Markov Chains II

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Hong Wan

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North Carolina State University

NC STATE UNIVERSITY

How do you define a DTMC from a given problem?

Markov Chain: First Passenger Time

Let $\{X_n, n \geq 0\}$ be a DTMC on $S = \{0, 1, 2, \cdots\}$, with transition probability matrix P, and initial distribution a. Let

$$T = \min\{n \geq 0 : X_n = 0\}.$$

The random variable T is called the first passage time into state 0 , since T is the first time the DTMC "passes into" state 0.

The same techniques can be used to study the first passage time into any set $A \subset S$.

What We Will Study

(1) Complementary cdf of T : $\mathrm{P}(T > n), \quad n \geq 0,$

(2) Probability of eventually visiting state 0 : $P(T < \infty)$,

eventually you will go to state 0.

Conditional Quantities

the conditional quantities for $i \in S$:

→ more than n times to go to state 0, given $X_0 = i$

$$v_i(n) = \mathrm{P}\left(T > n \mid X_0 = i\right)$$
$$u_i = \mathrm{P}\left(T < \infty \mid X_0 = i\right),$$

Then we can get the following expression:

$$\mathrm{P}(T > n) = \sum_{i \in S} a_i v_i(n)$$
$$\mathrm{P}(T < \infty) = \sum_{i \in S} a_i u_i$$

CDF of T

The following theorem illustrates how the first-step analysis produces a recursive method of computing the cumulative distribution of T. We first introduce the following matrix notation:

$$v(n) = [v_1(n), v_2(n), \cdots]^\top, \quad n \geq 0$$

$$B = [p_{i,j} : i, j \geq 1].$$

Thus B is a submatrix of P obtained by deleting the row and column corresponding to the state 0 .

Theorem

$$v(n) = B^n e, \quad n \geq 0,$$

where e is column vector of all ones.

Examples: Two-state DTMC

Consider the two state DTMC. The state-space is $\{1, 2\}$. Let T be the first passage time to state 1.

Examples

Consider a DTMC with state-space $\{1, 2, 3, 4, 5, 6\}$ and transition probability matrix given below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/16 & 1/4 & 1/8 & 1/4 & 1/4 & 1/16 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let

$$T = \min \{n \geq 0 : X_n = 1\}$$

Examples: Success Runs

Consider the success runs DTMC of with transition probabilities

$$p_{i,0} = q_i, \quad p_{i,i+1} = p_i, \quad i = 0, 1, 2, \cdots .$$

Let T be the first passage time to state 0 . Compute the complementary cdf of T starting from state 1 . (Hint: think about the special structure of this DTMC)

Example

Classification of State

- Accessible j is accesible to i then $p_{ij} > 0$
- Communicate

  if $p_{ij} \wedge p_{ji} > 0$
- irreducible

  "the graph is one piece"
- Recurrent

  ↳ If you go back to x_0, everything starts again. i.e. the whole process can restart.
- Transient

  ↳ not going back
- absorbing

  "black hole"

  can get out of that such state.

Absorbing Probability