Continuous Markov Chain: Poisson Process

ISE/OR 560 Fall 2022

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Overview

Markov Prop. future based on now.

. ctm discrete, no

Consider a continuous-time stochastic process $\{X(t), t \geq 0\}$ on a finite state space ______ with Markov Prop. ______ at each time $t \geq 0$. We shall call such a process continuous-time Markov Chain (CTMC).

A finite-state CTMC spends an exponential. ______ distributed amount of time in a given state before jumping out of it.

The associated poisson ______ processes form the foundation of many CTMC models.

Exponential Random Variables

Exponential random variable

Consider a **nonnegative** random variable with parameter $\lambda > 0$ with the following pdf:

pdf $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$ $\lambda \to$ only parameter you need to specify.

cdf $F(x) = 1 - e^{-\lambda x} = F(X < x)$ $\to$ this totally defines the exponential distribution

$P(X > x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$

$\mathrm{E}(X) = 1/\lambda = 1/\text{rate}$

$\mathrm{Var}(X) = 1/\lambda^2 \Rightarrow$ std dev $= \frac{1}{\lambda} = E(X)$

If the random variable X has units of time, the parameter λ has units of time $^{-1}$.

t^{-1}

Example: Time Failure

$\lambda \equiv$ parameter

$\lambda = 0.1$ hour

$E(X) = \frac{1}{\lambda} = \frac{1}{\frac{1}{10hr}} = 10hr$ $\wedge$ $var(X) = 100\ h^2$

std. dev. = 10hr

Suppose a new machine is put into operation at time zero. Its lifetime is known to be an $Exp(\lambda)$ random variable with $\lambda = .1$ / hour. What are the mean and variance of the lifetime of the machine?

$P(X > 24hrs) = e^{-(0.1)(24)} = 0.0907$

$P(X > 48 \mid X > 24) = P(X > 48, X > 24) / P(X > 24)$

for the first two days no problem.

$\Rightarrow \frac{P(X > 48, X > 24)}{P(X > 24)} = \frac{P(X > 48)}{P(X > 24)}$ Memoryless Property

$p(x)c(y)$

$$= \frac{e^{-(0.1)(0.48)}}{e^{-(0.1)(0.24)}} =$$

Memory

Memoryless Property

A random variable X on $[0, \infty)$ is said to have the memoryless property if

$$P(X > t + s \mid X > s) = P(X > t), s, t \geq 0.$$

The unique feature of an exponential distribution is that it is the **only** continuous nonnegative random variable with the memoryless property.

no other distribution have this

Formal Statement

Theorem (Memoryless Property of *Exp*(λ)): Let X be a continuous random variable taking values in $[0, \infty)$. It has the memoryless property if and only if it is an $\mathsf{Exp}(\lambda)$ random variable for some $\lambda > 0$.

I k=1, we get an exponential.

Next consider a random variable with parameters $k = 1, 2, 3, \ldots$ and $\lambda > 0$ taking values in $[0, \infty)$ with the following pdf:

$$f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{k-1}}{(k-1)!}, \quad x \geq 0$$

and

$$F(x) = 1 - \sum_{r=0}^{k-1} e^{-\lambda x} \frac{(\lambda x)^r}{r!}, \quad x \geq 0$$

$$\mathrm{P}(X > x) = e^{-\lambda x} \sum_{r=0}^{k-1} \frac{(\lambda x)^r}{r!}, \quad x \geq 0$$

Erlang Distribution: Erl(k, λ)

$$\mathrm{E}(X) = \frac{k}{\lambda}$$

and

$$\mathsf{Var}(X) = \frac{k}{\lambda^2}$$

If $k=1$, we get ~~poisson~~ exponential dist.

Example

We shall redo the previous example under the assumption that the lifetime of the machine is an $\mathsf{Erl}(k, \lambda)$ random variable with parameters $k = 2$ and $\lambda = .2/\mathrm{hr}.$

$E[X] = \frac{2}{\frac{2}{10}} = 10h$, $Var(X) = \frac{2}{0.2 \cdot 0.2} = 50 \text{ hrs}^2$

$$P(X>24) = e^{-(0.2)(24)} \cdot (1 + 0.2 \cdot 24) = 0.047$$

- Pr(no failure in the second 24 hrs | no failure in the first 24 hrs.) $= P(X>48 \mid X>24) = \dfrac{P(X>48)}{P(X>24)} = 0.0151$

Theorem

Sums of Exponentials: Suppose $\{X_i, i = 1, 2, \cdots, n\}$ are iid Exp(λ) random variables, and let

$$Z_n = X_1 + X_2 + \cdots + X_n$$

k=n

Then Z_n is an Erl(n, λ) random variable.

constant.

Minimum of Independent Exponential Random Variables

λ can be different for each of the variables

Let X_i be an $\mathrm{Exp}(\lambda_i)$ random variable ($1 \leq i \leq k$), and suppose $X_1, X_2, \ldots, X_k$ are independent. Let

$$X = \min\{X_1, X_2, \ldots, X_k\}.$$

We can think of X_i as the time when an event of type i occurs. Then X is the time when the first of these k events occurs.

The more complex the system
$\Rightarrow$ more likely something will wrong.

Theorem: $\underbrace{X = \min\{X_1, X_2, \ldots, X_k\}}$ is an $\underbrace{Exp(\lambda)}$ random variable, where

$$\lambda = \sum_{i=1}^{k} \lambda_i$$

Next, let X be as in previous two slides, and define $Z = i$ if $X_i = X$; i.e., if event i is the first of the k events to occur. Note that X_i 's are continuous random variables, and hence the probability that two or more of them will be equal to each other is zero. Thus there is no ambiguity in defining Z. The next theorem gives the joint distribution of Z and X.

Theorem (Distribution of (Z, X)): Z and X are independent random variables with

individual event

$$\mathrm{P}(Z = i; X > x) = \mathrm{P}(Z = i)\mathrm{P}(X > x) = \frac{\lambda_i}{\lambda} e^{-\lambda x}, 1 \leq i \leq k, x \geq 0.$$

summation

The time until the occurrence of the first of the k events is independent of which of the k events occurs first!

The conditional distribution of X, given that $Z = i$, is $\mathrm{Exp}(\lambda)$ and not $\mathrm{Exp}(\lambda_i)$ as we might have (incorrectly) guessed.

Example: Boy or a Girl?

A maternity ward at a hospital currently has seven pregnant women waiting to give birth. Three of them are expected to give birth to boys, and the remaining four are expected to give birth to girls. From prior experience, the hospital staff knows that a mother spends on average 6 hours in the hospital before delivering a boy and 5 hours before delivering a girl. Assume that these times are independent and exponentially distributed. What is the probability that the first baby born is a boy and is born in the next hour? ⇒ joint probability.

$X_i \equiv$ time to deliver of the ith expected mother.

mother 1-3 → boys mothers 4-7 → girls

Suppose we conduct n independent trials of an experiment. Each trial is successful with probability p or unsuccessful with probability $1-p$. Let X be the number of successes among the n trials. The state space of X is $\{0, 1, 2, \ldots, n\}$. The random variable X is called a binomial random variable with parameters n and p and is denoted as $\text{Bin}(n, p)$. The pmf of X is given by

$$p_k = \mathrm{P}(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n$$

Binomial in the Limit

as $n \to \infty$

In the limit, the Bin(n, p) random variable approaches a random variable with state space $S = \{0, 1, 2, \ldots\}$ and pmf

$$p_k = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in S.$$

The pmf has been given the special name **Poisson distribution**.

$$\mathrm{E}(X) = \lambda$$

and

$$\mathsf{Var}(X) = \lambda$$

Example: Counting Accidents:

Suppose accidents occur one at a time at a dangerous traffic intersection during a 24-hour day. We divide the day into 1440 minute-long intervals and assume that there can be only 0 or 1 accidents during each 1-minute interval. Let E_k be the event that there is an accident during the k th minute-long interval, $1 \leq k \leq 1440$. Suppose that the event E_k 's are mutually independent and that $\mathrm{P}\left(E_k\right) = .001$. Compute the probability that there are exactly k accidents during one day.

Sums of Poissons

Suppose $\{X_i, i = 1, 2, \ldots, n\}$ are independent random variables with $X_i \sim P\left(\lambda_i\right), 1 \leq i \leq n$. Let

$$Z_n = X_1 + X_2 + \cdots + X_n$$

Then Z_n is a $P(\lambda)$ random variable, where

$$\lambda = \sum_{i=1}^{n} \lambda_i$$

Poisson Process

Assumption: one event at a time, the successive inter-event times are iid exponential random variables.

Let S_n be the occurrence time of the nth event. Assume $S_0 = 0$, and define

$$T_n = S_n - S_{n-1}, \quad n \geq 1.$$

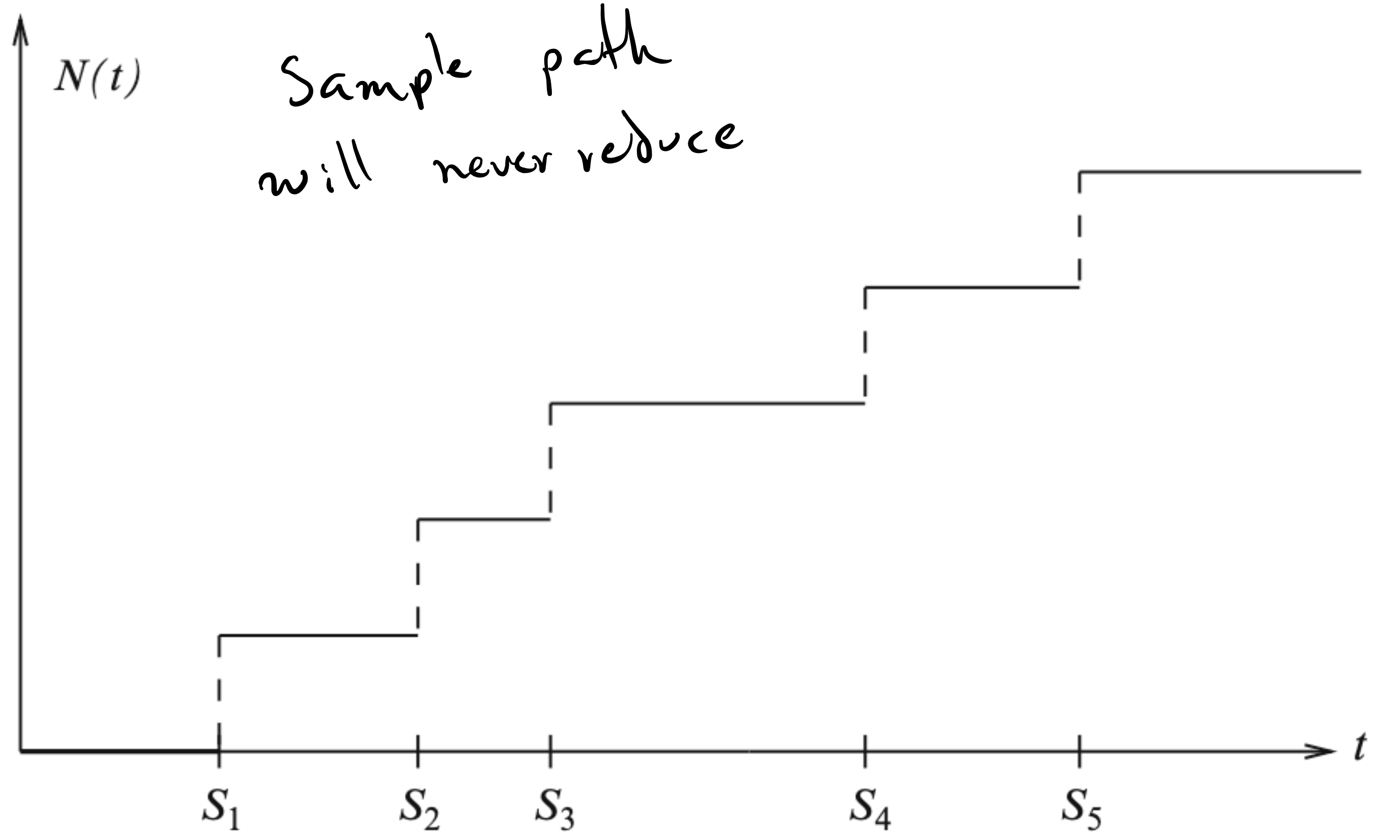
Thus T_n is the time between the occurrence of the nth and the $(n-1)$ st event. Let $N(t)$ be the total number of events that occur during the interval $(0, t]$. Thus an event at 0 is not counted, but an event at t, if any, is counted in $N(t)$. One can formally define it as

$$N(t) = \max\{n \geq 0 : S_n \leq t\}, \quad t \geq 0.$$

Poisson Process

The stochastic process $\{N(t), t \geq 0\}$, where $N(t)$ is as defined before, is called a Poisson process with rate λ (denoted by $\mathrm{PP}(\lambda)$) if $\{T_n, n \geq 1\}$ is a sequence of iid $\mathrm{Exp}(\lambda)$ random variables.

A Typical Sample Path



Poisson Process

Let $\{N(t), t \geq 0\}$ be a $\mathrm{PP}(\lambda)$. For a given t, $N(t)$ is a Poisson random variable with parameter λt; i.e.,

$$\mathrm{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k \geq 0.$$

Therefore,

$$\mathrm{E}(N(t)) = \lambda t, \quad \mathsf{Var}(N(t)) = \lambda t$$

Thus the expected number of events up to t increases linearly in t with rate λ. T

Markov Property of the Poisson Process

Let $\{N(t), t \geq 0\}$ be a $PP(\lambda)$. It has the Markov property at each time t; that is,

$$\mathrm{P}(N(t+s) = k \mid N(s) = j, N(u), 0 \leq u \leq s) = \mathrm{P}(N(t+s) = k \mid N(s) = j).$$

The distribution of the increment $N(t+s) - N(s)$ over any interval $(s, t+s]$ is independent of the events occurring outside this interval. In particular, the increments over non-overlapping intervals are independent.

the number of events on any time interval $(s, s+t]$ is given by

$$N(t+s) - N(s) \sim \mathrm{P}(\lambda t).$$

Thus the distribution of the increment $N(t+s) - N(s)$ over any interval $(s, t+s]$ depends only on t, the length of the interval. We say that the increments are stationary.

Example

Let $N(t)$ be the number of births in a hospital over the interval $(0, t]$. Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate 10 per day.

1. What are the mean and variance of the number of births in an 8-hour shift?
2. What is the probability that there are no births from noon to 1 p.m.?
3. What is the probability that there are three births from 8 a.m. to 12 noon and four from 12 noon to 5 p.m.?

Example