6.837 Linear Algebra Review

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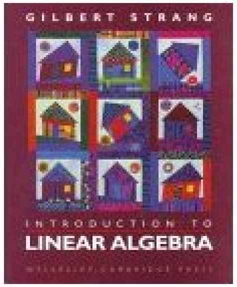
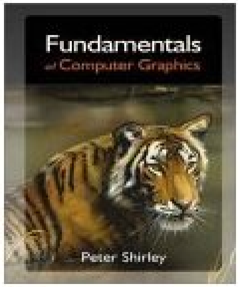
Overview

- Basic matrix operations (+, -, *)
- Cross and dot products
- Determinants and inverses
- Homogeneous coordinates
- Orthonormal basis

Additional Resources

- 18.06 Text Book
- 6.837 Text Book
- 6.837-staff@graphics.csail.mit.edu
- Check the course website for a copy of these notes





What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

$$m \times n \text{ matrix}$$

$$\begin{array}{rc} & \overbrace{\hphantom{a_{00} \quad\quad a_{01}}}^{n \text{ columns}} \\ m \text{ rows} \left\{ \vphantom{\begin{matrix} a \\ a \end{matrix}} \right. & \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \end{array}$$

Basic Operations

- Transpose: Swap rows with columns

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad M^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad V^T = \begin{bmatrix} x & y & z \end{bmatrix}$$

Basic Operations

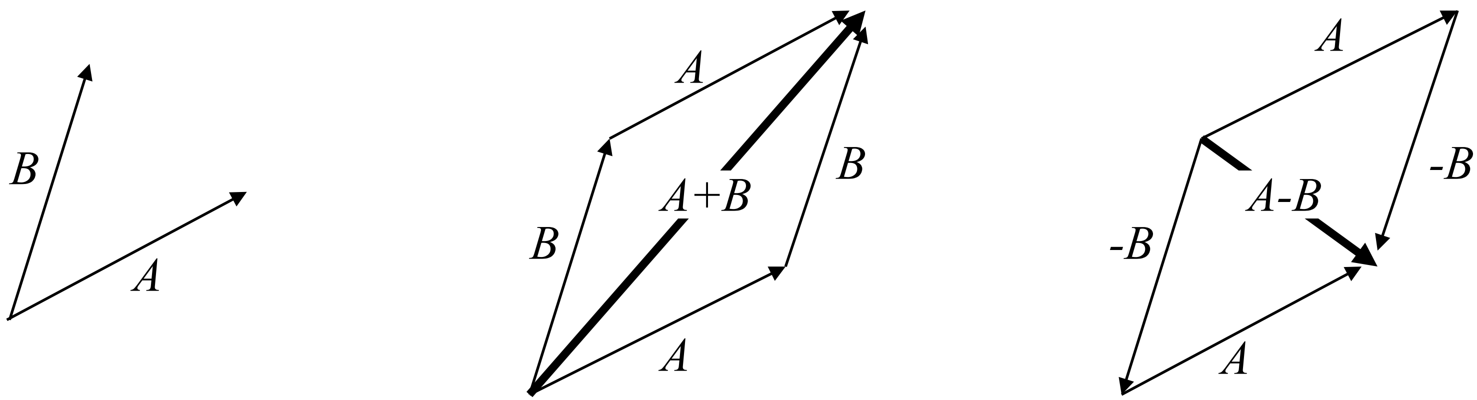
- Addition and Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements



Basic Operations

- Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

An $m \times n$ can be multiplied by an $n \times p$ matrix to yield an $m \times p$ result

Multiplication

- Is $AB = BA$? Maybe, but maybe not!

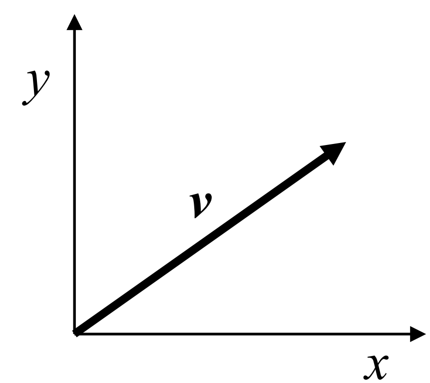
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \dots \\ \dots & \dots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & \dots \\ \dots & \dots \end{bmatrix}$$

- Heads up: multiplication is NOT commutative!

Vector Operations

- Vector: $n \times 1$ matrix
- Interpretation: a point or line in n-dimensional space
- Dot Product, Cross Product, and Magnitude defined on vectors only

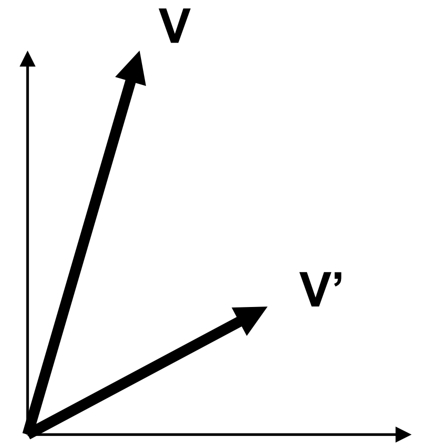
$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Interpretation

- Think of a vector as a line in 2D or 3D
- Think of a matrix as a transformation on a line or set of lines

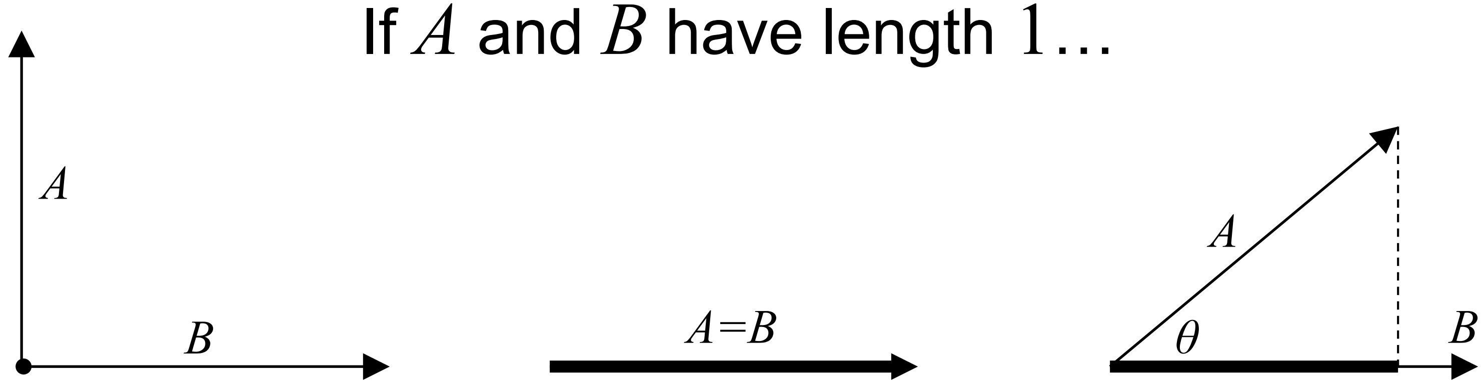
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Vectors: Dot Product

- Interpretation: the dot product measures to what degree two vectors are aligned

If A and B have length 1…



$A \cdot B = 0$ $A \cdot B = 1$ $A \cdot B = \cos\theta$

Vectors: Dot Product

$$A \cdot B = AB^T = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

$$\|A\|^2 = AA^T = aa + bb + cc$$

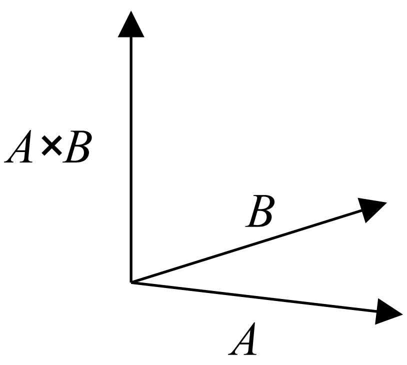
The magnitude is the dot product of a vector with itself

$$A \cdot B = \|A\| \, \|B\| \cos(\theta)$$

The dot product is also related to the angle between the two vectors

Vectors: Cross Product

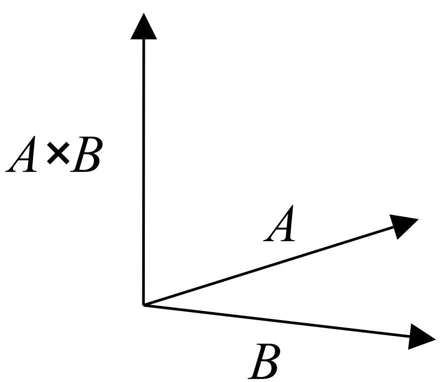
- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the sin of the angle between A and B
- The direction of C follows the **right hand rule** if we are working in a right-handed coordinate system



$$\|A \times B\| = \|A\| \; \|B\| \sin(\theta)$$

Vectors: Cross Product

The cross-product can be computed as a specially constructed determinant



$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Inverse of a Matrix

- Identity matrix:
  $AI = A$
- Some matrices have an inverse, such that:
  $AA^{-1} = I$
- Inversion is tricky:
  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

  Derived from non-commutativity property

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

- Used for inversion
- If $\det(A) = 0$, then A has no inverse
- Can be found using factorials, pivots, and cofactors!
- Lots of interpretations – for more info, take 18.06

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

For a 3×3 matrix:
Sum from left to right
Subtract from right to left

Note: In the general case, the determinant has $n!$ terms

Inverse of a Matrix

$$\begin{bmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f+0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{bmatrix}$$

1. Append the identity matrix to A
2. Subtract multiples of the other rows from the first row to reduce the diagonal element to 1
3. Transform the identity matrix as you go
4. When the original matrix is the identity, the identity has become the inverse!

Homogeneous Matrices

- Problem: how to include translations in transformations (and do perspective transforms)
- Solution: add an extra dimension

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & t_x \\ a_{10} & a_{11} & a_{12} & t_y \\ a_{20} & a_{21} & a_{22} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Orthonormal Basis

- Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis
- Orthogonal: dot product is zero
- Normal: magnitude is one
- Orthonormal: orthogonal + normal
- Most common Example: $\hat{x}, \hat{y}, \hat{z}$

Change of Orthonormal Basis

- Given:

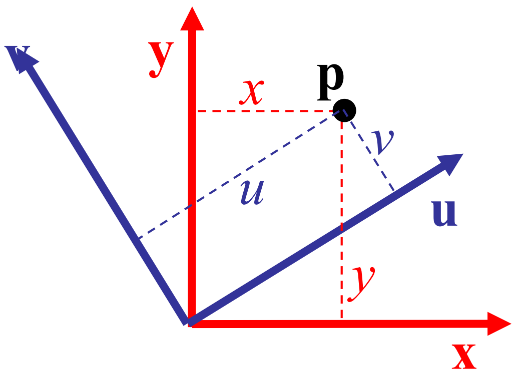
  coordinate frames

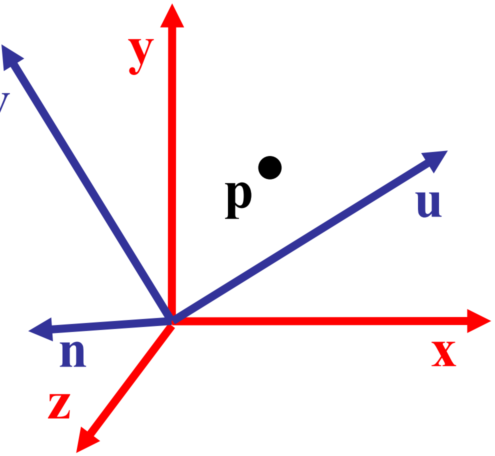
  xyz and **uvn**

  point $\mathbf{p} = (p_x, p_y, p_z)$

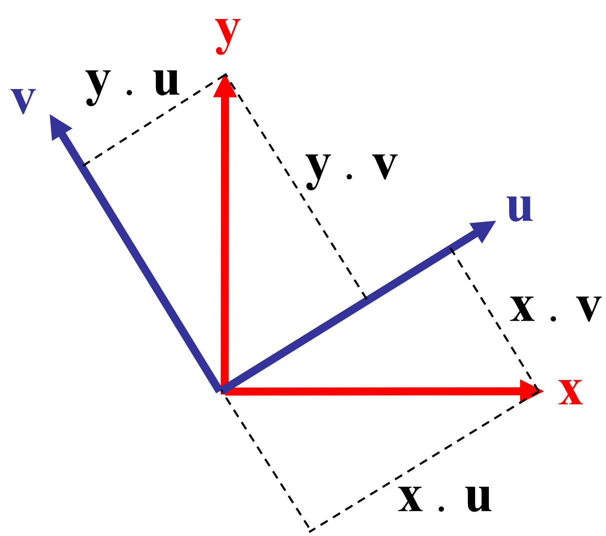
- Find:

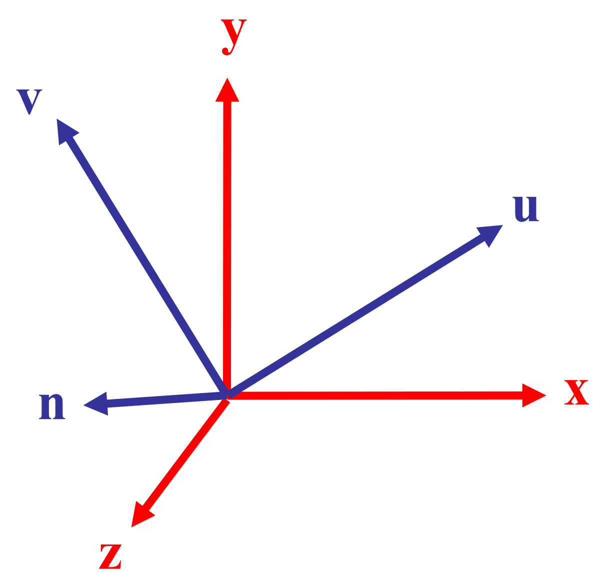
  $\mathbf{p} = (p_u, p_v, p_n)$





Change of Orthonormal Basis





$$\mathbf{x} = (\mathbf{x} . \mathbf{u})\,\mathbf{u} + (\mathbf{x} . \mathbf{v})\,\mathbf{v} + (\mathbf{x} . \mathbf{n})\,\mathbf{n}$$
$$\mathbf{y} = (\mathbf{y} . \mathbf{u})\,\mathbf{u} + (\mathbf{y} . \mathbf{v})\,\mathbf{v} + (\mathbf{y} . \mathbf{n})\,\mathbf{n}$$
$$\mathbf{z} = (\mathbf{z} . \mathbf{u})\,\mathbf{u} + (\mathbf{z} . \mathbf{v})\,\mathbf{v} + (\mathbf{z} . \mathbf{n})\,\mathbf{n}$$

Change of Orthonormal Basis

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}$$
$$\mathbf{z} = (\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}$$

Substitute into equation for *p*:

$$\mathbf{p} = (p_x, p_y, p_z) = p_x\,\mathbf{x} + p_y\,\mathbf{y} + p_z\,\mathbf{z}$$

$$\begin{aligned}\mathbf{p} = {} & p_x\,[\,(\mathbf{x} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{x} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & p_y\,[\,(\mathbf{y} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{y} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{y} \cdot \mathbf{n})\,\mathbf{n}\,] + \\ & p_z\,[\,(\mathbf{z} \cdot \mathbf{u})\,\mathbf{u} + (\mathbf{z} \cdot \mathbf{v})\,\mathbf{v} + (\mathbf{z} \cdot \mathbf{n})\,\mathbf{n}\,]\end{aligned}$$

Change of Orthonormal Basis

$$\begin{aligned}\mathbf{p} = p_x\,[\,(\mathbf{x}\,.\,\mathbf{u})\,\mathbf{u} &+ (\mathbf{x}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{x}\,.\,\mathbf{n})\,\mathbf{n}\,] + \\ p_y\,[\,(\mathbf{y}\,.\,\mathbf{u})\,\mathbf{u} &+ (\mathbf{y}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{y}\,.\,\mathbf{n})\,\mathbf{n}\,] + \\ p_z\,[\,(\mathbf{z}\,.\,\mathbf{u})\,\mathbf{u} &+ (\mathbf{z}\,.\,\mathbf{v})\,\mathbf{v} + (\mathbf{z}\,.\,\mathbf{n})\,\mathbf{n}\,]\end{aligned}$$

Rewrite:

$$\begin{aligned}\mathbf{p} = [\,p_x\,(\mathbf{x}\,.\,\mathbf{u}) &+ p_y\,(\mathbf{y}\,.\,\mathbf{u}) + p_z\,(\mathbf{z}\,.\,\mathbf{u})\,]\,\mathbf{u} + \\ [\,p_x\,(\mathbf{x}\,.\,\mathbf{v}) &+ p_y\,(\mathbf{y}\,.\,\mathbf{v}) + p_z\,(\mathbf{z}\,.\,\mathbf{v})\,]\,\mathbf{v} + \\ [\,p_x\,(\mathbf{x}\,.\,\mathbf{n}) &+ p_y\,(\mathbf{y}\,.\,\mathbf{n}) + p_z\,(\mathbf{z}\,.\,\mathbf{n})\,]\,\mathbf{n}\end{aligned}$$

Change of Orthonormal Basis

$$\mathbf{p} = [\, p_x(\mathbf{x}\cdot\mathbf{u}) + p_y(\mathbf{y}\cdot\mathbf{u}) + p_z(\mathbf{z}\cdot\mathbf{u})\,]\,\mathbf{u} + [\, p_x(\mathbf{x}\cdot\mathbf{v}) + p_y(\mathbf{y}\cdot\mathbf{v}) + p_z(\mathbf{z}\cdot\mathbf{v})\,]\,\mathbf{v} + [\, p_x(\mathbf{x}\cdot\mathbf{n}) + p_y(\mathbf{y}\cdot\mathbf{n}) + p_z(\mathbf{z}\cdot\mathbf{n})\,]\,\mathbf{n}$$

$$\mathbf{p} = (p_u, p_v, p_n) = p_u\,\mathbf{u} + p_v\,\mathbf{v} + p_n\,\mathbf{n}$$

Expressed in **uvn** basis:

$$p_u = p_x(\mathbf{x}\cdot\mathbf{u}) + p_y(\mathbf{y}\cdot\mathbf{u}) + p_z(\mathbf{z}\cdot\mathbf{u})$$
$$p_v = p_x(\mathbf{x}\cdot\mathbf{v}) + p_y(\mathbf{y}\cdot\mathbf{v}) + p_z(\mathbf{z}\cdot\mathbf{v})$$
$$p_n = p_x(\mathbf{x}\cdot\mathbf{n}) + p_y(\mathbf{y}\cdot\mathbf{n}) + p_z(\mathbf{z}\cdot\mathbf{n})$$

Change of Orthonormal Basis

$$
\begin{aligned}
p_u &= p_x\,(\mathbf{x} \,.\, \mathbf{u}) + p_y\,(\mathbf{y} \,.\, \mathbf{u}) + p_z\,(\mathbf{z} \,.\, \mathbf{u}) \\
p_v &= p_x\,(\mathbf{x} \,.\, \mathbf{v}) + p_y\,(\mathbf{y} \,.\, \mathbf{v}) + p_z\,(\mathbf{z} \,.\, \mathbf{v}) \\
p_n &= p_x\,(\mathbf{x} \,.\, \mathbf{n}) + p_y\,(\mathbf{y} \,.\, \mathbf{n}) + p_z\,(\mathbf{z} \,.\, \mathbf{n})
\end{aligned}
$$

In matrix form:

$$
\begin{bmatrix} p_u \\ p_v \\ p_n \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
$$

where:

$u_x = \mathbf{x} \,.\, \mathbf{u}$

$u_y = \mathbf{y} \,.\, \mathbf{u}$

etc.

Change of Orthonormal Basis

$$\begin{bmatrix} p_u \\ p_v \\ p_n \end{bmatrix} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

What's $\mathbf{M}^{-1}$, the inverse?

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_n \\ y_u & y_v & y_n \\ z_u & z_v & z_n \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_n \end{bmatrix}$$

$$u_x = \mathbf{x} \,.\, \mathbf{u} = \mathbf{u} \,.\, \mathbf{x} = x_u$$

$$\mathbf{M}^{-1} = \mathbf{M}^{T}$$

Caveats

- Right-handed vs. left-handed coordinate systems
  - OpenGL is right-handed
- Row-major vs. column-major matrix storage.
  - matrix.h uses row-major order
  - OpenGL uses column-major order

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 & 8 & 12 \\ 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \end{bmatrix}$$

row-major   column-major

Questions?

?