Continuous Markov Chain: Poisson Process

ISE/OR 560 Fall 2022

E(amount to be paid)

Hong Wan

November 7, 2022

North Carolina State University

NC STATE UNIVERSITY

Overview

Consider a continuous-time stochastic process $\{X(t), t \geq 0\}$ on a ____________ with ____________ at each time $t \geq 0$. We shall call such a process continuous-time Markov Chain (CTMC).

A finite-state CTMC spends an ____________ distributed amount of time in a given state before jumping out of it.

The associated ____________ processes form the foundation of many CTMC models.

Exponential Random Variables

Consider a **nonnegative** random variable with parameter $\lambda > 0$ with the following pdf:

$f(x) =$
F(x)=
P(X>x)=

$\mathbb{E}(X) =$
$\mathsf{Var}(X) =$

If the random variable X has units of time, the parameter λ has units of time $^{-1}$.

Example: Time Failure

Suppose a new machine is put into operation at time zero. Its lifetime is known to be an *Exp*(λ) random variable with $\lambda = .1$ /hour. What are the mean and variance of the lifetime of the machine?

Memoryless Property

A random variable X on $[0, \infty)$ is said to have the memoryless property if

$$\mathrm{P}(X > t + s \mid X > s) = \mathrm{P}(X > t), s, t \geq 0.$$

The unique feature of an exponential distribution is that it is the **only** continuous nonnegative random variable with the memoryless property.

Theorem (Memoryless Property of *Exp*(λ)): Let X be a continuous random variable taking values in $[0, \infty)$. It has the memoryless property if and only if it is an $\mathrm{Exp}(\lambda)$ random variable for some $\lambda > 0$.

Erlang Distribution

Next consider a random variable with parameters $k = 1, 2, 3, \ldots$ and $\lambda > 0$ taking values in $[0, \infty)$ with the following pdf:

$$f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}, \quad x \geq 0$$

and

$$F(x) = 1 - \sum_{r=0}^{k-1} e^{-\lambda x} \frac{(\lambda x)^r}{r!}, \quad x \geq 0$$

$$\mathrm{P}(X > x) = e^{-\lambda x} \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!}, \quad x \geq 0$$

Erlang Distribution: $\mathsf{Erl}(k, \lambda)$

$$\mathrm{E}(X) =$$

and

$$\mathsf{Var}(X) =$$

Example

We shall redo the previous example under the assumption that the lifetime of the machine is an $\mathsf{Erl}(k, \lambda)$ random variable with parameters $k = 2$ and $\lambda = .2/\mathrm{hr}$.

Theorem

Sums of Exponentials: Suppose $\{X_i, i = 1, 2, \cdots, n\}$ are iid $\text{Exp}(\lambda)$ random variables, and let

$$Z_n = X_1 + X_2 + \cdots + X_n$$

Then Z_n is an $\text{Erl}(n, \lambda)$ random variable.

Minimum of Independent Exponential Random Variables

Let X_i be an $\text{Exp}\,(\lambda_i)$ random variable ($1 \leq i \leq k$), and suppose $X_1, X_2, \ldots, X_k$ are independent. Let

$$X = \min\{X_1, X_2, \ldots, X_k\}\,.$$

We can think of X_i as the time when an event of type i occurs. Then X is the time when the first of these k events occurs.

Theorem: $X = \min\{X_1, X_2, \ldots, X_k\}$ is an $Exp(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^{k} \lambda_i$$

Next, let X be as in previous two slides, and define $Z = i$ if $X_i = X$; i.e., if event i is the first of the k events to occur. Note that X_i 's are continuous random variables, and hence the probability that two or more of them will be equal to each other is zero. Thus there is no ambiguity in defining Z. The next theorem gives the joint distribution of Z and X.

Theorem (Distribution of (Z, X)): Z and X are independent random variables with

$$\mathrm{P}(Z = i; X > x) = \mathrm{P}(Z = i)\mathrm{P}(X > x) = \frac{\lambda_i}{\lambda} e^{-\lambda x}, 1 \leq i \leq k, x \geq 0.$$

The time until the occurrence of the first of the k events is independent of which of the k events occurs first!

The conditional distribution of X, given that $Z = i$, is $\mathsf{Exp}(\lambda)$ and not $\mathsf{Exp}\,(\lambda_i)$ as we might have (incorrectly) guessed.

Example: Boy or a Girl?

A maternity ward at a hospital currently has seven pregnant women waiting to give birth. Three of them are expected to give birth to boys, and the remaining four are expected to give birth to girls. From prior experience, the hospital staff knows that a mother spends on average 6 hours in the hospital before delivering a boy and 5 hours before delivering a girl. Assume that these times are independent and exponentially distributed. What is the probability that the first baby born is a boy and is born in the next hour?

Suppose we conduct n independent trials of an experiment. Each trial is successful with probability p or unsuccessful with probability $1-p$. Let X be the number of successes among the n trials. The state space of X is $\{0,1,2,\ldots,n\}$. The random variable X is called a binomial random variable with parameters n and p and is denoted as $\mathrm{Bin}(n,p)$. The pmf of X is given by

$$p_k = \mathrm{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le k \le n$$

Binomial in the Limit

In the limit, the **Bin**(n, p) random variable approaches a random variable with state space $S = \{0, 1, 2, \ldots\}$ and pmf

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in S.$$

The pmf has been given the special name **Poisson distribution**.

$$\mathrm{E}(X) =$$

and

$$\mathsf{Var}(X) =$$

Example: Counting Accidents:

Suppose accidents occur one at a time at a dangerous traffic intersection during a 24-hour day. We divide the day into 1440 minute-long intervals and assume that there can be only 0 or 1 accidents during each 1-minute interval. Let E_k be the event that there is an accident during the k th minute-long interval, $1 \leq k \leq 1440$. Suppose that the event E_k 's are mutually independent and that $\mathrm{P}(E_k) = .001$. Compute the probability that there are exactly k accidents during one day.

Sums of Poissons

Suppose $\{X_i, i = 1, 2, \ldots, n\}$ are independent random variables with $X_i \sim P(\lambda_i), 1 \leq i \leq n$. Let

$$Z_n = X_1 + X_2 + \cdots + X_n$$

Then Z_n is a $P(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^{n} \lambda_i$$

Assumption: one event at a time, he successive inter-event times are iid exponential random variables.

Let S_n be the occurrence time of the nth event. Assume $S_0 = 0$, and define

$$T_n = S_n - S_{n-1}, \quad n \geq 1.$$

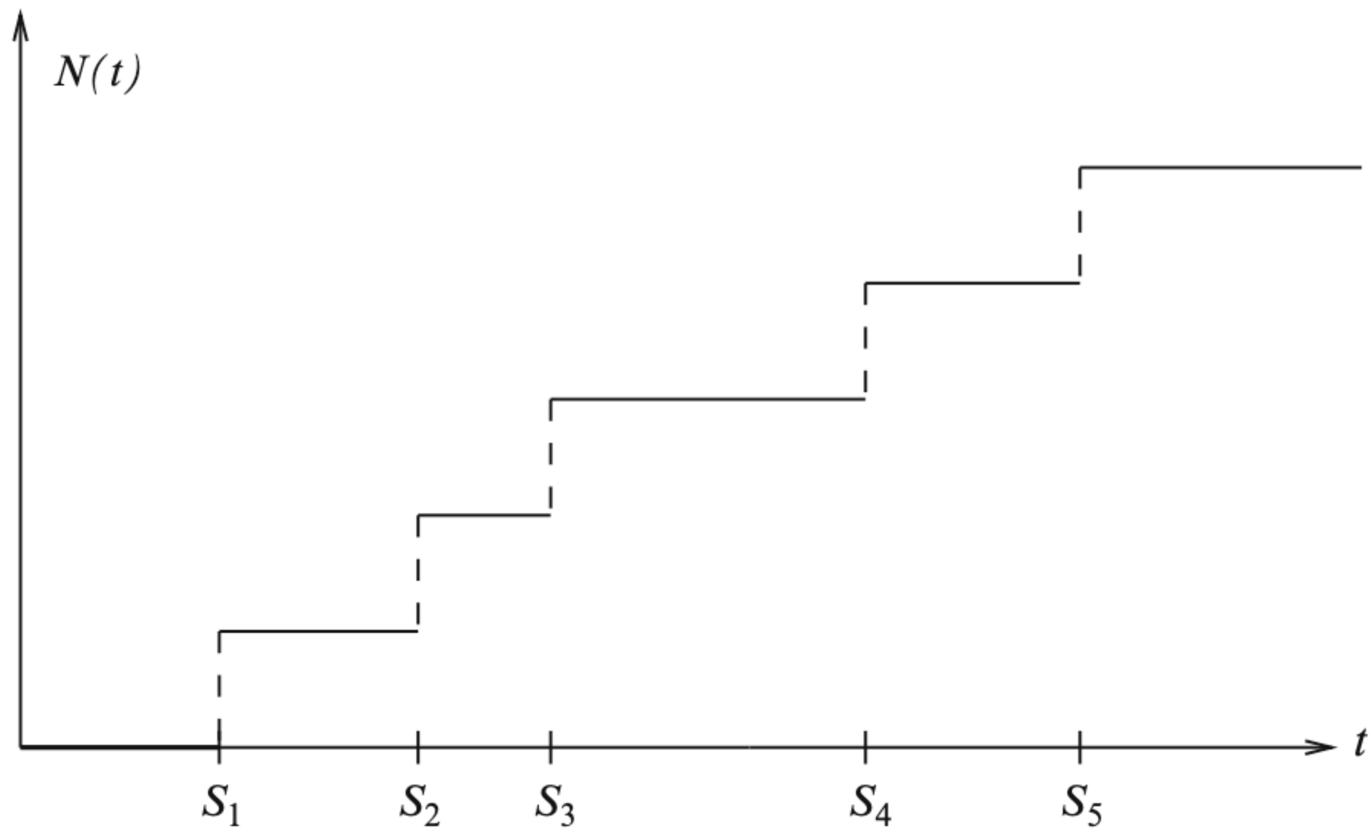
Thus T_n is the time between the occurrence of the nth and the $(n-1)$ st event. Let $N(t)$ be the total number of events that occur during the interval $(0, t]$. Thus an event at 0 is not counted, but an event at t, if any, is counted in $N(t)$. One can formally define it as

$$N(t) = \max\{n \geq 0 : S_n \leq t\}, \quad t \geq 0.$$

Poisson Process

The stochastic process $\{N(t), t \geq 0\}$, where $N(t)$ is as defined before, is called a Poisson process with rate λ (denoted by $\mathrm{PP}(\lambda)$) if $\{T_n, n \geq 1\}$ is a sequence of iid $\mathrm{Exp}(\lambda)$ random variables.

A Typical Sample Path



Poisson Process

Let $\{N(t), t \geq 0\}$ be a $\mathrm{PP}(\lambda)$. For a given $t, N(t)$ is a Poisson random variable with parameter λt; i.e.,

$$\mathrm{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k \geq 0.$$

Therefore,

$$\mathrm{E}(N(t)) = \lambda t, \quad \mathsf{Var}(N(t)) = \lambda t$$

Thus the expected number of events up to t increases linearly in t with rate λ. T

Markov Property of the Poisson Process

Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. It has the Markov property at each time t; that is,

$$\mathrm{P}(N(t+s) = k \mid N(s) = j, N(u), 0 \leq u \leq s) = \mathrm{P}(N(t+s) = k \mid N(s) = j).$$

The distribution of the increment $N(t+s) - N(s)$ over any interval $(s, t+s]$ is independent of the events occurring outside this interval. In particular, the increments over non-overlapping intervals are independent.

the number of events on any time interval $(s, s+t]$ is given by

$$N(t+s) - N(s) \sim \mathrm{P}(\lambda t).$$

Thus the distribution of the increment $N(t+s) - N(s)$ over any interval $(s, t+s]$ depends only on t, the length of the interval. We say that the increments are stationary.

Example

Let $N(t)$ be the number of births in a hospital over the interval $(0, t]$. Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate 10 per day.

1. What are the mean and variance of the number of births in an 8-hour shift?
2. What is the probability that there are no births from noon to 1 p.m.?
3. What is the probability that there are three births from 8 a.m. to 12 noon and four from 12 noon to 5 p.m.?

Example

Examples

1. (Two-Machine Workshop). Consider a workshop with two independent machines, each with its own repair person and each machine behaving as described in the previous example. Let $X(t)$ be the number of machines operating at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

2. (General Machine Shop). A machine shop consists of N machines and M repair persons. ($M \leq N$.) The machines are identical, and the lifetimes of the machines are independent $\text{Exp}(\mu)$ random variables. When the machines fail, they are serviced in the order of failure by the M repair persons. Each failed machine needs one and only one repair person, and the repair times are independent $\text{Exp}(\lambda)$ random variables. A repaired machine behaves like a new machine. Let $X(t)$ be the number of machines that are functioning at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

everytime it gets repaired, you gets ~~new~~ same ν.

3.

(Finite-Capacity Single Server Queue). Customers arrive at an automatic teller machine (ATM) according to a $\mathrm{PP}(\lambda)$. The space in front of the ATM can accommodate at most K customers. Thus, if there are K customers waiting at the ATM and a new customer arrives, he or she simply walks away and is lost forever. The customers form a single line and use the ATM in a first-come, first-served fashion. The processing times at the ATM for the customers are iid $\mathrm{Exp}(\mu)$ random variables. Let $X(t)$ be the number of customers at the ATM at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

Examples

(Finite-Capacity Single Server Queue). Customers arrive at an automatic teller machine (ATM) according to a $\text{PP}(\lambda)$. The space in front of the ATM can accommodate at most K customers. Thus, if there are K customers waiting at the ATM and a new customer arrives, he or she simply walks away and is lost forever. The customers form a single line and use the ATM in a first-come, first-served fashion. The processing times at the ATM for the customers are iid $\text{Exp}(\mu)$ random variables. Let $X(t)$ be the number of customers at the ATM at time t. Model $\{X(t), t \geq 0\}$ as a CTMC.

Finite Birth and Death Process

A CTMC on state space $\{0, 1, 2, \ldots, K\}$ with the rate matrix

$\to$ system capacity.

$$R = \begin{bmatrix} 0 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \mu_1 & 0 & \lambda_1 & \cdots & 0 & 0 \\ 0 & \mu_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_{K-1} \\ 0 & 0 & 0 & \cdots & \mu_K & 0 \end{bmatrix}$$

$\to$ single server queue.

is called a finite birth and death process. The transitions from i to $i+1$ are called the births, and the λ_i 's are called the birth parameters. The transitions from i to $i-1$ are called the deaths, and the μ_i 's are called the death parameters. It is convenient to define $\lambda_K = 0$ and $\mu_0 = 0$, signifying that there are no births in state K and no deaths in state 0 . A birth and death process spends an $\text{Exp}(\lambda_i + \mu_i)$ amount of time in state i and then jumps to state $i+1$ with probability $\lambda_i / (\lambda_i + \mu_i)$ or to state $i-1$ with probability $\mu_i / (\lambda_i + \mu_i)$

Example

(Telephone Switch): A telephone switch can handle K calls at any one time. Calls arrive according to a Poisson process with rate λ. If the switch is already serving K calls when a new call arrives, then the new call is lost. If a call is accepted, it lasts for an $Exp(\mu)$ amount of time and then terminates. All call durations are independent of each other. Let $X(t)$ be the number of calls that are being handled by the switch at time t . Model $\{X(t), t \geq 0\}$ as a CTMC.

Example

Example

You have a queue until someone can help you.

(Call Center): An airline phone-reservation system is called a call center and is staffed by s reservation clerks called agents. An incoming call for reservations is handled by an agent if one is available; otherwise the caller is put on hold. The system can put a maximum of H callers on hold. When an agent becomes available, the callers on hold are served in order of arrival. When all the agents are busy and there are H calls on hold, any additional callers get a busy signal and are permanently lost. Let $X(t)$ be the number of calls in the system, those handled by the agents plus any on hold, at time t. Assume the calls arrive according to a $PP(\lambda)$ and the processing times of the calls are iid $Exp(\mu)/$ random variables. Model $\{X(t), t \geq 0\}$ as a CTMC.

Draw a diagram:

- Do example of Leaky Bucket.
- $Z(t)$ can only have a value if $Y(t) = 0$