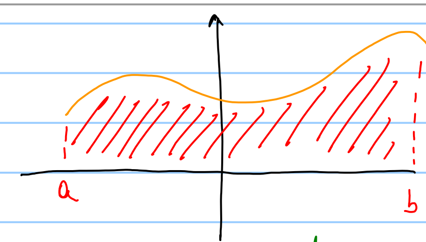
TUTORIAL WEEK 9

Note Title 09/03/2015

Numerical integration



$$I(f) = \int_a^b f(x)dx = F(b) - F(a)$$

sometimes I cannot compute the integral in closed form.

I can use quadrature formulas.

General quadrature formula:

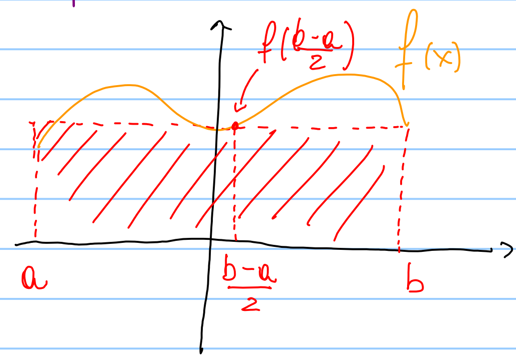
$$I_{approx}(f) = \sum_{i=0}^{n} w_i f(x_i) = w_0 f(x_0) + w_1 f(x_1) + \dots + w_n f(x_n)$$

w_i = WEIGHTS

x_i = NODES

you can look at the integral as a weighted sum.

1) MIDPOINT FORMULA



$\bar{I}$ approximate the area under $f(x)$ with a rectangle

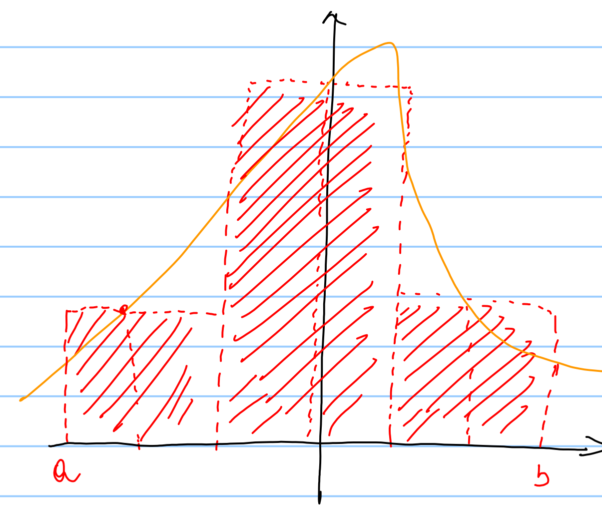
$$\bar{I}_{approx} = (b-a)\cdot f\left(\frac{b-a}{2}\right)$$

1a) COMPOSITE MIDPOINT FORMULA

divide $[a, b]$ into M subintervals of width H: $\frac{b-a}{M}$

define the midpoints of each subinterval $\bar{x}_k$

$$\bar{I}^{c}_{mp}(x) = H \sum_{j=1}^{n} f(\bar{x}_k)$$



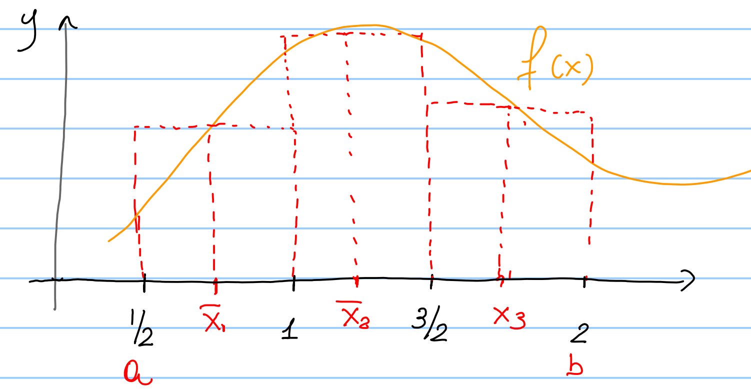
example : $\frac{2}{\sqrt{\pi}}\int_{1/2}^{2}\sqrt{4-x^3}\,dx \qquad M = 3$

$H = \frac{b-a}{m} = \frac{2-1/2}{3} = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}$ compute the width of each interval

$\bar{x}_1 = 3/4$

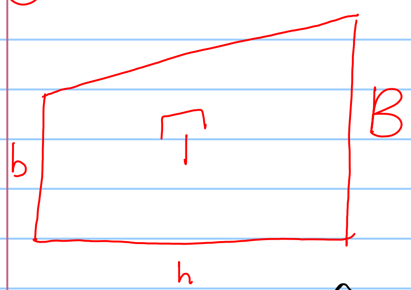
$\bar{x}_2 = 5/4$ compute the midpoints

$\bar{x}_3 = 7/4$

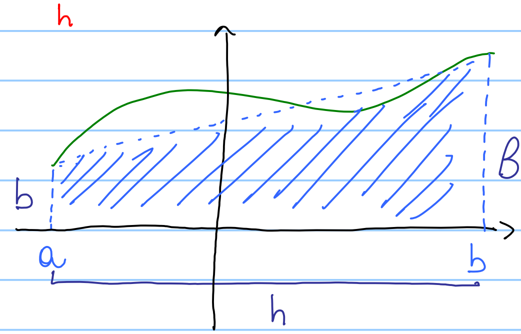


$$I = \frac{1}{2}\frac{2}{\sqrt{\pi}}\left(f(\bar{x}_1) + f(x_2) + f(x_3)\right) = \frac{1}{\sqrt{\pi}}\left(\sqrt{4-(3/4)^3} + \sqrt{4-(5/4)^3} + \sqrt{4-(7/4)^3}\right)$$

(2) TRAPEZOIDAL RULE



the area of the trapezoid is: $A = \frac{(B+b)h}{2}$



$$I(x) = \int_a^b f(x)\,dx \simeq (b-a) \cdot \frac{(f(b)+f(a))}{2}$$

2b COMPOSITE TRAPEZOIDAL RULE

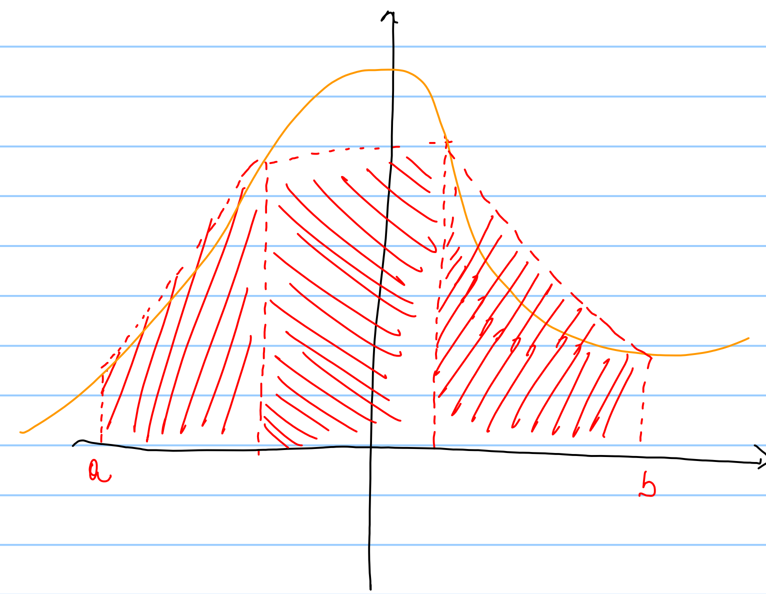
I want to compute $I(x) = \int_a^b f(x)dx$

divide $[a, b]$ into M subintervals of width $H = \frac{b-a}{M}$

the composite Trapezoidal rule is:

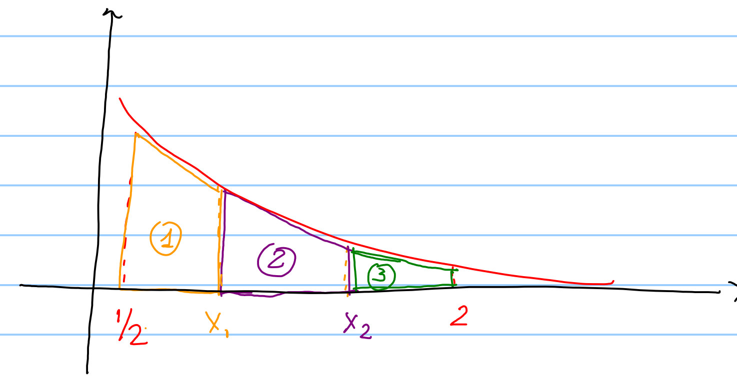
$$I_{tr}^{c}(f) = H\left(\frac{f(b) - f(a)}{2} + \sum_{k=1}^{m-1} f(x_k)\right)$$

To justify this $\neq$ weight for $f(a)$ and $f(b)$ think that you use these points just once in the sum.



example: COMPOSITE TRAPEZOIDAL RULE

$$\frac{1}{4}\int_{1/2}^{2} e^{-x^2}\,dx \quad \text{using } M = 3 \text{ intervals.}$$



width of the intervals: $H = \frac{b-a}{M} = \frac{2-1/2}{3} = \frac{1}{2}$

$\bar{x}_1 = a + 1 \cdot H = 1$

$\bar{x}_2 = a + 2 \cdot H = 3/2$

$$\bar{I} = 1 + 2 + 3 = H \cdot \frac{(f(a) + f(x_1))}{2} + H \cdot \frac{(f(x_1) + f(x_2))}{2} + H \frac{(f(x_2) + f(b))}{2} = \text{some } \#$$

Errors:

Basic quadrature rules : if $|f''(x)| < k$ then:

$$E_{mid} = \frac{k(b-a)^3}{24n^2}$$

$n = \#$ of nodes

$$E_{Trap} = \frac{k(b-a)^2}{12n^2}$$

Example:

$\int_0^1 e^{x^2}dx$ with $n = 10$ $f(x) = e^{x^2}$ $f'(x) = 2xe^{x^2}$ $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$

$|f''(x)| \to f''$ is increasing in $[0,1] \to$ so $\forall x$ in $[0,1]$

$$2e^{x^2} + 4x^2e^{x^2} \leq 2e^{(1)^2} + 4 \cdot 1^2 e^{(1)^2} \leq 6e \to k \text{ (is an upper bound)}$$

So the error is: $E_{mid} = \dfrac{k(b-a)^3}{24n^2} = \dfrac{6e(1-0)^3}{24 \cdot 10^2} = 0{,}0067\ldots$

$$E_{trap} = \frac{k(b-a)^3}{12n^2} = \frac{6e(1-0)^3}{12 \cdot 10^2} = 0{,}0135\ldots$$

the error for composite quadrature rules is:

$$E_{mp}^{c} = \frac{(b-a)}{24} H^2 f''(\xi)$$

find the upper bound for $f''(x)$

$$E_{TR}^{c} = \frac{(b-a)}{12} H^2 f''(\xi)$$