Contents

1 Notation

Firstly, define N_H and N_V to be the number of hidden and visible units respectively. H_h is a vector of weights for the hidden units, where $h = 1, 2, \ldots, N_H$. Similarly, V_v is a vector of weights for the visible units, where $v = 1, 2, \ldots, N_V$. W_{hv} is a matrix of weights. $C_{kl}^{(hv)}$ is the coupling matrix between the h^{th} hidden unit and the v^{th} visible unit. For simplicity, we will assume that C is a $d \times d$ matrix.

1.0.1 Note on the coordination number

$n_V^{(h)}$ is the coordination number for visible units of the h^{th} hidden unit. This is the number of connected visible units. Similarly, $n_H^{(v)}$ is the coordination number for hidden units of the v^{th} visible unit.

Throughout this section, we are operating on the last visible unit (the ${N_V}^{\text{th}}$ unit), which is connected to $n_H^{(N_V)}$ hidden units. Since it is clear which visible unit we are referring to, the superscript will be omitted.

In the general case, $n_H = N_H$ as the hidden and visible units form a complete bipartite graph. In the Snake RBM case, each visible unit is connected to exactly one hidden unit so $n_H = 1$.

2 Imposing the causality condition on general RBMs

Let $D_{k_1 k_2 \ldots k_{n_H}}$ be the contraction of the red network, and $|+\rangle$ to be a vector of 1s.

If the causality condition were to hold, then,

$$D = \bigotimes |+\rangle \,. \tag{1}$$

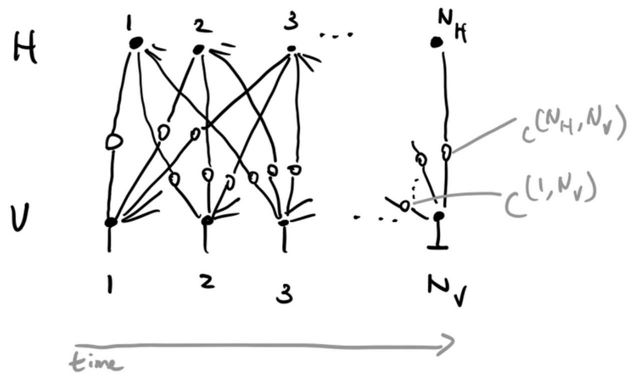


Figure 1: update this!

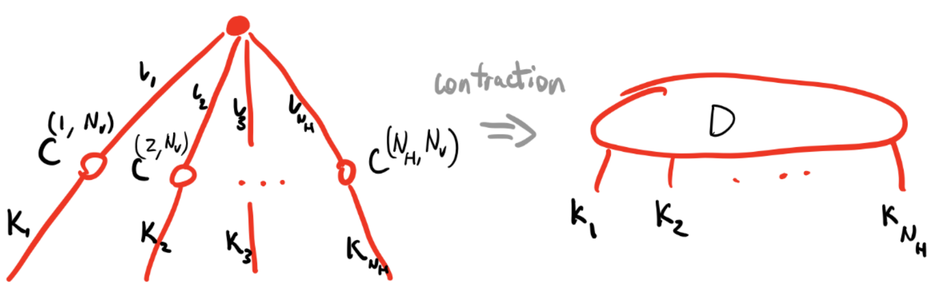
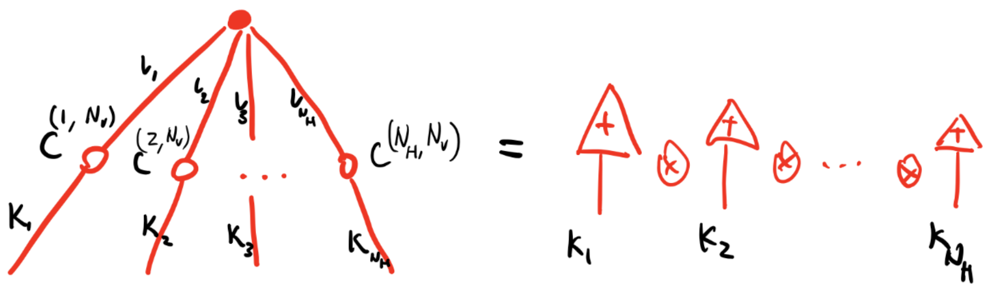


Figure 2: update this!

This is represented visually as



The RHS is then a tensor of 1s. This amounts to an a set of equations where each element of D must equal 1.

2.0.1 Preliminary steps

Let P be the contraction of the vectorised identity with the copy tensor, and let $k = \sqrt{d}$. The index at which a one appears in the vectorised identity is given by a sequence $S = (s_1, s_2, \dots, s_k)$ where $s_i = k(i-1)+i$ for $1 \leq i \leq k$.

$$P_{l_1 l_2 \dots l_{n_H}} = \begin{cases} 1 & l_1 = l_2 = \dots = l_{n_H} \text{ and } l_1 \in S \\ 0 & \text{otherwise} \end{cases}$$

2.0.2 Defining the coupling matrices

$C^{(hv)}_{kl}$ is a $d \times d$ coupling matrix between the $h^t h$ hidden and vth visible unit.

In our notation, we extend k and l to range over $[1, d]$, so a more useful definition could be as follows.

$$C^{(hv)}_{kl} = e^{W_{h,v}(k-1)(l-1) + \frac{H_h(k-1)}{N_H} + \frac{V_v(l-1)}{N_V}} \tag{2}$$

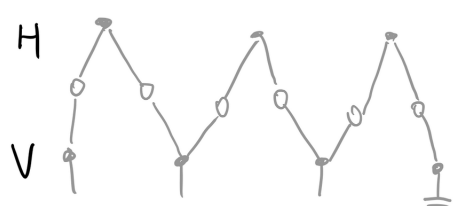
2.0.3 Solving for elements of the contracted tensor

In fact, we can solve for specific elements of the D tensor.

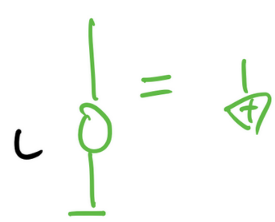
$$
\begin{aligned}
D_{k_1 k_2 \ldots k_{n_H}} &= \sum_{l_1 l_2 \ldots l_{n_H}} P_{l_1 l_2 \ldots l_{n_H}} C^{(1,N_V)}_{k_1 l_1} C^{(2,N_V)}_{k_2 l_2} \cdots C^{(n_H,N_V)}_{k_{n_H} l_{n_H}} && (3)\\
&= \sum_{s \in S} P_{ss\ldots s} C^{(1,N_V)}_{k_1 s} C^{(2,N_V)}_{k_2 s} \cdots C^{(n_H,N_V)}_{k_{n_H} s} && (4)\\
&= \sum_{s \in S} C^{(1,N_V)}_{k_1 s} C^{(2,N_V)}_{k_2 s} \cdots C^{(n_H,N_V)}_{k_{n_H} s} && (5)\\
&= \sum_{s \in S} \prod_{h=1}^{n_H} C^{(h,N_V)}_{k_h,s} && (6)\\
&= \sum_{s \in S} \prod_{h=1}^{n_H} e^{W_{h,N_V}(k_h-1)(s-1) + \frac{V_{N_V}(s-1)}{N_V} + \frac{H_h(k_h-1)}{N_H}} && (7)
\end{aligned}
$$

3 Imposing the causality condition on minimally-connected RBMs

Consider a subset of RBMs that are connected in a minimal way: the
We will call this the *Snake RBM*.
“Snake RBM” has this form.



Solving the Snake RBM amounts to solving the following simplified equation.



So we continue from Eq. (**??**), where $= 1$.

$$
\begin{aligned}
D_{1,1,\ldots,1} &= \sum_{s\in S} \prod_{h=1} C_{1,s}^{h,} && \text{(\textbf{??})} \\
&= \sum_{s\in S} e^{(s-1)\frac{V}{}} && (8)
\end{aligned}
$$

According to my code, it seems like in general there are solutions (for $d \neq 4$).