Student-T Process Instead of Gaussian Process for Empirical Kernel

資工21 周聖諺

Empirical Kernel

Normal-Wishart Distribution

Consider the randomness of the kernel matrix, with Bayesian rule, we can model the kernel matrix with **Inverse Wishart distribution**.

$$p(y|\phi, \nu, K) = \int p(y|\phi, \Sigma)p(\Sigma|\nu, K)d\Sigma = \mathcal{TP}(\phi, K, \nu)$$

Denote Inverse Wishart distribution as $\Sigma \sim \mathcal{IW}(\nu, K) = p(\Sigma|\nu, K)$ and the Gaussian process as $y|\phi, \Sigma \sim \mathcal{GP}(\phi, \Sigma) = p(y|\phi, \Sigma)$. Note that the random matrix $\Sigma \in \mathbb{R}^{N\times N}$ is generated by Inverse Wishart. The hyperparameters are $K \in \mathbb{R}^{N\times N}$, $\nu \in \mathbb{R}$, $\phi \in \mathbb{R}^N$.

Finally, $\mathcal{TP}(\phi, K, \nu)$ is the **Student-T process**.

Student-T Process

The Student-T process can be written as

$$y \sim \mathcal{TP}(\phi, K, \nu)$$

The hyperparameter ν is called **degrees of freedom**, it can control the covariance of the output $cov(y) = \frac{\nu}{\nu-2}K$. Thus, the relation between $\mathcal{TP}$ and $\mathcal{GP}$ is

$$\mathcal{GP}(\phi, K) = \lim_{\nu \to \infty} \mathcal{TP}(\phi, K, \nu)$$

Student-T Process

Given a training dataset $\{X_1, Y_1\}$ with n_1 samples and a testing dataset $\{X_2, Y_2\}$ with n_2 samples where $X_1 \in \mathbb{R}^{n_1 \times d}$, $Y_1 \in \mathbb{R}^{n_1}$, and $X_2 \in \mathbb{R}^{n_2 \times d}$, $Y_1 \in \mathbb{R}^{n_2}$.

Denote the mean function as z and the kernel function as k. Thus, $\phi_i = z(X_i)$ and $K_{ij} = k(X_i, X_j)$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{TP}(\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix})$$

Student-T Process

The posterior is

$$y_2|y_1 \sim \mathcal{TP}(\hat{\phi}, \frac{\nu + \beta + -2}{\nu + n_1 - 2}\hat{K}_{22}, \nu + n_1)$$

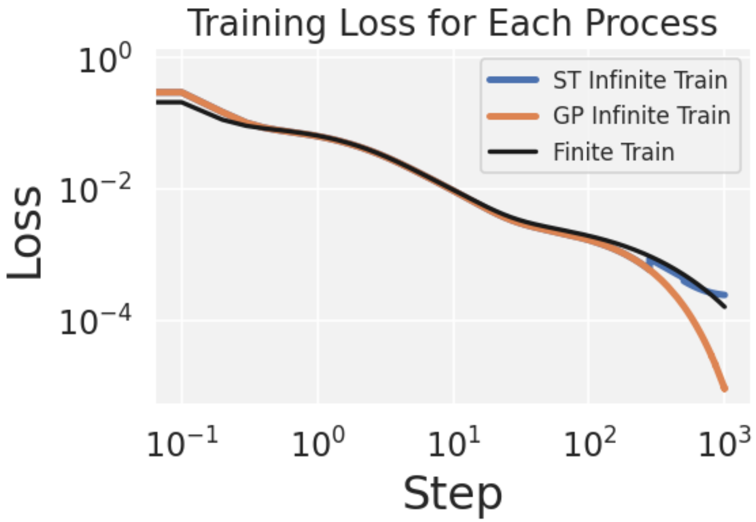
Where

$$\hat{\phi} = K_{21}K_{11}^{-1}(Y_1 - \phi_1) + \phi_2$$

$$\beta = (Y_1 - \phi_1)^\top K_{11}^{-1}(Y_1 - \phi_1)$$

$$\hat{K}_{22} = K_{22} - K_{21}K_{11}^{-1}K_{12}$$

Experiment



We've already known, if $\nu = \infty$, the student-T process will converge to Gaussian process. In the above figure, we compare the result of **fitting a $\sin()$ function with $\mathcal{TP}$ and $\mathcal{GP}$ respectively**. The **left** part of above figure shows the **training/testing progress of fitting**. The right part shows the fitting result of NTK.

As the **blue line** shows above, as the **training process goes**, **the ν gets lower**. It shows that we can **control ν of $\mathcal{TP}$ to achieve a better fitting**.