NOTES

Edited by **William Adkins**

On Homeomorphism Groups and the Compact-Open Topology

Jan J. Dijkstra

If X is a topological space, then we let $\mathcal{H}(X)$ denote the group of autohomeomorphisms of X equipped with the compact-open topology. For subsets A and B of X we define $[A, B] = \{h \in \mathcal{H}(X) : h(A) \subset B\}$, and we recall that the topology on $\mathcal{H}(X)$ is generated by the subbasis $\mathcal{S}_X = \{[K, O] : K \text{ compact}, O \text{ open in } X\}$. If X is a compact Hausdorff space, then $\mathcal{H}(X)$ is a topological group, that is, composition and taking inverses are continuous operations (see Arens [**2**]). It is well known that even for locally compact separable metric spaces the inverse operation on $\mathcal{H}(X)$ may not be continuous, thus $\mathcal{H}(X)$ is, in general, not a topological group (see the example to follow). However, it is a classic theorem of Arens [**2**] that if a Hausdorff space X is noncompact, locally compact, and locally connected, then $\mathcal{H}(X)$ is a topological group because the compact-open topology coincides with the topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$, where αX is the Alexandroff one-point compactification of X. We improve on this result as follows. (Recall that a *continuum* is a compact connected space. If A is a subset of X, then $\operatorname{int} A$ and ∂A denote the interior and boundary of A in X, respectively.)

Theorem. *Let X be a noncompact Hausdorff space. If every point in X has a neighbourhood that is a continuum, then the compact-open topology on $\mathcal{H}(X)$ coincides with the group topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$.*

Proof. The topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$ is generated by the subbasis

$$\mathcal{S}_X \cup \{[F, X \setminus K] : F \text{ closed}, K \text{ compact in } X\}$$

(see Arens [**2**], who calls this the *g-topology* on $\mathcal{H}(X)$). Let F be closed in X, and let O be the complement of a compact subset K of X. We show that $[F, O]$ is open in the compact-open topology. Let f be an arbitrary element of $[F, O]$, and consider the compactum $f^{-1}(K)$. Select for each x in $f^{-1}(K)$ a neighbourhood C_x of x in X that is a continuum. By compactness we can find a finite subset A of $f^{-1}(K)$ such that

$$f^{-1}(K) \subset \bigcup_{a \in A} \operatorname{int} C_a.$$

Define the compactum $C = \bigcup_{a \in A} C_a$ and construct in the same way a compact neighbourhood C' of C. Consider the obviously open subset

$$U = [C' \cap F, O] \cap [\partial C', f(X \setminus C)] \cap \bigcap_{a \in A} [\{a\}, f(\operatorname{int} C_a)]$$

of $\mathcal{H}(X)$ in the compact-open topology. It is clear that f belongs to U, and now we need verify only that U is a subset of $[F, O]$.

Assume to the contrary that h lies in $U \setminus [F, O]$. Thus there is an x in F with $h(x)$ in $K = X \setminus O$. Since $h \in U \subset [C' \cap F, O]$, we see that x cannot lie in C'. Note that

$$x \in h^{-1}(K) \subset h^{-1}(f(C)),$$

hence x lies in $h^{-1}(f(C_a))$ for some a in A. Since

$$h \in U \subset [\{a\}, f(\operatorname{int} C_a)],$$

we see that a lies in $h^{-1}(f(C_a))$. Observe that $h^{-1}(f(C_a))$ is a continuum that connects the point a inside C' with x outside C', hence there is a y in $h^{-1}(f(C_a)) \cap \partial C'$. Since $U \subset [\partial C', f(X \setminus C)]$, we infer that $h(y) \notin f(C)$, which contradicts the fact that y is a member of $h^{-1}(f(C_a))$. The proof is complete. ■

Example. For the sake of completeness we include an example of a locally compact metric space X such that the inverse operation on $\mathcal{H}(X)$ is not continuous. Let C be a Cantor set, and let a and b be distinct points of C. We consider the space $\mathcal{H}(X)$ with the compact-open topology, where $X = C \setminus \{a\}$. Select for a and b neighbourhood bases $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$, respectively, such that $U_1 \cap V_1 = \emptyset$, U_n and V_n are both closed and open, and $U_{n+1} \subsetneq U_n$ and $V_{n+1} \subsetneq V_n$ for each n. For instance, if we put

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0, 2\} \text{ for each } n \in \mathbb{N} \right\},$$

that is, C is the "middle third" Cantor set in the real line, then we can simply take $a = 0$, $b = 1$, $U_n = C \cap [0, 3^{-n}]$, and $V_n = C \cap [1 - 3^{-n}, 1]$.

Noting that nonempty closed and open subsets of C are also Cantor sets, we can find for each n in $\mathbb{N}$ an h_n in $\mathcal{H}(C)$ that satisfies the following conditions:

$$\begin{aligned} h_n(x) &= x \quad (x \in C \setminus (U_n \cup V_n)), \\ h_n(a) &= a, \\ h_n(U_{n+1}) &= U_n, \\ h_n(U_n \setminus U_{n+1}) &= V_{n+1}, \\ h_n(V_n) &= V_n \setminus V_{n+1}. \end{aligned}$$

If C is the middle third set, then the map h_n can be realized easily via a piecewise linear transformation of the interval [0,1].

Since $h_n(a) = a$, we see that the restriction $h_n \upharpoonright X$ is a member of $\mathcal{H}(X)$ for each n. Let the element $[K, O]$ of $\mathcal{S}_X$ be a neighbourhood of the identity element e of $\mathcal{H}(X)$. Thus K is a compact subset of O and X, which enables us to find an M such that U_M and K are disjoint. If b is not a member of K, then there is an N such that $N \geq M$ and $V_N \cap K = \emptyset$, so $h_n \upharpoonright K$ is an identity map and $h_n \upharpoonright X$ is a member of $[K, O]$ for each n satisfying $n \geq N$. If b is a point in K, then there is an N such that $N \geq M$ and V_N is contained in O. Consequently, whenever $n \geq N$ we have

$$h_n(K \setminus V_n) = K \setminus V_n \subset O, \quad h_n(K \cap V_n) \subset V_n \subset V_N \subset O,$$

whence $h_n \upharpoonright X$ is in $[K, O]$. We have shown that $\lim_{n\to\infty} h_n \upharpoonright X = e$ in $\mathcal{H}(X)$. On the other hand, $h_n^{-1}(b)$ lies in U_n for each n, so certainly $\lim_{n\to\infty}(h_n \upharpoonright X)^{-1} \neq e$.

Remarks. One of the more interesting (and challenging) problems in infinite-dimensional topology is the topological classification of $\mathcal{H}(M)$, where M is a manifold. Let $\mathbb{I}$ denote the interval [0, 1], and let $\mathcal{H}_\partial(\mathbb{I}^n)$ stand for the subgroup of $\mathcal{H}(\mathbb{I}^n)$ consisting of homeomorphisms that fix the boundary of the n-cube $\mathbb{I}^n$. Anderson [**1**] proved that $\mathcal{H}_\partial(\mathbb{I})$ is homeomorphic to the separable Hilbert space ℓ^2 (see [**3**, Proposition VI.8.1] or [**11**]). It was shown by Luke and Mason [**12**] that $\mathcal{H}_\partial(\mathbb{I}^2)$ is an absolute retract, that is, the space is homeomorphic to a retract of ℓ^2. This result in combination with the fact that $\mathcal{H}_\partial(\mathbb{I}^2)$ is a topological group guarantees that the space is homeomorphic to ℓ^2 (apply, for instance, Dobrowolski and Toruńczyk [**7**]). If $n \geq 3$ it is wide open whether $\mathcal{H}_\partial(\mathbb{I}^n)$ is an absolute retract. For the Hilbert cube Q, that is, for $n = \infty$, the corresponding problem was solved by Ferry [**10**] and Toruńczyk [**13**]. They proved that $\mathcal{H}(Q)$ is homeomorphic to ℓ^2 (observe that Q has no boundary).

If D is a countable dense subset of M, then we let $\mathcal{H}(M, D)$ signify the group $\{h \in \mathcal{H}(M) : h(D) = D\}$. The topological classification problem for $\mathcal{H}(M, D)$ was recently solved completely in joint work of the author and Jan van Mill [**4**], [**5**], [**6**]. If M is a one-dimensional topological manifold or a Cantor set, then $\mathcal{H}(M, D)$ is homeomorphic to $\mathbb{Q}^\infty$, the countable power of the space of rational numbers. If M is a topological manifold of dimension at least two, a Hilbert cube manifold, or a manifold modelled on a universal Menger continuum, then $\mathcal{H}(M, D)$ is homeomorphic to the famed Erdős space [**9**], which consists of the vectors in ℓ^2 whose coordinates are all rational. (Menger continua are higher dimensional analogues of the Cantor set; see, for instance, [**8**, sec. 1.11].)

REFERENCES

1. R. D. Anderson, Spaces of homeomorphisms of finite graphs (unpublished manuscript).
2. R. Arens, Topologies for homeomorphism groups, *Amer. J. Math.* **68** (1946) 593–610.
3. C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Polish Scientific Publishers, Warsaw, 1975.
4. J. J. Dijkstra and J. van Mill, Homeomorphism groups of manifolds and Erdős space, *Electron. Res. Announc. Amer. Math. Soc.* **10** (2004) 29–38.
5. ———, On the group of homeomorphisms of the real line that map the pseudoboundary onto itself, *Canad. J. Math.* (to appear).
6. ———, Erdős space and homeomorphism groups of manifolds (2004 preprint).
7. T. Dobrowolski and H. Toruńczyk, Separable complete ANRs admitting a group structure are Hilbert manifolds, *Topology Appl.* **12** (1981) 229–235.
8. R. Engelking, *Dimension Theory*, North-Holland, Amsterdam, 1978.
9. P. Erdős, The dimension of the rational points in Hilbert space, *Ann. of Math.* **41** (1940) 734–736.
10. S. Ferry, The homeomorphism group of a compact Hilbert cube manifold is an ANR, *Ann. of Math.* **106** (1977) 101–119.
11. J. E. Keesling, Using flows to construct Hilbert space factors of function spaces, *Trans. Amer. Math. Soc.* **161** (1971) 1–24.
12. R. Luke and W. K. Mason, The space of homeomorphisms of a compact two-manifold is an absolute neighborhood retract, *Trans. Amer. Math. Soc.* **164** (1972) 275–285.
13. H. Toruńczyk, Homeomorphism groups of compact Hilbert cube manifolds which are manifolds, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **25** (1977) 401–408.

Faculteit der Exacte Wetenschappen/Afdeling Wiskunde, Vrije Universiteit, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands
dijkstra@cs.vu.nl