RV Combinations and Conditional Expectations

ISE/OR 560, Hong Wan, Fall 2022

More than one random variables

- Joint Distribution Functions: CDF, pmf, pdf
- Covariance
- Independence
- Expectation for a function of a random variables.

Joint Distribution Functions

For two random variables, X and Y, the joint distribution is (cdf):

Joint CDF

- Discrete

- Continuous

Example

Suppose you toss four fair coins:

X: number of Hs in the first three coin tossed.

Y: number of Hs of the last two coin tossed.

Elementary Outcome	X	Y	Elementary Outcome	X	Y
0000	0	0	1000	1	0
0001	0	1	1001	1	1
0010	1	1	1010	2	1
0011	1	2	1011	2	2
0100	1	0	1100	2	0
0101	1	1	1101	2	1
0110	2	1	1110	3	1
0111	2	2	1111	3	2

Joint Probability Mass Function for (X, Y)

Y/X	0	1	2	3
0				
1				
2				

From the joint distribution to PMF

Y/X	0	1	2	3	
0					
1					
2					

Covariaance

Covariance (X,Y):

Going back to the example:

Y/X	0	1	2	3	
0					
1					
2					

Properties of Covariance

- $\text{Cov}(X, X) =$

- $\text{Cov}(X, Y) =$

- $\text{Cov}(cX, Y) =$

- $\text{Cov}(X, Y+Z)=$

Independent RV

Random variables X and Y are independent if, for all a, b

Joint CDF

Independent RVs

Discrete

Continuous

Independent RV

Expectation (X, Y)

Variance (X+Y)

Covariance (X, Y)

Example

You own a portfolio with 4 units of asset “a” and 3 units of asset “b”. If asset “a” has a return of Y_a and asset b has a return of Y_b, your total portfolio return is $4Y_a + 3Y_b$. Assume Y_a and Y_b are independent.

Suppose

Y_a is random with expected value 2 and variance .036

Y_b is random with expected value 3 and variance .25

What is the expected value and variance of your portfolio return?

Example

8% of the customers that walk into your store buy an item (for $1), and the rest leave without buying anything.

a). If only 1 customer walks in each day, what is the sample space, probability distribution, and mean of

R_1 = daily revenues?

b). If you earn 35 cents for every dollar spent, what are your expected daily earnings?

Five customers each day?

S = {xxxxx, xxxxb, . . xxbbb, . . . , bbbbb }

(there are 32 elementary outcomes in this sample space)

Since customers are independent, we can assign a probability to each outcome knowing only that each customer buys with probability .08

S = {xxxxx, xxxxb, . . xxbbb, . . . , bbbbb } with probs

Five customers each day

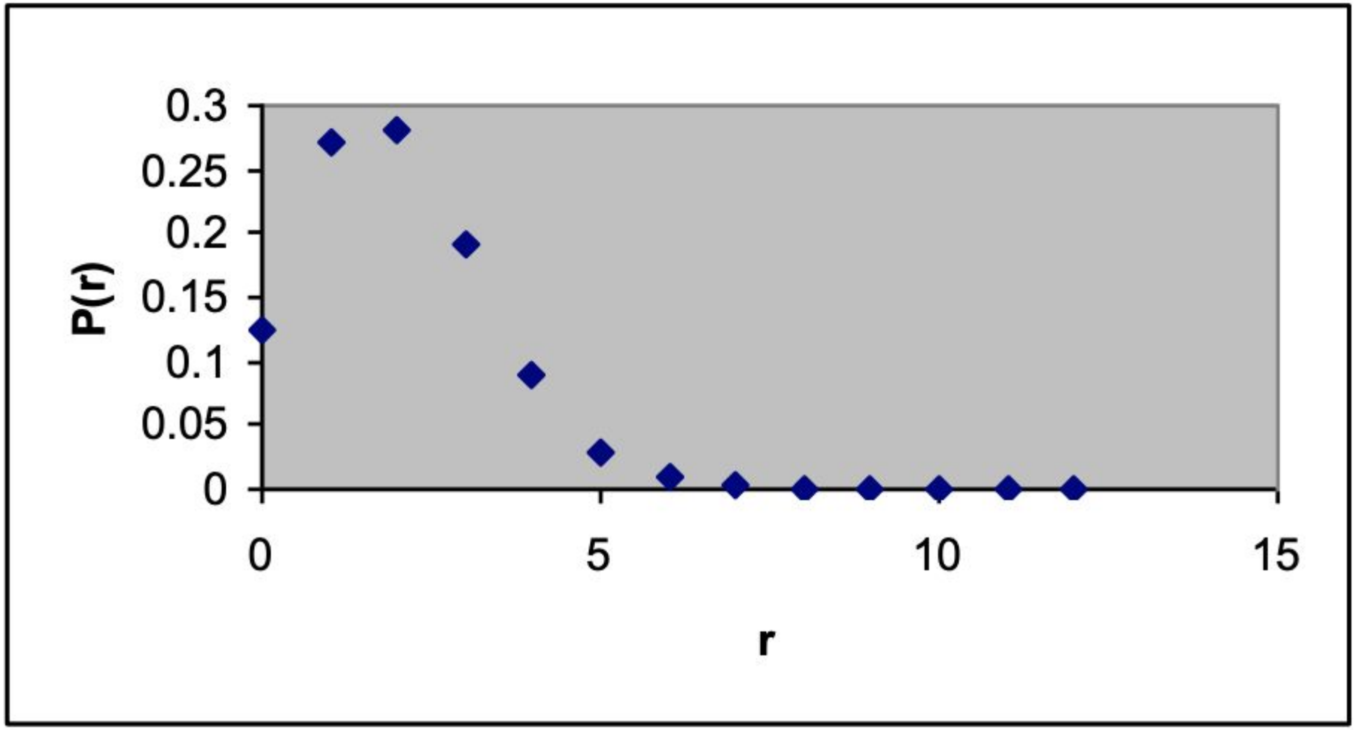
$R_5=k$ if there are exactly k “b”’s in the sequence. Adding these probabilities gives us the probability that $R_5=k$.

r	0	1	2	3	4	5
$P(R_5=r)$	.659	.287	.05	.0043	.0002	.00003

8% of the customers that walk into your store buy an item (for $1), and the rest leave without buying anything.

If 25 customers walk in each day, what is the expected value and variance of R_{25} = daily revenues?

r	0	1	2	3	4	5	6	7	8	9	10	etc
P(r)	.124	.27	.28	.19	.09	.03	.009	.002	.0004	.0001	0	0



$$\mu_{R_{25}} = 2.0$$

$$\sigma^2_{R_{25}} = 1.84$$

$$\sigma_{R_{25}} = 1.36$$

Distribution for daily revenues if 25 customers visit each day (33.6 million elementary outcomes xbbxx... etc.)?

Or 100 customers each day ($1.27*10^{30}$ elementary outcomes – The age of universe is about **5×10^{17} second** for reference)

200 customers each day? 1000?

25 Customers the Easy One

Independent

Each customer that walks in has a .08 probability of spending $1, and a .92 probability of spending $0.

n (number of customers	$E(R_n) = n \cdot (0.8)$	$Var(R_n)$ Σ all R_n together
1	.08	.0736
2	.16	.147
5	.40	.368
25	2.0	1.84
100	8.0	7.36
200	16.0	14.72
1000	80.0	73.6

Think about Bernoulli and Binomial relationship. The $\sum$ Bernoulli = Binomial.

Bernoulli R.V. $X = \begin{cases} 1 & \text{female} \\ 0 & \text{not femal} \end{cases}$ $\quad P(X=1) = 0.52 = P$

$P(X=0) = 0.48$

A large population is 52% female. If you draw 10 people at random from this population, what is the expected value and variance of the number of females in your sample of 10?

$\mathbb{E}(\# \text{ of females}) = 0.52$

$V(X) = \mathbb{E}(XX) - \mathbb{E}(X)^2 = 1 \cdot 0.52 - (0.52)^2 = 0.52 -$

Variance formulation for Bernoulli: $P - P^2 = P(1-P)$

Then $\mathbb{E}[N] = \mathbb{E}[X=1] = 10 \cdot (0.52) = n \cdot P(X=1) = 5.2$

$$Var(N) = \sum_{i=1}^{10} Var(x_i) = 10 \cdot P(1-P) = n \cdot P(1-P) \Big\} \text{Binomial distribution formulation}$$

$$= 10 \cdot (0.52) \cdot (0.48)$$

Review Combination random variable

- Joint Distribution Functions: CDF, pmf, pdf
  Covariance
- Independence
- Expectation for a function of a random variables.

Part II: Conditional Probability and Expectation

- Recall: For any two events A and B: $P(A|B) = P(A \cap B)/P(B)$

probability mass function

- For Discrete RVs, X and Y: $P_{(\tilde{X}|\tilde{Y})}(X|Y)$ → two specific realizations

$= P_r(\tilde{x} = x \mid \tilde{y} = y) = P(\tilde{x} = x, \tilde{y} = y) / P(\tilde{y} = y)$

joint dist.

Example $P(x=1 \mid y=2)$, $P(x=2 \mid y=2)$

Suppose p(x,y) is given by:

p(1,1) = 0.3, p(1,2) = 0.4, p(2,1) = 0.2, p(2,2) = 0.1

Calculate the conditional pmf of X given Y=2

$$\frac{P(x=1, Y=2)}{P_r(y=2)} = \frac{0.4}{0.5}, \quad P_r(\tilde{x}=2 \mid \tilde{y}=2) = 0.8$$

. Now, calculate

conditional PMF :

How do I write this into a function.

$$\sum_{\tilde{y} \le y} P_r(Y \mid x=1) = 1$$

$$F_{\tilde{Y}\mid\tilde{X}}(y \mid x) = \sum_{\tilde{y} \le y} P(y \mid x)$$

all of the y possible values

Conditional Probability Distribution of Y given X = x

$$\begin{cases} P_r(\tilde{Y}=1 \mid \tilde{X}=1) \\ P_r(\tilde{Y}=2 \mid \tilde{X}=1) \end{cases}$$

$P_r(\tilde{Y}=2 \mid \tilde{X}=2) = P_r(\tilde{Y} \mid X=1)$

$P_r(\tilde{Y}=2 \mid \tilde{X}=2) = P_r(\tilde{Y} \mid X=2)$

Conditional Expectation:

$$\mathbb{E}[Y \mid X=x] = \sum_{y \in \Theta_y} P(y \mid x) \cdot y$$

sample space

$$\sum y \cdot P_r(\tilde{y} \mid \tilde{x}=x)$$

$$\mathbb{E}[X \mid Y=2] = P_r(X=1 \mid Y\,2) \cdot 1 + 2P_r(X=2 \mid Y=2)$$

$1 \cdot 0.8 + 2 \cdot (0.2)$

Suppose p(x,y) is given by:

p(1,1) = 0.3, p(1,2) = 0.4, p(2,1) = 0.2, p(2,2) = 0.1

calculated in previous slide

Weighted avg

Calculate the E[X|Y=2] $= 1 \cdot P(X=1 \mid Y=2) + 2P(X=2 \mid Y=2)$

$= 0.8 + 2(0.2)$

What if we want $P(X=1) = 0.3 + 0.4 = 0.7$

$P(Y=2) = 0.5 = p(1,2) + p(2,2)$

For Continuous case $\sum \longrightarrow \int \sim P_r \longrightarrow f$

pdf of Y given X = x → joint dist.

$$f_{Y|X}(y|x) = \frac{f_Y(x, y)}{f_X(x)}$$

↳ marginal dist.

Conditional Expectation:

$$\mathbb{E}[Y \mid X = x] = \int_{-\infty}^{\infty} y \, f(y|x) \, dy$$

$f(x,y) \to$ joint distribution function. $\forall\ Y=y$, $X=x \iff x<y$

The joint density of X and Y is given by

$$f(x,y)=\frac{e^{-y}}{y},\quad 0<x<y, 0<y<\infty$$

until y bc. x is smaller than y

Compute $E[\tilde{X}^3 \mid \tilde{Y}=y] = \int_0^y x^3 f_{X|Y=y}(x|y)\,dx$ by the definition

$\to$ need to find this

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}/y}{f_Y(y)=\int_0^y f(x,y)\,dx = \int_0^y e^{-y}/y\,d\cancel{x}}$$

Then $\mathbb{E}[X^3 | Y = y] = \int_0^y x^3 f_{X|Y}(x|y)\, dx = \int_0^y x^3 \left(\frac{1}{y}\right) dx$

$$= \frac{1}{4} x^4 \left(\frac{1}{y}\right)\Big|_0^y = \frac{1}{y} \cdot \frac{y^4}{4} = \frac{y^3}{4} = \mathbb{E}[X^3 | Y = y]$$

study also for discrete

Conditional Continuous PDF

pdf of Y given X = x

It follows that

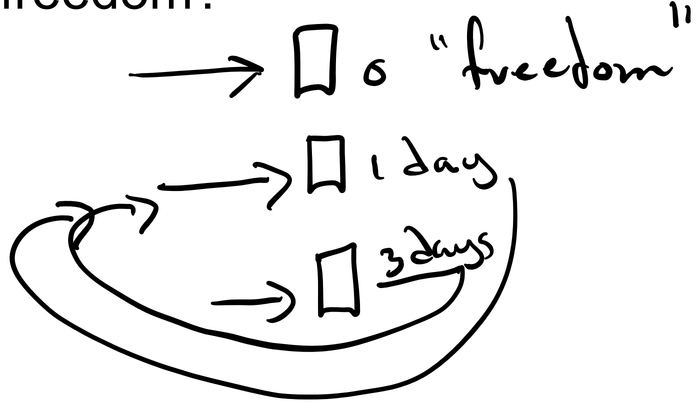
Independence

Recall that if X and Y are independent,

It follows that:

Thief in the Chamber

A prisoner is placed in a cell containing three doors. The first door leads immediately to freedom. The second door leads into a tunnel which returns him to the cell after one day's travel. The third door leads to a similar tunnel which returns him to his cell after three days. Assuming that the prisoner is at all times equally likely to choose any one of the doors, what is the expected length of time until the prisoner reaches freedom?



Use conditional expected value.

Example Workout

N: # of days to freedom. $E[N]$

X: door selected, $X = \begin{cases} 1 & 1/3 \\ 2 & 1/3 \\ 3 & 1/3 \end{cases}$, $E[N \mid X=1] = 0$,

$$\left.\begin{aligned} E[N \mid X=2] &= 1 + E[N] \\ E[N \mid X=3] &= 3 + E[N] \end{aligned}\right|$$

$$E[N] = E_X[E[N \mid X]]$$

$\rightarrow$ expected value accross all X

$$\therefore E[N] = \frac{1}{3}\cdot 0 + \frac{1}{3}(1 + E[N]) + \frac{1}{3}(3 + E[N])$$

$$E[N] = 4$$

Example Workout

Recall: E[Y] is a constant

However, E[Y|X] is a random variable that is a function of X

Ex. Y = profit and X = demand, suppose Y = 6X then

In general, $E[Y] = \sum E[Y \mid x=a] \cdot P_r(x=a)]$

$$E[Y \mid X=a] = 6a = 6E[x]$$

Conditions Probabilities

To solve problems where dependency is present

Process of first conditioning and then unconditioning

Computing Conditional Expectations

For all RVs X and Y: $E[Y] = E_X[E[Y \mid X]]$

$$= \sum E[Y \mid X = x] \Pr(x)$$

or $$\int E[Y \mid X = x] f(x) \, dx$$

Unconditioning

For Discrete X and Y:

For Continuous X and Y:

Conditional expectation of Y is a function of the RV X

Example 1

Binomial distribution

The probability an item is defective is P, which is a random variable with distribution F and mean p_0 Probability is a random variable

What is the expected number of defective items in a lot of size n? $n \cdot p$

$E[Y|P=p] = n \cdot p$ and $E[Y] = E[n \cdot P] = np_0$ conditional

random variable

Y : expected number of defective in a lot of size n.

discrete (event is discrete)

Example 2

Geometric distribution.

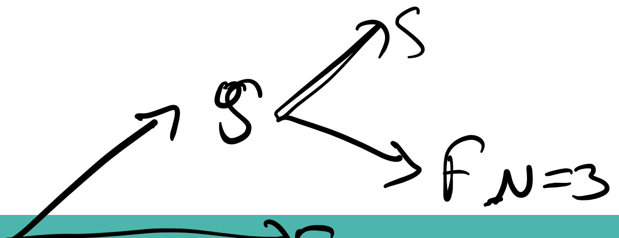
Given independent Bernoulli trials each resulting in success with probability *p* and failure with probability *1-p*. What is the mean time until the first failure?

→ continuous.

N = # of trials until the first failure → we have success until $n-1$ times

$$p(n) = P\{N = n\} = p^{n-1}(1-p)$$

$$\sum_{n=1}^{\infty} n \cdot p^{n-1}(1-p) = E[N]$$



at beginning of trials,

Direct Way

$\to q$ success $\to N=2$

$\to p$ failure

If we condition on the first trial, $X_1 = 1(S)$, $X_1 = 0(F)$

$$\left.\begin{array}{ll} X_1 = 1, & p \\ X_1 = 0, & 1-p \end{array}\right\} \quad \begin{array}{l} E[N] = E[N \mid X_1] \\ \text{and } E[N \mid X_1 = 1] = 1 + E[N] \end{array}$$

$E[N \mid X_1 = 0] = 1$, $E[N] = p \cdot (1 + E[N]) + (1-p) \cdot 1$

$\Rightarrow \quad E[N] = p + pE[N] + 1 - p$ (we have a function of p.

$\therefore E[N] = \dfrac{1}{1-p}$

Indirect Way

Example workout

Compute Probabilities by Conditioning

Summary

Combinations of Random Variables:

- Joint Distributions
- Joint Expectation
- Covariance
- Independence

Conditional Probability:

- Discrete & Continuous

Conditional Expectation:

- Discrete & Continuous