$$avg\_expr := \frac{\exp\left(-\left(\frac{n\cdot\text{Pi}}{L}\right)^2\cdot\text{D}\cdot t\right)}{L}\cdot int\left(-\frac{2\cdot M\cdot\left((-1)^n-1\right)}{\text{Pi}\cdot n}\cdot\sin\left(\frac{n\cdot\text{Pi}}{L}x\right), x=0..L\right)$$

$$avg\_expr := \frac{2M\left((-1)^n-1\right)\mathrm{e}^{-\frac{n^2\pi^2\text{D}\,t}{L^2}}\left(\cos(n\pi)-1\right)}{n^2\pi^2} \tag{6}$$

(d) The slowest decaying term occurs when n = 1. Then
$initial\_value := 100:$
$final\_value := 0.1\cdot initial\_value$

$$final\_value := 10.0 \tag{7}$$

$equ := subs(M=100, n=1, L=10, D=1, avg\_expr) - final\_value$

$$equ := -\frac{400\, e^{-\frac{\pi^2 t}{100}} (\cos(\pi) - 1)}{\pi^2} - 10.0 \tag{8}$$

$solve(equ, t)$

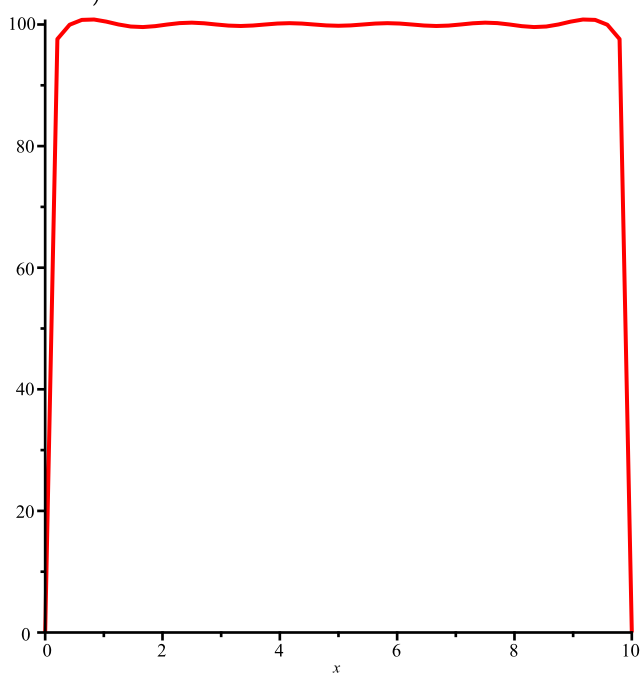
$$21.20213514 \tag{9}$$

(e) Creating Partial Sums:

$with(plots):$

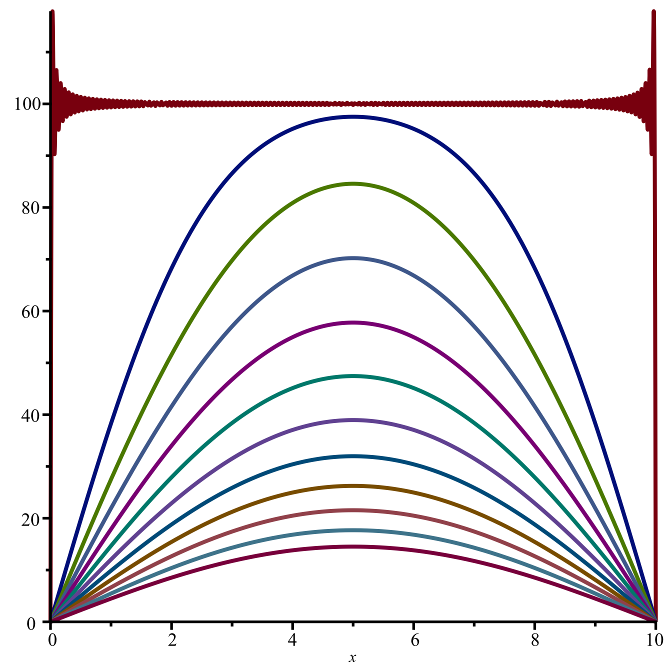
$psum := subs(M=100, D=1, L=10, sum(u, n=1..300)):$

$animate(psum, x=0..10, t=0..23)$



$curves := [seq(subs(t=2\cdot m, psum), m=0..11)]:$

$plot(curves, x=0..10)$



We can see that approaches 21.20213514 time units, the amplitud of the curves (temperature) get smaller. We can also see the Gibbs phenomenon for the t=0 curve.

For initial condition 1.ii:

$$f := piecewise\left(x > \frac{L}{2}, M, x > \frac{L}{2}, 0\right)$$

$$f := \begin{cases} M & \frac{L}{2} < x \\ 0 & \frac{L}{2} < x \end{cases} \tag{1}$$

(a) Calculating the solution u(x,t):
We know the family of solutions is as follows:

$$u := bn \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right) \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := -\frac{2\, M \left(\cos\left(\frac{n\,\pi}{2}\right) - 1\right) \sin\left(\frac{n\,\pi\,x}{L}\right) \mathrm{e}^{-\frac{n^2\,\pi^2\,\mathrm{D}\,t}{L^2}}}{n\,\pi} \tag{2}$$

Calculating the coefficients bn for t=0:

$$bn := \frac{int\left(M \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0\,..\frac{L}{2}\right)}{int\left(\sin^2\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right), x = 0\,..L\right)} \mathrm{assuming}(L > 0, n, integer)$$

$$bn := -\frac{2\, M \left(\cos\left(\frac{n\,\pi}{2}\right) - 1\right)}{n\,\pi} \tag{3}$$

Then, the family of solutions u(x,t) is:

$$u := bn \cdot \sin\left(\frac{n \cdot \mathrm{Pi} \cdot x}{L}\right) \cdot \exp\left(-\left(\frac{n \cdot \mathrm{Pi}}{L}\right)^2 \cdot \mathrm{D} \cdot t\right)$$

$$u := -\frac{2\, M \left(\cos\left(\frac{n\,\pi}{2}\right) - 1\right) \sin\left(\frac{n\,\pi\,x}{L}\right) \mathrm{e}^{-\frac{n^2\,\pi^2\,\mathrm{D}\,t}{L^2}}}{n\,\pi} \tag{4}$$

(b) Average temperature at time t=0:

$$avg\_t0 := simplify\left(\left(\frac{1}{L}\right) \cdot int(u, x = 0\,..L)\,\mathrm{assuming}(L > 0, n, integer)\right):$$
$$avg\_t0 := simplify(subs(M = 100, t = 0, L = 10, \mathrm{D} = 1, avg\_t0)):$$
$$avg\_t0 := evalf(sum(avg\_t0, n = 1\,..900))$$

$$avg\_t0 := 49.97748417 \tag{5}$$

(c) Average temperature expression:

$$avg\_temp := \left(\frac{1}{L}\right) \cdot int(u, x = 0\,..L)\,\mathrm{assuming}(L > 0, \mathrm{D} = 1, n, integer):$$
$$avg\_temp := subs(L = 10, M = 100, avg\_temp)$$

$$avg\_temp := \frac{200\left(\cos\left(\frac{n\,\pi}{2}\right)-1\right)e^{-\frac{n^2\pi^2 t}{100}}\left(-1+(-1)^n\right)}{n^2\,\pi^2} \quad \textbf{(6)}$$

(d) Estimating how long it takes for the average temperature to decrease to 10% of its initial value (for the slowest decaying term n=1):

$final\_value := 0.1 \cdot 50;$

$$final\_value := 5.0 \quad \textbf{(7)}$$

$equ := subs(n=1, avg\_temp - final\_value)$

$$equ := -\frac{400\left(\cos\left(\frac{\pi}{2}\right)-1\right)e^{-\frac{t\,\pi^2}{100}}}{\pi^2} - 5.0 \quad \textbf{(8)}$$

$equ := solve(equ, t)$

$$equ := 21.20213514 \quad \textbf{(9)}$$

It takes 21.20213514 times units (approximately) to go from the average temperature at t=0 of 50 to the average temperature of 5.

(e)Animation, partial sums, and plots

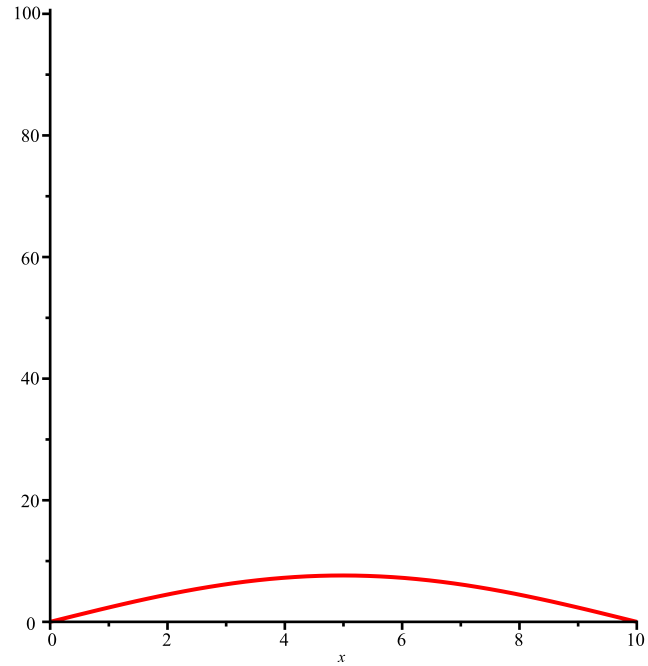
$u := subs(M=100, L=10, \mathrm{D}=1, u)$

$$u := -\frac{200\left(\cos\left(\frac{n\,\pi}{2}\right)-1\right)\sin\left(\frac{n\,\pi\,x}{10}\right)e^{-\frac{n^2\pi^2 t}{100}}}{n\,\pi} \quad \textbf{(10)}$$

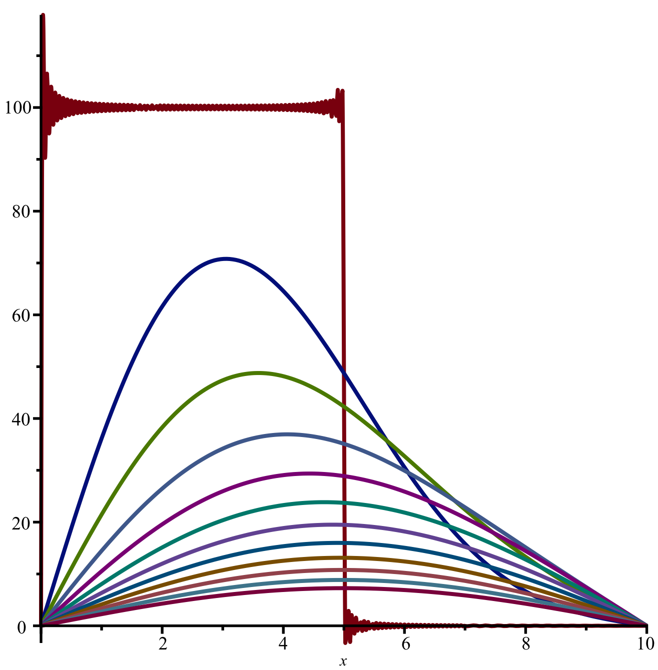
$psum := sum(u, n=1..300):$

$with(plots):$

$animate(psum, x=0..10, t=0..21.5)$



$curves := [seq(subs(t=2\cdot m, psum), m=0..11)]:$
$plot(curves, x=0..10)$



We can see that the average temperature amplitud is lower as t increases.