HW2-ISE560 Fall 2022, Due on Oct 14th, 2022

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Please note that this is a group homework so each group need to submit only one copy. Make sure that you have your group member's names on submitted homework

Problem 1

Consider a production inventory system in discrete time where production occurs in integer valued random batches and demands occur one at a time at times $n = 1, 2, \cdots$ The inventory of unsold items is maintained in a warehouse. We assume that demand at time n occurs before the production at time n, so the production at time n is not available to satisfy the demand at time n. Thus, if the warehouse is empty when a demand occurs, the demand is lost. Let X_0 be the initial number of items in the inventory and X_n be the number of items in the warehouse at time n, after accounting for the demand and production at time $n, n \geq 1$. Let Y_n be the size of the batch produced at time n (after the demand is satisfied at time n).

1. Describe X_{n+1} as a function of Y_{n+1} and X_n
2. (Extra credit) suppose $\{Y_n, n \geq 1\}$ is a sequence of iid random variables, with common pmf given by

$$\alpha_k = \mathrm{P}\left(Y_n = k\right), \quad k = 0, 1, 2, \cdots .$$

   Show that under this assumption $\{X_n, n \geq 0\}$ is a DTMC on state-space $S = \{0, 1, 2, \cdots\}$.
3. Derive the transition probability matrix

Problem 2

Now consider a production-inventory system where the demands occur in integer valued random batches and the production occurs one at a time at times $n = 1, 2, \cdots$ The inventory of unsold items is maintained in a warehouse. Unlike in the previous example, here we assume that the production occurs before demand. Thus the production (of one unit) at time n available to satisfy the demand at time n. Let X_0 be the initial inventory and X_n be the number of items in the warehouse at time n, after the production and demand at time n is accounted for. Let Y_n be the size of the batch demand at time $n \geq 1$. Any part of the demand that cannot be satisfied is lost.

Redo Q 1-3 of Problem 1

Problem 3

We have gone through the following clinical trials example in class. Suppose two drugs are available to treat a particular disease, and we need to determine which of the two drugs is more effective. This is generally accomplished by conducting clinical trials of the two drugs on actual patients. Here we describe a clinical trial setup that is useful if the response of a patient to the administered drug is sufficiently quick, and can be classified as "effective" or "ineffective." Suppose drug i is effective with probability $p_i, i = 1, 2$. In practice the values of p_1 and p_2 are unknown, and the aim is to determine if $p_1 \geq p_2$ or $p_2 \geq p_1$. Ethical reasons compel us to use the better drug on more patients. This is achieved by using the play the winner rule as follows.

The initial patient (indexed as patient 0) is given either drug 1 or 2 at random. If the nth patient is given drug $i (i = 1, 2)$ and it is observed to be effective for that patient, then the same drug is given to the $(n+1)$-st patient; if it is observed to be ineffective then the $(n+1)$-st patient is given the other drug. Thus we stick with a drug as long as its results are good; when we get a bad result, we switch to the other drug - hence the name "play the winner." Let X_n be the drug (1 or 2) administered to the n-th patient. If the successive patients are chosen from a completely randomized pool, then we see that

$$\mathrm{P}\left(X_{n+1} = 1 \mid X_n = 1, X_{n-1}, \cdots, X_0\right)$$

$=$ P(drug 1 is effective on the n-th patient) $= p_1$.

We can similarly derive $\mathrm{P}\left(X_{n+1} = j \mid X_n = i;\right.$ history$\left.\right)$ for all other (i, j) combinations, thus showing that $\{X_n, n \geq 0\}$ is a DTMC. Its transition probability matrix is given by

$$\left[\begin{array}{cc} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{array}\right]$$

If $p_1 > p_2$, the DTMC has a higher tendency to move to state 1 , thus drug 1 (the better drug) is used more often. Thus the ethical purpose is served by the play the winner rule.

Now consider the following extension of the above problem: suppose we follow the play the winner rule with k drugs $(k \geq 2)$ as follows. The initial player is given drug 1 . If the drug is effective with the current patient, we give it to the next patient. If the result is negative, we switch to drug 2 . We continue this way until we reach drug k. When we observe a failure of drug k, we switch back to drug 1 and continue. Suppose the successive patients are independent, and that drug i is effective with probability p_i. Let X_n be i if the n-th patient is given drug i.

1. Show that $\{X_n, n \geq 1\}$ is a DTMC.
2. Derive its transition probability matrix.

Problem 4

Let us review the Urn example we have gone through in class. Consider two urns labeled A and B, containing a total of N white balls and N red balls among them. An experiment consists of picking one ball at random from each urn and interchanging them. This experiment is repeated in an independent fashion. Let X_n be the number of white balls in urn A after n repetitions of the experiment. Assume that initially urn A contains all the white balls, and urn B contains all the red balls. Thus $X_0 = N$. Note that X_n tells us precisely the contents of the two urns after n experiments: if $X_n = i$, urn A contains i white balls and $N - i$ red balls; and urn B contains $N - i$ white balls and i red balls. That $\{X_n, n \geq 0\}$ is a DTMC on state-space $S = \{0, 1, \cdots, N\}$ can be seen from the following calculation. For $0 < i < N$

$$
\begin{aligned}
&\mathrm{P}\left(X_{n+1} = i+1 \mid X_n = i, X_{n-1}, \cdots, X_0\right) \\
&= \mathrm{P}(\text{ A red ball from urn } A \text{ and a white ball from urn } B \\
&\quad \text{are picked on the } n\text{-th experiment }) \\
&= \frac{N-i}{N} \cdot \frac{N-i}{N} = p_{i,i+1}.
\end{aligned}
$$

The other transition probabilities can be computed similarly to see that $\{X_n, n \geq 0\}$ is a random walk on $S = \{0, 1, \cdots, N\}$ with the following parameters:

$$
\begin{gathered}
r_0 = 0, \quad p_0 = 1, \\
q_i = \left(\frac{i}{N}\right)^2, \quad r_i = 2\left(\frac{i}{N}\right) \cdot \left(\frac{N-i}{N}\right), p_i = \left(\frac{N-i}{N}\right)^2, 0 < i < N, \\
r_N = 0, \quad q_N = 1.
\end{gathered}
$$

Note that in random walk, we have the following definition:

$$
p_{i,i+1} = p_i, \quad p_{i,i-1} = q_i, \quad p_{i,i} = r_i, \quad i \in S.
$$

Now Let $\{X_n, n \geq 0\}$ be the stochastic process of the urn with $N = 10$.

1. Write the transition matrix based on the given discription
2. Compute $\mathrm{E}\left(X_n\right)$ for $n = 0, 5, 10, 15$, and 20, starting with $X_0 = 10$.
3. Find the eigenvalues of the transition matrix .

Please do 2 and 3 with computer. Matlab, Python, Mathmatica, Maple, and Julia are some of the languages that should work. Or you can go more general coding language of C++, Java...(which I think is more complicated).