Probability Review

Hong Wan, Fall 2022

- Probability
  - is the likelihood or chance that a particular event will occur
  - is a proportion or fraction whose values range from 0 to 1.
    - An event that has no chance of occurring has a probability of 0,
    - while an event that is sure to occur has probability of 1.
- Probability is about information
- The probabilities that different events will occur will change as information accrues

34.1%
34.1%
13.6%
13.6%
2.1%
2.1%
0.1%
0.1%
−2σ
−1σ
0
1σ
2σ
3σ
VS
Probabaility
Statistics

Probability vs Statistics

Probability deals with the prediction of future events– theory branch of mathematics

statistics are used to analyze the frequency of past events - statistics is an applied branch of mathematics.

Cheat Sheet

Sample spaces, elementary outcomes, mutually exclusive, assigning appropriate probabilities, events

Basic Laws:	$P(A)$ $\quad$ $P(A') = 1 - P(A)$ $P(A \cup B) = P(A) + P(B) - P(AB)$ If A and B are mutually exclusive then $P(AB) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$
Conditional probability:	$P(A \mid B) = P(AB)/P(B)$
Bayes' Formula:	$P(A \mid B) = P(A)P(B \mid A) / [P(A)P(B \mid A) + P(A^c)P(B \mid A^c)]$
Independence:	$P(A \mid B) = P(A)$; $P(B \mid A) = P(B)$; $P(AB) = P(A)P(B)$

Probability Theory

1 Experiment:

- any process that yields a result or an observation
- Make an observation whose result is unpredictable

2 Elementary outcome:

- The most basic result of an experiment

3 Sample Space, “S”, or Ω:

- set of all possible elementary outcomes of an experiment (mutually exclusive and collectively exhaustive)

- An experiment consists of drawing one marble from a box that contains a mixture of red, yellow, and green marbles. List the Sample Space

- Suppose in this experiment, you are interested in the number of red marbles drawn. Is there another relevant sample space?

- How would this change if 2 marbles are drawn (with replacement)?

Probability Space: A sample space with probabilities assigned to each elementary outcome.

Define: A Probability Space is (Ω, a, P) where

Ω

a

P

Probability Space

A probability space is : where

Example Continued

Draw two marbles with replacement

Probability Theory

Once we have assigned probabilities to each elementary outcome, we can compute the probability of all manners of “events”

Events =

For any event A

P(A)=

EXAMPLE CONTINUED

Union

Example continued

Mutually exclusive   if A v B happens then C can't happen

Discrete Random Variables

Discrete
~~Discount~~ Random Variable: countable number of possible values

Probability Mass function: Probability(x_1), $Pr(x_2)$, ... $x \in S$

Cumulative distrib. function
CDF of the discrete random variable: $Pr(X < x_i)$

Exercise

PMF

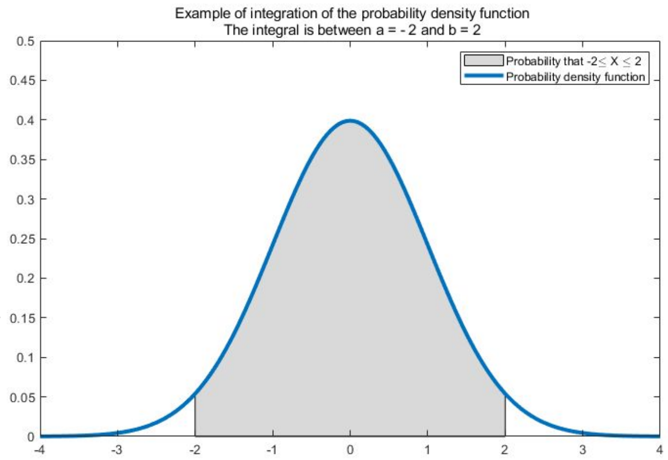
r	0	1	2	3	4	5
P(R=r)	.659	.28647	.05	.0043	.0002	.00003

For any event E (subset of values the random variable can take), the P(E) = sum of the probabilities of the elementary outcomes in E.

$$P(R = 1)$$

$$P(3 \leq R < 5)$$

Continuous Random Variables



Sample Space: for $N(0, \sigma^2)$, $(-\infty, \infty)$ is the sample space.

Event:

$\alpha_1 \leq RV \leq \lambda_1 \rightarrow$ this is the event.

Probability calculation:

Area under the ~~entire~~ range curve?

Area under a single point:

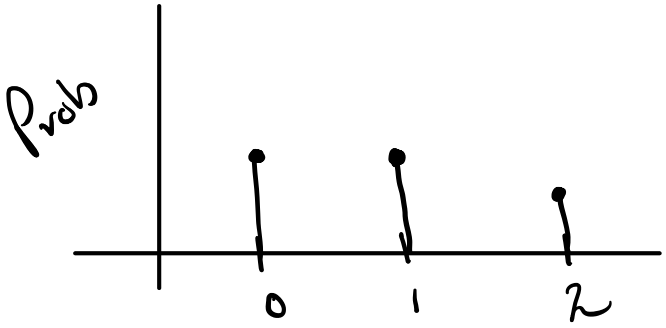
$X \to$ Random Variable

Example revisited:

Number of red marbles drawn: $\Omega = \{ \underset{4/9}{0}, \overset{4/9}{1}, \underset{1/9}{2} \}$ $P(\Omega) = 1$

Random variable:

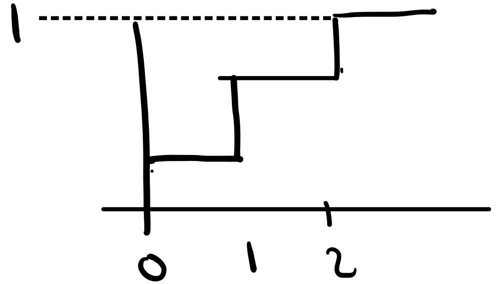
Its probability distribution:



$$P(x) = \begin{cases} x < 0, & 0 \\ x = 0, & 4/9 \\ 2, & 1/9 \\ x > 2, & \infty \end{cases}$$

Formula for CDF: $Pr(X \leq M) = \sum Pr(x = m \leq M)$

CDF:



Conditional Probability

- Consider a consumer electronics company that is interested in studying the intention of consumers to purchase a 4K Ultra HD television in the next 12 months and, as a follow up, whether they in fact actually purchase a television.

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	.20	.05	.25
No	.10	.65	.75
Total	.30	.70	1.0

If you randomly selected consumer says that she plans to purchase a 4K ultra HD TV in the next year, how likely is it that she will actually purchase one over the next 12 months

	B Actually Purchased		
Planned to Purchase	Yes	No	Total
A Yes	.20	.05	.25
No	.10	.65	.75
Total	.30	.70	1.0

A = event that consumer plans to buy

B = event that consumer actually buys

What is the probability of B <u>given that we know A has occurred</u>?

Let us step back and think about it:

		B		
		Actually Purchased		
Planned to Purchase		**Yes**	**No**	**Total**
A	**Yes**	**.20**	**.05**	**.25**
	No	**.10**	**.65**	**.75**
	Total	**.30**	**.70**	**1.0**

Information has accrued and we must update our sample space and probabilities!

A consumer has been chosen and found to be planning to buy a TV. WHat is an updated sample space for that consumer's eventual behavior?

Gender of Babies

Suppose a family has two boys, and they are expecting their third child. What is the probability that the third child will be a boy?

0.5, not conditional.

What is the probability that a three-children family has all boys?

Not conditional.

What is the probability a family has three boys.

$\Omega = \{\text{all families}\}$ → consider the distribution of all families

Bayes' formulation

Definition. *Baby Bayes Rule.* If $\mathrm{P}(B) > 0$, and $\mathrm{P}(E_1) > 0$, then

$$\mathrm{P}(E_1|B) = \frac{\mathrm{P}(B|E_1)\mathrm{P}(E_1)}{\mathrm{P}(B)}.$$

(Bayes)

Total Probability Rule

How do you correlate 2 events

Definition. *Total Probability Theorem.* (Applying this theorem is sometimes called *conditioning*).
For any events B and E_1,

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(B \cap E_1) + \mathrm{P}(B \cap E_1') \\ &= \mathrm{P}(B|E_1)\mathrm{P}(E_1) + \mathrm{P}(B|E_1')\mathrm{P}\left(E_1'\right). \end{aligned}$$

More generally, if events $E_1, E_2, \ldots, E_n$ partition Ω, then for any event B

$$\mathrm{P}(B) = \sum_{i=1}^{n} \mathrm{P}(B \cap E_i) = \sum_{i=1}^{n} \mathrm{P}(B|E_i)\mathrm{P}(E_i).$$

Independence

	Actually Purchased		
Planned to Purchase	Yes	No	Total
Yes	.20	.05	.25
No	.10	.65	.75
Total	.30	.70	1.0

Does knowledge of whether the consumer plans to purchase a TV change the likelihood of predicting whether the consumer actually will purchase a TV?

Independence

Two events are independent if the occurrence of the one event has no effect on the probability that the other event occurs:

1. Events A and B are independent.
2. $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.
3. $\mathrm{P}(A|B) = \mathrm{P}(A)$.
4. $\mathrm{P}(B|A) = \mathrm{P}(B)$.

(Statement 2 is often chosen to define event independence.)

Why Independence Matters

Because we can build up some complex, and useful, probability spaces and distributions from sequence of independent random variables.

In modern finance stock prices are often modeled as an accumulation of many independent random shocks.

(Brownian motion)

S(0) = stock price at time zero

Each time period the price can go up by a factor of u with probability .6, or down by a factor of d with probability .4

What is the probability space for S(3), the stock price in period 3?

S(0) = $100, u = 1.1, d = .9
You hold an option to buy the stock in period 3 for $110
What is the probability you will exercise the option?

Markov pocess

Sample space	Probability	Price S(3)
uuu		S(0)uuu=
uud	(0.6)(.6)(.4)	S(0)uud=
udu		S(0)udu=
udd		S(0)udd=
duu		S(0)duu=
dud		S(0)dud=
ddu		S(0)ddu=
ddd		S(0)ddd=

We care only period 3!

Expected Value $\to \mu$

Probability weighted sum of the random variable's possible outcomes

Discrete random variables X: weighted average for possible values

Continuous random variables X: $\int_{\Omega}$ PDF

Variance

= What is a distribution variance. Represents the spread of the distribution. For discrete $\sum [\bar{x} - x]^2 \cdot Pr($

easy to take the derivative.

Linear functions of Random Variables

If X is a random variable, and b and c are any constants, then for cX+b

The expected value: constant

$cX+b$

The variance: $c^2 \cdot$ var

$E(cX+b) = c \cdot E(X) + b$

$Var(cX+b) = c^2 Var(X)$

X is a random variable with expected value 73 and variance 25

$Y = X - 73 = 0$, var of $Y = 25$

$Z = 0.25Y = 0.25 \cdot 0 = 0$, variance $(0.25)^2 \cdot 25$

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If A and B are mutually exclusive then

$P(AB) = 0 \Rightarrow P(A \cup B) = P(A) + P(B)$

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Random Variables, PMF, CDF: X; $p(a) = P(X=a)$; $F(a) = P(X \leq a)$

Expected Value & Variance: $E[X]$; $Var[X]$