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# RV Combinations and Conditional Expectations

— ISE/OR 560, Hong Wan, Fall 2022 —

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## More than one random variables

- Joint Distribution Functions: CDF, pmf, pdf
- Covariance
- Independence
- Expectation for a function of a random variables.

# Joint Distribution Functions

For two random variables,  $X$  and  $Y$ , the joint distribution is (cdf):

# Joint CDF

- Discrete
- Continuous

## Example

Suppose you toss four fair coins:

X: number of Hs in the first three coin tossed.

Y: number of Hs of the last two coin tossed.

Elementary Outcome	X	Y	Elementary Outcome	X	Y
0000	0	0	1000	1	0
0001	0	1	1001	1	1
0010	1	1	1010	2	1
0011	1	2	1011	2	2
0100	1	0	1100	2	0
0101	1	1	1101	2	1
0110	2	1	1110	3	1
0111	2	2	1111	3	2

## Joint Probability Mass Function for $(X, Y)$

Y/X	0	1	2	3
0				
1				
2				

# From the joint distribution to PMF

Y/X	0	1	2	3	
0					
1					
2					

# Covariance

Covariance (X,Y):



## Going back to the example:

Y/X	0	1	2	3	
0					
1					
2					

# Properties of Covariance

- $\text{Cov}(X, X) =$
- $\text{Cov}(X, Y) =$
- $\text{Cov}(cX, Y) =$
- $\text{Cov}(X, Y+Z) =$

# Independent RV

Random variables  $X$  and  $Y$  are independent if, for all  $a, b$

Joint CDF

# Independent RVs

Discrete

Continuous

# Independent RV

Expectation (X, Y)

Variance (X+Y)

Covariance (X, Y)

# Example

You own a portfolio with 4 units of asset “a” and 3 units of asset “b”. If asset “a” has a return of  $Y_a$  and asset b has a return of  $Y_b$ , your total portfolio return is  $4Y_a + 3Y_b$ . Assume  $Y_a$  and  $Y_b$  are independent.

Suppose

$Y_a$  is random with expected value 2 and variance .036

$Y_b$  is random with expected value 3 and variance .25

What is the expected value and variance of your portfolio return?

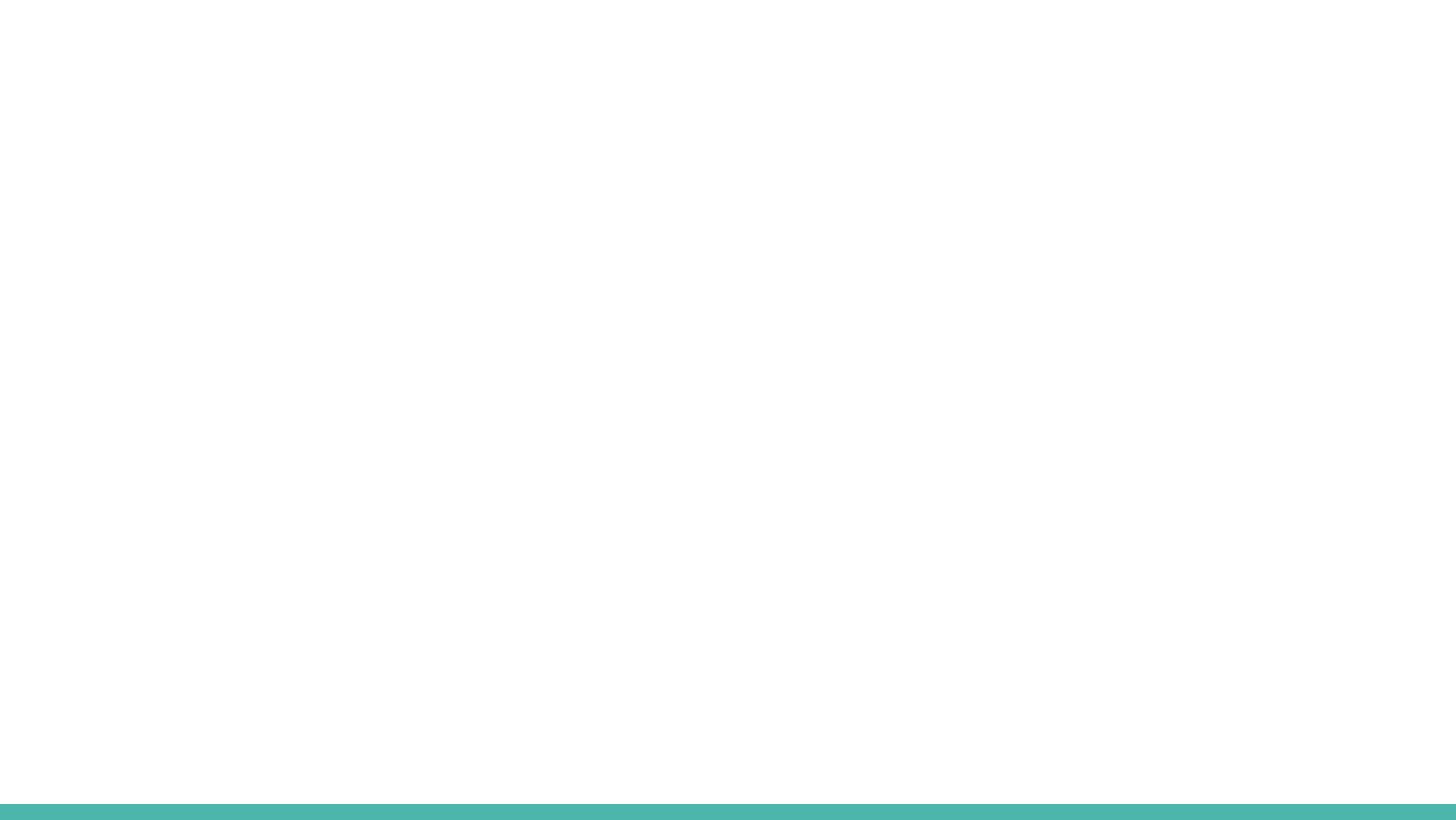
# Example

8% of the customers that walk into your store buy an item (for \$1), and the rest leave without buying anything.

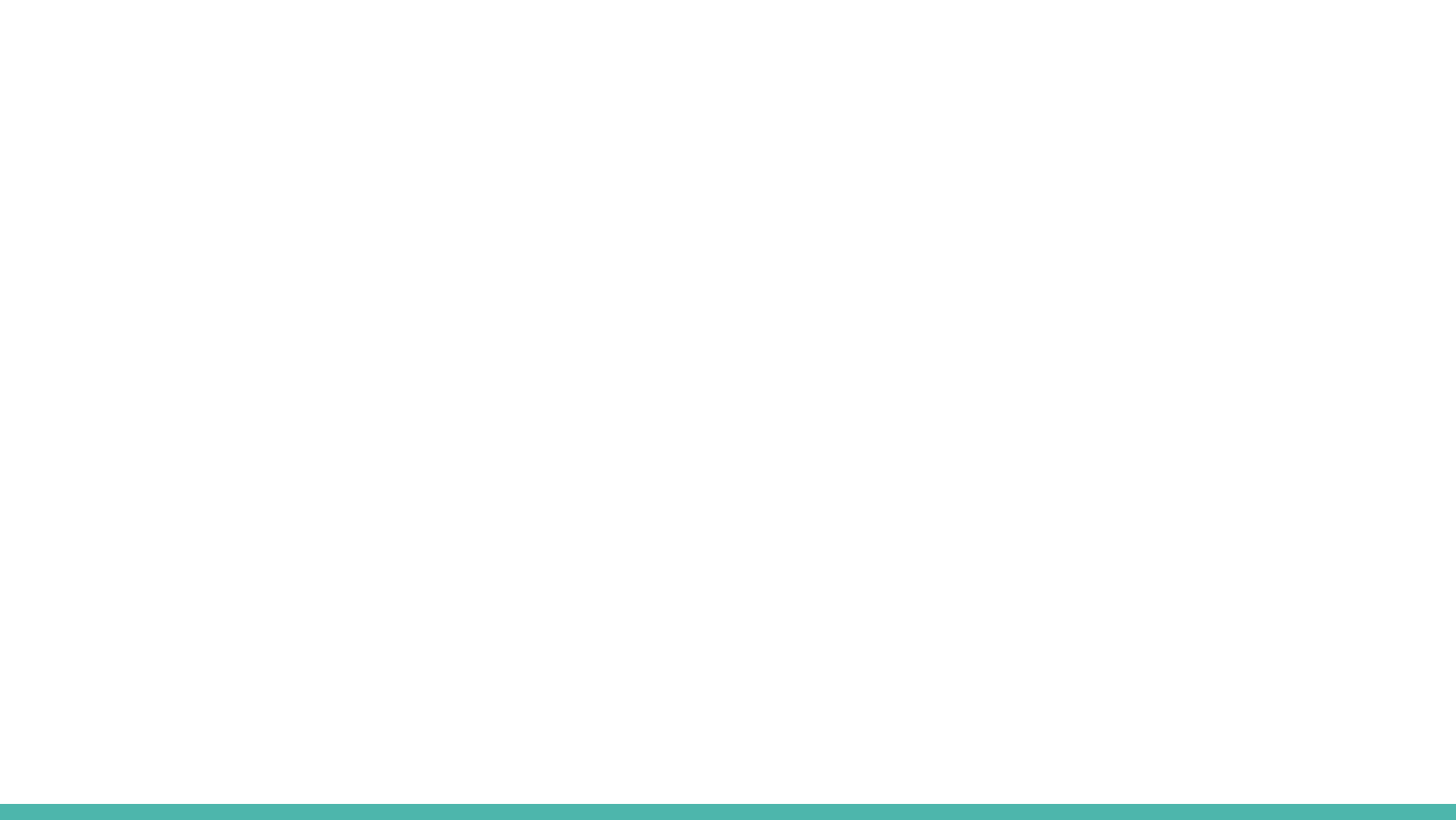
a). If only 1 customer walks in each day, what is the sample space, probability distribution, and mean of

$R_1$  = daily revenues?

b). If you earn 35 cents for every dollar spent, what are your expected daily earnings?







Five customers each day?

$S = \{xxxxx, xxxxb, \dots, xxxbb, \dots, bbbbb\}$

(there are 32 elementary outcomes in this sample space)

Since customers are independent, we can assign a probability to each outcome knowing only that each customer buys with probability .08

$S = \{xxxxx, xxxxb, \dots, xxxbb, \dots, bbbbb\}$  with probs

Five customers each day

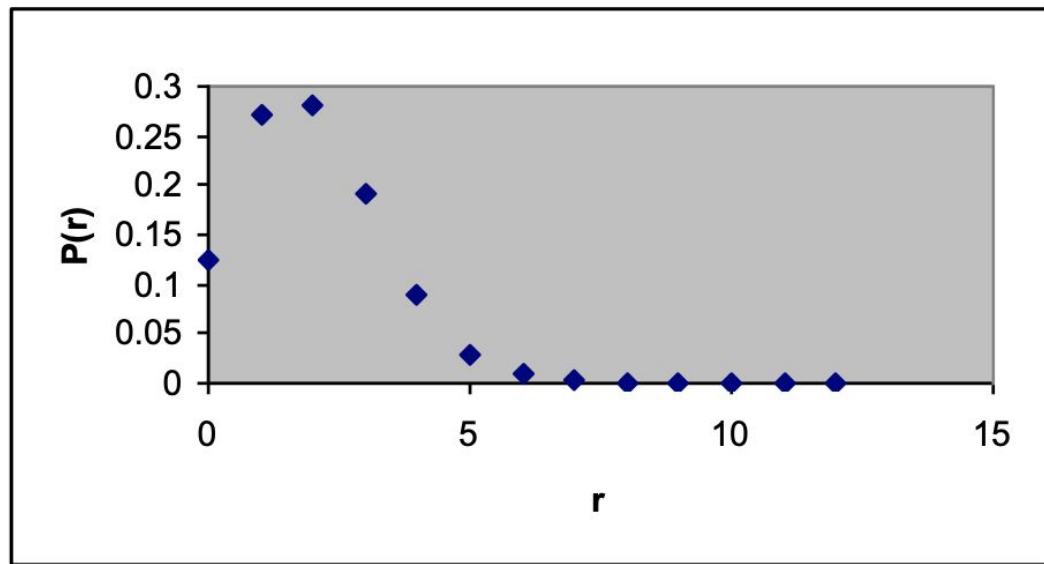
$R_5=k$  if there are exactly  $k$  “b”s in the sequence. Adding these probabilities gives us the probability that  $R_5=k$ .

r	0	1	2	3	4	5
$P(R_5=r)$	.659	.287	.05	.0043	.0002	.00003

8% of the customers that walk into your store buy an item (for \$1), and the rest leave without buying anything.

If 25 customers walk in each day, what is the expected value and variance of  $R_{25}$  = daily revenues?

r	0	1	2	3	4	5	6	7	8	9	10	etc
P(r)	.124	.27	.28	.19	.09	.03	.009	.002	.0004	.0001	0	0



$$\mu_{R_{25}} = 2.0$$

$$\sigma_{R_{25}}^2 = 1.84$$

$$\sigma_{R_{25}} = 1.36$$

Distribution for daily revenues if 25 customers visit each day (33.6 million elementary outcomes xbbxx... etc.)?

Or 100 customers each day (  $1.27 \times 10^{30}$  elementary outcomes – The age of universe is about  **$5 \times 10^{17}$  second** for reference)

200 customers each day? 1000?

## 25 Customers the Easy One

*Independent*

Each customer that walks in has a .08 probability of spending \$1, and a .92 probability of spending \$0.

n (number of customers)	$E(R_n) = n \cdot (.08)$	$\text{Var}(R_n)$ <i><math>\Sigma</math> all <math>R_n</math> together</i>
1	.08	.0736
2	.16	.147
5	.40	.368
25	2.0	1.84
100	8.0	7.36
200	16.0	14.72
1000	80.0	73.6

Think about Bernoulli <sup>relationship</sup> and Binomial. The  $\subseteq$  Bernoulli = Binomial.

$$\text{Bernoulli R.V. } X = \begin{cases} 1 & \text{female} \\ 0 & \text{not female} \end{cases} \quad \begin{aligned} P(X=1) &= 0.52 = p \\ P(X=0) &= 0.48 \end{aligned}$$

A large population is 52% female. If you draw 10 people at random from this population, what is the expected value and variance of the number of females in your sample of 10?

$$E(\# \text{ of females}) = 0.52$$

$$V(X) = E(X^2) - E(X)^2 = 1 \cdot 0.52 - (0.52)^2 = 0.52 -$$

$$\text{Variance formulation for Bernoulli: } p - p^2 = p(1-p)$$

$$\text{Then } E[N] = E[X=1] = 10 \cdot (0.52) = n \cdot P(X=1) = 5.2$$

$$\begin{aligned} \text{Var}(N) &= \sum_{i=1}^{10} \text{Var}(x_i) = 10 \cdot p(1-p) = n \cdot p(1-p) \quad \left. \begin{array}{l} \text{Binomial distribution} \\ \text{formulation} \end{array} \right\} \\ &= 10 \cdot (0.52) \cdot (0.48) \end{aligned}$$



## Review *Combination random variable*

- Joint Distribution Functions: CDF, pmf, pdf  
Covariance
- Independence
- Expectation for a function of a random variables.

## Part II: Conditional Probability and Expectation

- Recall: For any two events A and B:  $P(A|B) = P(A \cap B) / P(B)$

- For Discrete RVs, X and Y:  $P_{(\tilde{X}, \tilde{Y})}$   $\rightarrow$  probability mass function  
 $(x|y)$   $\rightarrow$  two specific realizations

$$= P_r(\tilde{X} = x | \tilde{Y} = y) = \underbrace{P(\tilde{X} = x, \tilde{Y} = y)}_{\text{joint dist.}} / P(\tilde{Y} = y)$$

## Example $P(X=1|Y=2), P(X=2|Y=2)$

Suppose  $p(x,y)$  is given by:

$$p(1,1) = 0.3, p(1,2) = 0.4, p(2,1) = 0.2, p(2,2) = 0.1$$

Calculate the conditional pmf of  $X$  given  $Y=2$

$$\frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0.4}{0.5}, \quad P(X=2|Y=2) = 0.8. \quad \text{Now, calculate}$$

conditional PMF.

How do I write this into a function.  $\sum_{\tilde{y} \leq y} P_r(Y|X=1) = 1$

Conditional Probability Distribution of Y given X = x

$$P_{\tilde{Y}|\tilde{X}}(y|x) = \sum_{\tilde{y} \leq y} P(Y|X)$$

$\begin{cases} P_r(\tilde{Y}=1 | \tilde{X}=1) & P_r(\tilde{Y}=2 | \tilde{X}=2) = P_r(\tilde{Y}|X=1) \\ P_r(\tilde{Y}=2 | \tilde{X}=1) & P_r(\tilde{Y}=2 | \tilde{X}=2) = P_r(\tilde{Y}|X=2) \end{cases}$

all of the y possible values

Conditional Expectation:

$$E[Y | X=x] = \sum_{\tilde{y} \in \Omega_Y} P(Y|X) \cdot \tilde{y}$$

sample space

$$\sum y \cdot P_r(\tilde{Y}|\tilde{X}=x)$$

$$\therefore E[X | Y=2] = P_r(X=1 | Y=2) \cdot 1 + 2P_r(X=2 | Y=2)$$

$$1 \cdot 0.8 + 2 \cdot (0.2)$$

Suppose  $p(x,y)$  is given by:

$$p(1,1) = 0.3, p(1,2) = 0.4, p(2,1) = 0.2, p(2,2) = 0.1$$

Weighted  
avg  $\uparrow$

Calculate the  $E[X|Y=2]$   $\rightarrow$  calculated in previous slide

$$= 1 \cdot p(x=1|Y=2) + 2p(x=2|Y=2)$$

$$= 0.8 + 2(0.2)$$

what if we want  $P(X=1) = 0.3 + 0.4 = 0.7$

$$P(Y=2) = 0.5 = p(1,2) + p(2,2)$$

For Continuous case  $\Sigma \rightarrow J \sim Pr \rightarrow f$

pdf of Y given  $X = x$   $\rightarrow$  joint dist.

$$f_{Y|X}(y|x) = \frac{f_J(x, y)}{f_X(x)}$$

$\downarrow$   
marginal dist.

Conditional Expectation:

$$E[Y | X = \bar{x}] = \int_{-\infty}^{\infty} y f(y|x) dy$$

$f(x,y) \rightarrow$  joint distribution function.  $\forall Y=y, X=x \Leftrightarrow x < y$

The joint density of  $X$  and  $Y$  is given by

$$f(x,y) = \frac{e^{-y}}{y}, \quad 0 < x < y, 0 < y < \infty$$

Compute  $E[\tilde{X}^3 \mid \tilde{Y}=y] = \int_0^y x^3 f(x,y) dx$  by the definition  
*until  $y$  bc.  $x$  is smaller than  $y$*   
 *$x|y=y \rightarrow$  need to find this*

$$f_{x|y}(x,y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}}{y}$$

$$f_y(y) = \int_0^y f(x,y) dx = \int_0^y \frac{e^{-y}}{y} dx$$

$$\text{Then } E[X^3 | Y=y] = \int_0^y x^3 f_{x,y}(x,y) dx = \int_0^y x^3 \left(\frac{1}{y}\right) dx$$

$$= \frac{1}{y} x^4 \left(\frac{1}{y}\right) \Big|_0^y = \frac{1}{y} \cdot \frac{y^4}{4} = \frac{y^3}{4} = E[X^3 | Y=y]$$

study also for discrete



# Conditional Continuous PDF

pdf of  $Y$  given  $X = x$

It follows that

# Independence

Recall that if  $X$  and  $Y$  are independent,

It follows that:

# Thief in the Chamber

A prisoner is placed in a cell containing three doors. The first door leads immediately to freedom. The second door leads into a tunnel which returns him to the cell after one day's travel. The third door leads to a similar tunnel which returns him to his cell after three days. Assuming that the prisoner is at all times equally likely to choose any one of the doors, what is the expected length of time until the prisoner reaches freedom?

# Example Workout

## Example Workout

Recall:  $E[Y]$  is a constant

However,  $E[Y|X]$  is a random variable that is a function of  $X$

Ex.  $Y$  = profit and  $X$  = demand, suppose  $Y = 6X$  then

In general,

# Conditions Probabilities

To solve problems where dependency is present

Process of first conditioning and then unconditioning

# Computing Conditional Expectations

For all RVs  $X$  and  $Y$ :

# Unconditioning

For Discrete X and Y:

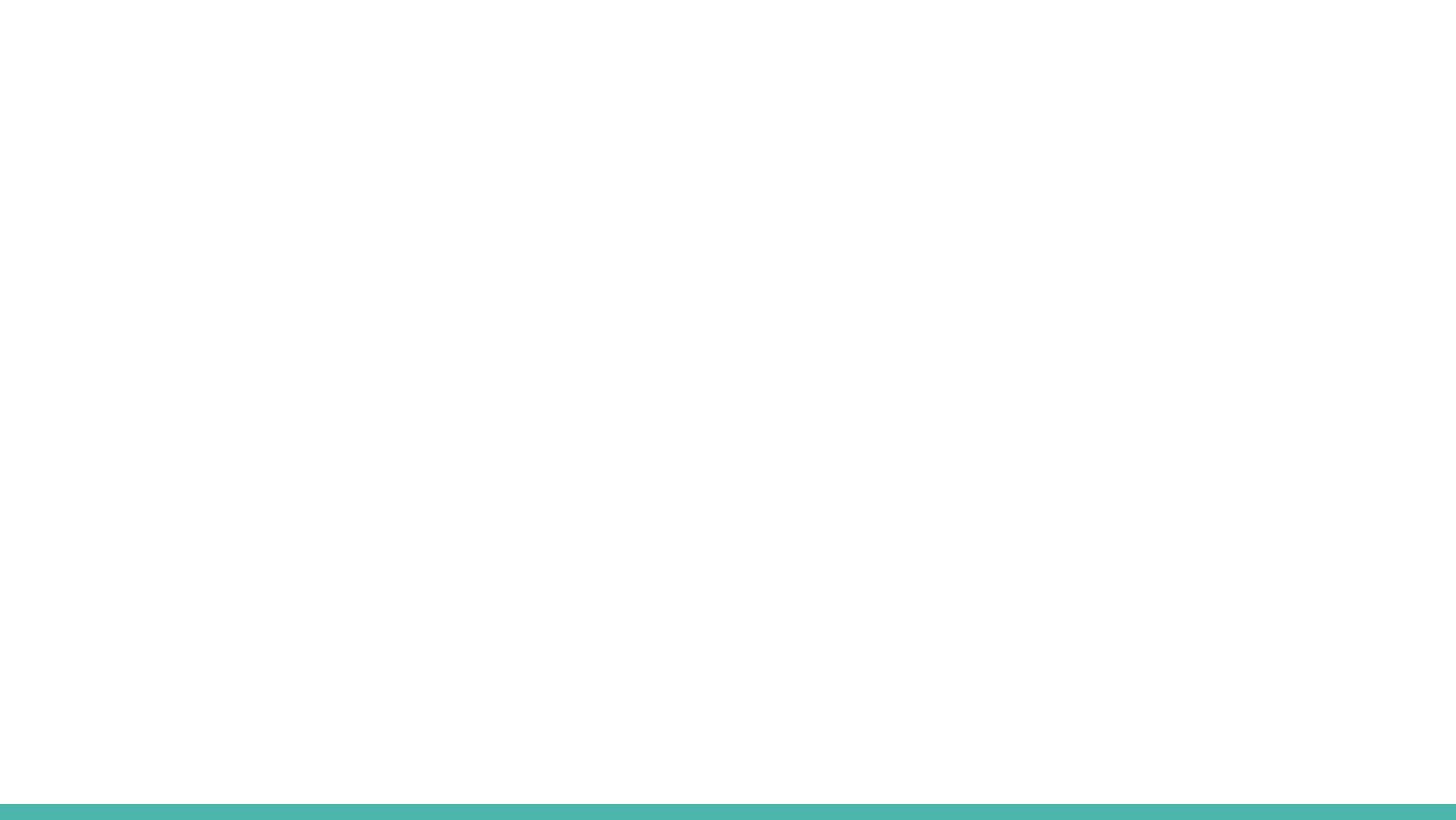
For Continuous X and Y:



# Example 1

The probability an item is defective is  $P$ , which is a random variable with distribution  $F$  and mean  $p_0$

What is the expected number of defective items in a lot of size  $n$ ?



## Example 2

Given independent Bernoulli trials each resulting in success with probability  $p$  and failure with probability  $1-p$ . What is the mean time until the first failure?

# Direct Way

# Indirect Way

# Example workout

# Compute Probabilities by Conditioning

# Summary

## Combinations of Random Variables:

- Joint Distributions

- Joint Expectation

- Covariance

- Independence

## Conditional Probability:

- Discrete & Continuous

## Conditional Expectation:

- Discrete & Continuous