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作业p143

5（2）

6（2）

7（2） （4） （6）

8（1） （3）

5. (2) $\int_0^1 dx\int_0^{\sqrt{1-x^2}} dy\int_0^{\sqrt{4-(x^2+y^2)}} dz.$

$$D:\ 0\le x\le 1,\quad 0\le y\le \sqrt{1-x^2},\quad 0\le z\le \sqrt{4-(x^2+y^2)}$$

$$I=\int_0^{\frac{\pi}{2}}d\theta\int_0^1 dr\int_0^{\sqrt{4-r^2}}dz$$

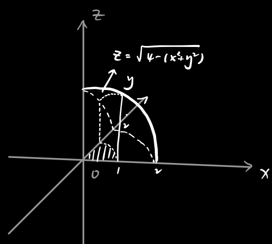
$$=\frac{\pi}{2}\int_0^1 r\sqrt{4-r^2}\,dr$$

$$=-\frac{\pi}{4}\int_0^1\sqrt{4-r^2}\,d(4-r^2)$$

$$=-\frac{\pi}{4}\cdot\frac{2}{3}\cdot(4-r^2)^{\frac{3}{2}}\Big|_0^1$$

$$=-\frac{\pi}{6}\cdot\left(3^{\frac{3}{2}}-4^{\frac{3}{2}}\right)$$

$$=-\frac{\pi}{6}\cdot(3\sqrt{3}-8)$$



6. (2) $\int_0^3 dy\int_0^{\sqrt{9-y^2}} dx\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz.$

$$\Omega:\begin{cases}0\le y\le 3\\ 0\le x\le\sqrt{9-y^2}\\ \sqrt{x^2+y^2}\le z\le\sqrt{18-(x^2+y^2)}\end{cases}$$

$$\Rightarrow\begin{cases}\varphi\in[0,\frac{\pi}{4}]\\ \theta\in[0,\frac{\pi}{2}]\\ r\in[0,3\sqrt{2}]\end{cases}$$

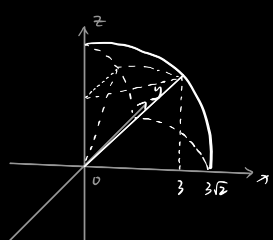
$$\therefore I=\int_0^{\frac{\pi}{4}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^{3\sqrt{2}}r^2\cdot r^2\sin\varphi\,dr$$

$$=-\cos\varphi\Big|_0^{\frac{\pi}{4}}\cdot\frac{\pi}{2}\cdot\frac{1}{5}r^5\Big|_0^{3\sqrt{2}}$$

$$=-\left(\frac{\sqrt{2}}{2}-1\right)\cdot\frac{\pi}{10}\cdot 243\cdot 4\sqrt{2}$$

$$=\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{486\sqrt{2}\pi}{5}$$

$$=\frac{486\pi}{5}(\sqrt{2}-1)$$



(2) $\iiint_\Omega(x^2+y^2+z^2)\,dxdydz$，$\Omega$ 为球面 $x^2+y^2+(z-1)^2\le 1$；

$$D:\begin{cases}\varphi\in[0,\frac{\pi}{2}]\\ \theta\in[0,2\pi]\\ r\in[0,2\cos\varphi]\end{cases}$$

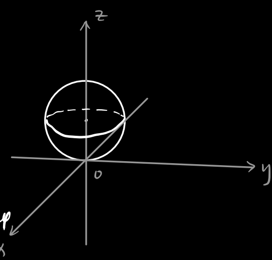
$$I=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2\pi}d\theta\int_0^{2\cos\varphi}r^2\cdot r^2\sin\varphi\,dr$$

$$=2\pi\cdot\int_0^{\frac{\pi}{2}}\frac{1}{5}\cdot 2^5\cdot\cos^5\varphi\sin\varphi\,d\varphi$$

$$=\frac{-64\pi}{5}\int_0^{\frac{\pi}{2}}\cos^5\varphi\,d\cos\varphi$$

$$=\frac{-64\pi}{5}\cdot\frac{1}{6}\cdot\cos^6\varphi\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{32\pi}{15}$$



(4) $\iiint_\Omega\frac{\sin\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}\,dxdydz$，其中 Ω：$x^2+y^2+z^2\le 1$，$x\ge 0$，$y\ge 0$，$z\ge 0$；

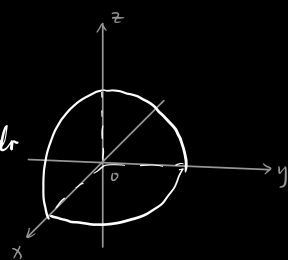
$$\Omega:\begin{cases}\varphi\in[0,\frac{\pi}{2}]\\ \theta\in[0,\frac{\pi}{2}]\\ r\in[0,1]\end{cases}$$

$$I=\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\pi}{2}}d\theta\int_0^1\frac{\sin r}{r^2}\cdot r^2\sin\varphi\,dr$$

$$=-\cos\varphi\Big|_0^{\frac{\pi}{2}}\cdot\frac{\pi}{2}\cdot(-\cos r)\Big|_0^1$$

$$=(0-1)\cdot\frac{\pi}{2}\cdot(\cos 1-1)$$

$$=\frac{\pi}{2}(1-\cos 1)$$



(6) $\iiint_\Omega z\,dxdydz$，Ω 由 $x^2+y^2+z^2=4$ 与 $z=\frac{1}{3}(x^2+y^2)$ 所围的闭区域；

$$I=\iint_D dxdy\int_{\frac{1}{3}(x^2+y^2)}^{\sqrt{4-(x^2+y^2)}}z\,dz$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}r\,dr\int_{\frac{1}{3}r^2}^{\sqrt{4-r^2}}z\,dz$$

$$=2\pi\cdot\int_0^{\sqrt{3}}r\cdot\frac{1}{2}\left(4-r^2-\frac{1}{9}r^4\right)dr$$

$$=\pi\int_0^{\sqrt{3}}\left(4r-r^3-\frac{1}{9}r^5\right)dr$$

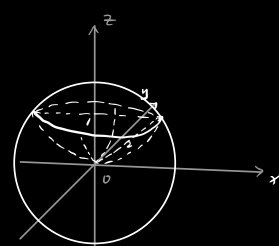
$$=\pi\left(2r^2-\frac{1}{4}r^4-\frac{1}{9}\cdot\frac{1}{6}r^6\right)\Big|_0^{\sqrt{3}}$$

$$=\pi\left(6-\frac{1}{4}\times 9-\frac{1}{54}\times 27\right)$$

$$=\pi\left(6-\frac{9}{4}-\frac{1}{2}\right)$$

$$=\frac{13\pi}{4}$$

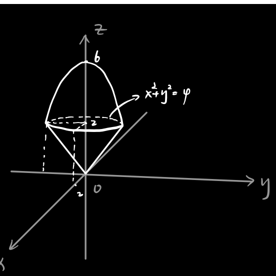
$$\frac{24-9-2}{4}$$



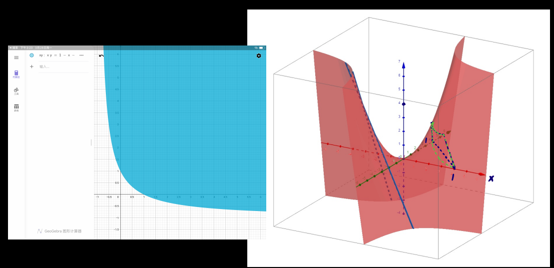
8. 利用三重积分计算下列由曲面所围成的立体的体积：

(1) $z=6-x^2-y^2$ 及 $z=\sqrt{x^2+y^2}$；

$$I=\iint_D dxdy\int_{\sqrt{x^2+y^2}}^{6-x^2-y^2}dz$$
$$=\int_0^{2\pi}d\theta\int_0^2 r\,dr\int_r^{6-r^2}dz$$
$$=2\pi\cdot\int_0^2 r\cdot(6-r^2-r)\,dr$$
$$=2\pi\left(3r^2-\frac{1}{4}r^4-\frac{1}{3}r^3\right)\Big|_0^2$$
$$=2\pi\left(12-4-\frac{8}{3}\right)$$
$$=\frac{32\pi}{3}$$



(3) $z=xy$，$x+y+z=1$ 及 $z=0$；



$$\begin{cases} z=xy \\ z=1-x-y \end{cases} \Rightarrow xy=1-x-y \Rightarrow y=\frac{1-x}{1+x}$$

Ω_1: $0\le x\le 1$　$0\le y\le\frac{1-x}{1+x}$　$0\le z\le xy$

Ω_2: $0\le x\le 1$　$\frac{1-x}{1+x}\le y\le 1-x$　$0\le z\le 1-x-y$

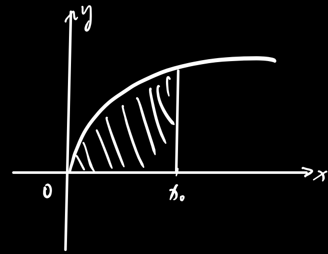
$$I=\int_0^1 dx\int_0^{\frac{1-x}{1+x}}dy\int_0^{xy}dz+\int_0^1 dx\int_{\frac{1-x}{1+x}}^{1-x}dy\int_0^{1-x-y}dz$$
$$=\int_0^1 dx\int_0^{\frac{1-x}{1+x}}xy\,dy+\int_0^1 dx\int_{\frac{1-x}{1+x}}^{1-x}(1-x-y)\,dy$$
$$=\int_0^1\frac{1}{2}x\left(\frac{1-x}{1+x}\right)^2dx+\int_0^1\left\{(1-x)\left(1-x-\frac{1-x}{1+x}\right)-\frac{1}{2}\left[(1-x)^2-\left(\frac{1-x}{1+x}\right)^2\right]\right\}dx$$
$$=\int_0^1\left[\frac{x(1-x)^2}{2(1+x)^2}+\frac{x(1-x)^2}{1+x}-\frac{1}{2}\cdot\left(1-x+\frac{1-x}{1+x}\right)\left(1-x-\frac{1-x}{1+x}\right)\right]dx$$
$$=\int_0^1\left[\frac{x(1-x)^2}{2(1+x)^2}+\frac{x(1-x)^2}{1+x}-\frac{(-x+1)^2\cdot x(x+2)}{2(1+x)^2}\right]dx$$
$$=\int_0^1\frac{(1-x)^2(x+2x+2x^2-x^2-2x)}{2(1+x)^2}dx$$
$$=\int_0^1\frac{(1-x)^2\cdot x}{2(1+x)}dx$$
$$\xlongequal[x=u-1]{1+x=u}\frac{1}{2}\int_1^2\frac{(u-1)(2-u)^2}{u}du$$
$$=\frac{1}{2}\int_1^2\left(u^2-5u+8-\frac{4}{u}\right)du$$
$$=\frac{1}{2}\left(\frac{1}{3}u^3-\frac{5}{2}u^2+8u-4\ln u\right)\Big|_1^2$$
$$=\frac{1}{2}\left(\frac{7}{3}-\frac{5}{2}\times3+8\times1-4\ln 2\right)$$
$$=\frac{17}{12}-2\ln 2$$

作业P149
1. (1)　(3)
2. (1)　(3)

1. 求由下列曲线所围成的均匀薄片的重心坐标：

(1) D 由 $y=\sqrt{2px}$，$x=x_0$，$y=0$ 所围成；

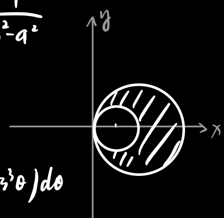
$$\bar{x}=\frac{\int_0^{x_0}dx\int_0^{\sqrt{2px}}x\,dy}{\int_0^{x_0}dx\int_0^{\sqrt{2px}}dy}$$
$$=\frac{\int_0^{x_0}x\sqrt{2px}\,dx}{\int_0^{x_0}\sqrt{2px}\,dx}$$
$$=\frac{\frac{2}{5}\cdot x_0^{\frac{5}{2}}}{\frac{2}{3}x_0^{\frac{3}{2}}}$$
$$=\frac{3}{5}x_0$$



$$\bar{y}=\frac{\int_0^{x_0}dx\int_0^{\sqrt{2px}}y\,dy}{\frac{2}{3}x_0^{\frac{3}{2}}\cdot\sqrt{2p}}=\frac{\frac{1}{2}\int_0^{x_0}2px\,dx}{\frac{2}{3}x_0^{\frac{3}{2}}\cdot\sqrt{2p}}=\frac{p\cdot\frac{1}{2}x_0^2}{\frac{2}{3}\sqrt{2p}\cdot x_0^{\frac{3}{2}}}$$
$$=\frac{3\sqrt{2p}}{8}\cdot x_0^{\frac{1}{2}}$$
$$\left(\frac{3}{5}x_0,\ \frac{3\sqrt{2px_0}}{8}\right)$$

(3) D 是介于两个圆 $r=a\cos\theta$，$r=b\cos\theta$ 之间的图形 $(0<a<b)$.

$$A=\pi\left(\frac{b^2}{4}-\frac{a^2}{4}\right)\quad\frac{1}{A}=\frac{4}{\pi}\cdot\frac{1}{b^2-a^2}$$



$$\bar{x}=\frac{1}{A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_{a\cos\theta}^{b\cos\theta}r^2\cos\theta\,dr$$
$$=\frac{1}{A}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{3}\cos\theta\left(b^3\cos^3\theta-a^3\cos^3\theta\right)d\theta$$
$$=\frac{b^3-a^3}{3A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$
$$=\frac{2(b^3-a^3)}{3A}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$
$$=\frac{2(b^3-a^3)}{3A}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$
$$=\frac{\cancel{2}(b^3-a^3)}{\cancel{3}A}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{2}}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$
$$=\frac{\pi(b^3-a^3)}{12}\cdot\frac{4}{\pi}\frac{1}{b^2-a^2}$$
$$=\frac{1}{3}\cdot\frac{b^3-a^3}{b^2-a^2}$$

$$\bar{y} = \frac{1}{A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{a\cos\theta}^{b\cos\theta} r^2\sin\theta\, dr$$

$$= \frac{1}{3A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,(b^3\cos^3\theta - a^3\cos^3\theta)\,d\theta$$

$$= \frac{b^3-a^3}{-3A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\, d\cos\theta$$

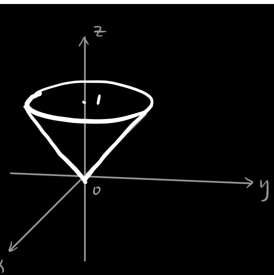
$$= \frac{b^3-a^3}{-3A}\cdot\frac{1}{4}\cos^4\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 0$$

$$\therefore \left(\frac{b^3-a^3}{3(b^2-a^2)},\ 0\right)$$

2. 求下列由曲面所围成的均匀立体的重心：

(1) $z^2 = x^2 + y^2$，$z = 1$；



$$\bar{x} = \bar{y} = 0$$

$$V = \int_0^1 \pi z^2\, dz$$

$$= \frac{\pi}{3}$$

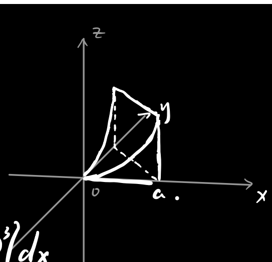
$$\bar{z} = \frac{1}{V}\cdot\int_0^1 dz\int_0^{2\pi} d\theta\int_0^z z r\, dr$$

$$= \frac{3}{\pi}\,2\pi\cdot\int_0^1 \frac{1}{2}\cdot z^3\, dz$$

$$= 6\cdot\frac{1}{2}\cdot\frac{1}{4}$$

$$= \frac{3}{4}$$

(3) $z = x^2 + y^2$，$x + y = a$，$x = 0$，$y = 0$，$z = 0(a > 0)$.



$$V = \iint_\Omega dxdy\int_0^{x^2+y^2} dz$$

$$= \int_0^a dx\int_0^{-x+a}(x^2+y^2)\,dy$$

$$= \int_0^a\left[x^2(-x+a) + \frac{1}{3}(-x+a)^3\right]dx$$

$$= \int_0^a -x^3 dx + \int_0^a ax^2 dx - \frac{1}{3}\int_0^a(-x+a)^3\, d(-x+a)$$

$$= -\frac{1}{4}a^4 + \frac{1}{3}a^4 - \frac{1}{3}\cdot\frac{1}{4}(-x+a)^4\Big|_0^a$$

$$= -\frac{1}{4}a^4 + \frac{1}{3}a^4 - \frac{1}{12}(-a^4)$$

$$= \frac{1}{6}a^4$$

$$\bar{x} = \frac{1}{V}\int_0^a dx\int_0^{-x+a} dy\int_0^{x^2+y^2} x\, dz$$

$$= \frac{1}{V}\int_0^a dx\int_0^{a-x} x(x^2+y^2)\,dy$$

$$= \frac{1}{V}\int_0^a\left[x^3(a-x) + \frac{x}{3}\cdot(a-x)^3\right]dx$$

$$= \frac{1}{V}\cdot\left(\int_0^a ax^3 - \int_0^a x^4 dx + \frac{1}{3}\int_0^a x(a^3+x^3+3ax^2-3a^2x)\,dx\right)$$

$$= \frac{1}{V}\left[\frac{1}{4}a^5 - \frac{1}{5}a^5 + \frac{1}{3}\left(\frac{1}{2}a^5 + \frac{1}{5}a^5 + \frac{3}{4}a^5 - a^5\right)\right]$$

$$= \frac{a^5}{5V} = \frac{a^5}{5}\cdot\frac{6}{a^4} = \frac{6a}{5}$$

$$\bar{x} = \bar{y}$$

$$\bar{z} = \frac{1}{V}\int_0^a dx\int_0^{-x+a} dy\int_0^{x^2+y^2} z\, dz = \frac{7}{30}a^2$$

$$\therefore \left(\frac{6a}{5},\ \frac{6a}{5},\ \frac{7a^2}{30}\right)$$