Problem 1

1. $X_{n+1} = \begin{cases} Y_{n+1}, & X_n = 0 \\ Y_{n+1} + X_n - 1, & X_n > 0 \end{cases}$

2. $S = \{0,1,2,3,4, \dots\}$, $\{X_n, n \geq 0\}$. We have that $\{X_n, n \geq 0\}$ is a DTMC if the following properties are true:

   (i) $\forall\ n \geq 0, X_n \in S$

   (ii) $\forall\ n \geq 0$ and $\forall$ i, j $\in$ S, $P(X_{n+1}|X_n) = P(X_{n+1}|X_n, X_{n-1}, X_{n-2}, \dots, X_0)$

Claim 1: $\{X_n, n \geq 0\}$ satisfies (i):
Proof:
It follows from the definition of X_{n+1} on exercise 1 that for all n $X_n \in S = \{0,1,2,3, \dots\}$.

Claim 2: $\{X_n, n \geq 0\}$ satisfies (ii):
Proof:
Without loss of generality, assume $X_n = Y_n + X_{n-1} - 1, X_n \geq 1$.

Let $n \geq 1$. Note that
$P(X_{n+1}|X_n) = P(X_{n+1} = Y_{n+1} + X_n - 1\ |X_n = Y_n + X_{n-1} - 1)$
$\rightarrow P(X_{n+1}|X_n) = P(Y_{n+1} = X_{n+1} - X_n + 1\ |Y_n = X_n - X_{n-1} + 1)$

Let $X_{n+1} - X_n + 1 = k_1$ and let $X_n - X_{n-1} + 1 = k_2$.
Then:
$\rightarrow P(X_{n+1}|X_n) = P(Y_{n+1} = \alpha_{k_1}|Y_n = \alpha_{k_2}) = P(Y_{n+1}|Y_n) = \frac{P(Y_{n+1} \cap Y_n)}{P(Y_n)} = P(Y_{n+1}) \quad (*)$,
because $\{Y_n, n \geq 1\}$ is a series of independent and identically distributed random variables. Also note that for any $n \geq 1$, we can write Y_n in terms of X_n.

Since $\{Y_n, n \geq 1\}$ are iid random variables, we also have
$P(Y_{n+1}|Y_n, Y_{n-1}, Y_{n-2}, \dots, Y_1) = \frac{P(Y_{n+1} \cap Y_n \cap Y_{n-1} \cap \dots \cap Y_1)}{P(Y_n \cap Y_{n-1} \cap \dots \cap Y_1)} = P(Y_{n+1}) \quad (**)$

$P(X_{n+1}|X_n, X_{n-1}, \dots, X_1) = P(Y_{n+1}|Y_n, Y_{n-1}, \dots, Y_1)$

From $(*)$ and from $(**)$, it follows that:
$P(X_{n+1}|X_n, X_{n-1}, \dots, X_1) = P(Y_{n+1}|Y_n, Y_{n-1}, \dots, Y_1) = P(Y_{n+1})$
But $P(X_{n+1}|X_n) = P(Y_{n+1})$
$\rightarrow P(X_{n+1}|X_n, X_{n-1}, \dots, X_1) = P(X_{n+1}|X_n) = P(Y_{n+1})$ where $Y_{n+1} = X_{n+1} - X_n + 1$
This proves claim ii.

Therefore $\{X_n, n \geq 0\}$ is a DTMC.

3. :

3. $P =$

$$
\begin{array}{c|cccccc}
 & X_n = 0 & 1 & 2 & 3 & 4 & \cdots \\
\hline
X_{n-1} = 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\
1 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\
2 & 0 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\
3 & 0 & 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\
4 & 0 & 0 & 0 & \alpha_0 & \alpha_1 & \cdots \\
5 & \vdots & \vdots & \vdots & 0 & \alpha_0 & \cdots \\
\vdots & & & & \vdots & 0 & \ddots \\
 & & & & & \vdots &
\end{array}
$$

Then we have the following for the transition matrix P:

$$P = [p_{i,j}] \text{ where } p_{i,j} = \begin{cases} \alpha_j, & i = 0 \text{ or } i = 1 \\ 0, & i \geq j + 2 \geq 2 \\ \alpha_{j+1-i}, & 2 \leq i \leq j + 1 \end{cases}$$

Where $\alpha_k = P(\text{production at time } n = k), k = 0,1,2,3 \ldots$

Note that $p_{i,j}$ is well defined since:

Case $i = 0$ or $i = 1$:

If the inventory of X_n is i, where $i = 0$ or $i = 1$, then by exercise 1 we have that $X_{n+1} = Y_{n+1}$ in either case.

Since $\alpha_k = P(Y_n = k)$, and production happens after the demand is either met or not met, we have that $p_{i,j} = P(Y_n = j) = \alpha_j$ for $i = 0$ or $i = 1$.

Case $i \geq j + 2$:

Note that $p_{i,j}$means going from $X_n = i$ to $X_{n+1} = j$, where $i \geq j + 2 > 0$. Note that this is impossible since if we have i items in the inventory at time X_n, then the demand is always met (since $i \geq j + 2 \geq 2$). Note that $X_{n+1} = j$ where $i \geq j + 2$ is a contradiction with the formula for X_{n+1} derived in exercise 1 , since if $X_n = i$,

then $X_{n+1} = Y_{n+1} + X_n - 1 = Y_{n+1} + i - 1 = j \rightarrow i \geq Y_{n+1} + i - 1 + 2 = Y_{n+1} + i + 1$

$\rightarrow 0 \geq Y_{n+1} + 1$ which is a contradiction since $Y_{n+1} \in \{0,1,2,3, \ldots\}$.

Hence $p_{i,j} = 0$ if $i \geq j + 2 > 0$

Case $2 \leq i \leq j + 1$:

Then $p_{i,j}$ means going from $X_n = i$ to $X_{n+1} = j$ where $2 \leq i \leq j + 1$ is satisfied. Since

$i \geq 2$, the demand at time n+1 is always met and $X_n = i \geq 2$.
By using the relation derived in problem 1, we have:
$X_{n+1} = Y_{n+1} + X_n - 1 = Y_{n+1} + i - 1 = j$
$\rightarrow Y_{n+1} = j - i + 1 \geq 0$

Then, by exercise 2, we have:
$\mathrm{p}_{(\mathrm{i,j})} = \mathrm{P}(\mathrm{X}_{\mathrm{n+1}} = \mathrm{j} | \mathrm{X}_{\mathrm{n}} = \mathrm{i}) = \mathrm{P}(\mathrm{Y}_{\mathrm{n+1}} = \mathrm{j} - \mathrm{i} + 1) = \alpha_{\mathrm{j-i+1}}$, where $\mathrm{j} + 1 \geq \mathrm{i} \geq 2$,,
This proves claim 2.

$\therefore \{X_n, n \geq 0\}$ *is a DTMC*

Problem 2

1. $\{\mathrm{Y_n}, \mathrm{n} \geq 1\}$, $\mathrm{Y_n}$ is the size of batch demand at discrete time $\mathrm{n} \geq 1$. $\mathrm{Y_n} \in \mathrm{S} = \{0,1,2,3, \dots, \mathrm{k}, \dots\}$

   $\rightarrow \mathrm{X_{n+1}} = \begin{cases} X_n + 1 - Y_{n+1}, & X_n + 1 > Y_{n+1} \\ 0, & otherwise \end{cases}$

2. $\mathrm{S} = \{0,1,2,3,4, \dots.\}$,
   $\{\mathrm{X_n}, \mathrm{n} \geq 0\}$.
   We have that $\{\mathrm{X_n}, \mathrm{n} \geq 1\}$ is a DTMC if the following properties are true:
   (i) $\forall\ \mathrm{n} \geq 0, \mathrm{X_n} \in \mathrm{S}$
   (ii) $\forall\ \mathrm{n} \geq 0$ and $\forall\ \mathrm{i}, \mathrm{j} \in \mathrm{S}$, $\mathrm{P}(\mathrm{X_{n+1}} | \mathrm{X_n}) = \mathrm{P}(\mathrm{X_{n+1}} | \mathrm{X_n}, \mathrm{X_{n-1}}, \mathrm{X_{n-2}}, \dots, \mathrm{X_0})$

   Claim 1: $\{\mathrm{X_n}, \mathrm{n} \geq 0\}$ satisfies (i):
   Proof:
   Since each $X_n, n \geq 0$ is a nonnegative integer, $X_n \in S = \{0,1,2,3,4, \dots\}$
   This proves claim 1.

   Claim 2: $\{\mathrm{X_n}, \mathrm{n} \geq 0\}$satisfies (ii):
   Proof:
   If $X_{n+1} = 0$, *then* $P(X_{n+1} = 0 | X_n) = P(Y_{n+1} \geq X_n + 1)$
   Let $k = X_n + 1$.Then:
   $P(Y_{n+1} \geq k) = \Sigma_{i=0}^{\infty} P(Y_{n+1} = k + i) = \Sigma_{i=0}^{\infty} \alpha_{k+i}$
   The above is equivalent to $P(X_{n+1} = 0 | X_n, X_{n-1}, \dots, X_0)$ since
   $P(X_{n+1} = 0 | X_n, X_{n-1}, \dots, X_0) = P(Y_{n+1} \geq X_n + 1) = \Sigma_{i=0}^{\infty} \alpha_{k+i}$

   If $X_{n+1} = X_n + 1 - Y_{n+1}$, then $Y_{n+1} = X_n + 1 - X_{n+1}$.
   Note $X_n + 1 - X_{n+1} \geq 0$ is true, since the inequality $X_n + 1 \geq X_{n+1}$ is always true.

   $\rightarrow P(X_{n+1} | X_n) = P(X_{n+1} = X_n + 1 - Y_{(n+1)} | X_n = X_{n-1} + 1 - Y_n)$
   $= P(Y_{n+1} = X_n + 1 - X_{n+1} | Y_n = X_{n-1} + 1 - X_n) =$
   $P(Y_{n+1} | Y_n) = \frac{P(Y_{n+1} \cap Y_n)}{P(Y_n)} = P(Y_{n+1})$.(*)

Similarly, $P(X_{n+1}|X_n, \dots, X_1) = P(Y_{n+1}|Y_n, \dots Y_1) = P(Y_{n+1})$ (**)
Hence by (*) and by (**) we get $P(X_{n+1}|X_n) = P(X_{n+1}|X_n, \dots, X_1) = P(Y_{n+1})$
This proves claim (ii)

$\therefore \{X_n, n \geq 0\}$ *is a DTMC*

3. Transition Probability Matrix:
Like problem 1, let $\alpha_k = P(Y_n = k), n \geq 1,\ k = 0,1,2,3 \dots$

Commented [FMV1]: COMPLETE THIS

$$\begin{array}{c|c} X_{n-1}|X_n & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{pmatrix} \alpha_1 & \alpha_0 & 0 & 0 & 0 & \cdots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \ddots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

$$p_{i,j} = \begin{cases} \alpha_{i+1}, & j = 0 \\ & \end{cases}$$

Problem 3

1. Show $\{X_n, n \geq 1\}$ *is a DTMC*.
   Claim 1: $X_n \in S = \{1,2,3, \dots, k\}\ \forall\ n \geq 1$:
   Proof:
   The definition of X_n= drug i administered to patient X at time n, $i \in S$.
   $\therefore X_n \in S$ *for all* $n \geq 1$.
   Claim proved.

   Claim 2: $P(X_{n+1}|X_n) = P(X_{n+1}|X_n, \dots, X_0)$:
   Proof:
   Let i, j be any two drugs from the state space $S = \{1,2, \dots, k\}$.

   Then $P(X_{n+1} = j|X_n = i) = \frac{P(X_{n+1} \cap X_n)}{P(X_n)} = P(X_{n+1})$ (*)
   since the successive patients $\{X_n, n \geq 1\}$ are independent of each other.
   Similarly, $P(X_{n+1} = j|X_n = i, X_{n-1}, \dots, X_0) = \frac{P(X_{n+1}, X_n, \dots, X_0)}{P(X_n, X_{n-1}, \dots, X_0)} = P(X_{n+1})$.(**)
   $\rightarrow P(X_{n+1} = j|X_n = i, X_{n-1}, \dots, X_0) = P(X_{n+1}|X_n)$ $(by (*) and (**))$
   This proves claim 2.

   $\therefore \{X_n, n \geq 1\}$ is a DTMC.

2. Derive the transition matrix.

$$
\begin{array}{c}
\\ P_{kbyk} =
\end{array}
\begin{array}{c}
i\,|j \\ 1 \\ 2 \\ 3 \\ \vdots \\ k-1 \\ k
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & k-1 & k \end{array} \\
\begin{pmatrix}
p_1 & 1-p_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & p_2 & 1-p_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & p_3 & 1-p_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & p_{k-1} & 1-p_{k-1} \\
1-p_k & 0 & 0 & 0 & \cdots & 0 & p_k
\end{pmatrix}
\end{array}
$$

In essence, we have the following:

$$
P = [p_{ij}] = \begin{cases} p_i, & i = j \\ 1-p_i, & j = i+1 \\ 1-p_k, & i = k \text{ and } j = 1 \\ 0, & otherwise \end{cases}
$$

Problem 4

1. $\{X_n, n \geq 0\} X_n: number\ of\ white\ balls\ at\ urn\ A$

For $0 < i < N$, we have:

$$
[p_{i,j}] = \begin{cases} \frac{(N-i)^2}{N^2}, & j = i+1 \\ 2\left(\frac{i}{N} * \frac{N-i}{N}\right), & j = i \\ \left(\frac{i}{N}\right)^2, & j = i-1 \\ 1, & i = N \text{ and } j = i-1 \\ 0, & otherwise \end{cases}
$$

$\rightarrow p_{N,N-1} = 1 \text{ and } p_{N,N} = 0 \text{ and } p_{0,0} = 0$

$\rightarrow p_{i,i+1} = \frac{(N-i)^2}{N^2} = p_i$

$\rightarrow p_{i,i} = 2 * \frac{i}{N} * \frac{N-i}{N} = r_i$

$\rightarrow p_{i,i-1} = \left(\frac{i}{N}\right)^2 = q_i$

$$
\begin{array}{c}
\\ P_{N+1byN+1} =
\end{array}
\begin{array}{c}
i/j \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots & N-2 & N-1 & N \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
q_i & r_i & p_i & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & q_i & r_i & p_i & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & q_i & r_i & p_i & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & q_i & r_i & p_i \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

Then for N=10 the transition probability P is:

	1	2	3	4	5	6	7	8	9	10	11
1	0	1.0000	0	0	0	0	0	0	0	0	0
2	0.0100	0.1800	0.8100	0	0	0	0	0	0	0	0
3	0	0.0400	0.3200	0.6400	0	0	0	0	0	0	0
4	0	0	0.0900	0.4200	0.4900	0	0	0	0	0	0
5	0	0	0	0.1600	0.4800	0.3600	0	0	0	0	0
6	0	0	0	0	0.2500	0.5000	0.2500	0	0	0	0
7	0	0	0	0	0	0.3600	0.4800	0.1600	0	0	0
8	0	0	0	0	0	0	0.4900	0.4200	0.0900	0	0
9	0	0	0	0	0	0	0	0.6400	0.3200	0.0400	0
10	0	0	0	0	0	0	0	0	0.8100	0.1800	0.0100
11	0	0	0	0	0	0	0	0	0	1.0000	0

NOTE: since this matrix was generated with MATLAB, for $i, j \in S = \{0,1,\dots,10\}$, then the location (i+1,j+1) in the matrix above will give you $p_{i,j}$.

2. Since $\mathrm{X}_0 = 10 \rightarrow \mathrm{N} = 10$ white marbles in urn A.
   For n=0:
   $E(X_0) = 10$ since is given.
   For n=5 transitions, X_5 have only five possible states i.e. $X_5 \in \{5,6,7,8,9,10\} \subset S$.
   Then
   $\mathrm{E}(\mathrm{X}_5) = \Sigma_{j=5}^{10}(X_5 = j) * P(X_5 = j) = \Sigma_{j=5}^{10} j * P(X_5 = j|X_0 = 10)$. where the marginal probability is:
   $\Sigma_{i \in S} P(X_5 = j|X_0 = i)$. But $X_0 = 10 \rightarrow P(X_5 = j) = P(X_5 = j|X_0 = 10)$.
   By theory, we have $a^n = a * P^n$ is the marginal probability of X_5.
   Hence: