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作业.周一. P65. 2.4.9.11.

2. 设 $u(x,y,z)=\dfrac{z}{x^2+y^2}$，求 du.

$$\frac{\partial u}{\partial x}=-\frac{z\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=-\frac{2zy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial z}=\frac{1}{x^2+y^2}$$

显然偏导函数连续

$$du=-\frac{2xz}{(x^2+y^2)^2}dx-\frac{2yz}{(x^2+y^2)^2}dy+\frac{1}{x^2+y^2}dz$$

4. 证明：函数 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ 在点(0,0)是不可微分的.

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

同理 $f'_y(0,0)=0$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}-0\cdot\Delta x-0\cdot\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$$

$$\because \lim_{\substack{\Delta x\to 0\\ \Delta y=\Delta x}}=\frac{\Delta x^2}{2\Delta x^2}=\frac{1}{2}\neq 0$$

$\therefore$ Δf 在(0,0)处不可表为 $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+o(\sqrt{\Delta x^2+\Delta y^2})$

$\therefore$ f在(0,0)处不可微

9. 函数 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ 在点(0,0)的两个偏导数是否存在？在点(0,0)是否可微？为什么？

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

同理 $f'_y(0,0)=0$

而 $\lim\limits_{\substack{x\to 0\\ y=x}}f(x,y)=\frac{1}{2}\neq 0$ $\therefore$ f在(0,0)不连续，$\therefore$ 更不可微

11. 若 $f'_x(x_0,y_0)$ 存在，且 $f'_y(x,y)$ 在点 (x_0,y_0) 连续，证明：$f(x,y)$ 在 (x_0,y_0) 处可微.

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$$dz=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$$

$$=f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)+f(x_0+\Delta x,y_0)-f(x_0,y_0)$$

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$$dz=f'_y(x_0+\Delta x,y_0+\theta_1\Delta y)\Delta y+f'_x(x_0+\theta_2\Delta x,y_0)\Delta x$$

$$\because \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}f'_y(x_0+\Delta x,y_0+\theta_1\Delta y)=f'_y(x_0,y_0)$$

$$\therefore f'_y(x_0+\Delta x,y_0+\theta_1\Delta y)=f'_y(x_0,y_0)+\varepsilon_1$$

其中 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\varepsilon_1=0$

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=f'_x(x_0,y_0)$$

则 $f(x_0+\Delta x,y_0)-f(x_0,y_0)=f'_x(x_0,y_0)\Delta x+\varepsilon_2\Delta x$

$\lim\limits_{\Delta x\to 0}\varepsilon_2=0$

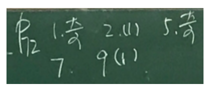
$$\therefore dz=f'_y(x_0,y_0)\Delta y+\varepsilon_1\Delta y+f'_x(x_0,y_0)\Delta x+\varepsilon_2\Delta x$$

故 $$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left|\frac{dz-(f'_y(x_0,y_0)\Delta y+f'_x(x_0,y_0)\Delta x)}{\rho}\right|$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left|\frac{\varepsilon_1\Delta y+\varepsilon_2\Delta x}{\sqrt{\Delta x^2+\Delta y^2}}\right|\le\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(|\varepsilon_1|+|\varepsilon_2|)=0.$$

$$\therefore \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{dz-(f'_y(x_0,y_0)\Delta y+f'_x(x_0,y_0)\Delta x)}{\rho}=0$$

$\therefore$ $f(x,y)$ 在 (x_0,y_0) 可微

作业.周三. 

1. 计算下列各题：

(1) 设 $z=\dfrac{t+s}{t-s}$，其中 $t=3x+y$，$s=x-3y$，求 $\dfrac{\partial z}{\partial x}$；

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial z}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\begin{tikzcd} & t \arrow[r, dash] \arrow[dr, dash] & x \\ z \arrow[ur, dash] \arrow[dr, dash] & & \\ & s \arrow[r, dash] \arrow[uur, dash] & y \end{tikzcd}$$

$$=\frac{-2s}{(t-s)^2}\cdot 3+\frac{2t}{(t-s)^2}\cdot 1$$

$$=\frac{2t-6s}{(t-s)^2}$$

(3) 设 $z=\arctan\dfrac{x+1}{y}$，$y=e^{(1+x)^2}$，求 $\dfrac{dz}{dx}$；

$$\frac{dz}{dx}=\frac{1}{1+(\frac{x+1}{y})^2}\cdot\frac{y-(x+1)y'}{y^2}$$
$$=\frac{y^2}{y^2+(x+1)^2}\cdot\frac{y-(x+1)\cdot 2(x+1)y}{y^2}$$
$$=\frac{y-2(x+1)^2y}{y^2+(x+1)^2}$$

$\because (x+1)^2=\ln y$

$$\therefore \frac{dz}{dx}=\frac{y-2y\ln y}{y^2+\ln y}$$

(5) 设 $z=\arctan(xy)$，而 $y=e^x$，求 $\dfrac{dz}{dy}$.

$x=\ln y$.

$$\therefore \frac{dz}{dy}=\frac{(1+\ln y)}{1+(xy)^2}.$$

2. 计算下列各题，其中 $f\in C^{(1)}$ 类函数：

(1) 设 $z=xf\left(\dfrac{y}{x}\right)+2y\varphi\left(\dfrac{x}{y}\right)$，式中 f，φ 均可导，求 $\dfrac{\partial z}{\partial x}$；

$$\frac{\partial z}{\partial x}=f(\tfrac{y}{x})+x\cdot f'(\tfrac{y}{x})(-\tfrac{y}{x^2})+2y\varphi'(\tfrac{x}{y})\cdot\frac{1}{y}$$
$$=f(\tfrac{y}{x})-\tfrac{y}{x}f'(\tfrac{y}{x})+2\varphi'(\tfrac{x}{y})$$

5. 求下列复合函数指定的偏导数：

(1) $z=(x^2+y^2)e^{-\arctan\frac{y}{x}}$，求 $\dfrac{\partial^2 z}{\partial x^2}$，$\dfrac{\partial^2 z}{\partial x\partial y}$，$\dfrac{\partial^2 z}{\partial y^2}$；

$$\frac{\partial z}{\partial x}=2xe^{-\arctan\frac{y}{x}}+(x^2+y^2)\cdot\frac{-1}{1+(\frac{y}{x})^2}(-\frac{y}{x^2})e^{-\arctan\frac{y}{x}}$$
$$=e^{-\arctan\frac{y}{x}}(2x+y)$$
$$\frac{\partial^2 z}{\partial x^2}=\frac{y\cdot e^{-\arctan\frac{y}{x}}}{x^2+y^2}(2x+y)+e^{-\arctan\frac{y}{x}}\cdot 2$$
$$=e^{-\arctan\frac{y}{x}}\left(\frac{2x^2+2xy+3y^2}{x^2+y^2}\right)$$
$$\frac{\partial^2 z}{\partial x\partial y}=\frac{-\frac{1}{x}}{1+(\frac{y}{x})^2}\cdot e^{-\arctan\frac{y}{x}}(2x+y)+e^{-\arctan\frac{y}{x}}$$
$$=e^{-\arctan\frac{y}{x}}\left(\frac{-x^2-xy+y^2}{x^2+y^2}\right)$$

$$\frac{\partial z}{\partial y}=2y\cdot e^{-\arctan\frac{y}{x}}+\frac{-\frac{1}{x}}{1+(\frac{y}{x})^2}e^{-\arctan\frac{y}{x}}(x^2+y^2)$$
$$=e^{-\arctan\frac{y}{x}}(2y-x)$$
$$\frac{\partial^2 z}{\partial y^2}=\frac{-x}{x^2+y^2}\cdot e^{-\arctan\frac{y}{x}}(2y-x)+2e^{-\arctan\frac{y}{x}}$$
$$=e^{-\arctan\frac{y}{x}}\left(\frac{3x^2-2xy+2y^2}{x^2+y^2}\right)$$

(3) $z=f(e^x\sin y,x^2+y^2)$，求 $\dfrac{\partial^2 u}{\partial x\partial y}$；

令 $u(x,y)=e^x\sin y$　　$v(x,y)=x^2+y^2$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$=f_1'\cdot e^x\sin y+f_2'\cdot 2x$$
$$\frac{\partial z}{\partial x\partial y}=e^x\sin y\left(f_{11}''\cdot e^x\cos y+f_{12}''\cdot 2y\right)+e^x\cos y f_1'$$
$$+2x\left(f_{21}''e^x\cos y+f_{22}''2y\right)$$

$f_{12}''=f_{21}''$

$$=e^x\cos y f_1'+e^{2x}\sin y\cos y f_{11}''+2e^x(x\cos y+y\sin y)f_{12}''+4xyf_{22}''$$

(5) 设函数 $z=F[\varphi(x)-y,x+\phi(y)]$，其中 $\varphi(x)$、$\phi(y)$ 都是可微函数，求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

令 $u(x,y)=\varphi(x)-y$　　$v(x,y)=x+\phi(y)$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$=F_1'\cdot\varphi'(x)+F_2'$$
$$\frac{\partial^2 z}{\partial x\partial y}=\varphi'(x)\left[F_{11}''\cdot(-1)+F_{12}''\cdot\phi'(y)\right]+F_{21}''\cdot(-1)+F_{22}''\cdot\phi'(y)$$
$$=-\varphi'(x)F_{11}''-F_{21}''+\varphi'(x)\phi'(y)F_{12}''+\phi'(y)F_{22}''$$

7. 证明：函数 $u=\varphi(x+at)+\varphi(x-at)$ 满足波动方程 $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$.

$$\frac{\partial u}{\partial t}=a\varphi'(x+at)-a\varphi'(x-at)$$
$$\frac{\partial^2 u}{\partial t^2}=a^2\varphi''(x+at)+a^2\varphi''(x-at)\quad\cdots ①$$
$$\frac{\partial u}{\partial x}=\varphi'(x+at)+\varphi'(x-at)$$
$$\frac{\partial u}{\partial x}=\varphi''(x+at)+\varphi''(x-at)\quad\cdots ②$$

联立 ① ② $\Rightarrow \dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}$

9. 设函数 $u=f(x,y)\in C^{(1)}$，$x=r\cos\theta$，$y=r\sin\theta$. 证明：

(1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$；

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}\cdot(-r\sin\theta)+\frac{\partial u}{\partial y}\cdot r\cos\theta$$

$$\therefore\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta+\sin^2\theta)+\frac{\partial u}{\partial y}(\sin^2\theta+\cos^2\theta)+2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\cos\theta\sin\theta-2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\sin\theta\cos\theta$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

作业. 同立. P80. 2. 4. 6. 7偏.

2. 函数 $y=y(x)$ 由方程 $y-\varepsilon\sin y=x\,(0<\varepsilon<1)$ 所确定，求 $\frac{d^2y}{dx^2}$.

两边对x求导：$y'-y'\cdot\varepsilon\cos y=1\Rightarrow y'=\frac{1}{1-\varepsilon\cdot\cos y}$

$$\therefore y''=-\frac{-\varepsilon\cdot(-\sin y)\cdot y'}{(1-\varepsilon\cdot\cos y)^2}$$

$$=-\frac{\varepsilon\cdot\sin y\cdot y'}{(1-\varepsilon\cdot\cos y)^2}$$

$$=-\frac{\varepsilon\cdot\sin y}{(1-\varepsilon\cdot\cos y)^2}\cdot\frac{1}{1-\varepsilon\cdot\cos y}$$

$$=\frac{-\varepsilon\cdot\sin y}{(1-\varepsilon\cdot\cos y)^3}=\frac{d^2y}{dx^2}$$

4. 函数 $z=z(x,y)$ 由方程 $F\left(x+\frac{z}{y},y+\frac{z}{x}\right)=0$ 所确定，其中 F 有连续的一阶偏导数，求证 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy$.

对x求导：$F_1'\cdot\left(1+\frac{1}{y}\cdot\frac{\partial z}{\partial x}\right)+F_2'\cdot\frac{\frac{\partial z}{\partial x}\cdot x-z}{x^2}=0$

$$\frac{\partial z}{\partial x}\left(\frac{F_1'}{y}+\frac{F_2'}{x}\right)+F_1'-\frac{z\cdot F_2'}{x^2}=0$$

$$x\cdot\frac{\partial z}{\partial x}=\frac{F_2'\cdot z-x^2F_1'}{\frac{x^2F_1'}{y}+F_2'}\cdot x=\frac{xz\cdot F_2'-x^3\cdot F_1'}{\frac{x^2}{y}\cdot F_1'+F_2'}$$

对y求导：$F_1'\cdot\frac{\frac{\partial z}{\partial y}\cdot y-z}{y^2}+F_2'\cdot\left(1+\frac{1}{x}\cdot\frac{\partial z}{\partial y}\right)=0$

$$\frac{\partial z}{\partial y}\cdot\left(\frac{F_1'}{y}+\frac{F_2'}{x}\right)-\frac{z}{y^2}\cdot F_1'+F_2'=0$$

$$y\cdot\frac{\partial z}{\partial y}=\frac{\frac{z}{y^2}F_1'-F_2'}{\frac{F_1'}{y}+\frac{F_2'}{x}}\cdot y=\frac{yz\cdot F_1'-y^3\cdot F_2'}{F_1'+\frac{y^2}{x}\cdot F_2'}$$

$$\therefore x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=\frac{yz\cdot F_2'-yx^2\cdot F_1'}{x\cdot F_1'+yF_2'}+\frac{xz\cdot F_1'-xy^2\cdot F_2'}{x\cdot F_1'+y\cdot F_2'}$$

$$=\frac{(yz-xy^2)F_2'+(-yx^2+xz)F_1'}{xF_1'+yF_2'}$$

$$=\frac{yF_2'(z-xy)+xF_1'(z-xy)}{xF_1'+yF_2'}$$

$$=z-xy$$

6. 设函数 $u=f(x,y,z)$，$z=g(x,y)$，求 $\frac{\partial^2u}{\partial x\partial y}$.

$$\frac{\partial u}{\partial x}=f_1'+f_3'\cdot g_1'$$

$$\frac{\partial^2u}{\partial x\partial y}=f_{12}''+f_{13}''\cdot g_2'+g_1'(f_{32}''+f_{33}''g_2')+g_{12}''\cdot f_3'$$

7. (2) 设 $\begin{cases}x^2+y^2=\frac{1}{2}z^2,\\ x+y+z=2\end{cases}$ 确定函数 $x=x(z)$，$y=y(z)$，求 $\frac{dx}{dz}$ 和 $\frac{dy}{dz}$ 在 $x=1$、$y=-1$、$z=2$ 处的值；

$$\begin{cases}2x\cdot\frac{dx}{dz}+2y\cdot\frac{dy}{dz}=z\\ \frac{dx}{dz}+\frac{dy}{dz}=-1\end{cases}$$

$$\frac{dx}{dz}=\frac{\begin{vmatrix}z&2y\\-1&1\end{vmatrix}}{\begin{vmatrix}2x&2y\\1&1\end{vmatrix}}=\left.\frac{z+2y}{2x-2y}\right|_{\substack{x=1\\y=-1\\z=2}}=0$$

$$\frac{dy}{dz}=\frac{\begin{vmatrix}2x&z\\1&-1\end{vmatrix}}{2x-2y}=\left.\frac{-2x-z}{2x-2y}\right|_{\substack{x=1\\y=-1\\z=2}}=\frac{-4}{4}=-1$$

(4) 函数 $u=u(x,y)$，$v=v(x,y)$ 由方程组 $\begin{cases}xu-yv=0,\\ yu+xv=1\end{cases}$ 所确定，求 dv.

$$\begin{cases}u+x\cdot\frac{\partial u}{\partial x}-y\cdot\frac{\partial v}{\partial x}=0\\ y\frac{\partial u}{\partial x}+v+x\cdot\frac{\partial v}{\partial x}=0\end{cases}\Rightarrow\frac{\partial v}{\partial x}=\frac{\begin{vmatrix}x&-u\\y&-v\end{vmatrix}}{\begin{vmatrix}x&-y\\y&x\end{vmatrix}}=-\frac{x\cdot v-y\cdot u}{x^2+y^2}$$

$$\begin{cases}x\cdot\frac{\partial u}{\partial y}-v-y\cdot\frac{\partial v}{\partial y}=0\\ u+y\cdot\frac{\partial u}{\partial y}+x\cdot\frac{\partial v}{\partial y}=0\end{cases}\Rightarrow\frac{\partial v}{\partial y}=\frac{\begin{vmatrix}x&v\\y&-u\end{vmatrix}}{\begin{vmatrix}x&-y\\y&x\end{vmatrix}}=-\frac{xu+yv}{x^2+y^2}$$

$$\therefore dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy=-\frac{x\cdot v-y\cdot u}{x^2+y^2}dx-\frac{xu+yv}{x^2+y^2}dy$$