Theory of Income I Computational Problem Set
Fall 2020

Prepared by
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October 29, 2020

Motivation: This is the first of the two computational problem sets we will assign this quarter. This week we keep things simple by asking you to find the value function and policy rule for a deterministic optimal saving problem. The goal of this problem set is to (i) help you get used to coding in Python or Julia (you can choose), and (ii) familiarise yourself with the basics of value function iteration. Also you may find it helpful to review the TA session notes and code I uploaded to Canvas, as a significant portion of questions 1 and 2 relate to the material in those documents.

What and when to submit: The problem set is due two weeks from now on Thursday the 12th of November. Please submit a Latex document with answers to the questions below. For the coding questions, we ask you to plot your results, please include those in your write-up. Additionally, also submit a Jupyter Notebook with your code. Try to follow good coding practices such as commenting appropriately.

1 Some maths before coding...

Consider a social planner who needs to decide how to allocate each periods returns optimally between consumption and accumulating capital. The sequential problem is,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to,

$$c_t + k_{t+1} = z k_t^{\alpha} + (1-\delta) k_t$$

$$c_t, k_{t+1} > 0$$

$$\text{given } k_0 > 0$$

where k_t is capital and c_t is consumption. Assume $0 < \alpha < 1$

(a) What is the maximum level of capital that can be accumulated? Hence what is the relevant state space?

(b) Write the Bellman for the sequential problem, substituting out consumption using the budget constraint. What are the state and control variables?

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(c) What is the Euler equation for this problem?

(d) Use the Euler equation and the social planners resource constraint to find the steady state level of capital and consumption.

(e) What is the Bellman operator for this problem?

(f) Show this operator satisfies Blackwell's sufficient conditions for a contraction mapping, and show that β is the modulus of contraction.[1] *(This is a lot easier than it sounds, and if you can't solve it you can quite easily find the proof online).*

2 Coding value function iteration, and speeding up convergence

For the remainder of the problem set use the following parameter values (unless stated otherwise): $\beta = 0.99, z = 1, \alpha = 0.3$, and $\delta = 0.1$. Also use a state space grid with a 500 points starting at 0.1 going up to the effective upper bound, and use an error tolerance of e^{-5}.

(a) Write code that implements value function iteration. Use an initial guess of $V_0(k) = 0, \forall k$.

  (i) Report the number of times you had to iterate, a plot with only the final value function, and a plot with all the iterations.

  (ii) Plot the policy rule $g(k)$. Briefly explain what this says about the agents behaviour.

  (iii) Use the policy rule to numerically approximate the steady state level of capital and consumption, and compare this to the actual steady state of the model.

(b) In part 1f we showed that β is the modulus of contraction.[2] Therefore, if we reduce β we should get faster convergence. To see this, repeat part 2ai for $\beta = 0.1$ and comment on your findings.

(c) As you can probably see from your plots of the iterations, our naive initial guess is quite far from the true value function. Therefore, lets try a better guess,

$$V_0(k) = \frac{\log(zk^\alpha - \delta k)}{1-\beta}$$

Repeat part 2ai for this guess and comment on your findings. *I would recommend using this as the initial guess for the remainder of the problem set, it will make your life easier.*

[1] **Blackwell's Sufficient Conditions**: these are sufficienct (but not necessary) for an operator to be a contraction mapping
Let $X \subseteq \mathcal{R}^l$ and let $C(X)$ be a space of bounded functions $f : X \to \mathcal{R}$, with the sup-metric. Let $B : C(X) \to C(X)$ be an operator satisfying:

(i) (Monotonicity) if $f, g \in C(X)$ and $f(x) \leq g(x) \forall x \in X$, then $(Bf)(x) \leq (Bg)(x) \forall x \in X$.

(ii) (Discounting) there exists some $\delta \in (0, 1)$ such that

$$[B(f + a)](x) \leq (Bf)(x) + \delta a \quad \forall f \in C(X), a \geq 0, x \in X$$

Then B is a contraction with modulus δ. [Note that a is a constant, and $f + a$ is the function generated by adding a constant to the function f]

[2] Reminder: A function $F : X \to X$ is a contraction mapping if for some $\beta \in (0, 1)$, we have,

$$\|F_x - F_y\| \leq \beta \|x - y\| \quad \forall z, y \in X$$

where β is the modulus of contraction of F.

3 Comparative statics: the effect of permanent productivity shocks

So far we have done a bunch of tedious mathematics and coding, but we are economists after all, so let's now do some economics. In this section, we will use our machinery from the previous parts to study how our economy responds to a permanent productivity increase (an increase in z).

(a) Plot the policy rules and approximate the steady states for (i) $z = 1$, and (ii) $z = 2$.

(b) We will now use the policy rules from the previous part to plot impulse response functions. Assume in period zero $z = 1$ and the economy is in steady state, then in period one there is a permanent productivity shock such that $z = 2$. Plot the path of capital from peiod 0 to period 29. Comment on your findings.