Bayes Seminar: Part I

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Statistical theory, minimally, tries to answer two questions:

- What can we learn from experience?
- How should we best learn from experience?

Bayesian statistical theory answers one additional question:

- How should we represent the knowledge we gain from experience?

- Since Hume, we've known that induction is problematic
- How do we use experiences that are not perfectly informative?

A motivating example:

- Imagine that you've gone to a new restaurant for the first time
- A mistake is made with your order
- Do you conclude that a mistake is made with every order?

- Most people will not reach this conclusion
- Some statistical methods will reach this conclusion
- We'd like to decide what is the *right* conclusion to draw
- How can we formalize the situation for mathematical analysis?

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- We assume that this binary variable is a random variable with a probability distribution over $\{0, 1\}$
- We assume that each separate order is an independent instantiation of this random variable

- Those assumptions are the core elements of most probabilistic models
- Having made them, we can provide a formal analysis

- Each order's accuracy is modeled as a Bernoulli variable
- n orders taken together are a binomial variable
- Either model has exactly one unknown parameter: p, the probability of a mistake
- We want to estimate this parameter

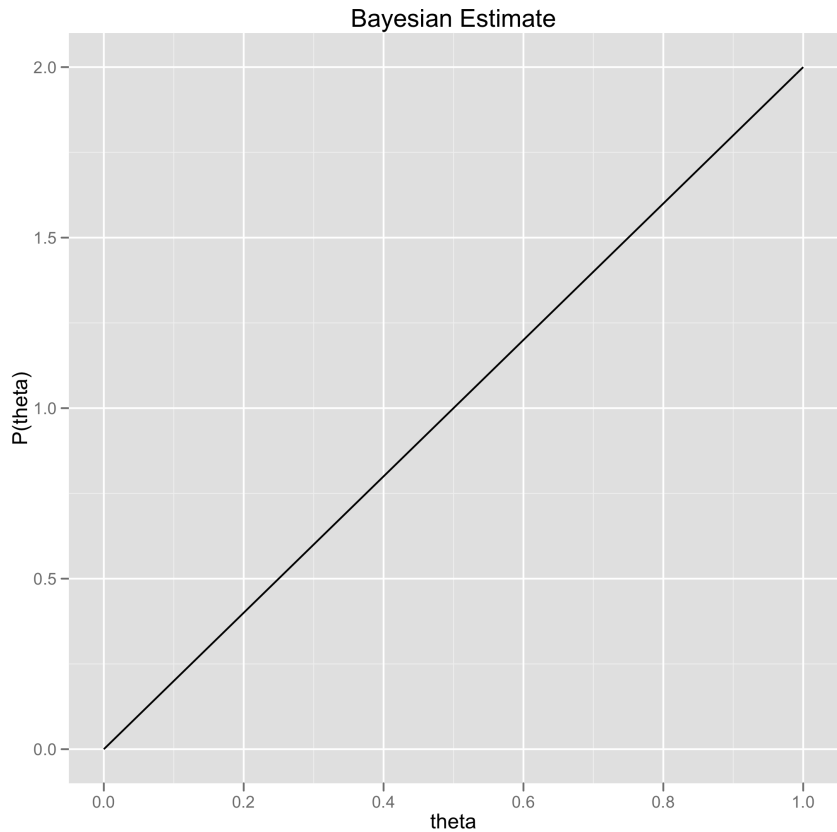
- In this context, we've answered the question of what we can learn
- p is all that we can learn
- We therefore only have to ask how to learn about p and how to represent what we learn

Some representations of p:

- Point estimation: our knowledge of p is a single value $\hat{p}$
- Interval estimation: our knowledge of p is a range, $[\hat{p}_l, \hat{p}_u]$
- Bayesian estimation: our knowledge of p is a probability distribution over $\hat{p}$'s

Some estimates given one order:

- Point Estimate: $\hat{p} = 1$ (MLE)
- Interval Estimate: $[\hat{p}_l, \hat{p}_u] = [0.025, 1.000]$ (95% CI)
- Bayesian Estimate:



- Throughout this seminar I'm going to focus on Bayesian estimation
- I'll contrast it with point and interval estimation

Generalizing our example:

- We have a data set, D, of n data points: $x_1, x_2, \ldots, x_n$
- We have a probabilistic model that generates data sets
- We write the probability of a data set as $p(D) = p(x_1, x_2, \ldots, x_n)$
- In this seminar, every model will be defined by a finite number of parameters
- Instead of the single parameter, p, we'll have a list of parameters, θ
- We then write $p(D|\theta)$

- If our data set is fixed, we can treat $p(D|\theta)$ as a function of θ
- We'll sometimes write $L(\theta; D)$ to describe this function
- This function is called the likelihood function
- L tells us the probability of seeing any specific data set if the parameters of the model were set to θ

Simplifying the likelihood function:

- Because we want to deal with arbitrarily large data sets easily using models we already have on hand, we'll typically simplify the models of data we use
- Specifically, we'll take a probabilistic model of individual data points and build up a model of data sets
- We do this by assuming that all data points in the data set have an identical probability distribution and that they are all independent of each other
- We call this the IID assumption

The IID Assumption:

- For all i, $p(x_i)$ is a single, unvarying probability distribution
- All i data points are independent samples from this constant underlying distribution
- With these assumptions, any data set has the property that

$$p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2)\ldots p(x_n) = \prod_{i=1}^{n} p(x_i)$$

- This factorization makes certain types of mathematical analysis very simple

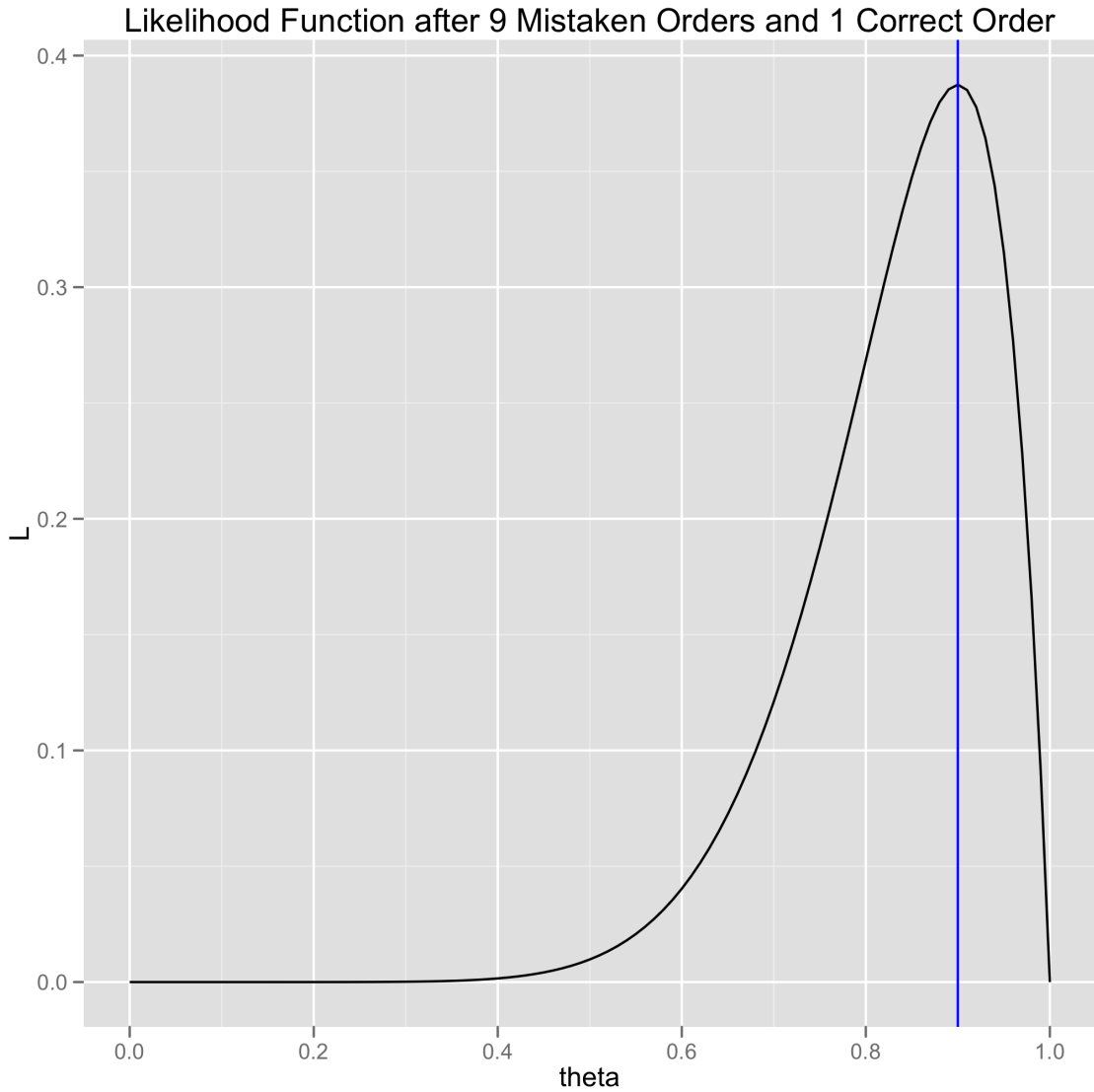
- Let's see how these assumptions play out for data about many orders' accuracy
- We'll assume we've seen n orders
- We'll assume a mistake occurred with $n - 1$ orders and 1 order was accurate

- The probability of a mistake in any given order is θ
- The occurrence of a mistake in each order is independent
- The probability of our data set given θ is therefore

$$\binom{n}{n-1}\theta^{n-1}(1-\theta)$$

- Given our data, how should we estimate θ?
- We'll start by graphing the likelihood function as we change θ
- For this example, let's assume that $n = 10$


Likelihood Function after 9 Mistaken Orders and 1 Correct Order
0.4
0.3
0.2
0.1
0.0
L
0.0
0.2
0.4
0.6
0.8
1.0
theta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theta

- We've seen mistakes in 90% of our orders
- The likelihood function has a single peak at $\theta = 0.9$
- We might guess that we can estimate θ by maximizing the likelihood function

- Estimating parameters by maximizing the likelihood function is a good general strategy
- It also seems intuitively reasonable:

  If we had to predict the future from our model, we should use parameter values that would increase our chances of predicting the data from the past

- For our example model, maximizing the likelihood function can be done analytically
- In other models, computational methods are required

- Note that the maximum of

$$\binom{n-1}{n}\theta^{n-1}(1-\theta)$$

- occurs at the same place as the maximum of

$$\theta^{n-1}(1-\theta)$$

- Then note that the maximum of

$$\theta^{n-1}(1-\theta)$$

- occurs at the same place as the maximum of

$$\log[\theta^{n-1}(1-\theta)] = (n-1)\log[\theta] + \log[1-\theta]$$

- The default parameter estimation strategy of maximizing the log likelihood function comes from Fisher
- Fisher demonstrated that it was a very powerful general strategy for estimating parameters

- Prior to Fisher's work, statistics often involved the ad hoc construction of methods for estimating parameters
- Let's review those ideas, because they're important for thinking critically about Bayesian estimation

- We'll call any function of our data a statistic
- Any statistic designed to give a point estimate of θ is an estimator

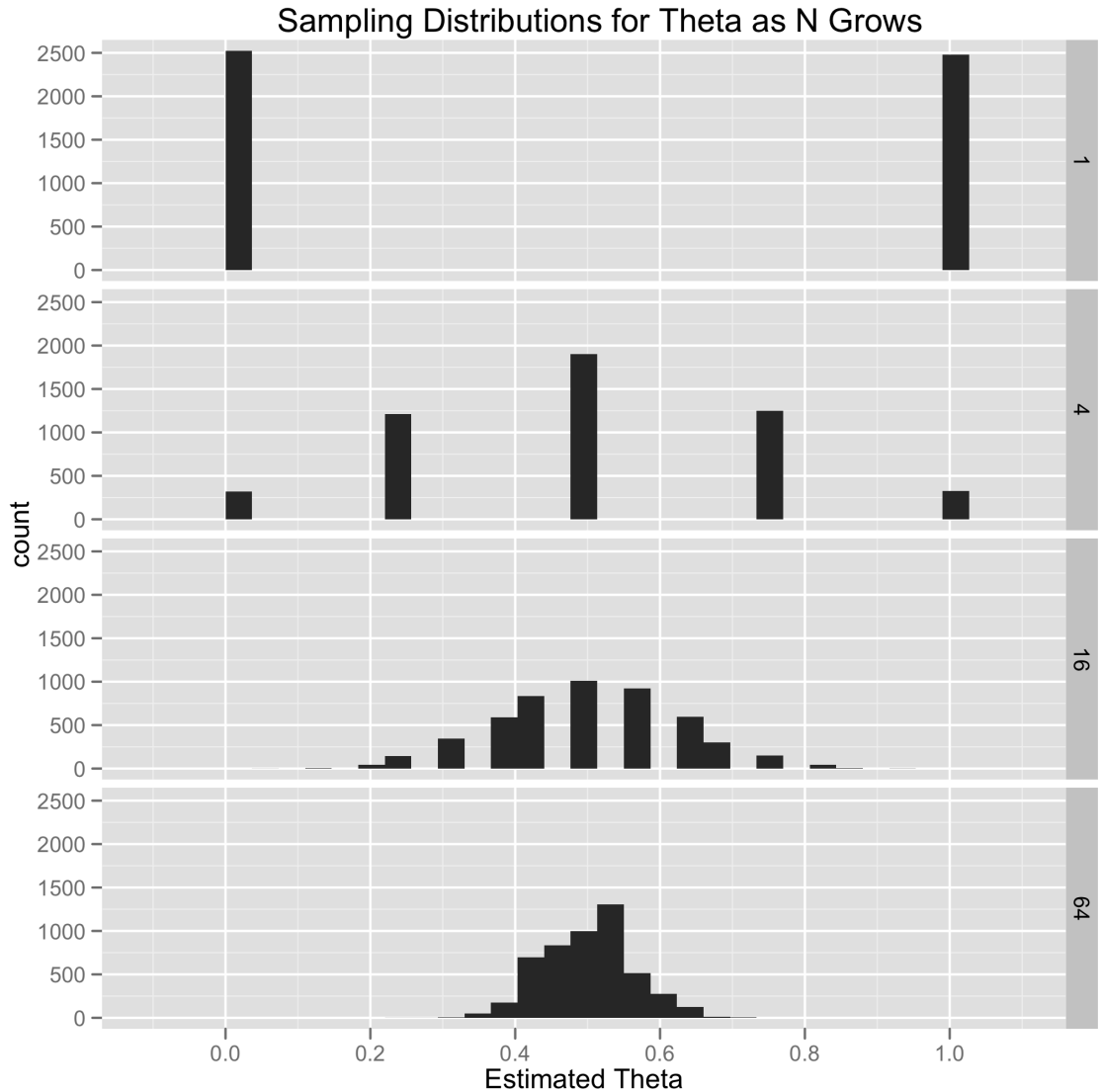
- Suppose we have data that comes from a normal distribution with mean μ
- How can we estimate μ?

- One estimator is the mean of the data
- Another is the median

- How can we tell which of these two estimators works better?

Sampling distribution analysis:

- Suppose that θ is fixed
- We repeatedly sample random data sets from our model
- We compute our estimator on the resulting data sets
- We look at the distribution of our estimator after repeated sampling


Sampling Distributions for Theta as N Grows
count
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0.2
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0.6
0.8
1.0
Estimated Theta

- We then analyze the quality of an estimator by analyzing its sampling distribution

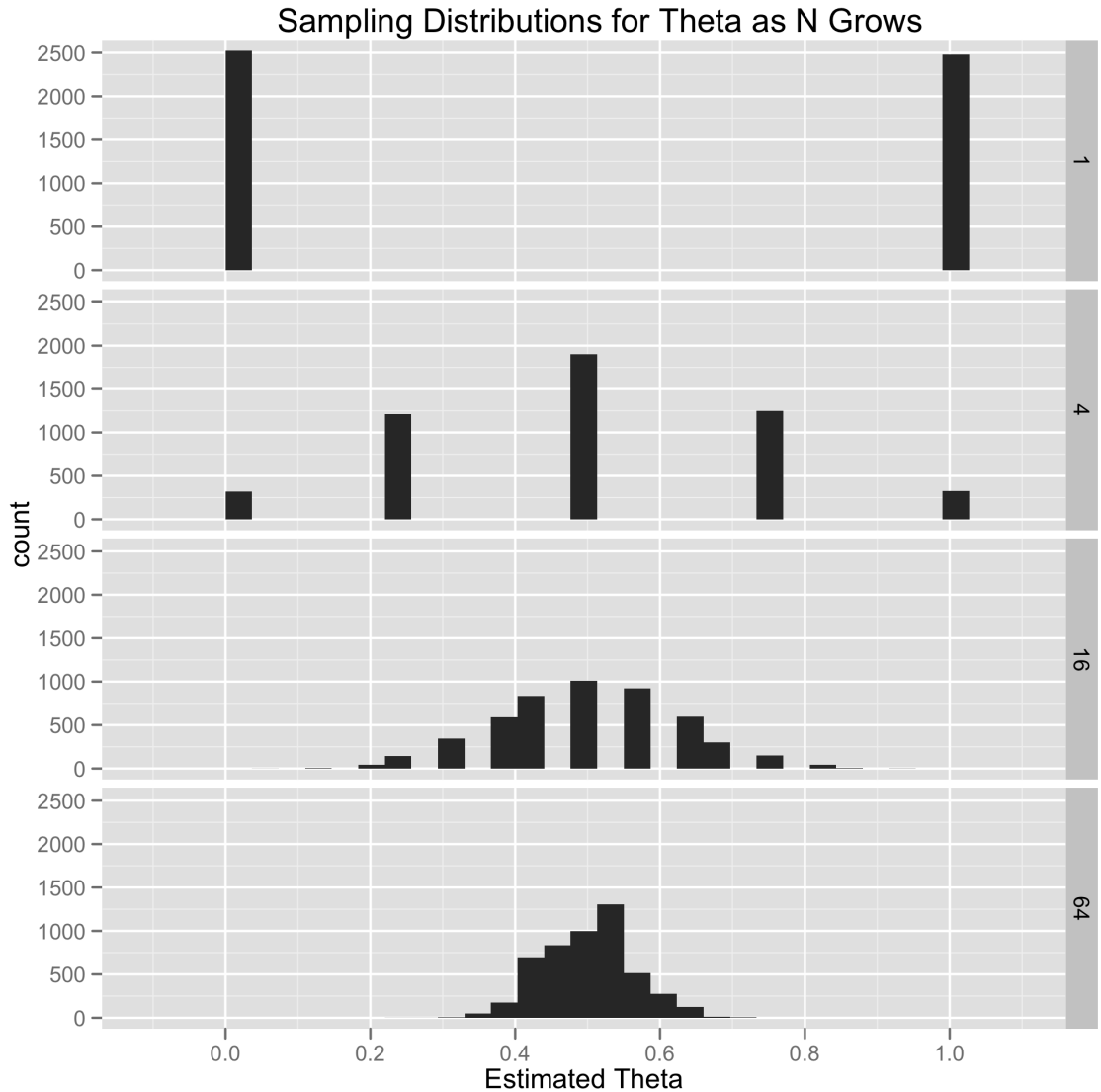
Three criteria for selecting estimators are particularly popular:

- Bias
- Variance
- Consistency

- The bias of an estimator, $\hat{\theta}$, is

$$\mathbb{E}[\hat{\theta} - \theta] = \mathbb{E}[\hat{\theta}] - \theta$$

- An unbiased estimator is one for which the mean of the sampling distribution is θ
- In short, an unbiased estimator's expected value is θ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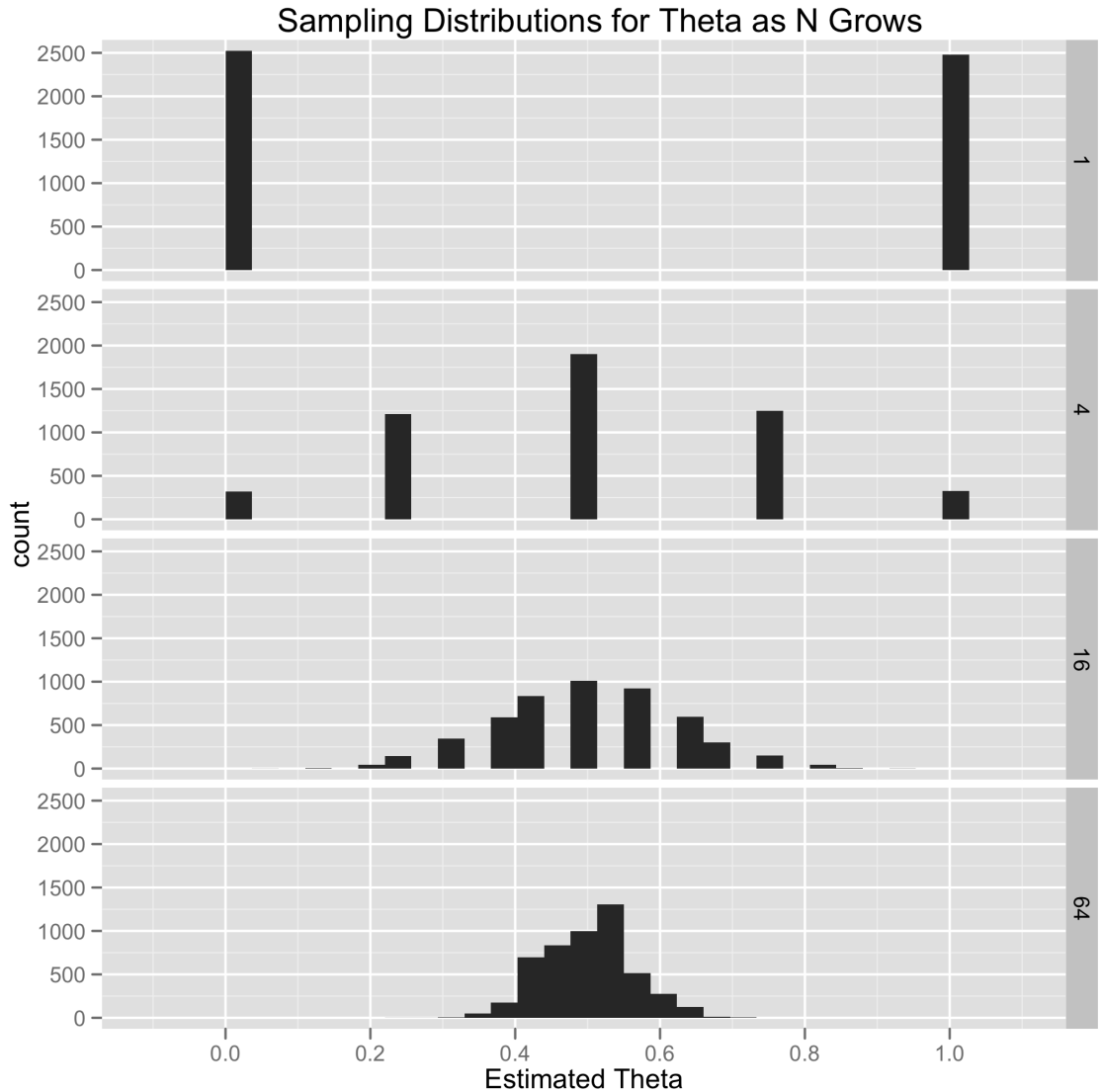
- The variance of an estimator, $\hat{\theta}$, is

$$\mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}])^2]$$

- If the estimator is unbiased, the variance is the expected Euclidean distance between $\hat{\theta}$ and θ
- In short, a low variance estimator is one that is typically close to $\mathbb{E}[\theta]$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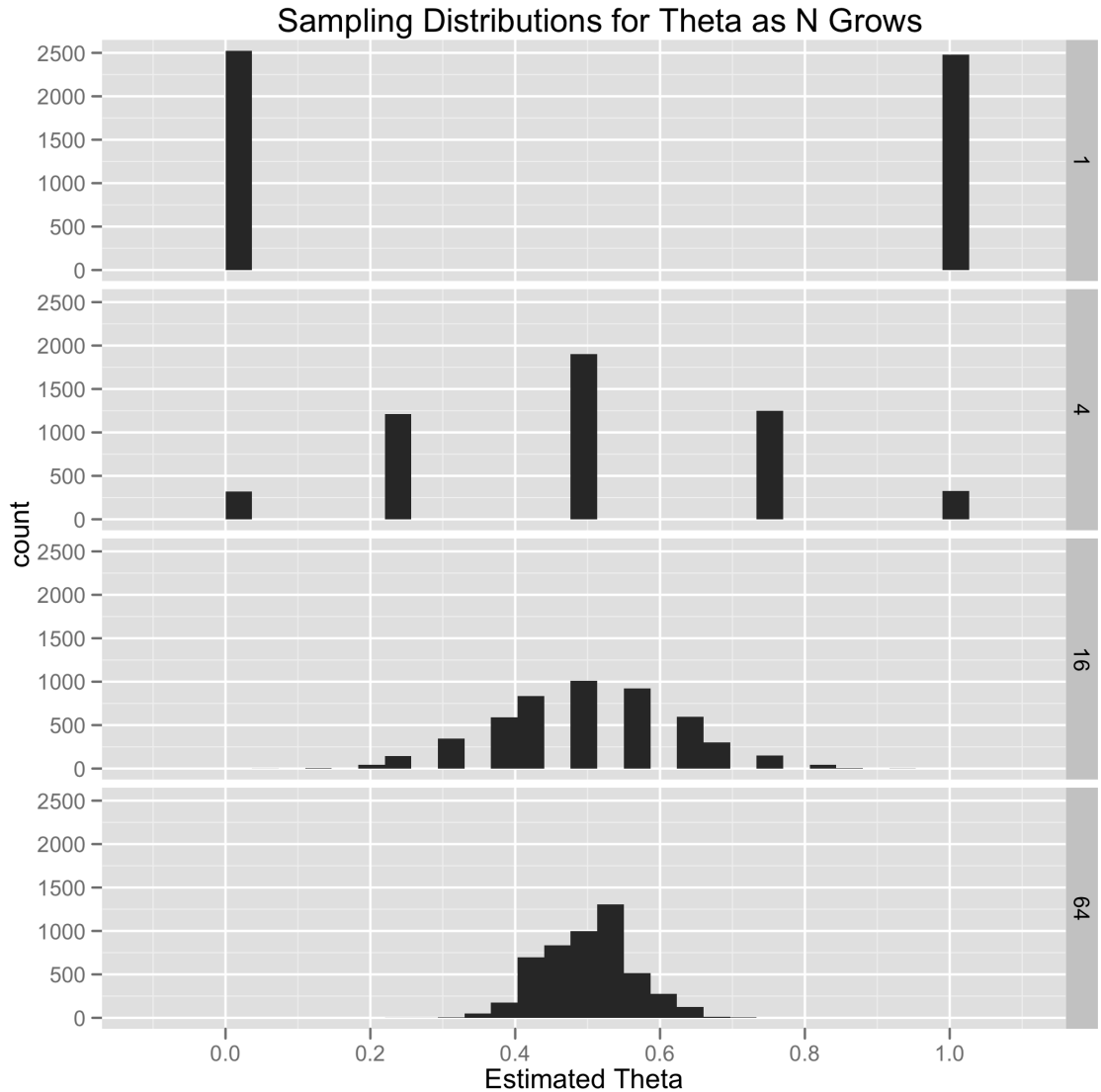
Estimated Theta

- An estimator, $\hat{\theta}$, is consistent if

$$\lim_{n \to \infty} \hat{\theta} = \theta$$

- A consistent estimator gets closer to θ as we get more data
- To be consistent, the bias and variance must both go to 0 as n grows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Estimated Theta

- Fisher made the MLE popular by showing that typically:
  - The MLE is unbiased
  - The MLE is consistent
  - The MLE is asymptotically the lowest variance estimator

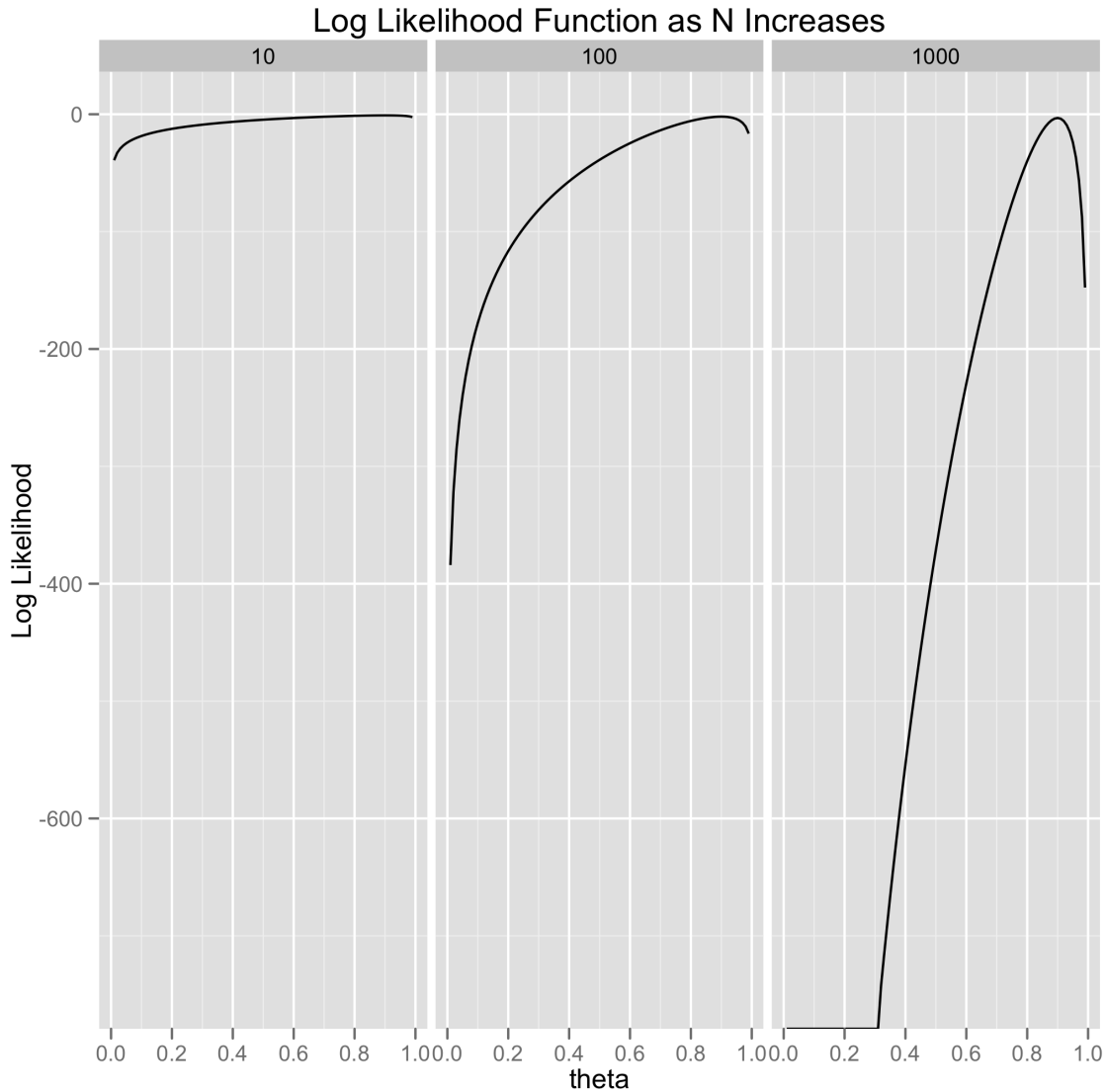
- From the 1920's until the 1950's, the MLE was king
- In the 50's, James-Stein published a simple problem for which the MLE was a bad estimator

- Since the 50's, interest has grown in alternative estimation strategies
- It is now clear that unbiased estimators are sometimes bad in practice
- It is now clear that finite sample behavior is not the same as asymptotic behavior

- Enter the Bayesian estimation strategy
- In many practical problems, Bayesian methods are biased
- But, in many practical problems, Bayesian methods have lower variance than MLE methods
- Moreover, bias and variance can often be explicitly traded off

- Let's go back to the likelihood function
- We've seen one way of using it to construct estimators
- What other information can we get from it?

- We can get a measure of our uncertainty about θ
- We do this by looking at the curvature of the likelihood function
- Formally, this is the Fisher information


Log Likelihood Function as N Increases
10
100
1000
0
-200
-400
-600
Log Likelihood
0.0
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0.4
0.6
0.8
1.0
theta

- We've now seen that we can learn a lot from the likelihood function
  - The peak gives us an estimator
  - The spread gives us a sense of uncertainty

- Pushing on these ideas, we might wish to have a single numeric language for representing our knowledge of θ that organizes the information in the likelihood function
- For a Bayesian that language is probability theory
- We'd like to say that we think that $\hat{\theta}$ is the most probable value for θ
- We'd also like to say that there is a 95% chance that θ's value is in some interval, $[a, b]$

- We start with a probability distribution over θ: $p(\theta)$
- We call this distribution our prior
- It is a prior because it represents our beliefs before we see data

- We wish to calculate our belief distribution after seeing data, D: $p(\theta|D)$
- We call this distribution our posterior
- It is a posterior because it represents our beliefs after we see data

- We can calculate our posterior using Bayes' theorem:

$$p(\theta|x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n|\theta)p(\theta)}{p(x_1, \ldots, x_n)}$$

- $p(D)$ is called the evidence. It is a constant
- Therefore

$$p(\theta|D) \propto p(D|\theta)p(\theta)$$

In words:

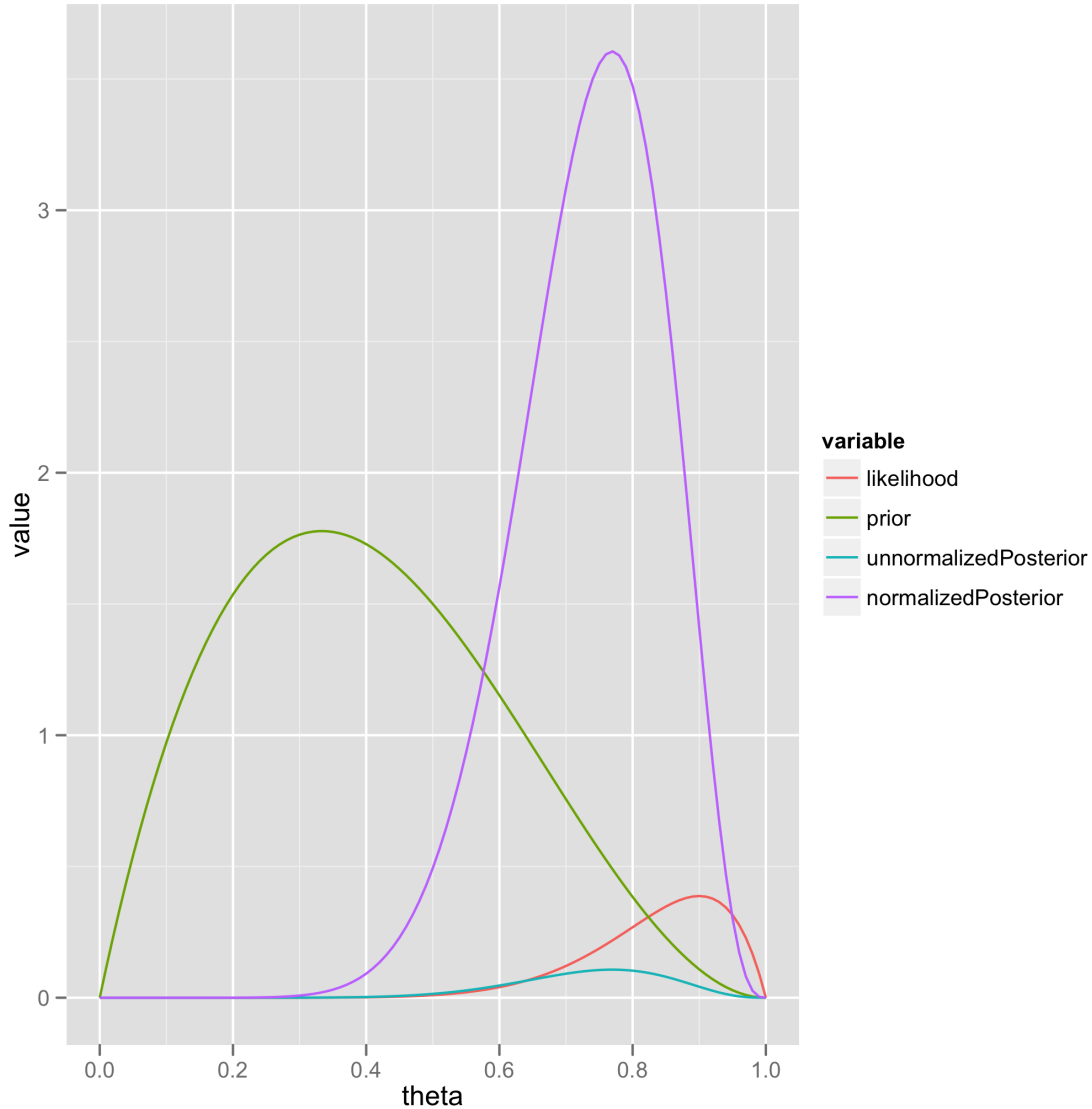
- Up to a scaling factor, the value of our posterior at a point θ^* is the product of the likelihood function evaluated at θ^* and the prior evaluated at θ^*
- If our prior is flat, the posterior's shape is the likelihood function's shape

- In practice calculating the posterior is hard because the evidence can be impossible to calculate analytically
- But we can usually make good approximations
- In some special circumstances, we can get analytic solutions

Approximation strategy 1:

- We want the posterior only at n points on a grid
- We find the unnormalized posterior by multiplying the likelihood and prior
- We evaluate the approximate evidence by summing the unnormalized posterior
- We divide the unnormalized posterior by the approximate evidence to get a proper probability distribution

- Suppose we start with a prior biased towards low values of θ
- Then we see data for which the likelihood function supports high values of θ
- The posterior describes our reconciliation of these two pieces of information


value
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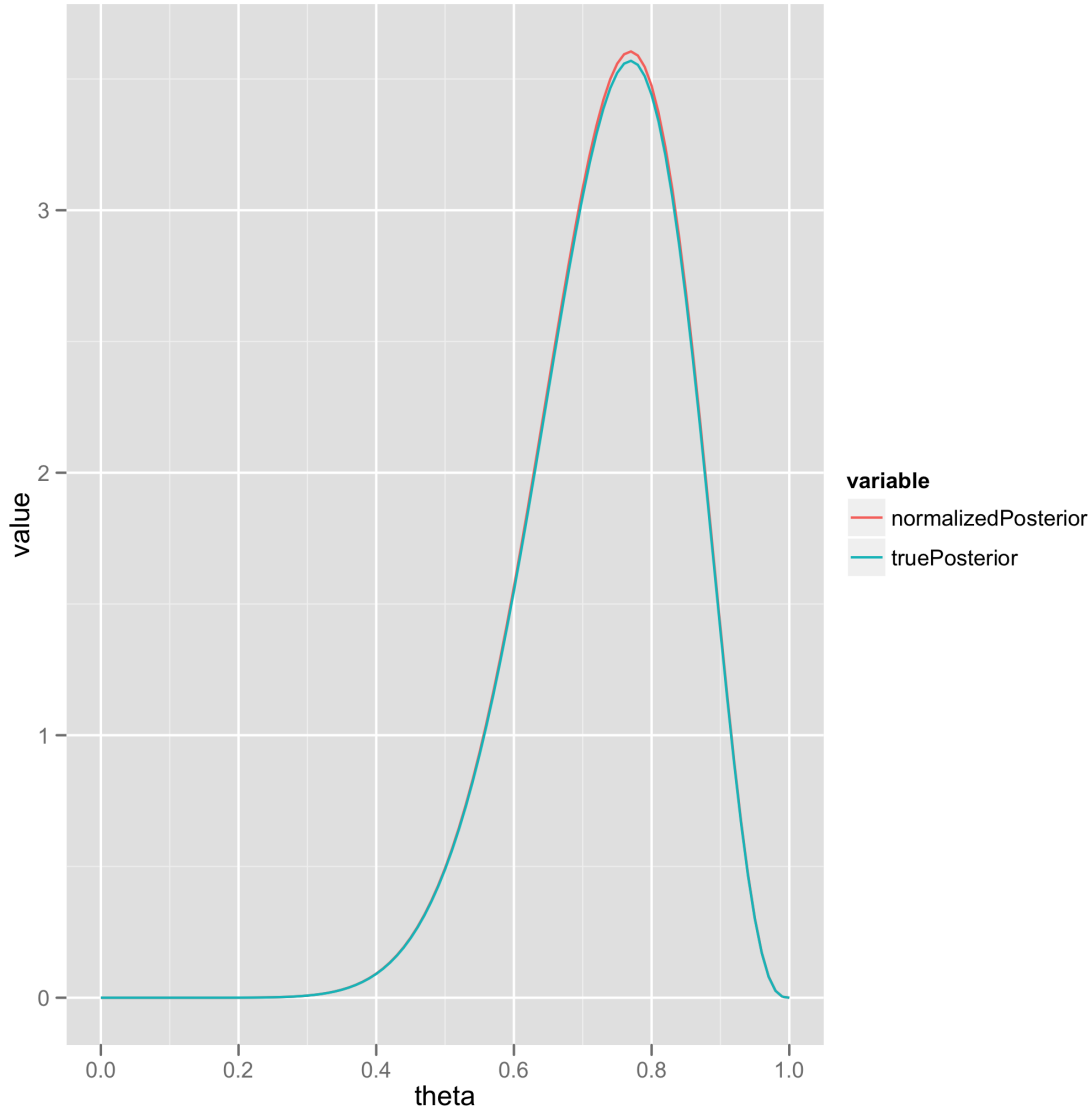
variable
likelihood
prior
unnormalizedPosterior
normalizedPosterior

In this example, we can calculate the exact posterior for comparison


value
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0.2
0.4
0.6
0.8
1.0
theta
variable
normalizedPosterior
truePosterior

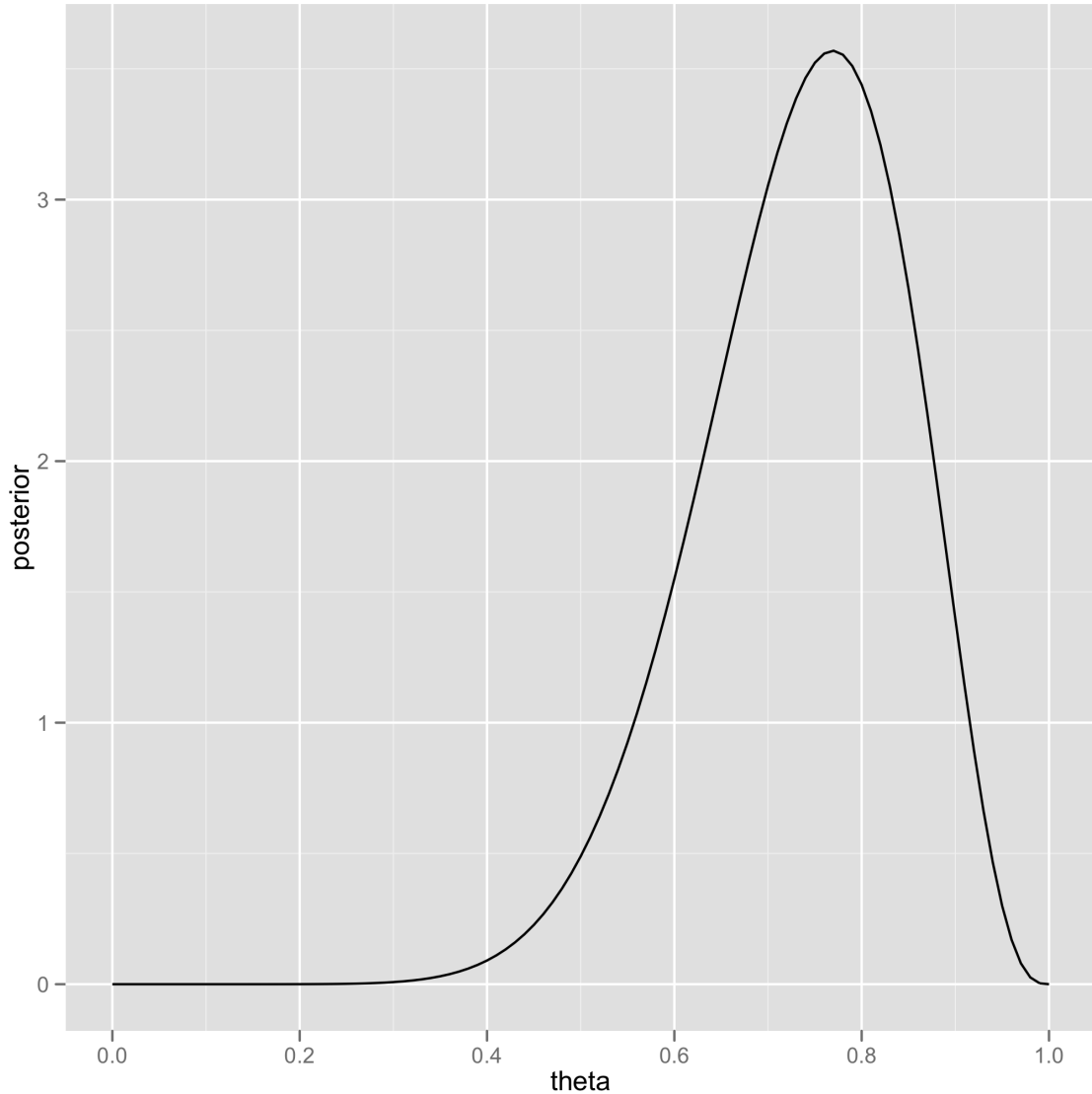
- Here grid approximation works perfectly
- So where does grid approximation go wrong?
- Computational time and space is exponential in number of parameters in θ
- Unclear how fine resolution of grid must be

- There are important cases where analytic techniques work
- Given an appropriate prior, the posterior can have a closed form solution
- If both the prior and posterior have a constant functional form F, the prior is called conjugate

- In our binomial example, a prior called the beta prior is conjugate to the binomial likelihood function
- The beta prior, $B(\alpha, \beta)$ has two parameters, α and β
- α is the number of 1's previously seen
- β is the number of 0's previously seen
- After x more 1's and y more 0's, the posterior has the form

$$B(\alpha + x, \beta + y)$$

- Let's work further with our example
- Our prior will be a $B(2, 3)$ distribution
- We'll suppose we've seen 9 more mistaken orders and 1 correct order
- Our posterior is then a $B(11, 4)$ distribution


posterior
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theta

- Visualizing the full posterior can tell us a lot
- What other things can we do?

- We can extract various point estimates from the posterior
  - Means
  - Medians
  - Modes
- We can compute interval estimates
  - 95% probability intervals using quantiles

- When we use the mode of the posterior as our estimate we call it the MAP
- MAP stands for Maximum A Posteriori
- It is the Bayesian analogue to the MLE

Using the MAP via maximizing its log value is like penalized maximum likelihood:

$$\log[p(\theta|D)] = \log[L(\theta|D)p(\theta)] = \log[L(\theta|D)] + \log[p(\theta)]$$

- Instead of finding the MAP, we can extract other values:
  - The mean of the posterior
  - The median of the posterior
  - Quantiles of the posterior
- To decide which value to use, we can use decision theory

Decision theory:

- We must decide what action to take
- We estimate the value of each action using our posterior
- We select the decision with the highest expected value

Pascal's Wager:

- Assume that our estimated probability that God exists is p
- Then the expected value of belief in God is $(1-p)*0+p*\infty$
- The expected value of disbelief in God is $(1-p)*0+p*0$
- We should therefore choose to believe

- In statistics, our decision is our estimate of θ
- We choose the estimate that minimizes our expected loss

- If the loss for an estimate $\hat{\theta}$ is $(\theta - \hat{\theta})^2$, the best estimate is the posterior mean
- If the loss for an estimate $\hat{\theta}$ is $|\theta - \hat{\theta}|$, the best estimate is the posterior median
- If the loss for an estimate $\hat{\theta}$ is 1 if $\theta \neq \hat{\theta}$, the best estimate is the highest posterior mode

- In practice, from now on, we're just going to use the mean and median of the posterior as point estimates

- In modern Bayesian statistics, we use conjugacy whenever we can
- Otherwise, we use MCMC techniques

- MCMC depends on a remarkable result
- We can draw samples from a probability distribution that's known only up to a constant
- The posterior is just such a distribution
- The constant is the evidence

- MCMC techniques draw many samples from the posterior
- There are therefore Monte Carlo methods
- As such, their quality increases when we take more samples
- To find each sample, a Markov chain is used
- Therefore MCMC = Markov chain Monte Carlo methods

- For many problems, MCMC analysis can be completely automatic
- Part II of the seminar will work through many examples of applied Bayesian computation
- We'll use an MCMC tool called BUGS that fully automates MCMC sampling
- If you want to learn how MCMC works under the hood, there are many good books