TIDAL THEORY AND COMPUTATIONS

J. J. DRONKERS[1]

Hydraulic Department of Deltaworks
The Hague, The Netherlands

I. Introduction

In the past, tidal phenomena have often drawn the attention of great scientists, such as Newton, Bernoulli, and Laplace in the seventeenth and eighteenth centuries, and Airy, Darwin, Thomson, Poincaré, and other scientists in the nineteenth century and in the beginning of this century. Important theoretical contributions have been made by Taylor and Proudman in the first part of this century. A synthesis of tidal theory up to about 1920 can be found in the classical book of Lamb (1932).

Practical methods of tidal prediction appeared first in the beginning of the nineteenth century: Laplace (1799) started in France and Whewell (1833–1836) in England; however their methods were different. The methods

[1] Soon after completion of the manuscript, Dr. Dronkers died on February 20, 1973 after years of distinguished service as Head of the Hydraulics Department. As this article is the last of his monumental works, its publication is respectfully dedicated to him for his invaluable contributions to the field of hydroscience particularly in the theory and computation of tides.

for the prediction of tides have been improved by Darwin and Thomson in the last part of the previous century, and by Doodson in this century.

The practical methods concerning the dynamical aspects of the tides started mainly in this century. The name of the great scientist H. A. Lorentz is connected with one of the earliest contributions. His work concerns the calculation of the changes of the tide in the case of the enclosing of the Zuiderzee in The Netherlands in 1932. An important publication in The Netherlands on the propagation of tides is the Schönfeld thesis (1951). A review of practical applications up to 1954 can be found in the work by Dronkers and Schönfeld (1954).

Since World War II the practical application of tidal theory has been influenced considerably by the introduction of the electronic computer. More complicated situations in reality can be analyzed better than before, and more refined schematization of tidal regions in nature can be considered. With the use of the computer, however, the methods for tidal prediction and for the determination of the propagation of tidal waves in seas and rivers have to be modified.

This article deals in particular with the development of tidal theory after 1964 when this author's book (Dronkers, 1964) on tidal computations for rivers, coastal areas, and seas was published. The main part of this article can be read without the need for references; for other parts, references are provided and listed at the end of the article.

This article consists of three major sections: Section II deals with the basic equations for the prediction of tides. The tidal potential is presented with two methods for tidal prediction: the harmonic method and the response method, recently developed by Munk and Cartwright; a practical application of the harmonic method also is given. Section III discusses the hydrodynamic tidal equations for the dynamic behavior of the tides, for which emphasis is given to the vertical velocity distribution in the derivation of the equations. The convective terms in the dynamic equations are studied separately by introducing terms containing the energy head and the rotor of the velocity vector. The relation between the rotor and the circulation of the water movement is also mentioned. Section IV deals with tidal computations in rivers, seas, and coastal waters. A brief historical review of various methods of computation is given. The harmonic method and the characteristic method are discussed in general terms. The theory of the Kelvin wave for seas with parallel coastal lines is given in more detail because of its theoretical interest for the explanation of the amphidromic points. The main part of this section is devoted to the finite-difference methods to be applied on an electronic computer. More detailed examples of explicit and implicit schemes are given for one- and two-dimensional tidal regions, and practical results are presented.

II. Analysis of Tides

A. Theory of Tidal Generation and the Harmonic Analysis of Tides

The *tide-generating forces*, which originate from the sun and the moon, consist of two parts: the centrifugal forces acting on the earth due to the motion of the earth around the resultant center of gravity of the moon and the earth; and the attractive forces due to the moon, which are most important, and to the sun. The development of the moon's potential will be dealt with here, while an analogous procedure can be followed for the sun's potential.

The tide-generating force at a point X on the earth is defined as the difference between the attractive force at X and that at the resultant center of gravity of the earth and the moon, where the attractive and centrifugal forces balance with each other. The vertical and horizontal components of the tide-generating force can immediately be derived from their potentials. The value of the moon's tide-generating potential V at a point X and at time t equals the difference between the attractive potential $(fM/r_{MX} - fM/r_M)$ and the potential of the constant vector field of the centrifugal forces $fMa \cos \vartheta_{MX}/r_M^2$ caused by the movement around the resultant center of gravity of the moon and the earth (see Dronkers, 1964). Hence,

$$V_M = fM\left[\frac{1}{r_{MX}} - \frac{1}{r_M} - \frac{a \cos \vartheta_{MX}}{r_M^2}\right] \tag{1}$$

in which $f = ga^2/E$ is the gravitational constant with E being the mass of the earth, M the mass of the moon, a the mean radius of the earth, r_{MX} the distance between the center of the moon and the point X, r_M the distance between the centers of the moon and the earth, and ϑ_{MX} the zenith distance of moon at a point X (the coordinate system is based on the horizon and the zenith). For the sun's tidal-generating potential, a similar formula applies well if M is replaced by S in the formula. The notation is shown in Fig. 1.

After expressing r_{MX} in terms of r_M and ϑ_{MX} according to $r_{MX}^2 = r_M^2 + a^2 - 2ar_M \cos \vartheta_{MX}$ (Fig. 1) and expanding $1/r_{MX}$ in powers of the parallax a/r_M by means of Taylor series, the well-known expansion in zonal-harmonic functions (Morse and Feshbach, 1953) is found to be

$$V_M = fM\frac{a^2}{r_M^3}\left[P_2(\vartheta_{MX}) + \frac{a}{r_M}P_3(\vartheta_{MX}) + \cdots\right] = V_2 + V_3 + \cdots \tag{2}$$

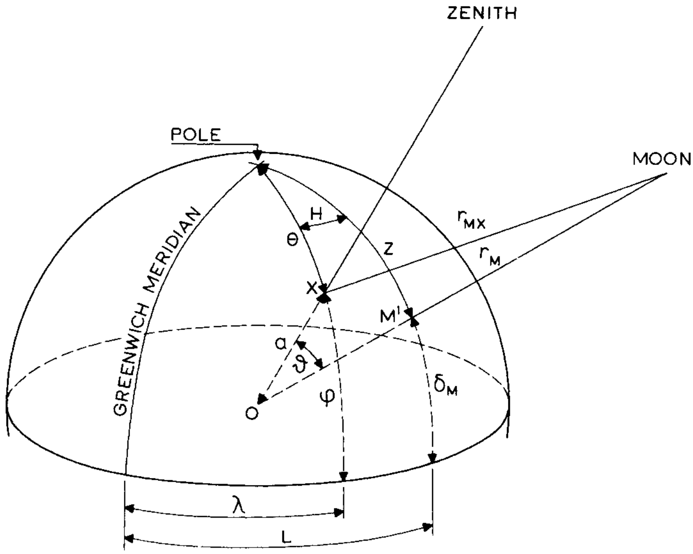

FIG. 1. Notation of celestial and local coordinates of point X on the Earth's surface and of the moon.

where

$$P_2(\vartheta_{MX}) = \tfrac{1}{2}(3\cos^2\vartheta_{MX} - 1),$$
$$P_3(\vartheta_{MX}) = \tfrac{1}{2}(5\cos^3\vartheta_{MX} - 3\cos\vartheta_{MX}), \text{ etc.}$$

The angle ϑ_{MX} changes in an irregular way. The most important change of ϑ_{MX} originates from the rotation of the earth in 24 hr. A smaller change is caused by the daily motion of the moon in its orbit (50 min/day). Further, a great number of smaller variations occur due to the irregular motion of the moon in its orbit. The period of the moon's motion in its orbit is 27.3 mean solar days, called the *lunar month.* Further, the lunar perigee changes with a period of 8.85 yr, and the regression of the lunar nodes with a period 18.6 yr. The basis of the method of harmonic analysis is the development of the tidal potential in harmonic terms, which describe all these periods.

The various steps for the further development of the tidal potential are mentioned here only in general terms. The first step is to transpose cos ϑ_{MX} in terms of the celestial coordinates based on the celestial equator: the declination δ of the moon, the hour angle H of the moon with respect to the meridian of the observer, and the geographical latitude φ of the observer. This is done by means of the cosine formula of spherical trigonometry. The second step is to express the cosine or sine of the hour angle and the declination, in terms of the motion of the moon of which the coordinates are the celestial longitude and latitude and the distance. These coordinates can be expressed by means of lunar theory in mean longitude s of the moon, mean longitude p of the moon's perigee, mean longitude N of the ascending node, and the variation of the distance. The various quantities are functions of time, and expressed in Julian centuries since 1 January 1900. The relative motion of the observer is expressed in solar time. Finally, Greenwich time is introduced. In this way the tidal potential is expanded into a great number of periodic constituents. This trigonometric expansion is

$$\frac{V}{g} = \sum_k C_k \cos(\omega_k t + \psi_k) \tag{3}$$

in which the ω_k denote the frequencies $2\pi/T_k$. The further basis of the mathematical representation of the tides in the oceans is the so-called *equilibrium tide*. The equilibrium conditions are satisfied when the free surface is normal to the resultant of the gravitational forces on the earth and the tide-generating forces. In the equilibrium theory it is assumed that the water level can respond instantaneously to these forces. The height of the water level h of the equilibrium tide is defined by V/g. Experience confirms that the real tide has constituents with the same periods as those of the tidal potential. However, every constituent lags behind its corresponding equilibrium constituent, and its amplitude is different.

The preceding procedure was followed by Thomson (Lord Kelvin) (1868–1876), and Darwin (1883–1886). It is a quasi-harmonic method since they retained phase shifts u and factors f in the amplitudes, which both have a period of the regression of the lunar nodes of 18.6 yr. An extension of Darwin's method is given in the U.S. Coastal and Geodetic Surveys "Manual of Harmonic Analysis and Predictions of Tides" (Schureman, 1958). Doodson (1921) published an extensive development which is based on Brown's formula of the moon's motion. He derived about 400 tidal constituents from the tide-generating potential. A description of tidal generation and the harmonic analysis of the tides is also found in Dronkers (1964).

Tidal constituents may be considerably magnified during their propagation. The propagation is modified in particular by its penetration into the continental shelves, coastal areas, and estuaries. Then distortion of the tidal wave takes place, e.g., its trough may be retarded more than the crest as a result of the difference in depth at high and low waters. In mathematical terms the propagation of the tide in shallow water is influenced by the nonlinear terms in the equations of motion and continuity. The distortion is caused by three effects: the friction of the bottom, the changes in the velocity due to variation in the depth, and the shape of the estuary. The deformation may be represented by terms of a Fourier series, e.g., the components M_4

and M_6 which are higher harmonics of the basic *semidiurnal moon constituent* M_2. Moreover, *compound tides* may occur due to mutual interaction of the tidal constituents during the propagation, e.g., the speed of the tidal component MS_4 is the sum of the speeds of M_2 and S_2, which is the basic *semidiurnal sun constituent* (see also Section IV,C,1). Semidiurnal tides mainly occur in the oceans. The most important ones are the M_2, S_2, K_2, and N_2 constituents. The constituent K_2 depends on the variation in the declination of the moon and the sun. Therefore K_2 is called the *lunar–solar declinational semidiurnal constituent.* The effect of the variation of the distance from the moon determines the N_2 constituent; this is called the *lunar elliptic semidiurnal constituent.* Diurnal tides, which are produced by the K_1, O_1, and P_1 constituents and which have one high water and low water each lunar day, occur mainly in the neighbourhood of the equator between the tropics. The O_1 constituent is called a *lunar declinational diurnal constituent,* K_1 the *lunar–solar declinational diurnal constituent,* and P_1 the *solar declinational diurnal constituent* (see Doodson and Warburg, 1941). The differences in height between successive high or low waters, occurring by combined constituents, are called the *daily inequalities.*

B. Tidal Prediction by the Harmonic Method

In the previous section it has been mentioned that frequencies of the tidal constituents are determined from the expansion of the tidal potential into harmonic functions, based on the motion of the earth, the moon, and the sun. The amplitudes and phases depend on the location at the seacoast or in the sea. They are determined by harmonic analysis of the tidal observations from a tidal gauge. The number of tidal constituents to be considered depends on the required accuracy of the tidal prediction.

It must be remarked that the theory of harmonic analysis and the theory of the Fourier series are different. The actual tide during a certain time interval, e.g., one day or a number of successive days, can be represented by means of a Fourier series, of which the basic frequency ω is that of the moon tide M_2 or of the sun tide S_2. The number of terms with frequencies $n\omega$ ($n = 1, 2$, etc.) depends again on the required accuracy of the representation. This representation applies to the time interval under consideration. This is in contradiction to the "harmonic" representation for a moon month, or year, etc., which is almost independent of a specific time interval. The Fourier representation may be very useful in practical applications, e.g., for the computation of the propagation of the tide for a certain time interval. On the other hand, Fourier series also occur in the harmonic analysis because certain irregularities in the earth's motion are developed into such series. The tidal wave may be deformed during the propagation into shallow water, due to the nonlinearity of the tidal equations. This distortion observed at a certain location may also be represented by the terms of a Fourier series. Such deformations make the tidal prediction more uncertain (Zetler and Cummings, 1967).

From the preceding discussion it follows that tidal frequencies appear to be centered at frequencies corresponding with one, two, or more cycles per lunar day, month, and year. In mathematical terms, the frequencies may be described by

$$k_1 \hat{f}_1 + k_2 \hat{f}_2 + k_3 \hat{f}_3 + k_4 \hat{f}_4 + k_5 \hat{f}_5$$

in which $\hat{f}_1$ is the group of frequencies of the earth's rotation, $\hat{f}_2$ that of the moon's orbital motion, $\hat{f}_3$ that of the sun's orbital motion, $\hat{f}_4$ that of the lunar perigee, and $\hat{f}_5$ that of the regression of the lunar nodes. Each group consists of a great number of constituents. The differences between the frequencies in a group may be very small, e.g., 0°.08 per hour in case of the K_1 and P_1 constituents and the K_2 and S_2 constituents.

According to the theory of harmonic analysis, the sea level h at any instant t is represented by

$$h = H_0 + \sum_{r=1}^{n} f_r H_r \cos[\omega_r t + (E + u)_r - g_r] \tag{4}$$

in which H_0 is the mean sea level, H_r the mean amplitude of the constituent r, f_r the reduction factor of H_r with respect to the year of prediction, ω_r the frequency, $(E + u)_r$ the astronomical argument at $t = 0$, t is time taken in hours from the beginning of the year of prediction (0 hr Greenwich Mean Time), and g_r the lag of phase behind the corresponding equilibrium tide at Greenwich.

Equation (4) can be rewritten in the form

$$h = H_0 + \sum_{r=1}^{n} A_r \cos \omega_r t + \sum_{r=1}^{n} B_r \sin \omega_r t \tag{5}$$

The constants H_0, A_r, and B_r are computed from the observations. A brief description of the equations in matrix notation according to Shipley (1967) follows here. A description, not in matrix notation, is given in Dronkers (1964). Let m consecutive hourly observations h_p at $t = 0, 1, \ldots, m - 1$ hours, be available. Then m equations are found for the determination of the quantities H_0, A_r, and B , $2n + 1$ in number, such that $m \gg 2n + 1$. The values of A_r and B_r are determined by the well-known procedure of Gauss, based on the least-square rule. The solution in matrix form is as follows: The set of Eqs. (5) is written in the form

$$H = CX \tag{6}$$

in which C is a matrix with m rows and $2n + 1$ columns. The $(p + 1)$th row of C is

$$c_{0,p} = 1; \qquad c_{1,p} = \cos \omega_1 p; \qquad \dots; \qquad c_{n,p} = \cos \omega_n p;$$
$$c_{n+1,p} = \sin \omega_1 p; \qquad \dots; \qquad c_{2n,p} = \sin \omega_n p$$

in which successively $p = 0, \dots, (m-1)\,\Delta t$; $\Delta t = 3600$ sec. The matrices X and H are column vectors:

$$X = \begin{pmatrix} H_0 \\ A_r \\ B_r \end{pmatrix} \quad (r = 1, 2, \dots, n),$$

$$H = \begin{pmatrix} h_0 \\ h_p \end{pmatrix} \quad (p = 1, 2, \dots, m-1)$$

Let C' be the transpose of C, defined by the terms $(c'_{rp}) = (c_{pr})$. Then the normal equations of Gauss are obtained by means of the operation

$$C'H = C'CX \tag{7}$$

in which the matrix $C'C$ is square. Let the inverse of $C'C$, or $(C'C)^{-1}$, be called D. Then the expression

$$D(C'C)X = X = DC'H \tag{8}$$

gives the solution X of the normal equations (7) in symbolic form. The transpose of X determines the coefficients H_0, A_r, and B_r. After that, H_r and $(E+u)_r$ can be obtained from A_r and B_r in the usual way.

It follows from the definitions of C and C' that the terms in the product $C'C$ are the sum of products of cosine or sine, e.g.,

$$S_{i,j} = \sum_{p=0}^{m-1} \cos(\omega_i p) \cos(\omega_j p)$$
$$\text{for} \quad 0 < i < n+1, \quad 0 < j < n+1$$

Similar expressions occur for the sums in which one or two cosine functions are replaced by a sine function. The sum in the expression of $S_{i,j}$ can be determined by replacing the product in each term by a sum. After that the sum of the series can be obtained. Thus, e.g.,

$$\sum_{p=0}^{m-1} \cos p(\omega_i + \omega_j) = \frac{1}{2}\left[1 + \frac{\sin[(2m-1)\frac{1}{2}(\omega_i + \omega_j)]}{\sin \frac{1}{2}(\omega_i + \omega_j)}\right]$$

The reader can determine all the separate terms that may occur in the square matrix $C'C$. Reference to this determination can be found in Shipley (1967) and van Ette and Schoemaker (1967).

The solution of the normal equations (7) is very elaborate and can now be processed on the computer, especially when n is large. Moreover, m must be much larger than $2n + 1$. Before the computer was available, this work had to be done by desk calculators. Then it was necessary to use skillful methods to solve the equations for a practical case. Especially, Doodson (1927, 1957) has devised such methods. After the introduction of the computer, solutions could be obtained much easier and quicker in a straightforward manner. However, much computer time is necessary for inverting the square matrix $(C'C)$, thus to determine $(C'C)^{-1}$. But for certain fixed lengths of records, e.g., the moon month which is 29 days, the matrix $(C'C)^{-1}$ needs to be inverted for a certain set of n constituents only once. After that the matrix can be applied to all records during moon months.

Another difficulty concerns the so-called "noise level," caused by disturbances in the records due to, for instance, the meteorological effects. It is often difficult to separate close constituents such as S_2 and K_2 of which the difference in speed per hour is $0°.082$, or K_1 and P_1 (the difference is also $0°.082$) from the records. For such cases, a considerably longer period than 29 days must be considered, e.g., observations during a year.

Solutions of the normal equations are moreover affected by the omitted constituents in the tidal analysis, the way of sampling of the observations, and nonastronomical effects. The influence of these factors were studied by van Ette and Schoemaker (1967) by applying the theory of stationary stochastic processes to the tidal phenomenon. It is not the aim here to follow their paper in detail. Briefly, they represent the observations by

$$g(t) = c_0 + \sum_{i=1}^{n} c_i \cos(\omega_i t) + d_i \sin(\omega_i t) + x(t)$$

in which $x(t)$ is the part of the record that has a nondeterministic character, e.g., meteorological influences, accidental failures of the tide gauge, and mistakes in the computer input. The function $x(t)$ is assumed to be a stationary stochastic process with zero average during a sufficiently long period of observations. The "noise level" $x(t)$ has a continuous spectrum in contradiction to the spectrum of the tidal constituents. It is especially difficult to compute constituents for which the frequencies differ very little from each other. Van Ette and Schoemaker discussed the criteria which enable them to decide which constituents may be computed in a reasonable way. These criteria, which concern the duration of the observed tide $g(t)$ and the time interval Δt between the successive observations, are found from a study of the matrix $C'C$ of the normal equations (7). The influence of $x(t)$ on the solution of the normal equations is studied in detail in the paper.

C. Tidal Prediction by the Response Method

A new method of tidal prediction is presented by Munk and Cartwright (1966). They apply the theory of time series to the tidal observations at a certain location to determine certain coefficients, which replace the amplitudes H and phases g of the tidal constituents in Eq. (4), which form the basis for tidal prediction by the harmonic method. The theory of this method is more involved than the theory of the harmonic method. Munk and Cartwright state that the response method gives a simpler and physically more meaningful representation of tides than the harmonic method. It is a more empirical modification of the equilibrium tide, based on the *theory of time series*, which was mainly developed after 1940. The principles of the response method in general, applied to time series, are described by, e.g., Bendat and Piersol (1971). In this particular case the word *response* refers to the sea level response at a location in the sea due to the tide-generating forces.

An advantage of the response method is that the total number of coefficients is less than the number of constituents used for the harmonic prediction of comparable accuracy. Cartwright (1967, 1968) presented some further results of the response method and gave more information about the practical execution. He also gives a method for computing high water and low water. At first the method was applied to deep sea ports like Honolulu and Newlyn (see Munk and Cartwright, 1966). In his paper Cartwright (1967, 1968) describes the application to ports around the coast of Britain, where the nonlinear shallow water effects are more important. His conclusion is that the prediction of these effects can also be dealt with by the response method, even better than by the harmonic method, because in the latter case many more constituents must be included.

In the following the principles of the response method applied to tidal observations will be discussed in a somewhat different way from that given in the mathematical treatment by Munk and Cartwright (1966). The equilibrium tide is written in the form [compare Eq. (2) and the definition of the coefficient f]

$$\frac{V}{g} = a\frac{M}{E}\sum_{n=2}^{\infty}\xi^{n+1}P_n(\vartheta) = \sum_{n=2}^{\infty}K_n\left(\frac{\bar{r}}{r}\right)^{n+1}P_n(\vartheta) \tag{9}$$

where $\xi = a/r$, $\bar{r}$ is the mean distance of earth and moon, $K_n = Ma^{n+2}/E(\bar{r})^{n+1}$, and the variables ϑ and r are functions of time. Then V/g is expressed in terms of the polar angle θ, the east longitude λ, the polar angle $Z(t)$, and the terrestrial longitude $L(t)$, where $Z(t)$ and $L(t)$ refer to the moon, and θ and λ refer to the observer (see Fig. 1). The notation is defined in the Greenwich system. The relation with the zenith distance ϑ is given by

$$\cos \vartheta = \cos \theta \cos Z + \sin \theta \sin Z \cos(\lambda - L) \tag{10}$$

The authors introduce the spherical coordinates θ, λ into the equation of the tidal potential [Eq. (9)] by means of the transformation formulas $x = r \sin \theta \cos \lambda$, $y = r \sin \theta \sin \lambda$, and $z = r \cos \theta$. Then the Laplace equation

$$\nabla V = \frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} + \frac{\delta^2 V}{\delta z^2} = 0$$

which is satisfied by the tidal potential, changes to

$$\nabla V = \frac{1}{r^2}\left[\frac{\delta}{\delta r}\left(r^2 \frac{\delta V}{\delta r}\right) + \frac{1}{\sin \theta}\frac{\delta}{\delta \theta}\left(\sin \theta \frac{\delta V}{\delta \theta}\right) + \frac{1}{\sin^2 \theta}\frac{\delta^2 V}{\delta \lambda^2}\right] = 0 \tag{11}$$

This equation may be solved by the well-known method of separation of variables (Morse and Feshbach, 1953):

$$V(r, \theta, \lambda) = R(r) \cdot F(\theta) \cdot G(\lambda)$$

in which $R(r)$ is a power series of r, according to Eq. (9). The function $G(\lambda)$ must be a sum of cosine and sine functions of $m\lambda$ according to Eqs. (2) and (10). Then $m = \pm 1, \pm 2, \ldots$, and $|m| \leq n$. Each term of $G(\lambda)$ satisfies an equation

$$\frac{1}{G}\frac{\delta^2 G}{\delta \lambda^2} = -m^2$$

After substitution of the function $G(\lambda)$ in Eq. (11) and of $R(r) = r^{-n-1}$ according to Eq. (9), the following equation for the function F is found:

$$\sin \theta \frac{d}{d\theta}\left(\sin \theta \frac{dF}{d\theta}\right) + [n(n+1)\sin^2 \theta - m^2]F = 0 \tag{12}$$

Substitution of $\cos \theta = x$ gives the well-known differential equation of Legendre, of which the solutions are the associated Legendre functions or *spherical harmonics of the second kind* $P_n^m(x)$ in which m may be a positive or negative integer. The functions $P_n(x)$ mentioned in Section II,A are called the *spherical harmonics of the first kind* or the *zonal harmonic functions*. They satisfy Eq. (12) if $m = 0$. These functions are given in many textbooks such as those by Hobson (1965) and Whittaker and Watson (1927, Sects. XV and XVIII), and also in textbooks on theoretical physics by, e.g., Morse and Feshbach (1953, Part II, Sect. 10.3). Ferrer's definition of $P_n^m(\cos \theta)$ is

$$P_n^m(\cos \theta) = \sin^m(\theta)\frac{d^m P_n(\cos \theta)}{d(\cos \theta)^m}, \qquad n > 0, \quad |m| \leq n$$

A very important property of the zonal and spherical harmonics is that they are orthogonal. It appears that

$$\int_{-1}^{+1} P_{n_1}^m(x) P_{n_2}^m(x)\, dx = \begin{cases} 0 & \text{if} \quad n_1 \neq n_2 \\ \dfrac{(n+m)!}{(n-m)!}\dfrac{2}{2n+1} & \text{if} \quad n = n_1 = n_2 \end{cases}$$

The sine and the cosine are also orthogonal functions. The normalized function of $P_n^m(x)$, $\overline{P_n^m}(x)$ is therefore defined by

$$\overline{P_n^m}(x) = \left[\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(x)$$

where $P_n^m(x) = 0$ if $|m| \geq n$; $\bar{P}_n(x) = [(2n+1)/2]^{1/2} P_n(x)$, $n > 0$; and $P_0(x) = 1$.

Consequently, any function $f(\theta, \lambda)$ defined on a sphere can be developed into a series of spherical harmonics in $\cos\theta$ ($0 \leq \theta \leq \pi$) and into a Fourier series in λ ($-\pi \leq \lambda \leq \pi$). Therefore the spherical function $P_n(\cos\vartheta)$ can also be expressed in a sum of functions, in which $P_n^m(\theta)$, $P_n^m(Z)$, $\sin m(\lambda - Z)$, $\cos m(\lambda - L)$ occur with θ and Z being abridgments for $\cos\theta$ and $\cos Z$.

From the Taylor series of $P_n(\cos\vartheta)$ (Whittaker and Watson, 1927)

$$P_n(\cos\vartheta) = \sum_{k=0}^{r} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k!)}(\cos\vartheta)^{n-2k}$$

where $r = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is an integer, and from Eq. (10), $P_n(\cos\vartheta)$ may be written in the form

$$P_n(\cos\vartheta) = \sum_{m=-n}^{n} F_n^m(\theta, Z) e^{im(\lambda - L)}$$

The function $F_n^m(\theta, Z)$ must satisfy Eq. (12) in θ, as well in Z (replace θ by Z) because $r^{-n-1}P_n(\cos\vartheta)$ satisfies the equation of Laplace [see Eq. (9)]. Consequently, $P_n(\cos\vartheta)$ can be written in the form

$$P_n(\cos\vartheta) = \sum_{m=-n}^{n} A_n^m P_n^m(\theta) P_n^m(Z) e^{im(\lambda - L)} \tag{13}$$

in which $A_n^m = (n-m)!/(n+m)!$. Equation (13) is called the *addition theorem for the Legendre polynomials.* In Whittaker and Watson (1927), a detailed proof is given for this equation, and in particular the coefficient A_n^m is determined.

From the preceding discussion it appears that functions of the form

$$V(r, \theta, \lambda) = r^{-n-1}P_n^m(\cos\theta)e^{im\lambda}$$

satisfy the equation of Laplace (11). The spherical harmonics that satisfy Eq. (11) are defined by

$$(Y_n^m)_c = \cos(m\lambda)P_n^m(\cos\theta) \qquad \text{and} \qquad (Y_n^m)_s = \sin(m\lambda)P_n^m(\cos\theta) \quad (14)$$

in which $|m| \leq n$, or $m = 0$. The normalization constant of these functions is defined by (Morse and Feshbach, 1953)

$$\left[\int_0^{2\pi} d\lambda \int_0^{\pi} \{Y_n^m(\theta, \lambda)\}^2 \sin\theta \, d\theta\right]^{1/2}$$
$$= \left(\frac{2\pi}{2n+1}\right)^{1/2}\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2}, \qquad n \geq 1, \quad 0 < |m| \leq n$$

where the factor $\pi^{1/2}$ refers to the normalization of the function $\cos m\lambda$ or $\sin m\lambda$, and the other factors refer to the normalization of $P_n^m(x)$, mentioned above in the definition of $P_n^m(x)$. In Eq. (13) the orthogonal functions Y_n^m are introduced in complex form:

$$Y_n^m(\theta, \lambda) = \left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\theta)e^{im\lambda} \quad (15)$$

In this formula in complex form the normalization factor $\pi^{1/2}$ has been replaced by $(2\pi)^{1/2}$. Hence Eq. (13) can be written in the form

$$P_n(\cos\vartheta) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}(Y_n^m(Z, L))^* Y_n^m(\theta, \lambda) \quad (16)$$

According to Hobson (1965, Sect. 60),

$$P_n^{-m}(\theta) = (-1)^m\frac{(n-m)!}{(m+n)!}P_n^m(\theta), \qquad m > 0$$

Then it follows from Eq. (15), in which m is replaced by $-m$, that

$$Y_n^{-m}(\theta, \lambda) = (-1)^m(Y_n^m(\theta, \lambda))^* \quad (17)$$

in which $(Y_n^m)^*$ is the conjugate formula of Y_n^m. Let

$$Y_n^m(\theta, \lambda) = U_n^m(\theta, \lambda) + iV_n^m(\theta, \lambda) \qquad \text{and} \qquad Y_n^0(\theta, \lambda) = U_n^0(\theta, \lambda)$$

for $m \neq 0$ and $m = 0$ respectively. Analogous formulas hold good for

$Y_n^m(Z, L)$. Then $P_n(\cos \vartheta)$ can be written as follows [see Eqs. (16) and (17)]:

$$P_n(\cos \vartheta) = \frac{4\pi}{2n+1}\Bigg[U_n^0(Z, L)U_n^0(\theta, \lambda) + 2\sum_{m=1}^{n}\{U_n^m(Z, L)U_n^m(\theta, \lambda) + V_n^m(Z, L)\cdot V_n^m(\theta, \lambda)\}\Bigg] \tag{18}$$

The negative values of m are rewritten as positive according to Eq. (17). Equation (18), applied by Munk and Cartwright (1966), is a modified form of the addition theorem given in Eq. (13). After introducing Eq. (18) into Eq. (9), the equilibrium tide is defined by

$$V = g\sum_{n=0}^{n}\sum_{m=0}^{n}[a_n^m(Z, L)U_n^m(\theta, \lambda) + b_n^m(Z, L)V_n^m(\theta, \lambda)] \tag{19}$$

In this equation Z and L are functions of time, due to the motion of the moon and the sun in their orbits. They can be expressed in terms of the orbital quantities in the same way as in the harmonic method by Munk and Cartwright (1966) and Dronkers (1964). The authors mention the formulas for the longitude of the moon and sun. The parameters are the solar time t, the mean ecliptic longitude s, the longitude of perigee p, and the longitude of the moon's node N. After introducing the formulas for Z and L into Eq. (19), the latter is rewritten in the form

$$V(\theta, \lambda, t) = g\sum_{n=0}^{n}\sum_{m=0}^{n}[a_n^m(t)U_n^m(\theta, \lambda) + b_n^m(t)V_n^m(\theta, \lambda)] \tag{20}$$

The solar potential is computed in the same way. Then the longitude s is called h, and the longitude of perigee, $p_s = 0$. Equation (20) is the basic formula for the equilibrium tide applied by Munk and Cartwright (1966). The coefficients $a_n^m(t)$ and $b_n^m(t)$ can be computed for any desired time interval at the location of the observer. A similar formula holds good for the solar radiational processes which cause variations in temperature, surface pressure, etc. Munk and Cartwright also give an expansion of these effects in spherical harmonics related to the motion of the sun.

In practical applications, $V(t)$ is computed hour by hour from the tidal potential after introducing the known orbital constants of the moon and sun into Eq. (19) or into Eq. (20). It is therefore not necessary to make a choice of constituents or to include f and u values as defined in Eq. (4).

Let hourly values $V(t)$ be computed for a given port. Munk and Cartwright then determine a prediction $\hat{h}(t)$ for time t as a weighed sum of the past and present values of the potential. The prediction is written in the form

$$\hat{h}(t) = \sum_s w(s)V(t - \tau_s) \tag{21}$$

in which τ_s is usually defined by $s\tau$ ($s = 1, 2$, etc.) where τ is a standard time lag. The weights $w(s)$ are determined such that the prediction error $\sum (h(t) - \hat{h}(t))^2$ is a minimum; where $h(t)$ are observations at a port from the past to the present. Equation (21) can be described in terms of time-varying linear systems (Bendat and Piersol, 1971) as follows: The function $V(t)$ is an input function, $\hat{h}(t)$ is the output function, and the sum denotes an impulse response relation.

After introducing Eq. (20) into Eq. (21), the prediction form for $\hat{h}(t)$, according to Munk and Cartwright (1966), is then

$$\hat{h}(t) = \sum_{m,n} \sum_{s} [u_n^m(s) a_n^m(t - s\tau) + v_n^m(s) b_n^m(t - s\tau)] \tag{22}$$

where the prediction weights $u_n^m(s)$ and $v_n^m(s)$ are determined by the least square rule as mentioned before.

The prediction $\hat{h}(t)$ can also be written in the form

$$\hat{h}(t) = \text{real} \sum_{m,n} \sum_{s=-S}^{S} c_n^{m*}(t - s\tau)[w_n^m(s)] \tag{23}$$

in which $w_n^m(s) = u_n^m(s) + iv_n^m(s)$; $c_n^{m*}(t - s\tau) = a_n^m(t - s\tau) - ib_n^m(t - s\tau)$. The values of the time series $a_n^m(t - s\tau)$ and $b_n^m(t - s\tau)$, in which t is mean solar time, are defined by the coefficients of the spherical harmonic expansion of V/g in Eqs. (19) and (20). These values are then directly introduced into the computer program. The weights $w_n^m(s)$ define the relation between the linear part of the tide and the equilibrium tide. The determination of $w_n^m(s)$ is an essential point in the response method. Therefore the theory of linear responses is applied.

The finite representation of the admittance function

$$Z_n^m(f) = \sum_{s=-S}^{S} w_n^m(s) \exp(-2\pi i f s)$$

in which f denotes the frequency (expressed in cycles per day) is closely related to the Fourier transform of the complex input gravitational potential

$$G(f) = \int_{-\infty}^{+\infty} c(t) \exp(2\pi i f t)\, dt$$

and the transform of the predicted tide

$$H(f) = \int_{-\infty}^{+\infty} \hat{h}(t) \exp(2\pi i f t)\, dt$$

in which

$$h(t) = \text{real part of} \int_{0}^{\infty} c^*(t - \tau) w(\tau)\, d\tau$$

where the function $c(t)$ is defined in Eq. (23); the indices n and m are omitted. Between the admittance function

$$Z(f) = \int_0^\infty w(\tau) \exp(-2\pi i f t)\, d\tau$$

$G(f)$ and $H(f)$, the following relation exists:

$$Z(f) = H(f)/G(f)$$

Bendat and Piersol (1971) deals with the theory of response functions of which some results are applied above. The relation for $Z(f)$ enables one to separate tidal and nontidal parts of the spectra $G(f)$ and $H(f)$. Moreover, it gives a useful guide to the choice of the time interval τ and the number of terms s in Eq. (23). Furthermore these functions are related to the input and output energy spectra.

For computation, the integrals are replaced by

$$G_r = \frac{1}{N} \sum_{-N}^{N} \left(1 + \cos \pi \frac{n}{N}\right) a(n\,\Delta t - t_0) \exp 2\pi i r \frac{n}{N}$$

and a similar expression for H_r ($r = 1, 2, \ldots, \frac{1}{2}N - 1$). The function $a(t)$ is defined in Eq. (23). The factor $(1 + \cos \pi n/N)$ is a " lag window," also called a *cosine taper function*, which is introduced for the rapid convergence of *side-band* effects. For this theory reference is made to Blackman and Tuckey (1959, Part I, Sect. 5). The prediction $\hat{h}(t)$ which occurs in the definition for $H(f)$ is not known before. Therefore $\hat{h}(t)$ must be replaced by the recorded tide $h(t)$. In that case the relation $Z(f) = H(f)/G(f)$ is an approximation, depending on the noise in $h(t)$. In practical applications, Cartwright considers a monthly period of 29.5 days and a period of 355 days. Then Δt is taken to be 1 hr and 3 hr respectively.

Further data used in the applications are: The standard time lag τ is taken as 2 days in practical application by Munk and Cartwright (1966). This interval appears to be the best compromise. It is about equal to the mean lag between the high water at a location in the North Sea and the high water of the equilibrium tide. The summation limit S is usually less than 3. The values of m correspond with the species of the tide [see Eqs. (13) and (20)]. Then $m \hat{=} 1$ for diurnal, $m = 2$ for semidiurnal, and n denotes the degree of the spherical harmonic. In practical applications $m \leq n$, and $n \leq 3$. The factors $c_2^1(t)$ and $c_2^2(t)$ are predominant. Cartwright (1967) further adds nonlinear terms for the prediction of the tide at shallow water ports. Also, he discusses practical results of the response method and the prediction of high and low waters. The theory of tidal prediction is further extended by Cartwright (1968) by considering the tides as well as the surges round North and East

Britain. The Munk–Cartwright method has been extended to evaluate the linear responses of the local sea level to sea weather disturbances over the sea area, and the linear interactions between surge and tide. This is of importance in shallow seas. The course of the atmospheric pressure gradients, which also determine the wind, is defined by a spatial Taylor expansion over the sea. The coefficients in this expansion depends on time. The result of the analysis is a set of empirical formulas, which could be used to predict the surge a few hours in advance. For a detailed treatment, reference is made to Cartwright's paper (1968).

In a further publication of Cartwright and Taylor (1971), a high precision harmonic expansion is determined for the gravitational tide potential. The authors introduced the most recent values of the astronomical constants of the motion of the moon and the sun. They compared this expansion with the very detailed Doodson's expansion (1921).

D. Application of Tidal Prediction

1. *General Remarks*

Tidal prediction is mainly applied to harbors situated at coasts where tidal records are often influenced by meteorological effects. At harbors along a river, additional disturbances occur as the runoff may change the water levels considerably.

Tidal prediction is difficult when the variation in the runoff is predominant, unless it is of a periodical nature as in tropical regions. Generally, however, the variation in the runoff is much smoother than that due to the meteorological effects, depending on the drainage of the land. Then the analysis can be carried out for periods during which the runoff does not vary considerably. Otherwise, the variation in high and low waters must be determined in a more or less empirical way. For example, the runoff may be predicted some time before and its influence on the tide may be evaluated by correlation.

The runoff and also an ice cover on the water hamper the propagation of the tide and causes a decrease of the tidal amplitude and a distortion of the tide. In the tidal analysis this is perceptible by the increase in the number of shallow water components. Munk and Cartwright (1966) also include the daily and seasonal variation of solar radiation in their method of tidal prediction by introducing the radiational function. Obviously the radiation from the sun depends also on the fundamental orbital constants.

The way of sampling the observations is also important. The instrument that measures the water-level variation continuously eliminates short waves and seiches of short duration. The analysis of the observations, which is the basis for the tidal prediction, is based on discrete observations taken at intervals Δt. This time interval is determined by the highest tidal frequency $\omega_{\max}$ to be considered. Then, $\Delta t < \pi/\omega_{\max}$. The frequency $\pi/\Delta t$ is the so-called *folding frequency* or *Nyquist frequency* (Bendat and Piersol, 1971). Usually $\Delta t = 1$ hr, expressed in radians, when M_8 is the highest frequency to be considered.

The accuracy of presenting the tide by such a chosen number of constituents can be judged by a spectral analysis of the residuals that are the differences between the observed tide and the computed tide. For the general theory of spectral analysis, reference can be made to Bendat and Piersol (1971). This spectral analysis is also used for wind-wave analysis (Kinsman, 1965). The spectral analysis determines the so-called "noise level" and gives an indication about further possible constituents. The spectrum shows cusps in the frequency ranges of the tide but not sharp lines because of the interaction of the noise and the tide. This method is used extensively by Munk and Cartwright (1966) to judge the validity of their method.

2. *Application of Harmonic Analysis*

The method of harmonic analysis discussed above has been applied to tidal observations at the Hook of Holland (the Netherlands) for a year's period. In this case, shallow water effects are considerable. At first, an analysis was carried out with 41 constituents. Then, 115 constituents were considered, including the 41 constituents of the former analysis. The most important constituents of the first group, containing 41 components, are an

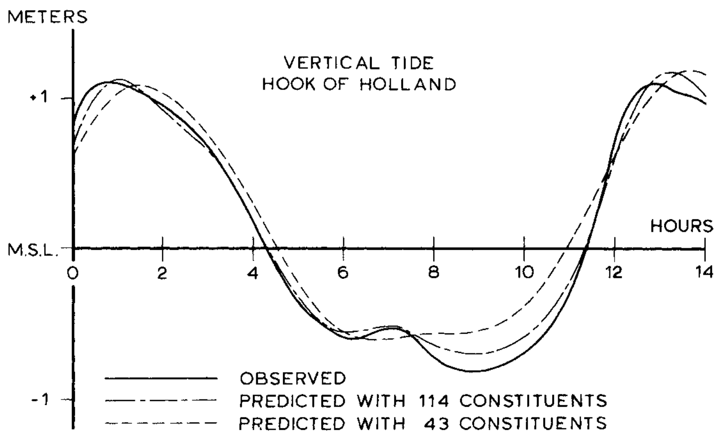

FIG. 2. Comparison of observed and predicted tides at a certain day.

M_2 constituent with an amplitude of 80.7 cm; S_2, 20 cm; M_4, 18.4 cm; MS_4, 11.8 cm; N_2, 11.8 cm; O_1, 10.5 cm; μ_2, 8.9 cm; K_1, 7.25 cm; K_2, 6 cm; and MN_4, 6.7 cm. The other 31 constituents have a smaller amplitude, of which 10 constituents have an amplitude even smaller than 1 cm. In the second group, 54 have an amplitude smaller than 1 cm. It appeared that the values of the amplitudes of the constituents mentioned above are modified very slightly by this extensive analysis; e.g., M_2, 80.7 cm; S_2, 20.1 cm; M_4, 18.3 cm; etc. The smaller constituents are influenced to a somewhat greater extent.

The improvement obtained by the more extensive analysis is shown in Fig. 2. The observed tide at a certain day is compared with the results obtained from the analysis. The meteorological effects in the North Sea were very small during the day. The improvement is not impressive in view of the fact that a much greater number of constituents have been taken into consideration.

3. *Some Remarks on Mean Sea Level*

In the preceding sections periodical variations in the sea level were considered. A very important problem is the determination of the mean sea level on which the periodical variations are superposed. A main problem is the definition of the mean sea level. A definition which states that mean sea level is equal to the water level that occurs if no waves, tides, wind influences, density, or temperature are present, has no practical sense. Mean sea level can be defined only for a time interval. Then it is equal to the mean value of sea level observations which are referred to a stable bench mark at a certain location. For different time intervals it has different values, even in cases where very long time intervals are considered. Long-period tidal variations, and long-term variations in climatological factors, which have partially a stochastical character and partially a trend, are the main causes.

The most commonly used unit for mean sea level is the day, after eliminating the diurnal and shorter period oscillations. The upper limit of the time interval of mean sea level is restricted by the data available from the gauge. Usually in practical applications, the hourly heights are considered in the averaging process. It must be noted that even in long-period averaging processes, contributions from the daily and monthly tides occur. For example, in a 365-day year, the contribution of the M_2 constituent is 0.035% of the amplitude, due to the fact that the phases are not completely distributed at random between 0 and 2π. Studies concerning the mean sea level are very extensive. For general considerations, reference is made to Rossiter (1962). A recent and very extensive publication on this subject is the "Report on the Symposium on Coastal Geodesy" (1971).

III. Tidal Equations

A. Tidal Equations for Two-Dimensional Regions and for a River

1. *Introduction*

The tidal equations, which describe the movement of the tidal wave in terrestrial waters, are derived from the hydrodynamic equations of motion and continuity. In applying the equations, time-smoothed velocity and pressure distributions must be considered because the flow is turbulent. The external forces are the gravity force, the Coriolis force, and the bottom friction together with Reynold's stresses. Wind forces on the surface of the water must be included when necessary.

The practical application of the tidal equations is related to regions where horizontal dimensions are much larger than the depth. The velocity components are mainly in the horizontal direction. This application is restricted to seas, which are relativly shallow in, for instance, continental shelves and coastal waters. Density currents caused by differences in temperature will be left out of consideration, but variations in density due to differences in salt concentration will be considered in some cases.

The vertical velocities will be ignored because of relative small slopes of the water surface during the tidal motion. The slope of the bottom also influences the vertical velocity components. Sudden changes in the bottom shape will however not be considered. In Section III,A,3, a discussion will be given on the relative significance of the vertical velocity components in the equations of motion and continuity. Mean values of the horizontal velocity components in the vertical direction are considered in the tidal equations. The velocity distribution in the vertical direction must be taken into account in the determination of these mean values. Many formulas are given for such distributions based on theoretical and empirical research. The distribution in the vertical direction may have a much more irregular shape in case of variable density, e.g., due to salt intrusion. The parabolic formula is used in case of constant density because the mean values can be so easily computed (Section III,A,5).

The variation in the bottom elevation is often larger than the variation in the water surface elevation. Thus, the variation in the bottom elevation as a function of x and y must also be included in the formulas. Further details on the distribution in the vertical direction and the derivation of the tidal equations are given in Section III,A,6.

The tidal equations for the sea are mentioned in Section III,A,5 and those for a river in Section III,A,7. The discharge per unit of time in a cross section of a river instead of the mean velocity is often considered as a dependent variable in these equations. The discharge is usually less dependent on the variation of the cross section than that of the velocity, and also the equation of continuity can be expressed in a more simple form.

2. *General Equations of Motion and Continuity*

In this section we assume that the density ρ is variable, due to the intrusion of the fresh water into the salt water. The equation of continuity is determined by writing a mass balance over a stationary volume element $\Delta x\ \Delta y\ \Delta z$ through which the fluid is flowing. The equation of continuity, which describes the rate of change of density at a fixed point resulting from the change in the mass velocity vector $\rho\mathbf{v}$, is

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) \tag{24}$$

in which u, v, and w are the velocity components in the three dimensional Cartesian coordinates x, y, and z. They vary with time t.

The equations of motion may be derived by writing a momentum balance for the volume element by considering the forces which act on fluid particles. This method corresponds to the derivation of the equation of continuity. The equation of motion in the x direction for a fluid element is then

$$\begin{aligned}\frac{\partial \rho u}{\partial t} + \frac{\partial \rho uu}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} + \frac{\partial p}{\partial x} - \rho g_x \\ + (x \text{ component of the Reynolds stresses}) \\ + (x \text{ component of the Coriolis force}) = 0\end{aligned}$$

in which p denotes the pressure and ρg_x the x component of the gravity force on the volume element. The Reynolds stresses τ are associated with the turbulent velocity fluctuation. The newtonian stresses, which are associated with the viscosity, may be neglected. Similar equations hold in the y and z directions. These equations of motion may be rearranged by means of Eq. (24) to give

$$\begin{aligned}\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] + \frac{\partial p}{\partial x} - \rho g_x \\ + (x \text{ component Reynolds stresses}) \\ + (x \text{ component Coriolis force}) = 0\end{aligned} \tag{25}$$

and similar equations for the y and z direction.

In the above form the equation of motion is related to the forces acting on a volume element which moves with the fluid. This form corresponds to the equation of continuity in the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{26}$$

Since water is considered incompressible,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{27}$$

The time-smoothed Eqs. (25) contain the turbulent momentum flux expressed by the Reynolds stresses τ'_{xx}, τ'_{xy}, and τ'_{xz}, which are the x components of the stress tensor. In practical applications these quantities must be handled empirically, e.g., by means of Boussinesq's eddy viscosity, *Prandtl*'s mixing length, etc. (Bird *et al.*, 1960).

3. *Estimation of Vertical Velocities*

The x and y axes are taken in the horizontal datum plane at a point of the water surface at mean water level (Fig. 3) and the z axis in the vertical direction. The time-smoothed vertical component of the velocity w at the water surface depends on the movement of the water surface $h(x, y, t)$ with respect to the horizontal datum level according to

$$w(h) = \frac{\partial h}{\partial t} + u(h)\frac{\partial h}{\partial x} + v(h)\frac{\partial h}{\partial y} \tag{28}$$

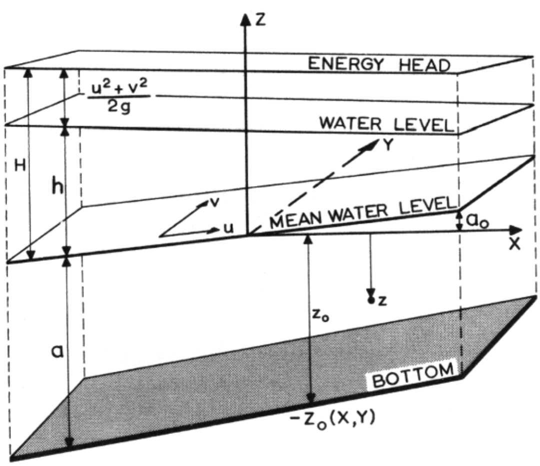

FIG. 3. Notation of water levels and energy head.

The mean water surface $a_0(x, y)$ is assumed to be a horizontal plane ($\partial a_0/\partial x = \partial a_0/\partial y = 0$) in this subsection. The slope of the mean water surface is usually much smaller than the maximum slope of the water surface due to the tide. Let $z_0(x, y)$ be the equation of the bottom with respect to the horizontal datum plane. Then, the vertical velocity $w(z_0)$ near the bottom is determined by

$$w(z_0) = u(z_0)\frac{\partial z_0}{\partial x} + v(z_0)\frac{\partial z_0}{\partial y} \tag{29}$$

In nature the slope of the bottom may have rather irregular values as a function of x and y. They are often much larger than the slope of the water surface and the slope of the mean water level. However, the values of $u(z_0)$ and $v(z_0)$ near the bottom are small with respect to the velocity at the surface, and $w(z_0)$ is still much smaller because $\partial z_0/\partial x$ and $\partial z_0/\partial y$ are of the order of 10^{-2} or smaller. The maxima of the smoothed values of $w(x, y, t)$ in the vertical are determined by Eq. (28) or (29). In any case $w(z)$ is much smaller than u or v because $\partial h/\partial x$ and $\partial h/\partial y$ have usually smaller values than 10^{-4}; $\partial h/\partial t$ is between 10^{-3} and 10^{-4} m/sec or smaller. Usually $w(h) \approx \partial h/\partial t$.

The values of the terms $w\ \partial u/\partial z$ and $w\ \partial v/\partial z$ in Eqs. (25) are small with respect to the values of $g\ \partial h/\partial x$, or $g\ \partial h/\partial y$ since u and v are small with respect to g, and $\partial u/\partial z$ is of the order of $u(h)/a$. However, near the bottom it is very difficult to make estimates about the values of these terms. Obviously, the terms $u\ \partial w/\partial z$ and $v\ \partial w/\partial z$ also have very small values.

4. *Equations of Motion, Neglecting the Vertical Velocity Component*

The change of the horizontal velocity components from bottom to surface, defined by $\partial u/\partial z$ and $\partial v/\partial z$, are usually more considerable than the change of the velocity components in the x and y directions. Consequently the derivatives of the shear components $\partial \tau_x/\partial z$ and $\partial \tau_y/\partial z$ are usually much greater than $\partial \tau_x/\partial x$, etc. It follows from these discussions and those of Section III,A,3 that the following equations of motion can be used as the basis for further derivation of the tidal equations:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \Omega v\right] + \frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial z} = 0 \tag{30}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \Omega u\right] + \frac{\partial p}{\partial y} + \frac{\partial \tau_y}{\partial z} = 0 \tag{31}$$

$$\frac{\partial p}{\partial z} + \rho g = 0 \tag{32}$$

in which p denotes the pressure, which is hydrostatic according to Eq. (32); τ_x and τ_y are the shear components in the x and y directions; $-\Omega v$ and Ωu are the components of the Coriolis force, and $\Omega = 2\omega \sin \varphi$, in which ω is the angular velocity of the earth around its axis and φ is the latitude. For the determination of the components of the Coriolis force in the horizontal plane, Dronkers (1964) follows a different method from that of Neumann and Pierson (1966). They consider also the vertical component, and they include the term $-2\omega \cos \varphi$, in which φ is the latitude, in Eq. (32). However, the value of this term is extremely small with respect to ρg.

Now the z axis is taken in the upward direction opposite to gravity; the x and y axes are taken anticlockwise. It follows from Eq. (32) in case of variable density that

$$p(z) = p_0 + g \int_z^h \rho \, dz \tag{33}$$

in which p_0 is the atmospheric pressure. Hence

$$\frac{\partial p}{\partial x} = g \int_z^h \frac{\partial \rho}{\partial x} dz + g\rho(h)\frac{\partial h}{\partial x}; \qquad \frac{\partial p}{\partial y} = g \int_z^h \frac{\partial \rho}{\partial y} dz + g\rho(h)\frac{\partial h}{\partial y} \tag{34}$$

5. *Equations of Motion and Continuity Containing Vertical Mean Velocities*

The mean values of the velocity components $\bar{u}$ and $\bar{v}$ are defined by

$$\bar{u} = \frac{1}{a+h}\int_{-a}^h u \, dz; \qquad \bar{v} = \frac{1}{a+h}\int_{-a}^h v \, dz \tag{35}$$

in which h is the height of the water surface with respect to mean water level, and a is the depth below mean water level (Fig. 3). In this section it is assumed that the density ρ is constant and that the variations due to the influence of the Coriolis force on the directions of the velocities in the vertical, the well-known *Ekman effect* (Ekman, 1905; see also Neumann and Pierson, 1966), can be ignored. The magnitude V of the velocity vector $\mathbf{v}$, of which the velocity components are u and v in the vertical, can be empirically represented by the parabolic formula (Dronkers, 1964)

$$V = V_1(z+a)^{1/q} \tag{36}$$

in which V_1 equals the magnitude of the velocity at $z + a = 1$; z is a point in the vertical; and $z_0 = -a$ is the bottom. The center of the coordinate system is thus taken at the mean sea level. It is provisionally assumed that the mean water surface is a horizontal plane. The mean value of the velocity V_{mean}

occurs at a depth z_{mean}:

$$z_{\text{mean}} = \left(1 + \frac{1}{q}\right)^{-q}(a + h) \qquad \text{and} \qquad V_{\text{mean}} = V_1 \frac{q}{1 + q}(a + h)^{1/q} \quad (37)$$

The mean values of the components u and v occur at the same depth because $u = V \cos \alpha$ and $v = V \sin \alpha$, where α is the angle between the vector $\mathbf{v}$ and the x axis.

The value of the exponent q may vary in the interval $5 < q < 7$, according to measurements of V in the North Sea. Then z_{mean} is about $0.4(a + h)$ and the quotient of V_{mean} to V_{max} is about 0.85. The formula for V given by Eq. (36) is not hydraulically valid in the immediate neighborhood of the bottom (Chow, 1961), particularly due to the occurrence of sand waves and irregularities in the bottom.

The mean values in the vertical of the various terms of the equations of motion (30) and (31), and of the equation of continuity (27) are computed by the application of the formula

$$\begin{aligned} \frac{1}{a + h}\int_{-a}^{h} \frac{\partial f(z)}{\partial x}\, dz = {} & \frac{\partial}{\partial x}\frac{1}{a + h}\int_{-a}^{h} f(z)\, dz \\ & + \frac{1}{(a + h)^2}\frac{\partial(a + h)}{\partial x}\int_{-a}^{h} f(z)\, dz \\ & - \frac{1}{a + h} f(h)\frac{\partial h}{\partial x} - \frac{1}{a + h} f(-a)\frac{\partial a}{\partial x} \end{aligned}$$

Then it can be shown that

$$\begin{aligned} \frac{1}{a + h}\int_{-a}^{h} \frac{\partial u}{\partial x}\, dz = \overline{\frac{\partial u}{\partial x}} = {} & \frac{\partial \bar{u}}{\partial x} \\ & + \frac{1}{a + h}\frac{\partial(a + h)}{\partial x}\bar{u} - \frac{u(h)}{a + h}\frac{\partial h}{\partial x} - \frac{u(-a)}{a + h}\frac{\partial a}{\partial x} \end{aligned}$$

and a similar expression holds for $\overline{\partial v/\partial y}$. Furthermore,

$$\int_{-a}^{h} \frac{\partial w}{\partial z}\, dz = w(h) - w(-a)$$

Then the equation of continuity (27) is transformed into

$$\frac{\partial(a + h)\bar{u}}{\partial x} + \frac{\partial(a + h)\bar{v}}{\partial y} + \frac{\partial h}{\partial t} = 0 \qquad (38)$$

after application of Eqs. (28) and (29) (read $-a$ instead of z_0). The equation of motion will be also considered in the form of the momentum balance (see Section III,A,2).

Then the mean values of $\partial u/\partial t$, $\partial uu/\partial x$, $\partial vu/\partial y$, $\partial wu/\partial z$ must be determined and the following expressions are thus found:

$$\overline{\frac{\partial uu}{\partial x}} = \frac{\partial}{\partial x}\overline{uu} + \frac{1}{a+h}\frac{\partial(h+a)}{\partial x}\overline{uu}$$

$$- \frac{1}{a+h}\left[\frac{\partial h}{\partial x}u^2(h) + \frac{\partial a}{\partial x}u^2(-a)\right] \tag{39a}$$

$$\overline{\frac{\partial vu}{\partial y}} = \frac{\partial}{\partial y}\overline{vu} + \frac{1}{a+h}\frac{\partial(h+a)}{\partial x}\overline{vu}$$

$$- \frac{1}{a+h}\left[\frac{\partial h}{\partial y}(vu)_h + \frac{\partial a}{\partial x}(vu)_{-a}\right] \tag{39b}$$

$$\overline{\frac{\partial wu}{\partial z}} = \frac{1}{a+h}[(wu)_h - (wu)_{-a}] \tag{39c}$$

$$\overline{\frac{\partial u}{\partial t}} = \frac{\partial \bar{u}}{\partial t} + \frac{\bar{u}}{a+h}\frac{\partial h}{\partial t} - \frac{u(h)}{a+h}\frac{\partial h}{\partial t} \tag{39d}$$

These equations determine the differences between the derivatives of the mean values of the functions in the vertical, and the mean values of the derivatives of the functions in the vertical.

In the Eqs. (39), the values of $\overline{uu}$ and $\overline{uv}$ can be expressed in terms of $\bar{u}^2$ and $\bar{u}\bar{v}$ by Eq. (36), using $u = V \cos \alpha$ and $v = V \sin \alpha$. Hence,

$$\bar{u}^2 = \frac{V_1^2 \cos^2 \alpha}{(a+h)^2}\left[\int_{-a}^{h}(z+a)^{1/q}\,dz\right]^2$$

and

$$\overline{u^2} = \frac{V_1^2 \cos^2 \alpha}{a+h}\int_{-a}^{h}(z+a)^{2/q}\,dz$$

After some calculation it is found that

$$\overline{uu} = \left(1 + \frac{1}{q^2 + 2q}\right)(\bar{u})^2 \tag{40}$$

and a similar formula holds for $\overline{vu}$ and $\bar{v}\bar{u}$, in which the same factor occurs. It appears that the value of this factor differs slightly from one, about 2–3%, if $5 < q < 7$, so that we can replace $\overline{uu}$ by $\bar{u}^2$ and $\overline{vu}$ by $\bar{v}\bar{u}$ in practical application. Hence,

$$\overline{\frac{\partial uu}{\partial x}} \approx \frac{\partial \bar{u}^2}{\partial x} \quad \text{and} \quad \overline{\frac{\partial vu}{\partial y}} \approx \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{u}\frac{\partial \bar{v}}{\partial y} \tag{41}$$

The mean value of the sum of the terms, mentioned in Eqs. (39), and according to Eqs. (28), (29) (read $-a$ instead of z_0), (38), and (41), is

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \overline{uu}}{\partial x} + \frac{\partial \overline{vu}}{\partial y} - \bar{u}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) \tag{42}$$

Finally, it is found that the expression (42) may be replaced by

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} \tag{43}$$

The various approximations applied in the derivation of expression (43) follow immediately from the preceding discussion. A similar expression holds for the $\bar{v}$ component of the inertia terms.

The mean value of $\partial p/\partial x$ in the vertical is determined from Eq. (34) as

$$\overline{\frac{\partial p}{\partial x}} = \frac{g}{a+h}\int_{-a}^{h}\int_{z}^{h}\frac{\partial \rho}{\partial x}\,dz^2 + g\rho(h)\frac{\partial h}{\partial x} \tag{44}$$

In case density variation does not occur, it is found that

$$\overline{\frac{\partial p}{\partial x}} = \rho g\frac{\partial h}{\partial x}; \qquad \overline{\frac{\partial p}{\partial y}} = \rho g\frac{\partial h}{\partial y} \tag{45}$$

It must be noted that the x and y axes are taken in the horizontal plane and that h is defined with respect to the mean water level. In the case that the mean water plane is not horizontal, Eq. (45) must be modified. Let $a_0(x, y)$, defined above, be the mean water level with respect to the datum level (Fig. 3). Then, Eq. (45) must be replaced by

$$\overline{\frac{\partial p}{\partial x}} = \rho g\frac{\partial(a_0 + h)}{\partial x} \tag{46}$$

and a similar expression holds for $\overline{\partial p/\partial y}$.

The determination of the mean sea level $a_0(x, y)$ is an important subject of oceanographic studies. Therefore, $a_0(x, y)$ is introduced separately into the formulas. Dronkers (1964) applies a different notation: Let a_0 be the depth with respect to the mean water level, a the depth below the water surface, and z_0 the height of the bottom with respect to the datum. Then, the mean sea level is defined by $z_0 + a_0$. Dronkers (1969b) also determined the relative importance of the terms in Eq. (44) from measurements in the Rotterdam Waterway in The Netherlands.

The mean value of the term $\partial\tau_x/\partial z$ in Eq. (30) is

$$\frac{1}{a+h}\int_{-a}^{h}\frac{\partial \tau_x}{\partial z}\,dz = \frac{1}{a+h}[\tau_x(h) - \tau_x(-a)]$$

and a similar expression holds for the mean value of $\partial\tau_y/\partial z$. The shear stress at the water surface $\tau_x(h)$ is mainly caused by wind effects; and at the bottom, $\tau_x(-a)$ by friction. Practical experience shows that $\tau(-a)$ is proportional to the squared mean velocity:

$$\tau_x(-a) = \frac{\rho g\bar{u}(\bar{u}^2 + \bar{v}^2)^{1/2}}{C^2}; \qquad \tau_y(-a) = \frac{\rho g\bar{v}(\bar{u}^2 + \bar{v}^2)^{1/2}}{C^2} \tag{47}$$

in which C is Chézy's coefficient expressed in $\mathrm{m}^{1/2}/\mathrm{sec}$. When Manning's coefficient is used, the relation between C and n is

$$C = \frac{1.49}{n}(h + a)^{1/6}$$

Let $\mathbf{W}$ be the velocity of the wind, and W_x and W_y its components. Then, it can be assumed that

$$\tau_x(h) = \frac{c}{a + h} \mid \mathbf{W} \mid W_x \quad \text{and}$$

$$\tau_y(h) = \frac{c}{a + h} \mid \mathbf{W} \mid W_y, \qquad \mid \mathbf{W} \mid = (W_x^2 + W_y^2)^{1/2} \tag{48}$$

in which c is an empirical coefficient and its value is of the order of 0.35×10^{-6} [meter is the unit of length] according to experience at sea in the case of long-continuing wind effects. It takes time before the wind forces exert the maximum influence on the water mass.

Introducing expression (43), Eqs. (45), (47), and (48), and the analogous expressions for the y coordinate into Eqs. (30) and (31), the following tidal equations are obtained for constant density:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \Omega v\right] = -\rho g\left(\frac{\partial h}{\partial x} + \frac{da_0}{dx}\right) - \rho\frac{gu(u^2 + v^2)^{1/2}}{C^2(a + h)} + \frac{c \mid \mathbf{W} \mid W_x}{a + h} \tag{49}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \Omega u\right] = -\rho g\left(\frac{\partial h}{\partial y} + \frac{da_0}{dy}\right) - \rho\frac{gv(u^2 + v^2)^{1/2}}{C^2(a + h)} + \frac{c \mid \mathbf{W} \mid W_y}{a + h} \tag{50}$$

The terms in Eqs. (49) and (50) are the mean values over the vertical; however the bars are omitted. The equation of continuity (38) must be added to Eqs. (49) and (50).

6. *Final Remarks on the Derivation of the Tidal Equations and on the Determination of the Coefficient of Friction*

For the derivation of the tidal equations an assumption is made that the velocity distribution in the vertical direction is of the parabolic type. It is also often assumed that this distribution is of the logarithmic type (Dronkers, 1964):

$$u = u_{\max} + \frac{(gia)^{1/2}}{k} \ln \frac{z}{a}$$

in which a is the depth, i is the slope of the water surface, and $k \approx 0.4$. Both distributions are compared in Fig. 4(a). The latter distribution is less convenient for the determination of $\bar{u}$ and $\overline{u^2}$ because it may not be applied near

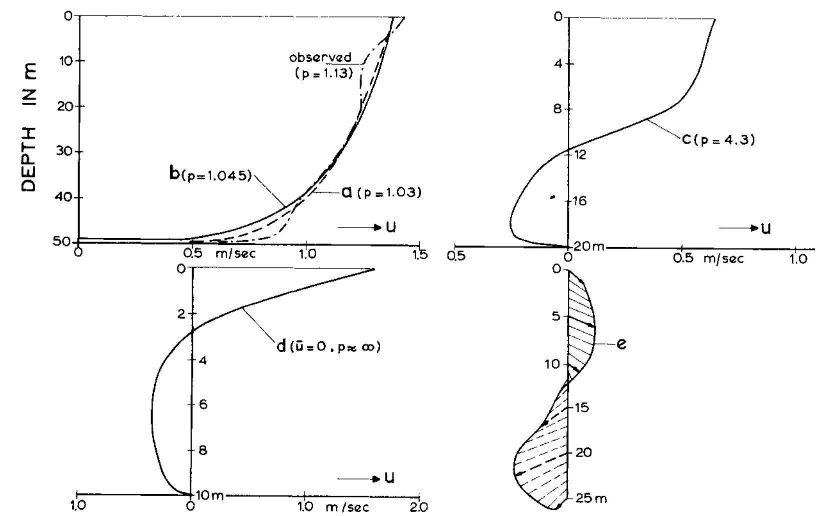

FIG. 4. Comparison of the parabolic (a), the logarithmic (b), and the observed velocity distributions. Examples of velocity distributions due to differences in density (c), wind influence (d), and combined effects of Coriolis force and bottom variation in the sea (e). The values p denote the quotient $\overline{u^2}/\bar{u}^2$.

the bottom. The measured velocity distribution has also been drawn in Fig. 4(a). The stochastical processes due to turbulence affect the shape of the curve.

There are other factors that may influence considerably the vertical distribution of velocity. Thus, it is very difficult to describe the distribution by an analytical formula, and the question is whether the tidal equations derived in Section III,A,5 are applicable. If this is not the case, the equations of motion for three dimensions must be applied, e.g., those of Navier–Stokes, modified for the influence of turbulence (see Bird *et al.*, 1960). The practical application of the theory in three dimensions is however in the beginning stage. By means of computers of very big capacity it is now possible to deal with such problems. In the following cases such elaborate research is necessary because the velocity distribution in the vertical deviates considerably from the parabolic or logarithmic type:

(a) the determination of density currents, due to the difference in density, e.g., intrusion of fresh water into salt water and conversely;
(b) the influence of wind forces on the water surface;
(c) the influence of swell or wind waves of long period in shallow regions.

In Fig. 4 various types of velocity distribution are shown, and the values of the quotient of $\overline{u^2}$ to $\bar{u}^2$, denoted by p, are mentioned. It appears that the mean values of the velocity in the vertical and the mean values of the squares $\overline{u^2}$ may differ considerably from $(\bar{u})^2$; e.g., it may be that $\bar{u} = 0$. This shows that the derivation of the equations in Section III,A,5 cannot be applied to such cases.

Irregularities in the bottom shape also influence the vertical distribution in an irregular way [Fig. 4(e)]. However, this influence together with the effects due to turbulence can often be accounted for by modification of the value of the coefficient of friction C. Generally, the value of this coefficient is obtained from experience, by comparing the results of measurement with those of computation. Then, it is possible to compute C by the logarithmic formula, which is derived on a more or less theoretical basis by von Kármán and others (Dronkers, 1964). At locations where a sudden jump in the bottom shape occurs, a separate equation at that location must be applied (Dronkers, 1969a).

Furthermore, the values of the friction coefficient C and the coefficient of the wind c in Eqs. (49) and (50) are not independent of each other because the wind forces modify the shape of the vertical velocity distribution. Then the bottom friction which determines the value of C is also affected. In this respect further research is necessary and in particular the application of the combined tidal motion and storm surge effects must be studied.

The considerations on the derivation of the tidal equations will be finished by making the remark that it is not always true that the vertical component of the velocity can be ignored in the equations that describe the water motion in the vertical direction z. According to Eqs. (27) and (28), it follows that

$$w(z) = \frac{\partial h}{\partial t} + u(h)\frac{\partial h}{\partial x} + v(h)\frac{\partial h}{\partial y} - \int_z^h \frac{\partial u}{\partial x}\,dz - \int_z^h \frac{\partial v}{\partial y}\,dz$$

At the bottom defined by $z_0(x, y)$, the vertical component of the velocity is defined by Eq. (29). Then a relation is found for the values of the integrals over the whole vertical.

It appears from detailed observations in nature that the values of the integrals at a point z in the vertical can be much more important than the values of the further terms of $w(z)$. This is especially true in case of density currents, e.g. in the Rotterdam Waterway. However in the tidal equations, which are applied to the vertical from surface to bottom, the values of $w(h)$ and $w(-a)$ can be ignored. For further discussions on this subject, reference is made to Dronkers (1973).

7. *Tidal Equations for a River*

The equations for a tidal wave in the river can be immediately derived from Eqs. (49) and (50) by setting $v = 0$. For wide rivers it may be possible that the Coriolis force or the wind force has some effect on the water levels in the transverse direction of the river. In that case, Eq. (50) applies, or

$$\Omega u = -\rho g \frac{\partial h}{\partial y} + \frac{c \mid \mathbf{W} \mid W_y}{a + h} \tag{51}$$

The width and depth of many rivers vary considerably. Moreover, the runoff from inland regions influences the mean water levels. The slopes of these mean levels are often much greater than in the sea. Often Q and h are considered as variables of x and t instead of u and h. Then Q denotes the discharge, i.e., the total quantity of water flowing through a cross section per unit of time. The variation of Q with distance is usually less than that of u. Let further $A(x, h, t)$ be the cross-sectional area of the stream bed of the water and $b_s(x, h, t)$ be the width of the stream bed. Both functions vary with the water level h. Then

$$Q = Au = (a + h)b_s u \tag{52}$$

After introducing Q instead of u into Eq. (49) and $v = 0$, the following equation of motion for a river is obtained (Dronkers, 1964):

$$\begin{aligned} &\frac{\partial Q}{\partial t} - \frac{Q}{a + h}\frac{\partial h}{\partial t} - \frac{Q}{b_s}\frac{\partial b_s}{\partial h}\frac{\partial h}{\partial t} \\ &\quad + \alpha \frac{Q}{A}\left[\frac{\partial Q}{\partial x} - \frac{Q}{a + h}\frac{\partial (a + h)}{\partial x} - \frac{Q}{b_s}\frac{\partial b_s}{\partial h}\frac{\partial h}{\partial x}\right] \\ &= -gA\left(\frac{da_0}{dx} + \frac{\partial h}{\partial x}\right) - \frac{g}{C^2 A(a + h)} \mid Q \mid Q + \frac{cb_s}{\rho}\frac{\mid \mathbf{W} \mid W_x}{a + h} \end{aligned} \tag{53}$$

In this equation the effect of the variation of the stream width of the river b_s as a function of the water level h has also been taken into account. The sign of the factor $|Q|Q$ reverses from ebb to flood. It is assumed that the variations of the stream width and depth are not abrupt. In case of sudden changes in these functions a separate equation must be applied (see Chow, 1961; Dronkers, 1964). A coefficient α has been introduced to account for the effect of the distribution of the velocities in the vertical, and in the transverse and lengthwise directions of the river.

The function $h(x, t)$ does not occur in Eq. (53) in the case of nonuniform steady flow upstream of the tidal region. The equation of continuity is

$$\frac{\partial Q}{\partial x} + b\frac{\partial h}{\partial t} + q = 0 \tag{54}$$

This equation shows that the differences in discharge between two cross sections x and $x + dx$, and the accumulation, or the loss, of water caused by rising and falling of the water surface, must balance. The storage width is $b(h, t)$. The function $q(h, t)$ represents the supplementary discharges per unit length, e.g., by overflow, etc. If important storage regions occur next to the river bed, the flow to and from these regions will influence the water motion in the main channel. For such influences, correction terms must be introduced into Eq. (53). Such details will not be considered here (see Dronkers, 1964).

It must be noted that in Eq. (53) the mean water level is also an unknown function. According to Fig. 3, it is

$$a(x, y) = z_0(x, y) + a_0(x, y) \tag{55}$$

in which $z_0(x, y)$ is a known function determined by the soundings. Therefore $a_0(x) + h(x, t)$ and $Q(x, t)$ are the unknown functions in Eqs. (53) and (54). In Eq. (54), $a_0(x)$ does not occur. After the computation of $a_0 + h$, a_0 can be obtained as the mean value of $a_0 + h$ over the tide.

B. Tidal Equations Expressed in Terms of Energy Head

1. *Introduction of the Energy Head and the Rotor of the Velocity Vector*

The tidal equations reduce to an interesting form when the energy head is considered instead of the water level h. This form may also be useful in practical applications concerning river flow. The energy head H at a point in the sea is defined by

$$H = h + \frac{u^2 + v^2}{2g} = h + \frac{V^2}{2g} \tag{56}$$

in which u and v are the mean velocities in the vertical. Introducing

$$\frac{\partial h}{\partial x} = \frac{\partial H}{\partial x} - \frac{1}{g}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]$$

and a similar expression for $\partial h/\partial y$ in the equations of motion (49) and (50),

$$\frac{\partial u}{\partial t} - v(\text{rot } \mathbf{v} + \Omega) = -g\frac{\partial H}{\partial x} - g\frac{\partial a_0}{\partial x} - Ru + \frac{\bar{W}_x}{\rho} \tag{57}$$

$$\frac{\partial v}{\partial t} + u(\text{rot } \mathbf{v} + \Omega) = -g\frac{\partial H}{\partial y} - g\frac{\partial a_0}{\partial y} - Rv + \frac{\bar{W}_y}{\rho} \tag{58}$$

in which rot $\mathbf{v} = \partial v/\partial x - \partial u/\partial y$, Ru and Rv are the friction terms, and $\bar{W}_x$ and $\bar{W}_y$ are the wind terms defined in the right-hand members of the Eqs. (49) and (50). The bottom is assumed horizontal and $\partial a_0/\partial x$, $\partial a_0/\partial y$ determine the slope of the mean water level. The derivatives $\partial H/\partial t$, $\partial H/\partial x$, and $\partial H/\partial y$ can be expressed in terms of h and V by Eq. (56).

After some calculation the equation of continuity (38) becomes

$$\frac{\partial H}{\partial t} - \frac{1}{2g}\frac{\partial V^2}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + \left(a + H - \frac{V^2}{2g}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
$$+ u\left(\frac{\partial a}{\partial x} - \frac{1}{2g}\frac{\partial V^2}{\partial x}\right) + v\left(\frac{\partial a}{\partial y} - \frac{1}{2g}\frac{\partial V^2}{\partial y}\right) = 0 \tag{59}$$

Multiplying Eq. (57) by u and Eq. (58) by v and adding the results, when the wind terms are ignored, give

$$\frac{1}{2}\frac{\partial V^2}{\partial t} = -gu\left(\frac{\partial H}{\partial x} + \frac{\partial a_0}{\partial x}\right) - gv\left(\frac{\partial H}{\partial y} + \frac{\partial a_0}{\partial y}\right) - RV^2 \tag{60}$$

Multiplying Eq. (57) by v and Eq. (58) by u and subtracting the latter result from the former give

$$\left(v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\right) - V^2(\text{rot } \mathbf{v} + \Omega)$$
$$= -gv\left(\frac{\partial H}{\partial x} + \frac{\partial a_0}{\partial x}\right) + gu\left(\frac{\partial H}{\partial y} + \frac{\partial a_0}{\partial y}\right) \tag{61}$$

In this equation the friction term does not occur. Equations (57)–(59) or (59)–(61) represent the tidal equations expressed in terms of the energy head H. Equation (60), in which the Coriolis force does not appear, can be used for the computation of the friction coefficient of Chézy C occurring in the definition of R [compare with Eq. (49)]. Then, measurements of u, v, and h

must be available (in that case H is also known). It follows from Eq. (60)

$$RV = -g \cos \alpha\left(\frac{\partial H}{\partial x} + \frac{\partial a_0}{\partial x}\right) - g \sin \alpha\left(\frac{\partial H}{\partial y} + \frac{\partial a_0}{\partial y}\right) - \frac{\partial V}{\partial t} \tag{62}$$

where α is the angle between the direction of the vector $\mathbf{v}$ and the x axis. The usefulness of this equation is that it is not necessary to consider the components of $\mathbf{v}$ and its spatial derivatives. Equation (61) may be used for checking the measurements of u, v, and h because the coefficient of Chézy does not appear. The equation of continuity, written in the form of Eq. (59), is much more complicated than that given by Eq. (38). Therefore applying Eq. (38) together with the definition of the energy head as defined in Eq. (56) is recommended.

2. *Importance of the Various Terms in the Tidal Equations for the Sea*

The importance of the main terms in the tidal equations are discussed by Dronkers (1968, 1969a). This investigation is based on tidal measurements in the North Sea. It shows that the main terms in Eqs. (49) and (50) are the local derivatives $\rho\ \partial u/\partial t$ and $\rho\ \partial v/\partial t$; the Coriolis terms $-v\Omega$ and $u\Omega$; the terms that determine the influence of gravity on the water motion $\rho g\ \partial h/\partial x$ and $\rho g\ \partial h/\partial y$; and the friction term having components Ru and Rv.

In this and the following sections a discussion will be given on the importance of the convective terms in the equations of motion (49) and (50), e.g., $u\ \partial u/\partial x$, $v\ \partial u/\partial x$, etc. These convective terms are rewritten by introducing the derivatives of the energy head, the terms v rot $\mathbf{v}$ and u rot $\mathbf{v}$ [see Eqs. (57) and (58)]. Then a discussion on the importance of the convective terms can be made more easily.

The factor $u^2 + v^2/2g$ that defines the energy term in the definition of H usually causes an effect of the second order on the tidal motion in the sea. Let $u_{\max} = v_{\max} = 1$ m/sec; then the values during the tide will be smaller than 0.1 m.

Let the vector $\mathbf{v}$ vary during the tide according to an ellipse:

$$u = u_1 \cos \omega t, \qquad v = v_1 \sin \omega t$$

in which ω is the frequency of the main tidal constituent M_2. Then

$$V^2 = \tfrac{1}{2}(u_1^2 + v_1^2) + \tfrac{1}{2}(u_1^2 - v_1^2) \cos 2\omega t - u_1 v_1 \sin 2\omega t$$

and according to Eq. (56), the main tidal constituent of the vertical tide M_2 is not influenced by the term $V^2/2g$. This term influences the tidal constituent M_4 and the mean water level a in a usually slight way. The inaccuracy in the measurements of h and the mean level is often of the same order as the value of the energy head. The additional terms found after the introduction of the energy head in the equation of continuity (59) are usually very small because of the small values of the energy head with respect to the total depth. The inaccuracy of determining the depth by echo-sounding is greater, and moreover the irregular shape of the bottom has more influence than the additional terms for the energy head in Eq. (59).

It follows from the preceding discussion that Eq. (59) may be approximated by

$$\frac{\partial(a+H)u}{\partial x} + \frac{\partial(a+H)v}{\partial y} + \frac{\partial H}{\partial t} = 0 \tag{63}$$

Thus, Eq. (63) can be used together with Eqs. (57) and (58). The preceding discussion refers to the tidal motion in the sea. The energy head is usually introduced into the equations of nonsteady flow in a river (see Dronkers, 1964). It is not introduced into the tidal equations of a river because the energy head is often much larger than in the sea. Otherwise the equation of continuity should become more complicated. The importance of the terms v rot $\mathbf{v}$ and u rot $\mathbf{v}$ in these equations and the physical meaning of rot $\mathbf{v}$ will be discussed in the next subsections.

3. *Calculation of Rotor* v *and the Circulation in Case of Linearized Terms in the Tidal Equations*

The definition of rotor $\mathbf{v}$ is given in Section III,B,1. This function, also called curl $\mathbf{v}$, is defined in two-dimensional regions by

$$\operatorname{rot} \mathbf{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{64}$$

The circulation C is defined along a closed curve in the fluid by the line integral (see Fig. 4):

$$C(x, y, t) = \quad v_s\, ds = \quad u\, dx + v\, dy \tag{65}$$

in which v_s is the tangential component of the velocity vector to the curve and ds is a line element of the curve. When the curve is moving with the fluid,

$$\frac{dC}{dt} = \frac{du}{dt}\, dx + \frac{dv}{dt}\, dy \tag{66}$$

which is the circulation acceleration.

It may be shown that the following relation exists between C and rot $\mathbf{v}$:

$$C(x, y, t) = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy \tag{67}$$

in which the integral is defined over the x, y plane enclosed by the curve C. It represents the two-dimensional case of Stokes' theorem. Equation (67) can be demonstrated by noting that rot $\mathbf{v}$ defines the circulation around an elementary rectangle in the x, y plane (see Fig. 5). The interior region of

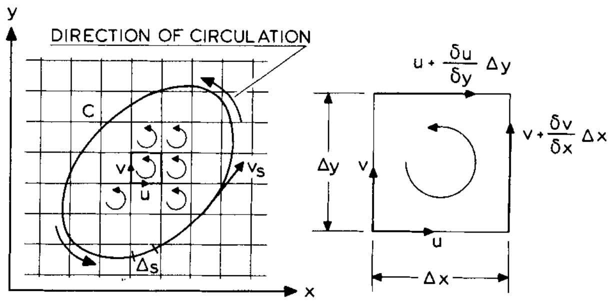

FIG. 5. Relation between the circulation along the curve C and rotor v.

curve C can be built up by means of such elementary rectangles. The algebraic sum of the circulations along these rectangles equals the circulation along the curve C. A more detailed treatment is found in Lamb (1932) and Neumann and Pierson (1966).

It is possible to calculate the value of rot $\mathbf{v}$ at a point in the fluid in the case of the following simplified equations [compare with Eqs. (49) and (50)]:

$$\frac{\partial u}{\partial t} - \Omega v = -g\frac{\partial h}{\partial x} - ku, \qquad \frac{\partial v}{\partial t} + \Omega u = -g\frac{\partial h}{\partial y} - kv \tag{68}$$

$$\frac{\partial h}{\partial t} = -a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{69}$$

in which the friction coefficient k is constant during the tide and the depth a does not depend on x and y.

After calculating $\partial^2 u/\partial t\,\partial y$ and $\partial^2 v/\partial t\,\partial x$ from Eqs. (68), subtracting the results, and applying Eq. (69), it is found that rot $\mathbf{v}$ satisfies the linear equation:

$$\frac{\partial}{\partial t}(\text{rot }\mathbf{v}) + k(\text{rot }\mathbf{v}) = \frac{\Omega}{a}\frac{\partial h}{\partial t} \tag{70}$$

The solution is

$$\text{rot }\mathbf{v} = D(x, y)e^{-kt} + \frac{e^{-kt}}{a}\int_0^t \frac{\partial h}{\partial t} e^{kt}\, dt \tag{71}$$

in which $D(x, y)$ equals the value of rot $\mathbf{v}$ at $t = 0$. For the main tidal constituent M_2,

$$h = h(x, y) \sin(\omega t - \varphi(x, y)) \tag{72}$$

Thus, from (71),

$$\begin{aligned} \text{rot } \mathbf{v} = D(x, y)e^{-kt} + \frac{\Omega\omega}{a(k^2 + \omega^2)} h(x, y) \\ \times \{[k \cos(\omega t - \varphi(x, y)) + \omega \sin(\omega t - \varphi(x, y))] \\ - [k \cos(\varphi(x, y)) - \omega \sin(\varphi(x, y))]e^{-kt}\} \end{aligned} \tag{73}$$

For sufficiently large values of t, the following approximation for rot $\mathbf{v}$ can be used by setting

$$k = p \cos \beta, \omega = p \sin \beta,$$

$$\text{rot } \mathbf{v} = \frac{\Omega \sin \beta}{a} h(x, y) \cos[\omega t - \varphi(x, y) - \beta] \tag{74}$$

It is therefore found from the preceding investigation that the maximum value of rot $\mathbf{v}$ during the tide may be approximated by

$$\text{rot } \mathbf{v} = \frac{\Omega h(x, y)}{a} \sin \beta \tag{75}$$

This equation shows that the deeper the sea, the smaller the value of rot $\mathbf{v}$ with respect to the Coriolis coefficient Ω. The value of the rotor is of more importance in shallow seas, and especially in coastal waters where the depth a decreases and the tidal range of h increases. The value of sin β decreases with increasing value of the resistance coefficient k because

$$\sin \beta = \omega/(\omega^2 + k^2)^{1/2} \tag{76}$$

so that rot $\mathbf{v}$ decreases with increasing resistance. The value is maximum when friction does not occur ($k = 0$). In that case decay of the value $D(x, y)$ of the rotor, present at $t = 0$, does not take place in the course of time. It may occur that rot $\mathbf{v}$, determined by the value of $D(x, y)$ at $t = 0$, has an important value, e.g., due to meteorological effects. Equation (71) shows that this influence decays according to e^{-kt}.

EXAMPLE: Let k be approximated by the Lorentz coefficient $8V_m^2/3\pi C^2 a$ (see Section IV,C,1), and $\omega = 1.4 \times 10^{-4}$, $V_m = 1$ m/sec, $C = 50$ m$^{1/2}$/sec, and $a = 10$. Then, $k = 3 \times 10^{-5}$ and the influence of $D(x, y)$ on rot $\mathbf{v}$ will decrease to $e^{-3} = 0.05$ of its original value after about 28 hr, or about 2.2 tidal periods [a tidal period $1/\omega$ is 44,700 sec]. In deeper seas the value of rot $\mathbf{v}$ will be smaller, but the decay is slower.

It follows from Eqs. (57), (58), and (75) that the factor rot **v** is usually small with respect to the Coriolis coefficient Ω; the maximum value of this relation is proportional to the quotient of tidal amplitude and depth. It must be noted that if rot **v** can be neglected in practical applications, the derivatives of $V^2/2g$ also can be ignored.

From Eqs. (67) and (74), the circulation along a curve C is

$$C(x, y, t) = \Omega \int_C \int \frac{\sin \beta}{a} h(x, y) \cos[\omega t - \varphi(x, y) - \beta] \, dx \, dy \qquad (77)$$

In the example mentioned above, it appears that $\sin \beta$ is nearly one and $\beta \approx \pi/2$, so that the circulation at an instant t is very slightly influenced by the friction. At smaller depths the friction increases. It follows from Eq. (77) that the circulation along a curve C can be determined from the vertical tide measurement in the region enclosed by the curve C.

4. *Modification of the Values of Rotor* **v** *through the Convective Terms in the Tidal Equations and the Boundary Conditions*

In the preceding paragraph it has been shown that rot **v** only depends on the Coriolis force in the case of linearized tidal equations (68) and (69). According to Eqs. (57) and (58), rot **v** occurs explicitly in the convective terms of the tidal equations. In that case rot **v** cannot be calculated analytically, and its values must be obtained from the results of tidal computation or from measurement of the tides.

Furthermore, rot **v** may depend in a considerable way on the boundary conditions. At increasing distances from the boundaries of the region the influence of the boundary conditions will usually decrease. Then rot **v** is mainly determined by the Coriolis force and the part of the convective terms in Eqs. (57) and (58) containing rot **v**. Variation in depth also influences the values of rot **v**.

The dependency of rot **v** on the boundary condition will be determined for the following example. Let the coastline be defined by $f(x, y) = 0$ and the direction by the angle α according to

$$\frac{df}{dx} + \tan \alpha \frac{df}{dy} = 0$$

Because the streamlines at the immediate neighborhood of the coast are usually parallel to the coastal line, the components of the velocity vector **v** in the immediate neighborhood of the coastal line can be described by

$$u = V \cos \alpha, \qquad v = V \sin \alpha \tag{78}$$

in which V is the magnitude of the velocity vector. After some calculation it is found that

$$\begin{aligned}
\operatorname{rot} \mathbf{v} &= V\left(\cos \alpha \frac{\partial \alpha}{\partial x} + \sin \alpha \frac{\partial \alpha}{\partial y}\right) + \sin \alpha \frac{\partial V}{\partial x} - \cos \alpha \frac{\partial V}{\partial y} \\
&= -\frac{V \cos \alpha}{\partial f/\partial y}\left[\cos^2 \alpha \frac{\partial^2 f}{\partial x^2} + \sin^2 \alpha \frac{\partial^2 f}{\partial y^2} + \sin 2\alpha \frac{\partial^2 f}{\partial x\, \partial y}\right] \\
&\quad + \sin \alpha \frac{\partial V}{\partial x} - \cos \alpha \frac{\partial V}{\partial y}
\end{aligned}$$

The function between brackets equals zero along a straight coastal line defined by $y = 0$. Then $\operatorname{rot} \mathbf{v} = -\partial V/\partial y$. In case the coastline is a part of a circle with radius r, defined by the equation $x^2 + y^2 - r^2 = 0$, then

$$\operatorname{rot} \mathbf{v} = -\frac{V}{r} - \frac{x}{r}\frac{\partial V}{\partial x} - \frac{y}{r}\frac{\partial V}{\partial y}$$

It appears that the components $-v \operatorname{rot} \mathbf{v}$ and $u \operatorname{rot} \mathbf{v}$ in the left-hand members of Eqs. (57) and (58) consist of two parts: the components of the centrifugal acceleration V^2/r directed perpendicular to the coastline, and the components of the acceleration due to the change of the velocity vector along the coastal line. These components are defined by

$$\frac{x^2}{2r^2}\frac{\partial V^2}{\partial x} + \frac{xy}{2r^2}\frac{\partial V^2}{\partial y} \qquad \text{and} \qquad -\frac{xy}{2r^2}\frac{\partial V^2}{\partial x} - \frac{y^2}{2r^2}\frac{\partial V^2}{\partial y}$$

From the theory of differential geometry it is known that at any point of a curve a circle of curvature exists. Thus, the radius of this circle at any point of the coastline determines the centrifugal acceleration at that point. The values of rot $\mathbf{v}$ near the coast may be much larger than those caused by the Coriolis force discussed in the previous section, e.g., when the radius r is smaller than 10,000 m and $V = 1$ m/sec [compare Eq. (75) in case $h = 1$ m, $\Omega \approx 10^{-4}$].

At a boundary line in the sea, the values of rot $\mathbf{v}$ must be determined from measurements of the vector $\mathbf{v}$. Considerations similar to those at the coastline can be set up for the points of the streamlines in the sea, and at the boundary provided the streamlines do not move.

5. *Tidal Equations at the Neighborhood of a Straight Coastal Line. The Equations of the Kelvin Wave*

Let α again be the direction of the coastal line. Because α is independent of time,

$$v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t} = 0$$

and Eq. (61) becomes

$$V(\text{rot } \mathbf{v} + \Omega) = g \sin \alpha\left(\frac{\partial H}{\partial x} + \frac{\partial a_0}{\partial x}\right) - g \cos \alpha\left(\frac{\partial H}{\partial y} + \frac{\partial a_0}{\partial y}\right) \tag{79}$$

In the case of a straight coastal line, α is independent of x and y. Let the x axis be coincident with the coastal line. Then $\alpha = 0$, $\mathbf{v} = u$, and Eq. (79) becomes

$$g\left(\frac{\partial H}{\partial y} + \frac{\partial a}{\partial y}\right) - u\frac{\partial u}{\partial y} = g\left(\frac{\partial h}{\partial y} + \frac{\partial a}{\partial y}\right) = -\Omega u \tag{80}$$

This equation follows also from Eq. (58) by putting $v = 0$. It determines the change of H in the direction perpendicular to the coastal line due to the Coriolis force. The equation of motion, from Eq. (62), is

$$\frac{\partial u}{\partial t} = -g\left(\frac{\partial H}{\partial x} + \frac{\partial a}{\partial x}\right) - Ru \tag{81}$$

Equations (80) and (81) together with the equation of continuity

$$\frac{\partial au}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{82}$$

determine the tidal flow in the immediate neighborhood of a straight coastal line. In general the velocity component v will increase seeward from the coastal line.

The *Kelvin wave* is a particular solution of Eqs. (80)–(82), in case the following assumptions are satisfied: The velocity component v remains equal to zero seaward from the coastal line, the Bernoulli term $u\,\partial u/\partial x$ is neglected, the depth is independent of x and y, and the resistance coefficient R is constant during the tide. Then the functions $u(x, y, t)$ and $h(x, y, t)$ must satisfy three equations:

$$\frac{\partial u}{\partial t} + ku = -g\frac{\partial h}{\partial x}; \qquad \Omega u = -g\frac{\partial h}{\partial y}; \qquad a\frac{\partial u}{\partial x} = -\frac{\partial h}{\partial t} \tag{83}$$

in which k is the constant resistance coefficient. In this case rot $\mathbf{v} = -\partial u/\partial y = (g/\Omega)\,\partial^2 h/\partial y^2$. In Section IV, it will be shown that harmonic solutions, the so-called *Kelvin waves*, satisfy these three equations.

IV. Tidal Computations

A. General

1. *General Consideration on Methods of Tidal Computations*

There are several methods for tidal computations of one-dimensional flow such as in a river. They are the harmonic method, the method of characteristics, and the finite-difference methods. The method of characteristics is also applied in the form of a finite-difference method; however, the grid in the x, t plane is determined in the course of the eomputation. The grid of finite-difference methods does not vary with time, but the finite-difference of x, Δx may vary along the river, depending on the schematization of the river. The finite-difference methods are more attractive for application than the method of characteristics, though the latter is more accurate. The finite-difference methods will be explained in more detail in this section. The harmonic and characteristic methods will be treated in more general terms.

The theory of tides in the sea is greatly complicated by the earth's rotation. Under actual circumstances on the earth the periods of the free oscillation and of the tidal force are not small compared with a day, so that the Coriolis force has a considerable influence on the tidal motion. Therefore, the water motion in the sea is in general more complex than that in rivers, and consequently the methods for tidal computation in the sea are still more limited than for rivers. In the sea the harmonic method and the characteristic method can be applied only for simplified tidal equations and boundary conditions. Only a few harmonic constituents can be considered. The finite-difference methods can be applied to complete tidal equations and to complex coastal lines.

2. *Review of Theoretical Studies on Tidal Theory*

A great number of theoretical studies have been carried out after those of Newton, who is the founder of tidal theory. Laplace (1799) dealt with the harmonic tidal oscillation of an ocean of relatively small depth covering the rotating earth. Further contributions were given by Thomson (1875), Hough (1897–1898), and Poincaré (1910). The influence of bottom friction has been ignored by all these investigators. Lord Kelvin (W. Thomson) considered tidal waves in an infinitely long uniform straight channel, subject to the Coriolis force. Such waves are now called *Kelvin waves* (Lamb, 1932, Ch. VIII).

The so-called *canal theory* was originated by Airy (1845). He considered a uniform canal which falls along the earth's equator and assumed that the moon describes a circular orbit in the same plane. Airy determined the free oscillation and the forced waves or tides by introducing the tide-generating forces, which are the derivatives of the tidal potential. Friction and Coriolis force are ignored in the derivation. According to expectation, the tide appears of a semidiurnal character. The case of a circular canal parallel to the equator can be dealt with in the same way. Then, diurnal as well as semidiurnal tidal constituents are found. A canal parallel to a meridian has also been considered. Obviously the propagation of tides in such cases deviates very much from the equilibrium tide defined in Section II,A, in particular when the depth of the canal is moderate. At an increasing depth the tide approaches more and more to the equilibrium tide. Airy dealt with the influence of the viscosity on the wave motion separately by introducing a linear friction term in the equations.

The propagation of tidal waves in two dimensions was considered by Airy for special cases in the absence of disturbing forces, e.g., free oscillations in a rectangular region. Then the eigen frequencies are determined by

$$\omega^2 = c^2\pi^2\left(\frac{m^2}{p^2} + \frac{n^2}{q^2}\right)$$

in which c is the propagation velocity equal to $(ga)^{1/2}$, a is the mean depth, p and q are the length and the width of the basin, and m and n are integers. An introduction to the theory of long waves in canals and standing waves in closed basins can be found in Defant (1960). He mentioned various practical applications to existing lakes.

Generally friction has been excluded in nearly all the tidal studies before the work of Reynolds (1883). It was found by Reynolds that the transition from the linear formula of resistance for laminar flow to the resistance formula for turbulent flow takes place at a critical velocity. In tidal motion, turbulent flow always occurs, and a nonlinear resistance formula must be applied (see Section III,A,5). For obtaining analytical solutions from the tidal equations, this nonlinear law is usually approximated by a linear law. Important work on the theory of turbulence has been done by Taylor. He calculated from the known velocities of tidal streams the rate of dissipation of energy in the Irish sea due to the friction (Taylor, 1919).

Canal cross sections of variable depth and width were also considered by various investigators: Proudman, (1913, 1935) and Defant (1960). Applications to tides in adjacent seas, which are enclosed or communicate with the open sea by means of openings, e.g., the North Sea, the Baltic, English Channel, Irish Sea, Adriatic, etc., are found in Defant (1960).

The theory of the Kelvin wave (Thomson, 1879) for an infinitely long uniform straight canal is also of interest for obtaining a general insight into the tidal motion along straight coasts and it gives an explanation of the well-known phenomenon of the *amphidromic points*. The influence of friction, which was not considered by Kelvin, complicates the formulas considerably (see Section IV,D).

Next to the Kelvin waves, *Poincaré waves* may occur in a rectangular region. They must be introduced to satisfy the boundary conditions at the closed end of a canal because the velocities perpendicular to the closed end and the sides must be zero. An explanation of this matter can be found in Defant (1960). The superposition of two Kelvin waves progressing in a rectangular canal in opposite directions cannot determine the complete reflection at the closed end during the tidal period in the case that the Coriolis force cannot be ignored. This problem has been solved by Taylor (1921) by introducing a series of transverse motions (see also Defant, 1960). At the Mathematical Center at Amsterdam (van Dantzig and Lauwerier, 1960–1961), this problem was also considered for the propagation of a storm flood in a rectangular basin open at one side.

B. Boundary Conditions

A general solution of the tidal equations mentioned in Section III, Eqs. (38), (49), and (50), do not exist, and solutions can be obtained only in case the boundary and initial conditions are added to the equations. Boundary conditions are functions of time and the spatial coordinates. They must be given at the boundaries of the region. Initial conditions, which are functions of x, or (x, y) in the case of two-dimensional regions, are given at a certain time t_0. In case periodical solutions must be determined, initial conditions are not necessary.

The tidal motion in a river is determined by two boundary conditions. The following are the most well-known examples of boundary conditions. For a canal closed at one end, the vertical tide at the mouth must be known, and the velocity at the closed end must be zero. In a river with fresh water flow, the tide dies out at infinity theoretically; in nature, however, it happens at a finite distance from the mouth. The boundary conditions are obtained from measurements of the vertical tide at the boundaries.

The initial conditions that must be introduced in case of nonperiodical motions, e.g., in case of storm surges, are the water levels and the velocities along the river at $t = 0$. These conditions must also be introduced in the case that finite-difference methods are applied even for periodical motions. Then it appears that the influence of the initial conditions on the solution in the course of the computation decreases and disappears when the solution becomes periodical after a sufficient long time. Therefore it is not necessary to determine the initial conditions from measurements of water levels and velocities. Usually the water levels at $t = 0$ are assumed to be constant, and the velocities are taken as zero.

The boundary conditions for the tidal motion in a region of the sea are more difficult to obtain from the theoretical than from the practical point of view. The flow normal to the coastal line must be zero; and along a boundary in the sea, the water levels must be known as a function of time. The initial conditions are analogous to those in a river, the water levels and velocities must be given within the region at $t = 0$. The question can be answered if these conditions are sufficient for the unique determination of the tide within the region. This is not the case when the convective terms must be included in the tidal computations. This appears, e.g., when the tidal equations are applied in the form of Eqs. (57) and (58), in which the functions H, u, and v are dependant variables. Then the energy heads H and thus the water levels h and the length of the velocity vector $\mathbf{v}$ must also be known at the boundary, and so must the derivatives occurring in rot $\mathbf{v}$. This mathematical problem can be answered by the application of the theory of characteristics to the two-dimensional tidal equations. This has been done by Daubert and Graffe (1967). They show that, besides the water levels, the velocity components at the boundary must be known during the tide. However, it depends on the direction of the flow. In case the convective terms can be ignored at the boundary, the water levels at the boundary are sufficient conditions.

For technical purposes, it is often necessary to carry out tidal computations for the determination of changes in the tidal propagation due to artificial or natural changes of the river dimensions. In that case it must be assumed that the boundary conditions are not affected by such modifications. Often separate tidal computations are required for boundary conditions at distant locations. It may occur that the vertical tide at the mouth of a river cannot be considered as an unchanged boundary condition and that boundary conditions must be assumed in the sea. In such cases combined one- and two-dimensional tidal computations must be carried out (see Dronkers, 1972).

C. Tidal Computations for Rivers

Tidal computations for rivers have been given in a textbook by Ippen (1966). They give a general review of the tidal dynamics in estuaries. Estuaries of rectangular sections are considered first. The harmonic description of the tide is mainly applied in the mathematical discussion. A practical application for real estuaries is given for the Delaware estuary. Harleman and Lee (1969) dealt in detail with the computation of tides and currents in estuaries and canals by using a finite-difference scheme. They mentioned various applications and made a comparison of the computation with experiments in a tidal flume.

In this section a review is given of the tidal computations in rivers applied in The Netherlands (see also Dronkers, 1964, 1969a). The harmonic, the characteristic, and the finite-difference methods for two different schemes are discussed subsequently. An application is mentioned for an extensive system of channels in an estuary.

1. *The Harmonic Method*

In the harmonic method the propagation of the periodical tidal motion is split up in the computation of a series of sinusoidal waves. For the linearized tidal equations, mentioned in Eqs. (36) and (37), the superposition principle applies. Then the propagations of the components are independent of each other, and the components can be computed separately. This procedure cannot be followed in the case of nonlinear tidal equations for a river using Eqs. (53) and (54). In that case the method becomes more complicated because the amplitudes and phases of the tidal constituents during the propagation mutually influence each other. Moreover, compound tidal components are generated (see also Section II,A). Therefore, this method, in which the tidal constituents are computed in common, gives an insight into the interaction of the tidal constituents during the propagation, in particular in shallow water eventually combined with upland discharge. In contradiction to the finite-difference methods, the tidal motion is computed as a continuous function in time and as a discrete function with respect to distance. In practical applications it is preferable to consider Eqs. (53) and (54) in which the discharge Q and the vertical tide h are dependent variables.

The harmonic method will be described in general. The following tidal equations are considered [compare with Eqs. (53) and (54)]:

$$\frac{\partial h}{\partial x} + \frac{da_0}{dx} + \frac{1}{gA}\frac{\partial Q}{\partial t} - \frac{b + b_s}{gA^2} Q \frac{\partial h}{\partial t} + \frac{1}{C^2 A^2 (a + h)} \,|\, Q \,|\, Q = 0$$

$$\frac{\partial Q}{\partial x} + b\frac{\partial h}{\partial t} = 0$$

in which the cross section $A = b_s(a + h)$. As an example, two semidiurnal tidal constituents are considered: the M_2 and the S_2 tides, or

$$h = h_1 \cos(\omega_1 t + \alpha_1(x)) + h_2 \cos(\omega_2 t + \alpha_2(x))$$

in which the indices 1 and 2 refer to the M_2 and the S_2 tides, respectively, $\omega_1 = 1.405 \times 10^{-4}$, and $\omega_2 = 1.454 \times 10^{-4}$. A similar expression holds for the discharge Q. Then h is replaced by Q and $\alpha(x)$ by $\beta(x)$. As an example, the substitution in one of the terms of the tidal equations is given. The functions h and Q are substituted in the term $(b + b_s/gA^2)Q\ \partial h/\partial t$. After some calculation, the following expression can be obtained:

$$\begin{aligned}\frac{1}{gA^2} Q \frac{\partial h}{\partial t} = -\frac{1}{2b_s^2 a^2}\,[&Q_1 h_1 \omega_1 \sin(\alpha_1 - \beta_1)\\
&+ Q_2 h_2 \omega_2 \sin(\alpha_2 - \beta_2)\\
&+ Q_1 h_1 \omega_1 \sin(2\omega_1 t + \alpha_1 + \beta_1)\\
&+ Q_1 h_2 \omega_2 \sin\{(\omega_1 + \omega_2)t + \alpha_2 + \beta_1\}\\
&+ Q_2 h_1 \omega_1 \sin\{(\omega_1 + \omega_2)t + \alpha_1 + \beta_2\}\\
&+ Q_2 h_2 \omega_2 \sin(2\omega_2 t + \alpha_2 + \beta_2)\\
&+ Q_1 h_2 \omega_2 \sin\{(\omega_2 - \omega_1)t + \alpha_2 - \beta_1\}\\
&+ Q_2 h_1 \omega_1 \sin\{(\omega_1 - \omega_2)t + \alpha_1 - \beta_2\}]\\
&\times \left[1 - \frac{h_1}{a}\cos(\omega_1 t + \alpha_1) - \frac{h_2}{a}\cos(\omega_2 t + \alpha_2) + \cdots\right]^2\end{aligned}$$

These formulas can be further expressed in terms of a sum of single harmonics. It appears that a great number of harmonics are generated next to the M_2 and S_2 tidal constituents. In principle, an infinite number of tidal constituents are generated, whose amplitudes decrease.

This procedure can also be applied to the other terms in the tidal equations. From the practical point of view the method becomes much more surveyable when the tide is developed into a Fourier series for a certain period, e.g., 12 hr 25 min, or 25 hr, etc., and complex functions are introduced. This is done as follows: The series for the vertical tide is written in complex form as

$$h(x, t) = a_0 + \sum_{k=1}^{n} a_k(x) e^{ik\omega t} + \text{conjugate function}$$

in which $a_k(x)$ is the complex amplitude, $h_k(x)e^{i\alpha_k x}$, where $\alpha_k x$ is the phase angle, and ω is the lowest frequency of the constituent to be considered. This series and the analogous series for the discharge Q are substituted in the two tidal equations. After this substitution the products and quotients of the complex functions of h and Q are replaced by a sum of terms of the form

$$P_0(x) + \sum_{k=1}^{n} P_k(x)e^{ik\omega t}$$

in which the $P_k(x)$ are algebraic, nonlinear relations of the complex amplitudes $a_k(x)$, $Q_k(x)$, and the derivatives with respect to x. Harmonic functions of frequency higher than $n\omega$ are omitted in the series.

A separate problem is the development in periodical functions of the factor $|Q|Q$ occurring in the friction term of the tidal equations in terms of Fourier series. This development can be found by application of the Fourier theory (see Dronkers, 1964). The corresponding equations are mentioned at the end of this section. After the substitution two equations are obtained from the two tidal equations for a river for each value of k in the factor $e^{ik\omega t}$ ($k = 1, 2, \ldots, n$). Thus, $2n$ equations are obtained for the $2n + 1$ unknown complex functions: $a_0(x)$, $a_k(x)$, and $Q_k(x)$. One equation is found for $P_0(x)$, in which the runoff discharge Q_0 must be assumed as known. Thus $2n + 1$ equations are found for the unknowns in complex form: $a_k(x)$ and $Q_k(x)$, and the real function $a_0(x)$.

Practical application can be made in the following way. Let the river be divided into a number of river sections, r in total, such that the mean depth, width, and storage width during the tide do not vary much in each section, and can be taken as constant values. Then, the values of $a_k(x)$ and $Q_k(x)$ are considered at both ends of each section. At the transition of each section x_i, $a_k(x)$, and $Q_k(x)$ are kept to change continuously. From the previous discussion it follows that $2n + 1$ equations are obtained between the a_k and Q_k values at both ends of a section. If the number of sections is r, the total number of equations to be solved is $r(2n + 1)$. Together with the boundary values for a_0 and a_k at the mouth of the river, and Q_k at the upstream end of the river, they form a sufficient set of equations for the determination of the unknown quantities a_0, a_k, and Q_k. After that, the real values of h_k, Q_k, α_k, and β_k can be determined. The equations should be solved simultaneously. Because these equations are nonlinear, they must be solved in an iterative way. These equations are very complicated in case more than two harmonics, e.g., the M_2 and M_4 constituents, must be considered. Then, finite-difference methods are more attractive for the solution of the tidal equations.

For some simple cases the harmonic method can be applied by desk calculators. Some particular cases based on the principles mentioned above are extensively dealt with by Schönfeld (1951), Dronkers and Schönfeld (1954), and Dronkers (1964). These cases will not be dealt with here. Only the equations will be mentioned; these are found for the development in harmonic functions of the resistance term because it is the most difficult part of the development in harmonic functions of the terms of the tidal equations.

The most simple case is when the upland discharge Q_0 is zero and the tide is represented by one constituent, e.g., the M_2 tide with frequency $\omega = 1.4/10^{-4}$, and $Q = Q_1(x)\cos(\omega t + \beta x)$. Then, the quadratic resistance factor $|Q|Q$ can be replaced by a linear term:

$$\frac{1}{\pi}\left[\left\{\int_{-\pi}^{+\pi} |Q|Q \cos \omega t \, dt\right\} \cos \omega t + \left\{\int_{-\pi}^{+\pi} |Q|Q \sin \omega t \, dt\right\} \sin \omega t\right]$$

$$= \frac{8}{3\pi} Q_1(x)[Q_1(x)\cos(\omega t + \beta x)]$$

This is the well-known *linearized friction factor* derived by Lorentz (see Ippen, 1966), in which $Q_1(x)$ is the amplitude of the velocity component of the first harmonic. Usually it refers to the M_2 tidal constituent.

The next case is the one when $Q_0 \neq 0$ and $k = 1$. This case is considered by Mazure (1937). When $Q_0 \geq Q_1(x)$, the factor $|Q|Q$ is replaced by

$$Q^2 = Q_0^2 + \tfrac{1}{2}Q_1(x)^2 + 2Q_0Q_1(x)\cos(\omega t + \beta x)$$

In case $Q_0 \leq Q_1(x)$, the runoff flow is smaller than the tidal flow, and then

$$|Q|Q = \frac{1}{4}\left[(2 + \cos 2j)\left(2 - \frac{4j}{\pi}\right) + \frac{6}{\pi}\sin 2j\right)\right]Q_1^2(x)$$

$$+ \frac{1}{2}\left[\frac{6}{\pi}\sin j + \frac{2}{3\pi}\sin 3j + \left(4 - \frac{8j}{\pi}\right)\cos j\right]$$

$$\times Q_1(x)[Q_1(x)\cos(\omega t + \beta x)]$$

in which $\cos j = Q_0/Q_1(x)$.

If the runoff flow is zero, $Q_0 = 0$ and $j = \pi/2$. Then the equation of Lorentz is obtained again. In case the upland discharge is equal to the maximum discharge of the tide, $Q_0 = Q_1(x)$ and $j = 0$ holds good. Then both equations for $|Q|Q$ become identical. In practical computation, one of the factors $Q_1(x)$ in the product $Q_1^2(x)$ must be estimated. The complete set of equations can be solved by an iteration method as mentioned above. The most complicated case considered to date is that the M_2 and M_4 tidal components are considered combined with runoff $Q_0 \neq 0$, $k = 1$, and $k = 2$ (Dronkers, 1964).

2. *The Characteristic Method*

In the characteristic method the tidal motion is considered to be the propagation of a succession of small disturbances from an initial state, determined by the initial conditions (see Section B). The periodic character of the tidal motion is not an essential factor in this method. Starting from the initial conditions the computation becomes periodical in the course of the computation due to the influence of the periodical boundary conditions. The theory of the method of characteristics is dealt with extensively by Schönfeld (1951). The application of this method for numerical computation of the propagation of tides is discussed by Dronkers (1964) based on the work of Schönfeld. Here only the basic equations are mentioned.

Two characteristic curves pass through each point of a fluid in motion. The forward and backward characteristic curves are defined by the equation

$$\left(\frac{dx}{dt}\right)_{1,2} = u \pm [g(a + h)]^{1/2}$$

in which dx/dt is the tangent to the characteristic curve at the point (x, t). In general, these curves are not straight lines. Further, the tidal equations are replaced by equations that describe the change of the water levels and the velocities along the characteristic curves. Then, the characteristic method describes the way in which these variables can be computed along the characteristic curves, usually by means of a finite-difference method.

The characteristic equations can be written in two approaches. In the first approach, h and u are the dependent variables. The corresponding equations are given by Schönfeld (1951) and Dronkers (1964). In the second approach, used by various authors, e.g., Liggett and Woolhiser (1967), the dependent variables are replaced by u and $c = [g(a + h)]^{1/2}$. Then the equations have the form

$$\left\{(u + c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right\}\left(u + 2c - g\frac{da}{dx}t\right) = -\frac{g}{C^2c^2}\,|u|\,u$$

and a second equation in which c is replaced by $-c$. The characteristic method is the fundamental method when the propagation of disturbances in a river, like a bore, must be studied.

3. *Finite-Difference Methods for a River*

a. INTRODUCTION. For tidal computations by a computer, the tidal equations are replaced by difference equations, and finite-difference methods are used in which grid points at intervals of time and distance are considered. The points are also taken from the curves which represent the boundary and initial conditions.

The finite-difference methods may be compared with the characteristic method, in which the solution also proceeds with small time steps. In both methods the grid points x, t where the water level or velocity must be computed, form a grid in the x, t plane. The grid is usually irregular in the characteristic method, and regular in the finite-difference methods. The differential equations can be replaced by finite-difference forms in various ways. However, some conditions must be fulfilled. The solution of the difference equations, which form a system of linear algebraic equations, must converge to the solution of the differential equations when the sides Δx and Δt of the grid approach zero. Even when this condition is fulfilled, the numerical solution of the difference equations may not necessarily approach the required solution of the tidal equations. It is possible that when the computation proceeds, waves are mathematically generated that overshadow the actual solution. These waves may be caused by the inevitable rounding errors of cumulative effect in the progress of practical computation. Then so-called spurious solutions exist. When such solutions occur, the difference method becomes *unstable*. Otherwise it is *stable*, and thus the effect of a rounding error decreases in the further steps of the computation. It may also occur that the computation scheme is stable for a certain condition of the relation $\Delta x/\Delta t$, and unstable when this condition is not fulfilled. Characteristic methods are usually stable and considered the most accurate. However, the computed data are distributed in an irregular way in time and distance, which is often an objection in practical applications. The mathematical theory for the determination of the stability conditions will not be discussed here. (For an introduction, see Dronkers, 1969a.)

Many finite-difference schemes for the computation of the propagation of long waves can be found in the literature. It is not the aim here to give a review of these methods. An example to show the explicit and implicit methods will be discussed in the practical application in the following subsections.

Let the points of a grid be defined by $t = n\tau$ and $x = mk$ ($n = 0, 1, 2, \ldots, N$; $m = 0, 1, 2, \ldots, M$), in which τ and k are the steps with respect to time and space. The values of the functions u and h at $t = n\tau$ and $x = mk$ are usually designated by u_m^n and h_m^n. Then, the differential quotient $\partial u/\partial t$ may be replaced by

$$\frac{u_m^{n+1} - u_m^n}{\tau} \qquad \text{or} \qquad \frac{u_m^{m+1} - u_m^{n-1}}{2\tau}$$

respectively called a *forward* or *central difference quotient*. Similar quotients may be written for the differentials in the x direction. By means of such operations the differential equations are reduced to a system of algebraic equations. Then two possibilities may be distinguished. In an *explicit scheme*, the unknown values at a grid point at a future instant $(n + 1)\tau$ are expressed as functions of those at a number of grid points at the instant $n\tau$. Given the initial conditions, the values at $t = 0$, as well as the boundary conditions at $x = 0$, $x = M$, one can proceed step by step to obtain the grid functions h_m^n and u_m^n, or Q_m^n, for all $t = n\tau$. In an *implicit scheme*, the values h_m^{n+1} and u_m^{n+1} also depend on these values at other points at time $(n+1)\tau$. This system of equations is therefore coupled. Then the calculation of h_m^{n+1} and u_m^{n+1} requires the solution of a whole system of linear simultaneous equations, so that a simple formula for the solution of individual points cannot be obtained. The boundary conditions must be added to the set of equations to obtain an equal number of unknowns.

Research of the stability problem shows that the explicit schemes commonly in use for the solution of the tidal equations are in general partially stable, such that the grid must satisfy a condition of the relation $\Delta x/\Delta t$, in which the wave velocity $c = (ga)^{1/2}$ is an important factor. The implicit schemes are usually unconditionally stable, so that no relation for Δx and Δt is required. Obviously the values of Δx and Δt must be chosen to be sufficiently small so that the solution for h and u satisfies the requirements of accuracy.

It is of interest to consider the concept of explicit and implicit schemes from a more general point of view. It is discussed by means of the finite-difference representation of the equation of continuity for rivers:

$$\frac{\partial h}{\partial t} + \frac{1}{b}\frac{\partial Au}{\partial x} = 0$$

After integration over the time step Δt and application of the mean value theorem of integral calculus on the integral of the term $\partial Au/\partial x$, it is found that

$$h(x, t+\Delta t) - h(x, t) + \Delta t\left(\frac{1}{b}\frac{\partial Au}{\partial x}\right)_{t+\theta_1 \Delta t} = 0$$

in which $0 \le \theta_1 \le 1$. This equation can be replaced by

$$h(x, t+\Delta t) - h(x, t) + \Delta t\left\{\theta\left(\frac{1}{b}\frac{\partial Au}{\partial x}\right)_{t+\Delta t} + (1-\theta)\left(\frac{1}{b}\frac{\partial Au}{\partial x}\right)_t\right\} = 0$$

in which $0 \le \theta \le 1$ (usually $\theta \neq \theta_1$). The value of θ depends on the values of $\partial Au/\partial x$ divided by b in the interval $(t, t+\Delta t)$ at the point x, and it cannot be determined in a tidal computation in advance. Then three choices can be made: $\theta = 0$, $\theta = \frac{1}{2}$, or $\theta = 1$. For accuracy, $\theta = \frac{1}{2}$ is usually the best approximation. The difference equation is an explicit representation in case $\theta = 0$, and implicit in case $\theta = 1$. Reference is made to Richtmeyer and Morton (1967).

b. EXAMPLE OF AN EXPLICIT FINITE-DIFFERENCE SCHEME FOR A RIVER. The following equations are considered: The equation of motion [see Eq. (49) in case $v = 0$]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} + \frac{da_0}{dx} - g\frac{|u|u}{C^2(a+h)} \tag{84}$$

and the equation of continuity [see Eq. (55), and $q = 0$]

$$\frac{\partial Au}{\partial x} + b\frac{\partial h}{\partial t} = 0 \tag{85}$$

in which $a = z_0 + a_0$, and $A = b_s(z_0 + a_0 + h)$.

The finite-difference scheme is defined by (for notation see Fig. 6)

$$\begin{aligned} u_{2m}^{2n+1} &= u_{2m}^{2n-1} - \frac{\tau}{2k}u_{2m}^{2n+1}(u_{2m+2}^{2n-1} - u_{2m-2}^{2n-1}) \\ &\quad - g\frac{\tau}{k}[(h+a_0)_{2m+1}^{2n} - (h+a_0)_{2m-1}^{2n}] \\ &\quad - \frac{2\tau g\,|u_{2m}^{2n-1}|}{C^2(a+h)_{2m}^{2n}}u_{2m}^{2n+1} \end{aligned} \tag{86}$$

$$h_{2m+1}^{2n+2} = h_{2m+1}^{2n} - \frac{\tau}{k}\left(\frac{1}{b}\right)_{2m+1}^{2n}(u_{2m+2}^{2n+1}A_{2m+2}^{2n} - u_{2m}^{2n+1}A_{2m}^{2n}) \tag{87}$$

$$A_{2m+2}^{2n} = \tfrac{1}{2}[A_{2m+1}^{2n} + A_{2m+3}^{2n}], \qquad \text{etc.}$$

In Fig. 6, the values of $h + a_0$ and u in Eq. (86) occur on the sides of the triangle PQR and those of Eq. (87) at the corners of the parallelogram $ABCD$. Thus, u at point P is computed by Eq. (86), and h at point A by Eq. (87). (See the note at the end of Section III,A,7 for the computation of $h + a_0$). This scheme is called the *leap frog scheme.* The computation is

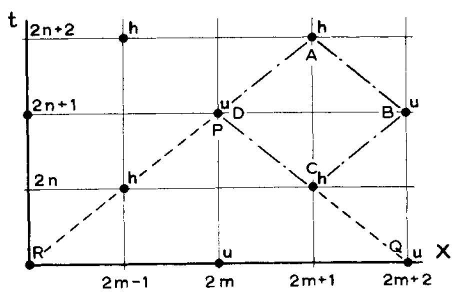

FIG. 6. Notation for an explicit difference scheme of a river.

performed in the following way: From the initial conditions for h at $t = 0$ and u at $t = 0$, the values of u at the grid points on the line $t = \tau$ are obtained by

$$u(x, \tau) = u(x, 0) - \frac{\tau}{2k} g[h(x + k, 0) - h(x - k, 0)] \tag{88}$$

After that the computation proceeds by use of Eq. (86) for the h values on $t = 2\tau$, and by Eq. (87) for the u values on $t = 3\tau$, etc. The river length must be divided into an odd number of sections when the boundary conditions are $h + a_0$ at one end and u at the other end. The stability condition determines the following condition for the time interval τ, when the friction and Bernoulli terms in Eq. (84) are ignored:

$$\frac{k}{\tau} > \left(g \frac{A}{b}\right)^{1/2} \tag{89}$$

The friction and Bernoulli terms would modify the right-hand number of the condition (89) (see Dronkers, 1969a). The stability problem for explicit schemes which is related to the error growth is dealt with by Richtmeyer and Morton (1967).

C. AN IMPLICIT SCHEME. Implicit schemes can be set up in various ways. Though any system of linear equations can be solved by computer, it is necessary to apply such schemes in order to obtain quick solutions. The so-called *sweep methods* can be used for this purpose. The elimination process which is used for the solution of the equations can be carried out by simple recurrence relations.

Two different implicit schemes of which the equations can be solved by a *double sweep method* can be considered here (see Dronkers, 1969a). In the first scheme, the water levels h are computed at both ends of a river section and the velocity u, or discharge Q at the midpoint of the section. In the second scheme, the h and u, or Q, values are computed at the both ends of a section. The second scheme can be more easily applied to a system of river branches, which will be dealt with below. The mathematical conditions at a junction of the river branches are more simple for the discharge Q than for the velocity u. Therefore the discharge Q instead of u will be considered in the following equations of motion and continuity [compare with Eqs. (53) and (54)]:

$$\frac{\partial h + a_0}{\partial x} = -\frac{1}{gA}\frac{\partial Q}{\partial t} - \frac{|Q|Q}{C^2A^2(a + h)} - \alpha\frac{B}{gA^2b} Q\frac{\partial Q}{\partial x}; \quad \frac{\partial Q}{\partial x} = -b\frac{\partial h}{\partial t} \tag{90}$$

in which $A = b_s(z_0 + a_0 + h)$, $B = b_s + b$. The value of the coefficient $\alpha \leq 1$ depends on the energy losses, due to the turbulence in the river and the flow to the storage regions. It is assumed that the terms containing $\partial b_s/\partial t$, $Q^2\, \partial(a + h)/\partial x$, and W can be neglected.

Let the tidal part of a river be divided into M sections, $(\Delta x)_m$ is the length of the mth section, and τ is the time step. The length $(\Delta x)_m$ is chosen in such a way that the variation in the cross sections is sufficiently small. It is not necessary that $(\Delta x)_m$ be a constant value for each section. The notation in this grid is shown in Fig. 7. In the following equations the notation h_m^n is

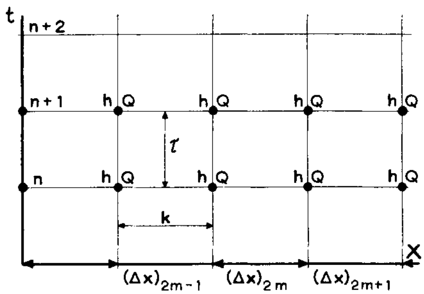

FIG. 7. Notation for an implicit difference scheme of a river.

replaced by h_m and h_m^{n+1} by h'_m for reasons of simplicity. The difference equations are applied to the mth section (rectangle $ABCD$ in Fig. 7). The equation of motion (90) is then replaced by

$$\begin{aligned} h'_{m+1} - h'_m = &-\frac{(\Delta x)_m}{2\tau g A_m}[(Q'_{m+1} - Q_{m+1}) + (Q'_m - Q_m)] \\ &-\frac{\alpha B_m}{2gA_m^2 b_m}(Q_{m+1} + Q_m)(Q'_{m+1} - Q'_m) \\ &-\frac{(\Delta x)_m}{4}\frac{|Q_{m+1} + Q_m|(Q'_{m+1} + Q'_m)}{C_m^2 A_m^2 a_m} \end{aligned} \tag{91}$$

and the equation of continuity by

$$Q'_{m+1} - Q'_m = -\frac{(\Delta x)_m}{2\tau} b_m[(h'_{m+1} - h_{m+1}) + (h'_m - h_m)] \tag{92}$$

In these equations the variable $h + a_0$ is replaced by h. Then h is defined with respect to the x, y plane in Fig. 3.

For M sections, there are $2M - 2$ equations (91) and (92). They can be rewritten in the form

$$h'_{m+1} - h'_m + \eta_m Q'_{m+1} + \theta_m Q'_m = \mu_m \tag{93a}$$

$$v_m(h'_{m+1} + h'_m) + Q'_{m+1} - Q'_m = \xi_m \tag{93b}$$

in which the definitions of the coefficients $\eta_m, \ldots, \xi_m$, for the mth section correspond to the notation in Eqs. (91) and (92) by comparison. Equations (93) contain four unknown functions h' and Q', and the group consists of $2M - 2$ equations, $2M$ unknowns, h, and Q. If $h_1(t)$ and $Q_M(t)$ are given as boundary conditions, then $2M - 2$ unknowns can be determined from $2M - 2$ linear equations. This system of equations can be solved as follows: Equations (93) are replaced by two equations, which contain three unknown functions instead of four. The term that contains h'_1 is taken separately unless h'_1 is known from the boundary condition. The equations for $m = 1$ are rewritten in the form

$$Q'_1 = -q_1 h'_2 - t_1 Q'_2 + s_1 + b_1 h'_1 \tag{94a}$$

and

$$h'_2 = -p_2 Q'_2 + r_2 + h'_1 \tag{94b}$$

in which the coefficients $q_1, \ldots, r_2$, and b_1 can be found from a comparison with Eqs. (93). Then, Q'_1 and h'_2 can be substituted in Eqs. (93) in case $m = 2$. Proceeding in this way, the equations for the $(m - 1)$th section $(m = 2, \ldots, M)$ can be written in the form

$$Q'_{m-1} = -q_{m-1} h'_m - t_{m-1} Q'_m + s_{m-1} + b_{m-1} h'_1 \tag{95a}$$

and

$$h'_m = -p_m Q'_m + r_m + a_m h'_1 \tag{95b}$$

in which three unknown functions of the mth section occur, and moreover the function h'_1 occurs explicitly. The coefficients in Eqs. (95) depend on those of the $(m - 1)$th section and the preceding sections. They are computed by recurrence equations which will be derived at the end of this section.

One of the two equations which hold for the Mth section

$$h'_M + p_M Q'_M - a_M h'_1 - r_M = 0 \tag{96}$$

gives a linear relation between the vertical tide at the mouth of the river and the tidal data at the end of the river. From Eq. (96), Q'_m (or h'_M) at time $t + \tau$ can be solved, when h'_M (or Q'_M) is known from the boundary condition for h or Q at $x_M = L$, which is the length of the tidal part of the river.

The water level h'_1 is known at the mouth of the river. The process for the successive computation of the coefficients of Eqs. (95) and (96) is called the *first sweep*. After completing this first sweep, the unknowns h' and Q' can be computed successively for the sections $M - 2$, $M - 3$, etc. up to $m = 2$.

For these computations, the following equations, which are similar to Eqs. (95), are applied in the downstream direction for successive sections n $(n = M, M-1, \ldots, 2)$:

$$Q'_n = -q_n^* h'_{n-1} - t_n^* Q'_{n-1} + s_n^* + b_n^* h'_M \tag{97a}$$

$$h'_{n-1} = -p_{n-1}^* Q'_{n-1} + r_{n-1}^* + a_{n-1}^* h'_M \tag{97b}$$

Then h'_{n-1} and Q'_{n-1} can be successively computed in the downstream direction. These computations are called the *second sweep*. The index n refers to the second sweep.

Applying Eq. (97b) to $n = 2$ gives

$$p_1^* Q'_1 + h'_1 - a_1^* h'_M - r_1^* = 0 \tag{98}$$

This equation and Eq. (96) provide two relations between the tidal functions h and Q at both ends of the river.

In the following the recurrence equations are used for the computation of the coefficients of Eqs. (95) and (97) $(m = 2, \ldots, M)$. The recurrence equations for Eqs. (95) are

$$\begin{aligned}
q_{m-1} &= \frac{1}{p_{m-1} + \theta_{m-1}}; \qquad t_{m-1} = \frac{\eta_{m-1}}{p_{m-1} + \theta_{m-1}}; \\
s_{m-1} &= \frac{r_{m-1} + \mu_{m-1}}{p_{m-1} + \theta_{m-1}} \\
p_m &= \frac{\sigma_{m-1} t_{m-1} + 1}{\sigma_{m-1} q_{m-1} + \nu_{m-1}}; \\
r_m &= \frac{\xi_{m-1} + \sigma_{m-1} s_{m-1} - \nu_{m-1} r_{m-1}}{\sigma_{m-1} q_{m-1} + \nu_{m-1}} \\
a_m &= \frac{\sigma_{m-1} b_{m-1} - \nu_{m-1} a_{m-1}}{\sigma_{m-1} q_{m-1} + \nu_{m-1}}; \\
b_{m-1} &= \frac{a_{m-1}}{p_{m-1} + \theta_{m-1}}
\end{aligned} \tag{99}$$

in which $\sigma_{m-1} = p_{m-1}\nu_{m-1} + 1$. Note that $a_1 = 1$, $p_1 = 0$, and $r_1 = 0$ according to Eqs. (94). It appears from Eqs. (99) that the coefficients of Eqs. (95) depend on the coefficients of Eqs. (93) when applied to the $(m-1)$th section. Furthermore, they depend on the coefficients of Eq. (95b) when applied to section $m - 2$. Thus,

$$h'_{m-1} = -p_{m-1} Q'_{m-1} + r_{m-1} + a_{m-1} h'_1 \tag{95c}$$

The coefficients of this equation are determined by the preceding computation of the coefficients of Eq. (95), which is applied to section $m - 2$. The recurrence equations (99) are derived as follows:

Substitute Q'_{m-1}, solved from Eq. (95c), in Eq. (95a). Then apply Eq. (93a) to section $m - 1$, in which h'_{m-1} is replaced by the right-hand member of Eq. (95c). Thus, the equations for q_{m-1}, t_{m-1}, s_{m-1}, and b_{m-1} are found by equating the coefficients of the modified Eqs. (93a) and (95a). In a similar way, the equations for p_m, r_m, and a_m are derived from Eq. (93b) when applied to section $m - 1$, and Eq. (95b). Then, h'_{m-1}, and Q'_{m-1} are eliminated, according to Eqs. (95c) and (95a).

The recurrence equations for q_n^*, t_n^*, s_n^*, and b_n^* are

$$q_n^* = \frac{1}{p_n^* - \eta_{n-1}}; \qquad t_n^* = \frac{-\theta_{n-1}}{p_n^* - \eta_{n-1}};$$

$$s_n^* = \frac{-\mu_{n-1} + r_n^*}{p_n^* - \eta_{n-1}}$$

$$b_n^* = \frac{a_n^*}{p_n^* - \eta_{n-1}}; \qquad p_{n-1}^* = \frac{\sigma_n^* t_n^* + 1}{\sigma_n^* q_n^* - \nu_{n-1}}$$

$$r_{n-1}^* = \frac{\sigma_n^* s_n^* + \nu_{n-1} r_n^* - \xi_{n-1}}{\sigma_n^* q_n^* - \nu_{n-1}};$$

$$a_{n-1}^* = \frac{\sigma_n^* b_n^* + \nu_{n-1} a_n^*}{\sigma_n^* q_n^* - \nu_{n-1}} \tag{100}$$

Furthermore, $p_M^* = 0$, $a_M^* = 1$, and $r_M^* = 0$. The derivation of these coefficients is similar to that mentioned above for q_{m-1}, p_m, etc. [compare with Eqs. (99)].

d. APPLICATION OF THE IMPLICIT SCHEME TO A SYSTEM OF RIVER BRANCHES. The application of this method to the system of river branches AC, BC, and CD as shown in Fig. 8 is demonstrated below. Let $h_1(t)$ and $h_2(t)$ be the boundary conditions at the mouths of the branches AC and BC, and let

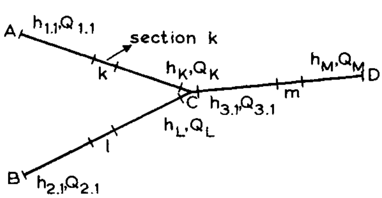

FIG. 8. Notation for the application of the implicit difference scheme on three river branches AC, BC, and CD.

Q_M be known at the upstream side of the river branch CD. Let the branch AC be divided into $K - 1$ sections, BC into $L - 1$ sections, and CD into $M - 1$ sections. Then, to each branch, a set of Eqs. (91) and (92), or (93) is applicable. The method discussed in the previous section can be applied to each river branch. Then the following relations between the tidal functions at A, B, C, and D hold. They are similar to Eqs. (96) and (98). For AC,

$$h'_K + p_{1,K}Q'_K - a_{1,K}h'_{1,1} + r_{1,K} = 0;$$
$$p^*_{1,K}Q'_{1,1} - a^*_{1,K}h'_K + h'_{1,1} - r^*_1 = 0 \tag{101a}$$

For BC,

$$h'_L + p_{2,L}Q'_L - a_{2,L}h'_{2,1} + r_{2,L} = 0;$$
$$p^*_{2,L}Q'_{2,1} - a^*_{2,L}h'_L + h'_{2,1} - r^*_2 = 0 \tag{101b}$$

For CD,

$$h'_M + p_{3,M}Q'_M - a_{3,M}h'_{3,1} + r_{3,M} = ;$$
$$p^*_{3,M}Q'_{3,1} - a^*_{3,M}h'_M + h'_M - r^*_M = 0 \tag{101c}$$

The notation is shown in Fig. 8. In the notation for the water levels and discharges at point C, the index 1 or 2, which refers to the number of the river branches AC and BC (e.g., in h'_K) respectively, are omitted. At the junction C,

$$h_K = h_L = h_{3,1} \qquad \text{and} \qquad Q_K + Q_L + Q_{3,1} = 0 \tag{101d}$$

holds, when the Bernoulli terms at the transitions of the river branches at C can be neglected. Thus, a set of nine linear equations must be solved for the nine unknown functions: $Q_{1,1}$, $Q_{2,1}$, h_K, Q_K, h_L, Q_L, $h_{3,1}$, $Q_{3,1}$, and h_M. The solution of these equations can be obtained in the usual way by Gaussian elimination of the unknowns.

The previous method can be extended to a much more complicated system of river branches. An example is shown in Fig. 9 for the estuary Brouwershavense Gat in the south western part of the Netherlands. This estuary is closed inland and the tidal motion enters this estuary from the North Sea. The gullies are schematized in a system of river branches. A great number of linear equations of the form (101a) and (101d), in which h and Q values at the junctions are the unknowns, must be solved. The method of elimination of the unknowns has been described in a computer program, using the particular form of the equations. In Fig. 10 the results of the vertical tide and discharge computations are compared with the measurements in nature, showing good agreement.

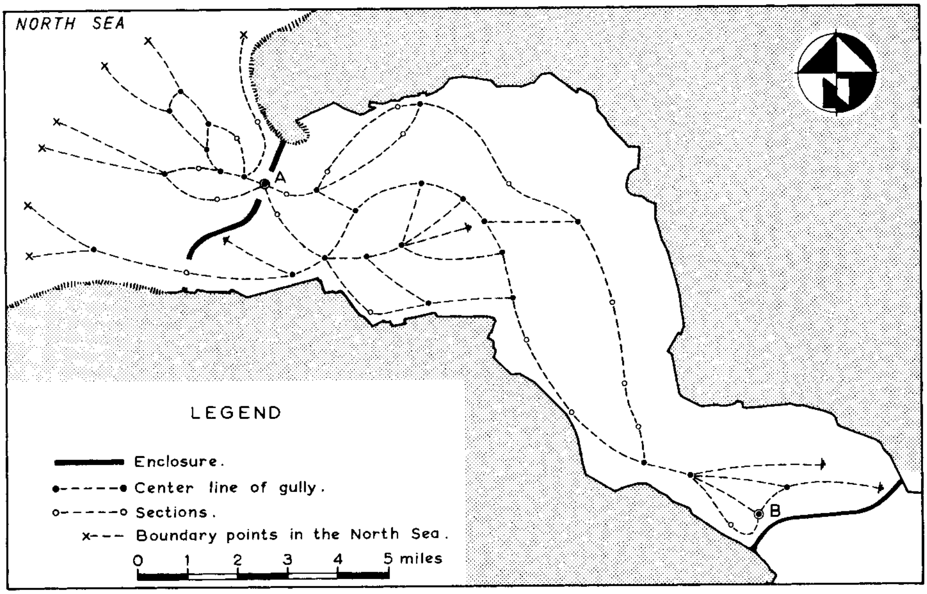

FIG. 9. The schematization of the gullies of Brouwershavense Gat for the application of the implicit difference scheme.

Ploeg and Kamphuis (1968) and Kamphuis (1970) applied the implicit scheme to the computation of the tidal motion in a part of the Saint Lawrence River. He used however a different method for a system of river branches. The equations of the river sections and those at the bifurcation points are solved together. Van de Kreeke (1971) dealt with the computation of the tidal motion in shallow lagoons. Then friction and the Bernoulli force are predominant in the entrance. He discussed these aspects in detail from the practical point of view.

The explicit method, described in Section IV,C,3,b can also be extended to a system of river branches. Then the equations that hold at the bifurcation points must be included. A computer program, called "Cherie," has been set up by Booy (1971).

e. SECOND IMPLICIT SCHEME. A different method for tidal computation in a system of river branches has been applied by Vreugdenhil (1971). He wrote the equation of continuity (90) as

$$2b\,\Delta x(h_{j+1}^{n+1} - h_{j+1}^{n}) + \theta\,\Delta t(Q_{j+2}^{n+1} - Q_j^{n+1}) + (1-\theta)\,\Delta t(Q_{j+2}^{n} - Q_j^{n}) = 0 \quad (102)$$

in which n denotes the time level $n\,\Delta t$; the values of h_{j+1} are taken halfway

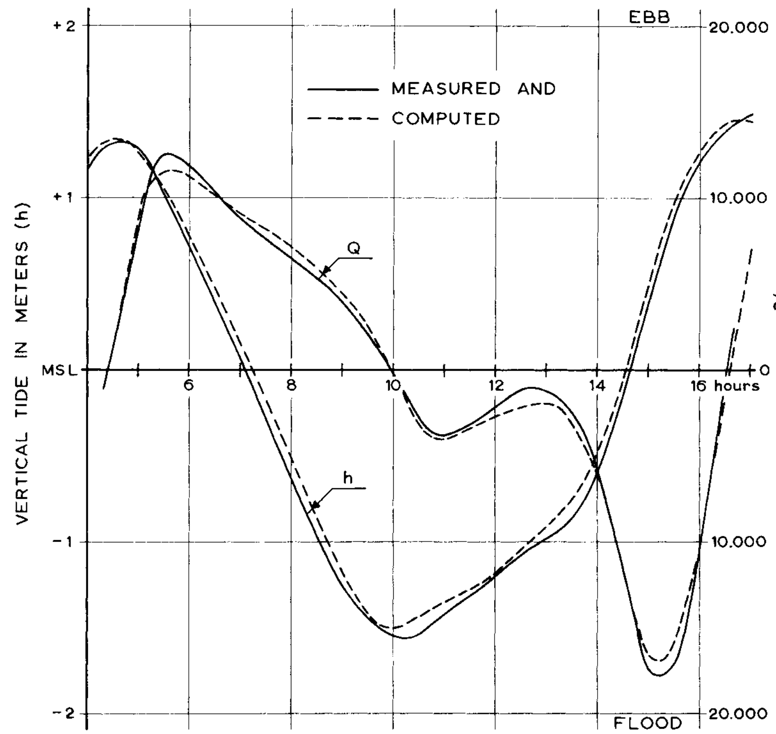

FIG. 10(a). Comparison of measured and computed vertical tides and discharges at point A in Fig. 9.

between the locations of values Q_j and Q_{j+2}. The significance of the factor θ has been explained in Section IV,C,3,a. The finite-difference representation for the equation of motion (90) is

$$2\,\Delta x(Q_j^{n+1} - Q_j^n) + \theta\,\Delta t g A_j^n(h_{j+1}^{n+1} - h_{j-1}^{n+1} + r_j^n Q_j^{n+1})$$
$$+ (1-\theta)\,\Delta t g A_j^n(h_{j+1}^n - h_{j-1}^n + r_j^n Q_j^n) = 0, \quad (103)$$

in which $r = Qg/C^2Aa$. The term which contains the factor $Q\,\partial h/\partial t$ in Eq. (90) has been left out of consideration.

In case $\theta = 0$, the finite-difference equations are of the explicit form, and in case $\theta \neq 0$ they are implicit. An important result of the study of Vreugdenhil is that the scheme is unconditionally stable if

$$\tfrac{1}{2} \leq \theta \leq 1$$

For $\theta = 1$, the stability is most favorable. However, it may occur that for a

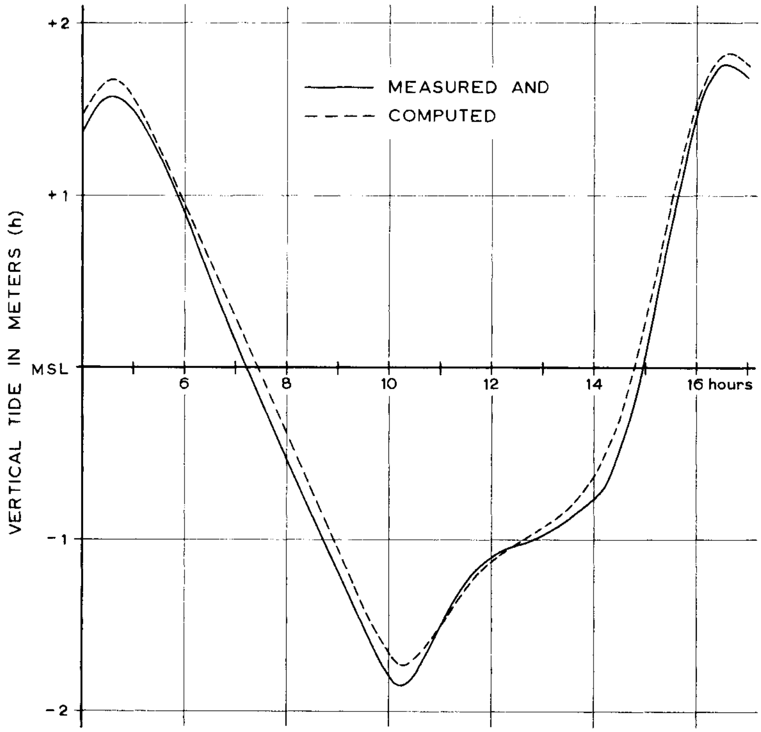

FIG. 10(b). Comparison of the measured and computed vertical tides at point B in Fig. 9.

value of θ in the interval $0 < \theta < \frac{1}{2}$, the finite-difference equations are accurate. Then a condition must be introduced to guarantee the stability of the scheme for error growth. The stability condition for an explicit system $(\theta = 0)$ is given by Eq. (89). Thus, it appears that the choice for the value of θ often has a different effect on the accuracy and stability of the scheme. However, the accuracy can be improved by taking smaller values of Δx and Δt.

Dronkers (1969a) gave an example of the application of an explicit $(\theta = 0)$ and an implicit system $(\theta = 1)$ for a tidal wave in a river. It appeared that the results are within the accuracy of the observations of the vertical and horizontal tide. For much shorter waves than the tidal waves, the accuracy decreases.

Vreugdenhil did not apply the "sweep" methods for the solution of the set of the linearized implicit difference equations [see previous Subsection c]. He applied the Gauss–Seidel iteration method (see, e.g., Goodwin, 1961) to the combined set of equations for a system of river branches. The applicability of this method is limited for reasons of the convergence of the iteration process.

An analogous implicit numerical operator was dealt with by Abbott and Ionescu (1967). The resistance term is neglected in the flow equations. The scheme is solved by a pair of simultaneous tridiagonal operators, analogous to the sweep methods (see Subsection c).

f. THE LAX–WENDROFF SCHEME FOR INTERNAL DISCONTINUITIES. In fluid flow internal discontinuities may occur. An example is the bore in rivers, which may exist in case of particular circumstances concerning the tide and the depths (see Dronkers, 1964). The mathematical treatment of such discontinuities is complicated when the discontinuity is in motion, which is not known in advance. This problem can be solved by introducing an artificial dissipative mechanism in the flow equations such that the discontinuity becomes a smooth one extending over a small distance. The finite-difference equations are set up in such a way that in the calculation the discontinuity changes automatically in a near discontinuity which has nearly the speed of the real discontinuity.

The explicit Lax–Wendroff scheme (1960) introduces the effect of an artificial viscosity. This scheme is mentioned below for the tidal equations, leaving the resistance out of consideration. More detailed information is found in Richtmeyer and Morton (1967). Vliegenthart (1968) has applied the scheme to the bore phenomenon.

The tidal equations (84) and (85) can be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial F_1}{\partial x} = 0, \qquad F_1 = hu$$

$$\frac{\partial u}{\partial t} + \frac{\partial F_2}{\partial x} = 0, \qquad F_2 = \tfrac{1}{2}u^2 + gh \tag{104}$$

or in matrix form

$$\begin{pmatrix} \partial h/\partial t \\ \partial u/\partial t \end{pmatrix} + \begin{pmatrix} a & h \\ g & u \end{pmatrix} \begin{pmatrix} \partial h/\partial x \\ \partial u/\partial x \end{pmatrix} = 0 \tag{105}$$

The Lax–Wendroff scheme, applied to the equation for the water level h, is

$$\begin{aligned} h_j^{n+1} = h_j^n &- \frac{1}{2}\frac{\Delta t}{\Delta x}(F_{1,j+1}^n - F_{1,j+1}^n) \\ &+ \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 [A_{j+1/2}^n(F_{1,j+1}^n - F_{1,j}^n) - A_{j-1/2}(F_{1,j}^n - F_{1,j-1}^n)] \end{aligned} \tag{106}$$

in which n is the time level, j the space coordinate, F_1 is defined above, and A

is the matrix

$$A = \begin{pmatrix} a & h \\ g & u \end{pmatrix}$$

The underlined term in the equation for h represents the dissipative term. The finite-difference equation for the velocity u has a similar form in which h is replaced by u and F_1 by F_2.

g. THE ACCURACY OF TIDAL COMPUTATIONS. Numerical tidal models are able to describe the tidal motion in rather complicated tidal regions. To what extent these methods can be used is still a point of discussion and experience. It depends on the phenomenon that must be studied, on the required data, and on the accuracy demanded.

It is difficult to determine the accuracy of the finite-difference solution by means of mathematical formulas. Some insight into the accuracy of the solution can be obtained by determining the modifications in the wave propagation obtained from finite-difference solutions in comparison with the physical wave determined by the differential equations. Leendertse (1967) dealt with this procedure for one- and two-dimensional tidal equations. Vreugdenhil (1971) applied this research to the scheme mentioned in Subsection e and Stroband (1971) to the scheme dealt with in Subsection c. The method can be carried out only for linearized tidal equations. In the following the method will be demonstrated for the one-dimensional tidal equations [compare Eqs. (90)]:

$$\frac{\partial Q}{\partial t} + gA\frac{\partial h}{\partial x} + rQ = 0$$

$$\frac{\partial Q}{\partial x} + b\frac{\partial h}{\partial t} = 0 \qquad (107)$$

in which A, r, and b are constant during the tide; $r = g(8/3\pi)Q_m/C^2A(a+h)$, Q_m being the amplitude of Q (see Section IV,C,1).

The wave propagation, which is derived from the finite-difference equations by substituting the finite-difference representation of the harmonic function

$$h(x, t) = \bar{h}e^{i(kx-\omega t)}, \qquad Q(x, t) = \bar{Q}e^{i(kx-\omega t)} \qquad (108)$$

into the finite-difference representation of Eqs. (107) must be compared with the solution of Eqs. (107), in which $\bar{h}$, $\bar{Q}$ are the amplitudes and ω is the frequency.

The finite-difference representations of the functions (108) are

$$h(\Delta x, \Delta t) = \bar{h} e^{i(\sigma \Delta x - \beta \Delta t)}, \qquad Q(\Delta x, \Delta t) = \bar{Q} e^{i(\sigma \Delta x - \beta \Delta t)} \tag{109}$$

in which σ and β are complex quantities of which the imaginary parts determine the modifications of $\bar{h}$ and $\bar{Q}$ in the solution of the finite-difference equations, and the argument of β determines the modification of the frequency ω of the physical wave defined by Eqs. (108).

The method is applied to the finite-difference scheme mentioned in Subsection e as an example. This scheme applied to the Eqs. (107) is

$$\begin{aligned}
2\,\Delta x(Q_j^{n+1} - Q_j^n) + \theta\,\Delta t[&gA(h_{j+1}^{n+1} - h_{j-1}^{n+1}) \\
&+ 2\,\Delta x r Q_j^{n+1}] \\
&+ (1-\theta)\,\Delta t[gA(h_{j+1}^n - h_{j-1}^n) \\
&+ 2\,\Delta x r Q_j^n] = 0 \\
2b\,\Delta x(h_{j+1}^{n+1} - h_{j+1}^n) + \theta\,\Delta t(&Q_{j+2}^{n+1} - Q_j^{n+1}) \\
&+ (1-\theta)\,\Delta t(Q_{j+2}^n - Q_j^n) = 0
\end{aligned} \tag{110}$$

After the substitution of the functions (109) into Eqs. (110), it follows that

$$\begin{aligned}
2\,\Delta x\bar{Q}(e^{-i\beta\Delta t} - 1) + \Delta t\bar{h}gAi[\theta e^{-i\beta\Delta t} + (1-\theta)]\sin\sigma\,\Delta x \\
+ 2r\,\Delta x\,\Delta t\bar{Q}[\theta e^{-i\beta\Delta t} + 1 - \theta] = 0 \\
2\,\Delta x b\bar{h}(e^{-i\beta\Delta t} - 1) + \Delta t\bar{Q}i[\theta e^{-i\beta\Delta t} + (1-\theta)]\sin\sigma\,\Delta x = 0
\end{aligned} \tag{111}$$

After the substitution of the functions (108) into Eqs. (107) the following relations are obtained for $\bar{h}$ and $\bar{Q}$:

$$-i\omega\bar{Q} + igAk\bar{h} + r\bar{Q} = 0 \qquad \text{and} \qquad -\omega b\bar{h} + k\bar{Q} = 0 \tag{112}$$

Solutions for $\bar{h}$ and $\bar{Q}$ can be determined if $k = (\omega/c)(1 + ri/\omega)^{1/2}$, in which $c = \pm(gA/b)^{1/2}$, is the propagation velocity of the physical wave (upriver and downriver). Application of Eqs. (112) shows that Eqs. (110) are identical in case resistance is negligible ($r = 0$). It follows that $\sigma = k$, and the complex quantity $\beta = p + iq$ is determined by

$$e^{-i\beta\Delta t} = \frac{2 - i(1-\theta)c(\sin k\,\Delta x)\,\Delta t\,\Delta x}{2 + i\theta c(\sin k\,\Delta x)\,\Delta t/\Delta x} \tag{113}$$

Then $e^{q\Delta t}$ equals the modulus of the right-hand member of (113), and $-p\,\Delta t$ is the argument. It appears that in each time step $\bar{h}$ and $\bar{Q}$ are multiplied by the factor $e^{q\Delta t}$, and the argument is modified by $-p\,\Delta t$. After a number of $T/\Delta t$ time steps a complete tidal period is passed. Then $\bar{h}$ and $\bar{Q}$ are modified by e^{qT}, and the argument of the numerical wave is changed by $-Tp$. The value of qT must be close to one, and the argument close to zero in case maximum accuracy of the finite-difference scheme is required. Such can be done by choosing the values of Δx and Δt in the proper way. For the case $\theta = 0$ (explicit scheme) and $\theta = 1$ (implicit scheme), formula (113) is simplified considerably.

In case of resistance ($r \neq 0$), Eqs. (111) are not identical after elimination of $\bar{h}$ and $\bar{Q}$ by means of Eqs. (112). Then it appears that $\sin \sigma \Delta x$ and $\exp(-i\beta \Delta t)$ can be calculated from these equations. In this case $\sigma \neq k$. For further details and practical evaluations, reference is made to Leendertse (1967). However, the treatment mentioned above is somewhat different. Moreover, in case of nonlinear tidal equations the propagation of the physical and numerical wave are distorted, and the theory of Section IV,C,1 must be applied for the determination of the distortion.

D. Tidal Computations in the Sea

1. *Introduction*

The theory of the Kelvin wave is explained in the next subsection because of the importance of this theory for getting an insight into the tidal motion along straight coastal lines and the existence of amphidromic points. The rest of this section is devoted to finite-difference methods.

Explicit and implicit finite-difference methods can also be distinguished in the numerical tidal computations in the sea. Various schemes of explicit methods are given in the literature such as by Hansen (1956), Fisher (1965), and Lauwerier and Damsté (1963). Heaps (1969) gave a review of various schemes. In his extensive paper he discussed the application of an explicit scheme to storm surges in the coastal waters of Great Britain and in the North Sea. Instead of the usual x, y coordinate system, he considered the terrestrial coordinate system (latitude and longitude) and the corresponding Proudman's equations of motion and continuity. Implicit systems were introduced by Leendertse (1967; see also Dronkers, 1969a). This method has been applied to the tidal motion in the North Sea and the coastal waters of the Netherlands.

In the following subsections the equations are given for explicit and implicit difference schemes when applied to simplified tidal equations. Section IV,D,4 discusses the computation of the convective terms in the complete tidal equations, and Section IV,D,5 the application of explicit and implicit schemes.

The implicit methods are more complicated than the explicit methods.

However, they have the advantage that conditions for the stability, which influence the size of the grid, need not be considered. The stability condition for the explicit method depends on the depth, the Coriolis coefficient, and the friction coefficient in relation to the time step (see, e.g., Heaps, 1969). In coastal waters in particular the variation in the depth may be very considerable due to the occurrence of shallows and gullies. Then, great care is necessary on the choice of the size of the grid and the time step with respect to the accuracy of the computation. This latter aspect applies to both methods. Leendertse (1967) has given in his thesis a discussion of this aspect by considering the relation between the mathematically generated wave in the computation and the physical wave. In case of one-dimensional tidal computations, see Section IV,C,3,g.

2. *Analytical Solutions of Simplified Tidal Equations in the Sea, the Kelvin Wave, and the Amphidromic Points*

In Section III,B,5, the tidal equations in the immediate neighborhood of a coastal line and the differential equations of the Kelvin wave in particular are presented. The equations for the Kelvin wave propagating in a channel between parallel coastal lines (Fig. 11) are determined. Solutions of

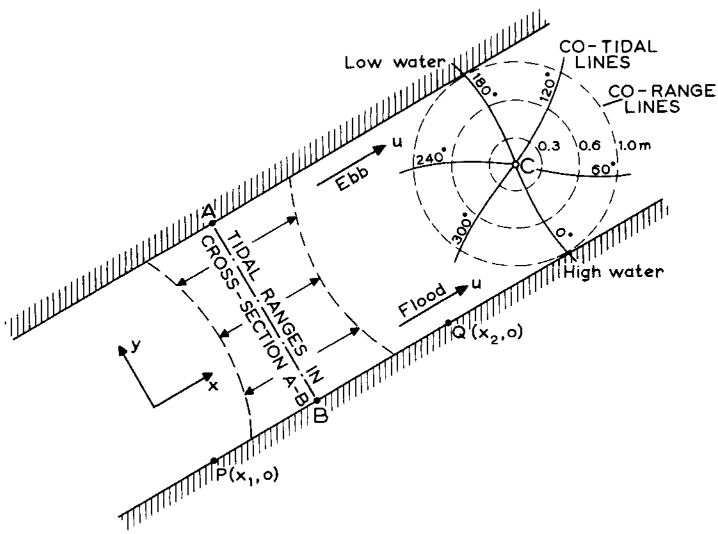

FIG. 11. The propagation of a Kelvin wave in a channel with parallel coastal lines and an amphidromic point at C. The change of the tidal range in a cross section AB due to the Coriolis force is also shown.

Eqs. (83) are obtained by the method of separation of variables. After elimination of the function $h(x, y, t)$ in the equations

$$\frac{\partial u}{\partial t} + ku = -g\frac{\partial h}{\partial x}, \qquad \Omega u = -g\frac{\partial h}{\partial y}, \qquad \text{and} \qquad a\frac{\partial u}{\partial x} = -\frac{\partial h}{\partial t} \tag{114}$$

the equations

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = ga\frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad \Omega\frac{\partial u}{\partial t} = ga\frac{\partial^2 u}{\partial x\, \partial y} \tag{115}$$

are obtained. Then solutions for $u(x, y, t)$ can be found having the form

$$u = f(t)p(x)q(y)$$

The functions f, p, and q can be determined after substitution of the function u in the differential equations. It appears that they must satisfy the following equations:

$$\frac{d^2 f}{dt^2} + k\frac{df}{dt} = c_1 f \tag{116a}$$

$$\frac{df}{dt} = c_2 f \tag{116b}$$

$$ga\frac{d^2 p}{dx^2} = c_1 p \tag{116c}$$

$$(ga)^{1/2}\frac{dp}{dx} = c_3 p \tag{116d}$$

$$(ga)^{1/2}\frac{dq}{dy} = c_4 q \tag{116e}$$

in which c_1, c_2, c_3, and c_4 are constants, which are not independent. From Eqs. (116a) and (116b), the following relation is found:

$$\left(\frac{c_2}{\Omega}\right)^2 + k\frac{c_2}{\Omega} = c_1$$

Similar relations follow from Eqs. (116c), (116d), and (116e):

$$c_3^2 = c_1 \qquad \text{and} \qquad c_3 c_4 = c_2$$

so that only one independent constant, e.g., c_1, occurs.

In case of a tidal wave, f is a harmonic function with frequency ω:

$$f(t) = d_1 e^{i\omega t}$$

in which d_1 is a constant. After substituting it in Eqs. (116a) and (116b), the following is obtained:

$$-\omega^2 + ik\omega = c_1; \qquad i\omega\Omega = c_2$$

Hence,

$$c_3 = \pm(-\omega^2 + ik\omega)^{1/2} \qquad \text{and} \qquad c_4 = \pm i\omega\Omega(-\omega^2 + ik\omega)^{-1/2}$$

Further Eqs. (116d) and (116e) can be solved for the functions $p(x)$, and $q(y)$. Then the function u can be written in a complex form:

$$u = Ce^{(\alpha x + \beta y) + i\omega t} \tag{117}$$

in which C is a complex constant, $\alpha = c_3/(ga)^{1/2}$, and $\beta = c_4/(ga)^{1/2}$. The function $h(x, y, t)$, after integration of equation $\partial h/\partial y = \Omega u$, becomes

$$h = \frac{\Omega}{\beta} u = \pm i\omega^{1/2}(ga)^{1/2}(-\omega + ik)^{1/2}u \tag{118}$$

The complex number $(-\omega + ik)^{1/2}$ can be written in the form $\pm(a^* + ib^*)$, in which

$$a^* = \tfrac{1}{2}(2)^{1/2}[-\omega + (\omega^2 + k^2)^{1/2}]^{1/2};$$
$$b^* = \tfrac{1}{2}(2)^{1/2}[\omega + (\omega^2 + k^2)^{1/2}]^{1/2}$$

The real functions h and u are determined from Eqs. (117) and (118) by taking the real parts of these equations. Then, h and u can be written in the form

$$h = h_1 \cos \omega t + h_2 \sin \omega t, \qquad u = \frac{1}{\Omega}\frac{\partial h}{\partial y}$$

in which h_1 has the form

$$\begin{aligned} h_1 = {} & e^{(a_1 x + a_2 y)}[A_1 \cos(b_1 x + b_2 y) + B_1 \sin(b_1 x + b_2 y)] \\ & + e^{-(a_1 x + a_2 y)}[C_1 \cos(b_1 x + b_2 y) + D_1 \sin(b_1 x + b_2 y)] \end{aligned} \tag{119}$$

A similar form exists for h_2, in which A_1 is replaced by $-A_1$, C_1 by $-D_1$, and D_1 by C_1. The four constants a_1, a_2, b_1, and b_2 are defined by

$$a_1 = \frac{\omega^{1/2}a^*}{(ga)^{1/2}}; \qquad b_1 = \frac{\omega^{1/2}b^*}{(ga)^{1/2}};$$
$$a_2 = \frac{\omega^{1/2}\Omega b^*}{(\omega^2 + k^2)(ga)^{1/2}}; \qquad b_2 = \frac{\omega^{1/2}\Omega a^*}{(\omega^2 + k^2)(ga)^{1/2}}$$

The four constants $A_1, \ldots, D_1$ are determined by the boundary conditions for h at two points $P(x_1, 0)$ and $Q(x_2, 0)$ (see Fig. 11).

It is a well-known fact that the tidal ranges increase in a line perpendicular to the coastline, which is to the right of the direction of propagation of the Kelvin wave (see Fig. 11). For an example, the tidal range at 10 km seaward of the Netherlands coast is 0.9 of that at the coast.

The amphidromic points (x^*, y^*) in the sea are defined such that the amplitude of the vertical tide at those points is zero. Hence, $h_1(x^*, y^*)$ and $h_2(x^*, y^*)$ must be zero. From Eq. (119) and a similar equation for h_2, it can be shown that such points may exist, when the following conditions are satisfied:

$$e^{2(a_1x+a_2y)} = \left[\frac{C_1^2 + D_1^2}{A_1^2 + B_1^2}\right]^{1/2}; \tan 2(b_1x + b_2y) = \frac{A_1D_1 + B_1C_1}{A_1C_1 - B_1D_1} \tag{120}$$

The velocity at an amphidromic point is not equal to zero, unless $\partial h/\partial y = 0$ at that point.

Dronkers (1964) has dealt with the combined Kelvin wave and Poincaré waves. The method described here for the determination of the Kelvin wave is different and more straightforward. The theory of the Kelvin wave can be applied to explain the amphidromic point in the southern part of the North Sea. The equations for the Poincaré waves cannot be derived by a method as described above (see Dronkers, 1964).

Bonnefille (1969) has made an extensive study of the influence of the Coriolis force on the tidal motion. He discussed the problem of reproduction in a physical model of the Coriolis force and described the reproduction of the tide on a rotating platform of 14-m diameter. The model has been applied to the tidal motion in the Bay of Saint-Malo (France). Furthermore tests were carried out for schematic basins. In the second part of his study the tidal oscillations in rotating rectangular basins were studied on a purely theoretical basis. This theoretical work is an extension of the work of Taylor (1921) (see also Section IV,A). The studies of van Dantzig and Lauwerier (1960–1961) dealt also with these phenomena extensively.

3. *Tidal Computations in the Sea by Finite-Difference Methods*

Explicit finite-difference schemes are often applied for tidal computations because the computational scheme is more simple than that of the implicit scheme, though for explicit schemes the time step is limited in connection with the stability. The stability condition is

$$\frac{\Delta t}{\Delta x}(2ga)^{1/2} < 1$$

in which a is the depth [compare with Eq. (89)].

Two explicit schemes are mentioned below. After that an implicit scheme will be given. Hansen (1956) was the first who applied a finite-difference scheme. His scheme will be given first. The convective terms are omitted in the finite-difference schemes represented in this section. Considerations on the finite-difference representation of the convective terms will be given in Subsection D,4.

The following simplified tidal equations will be considered [compare with Eqs. (38), (49), and (50)]

$$\frac{\partial u}{\partial t} = \Omega v - g\frac{\partial h}{\partial x} - \frac{g(u^2 + v^2)^{1/2}u}{C^2(a + h)} \tag{121}$$

$$\frac{\partial v}{\partial t} = -\Omega u - g\frac{\partial h}{\partial y} - \frac{g(u^2 + v^2)^{1/2}v}{C^2(a + h)} \tag{122}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial(a + h)u}{\partial x} - \frac{\partial(a + h)v}{\partial y} \tag{123}$$

a. FIRST EXPLICIT SCHEME (HANSEN). In Fig. 12 the notation for the grid is shown. The partial differential equations are replaced by the following finite-difference equations:

$$\begin{aligned} u_{m+1,n}^{t+\tau} = {} & (1 - 2R_u^{t-\tau})u_{m+1,n}^{t-\tau} + \frac{\Omega\tau}{2}(v_{m+2,n+1}^{t-\tau} + v_{m,n+1}^{t-\tau} \\ & + v_{m,n-1}^{t-\tau} + v_{m+2,n-1}^{t-\tau}) - \frac{g\tau}{k}(h_{m+2,n}^{t} - h_{m,n}^{t}) \end{aligned} \tag{124}$$

$$\begin{aligned} v_{m,n+1}^{t+\tau} = {} & (1 - 2R_v^{t-\tau}\tau)v_{m,n+1}^{t-\tau} - \frac{\Omega\tau}{2}(u_{m+1,n+2}^{t-\tau} + u_{m-1,n+2}^{t-\tau} \\ & + u_{m-1,n}^{t-\tau} + u_{m+1,n}^{t-\tau}) - \frac{g\tau}{k}(h_{m,n+2}^{t} - h_{m,n}^{t}) \end{aligned} \tag{125}$$

$$\begin{aligned} h_{m,n}^{t+2\tau} = {} & h_{m,n}^{t} - \frac{\tau}{k}[(u_{m+1,n}^{t+\tau} - u_{m-1,n}^{t+\tau} + v_{m,n+1}^{t+\tau} \\ & - v_{m,n-1}^{t+\tau})(a + h)_{m,n}^{t}] \end{aligned} \tag{126}$$

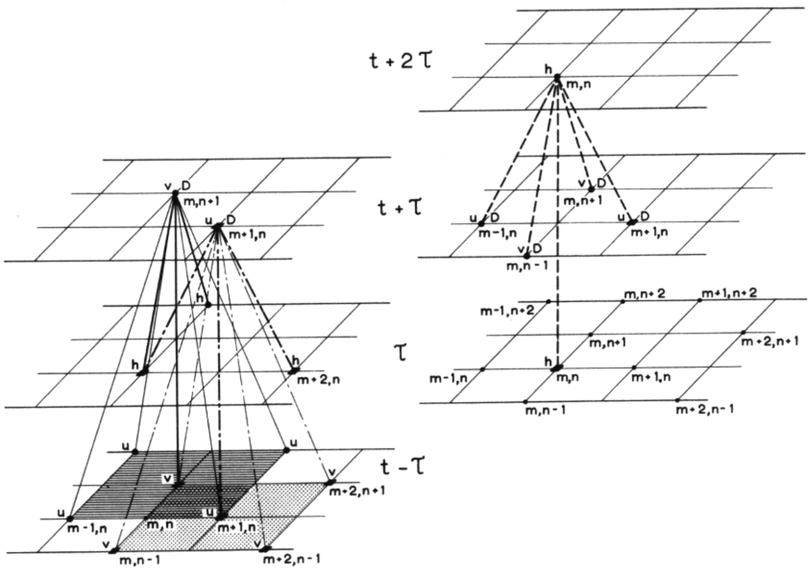

FIG. 12. Graphical representation of Hansen's two-dimensional explicit difference scheme.

in which

$$R_u = \frac{1}{C^2(a+h)_{m,n}}\left[u^2_{m+1,n} + \frac{1}{16}(v_{m+1,n+1} + v_{m-1,n+1} + v_{m-1,n-1} + v_{m+1,n-1})^2\right]^{1/2}_{t-\tau}$$

$$R_v = \frac{1}{C^2(a+h)_{m,n}}\left[\frac{1}{16}(u_{m+1,n+1} + u_{m-1,n+1} + u_{m-1,n-1} + u_{m+1,n-1})^2 + v^2_{m,n+1}\right]^{1/2}_{t-\tau}$$

In Fig. 12, a graphical representation is given of Hansen's scheme. This scheme has been applied by Sündermann (1966), Brettschneider (1967), and Ramming (1971).

b. SECOND EXPLICIT SCHEME. This explicit scheme, also mentioned by Dronkers (1964), has been applied by Lauwerier and Damsté (1963) and by Heaps (1969) who introduced some modifications. In this scheme the partial differential equations (121)–(123) are replaced by the following finite-difference equations:

$$u(t+\tau) = (1-\lambda)u(t) + \Omega v(t)\tau - g\tau D_1[h(t)] \tag{127}$$

$$v(t+\tau) = (1-\lambda)v(t) - \Omega u(t)\tau - g\tau D_2[h(t)] \tag{128}$$

$$h(t+\tau) = h(t) - \tau[a+h(t)]E_1(u(t)) - \tau[a+h(t)]E_2[v(t)] \tag{129}$$

in which $\lambda = g[u^2(t) + v^2(t)]^{1/2}/C^2[a+h(t)]$.

The grid is shown in Fig. 13; u and v are computed at the same point, and h at the intersection of the diagonals of the u, v points. Then $D_1(h)$, $D_2(h)$, $E_1(u)$, and $E_2(v)$ are defined as follows:

$$D_1[h(t)] = \frac{1}{4\,\Delta x}[h(x+\Delta x, y+\Delta y, t) - h(x-\Delta x, y+\Delta y, t) + h(x+\Delta x, y-\Delta y, t) - h(x-\Delta x, y-\Delta y, t)]$$

and a similar expression for $D_2(h(t))$ can be written by replacing x by y, Δx by Δy, etc.:

$$E_1[u(t)] = \frac{1}{4\,\Delta x}[u(x+2\,\Delta x, y+2\,\Delta y, t) - u(x, y+2\,\Delta y, t) + u(x+2\,\Delta x, y, t) - u(x, y, t)]$$

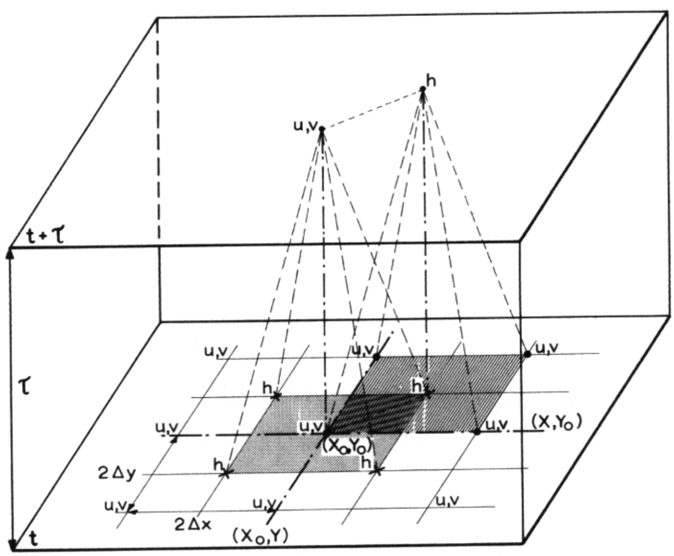

FIG. 13. Graphical representation of a modified two-dimensional explicit difference scheme applied by the Mathematical Centre, Amsterdam, and Heaps.

The expression for $E_2[v(t)]$ is obtained by replacing u by v because u and v are computed in the same grid points. Let h, u, and v be computed up to the time level $t = n\tau$. The computation of u, v, and h at the time level $t + \tau$ follows from Eqs. (127)–(129).

The grid points used in the computation are shown in Fig. 12. Equations (127)–(129) must be modified at the boundaries. Heaps (1969) discussed these modifications extensively. The scheme mentioned above can be modified in various ways such that it remains an explicit scheme. Heaps (1969) himself replaced Eq. (129) by

$$h(t + \tau) = h(t) - \tau[a + h(t)]E_1[u(t + \tau)] - \tau[a + h(t)]E_2[v(t + \tau)] \quad (130)$$

After the computation of $u(t + \tau)$ and $v(t + \tau)$ from Eqs. (127) and (128), and after introducing them in Eq. (130), $h(t + \tau)$ can be found. Another variation of the explicit scheme is that after the computation of the value of h in the $t + \tau$ plane from Eq. (129), the values in $D_1(h)$ and $D_2(h)$ can be replaced by the corresponding h values in the $t + \tau$ plane. Then, $u(t + \tau)$, $v(t + \tau)$ can be computed.

Fisher (1965) introduced a modification in the finite-difference representation of the Coriolis force by replacing $\Omega v(t)$ with $\frac{1}{2}\Omega[v(t) + v(t + \tau)]$ and Ωu with a corresponding expression. After that $u(t + \tau)$ and $v(t + \tau)$ can be computed from the modified equations (127) and (128). Then again an explicit scheme is obtained.

Considering Fig. 13, it is clear that the validity of the solution by Eqs. (127)–(129) is closely related to the theory of the characteristic cone, which is the extension in two dimensions of the theory of the characteristic curves for the one-dimensional tidal equations (see Daubert and Graffe, 1967; Dronkers, 1969a).

The finite-difference scheme described above has the disadvantage that the stability of the scheme is sensitive with respect to the Coriolis (geostrophic) terms.

In the schemes described above, forward differences are applied for the time differentials. In the interest of accuracy, centered space and time differences can also be applied. Such a scheme has been used by Gates (1966) for a numerical study of the wind-driven circulation in a homogeneous ocean, where three time levels are considered: $t + \tau$, t, and $t - \tau$. The time levels $t + \tau$ and $t - \tau$ are for the acceleration term. The pressure force is considered at time level t, the frictional dissipation mechanism is considered at time level $t - \tau$, and the convective terms are considered at time level t.

Heaps (1969) applied the explicit scheme extensively to the computation of the North Sea surges.

C. THE IMPLICIT SCHEME. The implicit scheme has been extensively described by Leendertse (1967). Here, a brief review of this implicit scheme for Eqs. (121)–(123) is given. For the notation of the grid net, see Fig. 14. The following notation is used in the equations:

$$\begin{aligned}
\bar{v}_{m,n} &= \tfrac{1}{4}[v_{m,n} + v_{m,n-1} + v_{m+1,n} + v_{m+1,n-1}] \\
\bar{a}_{m,n} &= \tfrac{1}{2}[a_{m,n} + a_{m,n-1} + h_{m,n} + h_{m+1,n}] \\
\bar{a}^{*}_{m,n} &= \tfrac{1}{2}[a_{m,n} + a_{m-1,n} + h_{m,n} + h_{m,n+1}] \\
\bar{u}_{m,n} &= \tfrac{1}{4}[u'_{m,n} + u'_{m-1,n} + u'_{m,n+1} + u'_{m-1,n+1}] \\
\bar{a}'_{m,n} &= \bar{a}_{m,n} \qquad \text{and} \qquad h \text{ is replaced by } h'
\end{aligned} \tag{131}$$

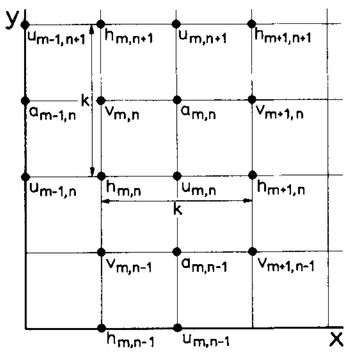

FIG. 14. Notation for a two-dimensional implicit difference scheme.

Let again the functions u, v, and h be computed at time $t = n\tau$. Then at time $t + \frac{1}{2}\tau$, u' and h' are found by using an implicit scheme, while v' is computed by an explicit scheme from u, v, and h. Subsequently, v'' and h'' at time level $t + \tau$ are found by an implicit scheme and u'' is determined explicitly from u', v', and h'. The implicit schemes are analogous to those described in Section IV,C,3,c for the one-dimensional case.

The equations for u' and h' follow from the finite-difference equations obtained from Eqs. (121) and (123) (for notation see Fig. 14). They are

$$u'_{m,n} = u_{m,n} + \frac{\tau\Omega}{2}\bar{v}_{m,n} - \frac{g\tau}{2k}(h'_{m+1,n} - h'_{m,n})$$
$$- \tau g \frac{(u^2_{m,n} + \bar{v}^2_{m,n})^{1/2}}{2C^2_{m,n}\bar{a}_{m,n}} u'_{m,n} \tag{132}$$

$$h'_{m,n} = h_{m,n} - \frac{\tau}{2k}\left(\bar{a}_{m,n}u'_{m,n} - \bar{a}_{m-1,n}u'_{m-1,n}\right)$$

$$- \frac{\tau}{2k}\left(\bar{a}^*_{m,n}v_{m,n} - \bar{a}^*_{m,n-1}v_{m,n-1}\right) \tag{133}$$

In Fig. 15(a), a graphical representation of Eqs. (132) and (133) is shown.

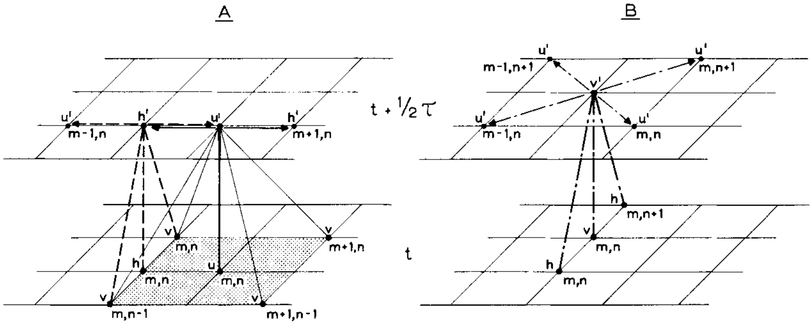

FIG. 15. Graphical representation of a two-dimensional implicit difference scheme for time steps t and $t + \frac{1}{2}\tau$, applied by Leendertse: (A) representations of formulas (132) and (133); (B) representation of formula (134).

Let n have a fixed value, and m pass from 1 to M. The boundary conditions are given at locations $(1, n)$ and (M, n). They determine $h'_{1,n}$ and $h'_{M,n}$, or $u'_{M,n}$. Then for each value of n, a set of $2M$ equations (132) and (133) must be solved for the underlined unknowns. This solution is similar to that described in the implicit scheme in Section IV,C,3,c where the double sweep method is used. When n passes from 1 to N, N sets of $2M$ equations must be solved. Then $u'_{m,n}$ and $h'_{m,n}$ are determined for all values of m and n at time level $t + \frac{1}{2}\tau$. After that the values of $v'_{m,n}$ are found from

$$v'_{m,n} = v_{m,n} - \frac{\tau\Omega}{2}\bar{u}'_{m,n} - \frac{g\tau}{2k}\left(h_{m,n+1} - h_{m,n}\right)$$

$$- \tau g \frac{[(\bar{u}'_{m,n})^2 + v^2_{m,n}]^{1/2}}{2C^2_{m,n}(\bar{a}^*_{m,n})'} v'_{m,n} \tag{134}$$

In Fig. 15(b), a graphical representation of Eq. (134) is shown. Subsequently, the values of v'' and h'' are determined at time level $t + \tau$ by a similar implicit difference scheme, which is applied to the y direction and in which n varies from 1 to N. Finally, Eqs. (122) and (123) are replaced by

$$v''_{m,n} = v'_{m,n} - \frac{\tau\Omega}{2}\bar{u}'_{m,n} - \frac{g\tau}{2k}(h''_{m,n+1} - h''_{m,n}) - \tau g \frac{[(\bar{u}'_{m,n})^2 + (v'_{m,n})^2]^{1/2}}{2C^2_{m,n}(\bar{a}^*_{m,n})'} v''_{m,n} \tag{135}$$

$$h''_{m,n} = h'_{m,n} - \frac{\tau}{2k}(\bar{a}'_{m,n}u'_{m,n} - \bar{a}'_{m-1,n}u'_{m-1,n}) - \frac{\tau}{2k}[\bar{a}'_{m,n}v''_{m,n} - (\bar{a}^*_{m,n-1})'v''_{m,n-1}] \tag{136}$$

The computation of the underlined values occurs in the same way as described above for the u' and h' values. In this case the boundary conditions are known at locations $(m, 1)$, and (m, N), e.g., $h''_{m,1}$ and $h''_{m,N}$, or $v''_{m,N}$. Moreover, when m passes from 1 to M, M sets of $2N$ equations must be solved consequently. Again, each set of $2N$ equations is solved by the double sweep method.

Finally, the u'' values in the $t + \tau$ level are determined explicitly from time level $t + \frac{1}{2}\tau$ to $t + \tau$ by

$$u''_{m,n} = u'_{m,n} + \frac{\tau\Omega}{2}\bar{v}''_{m,n} - \frac{g\tau}{2k}(h'_{m+1,n} - h'_{m,n}) - \tau g \frac{[u'^2_{m,n} + (\bar{v}''_{m,n})^2]^{1/2}}{2C^2_{m,n}\bar{a}'_{m,n}} u''_{m,n} \tag{137}$$

For the time steps $t + \frac{1}{2}\tau$ and $t + \tau$, Eqs. (135)–(137) can be graphically represented in the same way as shown in Fig. 15 for the time steps t and $t + \frac{1}{2}\tau$.

In the coefficients of the friction terms of Eqs. (134) and (137), u and v are taken at different time levels. From a physical point of view it is recommendable to take both coefficients at the same time level. Computation shows that the results do not change considerably by this assumption.

The set of equations which are analogous to Eqs. (93) can be solved by the double sweep method. Therefore, Eqs. (132) and (133) are rewritten in the form

$$\begin{aligned} u'_{m,n} + a_{m,n}(h'_{m,n} - h'_{m+1,n}) &= A_{m,n} \\ b_{m,n}u'_{m,n} + c_{m,n}u'_{m-1,n} + h'_{m,n} &= B_{m,n} \end{aligned} \tag{138}$$

in which the coefficients and $A_{m,n}$ and $B_{m,n}$ depend on the values of u, v, and h at $t = n\tau$. Then, $2M - 3$ equations for $2M - 3$ unknowns are obtained provided $h_{1,n}$ and $h_{M,n}$ are known from the boundary conditions.

In a way analogous to that discussed in Section IV,C,3,c, one of the unknowns in Eq. (138) can be eliminated, so that Eqs. (138) can be rewritten in the form

$$
\begin{aligned}
h'_{m,n} &= -P_{m,n}u'_{m,n} + Q_{m,n} \\
u'_{m-1,n} &= -R_{m-1}h'_{m,n} + S_{m-1,n}
\end{aligned}
\tag{139}
$$

These equations are similar to Eqs. (95). In Eqs. (138) and (139), however, $h'_{1,m}$ is not considered separately; $Q_{m,n}$ and $S_{m-1,n}$ contain $h'_{1,m}$. The recurrence equations for the coefficients are also similar to those of Eqs. (99). After the application of the first sweep discussed in Section IV,C,3,c, the h' and u' values at the grid points are obtained by the application of the second sweep.

In Fig. 16, a comparison is shown between the computed and measured velocity vectors seaward from the mouth of the Haringvliet, one of the

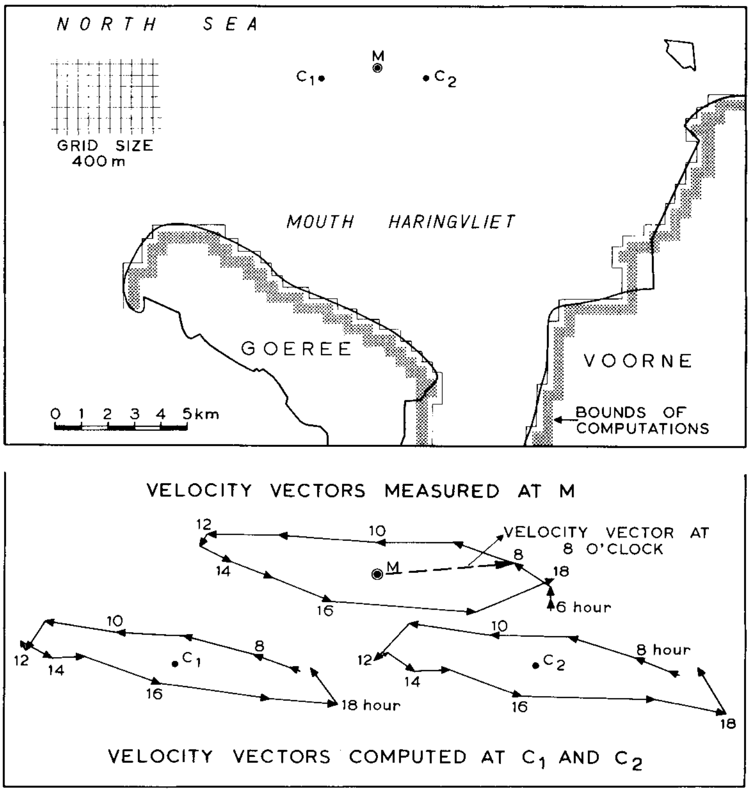

FIG. 16. Comparison of measured and computed velocity vectors in the mouth of Haringvliet, The Netherlands.

estuaries in the southwest part of The Netherlands, which is now enclosed from the sea. The implicit method is applied for these computations. The size of the squares of the grid is $\Delta x = \Delta y = 400$ m. The time step is 10 or 5 min. Further details about these computations can be found in Dronkers (1969a, 1970). The grid size is very small in comparison with the grid used by Heaps (1969) for the North Sea. The size varies between 16 and 23 miles. He applied the explicit methods described in Section IV,D,3,a. The dimensions of the grid are still much larger, e.g., $\Delta x = 160$ km, in computations for the oceans (Gates, 1966).

In practical applications it occurs that the tide in a coastal region and an estuary must be studied together. Dronkers (1972) considered the combined system of sea and river. He applied an implicit scheme dealt with in Sections IV,C,3,c and IV,D,3,c. Then the way of schematization of the grids in the transition zone is the main problem.

4. *Representation of the Convective Terms in the Tidal Equations by Finite Differences*

In the preceding subsection the finite-difference schemes for the linear or quasi-linear equations of the tidal motion in the sea have been considered. The expression *quasi-linear* refers to the approximation of the quadratic friction terms in the equation of motion and the space terms in the equation of continuity. Thus, one factor of these terms is considered at time level $t + \tau$, and the other factors at time level t.

In the deeper parts of the sea it is usually permitted to leave the convective terms out of consideration. Computations in the sea are mainly carried out by the application of quasi-linear equations. Near the coast the relative importance of the convective terms, which has been discussed in Sections III,B,2–4, increases mainly due to the variation in the depth and the shape of the coastline. The convective terms determine the circulation according to Eq. (67). The eddy systems, caused by coastal structures, e.g., harbor dams perpendicular to the coastline, are also described by the convective terms. However, they cannot describe the eddy streets because the derivatives of the velocities in the small eddies are undefined. The large eddy systems, which may occur behind harbor dams, are determined by the convective terms.

The introduction of the convective terms to the tidal equations (121)–(123) has two consequences: First, the boundary conditions are more complicated because the vertical tide and the velocity vectors must be known at the open sea boundary (see section IV,B). Often the measurement of the velocity vectors is a difficult affair from a practical point of view. It is recommended to choose the boundary in the sea, where the convective terms have small values. Second, the finite-difference representation of the convective terms and the solution of the difference equations may also give complications with respect to the stability of the solution and the accuracy.

Two different effects that may cause an instability in the computation can be distinguished: the "time" instability for error growth, which may occur in explicit schemes (see Section IV,C,3,c); and the "space" instability, which may occur in explicit and implicit schemes for two-dimensional flow in the case that the convective terms are important in the two-dimensional equations of motion. They are found in two-dimensional tidal computations applied to an estuary (see Dronkers, 1972). It appeared that this kind of instability does not occur when the convective terms in the tidal equations are left out of consideration. Space instabilities are discussed by Grammeltvedt (1969) for a barotropic fluid, and Kagan (1970) for tidal equations in case of explicit schemes, e.g., Hansen's scheme. Space instabilities can be avoided by introducing separate artificial viscosity terms $\varepsilon\, \partial^2 u/\partial x^2$ and $\varepsilon\, \partial^2 u/\partial y^2$ into the two-dimensional equations. This was found by Leendertse (1968). It must be remarked that the introduction of these viscosity terms into the tidal equations is different from the way in which the dissipative terms are introduced into the Lax–Wendroff scheme for describing the motion of a discontinuity (see section IV,C,3,e).

Further points of discussion are the time levels of the factors in the convective terms because the difference equations must remain linear with respect to the unknowns in the highest time level. The factors u and v in the terms $u\, \partial u/\partial x$, $v\, \partial u/\partial x$, etc. can be chosen at a higher time level than the derivatives. Thus, one may consider the convective terms in Eqs. (49) and (50) in the finite-difference schemes:

$$u(t+\tau)\left(\frac{\partial u}{\partial x}\right)_t \qquad \text{and} \qquad v(t+\tau)\left(\frac{\partial u}{\partial x}\right)_t, \qquad \text{etc.}$$

It is also possible to take both factors at the lower time level, $u(t)(\partial u/\partial x)_t$, $v(t)(\partial u/\partial x)$, etc.

However, both factors cannot be taken at the time level $t + \tau$. This is possible only for the factor u (or v). Otherwise, it is not possible to solve the system of simultaneous equations given in Section IV,D,3,c with additional convective terms by the double sweep method. Therefore, it is often recommended that the convective terms be considered completely for time level t so that these terms are known from the former computation. For the two-dimensional explicit scheme, mentioned in Section IV,D,3,a, the same conclusion holds when the convective terms are introduced. From the practical point of view, it appears from computations that the results of the computations are not much influenced by the choice of the time levels t or $t + \tau$ of the factors in the convective terms.

Leendertse (1967) introduced finite-difference representations for the convective terms in his method. They are not completely mentioned here, except the x component of the convective terms for the time step t and $t + \frac{1}{2}\tau$ (see Section IV,D,3,c):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \rightarrow u(t + \tfrac{1}{2}\tau)\left(\frac{\partial u}{\partial x}\right)_t + \bar{v}(t)\left(\frac{\partial u}{\partial y}\right)_t$$

in which $\bar{v}(t)$ is defined in Eq. (131).

The finite-difference representations of the space derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$, and $\partial v/\partial y$ in the convective terms and the size of the grid may have a great influence on the accuracy of the computation in case these factors vary considerably with distance. This is clear when the convective terms are written in the form given in Section III,B,1. The x components are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -v \operatorname{rot} \mathbf{v} + \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2)$$

The term $-v \operatorname{rot} \mathbf{v}$ is the x component of the centrifugal acceleration V^2/r, in which r is the radius of the circle of curvature at a point of a streamline (Section III,B,4), provided the acceleration along the streamline can be neglected. A comparison can be made with the friction term gV^2/C^2a in which C is the coefficient of Chézy and a is the depth. Let $V = 1$ m/sec, $C = 50$ m$^{1/2}$ sec, and $a = 10$ m. Then, the friction term equals 4×10^{-4}. If the radius r equals 2500 m, the centrifugal term has the same value. They occur in tidal rivers, or coastlines, with sharp bends. In such cases, a sufficiently fine grid that determines the centrifugal forces accurately must be used.

In his recent computations, Leendertse applied the inertia terms of motion in the form of the balance of momentum (see Section III,A,2). He adds the left-hand member of the equation of continuity (38) divided by $a + h$ to the left-hand member of Eqs. (49) and (50). Then he obtained for the left-hand member of Eq. (49)

$$\frac{1}{a + h}\left[\frac{\partial(a + h)u}{\partial t} + \frac{\partial(a + h)uu}{\partial x} + \frac{\partial(a + h)v\bar{u}}{\partial y}\right]$$

An analogous expression is found for the left-hand member of Eq. (50). Then the equations of continuity and forces denote balances, respectively, of water and of momentum.

5. *Discussion on the Use of Explicit or Implicit Schemes*

Many variations in the finite-difference schemes for the one- and two-dimensional equations can be found in the literature. The criteria for the choice of a finite-difference scheme must be based on the measure of stability and the accuracy of the solution of the scheme in comparison with the physical solution of the differential equations. The study of these criteria asks for a very extensive mathematical treatment. The stability criteria are often discussed, but mathematical criteria for the accuracy of the application of a finite-difference scheme are usually not found in the literature. At best, it is discussed by examples of numerical computation for particular problems.

Leendertse (1967) has given a more detailed discussion on the accuracy of the implicit difference scheme by comparing a mathematical wave, which is a solution of the finite-difference equations, with the physical wave, which is the solution of the differential equations. Such a comparison can only be obtained for linearized differential equations (see Section IV,C,3,g). For nonlinear tidal equations, it is not possible to obtain a solution in mathematical terms. For the one-dimensional nonlinear tidal equations, the best comparison can be made with the solution by the characteristic method (see Liggett and Woolhiser, 1967). However it is very difficult to obtain such a solution for the two-dimensional tidal equations, unless they are simplified. Also the schematization of profiles and depths influences the results of the computation. From a practical point of view, a comparison can be made between the measurements of the vertical tidal motion and the velocities in nature. The accuracy of these measurements is then a point for discussion. It is in general not possible to decide from this point of view about the accuracy of the various schemes.

The explicit schemes are simpler for application than the implicit schemes. However, the stability of the latter schemes is better than that of the explicit schemes. The stability criteria for explicit schemes depend on depth, friction, and the Coriolis force (see Heaps, 1969). In shallow waters, friction is usually of more importance than the Coriolis force, and moreover, considerable variation in the depth occurs. Then the implicit schemes have advantages from the point of view of the stability with respect to the time step. The time step depends on the size of the grid in case of explicit schemes and on the required accuracy of the computation. Implicit schemes are more independent of the size of the grid. Then, the accuracy of the computation is the most important criterion. Computations have revealed that the size of the grid for the implicit method can be several times larger than that for the explicit method, e.g., four times or more, depending on the particular problem. In case space instabilities occur, due to the convective terms, it is necessary to include artificial viscosity terms in the tidal equations (see the previous subsection).

Dronkers (1972) also compared tidal studies by means of tidal computations and by means of physical models. Furthermore, a discussion was given about modifications in the friction coefficients C which depend on the schematization of the tidal regions for tidal computations.

6. *Review of the Purpose of Tidal Computations*

Besides being interesting from the theoretical point of view, tidal computations are important for the prediction of tides in connection with navigation and for the execution of technical projects. Projects include land reclamation, safeguarding low countries from flooding by storm surges, enclosure of river arms or estuaries, building of new shipping channels, utilization of the energy of tides, etc.

The new developments in tidal computations by means of the electronic computer have increased the applications very considerably. Recently the pollution of water has become a very serious problem also in tidal regions. Therefore tidal computations are now carried out in combination with computations for the mass transport of the pollution. Then a separate equation or set of equations, which describe the mass balance of the pollution, must be added to the tidal equations. The equations for the pollution have, however, different mathematical aspects from those of the tidal equations because they have a parabolic structure. A recent publication of the Environmental Protection Agency and Water Quality Office (1971) deals with various techniques of water quality modeling in combination with tidal computations.

The computation of the salinity intrusion into rivers is an analogous problem if the salinity is completely mixed in the cross sections of the river. The vertical distribution of the velocity is modified in case of partial or completely stratified flow. Then equations must be added for the computation of the velocities in the vertical. To date this problem has not been solved in a satisfactory way, except for very simplified cases. For instance, approximate solutions are available for the determination of the salt intrusion over a tidal period in rivers that have partially mixed salinity distributions.

References

Abbott, M. B., and Ionescu, F. (1967). On the numerical computation of nearly horizontal flows. *J. Hydraul. Res.* **5**, 97–117.

Airy, G. B. (1845). "Tides and Waves." Encycl. Metropolitana, London.

BENDAT, J. S., and PIERSOL, A. G. (1971). "Random Data: Analysis and Measurement Procedures." Wiley (Interscience), New York.

BIRD, R. B., STEWART, W. E., and LIGHTFOOT, E. N. (1960). "Transport Phenomena." Wiley, New York.

BLACKMAN, R. B., and TUCKEY, J. W. (1959). "The Measurement of Power." Dover, New York.

BONNEFILLE, R. (1969). Contribution théorique et experimentale à l'étude du regime des marées. (Theoretical and experimental contribution on the study of the tidal processes.) *Bull. Dir. Etud. Rech., Elec. Fr.* **A1**, 1–352.

BOOY, N. (1971). "Stability and Accuracy of the Cherie Program." Univ. of Technology, Delft.

BRETTSCHNEIDER, G. (1967). Anwendung des hydrodynamisch-numerischen Verfahrens zur Ermittlung der M_2-Mitschwingungsgezeit der Nordsee. *Mitt. Inst. Meeresk. Univ. Hamburg* **VII**, 1–65.

CARTWRIGHT, D. E. (1967). Some further results of the response method of tidal analysis. *Proc. Symp. Tides, UNESCO, Int. Hydrogr. Bur., Monaco* pp. 195–202.

CARTWRIGHT, D. E. (1968). A unified analysis of tides and surges round North and East Britain. *Phil. Trans. Roy. Soc. London, Ser. A* **263**, 1–55.

CARTWRIGHT, D. E. (1971). On low frequency variations in sea level and the radiational tide. *Rep. Symp. Coastal Geod., Inst. Astron. Phys. Geod., Technical Univ., Munich, 1970.*

CARTWRIGHT, D. E., and TAYLOR, R. J. (1971). New computations of the tide-generating potential. *Geophys. J. Roy. Astron. Soc.* **23**, 45–74.

CHOW, V. T. (1959). "Open-Channel Hydraulics." McGraw-Hill, New York.

CHOW, V. T. (1961). Open channel flows. *In* "Handbook of Fluid Mechanics" (V. L. Streeter, ed.), Vol. 24, pp. 1–50. McGraw-Hill, New York.

DARWIN, G. H. (1883–1886). "Reports of a Committee for the Harmonic Analysis." Brit. Ass. Advan. Sci., London.

DAUBERT, A., and GRAFFE, O. (1967). Modèle mathématique de propagation de marée dans une zone litterale. *Proc. Congr. Int. Ass. Hydraul. Res., 12th, Fort Collins, Colo.* **4**, 307–317.

DEFANT, A. (1960). "Physical Oceanography," Vol. 2. Pergamon, Oxford. (An extensive bibliography on physical oceanographical studies is given in this volume.)

DOODSON, A. T. (1921). The harmonic development of the tide-generating potential. *Proc. Roy. Soc., Ser. A* **100**, 305–329.

DOODSON, A. T. (1927). The analysis of tidal observations. *Phil. Trans. Roy. Soc. London, Ser. A* **227**, 223–279.

DOODSON, A. T. (1957). The analysis and prediction of tides in shallow water. *Int. Hydrogr. Rev.* **34**, 5–46.

DOODSON, A. T., and WARBURG, H. D. (1941). "Admiralty Manual of Tides," H.M. Stationery Office, London.

DRONKERS, J. J. (1964). "Tidal Computations in Rivers and Coastal Waters." North-Holland Publ., Amsterdam.

DRONKERS, J. J. (1969a). Tidal computations for rivers, coastal areas and seas. *J. Hydraul. Div., Proc. Amer. Soc. Civil Eng.* **95 (HY1)**, 29–77.

DRONKERS, J. J. (1969b). Some practical aspects of tidal computations. *Proc. Congr. Int. Ass. Hydraul. Res., 13th, Tokyo* **3**, 11–17.

DRONKERS, J. J. (1970). Research for the coastal area of the delta region of The Netherlands. *Proc. Int. Conf. Coastal Eng., 12th, Washington, D.C.* **3**, 1783–1801.

DRONKERS, J. J. (1972). The schematization for tidal computations in case of variable bottom shape. *Proc. Int. Conf. Coastal Eng., 13th, Vancouver, B.C.*

DRONKERS, J. J. (1973). Considerations on the diffusivity of salt in a tidal river. *Proc. Congr., Int. Ass. Hydraul. Res., 15th, Istanbul.*

DRONKERS, J. J., and SCHÖNFELD, J. C. (1954). Tidal computations in shallow water. *J. Hydraul. Div., Proc. Amer. Soc. Civil Eng.* **81**, 1–49.

EKMAN, V. W. (1905). On the influence of the earth's rotation on ocean currents. *Ark. Mat., Astron. Fys.* **2**, 1–52.

ENVIRONMENTAL PROTECTION AGENCY, WATER QUALITY OFFICE (1971). "Estuarine Modeling: An Assessment." Environmental Protection Agency, Washington, D.C.

FISHER, G. (1965). A survey of finite difference approximations to the primitive equations. *Mon. Weather Rev.* **93**, 1–46.

GATES, W. L. (1966). "A Numerical Study of the Wind-Driven Transient Circulation in a Homogeneous Ocean," pp. 1–74. Rand Corp. and Univ. of California, Los Angeles.

GOODWIN, R. T. (1961). "Modern Computing Methods," Notes on Applied Science, No. 16. H.M. Stationery Office, London.

GRAMMELTVEDT, A. (1969). A survey of finite-difference schemes for the primitive equations for a barotropic fluid. *Mon. Weather Rev.* **97**, 384–404.

HANSEN, W. (1956). Theorie zur Errechnung des Wasserstandes und der Strömungen in Randmeeren und Anwendungen. *Tellus* **8**, 287–300.

HANSEN, W. (1961). Hydrodynamical methods applied to oceanographic problems. *Proc. Symp. Math.-Hydraul. Methods Phys. Oceanogr., Hamburg* pp. 25–34.

HARLEMAN, D. R. F., and LEE, C. H. (1969). The computations of tides and currents in estuaries and canals. *Mass. Inst. Technol., Hydraul. Lab., Tech. Bull.* **16**.

HEAPS, N. S. (1969). A two dimensional numerical sea model. *Phil. Trans. Roy. Soc. London, Ser. A* **265**, 93–137.

HOBSON, E. W. (1965). "The Theory of Spherical and Ellipsoidal Harmonics," 2nd Reprint. Chelsea, Bronx, New York.

HOUGH, S. S. (1897–1898). On the application of harmonic analysis to the dynamical theory of the tides, Parts I and II. *Phil. Trans. Roy. Soc. London, Ser. A* **189**, 201-257; **191**, 139–185.

IPPEN, A. T. (1966). "Estuary and Coast Line Hydrodynamics." McGraw-Hill, New York.

KAGAN, B. A. (1970). Properties of certain difference schemes used in the numerical solution of the equations for the tidal motion. *Izv. Acad. Sci., USSR Atmos. Oceanic Phys.* **6**, 707–714.

KAMPHUIS, J. W. (1970). Mathematical tidal study of the St. Lawrence river. *J. Hydraul. Div., Proc. Amer. Soc. Civil Eng.* **96(HY3)**, 643–664.

KINSMAN, B. (1965). "Wind Waves." Prentice-Hall, Englewood Cliffs, New Jersey.

LAMB, H. (1932). "Hydrodynamics," 6th Ed. Cambridge Univ. Press, London and New York.

LAPLACE, P. S. (1799). "Traité de la Mécanique Céleste," 5 Vols. L'imprimerie de Crapelet chez J. B. M. Duprat, Paris.

LAUWERIER, H. A., and DAMSTÉ, B. R. (1963). The North Sea problem, VIII. A numerical treatment. *Indagationes Math.* **25**, 167–184. Koninklijke Nederl. Akademie van Wetenschappen, Amsterdam.

LAX, P. D., and WENDROFF, B. (1960). Systems of conservation laws. *Communication on Pure and Appl. Math.*, **13**, 217–237.

LEENDERTSE, J. J. (1967). "Aspects of the Computational Model for the Long Period Wave Propagation." Rand Memo., Santa Monica, California; Ph.D. Thesis, Technical Univ., Delft.

LEENDERTSE, J. J. (1968). Use of a computational model for two-dimensional tidal flow. *Proc. Coastal Eng. Congr., 11th, London* Ch. 90, pp. 1403–1420.

LIGGETT, J. A., and WOOLHISER, D. A. (1967). Difference solutions of the shallow-water equation. *J. Eng. Mech. Div., Proc. Amer. Soc. Civil Eng.* **93(EM2)**, 39–71.

MORSE, P. M., and FESHBACH, H. (1953). "Methods of Theoretical Physics." McGraw-Hill, New York.

MUNK, W. H., and CARTWRIGHT, D. E. (1966). Time spectroscopy and prediction. *Phil. Trans. Roy. Soc. London, Ser. A* **259**, 533–581.

NEUMANN, G., and PIERSON, W. J. (1966). "Principles of Physical Oceanography." Prentice-Hall, Englewood Cliffs, New Jersey.

PLOEG, J., and KAMPHUIS, J. W. (1968). Comprehensive tidal study of the St. Lawrence River. *Proc. Int. Conf. Coastal Eng., 11th, London* Ch. 91, pp. 1421–1435.

POINCARÉ, H. (1910). "Leçons de Mécanique Céleste. Vol. 3: Théorie des Marées." Gauthier-Villars, Paris.

PROUDMAN, J. (1913). On some cases of tidal motion on rotating sheets of water. *Proc. London Math. Soc.* **35**, 453–473.

PROUDMAN, J. (1935). Tides in oceans bounded by meridians, Part. I. *Phil. Trans. Roy. Soc. London, Ser. A* **235**, 237–289.

RAMMING, H. G. (1971). Investigation of motion processes in shallow water areas and estuaries. *Rep. Symp. Coastal Geod., Inst. Astron. Phys. Geod., Technical Univ., Munich, 1970* pp. 439–452.

Report on the Symposium on Coastal Geodesy (1971). *Inst. Astron. Phys. Geod., Technical Univ., Munich, 1970*, 642 pp.

REYNOLDS, O. (1883). An experimental investigation on the circumstances which determine whether the motion of water will be direct or sinuous and of the law of resistance in parallel channels. *Phil. Trans. Roy. Soc. London* **179**, 935–982.

RICHTMEYER, R. D., and MORTON, K. W. (1967). "Difference Methods for Initial-value Problems." Wiley (Interscience), New York.

ROSSITER, J. R. (1962). Long term variations in sea level. *In* "The Sea" (M. N. Hill, ed.), Vol. 1, Wiley (Interscience), New York.

SCHÖNFELD, J. C. (1951). "Propagation of Tides and Similar Waves." Ph.D. Thesis, Technical Univ. of Delft.

SCHUREMAN, P. (1958). "Manual of Harmonic Analysis and Prediction of Tides." *U.S. Coast Geodet. Surv., Publ.* **98**.

SHIPLEY, A. M. (1967). Recent developments in tidal analysis in South Africa. *Proc. Symp. Tides, UNESCO, Int. Hydrogr. Bur., Monaco* pp. 59–74.

STROBAND, H. J. (1971). "Differentie Methoden voor Eendimensionale Getijberekeningen. (Finite-Difference Methods for One-Dimensional Tidal Computations.)" Rep. Hydraul. Branch, Delta Works, The Hague.

SÜNDERMANN, J. (1966). Ein Vergleich zwischen der analytischen und der numerischen Berechnung winderzeugter Strömungen und Wasserstände in einem Modellmeer mit Anwendungen auf die Nordsee. *Mitt. Inst. Meeresk. Univ. Hamburg* **IV**, 1–81.

TAYLOR, G. I. (1919). Tidal friction in the Irish Sea. *Phil. Trans. Roy. Soc. London, Ser. A* **220**, 1–33.

TAYLOR, G. I. (1921). Tidal oscillations in gulfs and rectangular basins. *Proc. London Math. Soc.* **20**, 148–181.

THOMSON, W. (Lord Kelvin) (1868–1876). "Reports of a Committee for the Harmonic Analysis of Tidal Observations." Brit. Ass. Advan. Sci., London.

THOMSON, W. (Lord Kelvin) (1875). On an alleged error in Laplace's theory of tides (note on the oscillations of the first species in Laplace's theory of the tides). *Phil. Mag.* **50**, 227–342.

THOMSON, W. (Lord Kelvin) (1879). On Gravitational oscillations of rotating water. *Proc. Roy. Soc. Edinburgh* **10**, 92–100.

VAN DANTZIG, D., and LAUWERIER, H. A. (1960–1961). The North Sea problem, I–VII. *Indagationes Math.* **22**, 170–180, 266–290, 423–438; **23**, 123–140, 418–431. Koninklijke Nederl. Akademie van Wetenschappen, Amsterdam.

VAN DE KREEKE, J. (1971). "Tide Induced Mass Transport in Shallow Lagoons." Tech. Rep. No. 8. Univ. of Florida, Gainesville.

VAN ETTE, A. C. M., and SCHOEMAKER, H. J. (1967). Harmonic analysis of tides-essential features and disturbing influences. *Proc. Symp. Tides, UNESCO, Int. Hydrogr. Bur., Monaco* pp. 79–107.

VLIEGENTHART, A. C. (1968). "Dissipative Difference Schemes for Shallow Water Equations," pp. 1–17. Rep. Math. Inst., Technical Univ. of Delft.

VREUGDENHIL, C. B. (1971). "Computational Methods for Channel Flow," pp. 1–44. Rep. Hydraul. Lab., Technical Univ. of Delft.

WHEWELL, W. (1833). Essay towards a first approximation to a map of cotidal lines. *Phil. Trans. Roy. Soc. London* Part 1, pp. 147–236.

WHEWELL, W. (1833–1836). Researches on the Tides (First–Sixth Series). *Phil. Trans. Roy. Soc. London* **1833**, **1834**, **1835**, **1836**.

WHITTAKER, E. T., and WATSON, G. W. (1927). "Modern Analysis." Cambridge Univ. Press, London and New York.

ZETLER, B. D. (1967). Computer applications to tide and current analysis in the Coast and Geodetic Survey. *Proc. Symp. Tides, UNESCO, Int. Hydrogr. Bur., Monaco*, pp. 75–76; Shallow water tide predictions. pp. 163–166.

ZETLER, B. D., and CUMMINGS, R. A. (1967). A harmonic method for predictions shallow-water tides. *J. Mar. Res.* **25**, 103–114.