Risk Aversion and Precautionary Savings in Dynamic Settings

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Abstract

We study how risk aversion affects precautionary savings when considering monotone recursive Kreps-Porteus preferences. In a general infinite-horizon setting, we prove that risk aversion unambiguously increases precautionary savings. The result is derived without specifying income uncertainty, which can follow any kind of stochastically monotone process, and accounting for possibly binding borrowing constraints.

Keywords: risk aversion, precautionary savings, recursive models, monotonicity.
JEL codes: D80, D91, E21.

1 Introduction

Analyzing the impact of risk on individual behaviors is a long standing line of research in decision sciences. However, understanding how individuals may take decisions in presence of risks in arbitrary situations is a very difficult question to tackle. Indeed, choices under uncertainty may depend on a number of factors, such as the risk appetite, the type of decisions that has to be made, the interaction between the various risks at stake or the insurance possibilities that are offered. To obtain a clear-cut characterization of individual behavior under risk, the problem needs to be further specified.

A particular specification is the so-called precautionary saving problem, which seeks to understand how future income uncertainty affects the individual current saving decision. Precautionary savings have received a considerable and steady attention in the literature, since the early works of Leland (1968) and Sandmo (1970). A potential explanation for this popularity is that, as emphasized by Carroll (1997) and Carroll and Samwick (1998), precaution –i.e., the sensitivity to future income uncertainty– could be, quantitatively speaking, one of

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the main motives for savings. In a recent study, Mody, Sandri, and Ohnsorge (2012) report that "at least two-fifths of the increase in households' saving rates between 2007 and 2009 was due to increased uncertainty about labour-income prospects." Understanding precautionary savings is thus very important to analyze household savings and consumption, but also to study their aggregate impact on macroeconomic variables, such as interest rates or GDP. This may have several implications for the design of monetary and fiscal policies.

Despite the constant interest for precautionary savings, few general results are available. Indeed, a large number of papers have focused on precautionary savings in two-period economies, in which, by construction, saving only occurs once (in the first period) and income is uncertain in a single period (in the second period). Among these papers, one can cite Drèze and Modigliani (1972), Kimball (1990), Courbage and Rey (2007), Kimball and Weil (2009), Eeckhoudt and Schlesinger (2008), Bommier, Chassagnon and LeGrand (2012), Jouini, Napp and Nocetti (2013), and Nocetti (2016). The extension to many periods or to an infinite horizon has rarely been addressed. We are aware of only two contributions where precautionary savings are analytically studied in an infinite horizon setting. These are those of van der Ploeg (1993) and Weil (1993). Both papers rely on specific fully parametrized income processes and on closed-form solutions. Besides these two analytical studies, the problem has also been tackled in infinite horizons using numerical techniques –which therefore also imply a specific parametric income process– such as in Wang, Wang, and Yang (2016).

In this paper, we address the question of precautionary savings in infinite horizon, without specifying income uncertainty, and accounting for possibly binding borrowing constraints. Our framework is therefore fully general and does embed any restriction, neither on the horizon nor on the uncertainty. We prove that whenever the income process is stochastically monotone,[1] an increase in risk aversion implies an increase in precautionary savings. Rather intuitively, more risk averse individuals will opt for larger amounts of savings in presence of income uncertainty. We provide therefore a simple and intuitive connection between risk aversion and saving decision. Up to our knowledge, this is the first general result that does not rely on a parametric specification of the income process and either a closed-form solution or a numerical simulation.

The machinery that we develop to prove our result relies on the concept of *comonotonicity* and more precisely of *conditional comonotonicity* introduced by Jouini and Napp (2004). Intuitively, two (real) processes will be conditionally comonotone if future large realizations for one process will be accompanied by large realizations for the other process. The first step of the proof consists in showing that income, consumption and continuation utilities are

[1]Loosely speaking, it means that good news for today's income cannot be bad news for income in future periods. A formal definition is provided below in Section XXX.

comonotone, even if borrowing constraints can be binding at any future date. This implies that a large income realization means a large consumption level and a large continuation utility. Therefore, a higher amount of savings today will be all the more valuable tomorrow if income realization tomorrow is low rather than if it is high. In consequence, increasing savings diminishes the dispersion of utilities and is thus a risk-reduction device. A more risk averse individual, who cares more about risk reduction by definition, will therefore choose higher amounts of savings. This builds up our main result.

As explained above, our result crucially relies on the fact that income, consumption and continuation utilities comove together. This latter relationship emphasizes the importance to consider utility functions representing preferences that are well-behaved in presence of risk. In particular, as we explain in Section 2XXX, monotone preferences play a decisive role when modelling choices under uncertainty.

The rest of the paper is organized as follows. In Section 2, we describe and motivate the preferences we consider in this paper. We present our setup and state our main result in Section 3. Section 4 concludes.

2 Monotonicity and risk-sensitive preferences

Preference monotonicity stipulates that an agent will not take an action if another available action is preferable in all circumstances. Consider for example a standard two-period consumption-saving problem, where uncertain second period income is distributed over an interval $[y_{min}, y_{max}]$. Monotonicity (combined with the usual assumptions of good normality and preference convexity) implies that an agent will not save more than what she would do if anticipating second period income y_{min} for sure, or less than what she would do if anticipating y_{max} for sure. Monotonicity can be viewed as a rather reasonable requirement to model individual behavior. The terminology "precautionary savings" seems itself to embed this assumption of monotonicity. Indeed, "precaution" indicates that individuals save to cope with the possible occurrence of low future incomes. But saving more than what would be optimal if the worst possible outcome would occur for sure –as this may happen with non-monotone preferences– could hardly be interpreted as a precaution.

Monotonicity, when combined with other structural assumptions such as recursivity and those of the Kreps-Porteus framework, drastically reduces the set of admissible representations. Bommier, Kochov and LeGrand (2016) prove that recursive monotone Kreps-Porteus preferences must be either Uzawa (1968) preferences or Hansen and Sargent (1995) risk-sensitive preferences. As Uzawa preferences prove incapable of disentangling risk aversion and elasticity of substitution, the only possibility to study the role of risk aversion with recursive monotone Kreps-Porteus preferences involves using risk-sensitive preferences.

With risk-sensitive preferences, utility U_t fulfills the following recursion:

$$U_t = \begin{cases} u(c_t) - \frac{\beta}{k} \log \left(E_t[e^{-kU_{t+1}}] \right) & \text{, if } k > 0, \\ u(c_t) + \beta E_t[U_{t+1}] & \text{, if } k = 0, \end{cases} \tag{1}$$

where c_t is the consumption level at date t, $u : \mathbb{R}^+ \to \mathbb{R}$ is an increasing and concave function and $\beta \in (0,1)$ and $k \geq 0$ are two parameters. Intuitively, these preferences are monotone because the instantaneous utility can be "entered" into the certainty equivalent:

$$U_t = \begin{cases} -\frac{\beta}{k} \log \left(E_t[e^{-\frac{k}{\beta}(u(c_t)+\beta U_{t+1})}] \right) & \text{, if } k > 0, \\ E_t[u(c_t) + \beta U_{t+1}] & \text{, if } k = 0, \end{cases}$$

which is typically impossible with other forms of certainty equivalents.

In absence of uncertainty, the recursion (1) becomes $U_t = u(c_t) + \beta U_{t+1}$, providing $U_0 = \sum_{t=0}^{\infty} \beta^t u(c_t)$. The function u and the parameter β respectively determine intertemporal substitutability and time preferences. The parameter k only plays a role in presence of uncertainty. More precisely, k governs risk aversion, with greater risk aversion related to larger values of k. The case $k = 0$ in equation (1) corresponds to the standard additively separable expected utility model.

3 Precautionary savings in dynamic settings

We consider infinitely long-lived agents endowed with an exogenous income process $(y_t)_{t \geq 0}$ which does not need to be specified to derive our results. We only assume that this process is *stochastically monotone* meaning that for all $t \geq 0$ and $x \in \mathbb{R}$ the function $(y_0, y_1, \ldots, y_t) \mapsto \text{Prob}(y_{t+1} \geq x | y_0, y_1, \ldots, y_t)$ is non-decreasing. The assumption of stochastic monotonicity formalizes the idea that a good income realization at a given date cannot convey bad information for subsequent periods. Most income processes used in the literature comply with such an assumption. This is for example the case of standard autoregressive processes, as those considered for instance in van der Ploeg (1993) and Weil (1993).

Agents have risk-sensitive preferences and only differ by the risk aversion parameter $k \geq 0$.[2] The function u is assumed to be twice continuously differentiable and concave. Technically speaking, the assumption of a stochastically monotone income process does not rule out

[2]By imposing $k \geq 0$, we restrict the study to agents that are at least as risk averse as a standard additively separable expected utility maximizer. Cases where $k < 0$ would be be difficult to address, due to potential non-convexity issues.

extremely rapid income growth or income decline, which could result in existence and convergence problems. Rather than introducing a set of technical assumptions, we simply assume that the income process and the preference parameters (in particular β and u) are such that convergence problems do not occur.

We consider the saving decision at time t of agents with wealth w and realized income trajectory denoted by $y^t = (y_0, \ldots, y_t)$.[3] The gross interest rate between dates t and $t+1$, denoted by R_{t+1}, is time-varying but deterministic. Let $w \to V_t^k(w, y^t)$ be the indirect utility at time t of the agent with risk aversion k. We have:

$$V_t^k(w, y^t) = \max_{s_t \in \mathbb{R}} u(c_t) - \frac{\beta}{k} \log E_t \left[e^{-kV_{t+1}^k(R_{t+1}s_t, y^{t+1})} \right], \tag{2}$$
$$\text{s.t. (i) } y_t + w - s_t = c_t, \text{ (ii) } c_t > 0,$$
$$\text{and (iii) } s_t \geq \underline{s}_t(y^t),$$

where $\underline{s}_t(y^t)$ is the borrowing limit at time t. This limit may reflect "natural" borrowing constraints due to the fact that an agent cannot borrow more than what she may able to repay in the worst scenario, or may be exogenous and related to market imperfections. Since the income process is not stationary, the borrowing limit is likely to depend on the information obtained through past income realization information. This is why $\underline{s}_t$ is explicit set as a function of y^t, the income trajectory. In line with the assumption of a stochastically monotone income process, we assume that $\underline{s}_t(y^t)$ is non-increasing in y^t. Intuitively, this means that good news concerning income cannot convey negative information about future borrowing constraints. Moreover, we assume that for all t and all income histories y^t at date t, and all income realizations y_{t+1} in period $t+1$, we have $R_t\underline{s}_t(y^t) + y_{t+1} > \underline{s}_{t+1}(y^{t+1})$, meaning that saving $\underline{s}_t(y^t)$ in period t is feasible.[4] If $w \leq \underline{s}_t(y^t) - y_t$, the above program is not well defined since no saving and consumption level fulfill the constraints (i) to (iii). For any $w > \underline{s}_t(y^t) - y_t$, we denote by $s_t^k(w, y^t)$ the solution of the optimization problem, that is the amount that the agent with wealth w and risk aversion k chooses to save. We can now state the following result:

Proposition 1 (Propensity to save) *For all $t \geq 0$, y^t, $w > \underline{s}(y^t) - y_t$, k, $k' \geq 0$, the*

[3]We indicate the whole income history, as the income process is not necessarily Markovian. Expectations regarding future income may then depend on the whole income history.

[4]Not assuming such an inequality would necessarily lead to introducing another borrowing constraint preventing the agent from taking a decision likely to yield an infeasible situation in the following period. Our formalization assumes that the feasibility constraints are already reflected in the $\underline{s}_t(y^t)$. There is a vast theoretical and empirical literature motivating exogenous borrowing constraints and studying their micro- and macroeconomic implications (see Bewley 1983 or Aiyagari 1994 for early references).

following implication holds:

$$k' \geq k \Rightarrow s_t^{k'}(w, y^t) \geq s_t^k(w, y^t).$$

Proof. The formal proof is in the Appendix. The main intuition of the proof can be summarized as follows. We first establish that, even in the presence of binding borrowing constraints, consumption, income and continuation utility fulfill a property of conditional comonotonicity introduced by Jouini and Napp (2004).[5] This means that, for a given history y^t, high income realizations at time $t+1$ will also correspond to high continuation utilities and high consumption levels. The latter aspect implies that the impact on continuation utility of a marginal increase in saving s_t is larger if information at date $t+1$ reveals a low income than if it reveals high income. Thus, an increase in savings induces a transfer of welfare from states with high continuation utility to states with low continuation utility, achieving a risk reduction. Highly risk averse agents value this risk reduction more than low risk averse agents and therefore end up saving more. ■

Proposition 1 makes a clear statement about the relationship between risk aversion and the propensity to save. The results show that the more risk averse agent saves more than the less risk averse one. This comparison is established for agents having the same amount of wealth in period t. In dynamic problems, however, agents with different degrees of risk aversion will hold different amounts of wealth at time t. The difference in their saving behaviors will therefore result both from differences in propensities to save and from differences in accumulated wealth. The function $w \mapsto s_t^k(w, y^t)$ being non-decreasing, both effects go in the same direction, providing the following result:

Corollary 1 (Precautionary savings) *Denote by w_t^k the wealth process that results from the optimal saving behavior of an agent with risk aversion k.*[6] *Then:*

$$k' \geq k \quad \Rightarrow \quad \forall t \geq 0,\ w_t^{k'} \geq w_t^k,$$

where the last inequality has to be understood almost surely.

Corollary 1 indicates that if we compare two agents receiving the same exogenous stochastic income and facing no other risks, the more risk averse agent will be wealthier than the less risk averse agent, at all ages and in all circumstances.

The overall message induced by Proposition 1 and Corollary 1 is very clear: greater risk aversion implies greater prudence. The key feature of our results is that they hold for any

[5] Conditional comonotonicity extends the concept of comonotonicity to dynamic settings.

[6] Formally speaking, w_t^k is defined by $w_0^k = 0$ and $w_{t+1}^k = R_{t+1} s_t^k(w_t^k, y^t)$ for $t \geq 0$.

stochastically monotone income process and also in presence of binding borrowing constraints. These results are thus more general than those of van der Ploeg (1993) and Weil (1993), who assume specific income processes in order to be able to derive closed-form solutions. To our knowledge, Proposition 1 and Corollary 1 provide the first general results on the determinants of precautionary savings in an infinite horizon model that are derived without assuming either a specific random income process or additive separability of preferences.

4 Related literature and concluding remarks

There is a huge theoretical literature on precautionary savings. However, most analytical studies rely on a two-period framework, as in Drèze and Modigliani (1972), Kimball (1990), Kimball and Weil (2009), Eeckhoudt and Schlesinger (2008), Bommier, Chassagnon and LeGrand (2012), Jouini, Napp and Nocetti (2013), among many others. The extension to many periods or to an infinite horizon has rarely been addressed. In most cases, papers tackling this issue are based on the standard additively separable life-cycle model where the role of risk aversion cannot be explored. In particular, the famous textbook result stipulating that prudence is related to the third derivative of the instantaneous utility function, while risk aversion is related to the second derivative, is restricted to such an additively separable setting. This result cannot be used to infer any relation between risk aversion and precautionary savings. Indeed, it is well known that in the standard additively separable expected utility model, risk aversion cannot be studied in isolation from intertemporal elasticity of substitution (see Epstein and Zin, 1989, for example). We are aware of only three contributions where precautionary savings are analytically studied in an infinite horizon setting with a framework that disentangles risk aversion and elasticity of substitution. These are those of van der Ploeg (1993) and Weil (1993) that we already mentioned and that of Wang, Wang, and Yang (2016). In all three cases, specific parametrized forms of uncertainty are assumed.[7] Risk aversion is then found to have a positive impact on precautionary savings. However, it is impossible to tell from these studies whether this positive correlation reflects a fundamental link between risk aversion and prudence, or whether this is a consequence of the specific risks considered.

A key assumption for our results is that of preference monotonicity. It is known from the two-period analysis of Kimball and Weil (2009) that there is no simple relation between risk aversion and precautionary savings for non-monotone Kreps-Porteus preferences, an ambiguous relation being found in the Epstein-Zin framework. As explained in Bommier, Kochov

[7]Van der Ploeg (1993) and Weil (1993) assume autoregressive income processes. Weil uses particular recursive preferences, while van der Ploeg considers a multiplicative expected utility model, with instantaneous quadratic utility functions. Wang, Wang and Yang (2016) provide numerical results about the impact of risk aversion for Epstein-Zin preferences and an income process following a geometric Brownian motion.

and LeGrand (2016), this directly derives from the fact that Epstein-Zin preferences may suggest an amount of savings that exceeds the one that would be chosen if the worst income realization were known to occur for sure.

The current paper confirms that with monotone preferences the role of risk aversion is quite simple and intuitive in problems where states of the world can be ranked independently of the agent's action.[8] Risk aversion can indeed be seen as the willingness to redistribute utility from good states of the world to bad states of the world, or in other words, to opt for a strategy "closer" to the one that would be chosen in bad states. Risk aversion leads to being more cautious about adverse circumstances, and the result that risk aversion increases precautionary savings comes with no surprise. However, for a number of reasons, such as the focus on the additively separable expected utility model, the use of non-monotone preferences, or the difficulty in finding closed form solutions, this fundamental relation between risk aversion and prudence remained largely unclarified. We hope that this short paper helps shed light on this relation, which might be key to understand heterogeneity in saving behaviors.

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[8] This exogenous ranking of states of the world holds in the precautionary saving problem with stochastically monotone income process, since receiving a high income in a given period is always preferable to receiving a low income in that period, independently of past savings.

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Appendix

A Proof of Proposition 1

A.1 Mathematical preamble

We consider that all our processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$. For any random variable X, we denote its expectation, under the probability $\mathbb{P}$, by $E[X] = \int_\Omega X(\omega)d\mathbb{P}(\omega)$, and its variance, under $\mathbb{P}$, $V[X] = E[X^2] - E[X]^2$ –when they exist. The covariance under $\mathbb{P}$ between two random variables X and Y is denoted by $cov(X,Y) = E[XY] - E[X]E[Y]$, when it exists. Let L be an a $\mathbb{P}-$almost surely (a.s., henceforth) nonnegative random variable such that $E[L] = 1$. Then, the function $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$, defined for any $A \in \mathcal{F}$ by

$$\mathbb{Q}(A) = \int_{\omega \in A} L(\omega)d\mathbb{P}(\omega)$$

is a probability measure. The expectation under $\mathbb{Q}$ denoted by $E^{\mathbb{Q}}[\cdot]$ verifies for any random variables X (when the expectation exists):

$$E^{\mathbb{Q}}[X] = E[LX]. \tag{3}$$

We also define (when they exist) the variance and the covariance under $\mathbb{Q}$:

$$V^{\mathbb{Q}}[X] = E[LX^2] - E[LX]^2, \tag{4}$$

$$cov^{\mathbb{Q}}(X,Y) = E[LXY] - E[LX]E[LY]. \tag{5}$$

The notations are straightforward to extend for conditional moments.

The proof will utilize a property of conditional comonotonicity and some important related results. Our presentation, below, is taken from Jouini and Napp (2004). Let $\mathcal{G}$ be a sub-sigma algebra of $\mathcal{F}$. We start by characterizing the comonotonicity conditionally to $\mathcal{G}$.

Definition 1 (Comonotonicity conditional to $\mathcal{G}$) *Two random variables X and Y defined on $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be comonotonic conditionally to $\mathcal{G}$ if and only if there exist a random variable ξ and two functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that: (i) $f(.,x)$ and $g(.,x)$ are $\mathcal{G}$-measurable; (ii) $f(\omega,.)$ and $g(\omega,.)$ are nondecreasing; and (iii) $(X,Y) = (f(\omega,\xi), g(\omega,\xi))$ almost surely.*

Jouini and Napp (2004) provide additional characterizations for the comonotonicity conditional to a sub-sigma algebra. The intuition is straightforward: X and Y are comonotonic conditionally to $\mathcal{G}$ if their "restrictions" to $\mathcal{G}$ are comonotononic. We now turn to the definition of conditional comonotonicity for processes.

Definition 2 (Conditional comonotonicity) *Two adapted random processes (X_t) and (Z_t) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ are said to be conditionally comonotonic if, for $t \geq 0$, the random variables X_{t+1} and Z_{t+1} are comonotonic conditionally to $\mathcal{F}_t$.*

Conditional comonotonicity therefore generalizes the concept of comonotonicity to stochastic processes.

We conclude this section with a useful result:

Proposition 2 (Jouini and Napp, 2004) *Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. If X and Y are comonotonic conditionally to $\mathcal{G}$, then $cov_{\mathcal{G}}^{\mathbb{Q}}(X, Y) \geq 0$ for all probability measures $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$.*

In the remainder, we will say that two random variables X and Y are *anticomonotonic* when X and $-Y$ are comonotonic.

A.2 Comonotonicity properties

As a first step in the proof of Proposition 1, we establish some comonotonicity properties that relate income, consumption and continuation utilities. In that part of the proof the degree of risk aversion k is fixed, the impact of an increase in k being analyzed in Section A.3.

The program (2) of the agent can be expressed as follows:

$$V_t^k(w_t, y^t) = \max_{\underline{s}_t \leq s_t < y_t + w_t} u(y_t - s_t + w_t) - \frac{\beta}{k} \log E_t \left[e^{-kV_{t+1}^k(w_{t+1}, y^{t+1})} \right], \tag{6}$$

$$\text{s.t. } w_{t+1} = R_{t+1} s_t,$$

where $E_t[\cdot]$ denotes the expectation conditional on $\mathcal{F}_t$.

To simplify the notation, we introduce W_t^k defined by:

$$W_t^k(w_t^k, y^t) = -kV_t^k(w_t^k, y^t) = \min_{\underline{s}_t \leq s_t < y_t + w_t} -ku(y_t - s_t + w_t^k) + \beta \log E_t e^{W_{t+1}^k(R_{t+1}s_t, y^{t+1})}. \tag{7}$$

To derive our result, we need to differentiate the value function. Following standard arguments of dynamic programming, we can show that since u is increasing and concave, so is the value function. Moreover, since u is assumed to be continuously derivable, so is the value function, as proved by Benveniste and Sheinkman (1979).[9] We will therefore denote

[9]Stokey and Lucas (1989, Theorem 4.11) also provide a proof of the differentiability of the value function in a slightly more general framework.

by $W^k_{t,w}$ and $W^k_{t,k}$ the derivatives of W^k_t with respect to wealth w_t and risk aversion k respectively. We respectively denote by c^k_t and s^k_t the optimal consumption and savings of an agent maximizing (6) endowed with the risk aversion parameter k. The strict concavity of u guarantees the unicity of s^k_t. The first-order condition provides (we recall that the interest R_{t+1} is deterministic):

$$ku'(c^k_t) \geq -\beta R_{t+1} \frac{E_t\left[W^k_{t+1,w}(w^k_{t+1}, y^{t+1}) e^{W^k_{t+1}(w^k_{t+1}, y^{t+1})}\right]}{E_t\left[e^{W^k_{t+1}(w^k_{t+1}, y^{t+1})}\right]}, \tag{8}$$

where equality holds if $s^k_t > \underline{s}_t$. The envelop theorem yields the following equalities (which are valid whether the constraint $s^k_t \geq \underline{s}_t$ is binding or not):

$$W^k_{t,w}(w^k_t, y^t) = -ku'(y_t - s^k_t + w^k_t) = -ku'(c^k_t) \tag{9}$$

$$W^k_{t,k}(w^k_t, y^t) = -u(c^k_t) + \beta E_t\left[W^k_{t+1,k} \frac{e^{W^k_{t+1}(w^k_{t+1}, y^{t+1})}}{E_t\left[e^{W^k_{t+1}(w^k_{t+1}, y^{t+1})}\right]}\right] \tag{10}$$

Note that as a corollary of the continuous differentiability of the value function, the consumption level and the saving choices are continuous in k. We now state the following lemma.

Lemma 1 (Comonotonicity of income, consumption and continuation utility) *In the setup of Proposition 1, at any date $t \geq 0$, the optimal consumption process $(c^k_t)_{t\geq 0}$, the income process $(y_t)_{t\geq 0}$ and the continuation utility process $(V^k_t)_{t\geq 0}$ are conditionally comonotonic.*

Proof. We prove the result assuming that there is a finite $T \in \mathbb{N}$ such that income in periods after T is deterministic. Since T can be any finite integer, and $\beta < 1$ (implying that what happen in the very long term ends up being negligible) a continuity argument implies that the result extends to $T = \infty$.

We prove by reverse induction on t that (i) c^k_t and y_t are comonotonic conditionally to $\mathcal{F}_{t-1}$ and that (ii) y_t and W^k_t are anticomonotonic conditionally to $\mathcal{F}_{t-1}$. At date $t = T$, there is no remaining uncertainty. Conditionally to the filtration $\mathcal{F}_{T-1}$, and in particular for a given history of income $(y_t)_{0\leq t\leq T-1}$ up to date $T-1$, let us consider two realizations y_T and y'_T of y_T (at date T) such that $y_T > y'_T$. Due to the stochastic monotonicity assumption, we know that $y_{T+\tau} \geq y'_{T+\tau}$ for any $\tau > 0$. Credit constraints being non-increasing in past income realizations, the (intertemporal) budget set when receiving y_T is therefore larger than the one obtained when receiving y'_T. The consumption profile $(c_{T+\tau})_{\tau\geq 0}$, chosen when receiving y_T is thus revealed preferred to the consumption profile $(c'_{T+\tau})_{\tau\geq 0}$ chosen when receiving y'_T. Assume that $c_T < c'_T$. Since $(c_{T+\tau})_{\tau\geq 0}$ is revealed preferred to $(c'_{T+\tau})_{\tau\geq 0}$ there must be at least one τ for which $c_{T+\tau} > c'_{T+\tau}$. Moreover since the borrowing limit $\underline{s}_t(y^t)$ is non-increasing in y^t, the borrowing constraint at time T cannot be binding when choosing c_T. We know that $u'(c'_T) \geq \beta^\tau \left(\prod_{k=1}^{\tau} R_{T+k}\right) u'(c'_{T+\tau})$, otherwise it would be optimal to decrease consumption c'_T by a small amount to increase $c'_{T+\tau}$. Since $c_T \leq c'_T$ implies $u'(c_T) \geq u'(c'_T)$ and $c_{T+\tau} > c'_{T+\tau}$ implies that $u'(c'_{T+\tau}) > u'(c_{T+\tau})$, we obtain $u'(c_T) > \beta^\tau \left(\prod_{k=1}^{\tau} R_{T+k}\right) u'(c_{T+\tau})$, contradicting the optimality of $(c_{T+\tau})_{\tau \geq 0}$. We deduce that $c_T \geq c'_T$. We conclude that c_T^k and y_T are comonotonic conditionally to $\mathcal{F}_{T-1}$. Since $W_T^k = -k \sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau}^k)$ (no uncertainty is left after T) and u increasing, we can also conclude that y_T and W_T^k are anticomonotonic conditionally to $\mathcal{F}_{T-1}$, and thus that y_T and V_T^k are comonotonic.

We have shown that points (i) and (ii) hold for $t = T$. We now proceed by induction showing that if they hold for $0 < t \leq T$, they also hold for $t-1$. When the borrowing constraint does not bind at time $t-1$, the Euler equation (8) together with (9) implies:

$$u'(c_{t-1}^k) = (\beta R_t) E_{t-1}\left[u'(c_t^k) \frac{e^{W_t^k(w_t^k, y^t)}}{E_{t-1}[e^{W_t^k(w_t^k, y^t)}]}\right]. \tag{11}$$

Using the induction hypothesis, we know that $u'(c_t^k)\frac{e^{W_t^k(w_t^k, y^t)}}{E_{t-1}[e^{W_t^k(w_t^k, y^t)}]}$ and y_t are anticomonotonic conditionally to $\mathcal{F}_{t-1}$. Since the income process is stochastically monotone, we deduce that $u'(c_{t-1}^k)$ is non-increasing with y_{t-1}, meaning that c_{t-1}^k and y_{t-1} are comonotonic conditionally to $\mathcal{F}_{t-2}$. Since the income process is stochastically increasing, while the borrowing constraints are non-increasing, the definition (7) of W_{t-1}^k implies that W_{t-1}^k is decreasing with y_{t-1}. The induction hypothesis and the comonotonicity of c_{t-1}^k and y_{t-1} conditionally to $\mathcal{F}_{t-1}$ allow us to conclude that W_{t-1}^k and y_{t-1} are anticomonotonic conditionally to $\mathcal{F}_{t-2}$, and thus that y_{t-1} and V_{t-1}^k are comonotonic.

When borrowing constraints bind, denote by $\underline{y}_{t-1}$ the cut-off value of y_{t-1} below which the Euler equation does not hold. For any realization of y_{t-1} below $\underline{y}_{t-1}$, the borrowing constraint at date $t-1$ is binding: the consumption at date $t-1$ varies exactly as y_{t-1}. Therefore, when the Euler equation does not hold, c_{t-1}^k, y_{t-1}, and V_T^k are also comonotonic conditionally to $\mathcal{F}_{t-2}$. ■

A.3 Increasing risk aversion

To complete the proof of Proposition 1, we consider the impact of an increase in risk aversion. Ideally, we would wish to differentiate the Euler equation with respect to k and show that $\frac{\partial s_t^k}{\partial k} > 0$. However, this supposes that $k \mapsto s_t^k$ is differentiable, or equivalently that $w \mapsto W_t^k(w, y^t)$ is twice differentiable. Unfortunately, this is not trivial to prove in our setup and standard arguments as in Araujo (1991) or in Santos (1991) do not apply.[10] The basic reason is that $(s, s') \mapsto u(y + s' - Rs)$ is not strictly concave. We will therefore use a method that avoids differentiation of the savings function.

Consider two agents endowed with risk aversion parameters $k' > k$. In order to show that $s_t^k \geq s_t^{k'}$, we distinguish two cases, depending on whether the Euler equation for k holds with equality or not.

First case: Euler equations (8) and (11) hold with equality for agent k. Euler equation for agent k' may not hold with equality. Dropping the dependence in y^t, we deduce from Euler equation (8) for k and k'

$$k'u'(c_t^{k'}) - ku'(c_t^k) \geq \beta R_{t+1} \left[\frac{E_t \left[W_{t+1,w}^k(w_t^k) e^{W_{t+1}^k(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^k(w_{t+1}^k)} \right]} - \frac{E_t \left[W_{t+1,w}^{k'}(w_t^{k'}) e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]} \right],$$

and after some rearrangement, we obtain:

$$\begin{aligned} \frac{k'u'(c_t^{k'}) - ku'(c_t^k)}{\beta R_{t+1}} \geq & - \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^{k'}) e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]} + \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^k) e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]} \\ & - \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^k) e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]} + \frac{E_t \left[W_{t+1,w}^k(w_{t+1}^k) e^{W_{t+1}^k(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^k(w_{t+1}^k)} \right]}, \end{aligned}$$

which yields:

$$\frac{k'(u'(c_t^{k'}) - u'(c_t^k))}{\beta R_{t+1}} + \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^{k'}) e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})} \right]} - \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^k) e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]} \geq \tag{12}$$

$$\frac{(k - k')u'(c_t^k)}{\beta R_{t+1}} - \frac{E_t \left[W_{t+1,w}^{k'}(w_{t+1}^k) e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^{k'}(w_{t+1}^k)} \right]} + \frac{E_t \left[W_{t+1,w}^k(w_{t+1}^k) e^{W_{t+1}^k(w_{t+1}^k)} \right]}{E_t \left[e^{W_{t+1}^k(w_{t+1}^k)} \right]}. \tag{13}$$

To prove that the inequality made of equations (12) and (13) implies that $s_t^k \geq s_t^{k'}$, we proceed in two steps. First, we prove that the left-hand side (equation (12)) has the sign of $s_t^{k'} - s_t^k$. Second, we show that the right-hand side (equation (13)) is positive.

Let us start with the left-hand side of equation (12). Since u is concave and $k' > 0$,

[10] Blot and Crettez (2004) provide another set of conditions and another proof for the differentiability of the policy function, but their results do not apply here.

$k'(u'(c_t^{k'}) - u'(c_t^k))$ has the sign of $s_t^{k'} - s_t^k$. Moreover, the second part in (12) becomes:

$$\frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k'})e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} - \frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k})e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]} =$$

$$\frac{E_t\left[\left(W_{t+1,w}^{k'}(w_t^{k'}) - W_{t+1,w}^{k'}(w_t^{k})\right)e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}$$

$$+ E_t\left[W_{t+1,w}^{k'}(w_t^k)\left(\frac{e^{W_{t+1}^{k'}(w_{t+1}^{k'})}}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} - \frac{e^{W_{t+1}^{k'}(w_{t+1}^{k})}}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}\right)\right].$$

Since $w \mapsto W_{t+1,w}^{k'}(w)$ is increasing (indeed $V_{t+1}^{k'} = -W_{t+1}^{k'}/k'$ is concave), $W_{t+1,w}^{k'}(w_t^{k'}) - W_{t+1,w}^{k'}(w_t^k)$ has the sign of $w_t^{k'} - w_t^k$ and therefore of $s_t^{k'} - s_t^k$. Moreover, the term

$$E_t\left[W_{t+1,w}^{k'}(w_t^k)\left(\frac{e^{W_{t+1}^{k'}(w_{t+1}^{k'})}}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} - \frac{e^{W_{t+1}^{k'}(w_{t+1}^{k})}}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}\right)\right]$$

has also the sign of $w_t^{k'} - w_t^k$ if $w \mapsto E_t\left[W_{t+1,w}^{k'}(w_t^k)\frac{e^{W_{t+1}^{k'}(w)}}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]}\right]$ is increasing. This is indeed the case, since $W_{t+1}^{k'}$ being continuously derivable, we have

$$\begin{aligned}\frac{\partial}{\partial w}E_t\left[W_{t+1,w}^{k'}(w_t^k)\frac{e^{W_{t+1}^{k'}(w)}}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]}\right] &= \frac{E_t\left[W_{t+1,w}^{k'}(w_t^k)W_{t+1,w}^{k'}(w)e^{W_{t+1}^{k'}(w)}\right]}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]} \\ &\quad - \frac{E_t\left[W_{t+1,w}^{k'}(w_t^k)e^{W_{t+1}^{k'}(w)}\right]E_t\left[W_{t+1,w}^{k'}(w)e^{W_{t+1}^{k'}(w)}\right]}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]^2} \\ &= cov_t^{\mathbb{Q}_{k'}}(W_{t+1,w}^{k'}(w_t^k), W_{t+1,w}^{k'}(w)),\end{aligned} \tag{14}$$

where the probability $\mathbb{Q}_{k'}$ is defined by its Radon-Nikodym derivative as: $\frac{d\mathbb{Q}_{k'}}{d\mathbb{P}} = \frac{e^{W_{t+1}^{k'}(w)}}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]}$.

The term (14) is positive when $w = w_t^k$ (the covariance is then a variance) and therefore is, by continuity, also positive for any w close to w_t^k. Since $k \mapsto s_t^k$ is continuous, we obtain that the term (14) is positive whenever k' is close to k. We thus deduce that the left-hand side (equation (12)) has the sign of $s_t^{k'} - s_t^k$, as soon as k' is close to k.

We now focus on equation (13), which is the RHS of the inequality. Let us now introduce the function h:

$$h:\ (k,s_t^k)\mapsto \frac{ku'(c_t^k)}{\beta R_{t+1}}+\frac{E_t\left[W_{t+1,w}^k(w_{t+1}^k)e^{W_{t+1}^k(w_{t+1}^k)}\right]}{E_t\left[e^{W_{t+1}^k(w_{t+1}^k)}\right]},$$

where we recall that $w_{t+1}^k=R_{t+1}s_t^k$. With this notation, the equation (13) can be expressed as $h(k,s_t^k)-h(k',s_t^k)$. When $k'>k$ remains close to k, the sign of $h(k,s_t^k)-h(k',s_t^k)$ is the opposite of the one of $\frac{\partial h(k,s_t^k)}{\partial k}\Big|_{s_t^k=cst}$. Since equation (9) implies that $\frac{\partial W_{t+1,w}^k}{\partial k}\Big|_{s_t^k}=-u'(c_{t+1}^k)$, we have:

$$\begin{aligned}\frac{\partial E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^k\right]}{\partial k}\Bigg|_{s_t^k} &= -E_t^{\mathbb{Q}_k}\left[u'(c_{t+1}^k)\right]+E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^kW_{t+1,k}^k\right]\\ &\quad -E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^k\right]E_t^{\mathbb{Q}_k}\left[W_{t+1,k}^k\right],\end{aligned}$$

where the probability $\mathbb{Q}_k$ is defined similarly to $\mathbb{Q}_{k'}$. Using the Euler equation (8) and equality (9) we get:

$$\frac{\partial h(k,s_t^k)}{\partial k}\Bigg|_{s_t^k} \;=\; -k\beta R_{t+1}\,cov_t^{\mathbb{Q}_k}(u'(c_{t+1}^k),W_{t+1,k}^k). \tag{15}$$

Observe now that, from equation (10), we can derive by iteration:

$$\begin{aligned}W_{t+1,k}^k &= -\sum_{\tau=1}^{\infty}\beta^{\tau-1}E_{t+1}\left[u(c_{t+\tau}^k)\frac{e^{\sum_{j=2}^{\tau}W_{t+j}^k}}{\prod_{j=2}^{\tau}E_{t+j-1}\left[e^{W_{t+j}^k}\right]}\right]\\ &= -\sum_{\tau=1}^{\infty}\beta^{\tau-1}E_{t+1}^{\widehat{\mathbb{Q}}_{t+\tau}}\left[u(c_{t+\tau}^k)\right],\end{aligned} \tag{16}$$

where for any $\tau\geq 1$, the probability $\widehat{\mathbb{Q}}_{t+\tau}$ is defined by its Radon-Nikodym derivative as: $\frac{d\widehat{\mathbb{Q}}_{t+\tau}}{d\mathbb{P}}=\frac{e^{\sum_{j=2}^{\tau}W_{t+j}^k}}{\prod_{j=2}^{\tau}E_{t+j-1}\left[e^{W_{t+j}^k}\right]}$. Using this notation, we finally obtain:

$$\frac{\partial h(k,s_t^k)}{\partial k}\Bigg|_{s_t^k}=k\beta R_{t+1}\sum_{\tau=1}^{\infty}\beta^{\tau-1}cov_t^{\widehat{\mathbb{Q}}_{t+\tau}}\left(u'(c_{t+1}^k),u(c_{t+\tau}^k)\right). \tag{17}$$

From Lemma 1, we know that $(c_t^k)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are conditionally comonotonic. Since the income process is stochastically monotone, Proposition 2 then implies that we have $cov_t^{\widehat{\mathbb{Q}}_{t+\tau}}\left(u'(c_{t+1}^k),u(c_{t+\tau}^k)\right)<0$. From Equation (17), we get $\frac{\partial h(k,s_t^k)}{\partial k}\Big|_{s_t^k}<0$. Recalling that (13) can be expressed as $h(k,s_t^k)-h(k',s_t^k)$, we deduce that for $k'>k$ close to k, equation

(13) is positive.

Since we showed that the left-hand side in equation (12) had the same sign as $s_t^{k'} - s_t^k$, we deduce that $s_t^{k'} \geq s_t^k$, whenever $k' \geq k$ is close to k.

Second case: Euler equations (8) and (11) hold with strict inequality for agent k. We have $s_t^k = \underline{s}_t$. The budget constraint implies that $s_t^{k'} \geq \underline{s}_t = s_t^k$.

Conclusion. We have shown that $k \mapsto s_t^k$ was locally non-decreasing for any $k \geq 0$, which implies that $k \mapsto s_t^k$ is globally non-decreasing on $\mathbb{R}^+$. This concludes the proof of Proposition 1.