Markov Chains II

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Let $\{X_n, n \geq 0\}$ be a DTMC on $S = \{0, 1, 2, \cdots\}$, with transition probability matrix P, and initial distribution a. Let

$$T = \min \{n \geq 0 : X_n = 0\}.$$

The random variable T is called the first passage time into state 0 , since T is the first time the DTMC "passes into" state 0.

The same techniques can be used to study the first passage time into any set $A \subset S$.

What We Will Study

(1) Complementary cdf of T : $\mathrm{P}(T > n), \quad n \geq 0$,

(2) Probability of eventually visiting state 0 : $P(T < \infty)$,

Conditional Quantities

the conditional quantities for $i \in S$:

$$\begin{aligned} v_i(n) &= \mathrm{P}\left(T > n \mid X_0 = i\right) \\ u_i &= \mathrm{P}\left(T < \infty \mid X_0 = i\right), \end{aligned}$$

Then we can get the following expression:

$$\mathrm{P}(T > n) = \sum_{i \in S} a_i v_i(n)$$

$$\mathrm{P}(T < \infty) = \sum_{i \in S} a_i u_i$$

CDF of T

The following theorem illustrates how the first-step analysis produces a recursive method of computing the cumulative distribution of T. We first introduce the following matrix notation:

$$v(n) = [v_1(n), v_2(n), \cdots]^\top, \quad n \geq 0$$

$$B = \left[p_{i,j} : i, j \geq 1\right].$$

Thus B is a submatrix of P obtained by deleting the row and column corresponding to the state 0 .

Theorem

$$v(n) = B^n e, \quad n \geq 0,$$

where e is column vector of all ones.

Examples: Two-state DTMC

Consider the two state DTMC. The state-space is $\{1, 2\}$. Let T be the first passage time to state 1.

Examples

Consider a DTMC with state-space $\{1, 2, 3, 4, 5, 6\}$ and transition probability matrix given below:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/16 & 1/4 & 1/8 & 1/4 & 1/4 & 1/16 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let

$$T = \min\{n \geq 0 : X_n = 1\}$$

Examples: Success Runs

Consider the success runs DTMC of with transition probabilities

$$p_{i,0} = q_i, \quad p_{i,i+1} = p_i, \quad i = 0, 1, 2, \cdots .$$

Let T be the first passage time to state 0 . Compute the complementary cdf of T starting from state 1 . (Hint: think about the special structure of this DTMC)

Example

Classification of State

- Accessible
- Communicate
- irreducible
- Recurrent
- Transient
- absorbing

Absorption Probability

Since $v_i(n)$ is a monotone bounded sequence we know that

$$v_i = \lim_{n\to\infty} v_i(n)$$

exists. The quantity

$$u_i = 1 - v_i = \mathrm{P}\left(T < \infty \mid X_0 = i\right)$$

is the probability that the DTMC eventually visits state 0 starting from state i, also called the absorption probability into state 0 starting from state i when $p_{0,0} = 1$. In this section we develop methods of computing u_i or v_i. The main result is given by the next theorem.

Theorem 3.2 The vector $v = \lim_{n\to\infty} v(n)$ is given by the largest solution to

$$v = Bv$$ } for the largests

such that $v \leq e$, where B is as defined in Equation 3.3, and e is a vector of all ones.

Examples

- Genotype
- Gambler's ruin

Expectation of T

Let $\{X_n, n \geq 0\}$ and T be as defined in Section 3.1. In this section we assume that $u_i = \mathrm{P}(T < \infty \mid X_0 = i) = 1$ for all $i \geq 1$ and compute the expected time to reach state 0 starting from state i as

$$m_i = \mathrm{E}(T \mid X_0 = i), \quad i \geq 1.$$

Notice that $u_i < 1$ would imply that $m_i = \infty$. Let

$$m = [m_1, m_2, m_3, \cdots]^\top,$$

and B be as defined in Equation 3.3. The main result is given by the following theorem.

Theorem 3.3 Suppose $u_i = 1$ for all $i \geq 1$. Then m is given by the smallest nonnegative solution to

$$m = e + Bm,$$

where e is a column vector of all ones.