E40201 Problem Set 1

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Problem 1

Part 1

Let Xi with i = 1, ...,N be a sequence of independent type 1 extreme value random variables with location parameter μ_i and scale parameter $\sigma > 0$ $(T1EV(\mu_i, \sigma))$. The c.d.f is given by:

$$Pr(X_i \leq x|\mu_i, \sigma) = \exp(-\exp(-\frac{x - \mu_i}{\sigma}))$$

Derive the distribution of $Y = \max_i\{X_i\}$

Proof. We know that $Y = \max_i\{X_i\}$ only if $Y \in set(X)$ and $Y > x \forall x \in set(X)$. So, the probability of Y is $P(Y \leq x) = P(X_1 \leq x, \ldots, X_n \leq x|\mu_1, \mu_2, \ldots \mu_n, \sigma) = \prod_{i=1}^{n} P(X_i \leq x) = \prod_{i=1}^{n} \exp(-\exp(-\frac{x-\mu_i}{\sigma})) = \exp(-\sum_{i=1}^{n}(\exp(-\frac{x-\mu_i}{\sigma})))$ where the last two equalities holds because X's are iid.

We can further simplify this equation as follows:

$$\begin{aligned}
\exp(-\sum_{i=1}^{n}(\exp(-\frac{x-\mu_i}{\sigma}))) &= \exp\left[-\exp(-\frac{x}{\sigma})\sum_{i=1}^{n}\left(\exp(\frac{\mu_i}{\sigma})\right)\right] \\
&= \exp\left[-exp\left(-\frac{x-\mu}{\sigma}\right)\right] \\
&= F_{\mu,\sigma}(x)
\end{aligned}$$

where we let $\frac{\mu}{\sigma} = ln\left[\sum_{i=1}^{n}\exp\left(\frac{\mu_i}{\sigma}\right)\right]$, and $F_{\mu,\sigma}$ the c.d.f of a T1EV distribution with parameters μ and σ.

□

Part 2

Let X and Y two independent T1EV random variables with location parameters μ_x and μ_y respectively and common scale parameter $\sigma > 0$. Derive the distribution of X − Y.

Proof. First we note that the pdf of the T1EV distribution with parameter a is given by

$$f_a(u) = \exp(a - u)F_a(u)$$

Thus,

$$\begin{aligned}
Pr(X-Y\le u) &= Pr(X\le Y+u)\\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{y+u} f_{\mu_x,\sigma}(x)dx\right] f_{\mu_y,\sigma}(y)dy\\
&= \int_{-\infty}^{\infty} f_{\mu_y,\sigma}(y)F_{\mu_x,\sigma}(y+u)dy\\
&= \int_{-\infty}^{\infty} \exp[-(y-\frac{\mu_y}{\sigma})]F_{\mu_y,\sigma}(y)F_{\mu_x,\sigma}(y+u)dy\\
&= \int_{-\infty}^{\infty} \exp[-(y-\frac{\mu_y}{\sigma})]\exp[-\exp[-(y-\frac{\mu_y}{\sigma})]]\exp[-\exp[-(y+u-\frac{\mu_x}{\sigma})]]dy\\
&= \int_{-\infty}^{\infty} \exp\Big[-(y-\frac{\mu_y}{\sigma})]\exp[-\exp[-(y-\frac{\mu_y}{\sigma})]\{1+\exp[-(u+\frac{y-x}{\sigma})]\}\Big]dy
\end{aligned}$$

Denote $a=\{1+\exp[-(u+\frac{y-x}{\sigma})]\}$, the above equation can be simplified as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\big[-(y-\frac{\mu_y}{\sigma})]\exp[-a\exp[-(y-\frac{\mu_y}{\sigma})]\big]dy\\
&= \frac{1}{a}\int_{-\infty}^{\infty} a\exp[-(y-\frac{\mu_y}{\sigma})]\exp[-\exp[-(y-\frac{\mu_y}{\sigma})]]ady\\
&= \frac{1}{a}\\
&= \frac{1}{1+\exp[-(u+\frac{y-x}{\sigma})]}\\
&= \frac{\exp[u+\frac{y-x}{\sigma}]}{1+\exp[u+\frac{y-x}{\sigma}]}
\end{aligned}$$

Note that the second line, within the integral sign is the pdf of a T1EV distribution with parameter $\frac{1}{a}$. Thus it integrates to 1. The last line is the pdf of a logistic regression with parameter $-\frac{y-x}{\sigma}$.

Part 3

Note that

$$\begin{aligned}
Pr(\text{chooses j}) &= Pr(u_j > u_i \quad \forall i\neq j)\\
&= Pr(\mu_j+\epsilon_j > \max_{i\neq j}(\mu_i+\epsilon_i))
\end{aligned}$$

We assume that errors are independent and identically distributed. For simplicity, denote $u_{-j} = \max_{i\neq j}\mu_i+\epsilon_i$. Let $\mu_{-j} = ln[\sum_{i\neq j}\exp(\mu_i)]$. Since ϵ is distributed $T1EV(0,1)$, by part 1, we know that $u_{-j}\sim T1EV(\mu_{-j},1)$ and $u_j\sim T1EV(\mu_j,1)$. Thus,

$$\begin{aligned}
Pr(\text{chooses j}) &= Pr(u_j > u_{-j})\\
&= Pr(u_{-j}-u_j<0)\\
&= \frac{exp(\mu_j)}{exp(\mu_j)+exp(\mu_{-j})} \quad \text{by part 2}\\
&= \frac{exp(\mu_j)}{\sum_{i=1}^{N} exp(\mu_i)}
\end{aligned}$$

Part 4

(4-i)

From part 3, we have

$$\begin{aligned} Pr(\text{i chooses j}) &= \frac{\exp(\alpha y_i - \alpha p_i)}{\sum_{k\in J}\exp(\alpha y_k - \alpha p_k)} \\ &= \frac{\exp(-\alpha p_i)}{\sum_{k\in J}\exp(-\alpha p_k)} \end{aligned}$$

Since income does not enter $s_j(i)$, the partial derivative of $s_j(i)$ with respect to y_i is 0. Note that income enters consumer utility linearly and is common to all alternatives. The consumer's objective function is

$$\max_{k\in J}\{u_{i_k}\} = \max_{k\in J}\{u_{i_k} - \alpha y_i\}$$

The solution to the unconstrained utility maximization problem is invariant to location adjustments. We can make location adjustments to eliminate the income effect.

(4-ii)

Since the parameter α is common to all individuals, we are removing all structural heterogeneity over individuals in the utility function. Thus, we can apply the multivariate logit model in this case.

With a large number of consumers in this market, we may equate the individual choice probability with the market share. Thus, we have

$$s_j \approx \frac{\exp(-\alpha p_i)}{\sum_{k\in J}\exp(-\alpha p_k)}$$

The price elasticities are given by $\frac{\partial s_j p_k}{\partial p_k s_j}$. The own price elasticity is

$$\epsilon_{jj} = -\alpha p_j(1 - s_j)$$

and the cross-price elasticity is

$$\epsilon_{jk} = \alpha p_k s_k$$

The cross price elasticity does not seem reasonable to me because it only depends on the prices and market shares. It restricts the substitution pattern to the IIA property, which may not be realistic in lots of cases, such as the red bus blue bus case.

(4-iii)

Note that

$$\begin{aligned}
Pr(\text{i chooses j}) &= Pr(u_{ij} > u_{ik}) \quad \forall k \neq j \\
&= Pr(\alpha(y_i - p_j) + \beta_i x_j > \alpha(y_i - p_k) + \beta_i x_k) \quad \forall k \neq j \\
&= Pr(\alpha(p_j - p_k) > \beta_i(x_k - x_j)) \quad \forall k \neq j \\
&= Pr\left(\frac{p_j - p_k}{x_j - x_k} < \frac{\beta_i}{\alpha}\right) \quad \forall k \neq j \\
&= \prod_{k \neq j} Pr\left(\alpha \frac{p_j - p_k}{x_j - x_k} < \beta_i\right)
\end{aligned}$$

First we assume that β is common to all individuals. Note that the choice probability is now common to all individuals as well, so the market share is equal to the choice probability when the number of individuals is large. Thus, the market share is given by

$$s_j = \prod_{k \neq j} Pr\left(\alpha \frac{p_j - p_k}{x_j - x_k} < \beta\right)$$

which can be seen as the probability that taste shock is greater than price advantage weighted by quality differences for all products.

Next we assume that β_i is i.i.d uniform on $[0, \overline{\beta}]$. This is very similar to the specification of Bresnahan (1987). In this model, the only reason that different consumers make different choices comes from the taste shock β_i. Therefore, we want to compute the set of $\beta's$ that will induce the choice of each product and then integrate over its distribution on this region to get the market share. In the vertical differentiation model, the sets are defined by cutoffs in β. A consumer is willing to pay β_i for peoduct j if $\beta^*_{j+1} > \beta_i > \beta^*_j$, with products ranked from the lowest quality $j = 1$ to the highest quality and the cutoff β^*_j is defined as the indifference point for consumers. This implies that the market share for product j is

$$s_j = F(\beta^*_{j+1}) - F(\beta^*_j)$$

With the assumed distribution for β_i, we have the cross-price elasticity

$$\frac{\partial s_j}{\partial p_r} \begin{cases} \frac{1}{x_j - x_r} & r = j-1, j+1 \\ 0 & \text{otherwise} \end{cases}$$

This model is also restrictive but in another fashion: competition is highly localized and only applies to the product above or below it in terms of the "quality".

(4-iv)

The utility function becomes

$$u_{ij} = \alpha(y_i - p_j) + \beta_i x_j + v_{ij}$$

For individual i, the choice probability for product j is

$$Pr(\text{i chooses j}) = \frac{\exp(-\alpha p_i + \beta_i x_j)}{\sum_{k \in J} \exp(-\alpha p_k + \beta_i x_k)}$$

The market share of alternative j is given by the integration of individual choice probability over the taste shocks β. Thus, the market share is given by

$$\begin{aligned} s_j &\approx E[Pr(\text{i chooses j})] \\ &= \int_{\beta_i} Pr(\text{i chooses j}) dF(\beta_i) \\ &= \int_{\beta_i} \frac{\exp(-\alpha p_i + \beta_i x_j)}{\sum_{k\in J} \exp(-\alpha p_k + \beta_i x_k)} dF(\beta_i) \end{aligned}$$

The cross-price elasticity is given by

$$\frac{\partial s_j}{\partial p_k}\frac{p_k}{s_j} = \begin{cases} -\frac{p_j}{s_k}\int_{\beta_i} \alpha s_j(1-s_j) dF(\beta_i) & if j = k \\ \frac{p_k}{s_j}\int_{\beta_i} \alpha s_j s_k dF(\beta_i) & \text{otherwise} \end{cases}$$

Compared to (i), the IIA issue is solved by adding in taste shocks. However, it might be computational challenging to estimate the integral with respect to β because the c.d.f is generic.

(4-v)

Assume we are in the case of homogeneous preference again. We want to rewrite

$$W \equiv E\left[\max_{j=0,\dots,J} u_{ij}\right]$$

From part 1 and the assumptino that $\epsilon_{ij} \sim T1EV(0,1)$, we have

$$E\left[\max_{j=0,\dots,J} \alpha(y_i - p_j) + \epsilon_{ij}\right] = ln\left[\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))\right] + k$$

where k is the Euler-Mascheroni constant.

Note that we normalize everything of the outside goods to be zero, so that $E[u_{i,0}] = 0$. Thus, the probability that alternative 0 is chosen in this case is

$$\frac{\exp(0)}{\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))} = \frac{1}{\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))}$$

Therefore, the welfare function above can be written as $ln(\frac{1}{s_0})$.

When a new good $J+1$ is introduced, the change in welfare is given by

$$\begin{aligned} ln\left[\sum_{j=1}^{J+1} \exp(\alpha(y_i - p_j))\right] - ln\left[\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))\right] &= ln\left[\frac{\sum_{j=1}^{J+1} \exp(\alpha(y_i - p_j))}{\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))}\right] \\ &\geq ln\left[\frac{\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))}{\sum_{j=1}^{J} \exp(\alpha(y_i - p_j))}\right] \\ &= 0 \end{aligned}$$

Thus, with the introduction of the new goods, welfare unambiguously increases. This is because consumers now have one more option without compromising the original options, so their utility is weakly higher. □

Problem 2

Part 1

The coefficients in β capture how sensitive is consumer i over attributes of product j. It can be prices or any other characteristics.

Coefficients in Γ correspond to the sensitivity of the match effect between individual (characteristics) demographics and product attributes.

Part 2

$$\text{lik}(Y, X|\delta, \Gamma) = \prod_{i=1}^{N}\prod_{j=1}^{J}\left[\frac{\exp\{\delta_j + d_i\Gamma x_j + \epsilon_{ij}\}}{1+\sum_{k=1}^{J}\exp\{\delta_k + d_i\Gamma x_k + \epsilon_{ik}\}}\right]^{\mathbb{1}\{j=y_i\}}$$

Note that the log likelihood version of this is:

$$\text{loglik}(Y.X|\delta, \Gamma) = \sum_{i=1}^{N}\sum_{j=1}^{J}\mathbb{1}\{j=y_i\} \times \left(\delta_j + d_t\Gamma x_j + \epsilon_{ij}) - \log(1 + \sum_{k=1}^{J}\exp(\delta_k + d_i\Gamma x_k + \epsilon_{ik})\right)$$

Part 3

The F.O.C for δ_j is given by

$$\sum_{i=1}^{N}\left[1 - \frac{exp(\delta_j + d_i\Gamma x_k + \epsilon_{ik})}{1+\sum_{k=1}^{J}\exp\{\delta_k + d_i\Gamma x_k + \epsilon_{ik}\}}\right] = 0$$

At optimality, the people who choose this product j contribute to j's whole market share. In other words, people choose the product that maximizes their utility.

The F.O.C for Γ is given by

$$\sum_{i=1}^{N}\sum_{j=1}^{J}\left[d_t x_j - \frac{\sum_{k=1}^{J} d_i x_k \exp(\delta_j + d_i\Gamma x_k + \epsilon_{ik})}{1+\sum_{k=1}^{J}\exp\{\delta_k + d_i\Gamma x_k + \epsilon_{ik}\}}\right] = 0$$

At optimality, the interaction of individual i's demographics and characteristics of product j should be equal to the sum of the interaction of individual i's demographics and characteristics of all product, weighted by the latent utility from all products for this individual.

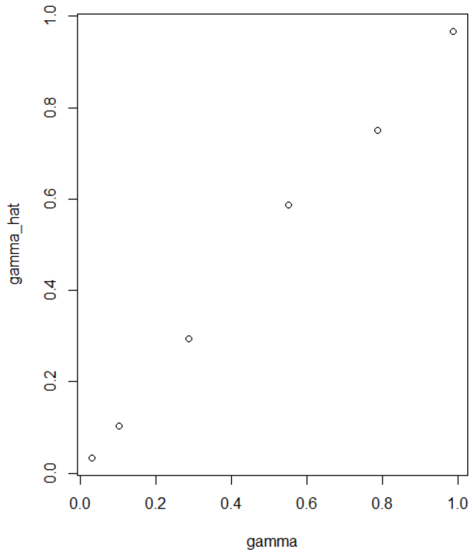

Figure 1: True Γ V.S. Estimated Γ

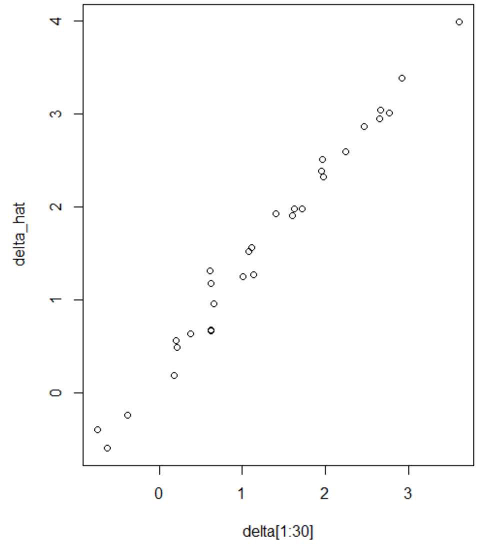

Figure 2: True δ V.S. Estimated δ

Part 4

To test the validity of our MLE estimation method, we first generate a fake data with known xi and ϵ terms. Then, we estimate each entry of Γ and δ using the observables (D's and x's). The graph below provides a straightforward illustration of the true parameter values and the estimated parameter values. We believe our estimates are pretty close to the true values. Thus, we have confidence on our MLE estimation.

Our estimated Γ is $\begin{bmatrix} 0.6864496 & 0.3003907 & 0.7147353 \\ 0.5008473 & 0.2629633 & 0.8223779 \end{bmatrix}$

Our estimated δ is [2.1057683, 1.1717959, 1.3349105, 0.8824899, 3.7209914, 3.0090236, -0.1282324, 2.9721350, 2.6495249, 1.9243542, 1.5680791, 3.5276537, 2.7965611, 0.4853798, 1.2921894, 2.4317173, 1.8596784, 2.4349638, 2.8349584, 0.3678059, 3.0319167, 2.7998421, 2.3694208, 4.4863552, 1.9696869, 2.2300424, 0.5617490, 0.8628005, 4.6200726, 2.4309534].

Part 4

Here the most simple case would be to propose $E[X\xi] = 0$. If this is true, then OLS can do the work to estimate β.

$$\delta_j = X_j\beta + \xi_j$$

Part 5

The estimated β is $\begin{bmatrix} 0.06431 & 0.57582 & 0.31705 \end{bmatrix}'$

Problem 3:

Part 1

Again consider the mean utility:

$$\delta_{jt} = -\alpha p_{jt} + \beta x_{jt} + \xi_{jt}$$

In this case the moment condition would be to have Z, an instrument for p_{jt} uncorrelated with ξ_{jt} and x also to be uncorrelated with ξ. Or more precisely, define $W = [X, Z]$, then the moment condition requires: $E[W_{jt}\xi_{jt}] = 0_{rank(X)+rank(Z)}$, or $E[\xi_j|z_j] = 0$.

Part 2

First note that $\hat{s}_{jt}^{BLP} = \int_{v_i} \frac{exp(\delta_{jt})}{1+\sum_j \exp(\delta_{jt})} dF(v_i) = \frac{exp(\delta_{jt})}{1+\sum \exp(\delta_{jt})}$. This is because the choice probability is the same across individuals in a single market. This is because there's no interaction with individual characteristics. Therefore, to obtain δ_{jt} we can just compute the recursion until δ_{jt} converges:

$$\delta_{jt}^{(k+1)} = \delta_{jt}^{(k)} + log(s_{jt}) - log(\hat{s}_{jt}^{BLP}(\delta_{jt}^{k}))$$

Once we have that, we can use the moment condition to estimate price sensititivy parameter in a TSLS framework. That brings us α. Then, according to Nevo (2000) we can obtain the elasticities as:

$$\eta_{jkt} = \frac{\partial s_{jt} p_{kt}}{\partial p_{kt} s_{jt}} = \begin{cases} -\alpha p_{jt}(1 - s_{jt}) & \text{if } j = k \\ \alpha p_{kt} s_{kt} & \text{otherwise} \end{cases}$$

Table 1: Average own and cross elasticities η_{ij}

	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6
j = 1	-1.25	0.323	0.129	0.127	0.126	0.129
j = 2	0.321	-1.251	0.129	0.127	0.126	0.129
j = 3	0.321	0.323	-1.289	0.127	0.126	0.129
j = 4	0.321	0.323	0.129	-1.294	0.126	0.129
j = 5	0.321	0.323	0.129	0.127	-1.291	0.129
j = 6	0.321	0.323	0.129	0.127	0.126	-1.291

Part 3

Now, note that the marginal cost can be obtained with the following formula.

$$\mathbb{mc}_{\mathrm{m}} = \mathbb{p}_{\mathrm{m}} + \mathrm{diag}^{-1}(\eta) \cdot \mathbb{p}_{\mathrm{m}}$$

Results indicate that the average marginal costs across market per product are:

	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6
MC_j	0.66712	0.67189	0.67868	0.68890	0.68263	0.68268

Below I show a figure on its relationship with prices.

Figure 3: Relation Between Prices and Marginal Cost

Part 4

Now we have to consider that prices move endogenously, and marginal cost is exogenous. Then prices that satisfy FOC are:

$$\mathbb{p}_{\mathrm{m}} = (\mathbb{I} + \mathrm{diag}^{-1}(\eta))^{-1} \mathbb{cm}_m$$

In each market, we know α, β, X, and the marginal cost calculated in part 3. We do not know the market shares and the prices of the five remaining goods, so we have in total of 10 unknowns. This means we need 10 equations to jointly estimate them.

Our first five equations come from the marginal cost. In this simple case, we have

$$MC_j = p_{jt} - \frac{p_{jt}}{|\epsilon_{jj}|}$$

where ϵ is the own price elasticity of goods j. We know that ϵ is a function of α, price and market share.

Our second five equations come from the calculation for market share s_{blp}. It is an equation with parameters α, β, ξ and price. Note that ξ can be backed out using the data we have. As a way to make sure we are backing out the right values for the system of non linear equations, we computed the solution for the benchmark case assuming prices are endogenous and we were able to back out the prices and shares in almost all markets as can be seen in the figures:

For the remaining product 2 to product 6, we find the counterfactual price vector is

$$\begin{bmatrix} 3.5168886 & 3.09424396 & 3.0974508 & 3.0909500 & 3.0995115 \end{bmatrix}$$

and the counterfactual share vector is

$$\begin{bmatrix} 0.24386 & 0.11286 & 0.11079 & 0.11034 & 0.11321 \end{bmatrix}$$

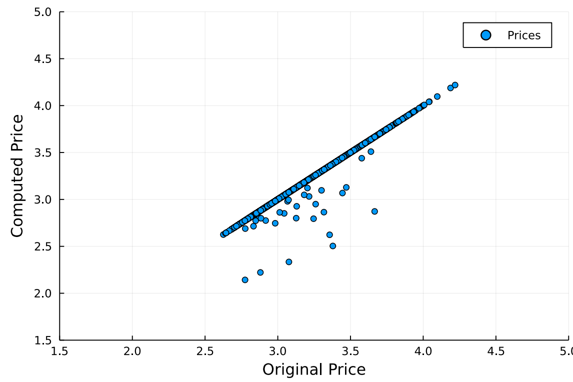

Figure 4: Prices

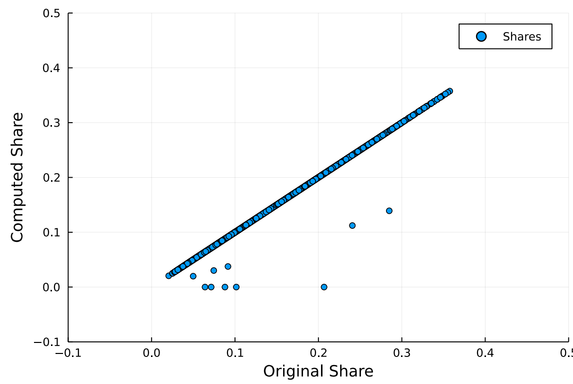

Figure 5: Shares

Part 5

For firm 2 to 6, the average increase in profits per producer is

$$\begin{bmatrix} 0.27776 & 0.29408 & 0.29787 & 0.29217 & 0.29406 \end{bmatrix}$$

Except for firm 3, all other firms obtain higher profits after firm 1 exits.

As for the consumer welfare, the total utility was reduced in 20.38%.

Problem 4

Part 1

After implementing the BLP routine, we get $\alpha = 0.6400457$, $\beta = 0.3342671$, and

$$\Gamma = \begin{bmatrix} -2.0019 & 0 \\ -0.0438141 & 0.143767 \end{bmatrix}$$

Part 2

The following table provides the average elasticities across products:

$$\begin{bmatrix} -0.0011969749 & 1.762364e-04 & 0.02237445 & 0.01777501 & 0.11675806 & 0.1206596 \\ 0.0002661128 & -1.186215e-03 & 0.02662847 & 0.01781757 & 0.09670868 & 0.1371771 \\ 0.0002346572 & 1.590419e-04 & -0.93202315 & 0.05249452 & 0.28387336 & 0.3779566 \\ 0.0002347152 & 1.614086e-04 & 0.05054603 & -0.89481476 & 0.25683615 & 0.3039772 \\ 0.0001194746 & 1.374794e-04 & 0.13234212 & 0.06339285 & -0.96216626 & 0.6409637 \\ 0.0001062859 & 7.879856e-05 & 0.09714398 & 0.09309518 & 0.54496856 & -0.8391673 \end{bmatrix}$$

The main difference is that the cross price elasticity in each column is no longer the same. In question 3, because we eliminates heterogeneity, the cross price elasticity for all other products with product k will be the same. However, in this question we introduces heterogeneity: different people have different tastes for products. This leads to the fluctuation in cross price elasticities even with one particular goods.

Part 3

Prices and market shares in this dataset are weird. The average share of outside goods is 0.385. The average price vector is [0.0024 0.0022 2.019 1.751 3.576 4.442], and the quality vector [-0.0193; -0.02603; -0.08125; -0.1801; 1.6926; 2.0023]. Also, given the elasticity matrix in part 2, we can see that that firms 1 and 2 are already pricing at the inelastic part of the demand (since η_{ii} is close to 0) at very small prices. This is weird but makes sense given that the preference of people over quality are positive and that quality for products 1 and 2 is negative. Therefore, there's no much more scope for those firms to increase prices as long as they do not increase quality relative to the other products.

Part 4