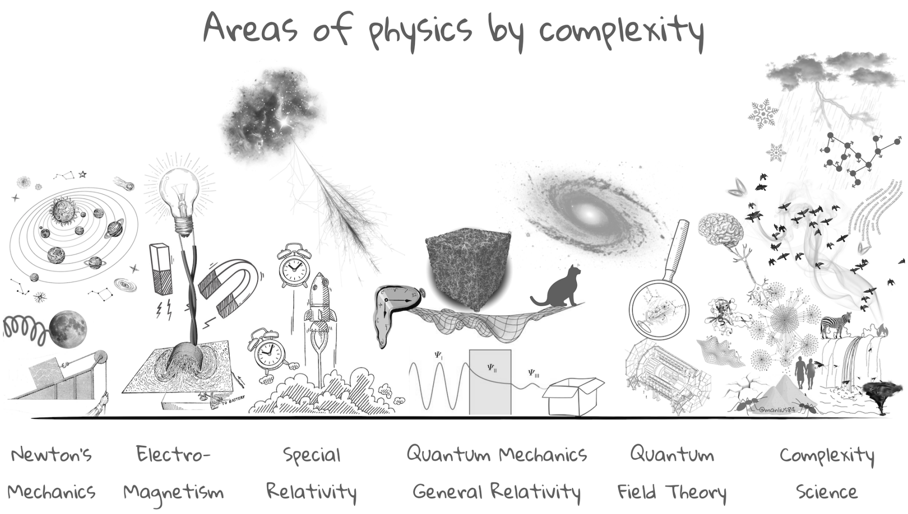
Final Report

Physics of Complex Networks: Structure and Dynamics

Università degli Studi di Padova



Project # X: ....

Aquistapace Tagua, Franco

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Contents

1 | Song-Havlin-Makse self-similar model

1.1 | Introduction and methods

The work by Song, Havlin and Makse [1] aims to explain the characteristic features of empirical scale–free fractal and non–fractal networks by means of a single growth model, that has the concept of renormalisation at its core. More specifically, the proposed growth mechanism works as the inverse of the renormalisation procedure. The goal of this task is then to reproduce their growth and renormalisation procedures, and to recover the characteristic behaviour of fractal and non–fractal networks through relevant descriptors.

When performing the growth process of a network, each existing node is considered as a future hub. For a node with degree k, mk offspring nodes are attached to it at each growth step. Then, each edge between two of the original nodes is removed with probability $p = 1 - e$, and replaced by an edge between two of their offspring. The parameter e then controls the hub–hub attraction/repulsion behaviour of the network. Mode I growth, given by $e \to 1$, leads to hub–hub attraction and a non–fractal topology. Meanwhile, Mode II growth, given by $e \to 0$, leads to hub–hub repulsion and a fractal network. This growth process is referred to as the minimal model. The renormalisation of a network is then done as described in the source material, by covering its N nodes with $N_B(L_B)$ boxes, where the maximum shortest path between any two nodes in a box must be L_B.

For each case $e = 1.0, 0.8$, a set of 10 graphs is generated with the minimal model. In all cases, the network is initialised as a star graph with $N_{init} = 5$ nodes. Then, the growth mechanism is applied $n_{growth} = 4$ times, with a growth factor of $m = 2$. Once obtained, each graph is then subjected to the normalisation procedure for distances $1 \leq L_B \leq 32$. Furthermore, for each minimal model graph an uncorrelated version is generated by random swapping of links, while maintaining the degree distribution. Additional details on the methodology can be found in section 4.1 of the Supplementary Material.

1.2 | Results and discussion

The correlation profile, $R(k_1, k_2)$, can be used to visualise the correlated topological structure of the minimal model, in comparison to its random uncorrelated counterpart.

Fig. 1.1 presents the $R(k_1, k_2)$ matrices for both cases $e = 1.0, 0.8$. It can be seen that $e = 1.0$ leads to hub–hub attraction, while hub–hub repulsion is observed for the case $e = 0.8$. This descriptor is however insufficient to define the fractality of a network, as it cannot describe how the topology of the network changes under renormalisation.

In order to complement this, the scaling relations $N_B(L_B)/N$ and $\mathcal{S}(L_B)$ are estimated for the minimal model graphs, where $\mathcal{S}(L_B)$ is the maximum degree scaling. The results of $N_B(L_B)/N$ and $\mathcal{S}(L_B)$ are presented in Fig. 1.2 with respect to L_B, for both cases $e = 1.0, 0.8$. As in the source material, an exponential decrease of both descriptors can be seen for the non–fractal topology resulting from the case $e = 1.0$. Meanwhile, the power–law behaviour of fractal networks is observed in both descriptors for the case $e = 0.8$. A finite–size effect can be observed for large values of L_B, more noticeably for the case $e = 1.0$, which appears as a constant value of the scaling relations.

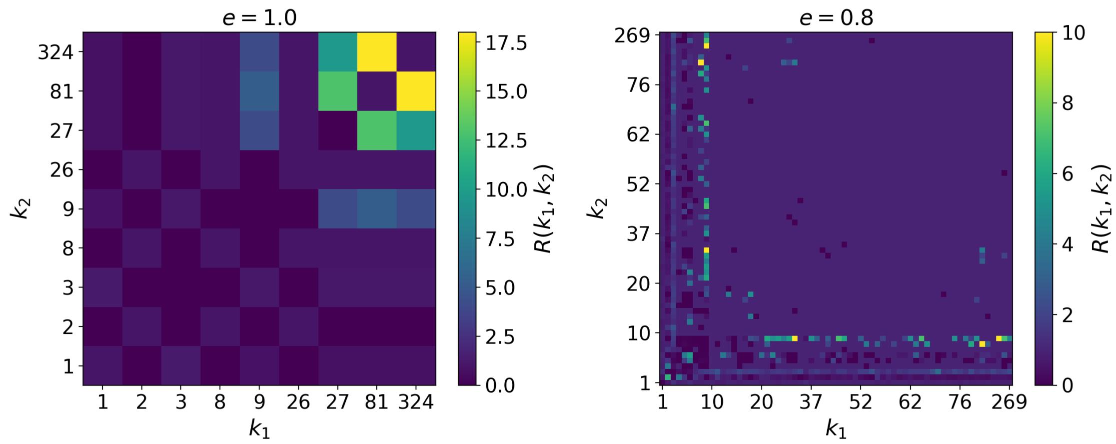

Figure 1.1: Correlation profile $R(k_1, k_2)$ of networks generated with the minimal model, for the cases $e = 1.0$ (left) and $e = 0.8$ (right).

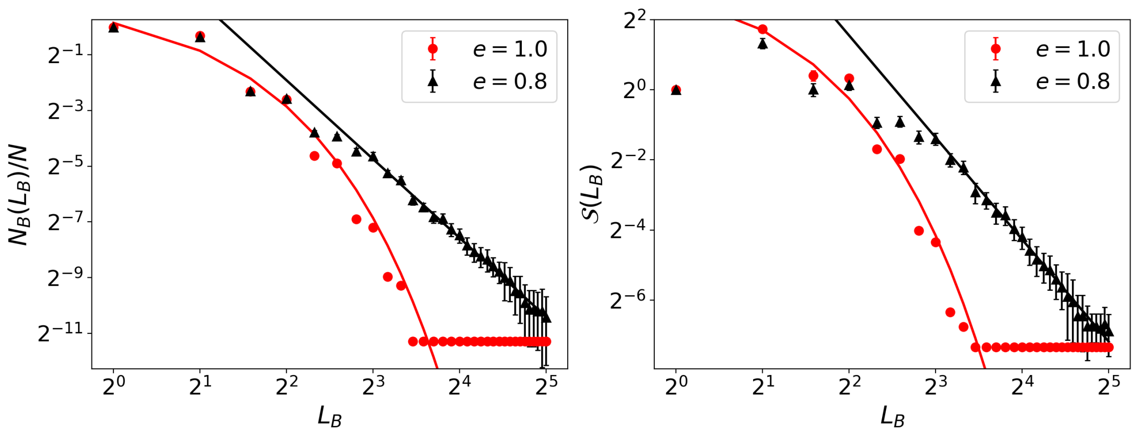

Figure 1.2: Scaling relations $N_B(L_B)/N$ (left) and $\mathcal{S}(L_B)$ (right) with respect to box distance L_B. Results are shown for networks generated with the minimal model, for the cases $e = 1.0$ and $e = 0.8$. The fitted theoretical models are shown as solid lines.

2 | Task title...

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2.1 | A section...

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3.1 | A section...

Reference examples:

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4 | Supplementary material

4.1 | Song-Havlin-Makse self-similar model

Algorithmic implementation of the renormalisation procedure

The renormalisation of a given network is performed algorithmically as follows:

1. Initialise a box with a randomly picked node.
2. For every remaining node i, add it to the box if $\max_{j \in B} d(i,j) \leq L_B$, where B contains the nodes currently in the box, and $d(i,j)$ is the shortest path between nodes i and j.
3. Repeat steps 1. and 2. with the remaining available nodes until all nodes are assigned to a box.

Correlation profiles

The correlation profile $R(k_1, k_2) = P(k_1, k_2)/P_r(k_1, k_2)$ is defined as the ratio between the joint degree–degree probability distribution of the network of interest, $P(k_1, k_2)$, and the joint degree–degree probability distribution of a randomised uncorrelated version of the same network, $P_r(k_1, k_2)$, where the links have been randomly rewired while preserving the degree distribution.

In this work, the joint degree–degree probability distribution is estimated in a frequentist approach, by considering all equivalent instances as observations. For example, to obtain $P(k_1, k_2)$ for the minimal model graph with $e = 1.0$, the degree–degree pairs of the 10 graphs generated for $e = 1.0$ are collected. Then, the joint probability for a given pair (k_1, k_2) is calculated as in Eq. 4.1:

$$P(k_1, k_2) = \frac{N_{obs}(k_1, k_2)}{N_{obs,tot}} \quad , \tag{4.1}$$

where $N_{obs}(k_1, k_2)$ is the number of occurrences of the pair (k_1, k_2), and $N_{obs,tot}$ is the total number of observations. The same methodology is applied for the calculation of $P_r(k_1, k_2)$. No interpolation is done for pairs (k'_1, k'_2) that are not observed, and their probability is thus taken as $P(k'_1, k'_2) = 0$.

Scaling relations

The descriptor $N_B(L_B)/N$ is defined as the ratio between the number of original nodes, N, and the number of boxes, N_B, obtained after performing a renormalisation step with box size L_B. Meanwhile, the descriptor $\mathcal{S}(L_B) = k_B(L_B)/k_{hub}$ is the ratio between the maximum degree of the boxes after a renormalisation step with box size L_B, $k_B(L_B)$, and the maximum degree of the among the original nodes, k_{hub}. For both scaling relations, $N_B(L_B)/N$ and $\mathcal{S}(L_B)$, the mean value and standard deviation is obtained for each L_B from the 10 networks generated for both cases $e = 1.0, 0.8$. From the obtained results, the theoretical models proposed in the source material [1] are fitted.

For the fractal case $e = 0.8$, $N_B(L_B)/N$ and $\mathcal{S}(L_B)$ are described, respectively, by Eqs. 4.2 and 4.3:

$$N_B(L_B)/N = C_N(L_B + L_0)^{-d_B} \tag{4.2}$$

$$\mathcal{S}(L_B) = C_S(L_B + L_0)^{-d_k} \tag{4.3}$$

where L_0, d_B, d_k, C_N and C_S are parameters fitted from the data. The data points with $L_B < 8$ were excluded from the fit of $N_B(L_B)/N$, and those with $L_B < 12$ were excluded from the fit of $\mathcal{S}(L_B)$. From the fit, the values $d_B = 2.83$ and $d_k = 2.91$ are obtained.

On the other hand, for the non–fractal case $e = 1.0$, $N_B(L_B)/N$ and $\mathcal{S}(L_B)$ are described, respectively, by Eqs. 4.4 and 4.5:

$$N_B(L_B)/N = C_N \exp(-L_B/L_0) \tag{4.4}$$

$$\mathcal{S}(L_B) = C_S \exp(-L_B/L_0) \tag{4.5}$$

where again L_0, C_N and C_S are fitted from the data. In this case, the data points with $L_B > 12$ are excluded from the fit of both $N_B(L_B)/N$ and $\mathcal{S}(L_B)$.

4.2 | Another section...

5 | Bibliography

[1] Chaoming Song, Shlomo Havlin, and Hernán A Makse. Origins of fractality in the growth of complex networks. *Nature physics*, 2(4):275–281, 2006.