Tutorial week 6

Note Title — 24/02/2015

① LU decomposition with partial pivoting

Given the system $Ax = B$ I want to decompose A so that: $P \cdot A = LU$ where

P = permutation matrix
L = lower triangular matrix
U = upper triangular matrix

If A is $n \times n$ I have to multiply A on the left by a permutation matrix P and an lower triangular L so to obtain an upper triangular matrix U

$$L_{n-1} P_{n-1} \cdots L_2 P_2 L_1 P_1 A \Rightarrow U,\ L,\ U \text{ of my decomposition}$$

Example:

Given $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$

I want to choose a pivot in the <u>first column</u> so to minimise the # of operations.
Once I chosed the pivot I want to interchange the rows of A so to have the pivot in A_{11}.

the right pivot to choose is 8 so I need to interchange row 1 and 3 in A:

I can do that by multiplying A by an appropriate permutation matrix P_1:

$$\overbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{P_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix} = A_1$$

Now I want to multiply A_1 on the left by L_1. L_1 is lower triangular, with ones on the main diagonal and entries on the first column so to eliminate the first column of A_1

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 0 & 1 & 0 \\ 3/4 & 0 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 8 & 7 & 9 & 8 \\ 4 & 3 & 3 & 1 \\ 2 & 1 & 1 & 0 \\ 6 & 7 & 9 & 8 \end{bmatrix}}_{A_1} = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & -1/2 & -3/2 & -3/2 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & 7/4 & 9/4 & 17/4 \end{bmatrix} = A_2$$

Now I have to do the same thing to eliminate the 2^{nd} row of A_2

$$A_2 = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & -1/2 & -3/2 & -3/2 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & 7/4 & 9/4 & 17/4 \end{bmatrix}$$

I need to choose the pivot in the 2^{nd} column.

the right pivot is $7/4$ so I need to change the position of the fourth and second row of A_2

the permutation matrix P_2 looks like this:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{P_2} \underbrace{\begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & -1/2 & -3/2 & -3/2 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & 7/4 & 9/4 & 17/4 \end{bmatrix}}_{A_2} = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & -3/4 & -5/4 & -5/4 \\ 0 & -1/2 & -3/2 & -3/2 \end{bmatrix} = A_3$$

Now I need to multiply A_3 by a matrix L_2 of the same kind of L_1, but with coefficients in the second column so to eliminate the 2nd column of A_3

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3/7 & 1 & 0 \\ 0 & 2/7 & 0 & 1 \end{bmatrix}}_{L_2} \cdot A_3 = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -2/4 & 4/7 \\ 0 & 0 & -6/7 & -2/7 \end{bmatrix} = A_4$$

I multiply A_4 by a permutation matrix P_3 so to have the pivot $-6/7$ in the right position:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P_3} \cdot A_4 = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & \underset{\alpha}{-2/4} & 4/7 \end{bmatrix} = A_5$$

I need now to multiply A_5 by a matrix L_3 so to eliminate the entry α:

$$L_3 \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & 1 \end{bmatrix} \right. \cdot A_5 = \begin{bmatrix} 8 & 7 & 9 & 8 \\ 0 & 7/4 & 9/4 & 17/4 \\ 0 & 0 & -6/7 & -2/7 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} = \mathcal{U}$$

: this is my matrix U for the decomposition the matrix $\mathbf{P}$ for the decomposition is given by:

$$P = P_3 \cdot P_2 \cdot P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

the matrix $\mathbf{L}$ is lower triangular, with ones on the main diagonal and entries under it equals to the coefficients in L_1, L_2, L_3

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/4 & 3/7 & 1 & 0 \\ 3/4 & 2/7 & -1/3 & 1 \end{bmatrix}$$

Once you have P, L, U you can use Gaussian elimination to solve $Ax = B$ as we saw in the last tutorial.

2) Flops counting

1 FLOP = 1 FP operation $(+ ; - ; \times ; \div)$

Adding the entries of 1 vector:

$\bar{x} \in \mathbb{R}^n \qquad \bar{x} = [x_1, x_2, x_3, \dots, x_n]$

$\sum_{k=1}^{n} x_k = n - 1$ (because you initialize the sum with x_1)

- factorial $(n > 2)$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

$$\prod_{k=1}^{n} n = n-2$$ (because the last multiplication is always by 1!)

$$5! = \underbrace{5 \cdot 4}\underbrace{\cdot 3}\underbrace{\cdot 2} \cdot 1$$ $n=5$ but I do 3 multiplication

Matrix - vector multiplication:

$A_{n \times n}$ $\quad \bar{x} \in \mathbb{R}^n$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$$

for $j = 1:n$

$\quad c_j = A_{j1} \cdot x_1 \rightarrow n$ flops

$\quad$ for $k = 2:n$

$\quad\quad c_j = c_j + A_{jk} \cdot x_k \rightarrow 1 + 2(n-1)$ flops

j is on the row of A, k on the colums

$$\sum_{j=1}^{n} \left(1 + \sum_{k=2}^{n} 2 \right) = n\left(1 + 2(n-1)\right) = n\left(1 + 2n - 2\right) = 2n^2 - n$$

matrix - matrix multiplication

$A_{n\times n}$ $B_{n\times n}$

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \end{array}$$

$$\left.\begin{array}{l} \text{for } j = 1:n \\ \quad \text{for } \ell = 1:n \\ \quad\quad c_{j\ell} = A_{j1} \cdot B_{1\ell} \end{array}\right\} n \cdot n \text{ flops}$$

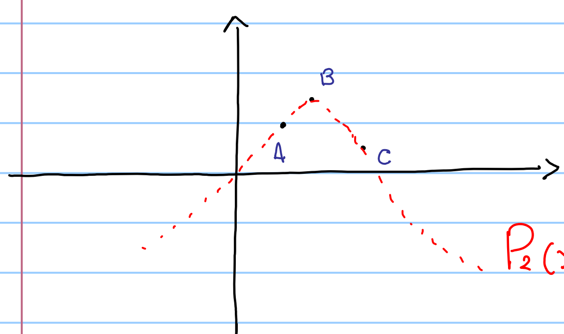
$$\begin{array}{l} \quad\quad \text{for } k = 2:n \\ \quad\quad\quad \left. c_{j\ell} = c_{j\ell} + A_{jk} \cdot B_{k\ell} \right\} 1 + 2(n-1) \text{ flops} \end{array}$$

j is on the row of A
ℓ is on the colums of B
k is on the column of A and rows of B

$$\sum_{j=1}^{n}\sum_{\ell=1}^{n}\left(1 + \sum_{k=2}^{n} 2\right) = n \cdot n\,\big(1 + 2(n-1)\big) = n^2(1 + 2n - 2) = 2n^3 - n^2$$

Polynomial interpolation using vandermonde matrix

Given A (1, 2) B (1.4, 2.2) C (1.7, 1.6) I want to find the interpolating polynomial:



$P_2(x) = a_0 + a_1 x + a_2 x^2$ polynomial of degree 2 (I have 3 points)

To find a_0, a_1 and a_2 I need to solve : $VA = Y$ where:

V is the vandermonde matrix
A is $[a_0, a_1, a_2]^T$
Y is $[2, 2.2, 1.6]^T$ (y components of my given points)

V looks like this:

$$\begin{bmatrix} 1{,}0 & (1{,}0)^2 & (1{,}0)^3 \\ 1{,}4 & (1{,}4)^2 & (1{,}4)^3 \\ 1{,}7 & (1{,}7)^2 & (1{,}7)^3 \end{bmatrix}$$

So the system I need to solve to find a_0, a_1 and a_2 is:

$$\begin{bmatrix} 1{,}0 & 1{,}0 & 1{,}0 \\ 1{,}4 & (1{,}4)^2 & (1{,}4)^3 \\ 1{,}7 & (1{,}7)^2 & (1{,}7)^3 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2.0 \\ 2.2 \\ 1.6 \end{pmatrix}$$

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