Continuous-Time Markov Chain

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Definition

(Continuous-Time Markov Chain (CTMC)). A stochastic process $\{X(t), t \geq 0\}$ on state space S is called a CTMC if, for all i and j in S and $t, s \geq 0$,

$$P(X(s+t) = j \mid X(s) = i, X(u), 0 \leq u \leq s) = P(X(s+t) = j \mid X(s) = i)$$

The CTMC $\{X(t), t \geq 0\}$ is said to be time homogeneous if, for $t, s \geq 0$,

$$P(X(s+t) = j \mid X(s) = i) = P(X(t) = j \mid X(0) = i)$$

constant

no change of speed of the system as it evolves.

Unless otherwise mentioned, that all CTMCs are time homogeneous and have a finite state space $\{1, 2, \dots, N\}$. For such CTMCs, we define

$$p_{i,j}(t) = \mathrm{P}(X(t) = j \mid X(0) = i), \quad 1 \le i, j \le N.$$

The N^2 entities $p_{i,j}(t)$ are arranged in matrix form as follows:

which state how long the time is

t is a parameter

$$P(t) = \begin{bmatrix} p_{1,1}(t) & p_{1,2}(t) & p_{1,3}(t) & \cdots & p_{1,N}(t) \\ p_{2,1}(t) & p_{2,2}(t) & p_{2,3}(t) & \cdots & p_{2,N}(t) \\ p_{3,1}(t) & p_{3,2}(t) & p_{3,3}(t) & \cdots & p_{3,N}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{N,1}(t) & p_{N,2}(t) & p_{N,3}(t) & \cdots & p_{N,N}(t) \end{bmatrix}.$$

The matrix above is called the transition probability matrix of the CTMC $\{X(t), t \ge 0\}$. Note that $P(t)$ is a matrix of functions of t; i.e., it has to be specified for each t. The following theorem gives the important properties of the transition probability matrix $P(t)$.

Properties of $P(t)$

A transition probability matrix $P(t) = \left[p_{i,j}(t)\right]$ of a time-homogeneous CTMC on state space $S = \{1, 2, \ldots, N\}$ satisfies the following:

1. $p_{i,j}(t) \geq 0, \quad 1 \leq i, j \leq N; t \geq 0,$
2. $\sum_{j=1}^{N} p_{i,j}(t) = 1, \quad 1 \leq i \leq N; t \geq 0,$
3. $p_{i,j}(s+t) = \sum_{k=1}^{N} p_{i,k}(s) p_{k,j}(t) = \sum_{k=1}^{N} p_{i,k}(t) p_{k,j}(s), \quad 1 \leq i \leq N; t, s \geq 0.$

Property 3 states the Chapman-Kolmogorov equations for CTMCs and can be written in matrix form as

$$P(t+s) = \overbrace{\underbrace{P(s)P(t)}_{\text{matrix multiplication}} = P(t)P(s)}^{\text{they commute!!!}}.$$

Thus the transition probability matrices $P(s)$ and $P(t)$ commute! Since AB is generally not the same as BA for square matrices A and B, the property above implies that the transition probability matrices are very special indeed!

Example

Two-State CTMC:Suppose the lifetime of a high-altitude satellite is an Exp(μ) random variable. Once it fails, it stays failed forever since no repair is possible. Let $X(t)$ be 1 if the satellite is operational at time t and 0 otherwise.

$0 \to$ absorbing state.

Example

$$P_{10}(t) = 1 - e^{-\mu t}$$

$$\left.\begin{array}{cc} & \text{states} \equiv \; 0 \qquad 1 \\ P(t) = & \begin{bmatrix} 1 & 0 \\ 1 - e^{-\mu t} & e^{-\mu t} \end{bmatrix} \end{array}\right\} \text{fully characterizes the CTMC.}$$

Giving $P(t)$ for each t is too complicated for most CTMCs. We need a simpler method of describing a CTMC.

We separate the jumping probability with transition time t.

Example

The stochastic process $\{X(t), t \geq 0\}$ of the above example can be a CTMC if and only if the lifetime of the satellite has the memoryless property.

Details

Let $X(t)$ be the state of a system at time t. Suppose the state space of the stochastic process $\{X(t), t \geq 0\}$ is $\{1, 2, \ldots, N\}$. The random evolution of the system occurs as follows.

Suppose the system starts in state i. It stays there for an $\text{Exp}(r_i)$ amount of time, called the sojourn time in state i.

or staying time

At the end of the sojourn time in state i, the system makes a sudden transition to state $j \neq i$ with probability $p_{i,j}$, independent of how long the system has been in state i.

(we have memoryless prob)

Once in state j, it stays there for an $\text{Exp}(r_j)$ amount of time and then moves to a new state $k \neq j$ with probability $p_{j,k}$, independently of the history of the system so far. And it continues this way forever.

$r_i \to$ how fast the transition will happen

To summarize

The stochastic process $\{X(t), t \geq 0\}$ with parameters $r_i, 1 \leq i \leq N$, and $p_{i,j}, 1 \leq i, j \leq N$, as described above is a CTMC.

what state you are currently in.

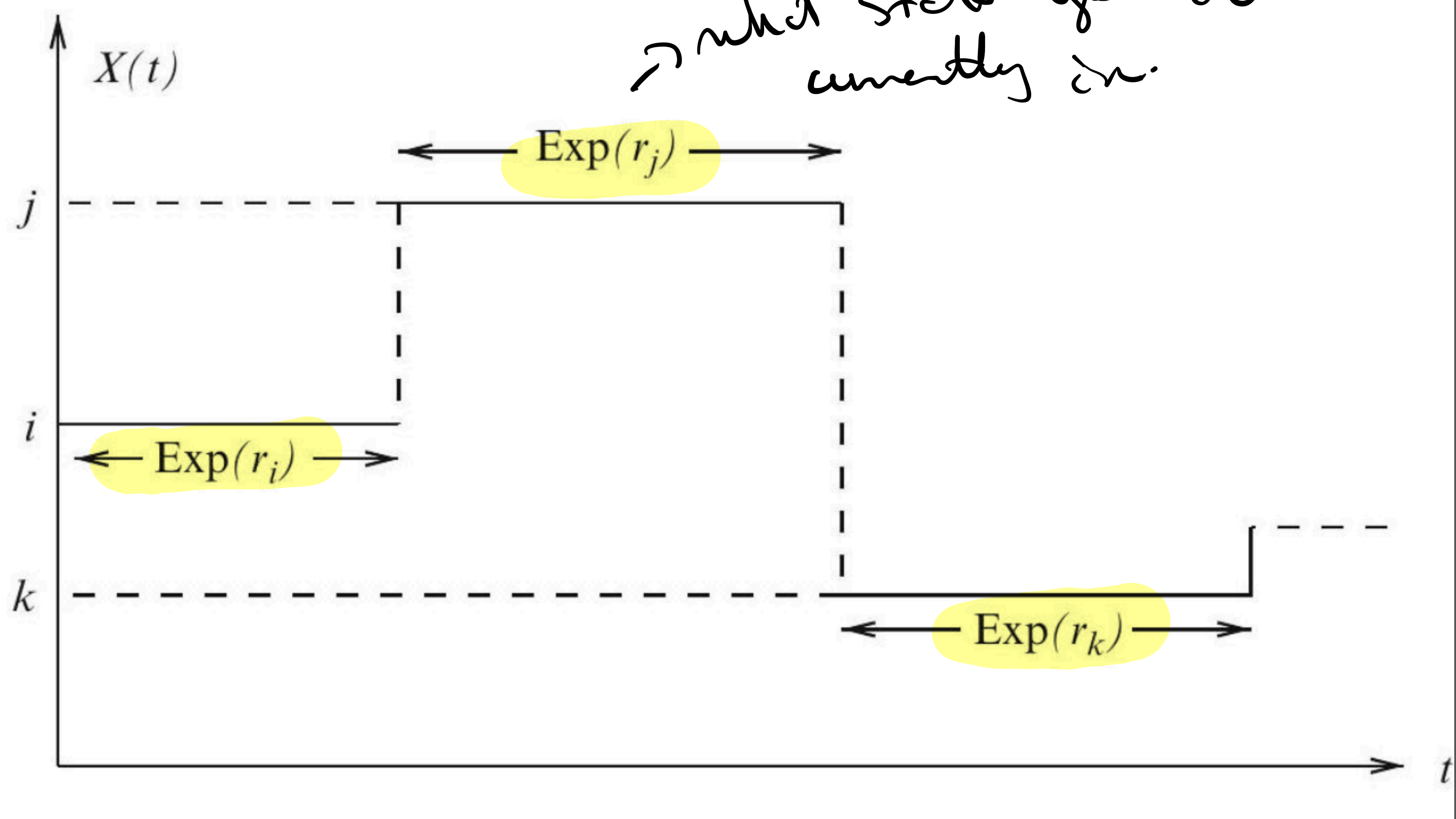


Figure 1: A Sample Path

Rate Diagram

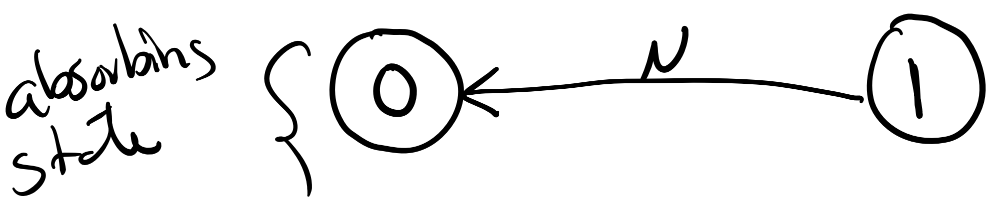
can be represented as a directed graph.

Hence we can describe a CTMC by giving the parameters $\{r_i, 1 \leq i \leq N\}$ and $\{p_{i,j}, 1 \leq i, j \leq N\}$. This is a much simpler description than giving the transition probability matrix $P(t)$ for all $t \geq 0$.

The rate diagram of the CTMC: a CTMC can also be represented graphically by means of a directed graph as follows. The directed graph has one node (or vertex) for each state. There is a directed arc from node i to node j if $p_{i,j} > 0$. The quantity $r_{i,j} = r_i p_{i,j}$, called the transition rate from i to j, is written next to this arc.

Note that there are no self-loops (arcs from i to i) in this representation.

We can understand the dynamics of the CTMC by visualizing a particle that moves from node to node in the rate diagram as follows: it stays on node i for an $\mathsf{Exp}\,(r_i)$ amount of time and then chooses one of the outgoing arcs from node i with probabilities proportional to the rates on the arcs and moves to the node at the other end of the arc. This motion continues forever. The node occupied by the particle at time t is the state of the CTMC at time t.



$r_1 = \sum_{j=0}^{1} r_{ij} = r_{10} + r_{11}$

~~$p_{10} = 1$~~ $r_1 = N =$ prob + to 0

$r_{10} =$

$r_1 = N$

Speed at any state

$$r_i = \sum_{j=1}^{N} r_{i,j},$$

$$p_{i,j} = \frac{r_{i,j}}{r_i} \text{ if } r_i \neq 0.$$

rate I am getting out of state 1

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} & \cdots & r_{1,N} \\ r_{2,1} & r_{2,2} & r_{2,3} & \cdots & r_{2,N} \\ r_{3,1} & r_{3,2} & r_{3,3} & \cdots & r_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N,1} & r_{N,2} & r_{N,3} & \cdots & r_{N,N} \end{bmatrix}_{N \times N}.$$

for our problem

$R = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}$ = rate matrix.

Note that $r_{i,i} = 0$ for all $1 \le i \le N$, and hence the diagonal entries in R are always zero. R is called the rate matrix of the CTMC. It is closely related to $Q = [q_{i,j}]$, called the generator matrix of the CTMC, which is defined as

$$\begin{bmatrix} 0 & 0 \\ N & -N \end{bmatrix} =$$

$-N = -r_1$

$$q_{i,j} = \begin{cases} -r_i \text{ if} & i = j \\ r_{i,j} \text{ if} & i \neq j \end{cases}$$

$$q_{ij} = \begin{cases} 0 & , i = j \end{cases}$$

Thus the generator matrix Q is the same as the rate matrix R with the diagonal elements replaced by $-r_i$'s. It is common in the literature to describe a CTMC by the Q matrix. However, we shall describe it by the R matrix. Clearly, one can be obtained from the other.

$r_1 \to$ sum of each row.

Obtain Q from R $\longleftrightarrow$ Obtain R from Q.

Example Revisited

Two-State Machine

Rate diagram :

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$$R = \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix}$$

Two-State Machine: Consider a machine that operates for an $\text{Exp}(\mu)$ amount of time and then fails. Once it fails, it gets repaired. The repair time is an $\text{Exp}(\lambda)$ random variable and is independent of the past. The machine is as good as new after the repair is complete. Let $X(t)$ be the state of the machine at time t, 1 if it is up and 0 if it is down. Model this as a CTMC.

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$