Benchmark : Finite volume methods for the Stokes and Navier-Stokes equations

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Motivation

- Several recent methods for the Stokes/Navier-Stokes equations were developed in the Finite Volume community
- Test their ability to handle various families of meshes, including distorted and non-conforming, locally refined ones
- Robustness with respect to low viscosity values, with respect to the invariance property
- Compare accuracies with respect to complexity / number of unknowns

Test cases

https://github.com/FranckBoyer/FVCA8_Benchmark

- 2D and 3D steady Stokes tests
- 2D and 3D steady Navier-Stokes tests and robustness with respect to low viscosity values
- 2D and 3D unsteady Navier-Stokes tests
- 2D steady Stokes and Navier-Stokes robustness with respect to the invariance property
- 2D steady lid-driven cavity tests

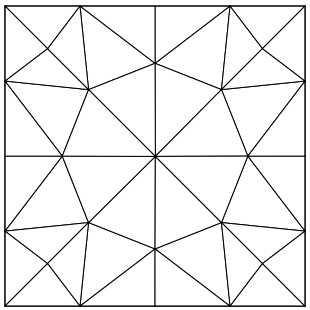
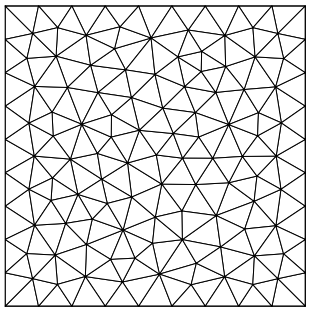
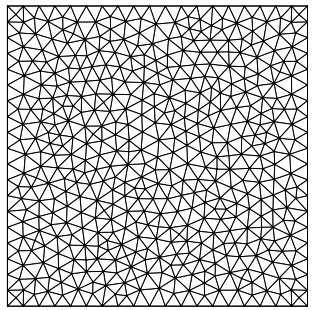
Participating methods

- 2 MAC-type schemes extended to non-conforming meshes
  Eymard, Chenier, Herbin
  Vittoz, Oger, Li, De Leffe, Le Touzé
- Hybrid Mimetic Mixed schemes
  Droniou, Eymard
- 2 Discrete Duality Finite Volume methods
  Delcourte, Omnes
  Boyer, Krell, Nabet
- 1 Hybrid High Order Method
  Di Pietro, Krell
- 1 Finite Volume Element scheme 3D !
  Angeli, Puscas, Fauchet, Cartalade

+ A handful of interested teams that finally did not contribute

2D Meshes

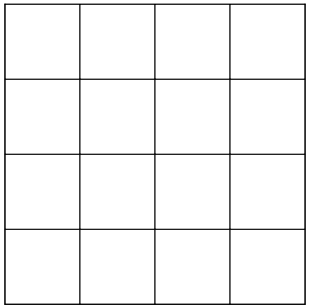
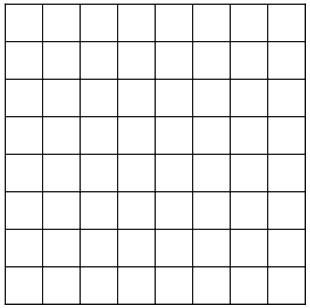
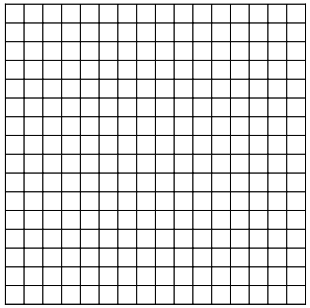
- Triangular meshes

mesh nb	Vertices	Volumes	Edges
1	29	40	68
2	131	224	354
3	506	934	1439
4	3310	6422	9731
5	13135	25872	39006
6	52609	104420	157028

2D Meshes

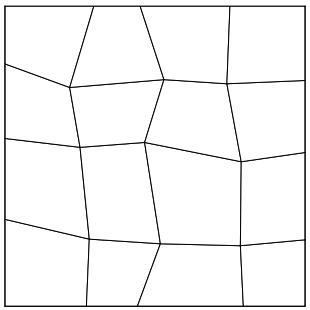
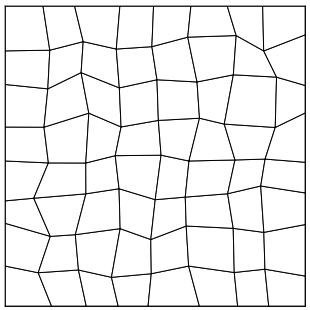
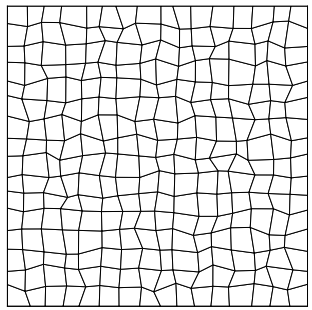
- Rectangular meshes

mesh nb	Vertices	Volumes	Edges
1	25	16	40
2	81	64	144
3	289	256	544
4	1089	1024	2112
5	4225	4096	8320
6	16641	16384	33024
7	66049	65536	13154

2D Meshes

- Distorted quadrangular meshes

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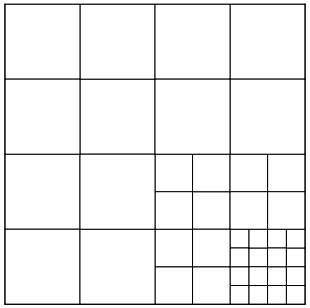
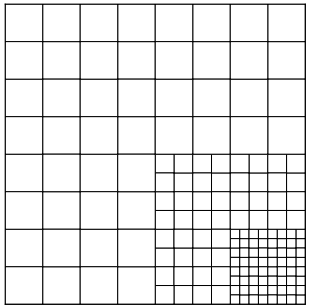
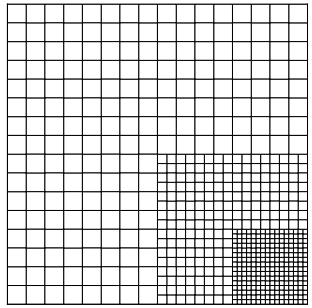
2D Meshes

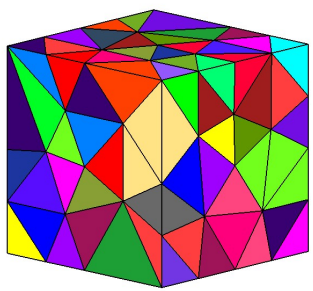
- Locally refined non-conforming meshes

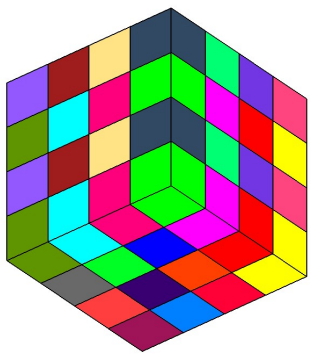




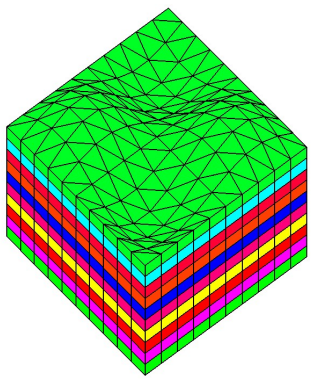
mesh nb	Vertices	Volumes	Edges
1	57	40	96
2	193	160	352
3	705	640	1344
4	2689	2560	5248
5	10497	10240	20736
6	41473	40960	82432
7	164865	163840	328704

3D Meshes

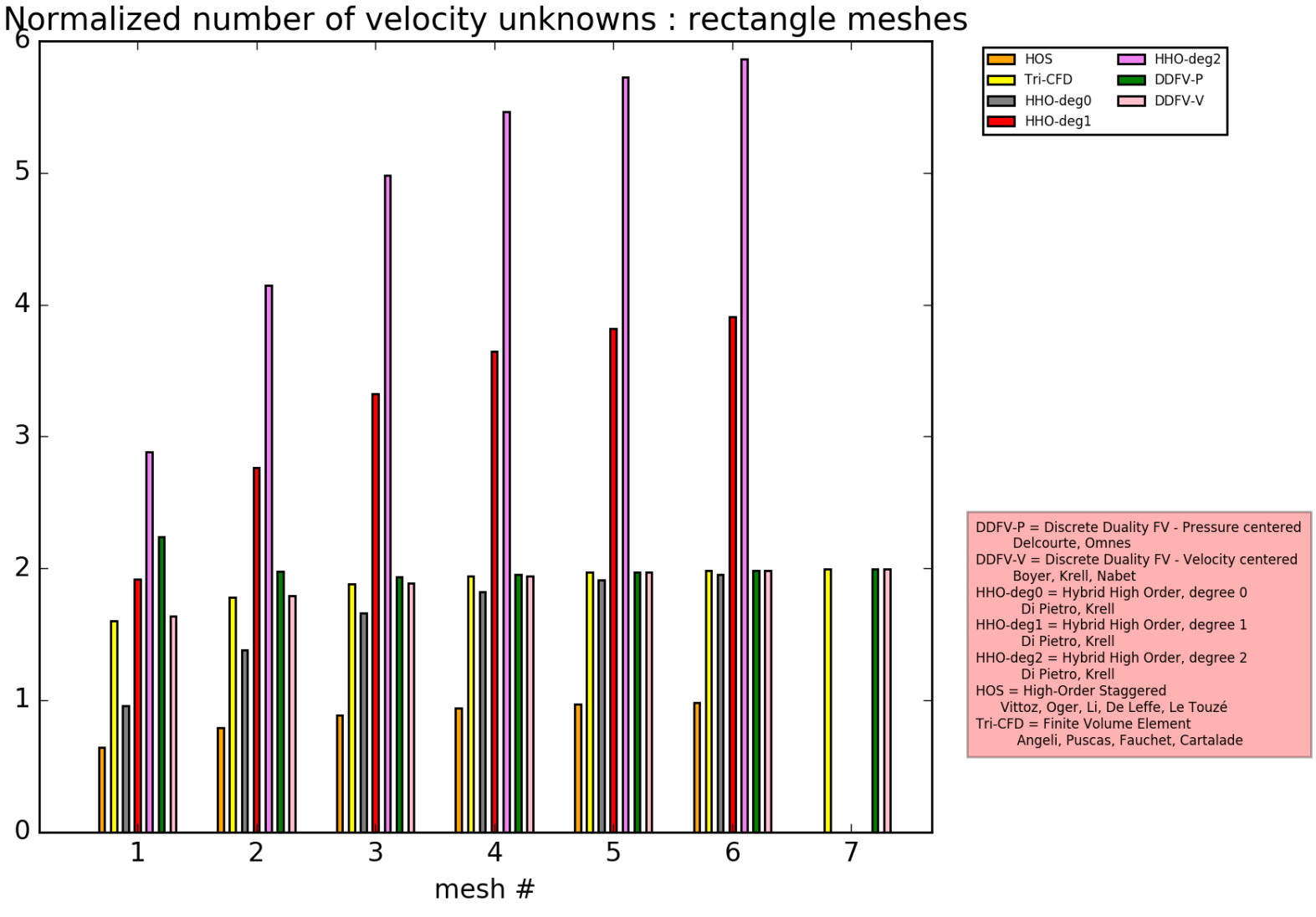
- tetrahedral meshes
- hexahedral meshes
- prismatic meshes



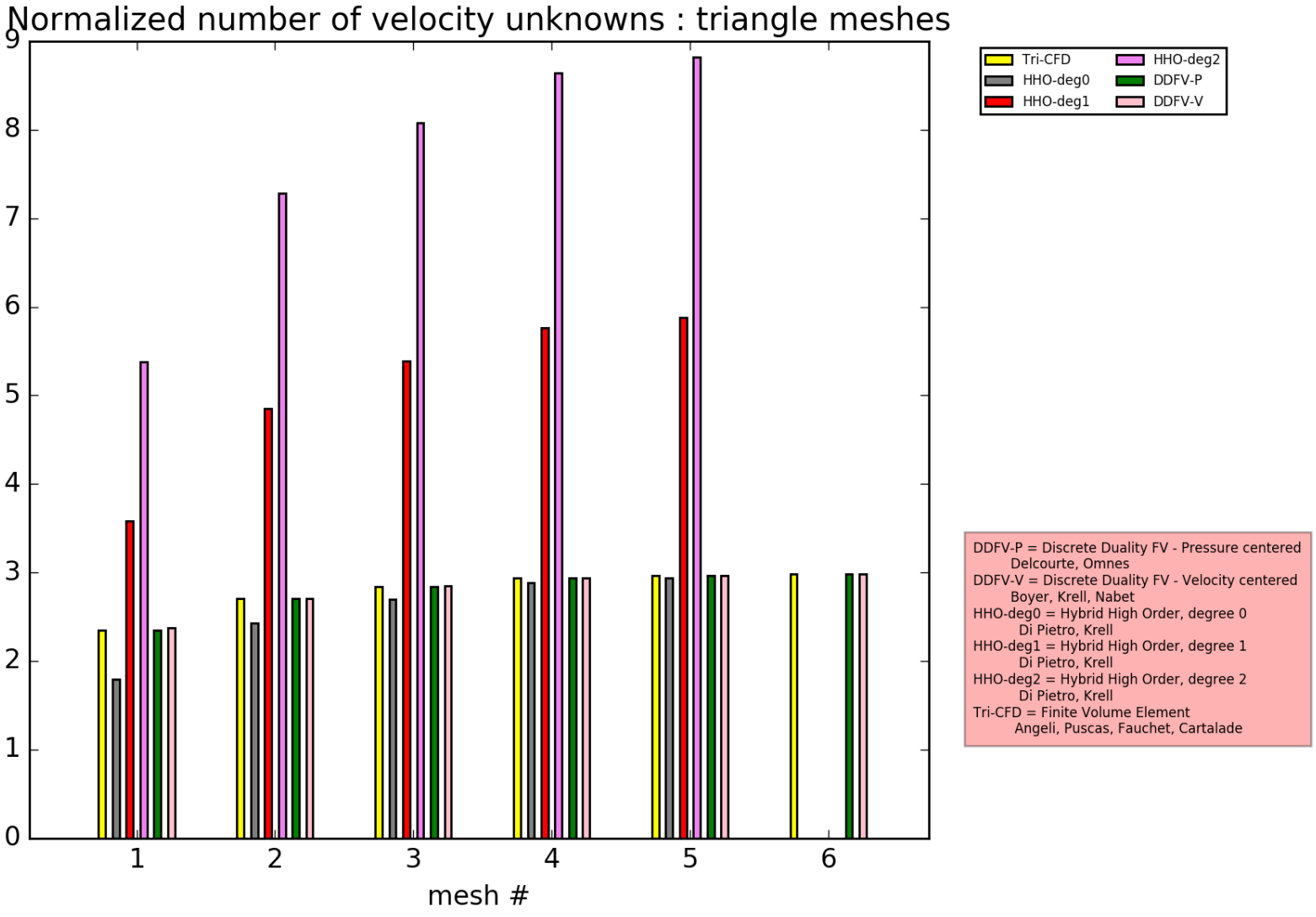




2D schemes complexity



2D schemes complexity



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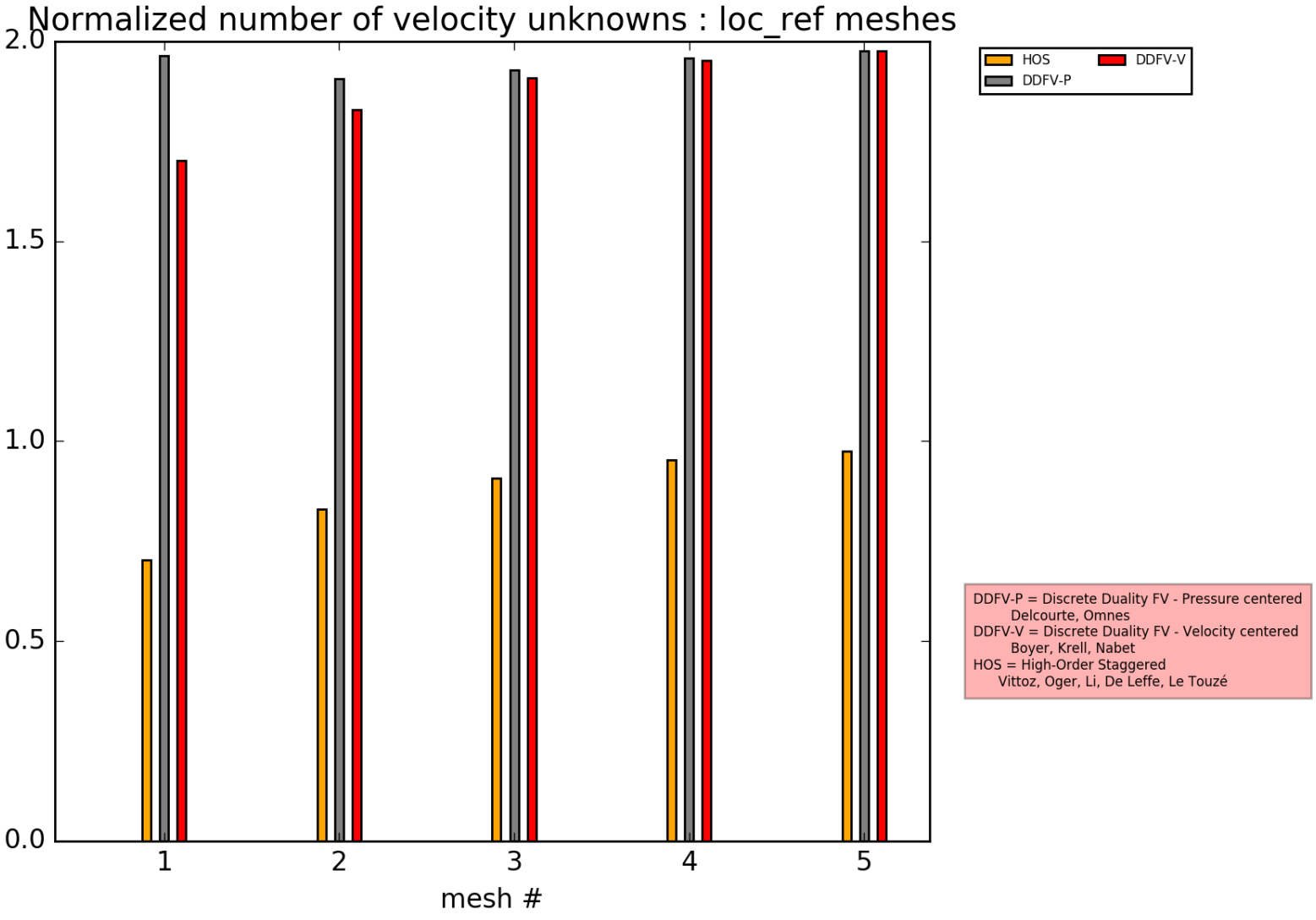


2D Steady Stokes tests

Case 2.1 : 2D Bercovier-Engelman

Exact solution:

$$\boldsymbol{u}_{\text{ex}} = (u_1(x,y), -u_1(y,x))^T,$$

$$p_{\text{ex}} = (x-1/2)(y-1/2),$$

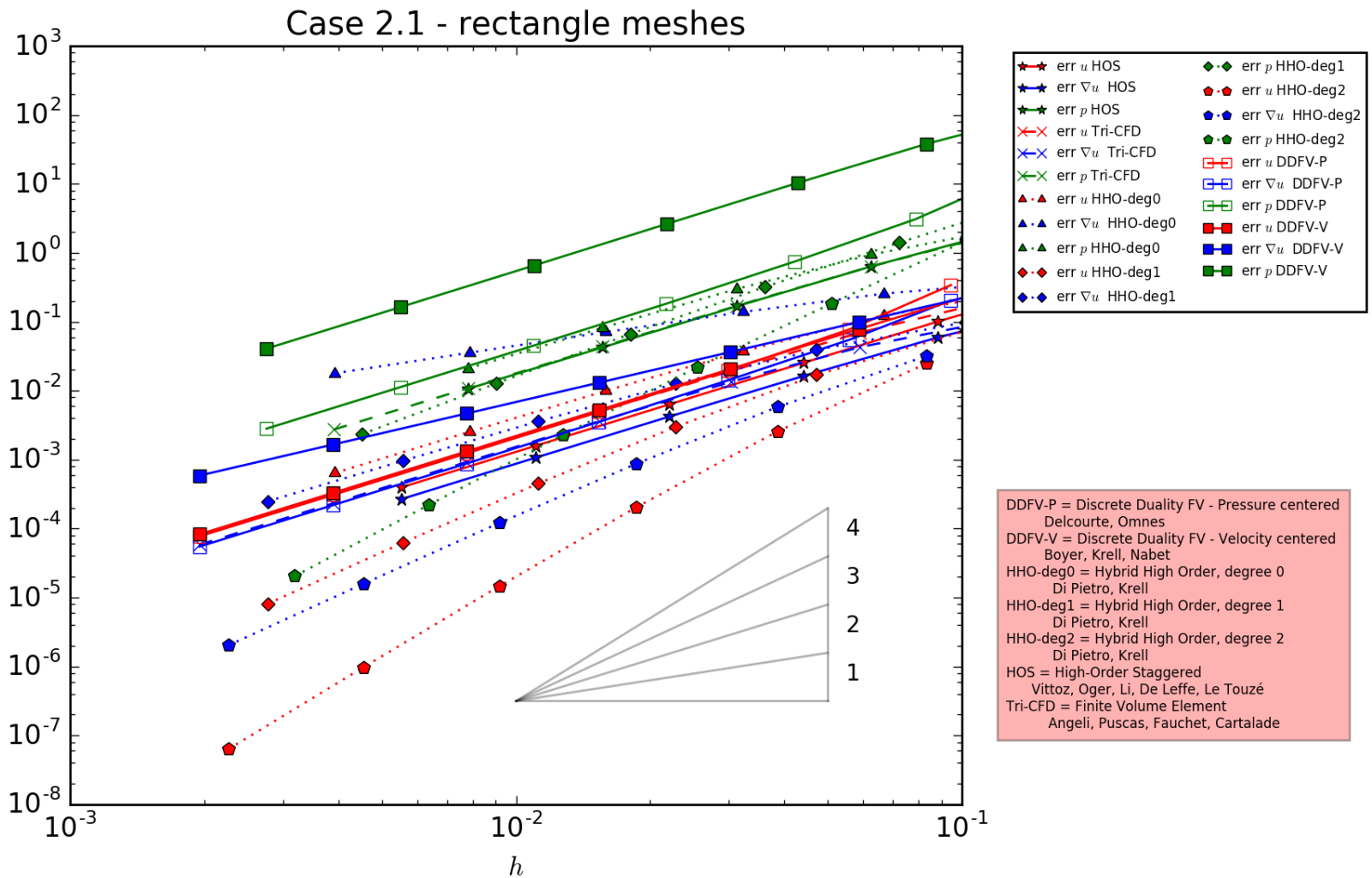
with $u_1(x,y) = -256x^2(x-1)^2y(y-1)(2y-1)$

Viscosity $\nu = 1$.

Important information : in all the graphs, we took

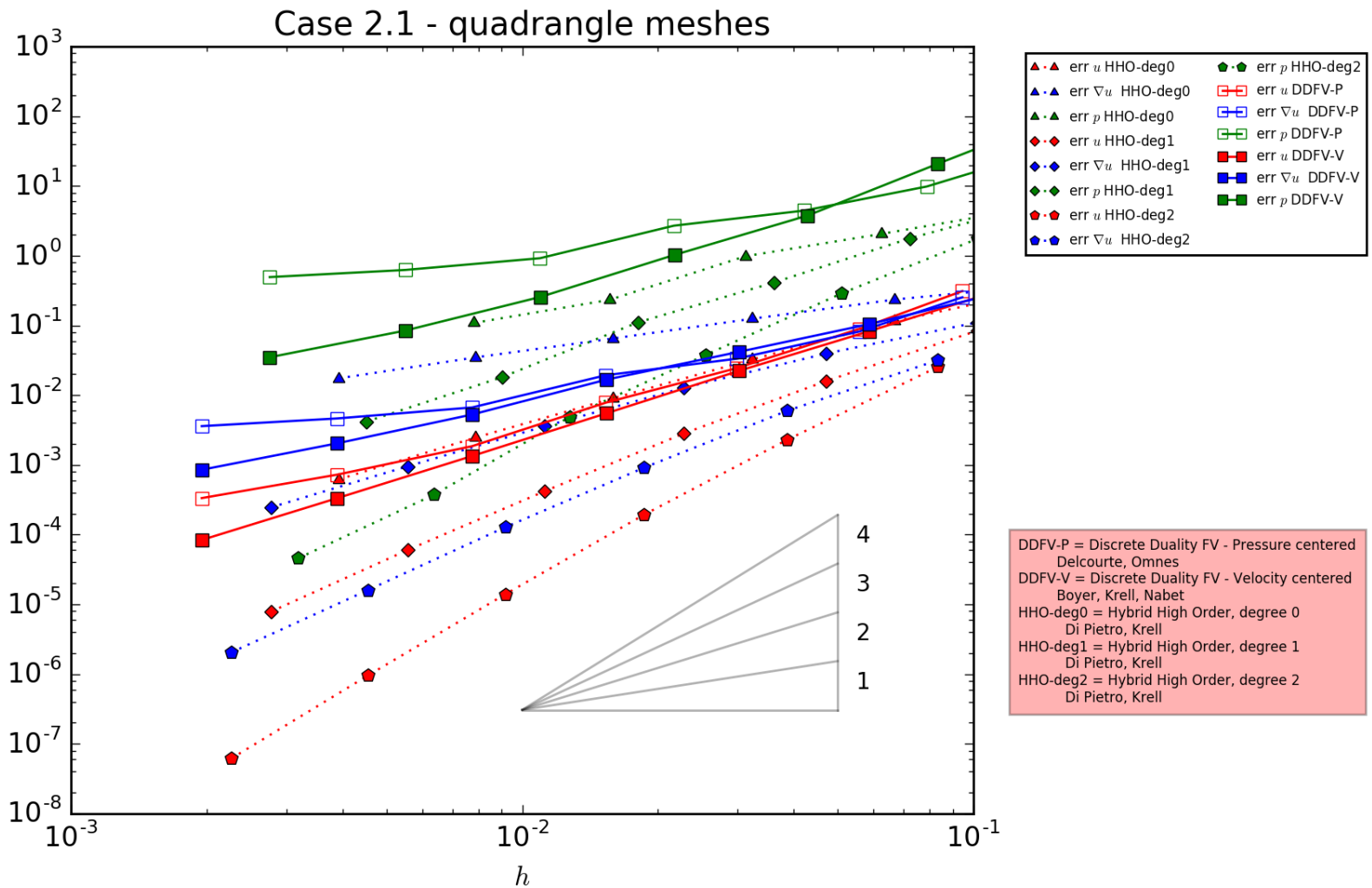
$$h = \frac{1}{\sqrt{\text{nuu}}}, \text{ or } h = \frac{1}{\sqrt{\text{nup}}}.$$

2D Steady Stokes tests



DDFV-P = Discrete Duality FV - Pressure centered
Delcourte, Omnes
DDFV-V = Discrete Duality FV - Velocity centered
Boyer, Krell, Nabet
HHO-deg0 = Hybrid High Order, degree 0
Di Pietro, Krell
HHO-deg1 = Hybrid High Order, degree 1
Di Pietro, Krell
HHO-deg2 = Hybrid High Order, degree 2
Di Pietro, Krell
HOS = High-Order Staggered
Vittoz, Oger, Li, De Leffe, Le Touzé
Tri-CFD = Finite Volume Element
Angeli, Puscas, Fauchet, Cartalade

2D Steady Stokes tests



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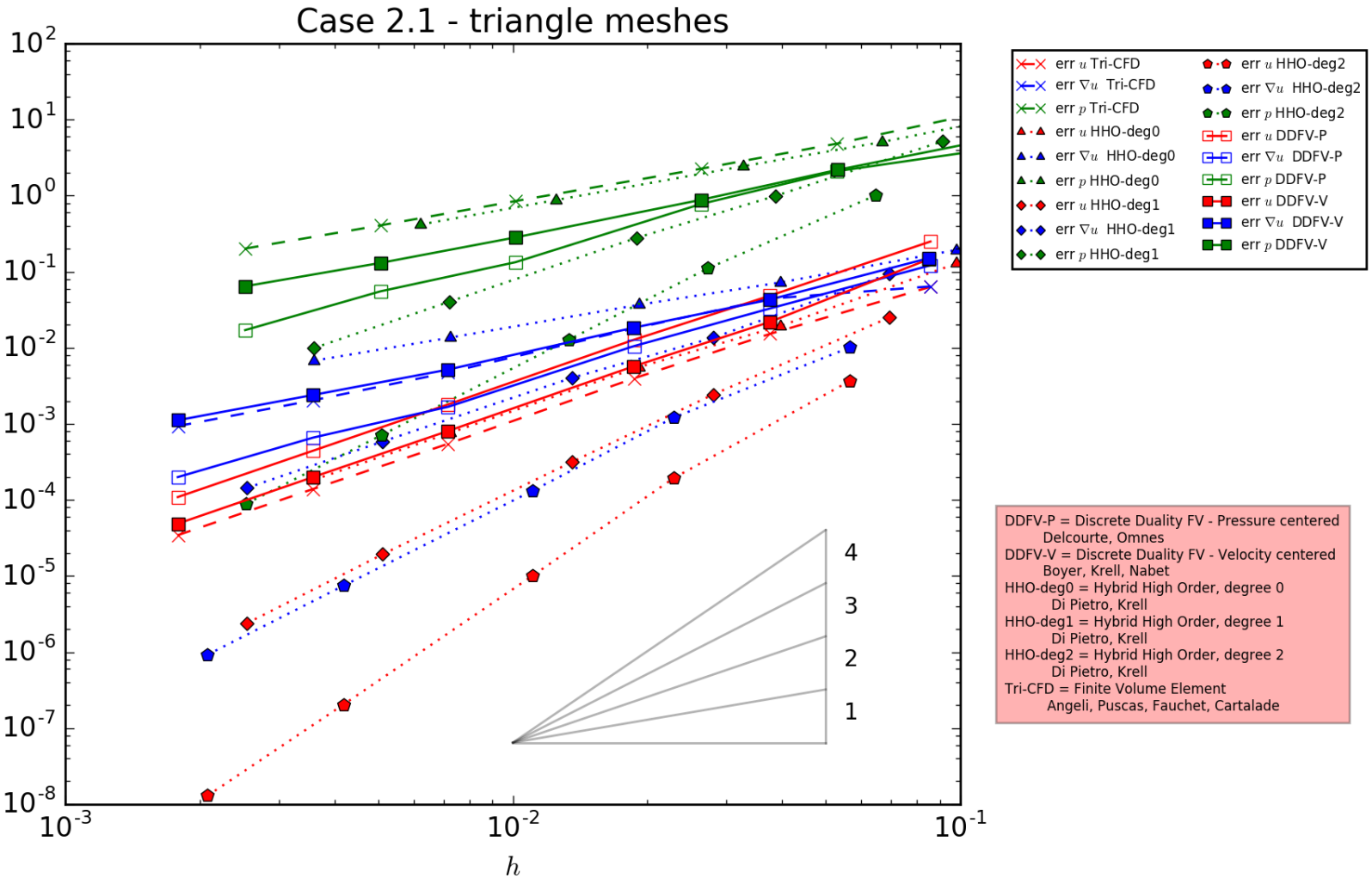
2D Steady Stokes tests



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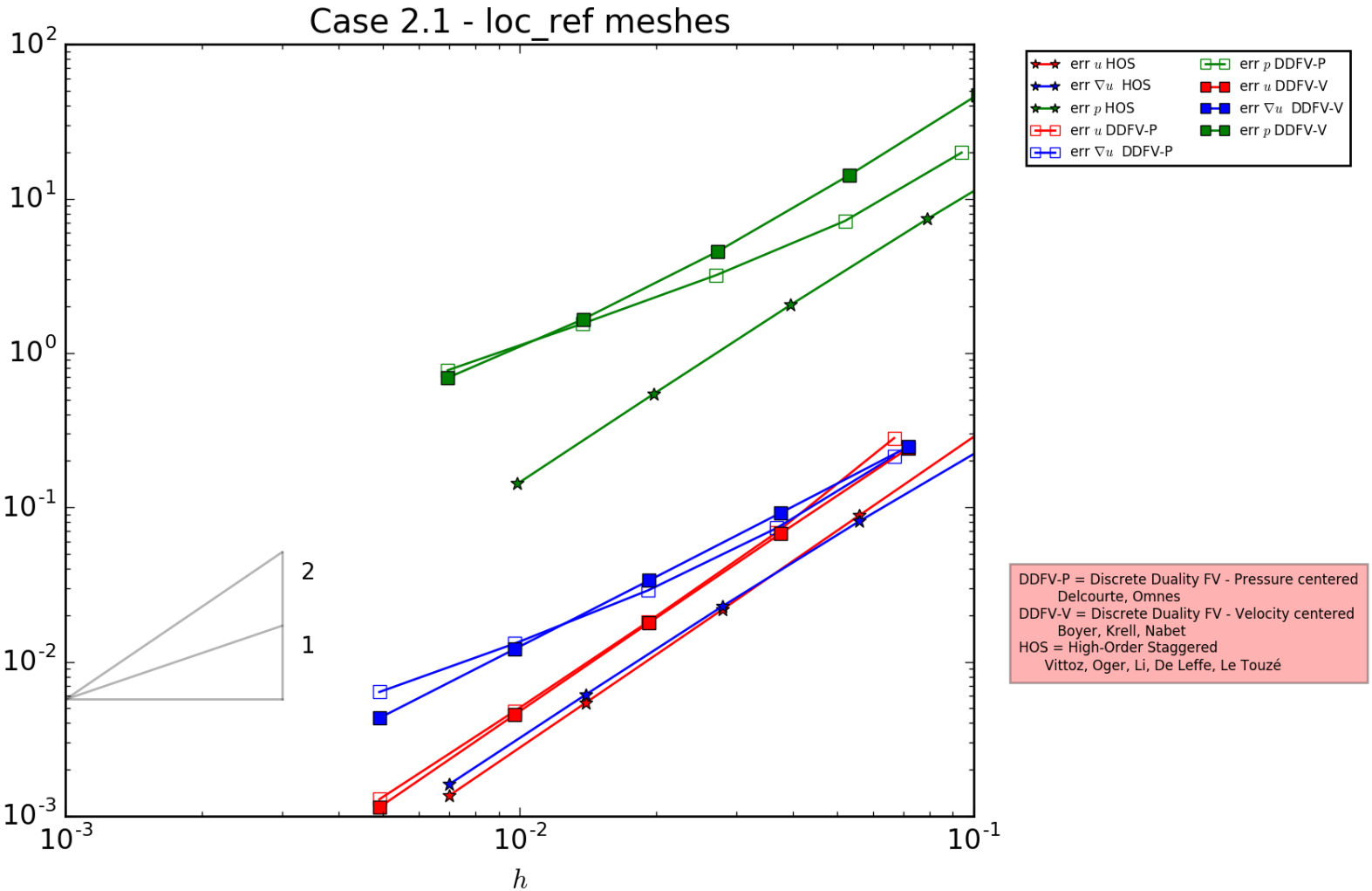
2D Steady Stokes tests



3D steady Stokes tests

The 3D Taylor Green Vortex test-case.

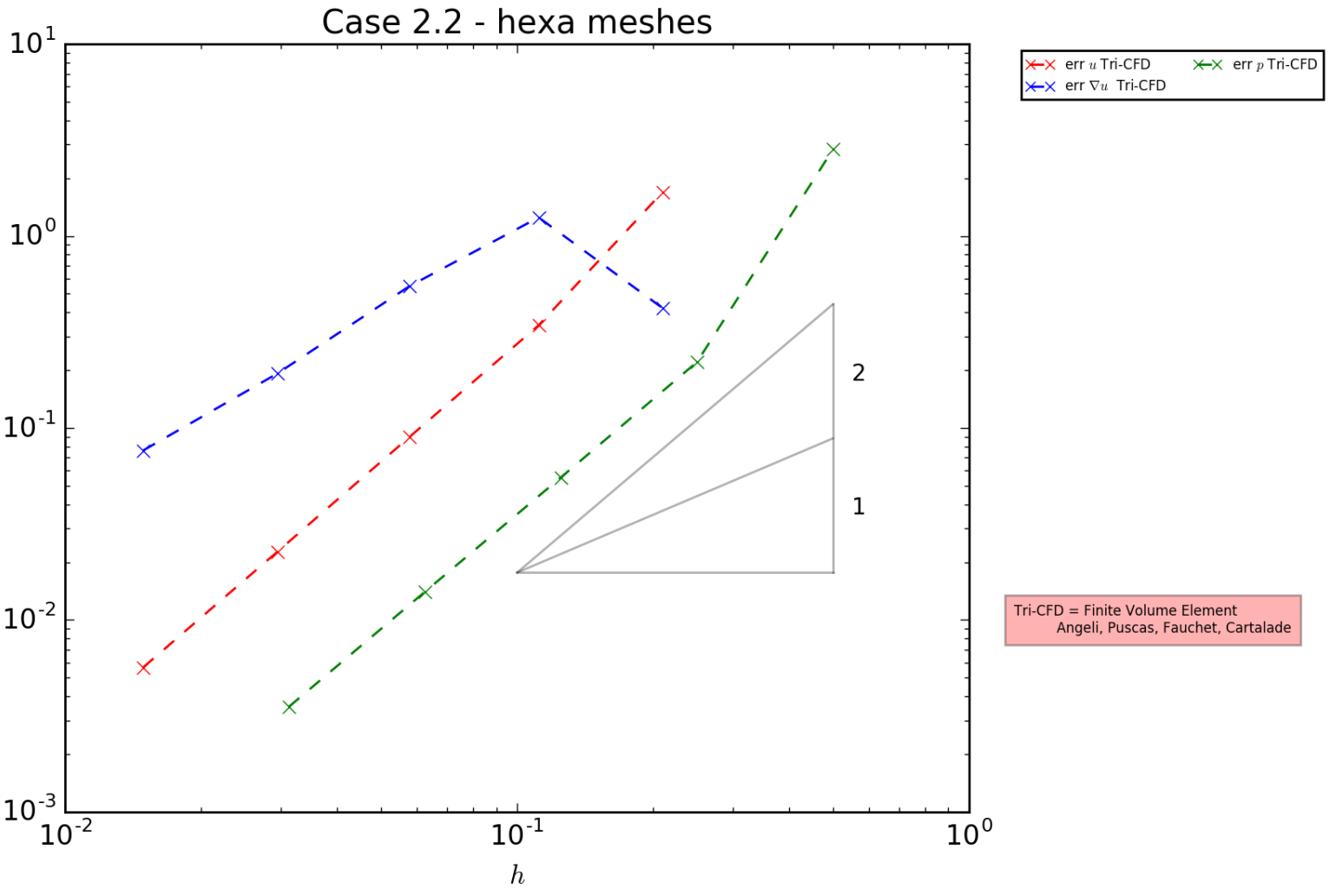
Exact solution:

$$\boldsymbol{u}_{\text{ex}} = \begin{pmatrix} -2\cos(2\pi x)\sin(2\pi y)\sin(2\pi z) \\ \sin(2\pi x)\cos(2\pi y)\sin(2\pi z) \\ \sin(2\pi x)\sin(2\pi y)\cos(2\pi z) \end{pmatrix}$$

and $p_{ex} = -6\pi \sin(2\pi x)\sin(2\pi y)\sin(2\pi z)$.

Viscosity $\nu = 1$.

3D steady Stokes tests



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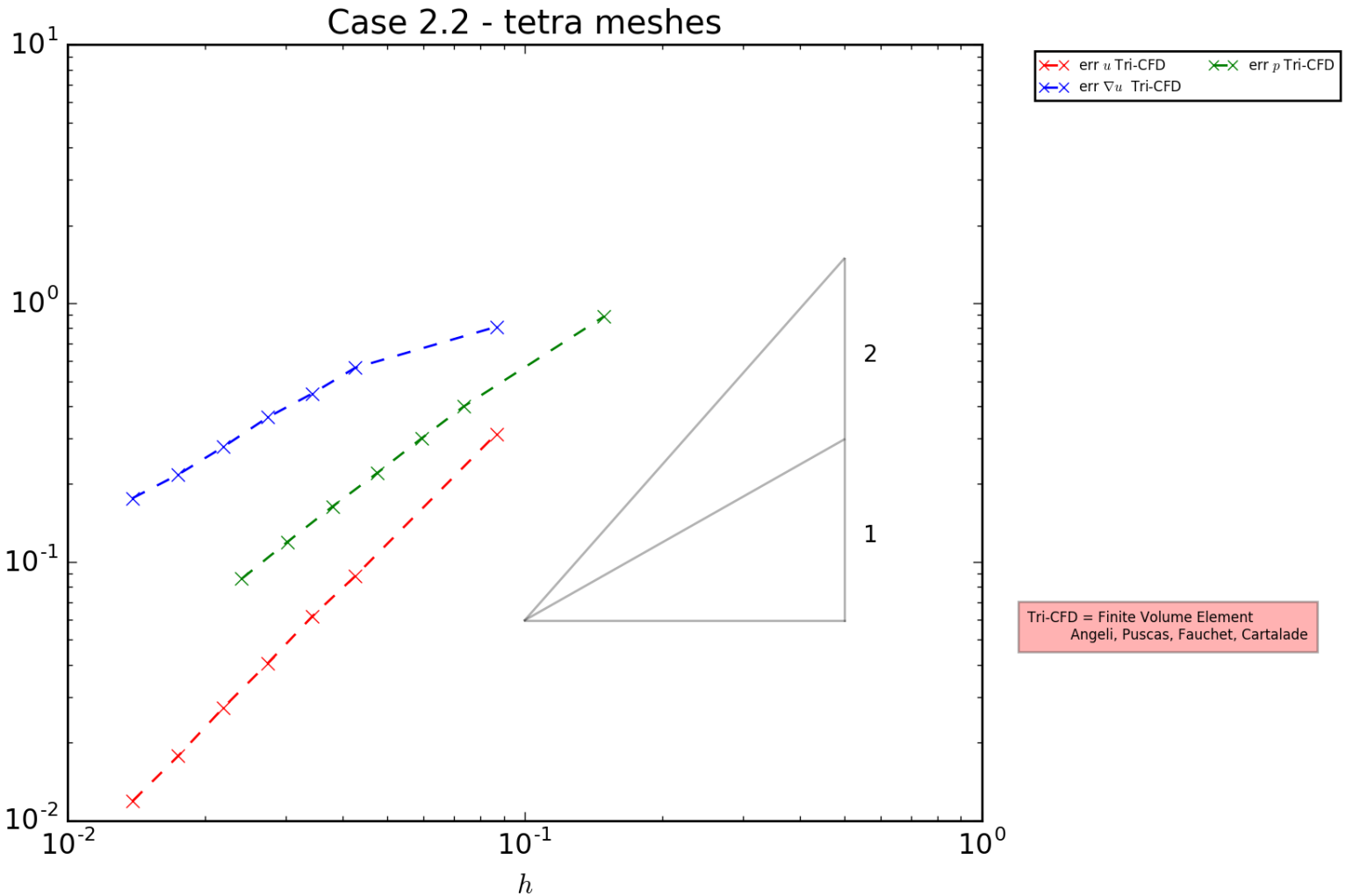
3D steady Stokes tests



2D steady Navier-Stokes tests and robustness with respect to low viscosity values

Case 3.1 : Simple vortex balanced by the pressure gradient.

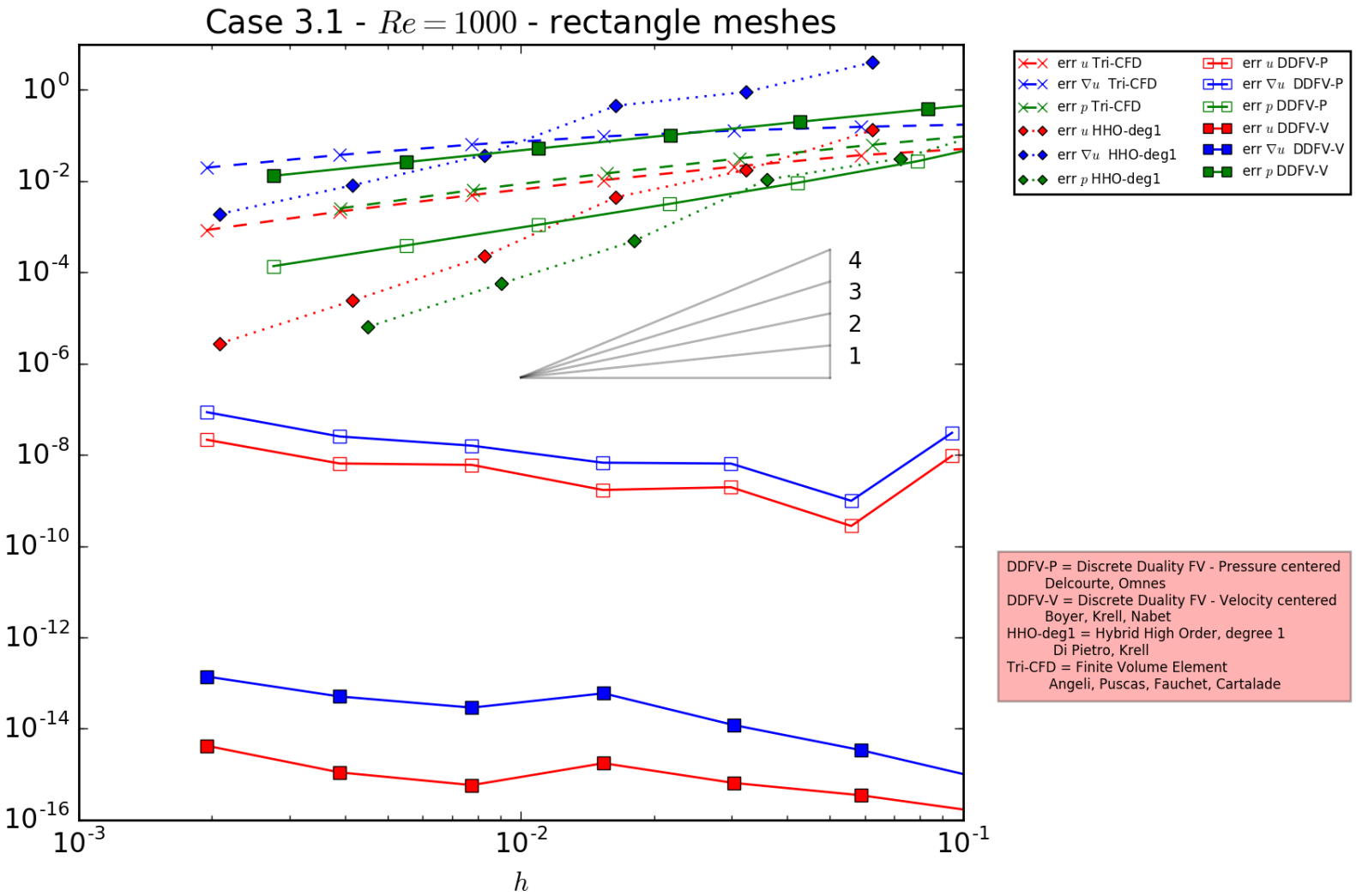
Exact solution:

$$\boldsymbol{u}_{\text{ex}} = (y, -x)^T,$$

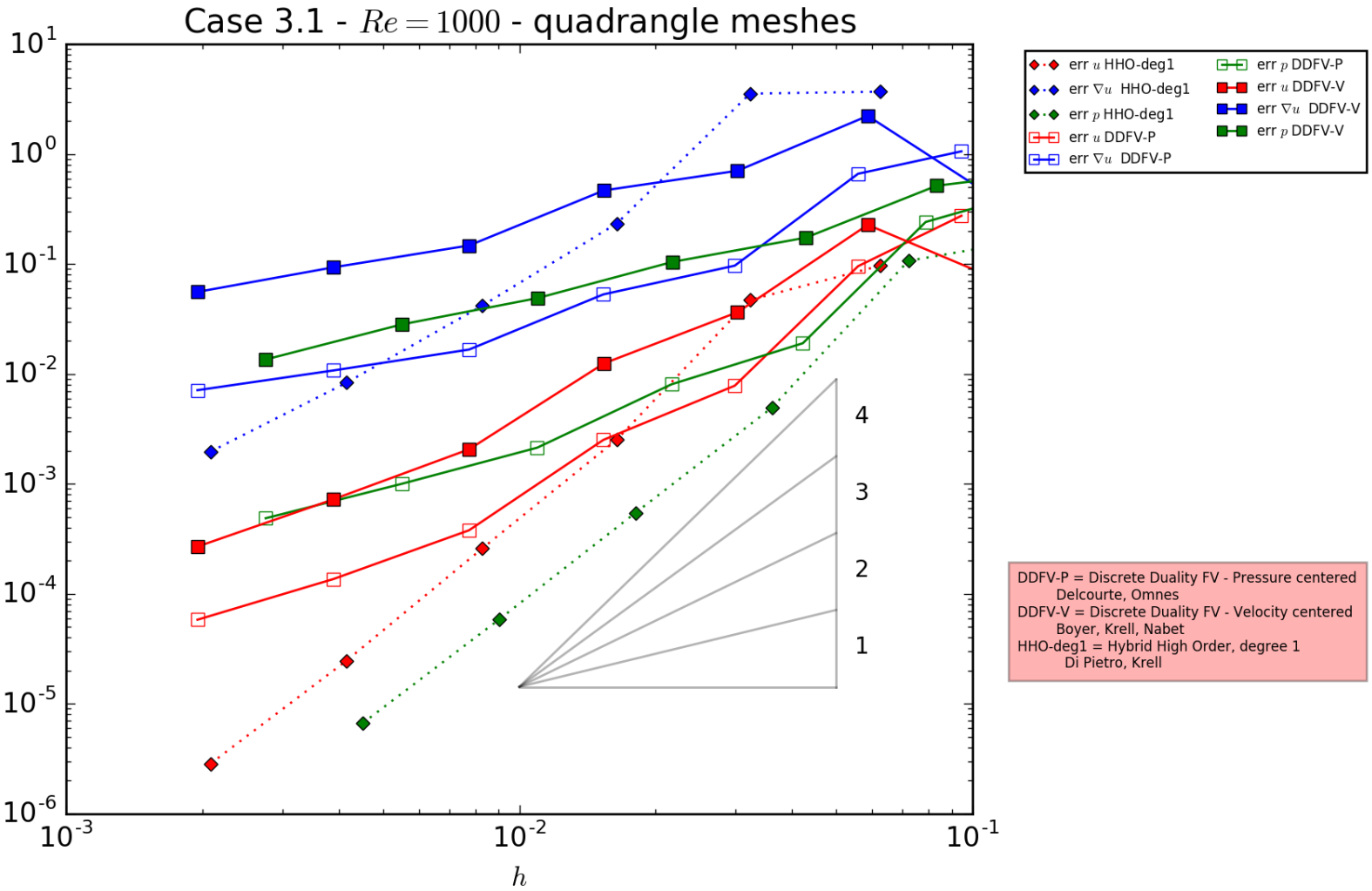
$$p_{\text{ex}} = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}$$

Viscosity: $\nu = 10^{-1}$, $\nu = 10^{-2}$ and $\nu = 10^{-3}$.

2D steady Navier-Stokes tests and robustness with respect to low viscosity values



2D steady Navier-Stokes tests and robustness with respect to low viscosity values



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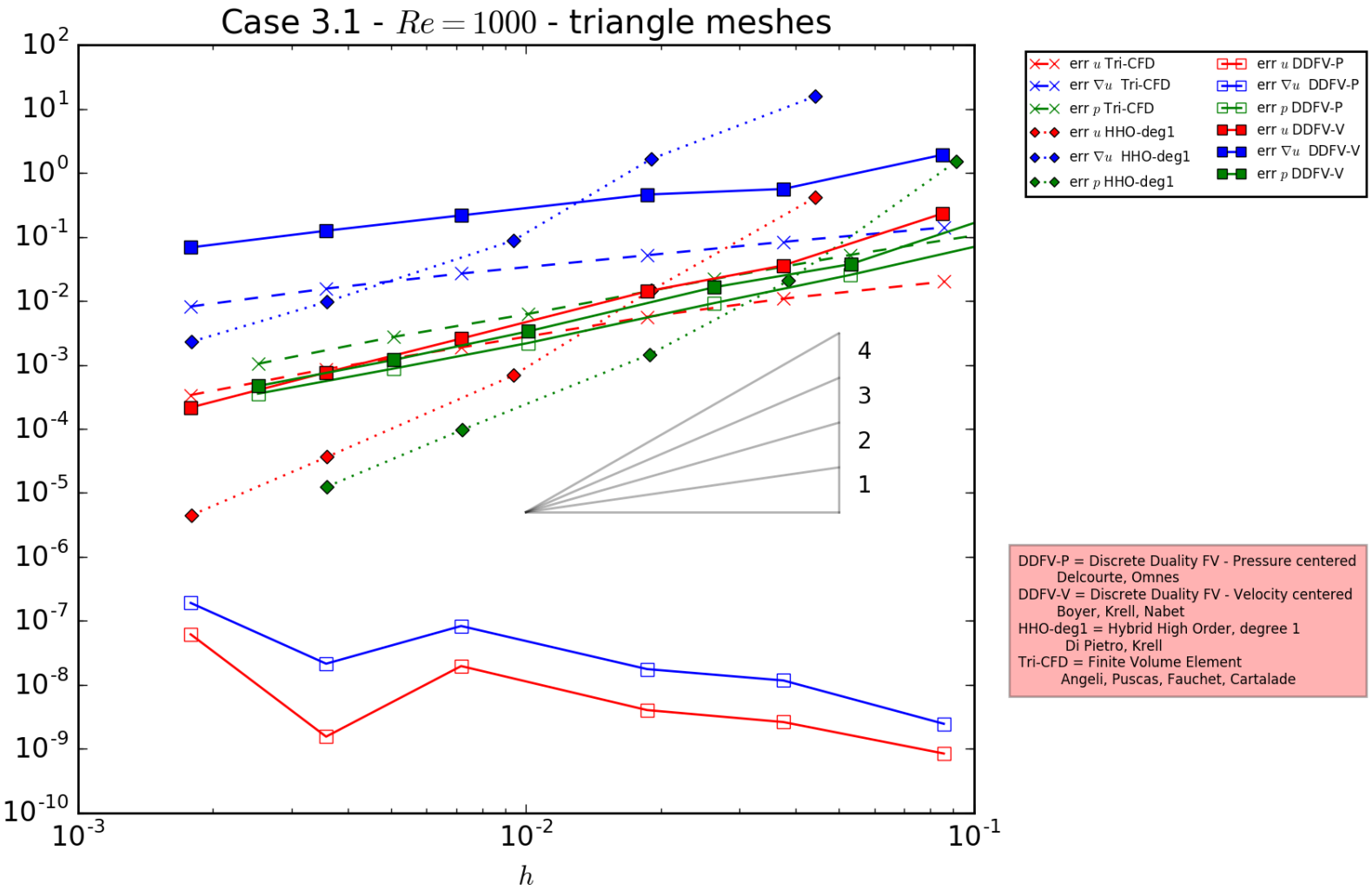
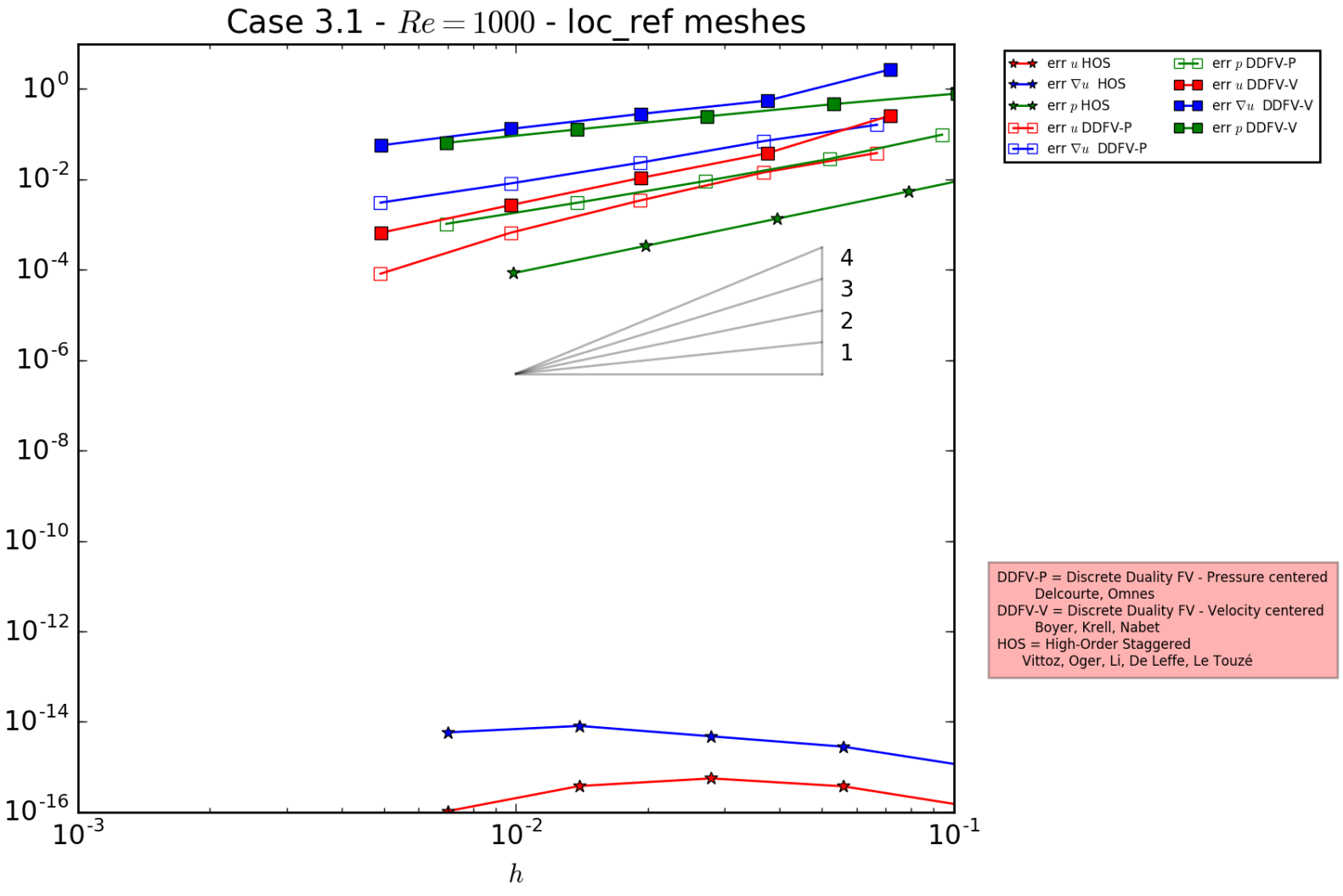


2D steady Navier-Stokes tests and robustness with respect to low viscosity values



3D steady Navier-Stokes tests and robustness with respect to low viscosity values

Simple vortex balanced by the pressure gradient.

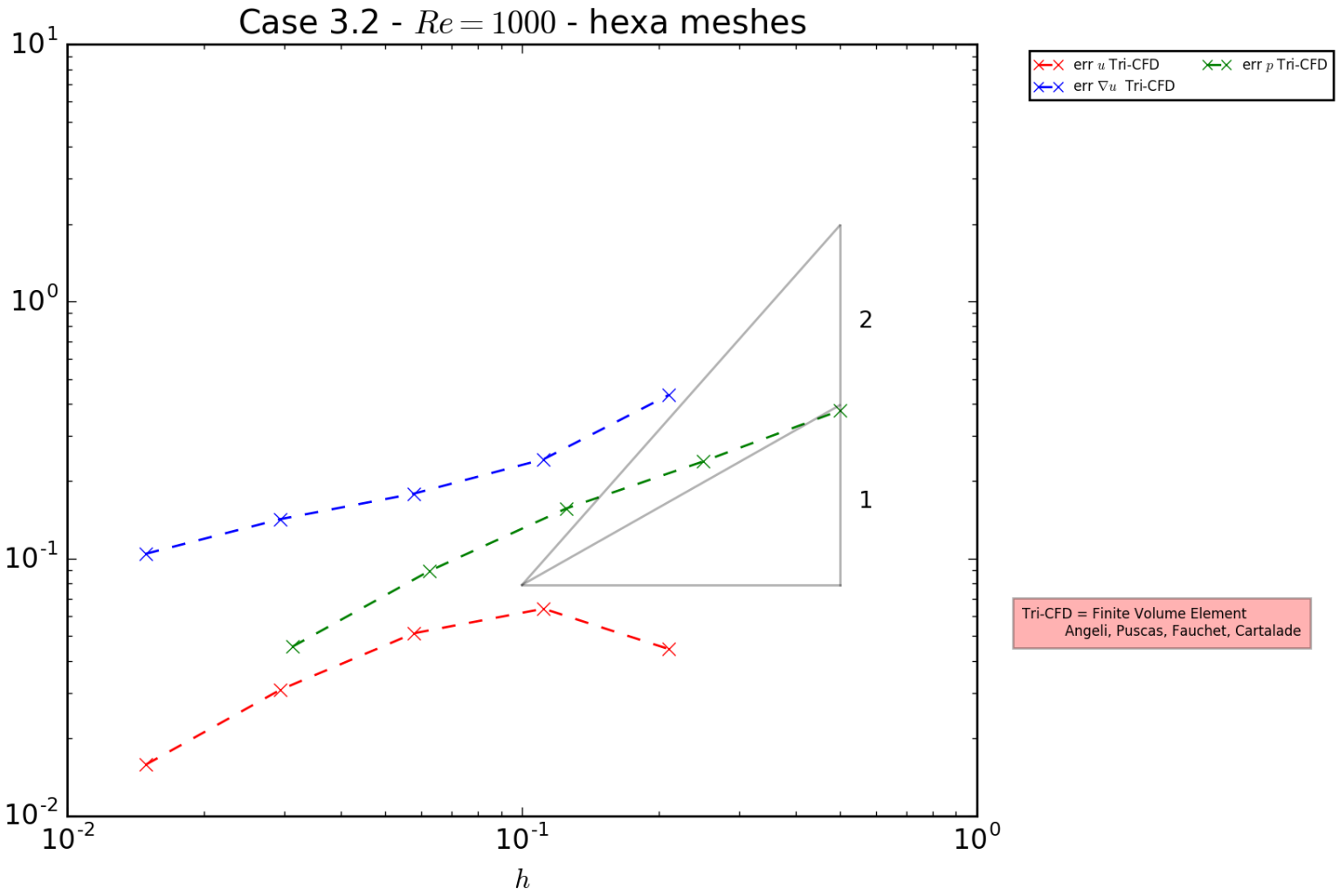
Exact solution:

$$\boldsymbol{u}_{\mathrm{ex}} = (y - z, z - x, x - y)^T,$$

$$p_{\mathrm{ex}} = (x^2 + y^2 + z^2) - xy - xz - yz - \frac{1}{4}.$$

Viscosity: $\nu = 10^{-1}$, $\nu = 10^{-2}$ and $\nu = 10^{-3}$.

3D steady Navier-Stokes tests and robustness with respect to low viscosity values



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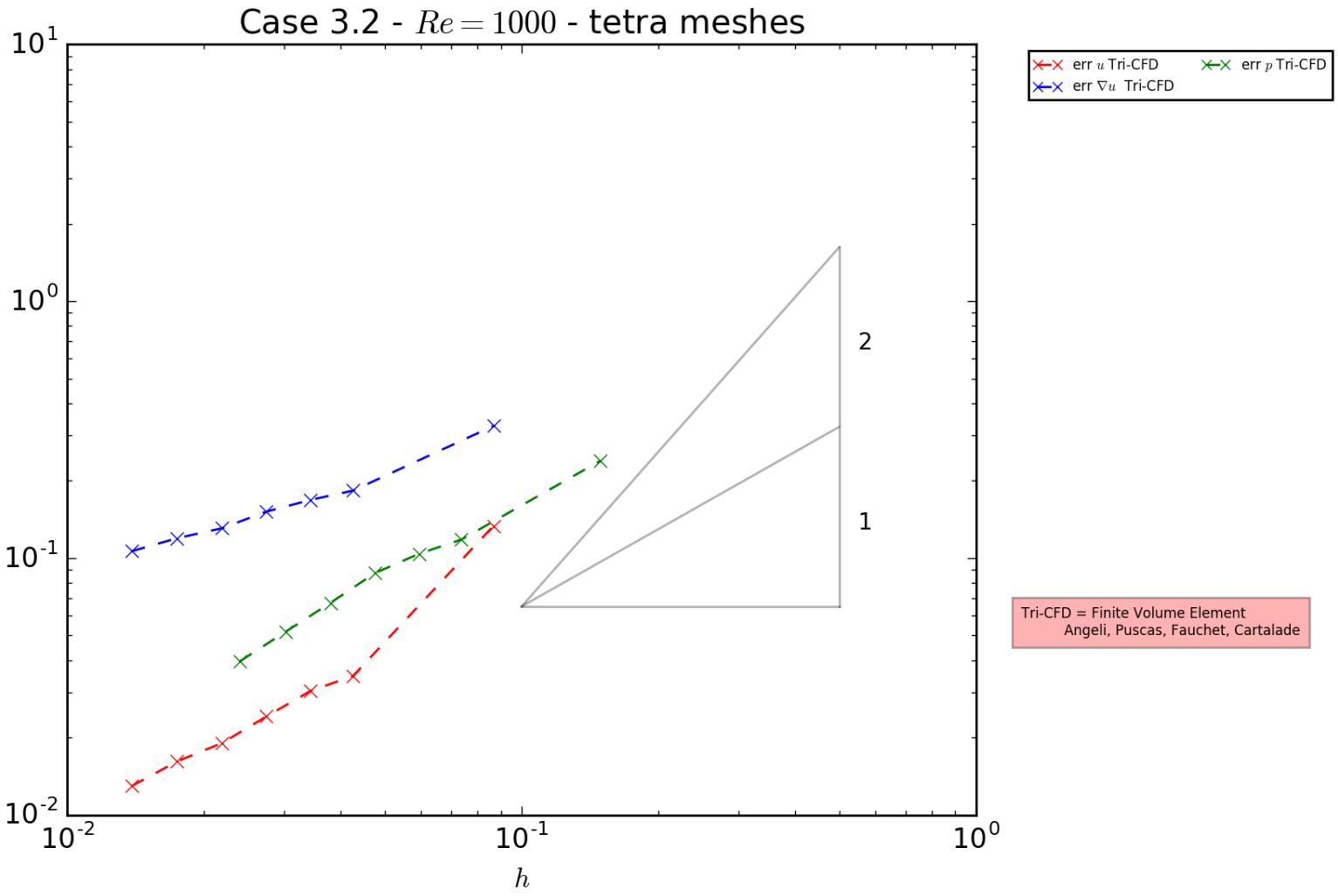


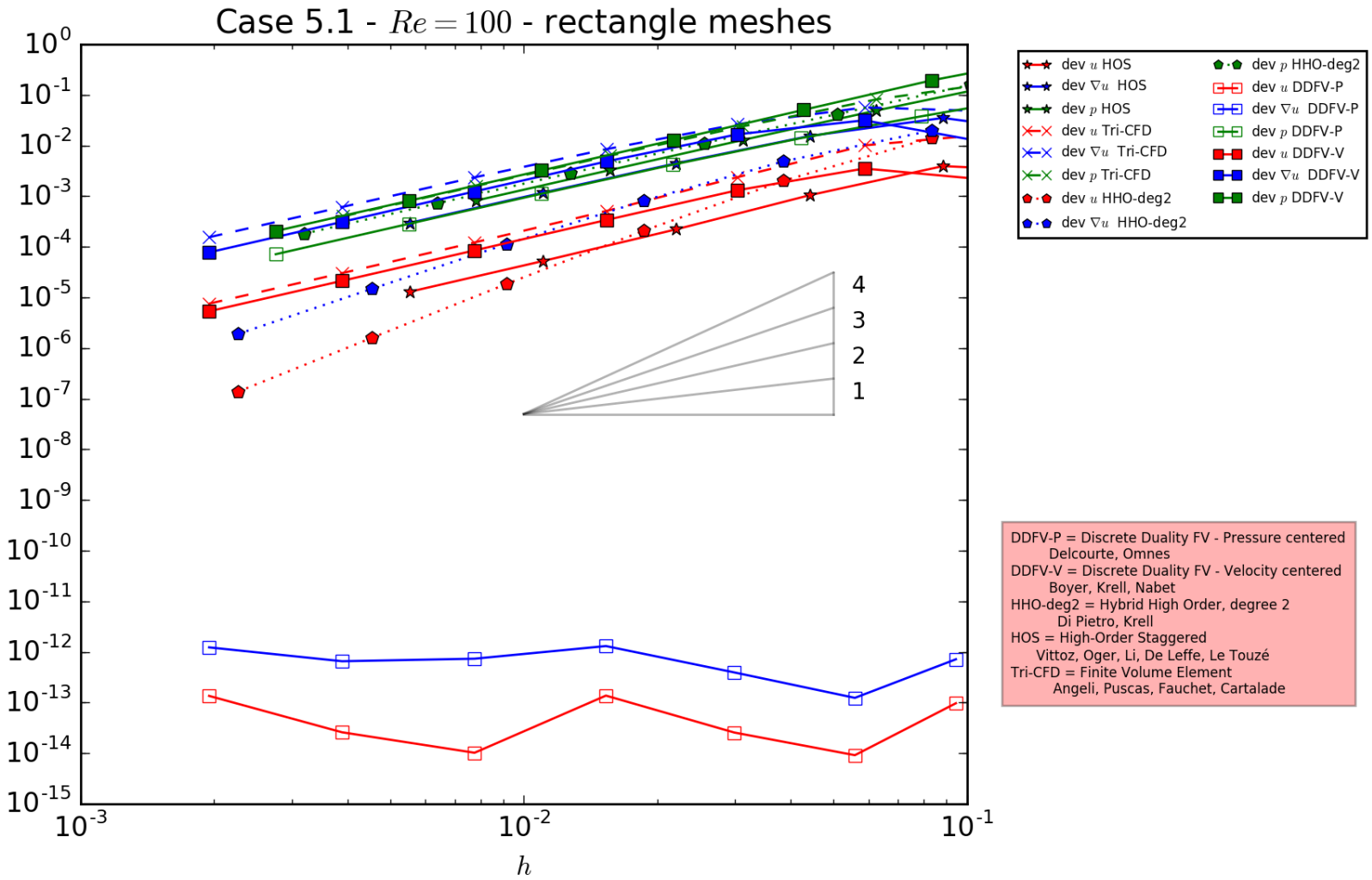
2D steady Stokes and Navier-Stokes robustness with respect to the invariance property

If $(\boldsymbol{u}, p)$ is solution of the equations with right-hand side $\mathbf{f}$, then $(\boldsymbol{u}, p + \psi)$ is solution of the equations with right-hand side $\mathbf{f} + \nabla\psi$.

Case 5.1 - Stokes : reference : $\boldsymbol{u} = 0$, $p = 0$, $\mathrm{Re} = 100$

Case 5.2 - Navier-Stokes : reference : lid-driven cavity $\mathrm{Re} = 400$

2D steady Stokes and Navier-Stokes robustness with respect to the invariance property



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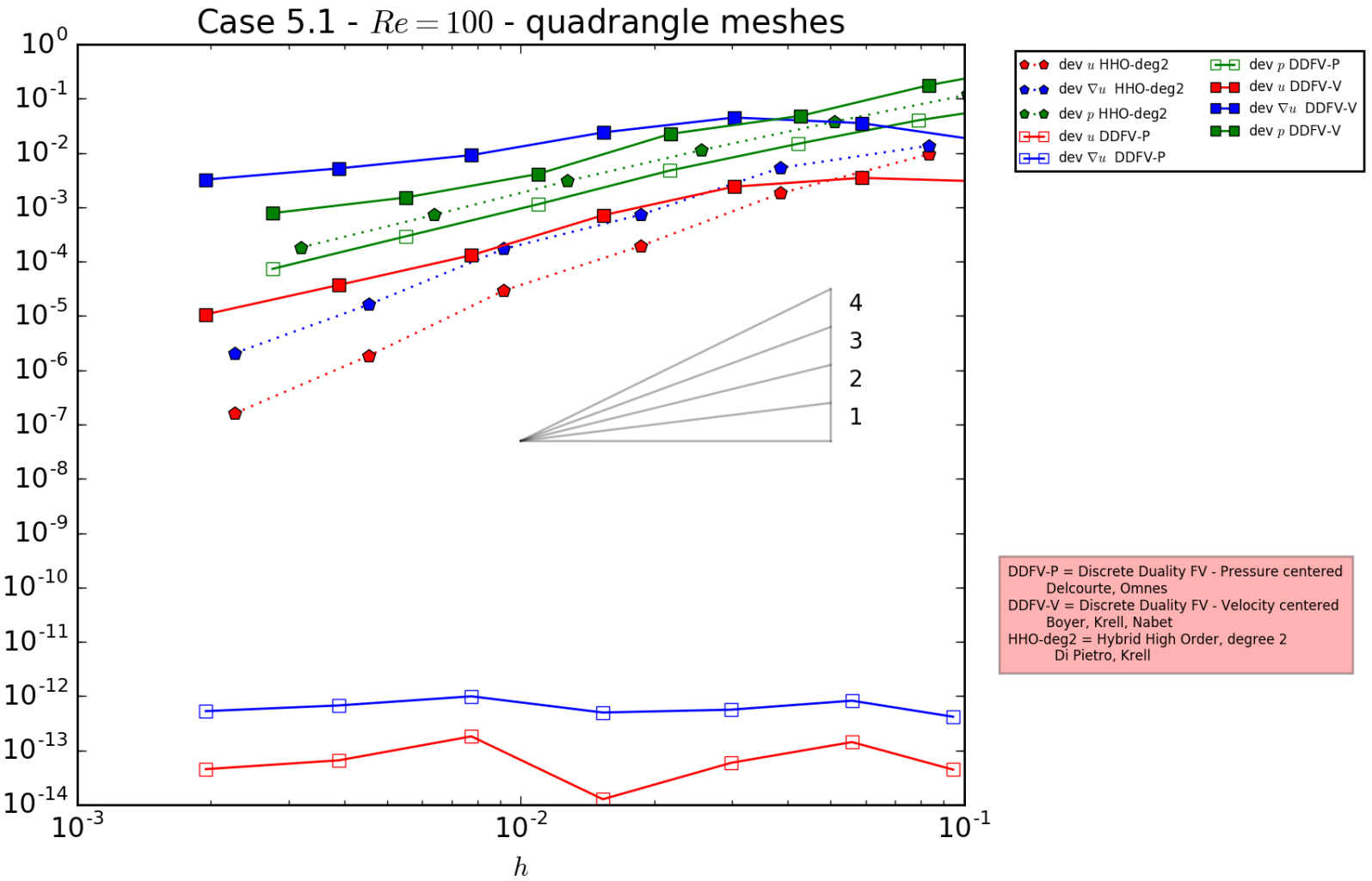

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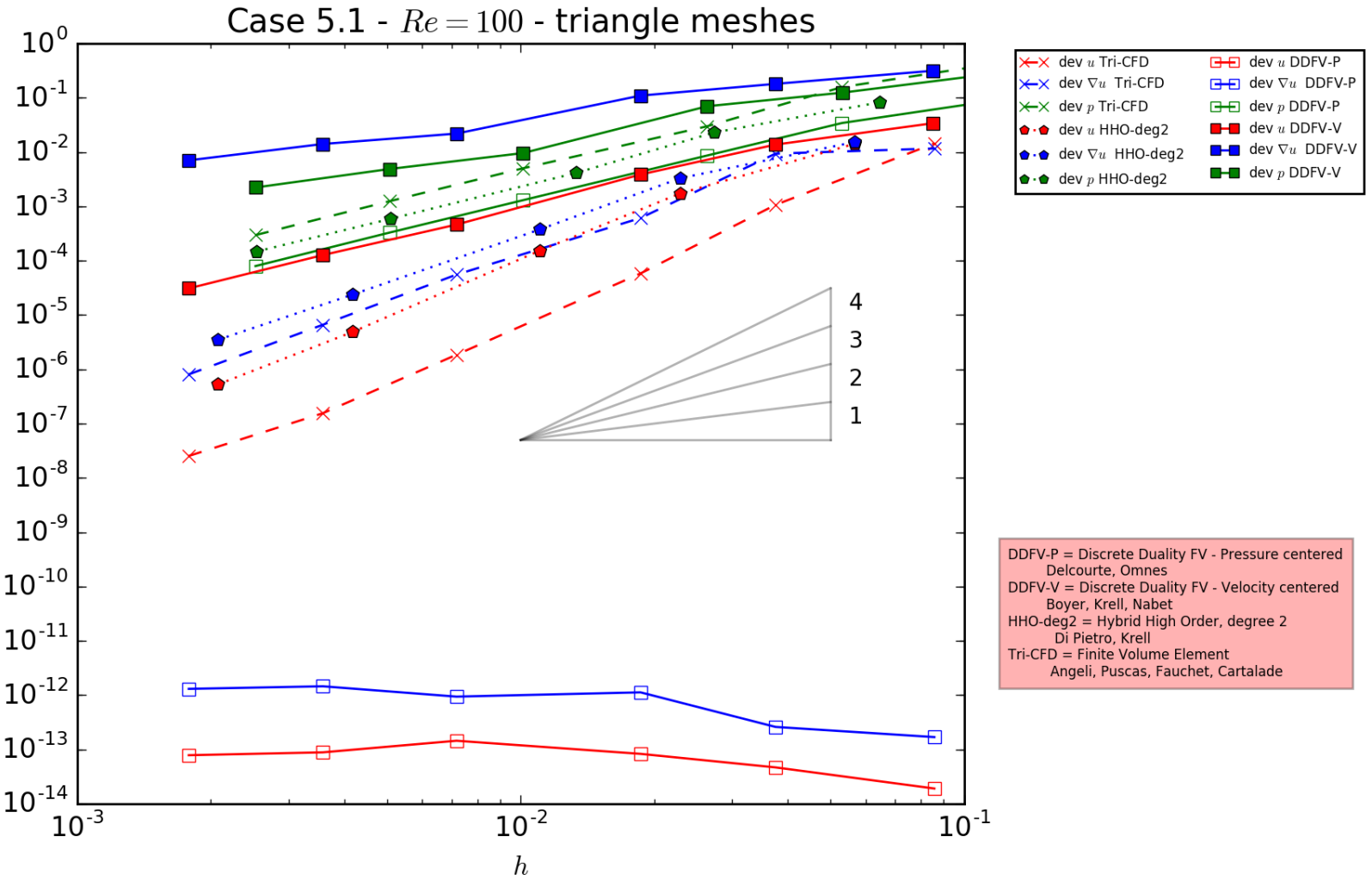

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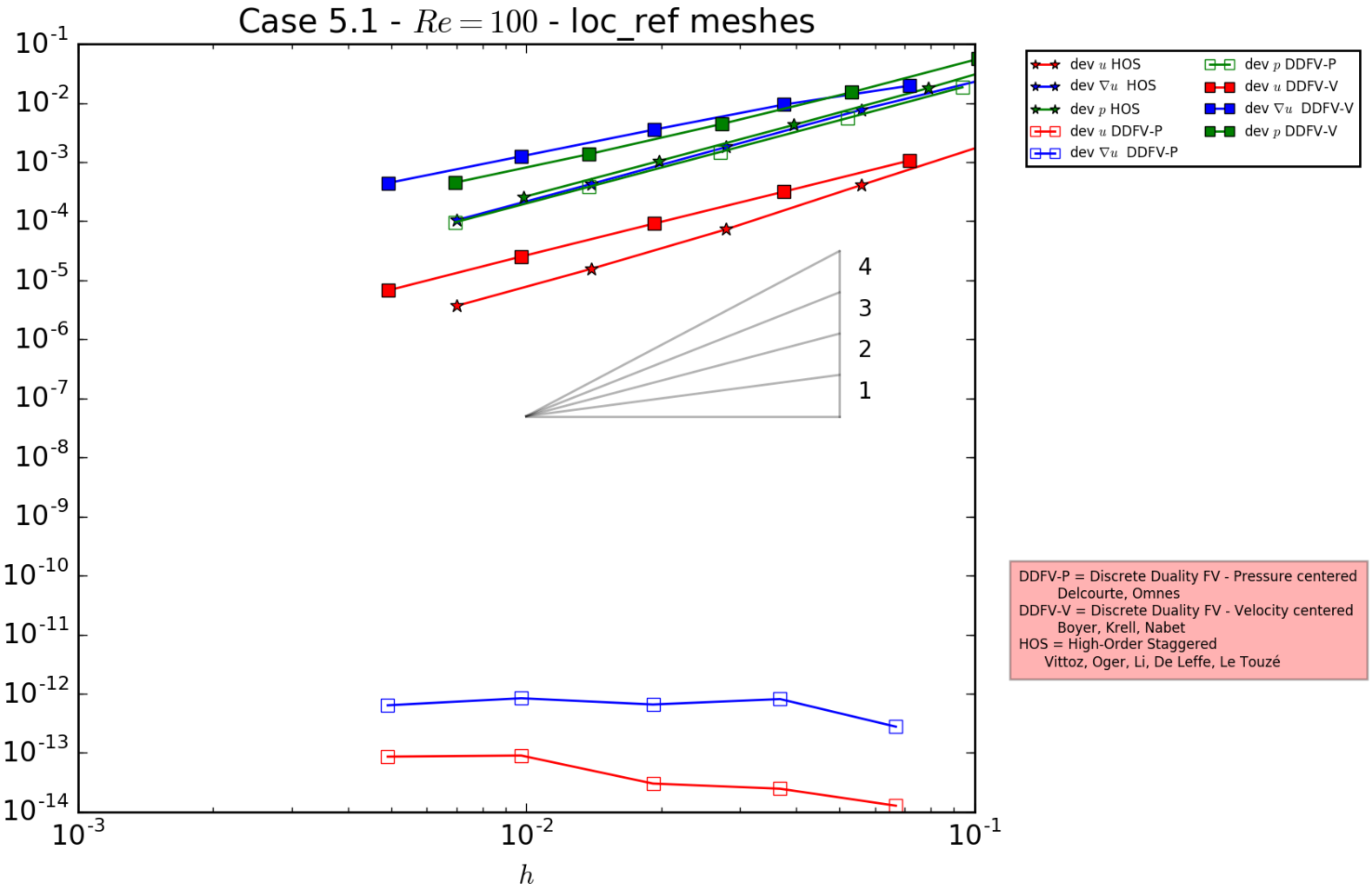


2D steady lid-driven cavity tests

Parameters

- Reynolds : 100, 400, 1000, 5000

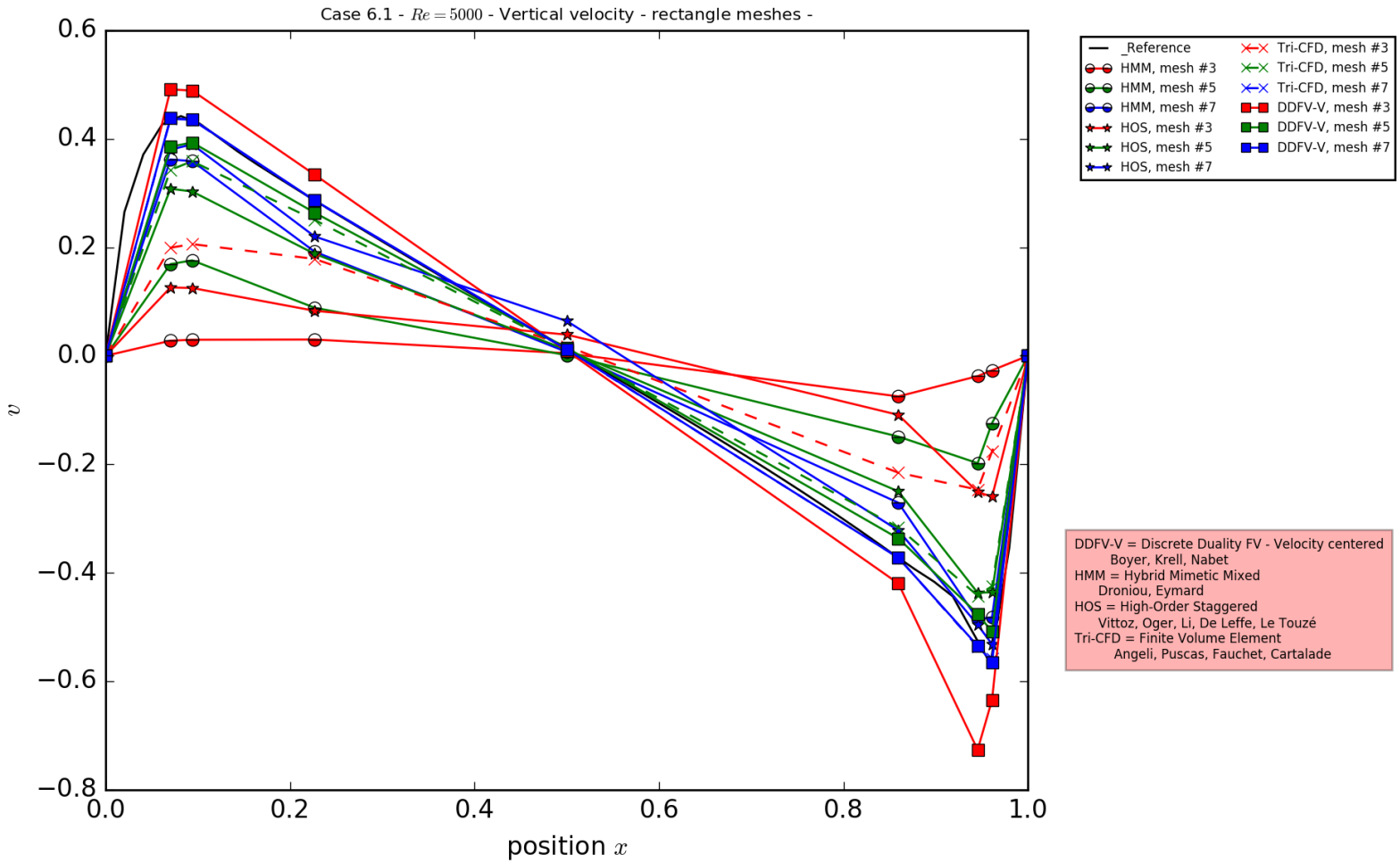
Outputs

- Velocity profiles on $\{x = 0.5\}$ and $\{y = 0.5\}$
- Primary and secondary vortices : position and amplitude

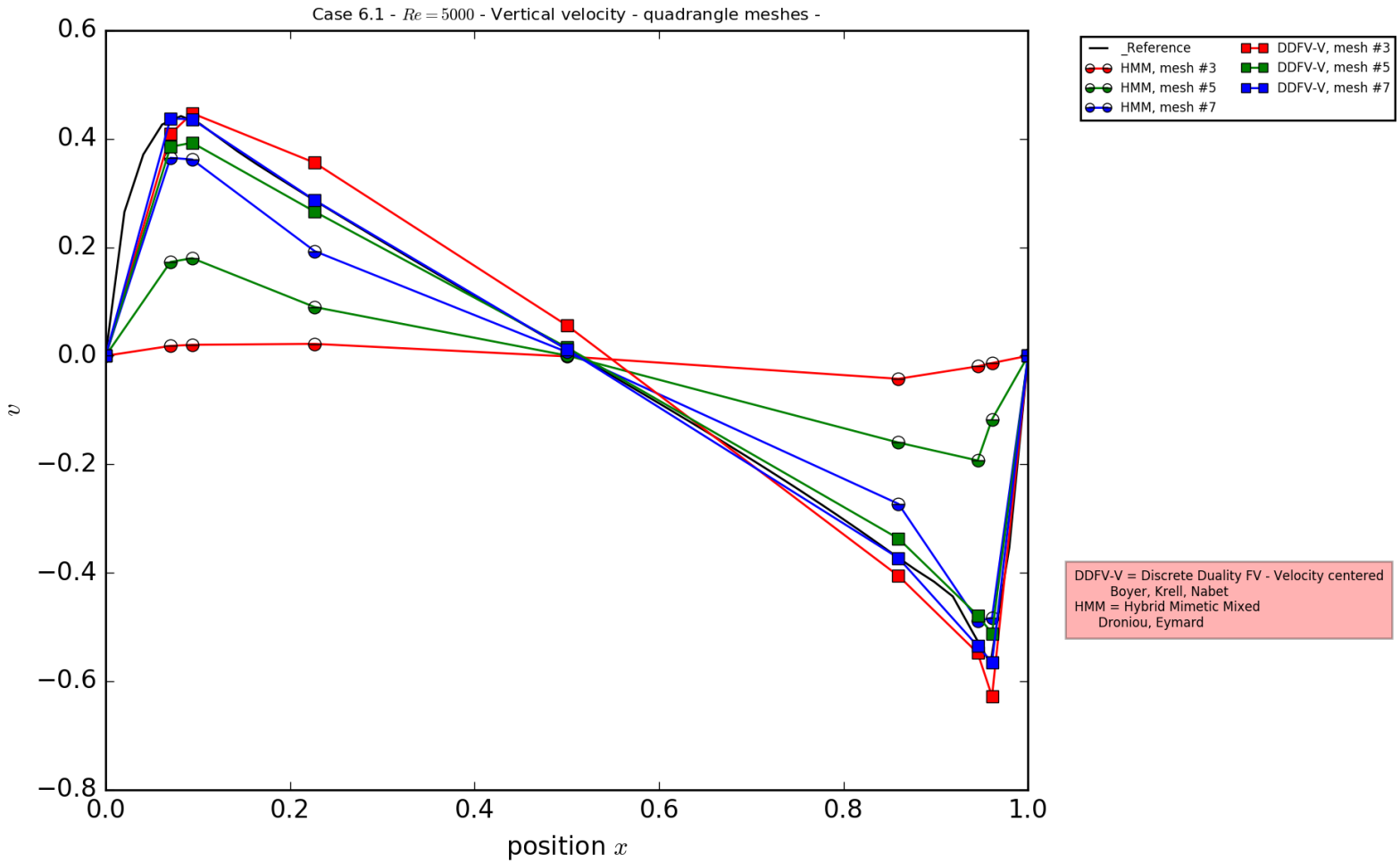
References

- Ghia, Ghia, Shin, 1982
- Bruneau, Saad, 2006

2D steady lid-driven cavity tests



2D steady lid-driven cavity tests



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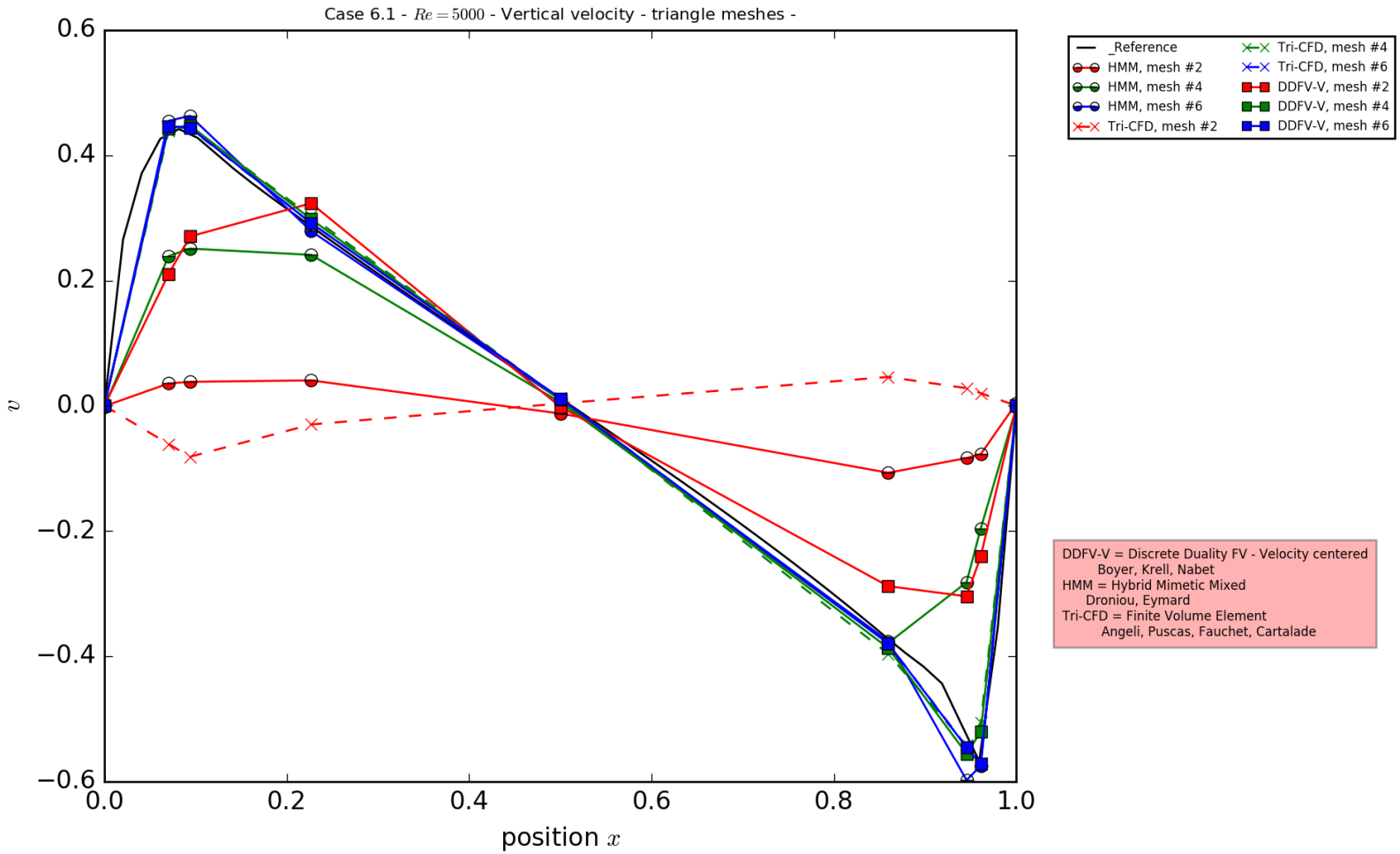

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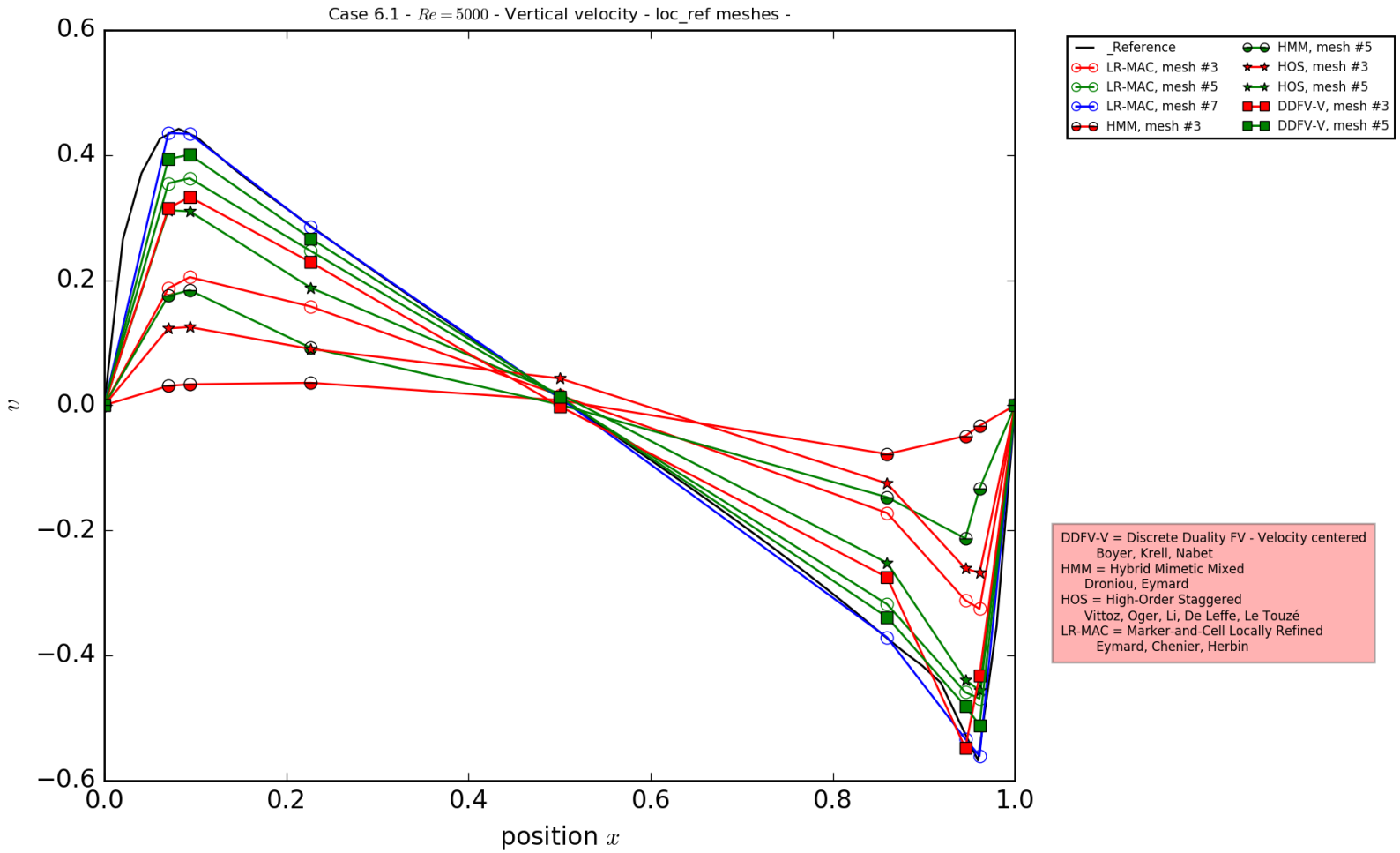

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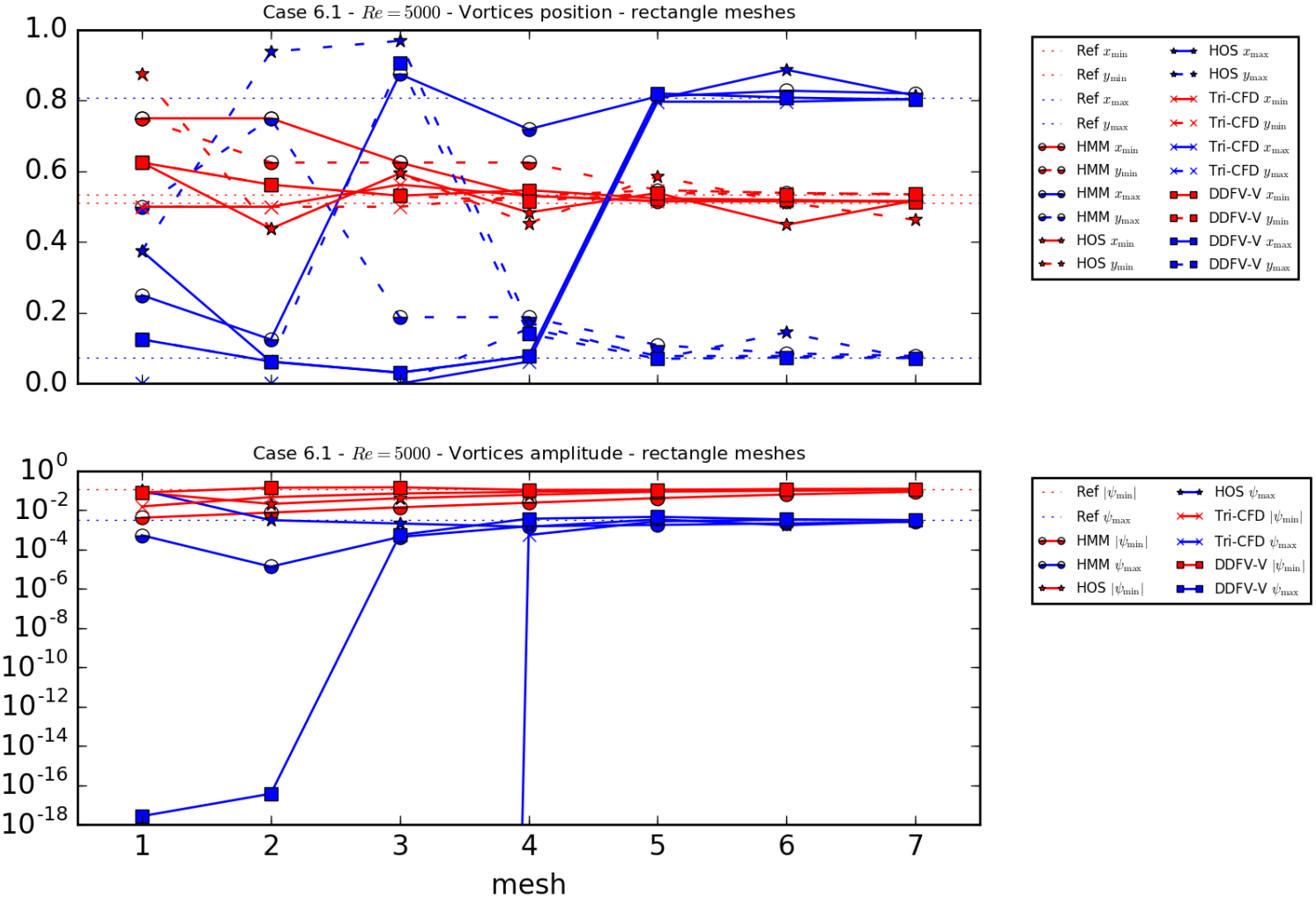

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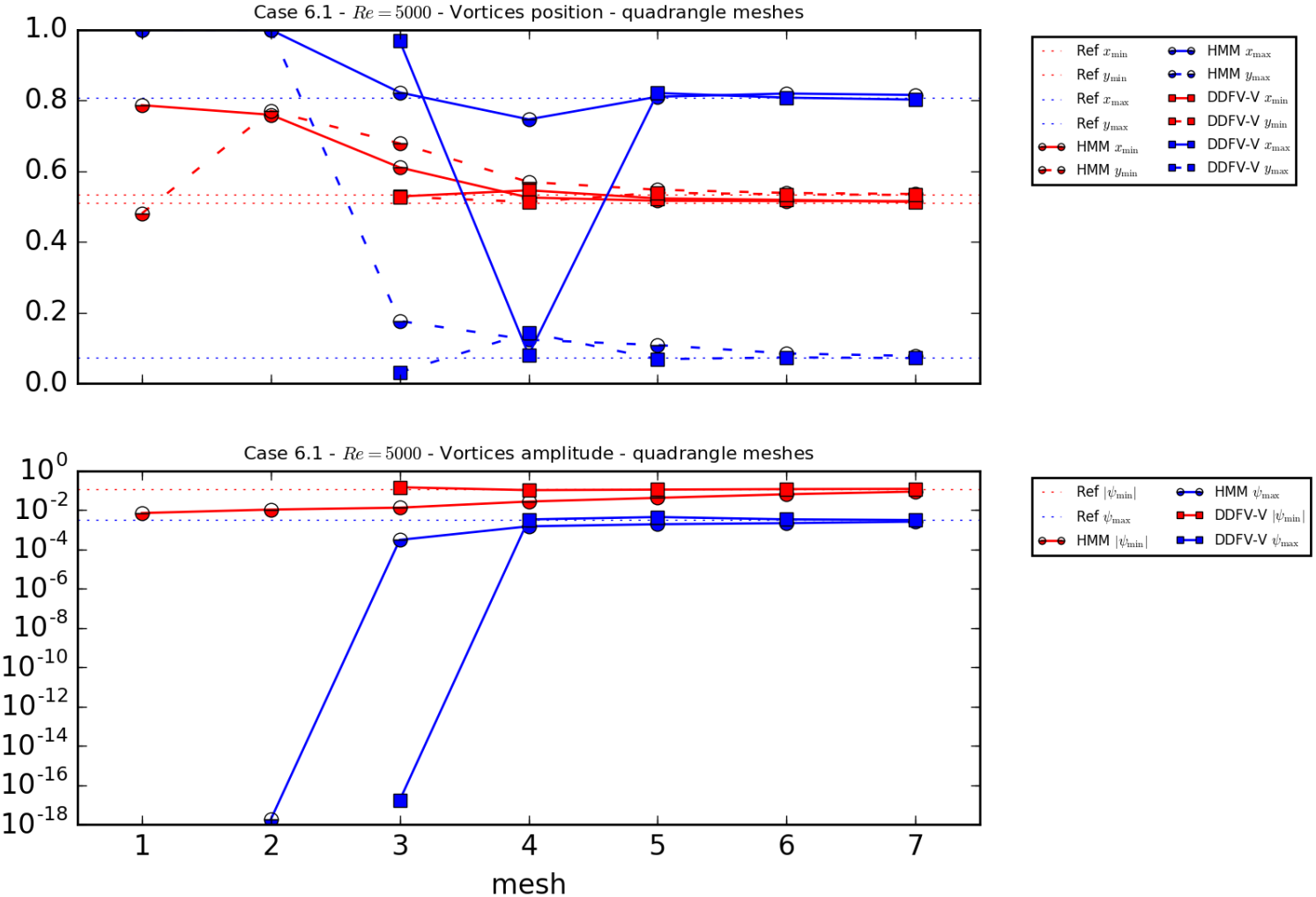
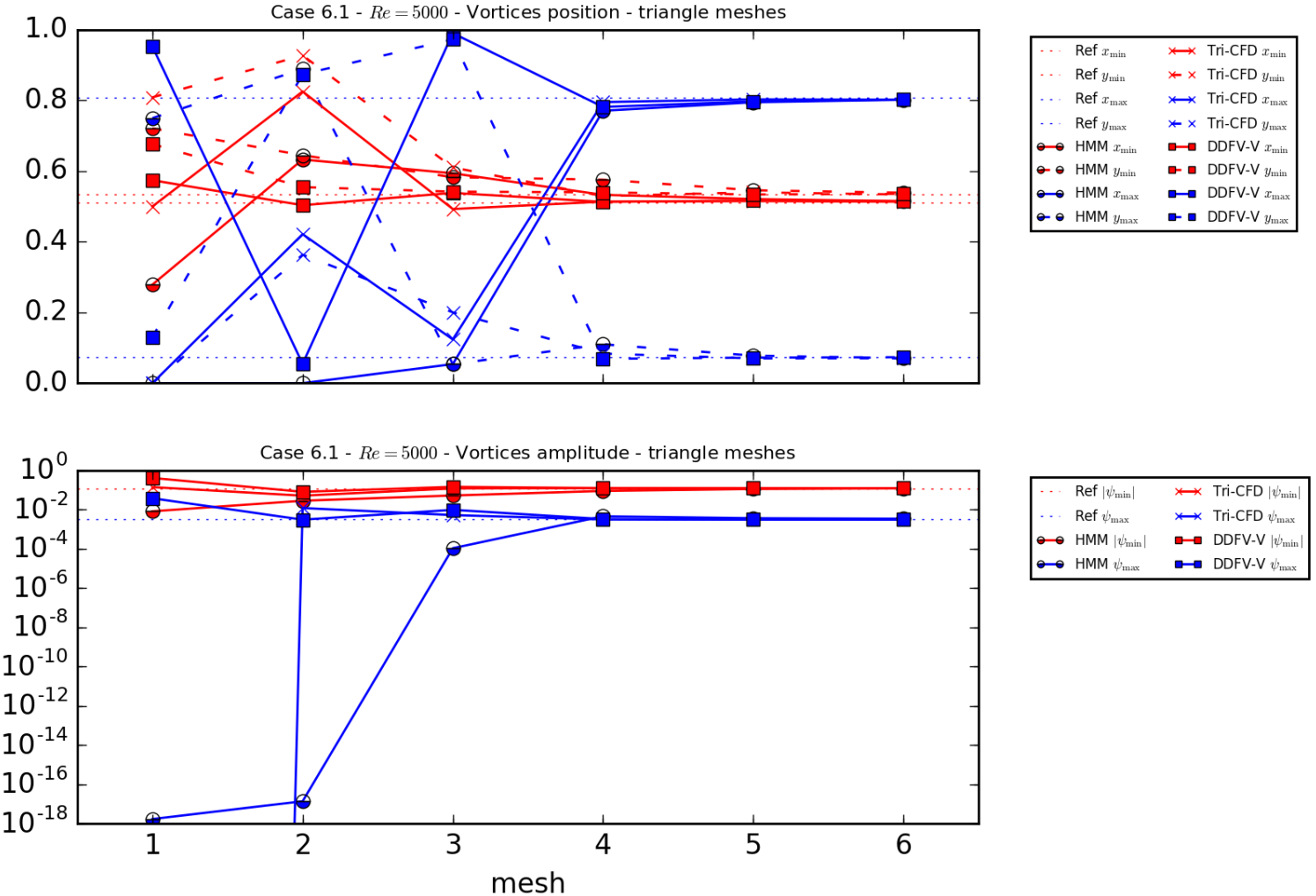
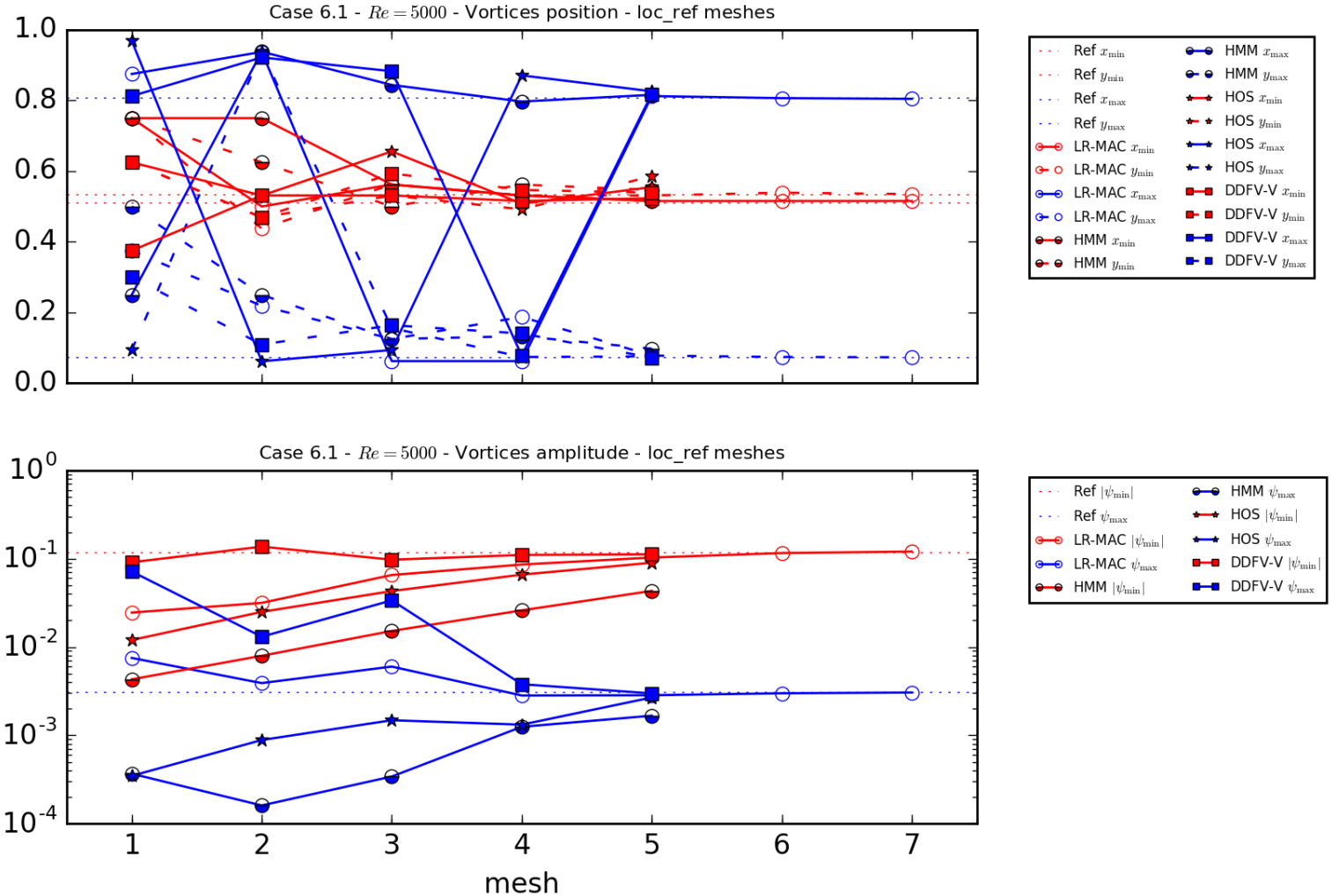


2D steady lid-driven cavity tests



2D steady lid-driven cavity tests



Poster presentations

And now ...

- R. Eymard
- J. Droniou
- L. Vittoz
- M.A. Puscas
- S. Krell
- P. Omnes
- F. Nabet

Then ... discussions !