Tutorial week 10: ODEs & PDEs

Note Title 16/03/2015

Differential equations are really important when it comes to describe and model physic's laws. In fact, many times, when we try to describe a Physic's problem two or more related quantities (such as forces and accelaration for Newton's law, or the heat and the temperature profile in the Heat equation) interact and influence each other . This means that a change of one variable might influence how the other behave.

Newton's law and the spring

In general, Newton's law can be written as: $\vec{F} = m\vec{a}$

In this law, the applied force influences the accelaration which is, the derivative of the velocity.

$$\vec{a} = d\vec{v}/dt \quad (1)$$

Moreover, from Physic, we could observe that the velocity itself is just the derivative of the displacement.

$$\vec{v} = d\vec{x}/dt \quad (2) \qquad \text{therefore: } \vec{a} = d\vec{x}^2/dt^2$$

Let's consider a spring. We know already, that the force F applied to the spring is proportional to the displacement the spring experience. This can be written mathematically as:

$$\vec{F} = -k\vec{x}$$

where k is the so called Hook's constant and it describes some material properties of the spring.

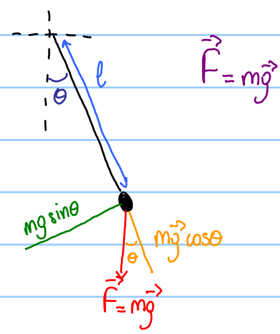
Putting together Newton's law and what we have just written to describe the behaviour of a spring we get our first Ordinary Differential Equation (ODE):

$$-kx = m\frac{d\vec{x}^2}{dt}$$

The above equation describes the displacement as a function of time, and it relates the displacement x itself to the accelation.

The fact that we state that the displacment in the above equation is a function of time allows us the define the time t as independent variable and x as dependent variable. This means also, that a solution to the differential equation will be expressed as x(t).

II Pendulum:



Because we are only concerned with changes in speed, and because the bob is forced to stay in a circular path, we apply Newton's equation to the tangential axis only. g is the gravitational accelaration. It's a quantity similar to a in the previous example.

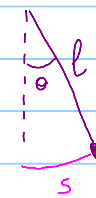
Therefore we can rewrite the force acting on a pendulum as:

$$F = -mg\sin\theta = ma \rightarrow a = -g\sin\theta$$

why there is a minus sign?

This linear acceleration a along the green axis can be related to the change in angle theta by the arc length formulas; s is arc length given by:

$s = \theta l$ -> s is the displacement here



so the velocity v is given by: $\vec{v} = \frac{ds}{dt} = l\frac{d\theta}{dt}$

and the acceleration $\vec{a}$ is: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2 s}{dt^2} = l\frac{d^2\theta}{dt^2}$

Therefore the Ordinary Differential Equation for the simple pendulum is:

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

The above equation relates the angle theta created by the pendulum in motion to the angular accelaration it experience during its oscillation.

Introduction to Partial Differential Equations

A more complex and usefull class of differential equation is given by the so called Partial Differential Equations. In fact, most physical phenomena, wheter in the domain of fluid dynamics, elctricity, diffusion, optics an so on, can be described in general by PDEs.

A Partial Differential Equation (PDE) is an equation that contains partial derivatived. In contrast to ODEs, where the unknown function depends only on one variable, in PDEs, the independent variable depends on several variables such as location and time for example.

A very well known PDE is the heat equation in one dimension: $\frac{du}{dt} = k\frac{d^2u}{dx^2}$

In the previous, the unknown variable u depends always on more that one variable. It is also clear, from the previous that there are two independent variables: the time t and the displacement x.

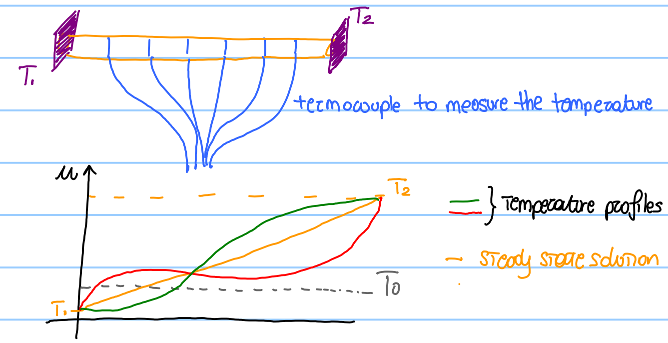
u is a function of x and t : $u(x,t)$

Understanding the heat equation:

Let's suppose we have the following experiment: a long insulated rod in copper is placed is a room at some temperature T0 for a sufficently long time so that the rod comes to a steady-state temperature similar to the enviroment.

$T = T_0 = 10°C$

Next we take the rod out of the enviroment and we attach two temperature elements at the ends of the rod. This means that if the temperature of the rod differs from one of the two prescribed by the temperature elements this will start cooling or heating the rod until it reaches the desired temperature.

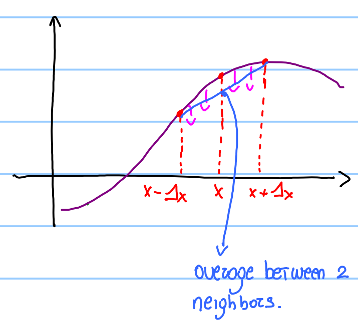


$$\underbrace{\frac{du}{dt}}_{\text{This is the rate of change in temperature with respect to time}} = k\underbrace{\frac{d^2u}{dx^2}}_{\text{The second derivative of a function describes its concavity. Here essentially it compares the temperature at one point to the temperature at neighboring points.}}$$

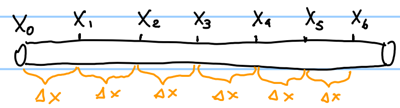
This equaiton simply sais that the temperature u(x,t), at some point on the rod, at some time t in time, is increasing $(u_t > 0)$ or decreasing $(u_t < 0)$ according to wheter u_{xx} is positive or negative.

To see how the RHS of the heat equation can be interpreted to measure the heat flow, suppose we approximate the second derivative of the temperature with respect to x using finite difference:

$$\underbrace{u_{xx}(t,x) = \frac{1}{\Delta x^2}\left[u(x+\Delta x) - 2u(x,t) + u(x-\Delta t)\right]}_{\text{second order central finite difference scheme}}$$



This is the same as dividing the rod in (n+1) equal parts of amplitude Δx. We now have a set of grid points on the rod that defines all the sub intervals.



We can now associate each nodes to a variable u_j which describes the change in temperature at that point in time.

the central finite difference scheme now can be written as:

$$\frac{d^2u}{dx^2}(x_j,t) \sim \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{\Delta x^2}$$

Using the expression above, the initial PDE can be written as a system of ODEs!!

$$\frac{du_j}{dt} = \frac{k}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1}) \qquad 1 \leq j \leq n$$

Some practical examples

① Separable D.E. : it's in the form: $\frac{dy}{dx} = g(y)f(x)$

each of these functions is a function of just ① variable

$$\frac{dy}{dx} = g(y)f(x) \rightarrow \underbrace{\frac{dy}{g(y)} = f(x)dx}_{\text{separate}} \rightarrow \underbrace{\int \frac{dy}{g(y)} = \int f(x)dx}_{\text{integrate}} \rightarrow \underbrace{y(x) = \dots}_{\text{isolate for the dependent variable}}$$

① $\frac{dy}{dx} = -\frac{x}{y} \rightarrow y\,dy = -x\,dx = \int y\,dy = -\int x\,dx = \frac{y^2}{2} = -\frac{x^2}{2} + c = \frac{y^2}{2} + \frac{x^2}{2} = c$

② $(1-x)dy + y\,dx = 0 \rightarrow \frac{dy}{y} = \frac{dx}{(1-x)} = \int \frac{dy}{y} = \int \frac{dx}{(1-x)} = \ln|y| = \ln|1-x| + c \rightarrow e^{\ln|y|} = e^{\ln|1-x|+c} \rightarrow$

$\rightarrow y(x) = c|1-x|$

Homogeneous second order D.E.

$ay'' + by' + cy = 0$ this is an homogeneous second order type of D.E.

$y = e^{mx} \quad y' = me^{mx} \quad y'' = m^2e^{mx}$ I apply this substitution to the DE

$a m^2 \cancel{e^{mx}} + bm\cancel{e^{mx}} + c\cancel{e^{mx}} = 0 \rightarrow am^2 + bm + c = 0 \rightarrow$ is a quadratic equation, I can find the roots

using $m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Depending on the nature of the roots I can write the solution of the D.E

a) m_1 and m_2 are real and distinct $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

b) m_1 and m_2 are real and repeated: $y(x) = C_1 e^{m_1 x} + C_2 x e^{m_2 x}$

c) m_1 and m_2 are complex $y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ $\alpha = \mathrm{Re}\{m_1\}$ $\beta = \mathrm{Im}\{m_1\}$

Example ①

$2y'' - 5y' - 3y = 0 \rightarrow 2m^2 - 5m - 3 = 0 \rightarrow m_{1,2} = \frac{5 \pm \sqrt{25+24}}{4} = \frac{5 \pm 7}{4}$ $m_1 = 3$, $m_2 = -\frac{1}{2}$

roots are real and distinct!

$y(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$

Example ②

$y'' - 10y' + 25y = 0 \rightarrow m^2 - 10m + 25 = 0 \rightarrow m_{1,2} = \frac{10 \pm \sqrt{100 - 100}}{2} \Rightarrow m_1 = m_2 = 5$

roots are real and repeated

$y(x) = C_1 e^{5x} + C_2 x e^{5x}$

Example ③

$y'' + 4y' + 7y = 0 \rightarrow m^2 + 4m + 7 = 0 \rightarrow m_{1,2} = \frac{-4 \pm \sqrt{16-28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm \sqrt{3}i$

roots are complex (and conjugate!)

$\alpha = \mathrm{Re}\{m_1\} = -2$; $\beta = \mathrm{Im}\{m_1\} = \sqrt{3}$

$y(x) = e^{-2x}\left(C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)\right)$