ELSEVIER

European Journal of Operational Research 99 (1997) 89–112

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Optimization of computer simulation models with rare events ★

Reuven Y. Rubinstein

Faculty of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa 32000, Israel

Received December 1995; revised manuscript received October 1996

Abstract

Discrete event simulation systems (DESS) are widely used in many diverse areas such as computer-communication networks, flexible manufacturing systems, project evaluation and review techniques (PERT), and flow networks. Because of their complexity, such systems are typically analyzed via Monte Carlo simulation methods. This paper deals with optimization of complex computer simulation models involving rare events. A classic example is to find an optimal (s, S) policy in a multi-item, multicommodity inventory system, when quality standards require the backlog probability to be extremely small. Our approach is based on change of the probability measure techniques, also called likelihood ratio (LR) and importance sampling (IS) methods. Unfortunately, for arbitrary probability measures the LR estimators and the resulting optimal solution often tend to be unstable and may have large variances. Therefore, the choice of the corresponding importance sampling distribution and in particular its parameters in an optimal way is an important task. We consider the case where the IS distribution comes from the same parametric family as the original (true) one and use the *stochastic counterpart* method to handle simulation based optimization models. More specifically, we use a two-stage procedure: at the first stage we identify (estimate) the optimal parameter vector at the IS distribution, while at the second stage we estimate the optimal solution of the underlying constrained optimization problem. Particular emphasis will be placed on estimation of rare events and on integration of the associated performance function into stochastic optimization programs. Supporting numerical results are provided as well. © 1997 Elsevier Science B.V.

Keywords: Inventory; Score function; Sensitivity analysis; Optimization; Simulation

1. Background

1.1. Introduction

The objective of this paper is to present a brief overview on discrete event simulation systems (DESS), and some new results in this area with particular emphasis on optimization involving rare events. Both estimation of rare event probabilities of DESS and simulation-based optimization have attracted considerable attention during the past decade. Their integration is of crucial importance for both theory and practice and it is far from being trivial or easy.

★ Research supported by the fund for promotion of research at the Technion.

PII S0377-2217(96)00385-2

A classical approach to simulation-based optimization is the stochastic approximation (SA) method, which presents a *recursive procedure* of steepest descent type (see, e.g., [7,8,16,17,19,21,24,29–32]). In order to apply SA, only estimation of the gradient (in the convex case estimation of a subgradient) of the objective function at each iteration points is required. Drawbacks of the SA method are well known – slow convergence, absence of a good stopping rule and difficulty in handling constraints. An alternative approach to the above optimization is the so-called *stochastic counterpart* (SC) (or sample path) method which, opposite to SA, is *non-recursive*. It is shown in [29] that for smooth unconstrained problems, the SA estimators based on an optimal choice of the stepsizes are asymptotically equivalent to the stochastic counterpart estimators. In the SC method the optimal solution(s) of the original optimization problem is (are) estimated by solving a stochastic counterpart of the original program by deterministic optimization techniques. For such an approach to be applicable, one needs a mechanism to evaluate the approximating functions (and their derivatives) for all values of decision variables in a given region. The decision vector might belong either to the underlying distribution, or to the sample performance function, or to both. They are called the *distributional* and *structural* decision vectors, respectively. An example of the former is the parameter vector of the service rates in the service time distribution in a queueing network, and an example of the latter is the vector of buffer sizes at different nodes of the network. Optimization with respect to the distributional parameters can be performed by the *score function* (SF) method (also called the *likelihood ratios* (LR), and the *importance sampling* (IS) method [26]), while optimization with respect to the structural parameters can be performed by the so-called 'push-out' method [25]. Although the 'push-out' has been recently applied in optimization context to a broad variety of queueing, inventory and insurance risk models [5,22,25], one has to take into account that it is model dependent and requires some skill. We deal below with constrained optimization involving distributional decision parameters only. We assume that the IS distribution comes from the same parametric family as the original (true) one and use the SC method to handle such simulation-based optimization models. More specifically, we use a two-stage procedure: at the first stage we identify (estimate) the optimal parameter vector at the IS distribution, while at the second stage we estimate the optimal solution of the constrained optimization problem. Particular emphasis will be placed on estimation of rare events and integration of the associated performance function into stochastic optimization programs.

To give some background on the SC method, consider the following optimization problem associated with a static (not evolving with time) simulation model:

$$\min_{\mathbf{v}\in V} \ell(\mathbf{v}) = \min_{\mathbf{v}\in V} \mathbb{E}_{\mathbf{v}}\{L(\boldsymbol{Y})\}, \tag{1.1}$$

where $L(\boldsymbol{Y})$ is the sample performance function, $\boldsymbol{Y}$ is a random vector whose distribution is known and defined by the probability density function (pdf) $f(\mathbf{y},\mathbf{v})$ depending on the parameter vector $\mathbf{v}$ representing, in fact, the controllable (decision) vector.

Suppose that the objective function $\ell(\mathbf{v})$ is not given explicitly and therefore must be estimated by Monte Carlo simulation techniques. That is, for a given $\mathbf{v} \in V$, a sequence of random variables $L_i = L(\boldsymbol{Y}_i)$, $i = 1,\ldots,$ is generated and $\ell(\mathbf{v})$ is approximated, say by the sample averages

$$\hat{\ell}_N(\mathbf{v}) = N^{-1}\sum_{i=1}^{N} L(\boldsymbol{Y}_i). \tag{1.2}$$

Using the likelihood ratio method [26], we can write $\ell(\mathbf{v})$ and $\bar{\ell}_N(\mathbf{v})$ in (1.1) and (1.2) as

$$\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}_0}\{L(\boldsymbol{Z})W(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0)\} \tag{1.3}$$

and

$$\bar{\ell}_N(\mathbf{v}) = N^{-1} \sum_{i=1}^{N} L(\mathbf{Z}_i) W(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0), \tag{1.4}$$

respectively, and similarly for dynamic (evolving with time) models [26]. Here

$$W(\mathbf{z}, \mathbf{v}, \mathbf{v}_0) = f(\mathbf{z}, \mathbf{v}) / f(\mathbf{z}, \mathbf{v}_0)$$

is called the *likelihood ratio* function and $\mathbf{v}_0$ is called the *reference parameter* vector.

With the function $\bar{\ell}_N(\mathbf{v})$ at hand, one can estimate the optimal solutions of the original (true) optimization problem (1.1) by solving the following program:

$$\min_{\mathbf{v} \in V} \bar{\ell}_N(\mathbf{v}) = \min_{\mathbf{v} \in V} \left\{ N^{-1} \sum_{i=1}^{N} L(\mathbf{Z}_i) W(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0) \right\}. \tag{1.5}$$

We refer to (1.5) as the *stochastic counterpart* (SC) program.

Note that, given a random sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_N \sim f(\mathbf{z}, \mathbf{v}_0)$, the approximating function $\bar{\ell}_N(\cdot)$ can be calculated explicitly, through the corresponding LR function $W(\mathbf{z}, \cdot, \mathbf{v}_0)$. Note also that $\bar{\ell}_N(\cdot)$ is smooth (differentiable), provided the pdf $f(\mathbf{y}, \mathbf{v})$ is a smooth function of $\mathbf{v}$. Note finally that solving the program (1.5) by (deterministic) techniques of nonlinear programming we can estimate the optimal solution $\mathbf{v}^*$ of the program (1.1).

Let $\bar{\mathbf{v}}_N^*$ be the optimal solution of the program (1.5). It is well known (see, e.g., [26]) that, under mild regularity conditions, the optimal solution $\bar{\mathbf{v}}_N^*$ and the optimal value $\bar{\ell}_N(\bar{\mathbf{v}}_N^*)$ of the program (1.5) converge with probability one, as $N \to \infty$, to their counterparts of the limiting program (1.1). However, this is not exactly what we need in order to justify consistency of the method. Typically, in order to solve the optimization problem (1.5), one runs a (deterministic) algorithm until some stopping rule is satisfied. For example (see [29]), if the problem is smooth and unconstrained, one can use the gradient of the objective function as an indication of how close a current iteration point $\hat{\mathbf{v}}$ can be to an optimal solution. That is, if $\|\nabla \bar{\ell}_N(\hat{\mathbf{v}})\| \leqslant \varepsilon$, for a chosen precision value ε, the algorithm is stopped. For small values of ε this stopping rule guarantees, under mild regularity conditions, that the iteration point $\hat{\mathbf{v}}$ is close to a stationary point of $\bar{\ell}_N(\cdot)$ where the gradient is zero. Moreover, if the gradients of $\bar{\ell}_N(\cdot)$ are uniformly close to the gradients of the limiting function $\ell(\cdot)$, then $\hat{\mathbf{v}}$ should be close to a stationary point of the limiting function as well. For more details see [29].

In many cases the feasible set V is defined by constraints which need to be estimated from simulation as well. The constrained (extended) optimization programs (1.1) and (1.5) can be written as

(P_0)

$$\begin{aligned} &\text{Minimize} && \ell_0(\mathbf{v}), \quad \mathbf{v} \in V, \\ &\text{subject to} && \ell_j(\mathbf{v}) \leqslant 0, \quad j = 1, \ldots, k, \\ &&& \ell_j(\mathbf{v}) = 0, \quad j = k+1, \ldots, M, \end{aligned} \tag{1.6}$$

and

$(\bar{\mathrm{P}}_N)$

$$\begin{aligned} &\text{Minimize} && \bar{\ell}_{0N}(\mathbf{v}), \quad \mathbf{v} \in V, \\ &\text{subject to} && \bar{\ell}_{jN}(\mathbf{v}) \leqslant 0, \quad j = 1, \ldots, k, \\ &&& \bar{\ell}_{jN}(\mathbf{v}) = 0, \quad j = k+1, \ldots, M, \end{aligned} \tag{1.7}$$

Here, as before (for static models),

$$\ell_j(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L_j\} = \mathbb{E}_{\mathbf{v}_0}\{L_j(\mathbf{Z}) W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_0)\} \tag{1.8}$$

are the expected performances associated with the sample performances, L_j, $j = 0, 1, \ldots, M$, and similarly for dynamic models; see below.

Notice that either the objective function or some of the constraints, or both, might involve rare events. Examples of such constrained optimization (simulation based) programs involving rare events are abundant. Below are a few:

Example 1.1. *Production inventory model.* Consider a single-commodity production-inventory model (PIM), which operates under the following (base-stock) policy: in each *fixed-length* period, production is set to a target level S (not exceeding the capacity s of the production facility), in order to satisfy the demand in that period. It is assumed that the demand is either filled immediately or back-ordered. Let N_t denote the net inventory (on-hand inventory minus back orders) at the start of period t, and let D_t denote the demand in period t (see [9]). Then, the production in period t is set to $\min\{s, S - N_t + D_t\}$ and the inventory at the end of period t is

$$N_{t+1} = \Big\{ N_t - D_t + \min\{s, S - N_t + D_t\} = \min\{N_t + s - D_t, S\} \Big\}.$$

If, instead of net inventory, N_t, we consider $L_t = S - N_t$ (the amount by which the target inventory exceeds the net inventory, called *shortfall*), then

$$L_{t+1} = \Big\{ S - \min\{N_t + s - D_t, S\} = \max\{0, L_t + D_t - s\} \Big\}. \tag{1.9}$$

Thus, the shortfall process satisfies Lindley's equation for the waiting time process in a D/G/1 queue with fixed interarrival times s and service times D_t.

Performance measures are, for example, the expected steady-state *backlog* and the *fill rate* (the expected steady-state proportion of demands met from stock), defined as

$$\ell^b = \mathbb{E}\{(L - S)^+\} \quad \text{and} \quad \ell^f = 1 - \frac{\mathbb{E}\{[\min(L - (S - s))^+, s]\}}{\mathbb{E}\{D\}},$$

respectively. Here S is the target inventory level and L and D are random variables having the steady state distribution of the corresponding processes L_t and D_t, respectively. One might be interested in minimizing the expected backlog with respect to parameter vector (s, S), subject to the constraint that the expected fill rate $\ell^f \geqslant 1 - 10^{-6}$.

For additional examples, consider:

(a) Maximizing an expected lifetime of a highly reliable coherent reliability system subject to some cost constraints.
(b) Designing a buffer for a communication system such that the probability of losing a message (packet) is less than 10^{-10}.
(c) Finding an optimal (s, S) policy in a classic multiitem, multicommodity inventory system, (under general distributions of the interdemand time and demand size), provided the backlog probability is less than 10^{-10}.

It is well known (e.g., [26]) that the accuracy (variance) of the estimates $\bar{\mathbf{v}}_N^*$ and the optimal value $\bar{\ell}_N(\bar{\mathbf{v}}_N^*)$ of the SC program (1.5) depends on the choice of the reference parameter $\mathbf{v}_0$. Section 2 and Section 3 discuss finding a 'good' (optimal) reference parameter vector $\mathbf{v}_0$ in the IS distribution $f(\mathbf{y}, \mathbf{v}_0)$ for sample performance functions $\bar{\ell}_{jN}$ involving rare events. Before closing this section we would like to make several remarks on estimation of probabilities of rare events (the vector $\mathbf{v}$ is fixed), or shortly rare events, and their relationship to importance sampling. Importance sampling has been applied to a variety of problems arising in the analysis of rare events in static, queueing, inventory, insurance risk and storage systems (see, e.g., [4,9–15,20,33]).

The main idea of IS, when applied to rare events, is to make their occurrence more frequent, or in other words to 'speed up' the simulation. To achieve this goal, large deviation theory and the so-called *exponential change of measure* (ECM) techniques have been extensively used for the past decade. Let

$$\hat{F}[\theta] = \mathbb{E}\{e^{\theta Y_k}\} = \int_{-\infty}^{\infty} e^{\theta x} F(dx) \tag{1.10}$$

be the *moment generating function* (mgf) of the random variable Y_k with cdf $F(\mathbf{y})$. An *exponential change of measure* (ECM) corresponds to one where the original cdf F is replaced by the cdf F_θ, given by

$$F_\theta(x) = \frac{1}{\hat{F}[\theta]} \int_{-\infty}^{x} e^{\theta y} F(dy), \tag{1.11}$$

where θ is a parameter. Following are some examples of ECM's $F_\theta(x)$ for various cdf's F:

(a) For $F = \mathrm{Exp}(v)$, $F_\theta = \mathrm{Exp}(v - \theta)$.
(b) For $F = \mathrm{G}(\lambda, \beta)$, $F_\theta = \mathrm{G}(\lambda - \theta, \beta)$.
(c) For $F = N(\mu, \sigma^2) F_\theta = N(\mu + \theta, \sigma^2)$.
(d) For $F = \mathrm{Ge}(p)$, $F_\theta = \mathrm{Ge}(1 - (1-p)e^\theta)$.

It is readily seen that for the above examples, the original cdf, F, and the ECM, F_θ, *differ only by a single parameter* θ; in fact, the original cdf, $F \equiv F_{\theta=0}$ is a special case of F_θ with $\theta = 0$. Although this is not always the case (e.g., the Weibull distribution), we shall assume below that the ECM F_θ (for some $\theta > 0$) is the IS (dominating) cdf for the original one, F, so that θ will always be the reference parameter. Moreover, in order to be consistent with the previous notations, we shall restrict ourselves to the exponential family distribution in the following canonical form (e.g., [26]):

$$f(\mathbf{y}, \mathbf{v}) = c(\mathbf{v}) \exp\Big(\sum_{k=1}^{m} v_k t_k(\mathbf{y}) \Big) h(\mathbf{y}), \tag{1.12}$$

where $c(\mathbf{v}) > 0$ is a real-valued function of the parameter vector $\mathbf{v}$ and $t_k(\mathbf{y})$, and $h(\mathbf{y})$ are real-valued functions of $\mathbf{y}$, such that $F \equiv F_{\theta=0}$ is a special case of F_θ with $\theta = 0$ (see above examples (a)–(d)). This allows us to make the following convenient identification:

$$F_\theta(\mathbf{y}, \mathbf{v}) \equiv F(\mathbf{y}, \mathbf{v}_0) \quad \text{and} \quad F_{\theta^*}(\mathbf{y}, \mathbf{v}^*) \equiv F(\mathbf{y}, \mathbf{v}_0^*) \tag{1.13}$$

for ECM and *optimal exponential change of measure* (OECM), respectively (see the program (2.4) below). In short, we assume that the IS distribution $G(\mathbf{y})$ belongs to the same *parametric family* as the original distribution, $F(\mathbf{y}, \mathbf{v})$, so that $G(\mathbf{y}) = F(\mathbf{y}, \mathbf{v}_0)$. Henceforth, we shall use both (equivalent) notations. For example:

(a′) For $Y \sim \mathrm{Exp}(\mathbf{v})$, $v_0 \equiv v - \theta$ and $v_0^* \equiv v - \theta^*$.
(b′) For $Y \sim \mathrm{G}(\lambda, \beta)$, $\mathbf{v}_0 \equiv (\lambda - \theta, \beta)$ and $\mathbf{v}_0^* \equiv (\lambda - \theta^*, \beta)$.

With $F_\theta(\mathbf{y}, \mathbf{v}) \equiv F(\mathbf{y}, \mathbf{v}_0)$ at hand we can minimize the variance of the IS estimator $\bar{\ell}_N(\mathbf{v}, \mathbf{v}_0)$ with respect to the parameter $\mathbf{v}_0$, rather than with respect to the distribution $G(\mathbf{y})$ itself. Fortunately, it appears that such parameterization of $F(\mathbf{y}, \mathbf{v})$ based on ECM are extremely useful for estimation of rare-event probabilities. For static systems and reasonably simple queueing models, such as the GI/G/1 queue, the optimal choice of ECM parameters, say $\mathbf{v}_0^*$, leads to accurate (polynomial time) rare-event probability estimators (see [15] for definition and background on *polynomial time* or as *exponential time* IS estimators). It is shown in [18] that for a static IS estimator of the type

$$\bar{\ell}_N(x) = N^{-1} \sum_{i=1}^{N} I_{\{L(Z_i)>x\}} W(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0),$$

to be polynomial time, it suffices that its relative error is bounded in x by some polynomial function, $p(x)$, and similarly for an estimator of a dynamic type [9,15]. Here I_A is the indicator of the event A. Moreover, it turns out that for complex DES, both static and dynamic, there is *no need to parameterize the entire set of input distributions*, $F(\mathbf{y}, \mathbf{v})$. In fact, it is shown in [18] that for many interesting applications, it suffices to change the probability measure from $F(\mathbf{y}, \mathbf{v})$ to $F(\mathbf{y}, \mathbf{v}_{0b})$, rather than to $F(\mathbf{y}, \mathbf{v}_0)$, where $\mathbf{v}_{0b}$ is selected from a *small subset* of $\mathbf{v}_0$. This subset is associated with the notions of the so-called *bottleneck cut* in static models and the *bottleneck queueing module* in queueing networks. The size of the vector $\mathbf{v}_{0b}$ is typically in the range 3–10, while the size of $\mathbf{v}_0$ might be on the order of hundreds. To summarize, highly accurate (polynomial time) estimators of rare-event probabilities can be derived by using a *parametric* ECM $F(\mathbf{y}, \mathbf{v}_{0b})$, where $\mathbf{v}_{0b} \in V$ is from a small subset of V.

The rest of the paper is organized as follows. Section 2 shows how to choose the optimal reference parameter vector $\mathbf{v}_0^*$ for rare events in both static and dynamic settings, provided $\mathbf{v}$ is fixed. Section 3 deals with stochastic optimization of DESS involving rare events. Here we present a two-stage procedure: at the first stage we identify (estimate) the optimal parameter vector $\mathbf{v}_0^*$ of the IS distribution, and at the second stage we estimate the optimal solution $\mathbf{v}$ of an optimization program involving rare events. Section 4 presents supporting numerical results and in Section 5 concluding remarks and some ideas for further research are given.

2. Optimal choice of the reference vector $\mathbf{v}_0^*$

We now present a method for estimating the optimal value $\mathbf{v}_0^*$ of the reference parameter in details, assuming that the vector $\mathbf{v}$ in the underlying pdf $f(\mathbf{y}, \mathbf{v})$ is *fixed*. The following section deals with estimating the optimal reference vector $\mathbf{v}_0^*$ when $\mathbf{v}$ *varies*, namely $\mathbf{v} \in V$. Static and dynamic models are considered separately.

2.1. Static models

Let the expected performance of the system be given as

$$\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{\varphi[L(\boldsymbol{Y})]\},$$

where φ is a given function. Examples are (i) $\varphi(L) = L$ and (ii) $\varphi(L) = I_{\{L>x\}}$. Assuming that x is large, we refer to cases (i) and (ii), namely to $\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L(\boldsymbol{Y})\}$ and $\ell(x) = P\{L(\boldsymbol{Y}) > x\} = \mathbb{E}_{\mathbf{v}}\{I_{\{L>x\}}\}$ as the *standard performance* (SP) and *performance with rare events* (PRE), respectively.

(i) *Standard performance*

For a fixed $\mathbf{v} \in V$, consider the problem of minimizing the variance of $\bar{\ell}_N(\mathbf{v}, \mathbf{v}_0)$ with respect to $\mathbf{v}_0$, which can be written as

$$\min_{\mathbf{v}_0 \in V} \left\{ \mathcal{L}(\mathbf{v}, \mathbf{v}_0) = \mathbb{E}_{\mathbf{v}_0} \left[L^2(\boldsymbol{Z}) W^2(\boldsymbol{Z}, \mathbf{v}, \mathbf{v}_0) \right] \right\}, \tag{2.1}$$

or alternatively as

$$\min_{\mathbf{v}_0 \in V} \left\{ \mathcal{L}(\mathbf{v}, \mathbf{v}_0) = \mathbb{E}_{\mathbf{v}} \left[L^2(\boldsymbol{Y}) W(\boldsymbol{Y}, \mathbf{v}, \mathbf{v}_0) \right] \right\}, \tag{2.2}$$

where $\boldsymbol{Z} \sim f(\mathbf{y}, \mathbf{v}_0)$ and $\boldsymbol{Y} \sim f(\mathbf{y}, \mathbf{v})$, respectively.

Given a sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_N$ from $f(\mathbf{z}, \mathbf{v}_0)$, we can estimate the optimal solution $\mathbf{v}_0^*$ of (2.1) by an optimal solution of the program

$$\min_{\mathbf{v}_0 \in V} \left\{ \bar{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0) = \sum_{i=1}^{N} L^2(\mathbf{Z}_i) W^2(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0) \right\}. \tag{2.3}$$

To obtain a reasonable approximation of $\mathbf{v}_0^*$ of the program (2.3) we can use gradient discent or Newton's type recursive algorithms starting from $\mathbf{v}_0 = \mathbf{v}$.

We next resort to a slightly different approach for estimating $\mathbf{v}_0^*$, which is based on (2.2) and as will be seen below has some attractive properties. Since $\mathcal{L}(\mathbf{v}, \mathbf{v}_0)$ in (2.2) is represented under the original probability measure $f(\mathbf{y}, \mathbf{v})$, using the standard LR approach we can rewrite the program (2.2) as

$$\min_{\mathbf{v}_0 \in V} \left\{ \mathcal{L}(\mathbf{v}, \mathbf{v}_0) = \mathbb{E}_{\mathbf{v}_1} \left[L^2(\mathbf{Z}) W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_0) W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_1) \right] \right\}. \tag{2.4}$$

Here, as usual, $f(\mathbf{y}, \mathbf{v}_1)$ dominates the pdf $f(\mathbf{y}, \mathbf{v})$ in the absolutely continuous sense,

$$W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_1) = f(\mathbf{Z}, \mathbf{v}) / f(\mathbf{Z}, \mathbf{v}_1),$$

and $\mathbf{Z} \sim f(\mathbf{y}, \mathbf{v}_1)$. Note also that the program (2.1) presents a particular case of the program (2.4) since it can be written as

$$\min_{\mathbf{v}_0 \in V} \left\{ \mathcal{L}(\mathbf{v}, \mathbf{v}_0) = \mathbb{E}_{\mathbf{v}_0} \left[L^2(\mathbf{Z}) W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_0) W(\mathbf{Z}, \mathbf{v}, \mathbf{v}_0) \right] \right\}. \tag{2.5}$$

Given a sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_N$ from $f(\mathbf{z}, \mathbf{v}_1)$, we can estimate the optimal solution (2.4) (and therefore that of $\mathbf{v}_0^*$ of (2.1)) by an optimal solution of the program

$$\min_{\mathbf{v}_0 \in V} \left\{ \hat{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1) = \sum_{i=1}^{N} L^2(\mathbf{Z}_i) W(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0) W(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_1) \right\}. \tag{2.6}$$

From the results of Rubinstein and Shapiro [26] it follows that when the considered pdf, $f(\mathbf{y}, \mathbf{v})$, is from an exponential family in the canonical form, then both functions, $\mathcal{L}(\mathbf{v}, \mathbf{v}_0)$ and $\hat{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1)$, are convex in $\mathbf{v}_0$; consequently, the stochastic counterpart (2.6) is a convex program, provided the parameter set, V, is also convex. Due to the convexity of (2.6), one can find its optimal solution, $\bar{\mathbf{v}}_{0N}^*$, from the solution of the following set of equations:

$$\nabla_{\mathbf{v}_0} \hat{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1) = -\sum_{i=1}^{N} L^2(\mathbf{Z}_i) W^2(\mathbf{Z}_i, \mathbf{v}, \mathbf{v}_0) \nabla_{\mathbf{v}_0} W(\mathbf{Z}_i, \mathbf{v}_0, \mathbf{v}_1) = \mathbf{0}.$$

Clearly, the accuracy of the optimal solution $\bar{\mathbf{v}}_{0N}^*$ depends on the particular choice of $\mathbf{v}_1$. Let $\bar{\mathbf{v}}_{0N}^*$ be the optimal solution of (2.6). Under mild regularity conditions (see [26]) the vector $N^{1/2}(\bar{\mathbf{v}}_{0N}^* - \mathbf{v}_0^*)$ converges in distribution to multivariate normal with zero mean and the covariance matrix $\boldsymbol{B}^{-1} \boldsymbol{\Sigma} \boldsymbol{B}^{-1}$, where $\boldsymbol{B}$ is the Hessian of the function $\mathcal{L}(\mathbf{v}, \cdot)$ calculated at $\mathbf{v}_0 = \mathbf{v}_0^*$, and $\boldsymbol{\Sigma}$ is the covariance matrix given by

$$\boldsymbol{\Sigma} = \mathbb{E}_{\mathbf{v}_1} \{ L^4(\mathbf{Z}) W^4(\mathbf{Z}, \mathbf{v}, \mathbf{v}_0^*) \nabla_{\mathbf{v}_0} W(\mathbf{Z}, \mathbf{v}_0^*, \mathbf{v}_1) \nabla_{\mathbf{v}_0} W(\mathbf{Z}, \mathbf{v}_0^*, \mathbf{v}_1)' \}, \tag{2.7}$$

where $\mathbf{Z} \sim f(\mathbf{z}, \mathbf{v}_1)$. Both $\boldsymbol{B}$ and $\boldsymbol{\Sigma}$ can be estimated from the generated sample, and an approximate confidence region for $\mathbf{v}_0^*$ can be constructed.

We shall now show that (2.6) has certain advantages over (2.3). First, note that the choice of $\mathbf{v}_1$ in (2.6) is not as crucial as that of the reference parameter vector $\mathbf{v}_0$ in (2.3), since (2.6) serves only as an auxiliary tool for identifying a 'good' value of $\mathbf{v}_0$, and not for sensitivity analysis or optimization of $\ell(\mathbf{v})$. Note next that

'good' choices of $\mathbf{v}_1$ and $\mathbf{v}_0$ in (2.6) and (2.3) would be those that render, say, the trace of their corresponding asymptotic covariance matrix $\boldsymbol{B}^{-1}\boldsymbol{\Sigma}\boldsymbol{B}^{-1}$ of the estimator $\bar{\mathbf{v}}_{0N}^*$, as small as possible. Program (2.6) is, however, more flexible and better suited for minimizing the trace than (2.3). To see this, note first that the covariance matrix $\boldsymbol{\Sigma}$ associated with the optimal solution $\bar{\mathbf{v}}_{0N}^*$ of (2.3) is given in analogy to (2.7) by

$$\boldsymbol{\Sigma} = \mathbb{E}_{\mathbf{v}_0}\{L^4(\boldsymbol{Z})W^4(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0^*)\nabla_{\mathbf{v}_0}W(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0^*)\nabla_{\mathbf{v}_0}W(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0^*)'\}, \tag{2.8}$$

where $\boldsymbol{Z} \sim f(\mathbf{z},\mathbf{v}_0)$.

Assume next that $\mathbf{v}_1 = \mathbf{v}_0^*$. Taking into account (see [26]) that

$$\nabla_{\mathbf{v}_0}W(\boldsymbol{Z},\mathbf{v}_0^*,\mathbf{v}_1) = W(\boldsymbol{Z},\mathbf{v}_0^*,\mathbf{v}_1)\cdot S(\boldsymbol{Z},\mathbf{v}_0^*),$$

where $S(\boldsymbol{Z},\mathbf{v}) = \nabla_{\mathbf{v}}\log f(\boldsymbol{Z},\mathbf{v})$ is called the *score function* (SF), we obtain for $\mathbf{v}_1 = \mathbf{v}_0^*$

$$\nabla_{\mathbf{v}_0}W(\boldsymbol{Z},\mathbf{v}_0^*,\mathbf{v}_1) = S(\boldsymbol{Z},\mathbf{v}_1).$$

It follows from the above that if variability of $L(\boldsymbol{Z})$ is not large compared with the variability of the corresponding likelihood ratio functions, then $\mathbf{v}_1 = \mathbf{v}_0^*$ could be a good choice of $\mathbf{v}_1$ in the sense that the trace of $\boldsymbol{\Sigma}$ is small. Clearly, the stochastic counterpart (2.3) does not possess such flexibility. However, $\mathbf{v}_0^*$ is not known. This motivates us to present the following iterative procedure for choosing a 'good' (converging to $\mathbf{v}_0^*$) parameter vector $\mathbf{v}_1$.

Algorithm 2.1. *Choice of a good* v_1 *and optimal* v_0^* *for* $\ell(v) = \mathbb{E}_v\{L(\boldsymbol{Y})\}$.

Step 1. Choose an initial point $\mathbf{v}_1(0)$, say $\mathbf{v}_1(0) = \mathbf{v}$.

Step 2. Generate a sample $\boldsymbol{Z}_1,\ldots,\boldsymbol{Z}_N$ from the pdf $f(\mathbf{y},\mathbf{v}_1)$ and solve the stochastic counterpart (2.6). Denote the solution as $\bar{\mathbf{v}}_0^*(0)$.

Step 3. Set $\mathbf{v}_1(1) \equiv \mathbf{v}_0^*(0)$ and repeat Step 2. Denote the solution by $\bar{\mathbf{v}}_0^*(1)$.

Step 4. Proceed with Steps 2 and 3 until the desired precision $\epsilon \geqslant ||\mathbf{v}_1(k) - \mathbf{v}_1(k-1)||$ is achieved.

Our simulation runs with rather general static and queueing models give reasonable good results with Algorithm 2.1, after 2–3 iterations, in the sense that $\mathbf{v}_1$ converges fast to $\mathbf{v}_0^*$. In our studies we used samples of size $N = 1000$ and the same stream of random numbers at each step of Algorithm 2.1.

As far as the initial point $\mathbf{v}_1(0)$ is concerned, one can either take $\mathbf{v}_1(0) = \mathbf{v}$ (as suggested in Step 1), or again use Newton's algorithm with $\mathbf{v}_0(1)$ replaced by $\mathbf{v}_1(0)$, or finally carry out a line search (with respect to α) in the following one step gradient procedure:

$$\mathbf{v}_1(0) = \mathbf{v} - \alpha\nabla_{\mathbf{v}_1}\hat{\mathcal{L}}_N(\mathbf{v},\mathbf{v}_0,\mathbf{v}_1),$$

with $\mathbf{v} = \mathbf{v}_0 = \mathbf{v}_1$.

(ii) *Performance with rare events*

Consider the program (2.1) with $L^2(\boldsymbol{Y})$ replaced by the indicator $I_{\{L(\boldsymbol{Y}>x)\}}$. How should one choose a good IS density in this case? It follows directly from the expression $\mathcal{L}(\mathbf{v},\mathbf{v}_0)$ in (2.1) that in order to reduce the variance, the likelihood ratio $W(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0)$ must be small on the set $A = \{L(\boldsymbol{Y}) > x)\}$. Since $A = \{L(\boldsymbol{Y}) > x)\}$ is a 'rare-event' set , typically we expect that $f(\mathbf{y},\mathbf{v})$ is small on A. Thus, to make the likelihood ratio $W(\boldsymbol{Z},\mathbf{v},\mathbf{v}_0)$ small on A, one should choose the IS density $f(\mathbf{y},\mathbf{v}_0)$ so that $f(\mathbf{y},\mathbf{v}_0)$ is large on A. By doing so, the event A will be likely to occur, i.e., A will not be a rare event under $f(\mathbf{y},\mathbf{v}_0)$, and the rare-event simulation problem will be transferred into a nonrare-event simulation.

Let us return for a moment to Algorithm 2.1 suitable for estimation of $\mathbf{v}_0^*$ in the standard sample performance $L(\boldsymbol{Y})$. It is readily seen that, as stated, Algorithm 2.1 is useless for estimating $\mathbf{v}_0^*$ involving rare events. The

reason is that, owing to the rarity of the event $\{L(\mathbf{Y}_i) > x\}$, the random variables $I_{\{L_i > x\}}$, $i = 1, \ldots, N$, and the associated derivatives of $\hat{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1)$ (see (2.6)) will be zero with very high probability, provided the sample size is not too large, say $N < 10^4$. With (2.12) in mind we can, however, modify Algorithm 2.1 by iterating *adaptively* in both parameters, x and $\mathbf{v}_1$. We start the algorithm by choosing, say $\mathbf{v}_1(0) = \mathbf{v}$ (see Step 1) and some initial x_0 ($x_0 < x$), (see program (2.10) below) such that under the original pdf $f(\mathbf{y}, \mathbf{v})$, the probability $\ell(x_0) = \mathbb{E}_{\mathbf{v}}\{I_{\{L > x_0\}}\}$ is not too small, say $\ell(x_0) \equiv \alpha_0 \approx 10^{-2}$ and then iterate in both x and $\mathbf{v}_1$.

Algorithm 2.2. *Choice of a good $\mathbf{v}_1$ and optimal $\mathbf{v}_0^*$ for $\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{I_{\{L(\mathbf{Y}) > x\}}\}$.*

Step 1. Choose an initial point $\mathbf{v}_1(0)$, say $\mathbf{v}_1(0) = \mathbf{v}$.

Step 2. Generate a sample $\mathbf{Z}_1, \ldots, \mathbf{Z}_N$ from the pdf $f(\mathbf{y}, \mathbf{v}_1(0))$, and solve the stochastic counterpart

$$\text{Max } y \quad \text{s.t.} \quad N^{-1} \sum_{i=1}^{N} I_{\{L(\mathbf{Z}_i) > y\}} \geqslant \alpha_0 \tag{2.9}$$

of the program

$$\text{Max } y \quad \text{s.t.} \quad \mathbb{E}\left\{I_{\{L(\mathbf{Z}) > y\}}\right\} \geqslant \alpha_0. \tag{2.10}$$

Denote the solution of (2.10) and its stochastic counterpart (2.9) by x and $\bar{x}_0$, respectively. It is readily seen that, in fact,

$$\bar{x}_0 = L_{([(1-\alpha_0)N])},$$

where $L_{(i)}$ is the i-th order statistics of the sequence $\{L_i \equiv L(\mathbf{Z}_i),\ i = 1, \ldots, N\}$. For $\alpha_0 = 10^{-2}$ this reduces to

$$\bar{x}_0 = L_{([\frac{99}{100}N])}.$$

Step 3. Estimate the optimal solution $\mathbf{v}_0^*$ of program (2.1) (with $L^2 \equiv I_{\{L(\mathbf{Z}_i) > \bar{x}_0\}}$) by solving the stochastic counterpart (2.6) for $\bar{x}_0$. Denote the solution as $\bar{\mathbf{v}}_0^*(\bar{x}_0)$.

Step 4. Set $\mathbf{v}_1(1) \equiv \bar{\mathbf{v}}_0^*(\bar{x}_0)$, and repeat Step 2 again with $\mathbf{v}_1(0)$ replaced by $\bar{\mathbf{v}}_0^*(\bar{x}_0)$. Denote the solution as $\bar{x}_1$. Repeat Step 3, with $\mathbf{v}_1(0)$ and $\bar{x}_0$ replaced by $\bar{\mathbf{v}}_0^*(\bar{x}_0)$ and $\bar{x}_1$, respectively. Denote the solutions as $\bar{\mathbf{v}}_0^*(\bar{x}_1)$. Denote the solutions at stage k from Step 2 and Step 3 as $\bar{\mathbf{v}}_0^*(\bar{x}_k)$ and $\bar{x}_k$, respectively.

Step 5. If $\bar{x}_k \geqslant x$, $k = 0, 1, 2, \ldots$, set $\bar{x}_k = x$, perform Step 3 once more and take the obtained solution $\bar{\mathbf{v}}_0^*(x)$ as an estimate of the optimal solution of the program (2.1) (with $L^2 \equiv I_{\{L(\mathbf{Z}_i) > x\}}$). Otherwise repeat Steps 2–4 again.

It is readily seen that Algorithm 2.1 presents a particular case of Algorithm 2.2 when x does not change.

Observe that the quantile x_0 of $\ell(x_0) = \alpha_0 \approx 10^{-2}$ can be readily estimated from a pilot run. The auxiliary (random) sequence $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots$ was designed here to be monotonic increasing with high probability, that is,

$$\bar{x}_0 > \bar{x}_1 > \bar{x}_2 > \cdots > \bar{x}_i > \cdots = x \tag{2.11}$$

with high probability; furthermore, the target value x is reached after just a few iterations.

The monotonicity of the sequence $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots$ is an inherent part of Algorithm 2.2. Simulation experiments with static models, such as shortest path models and coherent reliability models, show that this is indeed the case. The following proposition validates formula (2.11) for the gamma $G(\lambda, \beta)$ pdf.

Proposition 2.1. *Let $Y \sim G(\lambda, \beta)$. Suppose that $L^2(y)$ is a monotonically increasing function on the interval $[0, \infty)$. Then:*

1) *The random sequence* $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots$ *generated by Algorithm* 2.2 *is monotonic increasing, i.e.* (2.11) *holds with high probability.*
2) *The target value* x *is reached with high probability in a finite number of iterations.*

Proof. We prove Part 1) of Proposition 2.1 first. Consider Algorithm 2.2, but where the stochastic counterparts (2.6) and (2.9) are replaced by the original programs (2.1) and (2.10), respectively. In this case the random sequence $\bar{x}_0, \bar{x}_1, \bar{x}_2, \ldots$ becomes deterministic, say $x_0, x_1, x_2, \ldots$, and by virtue of Proposition 4.2.2 of [26] the sequence $x_0, x_1, x_2, \ldots$ is monotonic increasing, since for any fixed λ and x_k, $k = 0, 1, \ldots$ (see formula (4.6.7), p. 201 of [26]),

$$\lambda_0^*(x_0) < \lambda, \quad \lambda_0^*(x_1) < \lambda_0^*(x_0), \ldots, \quad \lambda_0^*(x_k) < \lambda_0^*(x_{k-1}), \ldots$$

The proof of Part 1) of Proposition 2.1 follows now, since at each stage $k = 0, 1, \ldots$ we have that asymptotically in N, the optimal solutions $\bar{\boldsymbol{v}}_k^*$ and $\bar{x}_k$ of the stochastic counterparts (2.6) and (2.9) converge almost surely to $\boldsymbol{v}_k^*$ and x_k of the original programs (2.1) and (2.10), respectively.

To prove Part 2) of Proposition 2.1, consider again the original programs (2.1) and (2.10) first. Taking into account formula (2.11) and the facts that $\ell(\bar{x}_0) = \alpha_0 < 1$, $\ell(x) = \alpha > 0$ and the interval (x_0, x) is finite, the result follows. The proof of Part 2) using stochastic counterparts (2.6) and (2.9) is similar to Part 1).

Proposition 2.1 can be readily extended to the multidimensional gamma distribution with independent components and to some other pdf like multidimensional normal pdf, since Proposition 4.2.2 in [26] readily extends too (see [26]).

As we mentioned earlier, there is no need to solve the programs (2.6) and (2.9) very accurately, at least at the first stages. In particular one can perform several steps of Newton's algorithm. This is the reason we use in Proposition 2.1 the term 'with high probability' rather than 'with probability one', which assumes an infinity sampling.

It is interesting to note that under the IS pdf $f(\mathbf{y}, \mathbf{v}_0^*)$ one has

$$P\{L(\boldsymbol{Z}^*) > x\} = \mathbb{E}_{\mathbf{v}_0^*}\{I_{\{L(\boldsymbol{Z}^*)>x\}}\} = \mathrm{O}(1), \tag{2.12}$$

where $\boldsymbol{Z}^* \sim f(\mathbf{y}, \mathbf{v}_0^*)$. This is so since at each stage of Algorithm 2.2 we estimate a probability of order $\ell(x_0) = \mathbb{E}_{\mathbf{v}}\{I_{\{L>x_0\}}\}$ which by itself is of order $\mathrm{O}(1)$.

We show next how much the optimal solution $\mathbf{v}_0^*$ differs in cases (i) and (ii).

Example 2.1. Let $L(Y) = Y$, where $Y \sim \exp(v)$. Simple calculations yield (see [26])

$$\mathbb{E}_v\{Y\} = 1/v, \quad v_0^* = 0.5v, \quad \text{and} \quad P\{Y > x\} = \mathrm{e}^{-vx}, \quad v_0^* \approx 1/x, \tag{2.13}$$

for cases (i) and (ii) respectively, provided x is large. For $v = 1$ and $x = 100$ (which corresponds to $\ell(x) = \mathrm{e}^{-100}$), we obtain $v_0^* = 0.5$ and $v_0^* \approx 0.01$, respectively.

Note that the aim of Algorithms 2.1 and 2.2, if applied to Example 2.1, is to estimate consistently the optimal parameters $v_0^* = 0.5$ and $v_0^* \approx 0.01$.

Assume for a moment that we seek to estimate an expected performance of the form

$$\ell(v) = \mathbb{E}_v\{Y\} + \mathbb{E}_v\{I_{\{Y>x\}}\}. \tag{2.14}$$

As the reference parameter vector $\mathbf{v}_0^*$ (for fixed $\mathbf{v}$) in the standard performance function $\mathbb{E}_{\mathbf{v}}\{Y\}$ and the performance function with rare events $\mathbb{E}_{\mathbf{v}}\{I_{\{Y>x\}}\}$ are quite dissimilar, one can use two different simulations. In the following example we show that, in fact, a *single* simulation with the reference parameter $v_0^* \approx 1/x$

(optimal value for part $\mathbb{E}_{\mathbf{v}}\{I_{\{Y>x\}}\}$) is sufficient. More precisely, we show that, if we replace the optimal value $v_0^* = 0.5$ (for part $\mathbb{E}_v\{Y\}$) by $v_0^* \approx 1/x$, the accuracies of both the *single* run and the *double* run LR estimators remain the same in terms of orders of their magnitude.

Example 2.2. (Example 2.1 continued.) Consider (2.14), where $Y \sim \exp(v)$. Recall that the optimal values of the reference parameter for parts $\mathbb{E}_v\{Y\}$ and $\mathbb{E}_v\{I_{\{Y>x\}}\}$ are $v_0^* = 0.5$ and $v_0^* \approx 1/x$, respectively. Consider first the part $\mathbb{E}_v\{I_{\{Y>x\}}\}$. Under $v_0^* \approx 1/x$ the squared coefficient of variation (relative error), defined as

$$\kappa_1^2(\mathbf{v}_0, x) = N \operatorname{Var}\{\bar{\ell}_N(\mathbf{v}_0, x)\}/\ell^2(x), \tag{2.15}$$

is *linear* in x (see [18]), provided x is large. More precisely, under $v_0^* \approx 1/x$ (see [18]),

$$\kappa_1^2(v_0^*, x) \approx 0.5xve. \tag{2.16}$$

Consider next the part $\mathbb{E}_v\{Y\}$. Under arbitrary v_0 simple calculations yield

$$\kappa_2^2(v_0) = \frac{2v^4}{v_0(2v - v_0)^3} - 1\ . \tag{2.17}$$

Substituting $v_0^* \approx 1/x$ into (2.17) we obtain (for large x)

$$\kappa_2^2(v_0^*, x) \approx \tfrac{1}{4}xv - 1. \tag{2.18}$$

Thus, again $\kappa_2^2(v_0^*, x)$ is a *linear* function in x. Clearly, the resulting relative error $\kappa^2(v_0^*, x) = \kappa_1^2(v_0^*, x) + \kappa_2^2(v_0^*, x)$ is linear in x for both (single and double) runs.

2.2. Dynamic models

As for static models we consider separately the choice of the optimal reference parameter vector $\mathbf{v}_0^*$ for the (i) standard performance $\ell(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L(\boldsymbol{Y})\}$ and (ii) performance $\ell(x) = P\{L(\boldsymbol{Y}) > x\}$ with rare events. If not stated otherwise, we assume that the output process $\{L_t\}$ is regenerative, say $\{L_t\}$ is the waiting time process in a GI/GI/1 queue.

(i) *Standard performance*

According to the regenerative LR approach, $\ell(\mathbf{v})$ can be written [26] as

$$\ell(\mathbf{v}) = \frac{\mathbb{E}_g\{\sum_{t=1}^{\tau} L_t \widetilde{W}_t\}}{\mathbb{E}_g\{\sum_{t=1}^{\tau} \widetilde{W}_t\}}, \tag{2.19}$$

where

$$\widetilde{W}_t = \prod_{k=1}^{t} W_1(\mathbf{Z}_{1k}, \mathbf{v}_1) W_2(\mathbf{Z}_{2k}, \mathbf{v}_2) = \prod_{k=1}^{t} \frac{f_1(\mathbf{v}_1, Z_{1k})}{f_1(\mathbf{v}_{01}, Z_{1k})} \cdot \frac{f_2(\mathbf{v}_2, Z_{2k})}{f_2(\mathbf{v}_{02}, Z_{2k})} \tag{2.20}$$

is the *likelihood ratio process*, $\{Z_{1k}\}$ and $\{Z_{2k}\}$ are the sequences of interarrival and service times from the IS pdf's $f_1(\mathbf{y}, \mathbf{v}_{01})$ and $f_2(\mathbf{y}, \mathbf{v}_{02})$, respectively, and τ is the length of the regenerative cycle. A consistent estimator of $\ell(x)$ is

$$\bar{\ell}_N(\mathbf{v}) = \frac{\sum_{i=1}^{N} \sum_{t=1}^{\tau_i} L_{ti} \widetilde{W}_{ti}(\mathbf{v}, \mathbf{v}_0)}{\sum_{i=1}^{N} \sum_{t=1}^{\tau_i} \widetilde{W}_{ti}(\mathbf{v}, \mathbf{v}_0)} = \frac{\bar{\ell}_{1N}}{\bar{\ell}_{2N}}, \tag{2.21}$$

where

$$\widetilde{W}_{ti} = \prod_{k=1}^{t} W_1(\mathbf{Z}_{1ki}, \mathbf{v}_1) W_2(\mathbf{Z}_{2ki}, \mathbf{v}_2) = \prod_{k=1}^{t} \frac{f_1(\mathbf{v}_1, Z_{1ki})}{f_1(\mathbf{v}_{01}, Z_{1ki})} \cdot \frac{f_2(\mathbf{v}_2, Z_{2ki})}{f_2(\mathbf{v}_{02}, Z_{2ki})}, \tag{2.22}$$

and N is the number of regenerative cycles.

To proceed with $\mathbf{v}_0^*$ we extend the program (2.1) for static models to queueing models. Consider first the numerator $\bar{\ell}_{1N}$ of the regenerative estimator (2.21)–(2.22). We have in analogy to the program (2.1)

$$\min_{\mathbf{v}_0 \in V} \mathcal{L}_1(\mathbf{v}, \mathbf{v}_0) = \min_{\mathbf{v}_0 \in V} \mathbb{E}_{\mathbf{v}_0} \Big\{ \sum_{t=1}^{\tau} L_t(\underline{\mathbf{Z}}_t) \widetilde{W}_t(\underline{\mathbf{Z}}_t, \mathbf{v}, \mathbf{v}_0) \Big\}^2. \tag{2.23}$$

Consider next the regenerative estimator $\bar{\ell}_N$ in (2.21)–(2.22). Its asymptotic variance (see [26]) is $1/N$ times the quantity

$$\sigma^2(\mathbf{v}, \mathbf{v}_0) = a^2 \mathrm{Var}_{\mathbf{v}_0} \Big\{ \sum_{t=1}^{\tau} M_t(\underline{\mathbf{Z}}_t) \widetilde{W}_t(\underline{\mathbf{Z}}_t, \mathbf{v}, \mathbf{v}_0) \Big\},$$

where $a = 1/\mathbb{E}_{\mathbf{v}}\{\tau\}$, and $M_t = L_t - \bar{\ell}_N(\mathbf{v})$. That is, the optimal solution, $\mathbf{v}_0^*$, can be obtained by solving the program

$$\min_{\mathbf{v}_0 \in V} \mathcal{L}(\mathbf{v}, \mathbf{v}_0) = \min_{\mathbf{v}_0 \in V} \mathbb{E}_{\mathbf{v}_0} \Big\{ \sum_{t=1}^{\tau} M_t(\underline{\mathbf{Z}}_t) \widetilde{W}_t(\underline{\mathbf{Z}}_t, \mathbf{v}) \Big\}^2. \tag{2.24}$$

The stochastic counterpart of (2.24) can be written in analogy to (2.3) as

$$\min_{\mathbf{v}_0 \in V} \bar{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0) = \min_{\mathbf{v}_0} \Big\{ \sum_{i=1}^{N} \Big[\sum_{t=1}^{\tau_i} M_{ti}(\underline{\mathbf{Z}}_{ti}) \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}, \mathbf{v}_0) \Big]^2 \Big\}, \tag{2.25}$$

where $\{\mathbf{Z}_{11}, \ldots, \mathbf{Z}_{\tau_1 1}, \ldots, \mathbf{Z}_{1N}, \ldots, \mathbf{Z}_{\tau_N N}\}$ is a sample of N regenerative cycles generated from the dominating pdf, $f(\mathbf{y}, \mathbf{v}_0)$.

Note that the optimal solution $\bar{\mathbf{v}}_{0N}^*$ of the program (2.25) can be estimated either by using recursive algorithms of stochastic approximation or using a non-recursive approach, say by solving the program (2.25) by standard mathematical programming methods. Note also that Asmussen and Rubinstein [3] calculated analytically the optimal solution, $\mathbf{v}_0^*$, for the program (2.24) with $\{L_t\}$ being the waiting time process in the M/M/1 queue and v_0^* (the decision variable) being the service rate.

As for static models we shall use here a different approach, namely an approach involving an auxiliary IS pdf $f(\mathbf{y}, \mathbf{v}_1)$. Arguing as in (2.4) and taking into account (2.25) the stochastic counterpart, based on $f(\mathbf{y}, \mathbf{v}_1)$, can be written (similarly to (2.25)) as

$$\min_{\mathbf{v}_0 \in V} \bar{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1) = \min_{\mathbf{v}_0 \in V} \Big\{ \frac{1}{N} \sum_{i=1}^{N} \Big[\sum_{t=1}^{\tau_i} M_{ti}(\underline{\mathbf{Z}}_{ti}) \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}, \mathbf{v}_0) \Big]^2 \widetilde{W}_{\tau i i}(\underline{\mathbf{Z}}_{\tau_i i}, \mathbf{v}_0, \mathbf{v}_1) \Big\}, \tag{2.26}$$

where

$$\widetilde{W}_{\tau i i}(\underline{\mathbf{Z}}_{\tau_i}, \mathbf{v}_0, \mathbf{v}_1) = \prod_{t=1}^{\tau_i} \frac{f(\mathbf{Z}_{ti}, \mathbf{v}_0)}{f(\mathbf{Z}_{ti}, \mathbf{v}_1)}.$$

Here $\{\mathbf{Z}_{11}, \ldots, \mathbf{Z}_{\tau_1 1}, \ldots, \mathbf{Z}_{1N}, \ldots, \mathbf{Z}_{\tau_N N}\}$ is a sample of N regenerative cycles, generated from the auxiliary pdf $f(\mathbf{z}, \mathbf{v}_1)$.

Taking into account (see [26]) that

$$\mathbb{E}_{\mathbf{v}}\Big\{\sum_{t=1}^{\tau} M_t \sum_{t=1}^{\tau} \widetilde{W}_t\Big\} = \mathbb{E}_{\mathbf{v}}\Big\{\sum_{t=1}^{\tau} M_t \widetilde{W}_t\Big\},$$

we obtain, after some simple manipulation with (2.26), the following alternative:

$$\begin{aligned}
&\min_{\mathbf{v}_0 \in V} \hat{\mathcal{L}}_N(\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1) \\
&= \min_{\mathbf{v}_0 \in V} \Big\{ \frac{1}{N} \sum_{i=1}^{N} \Big[\sum_{t=1}^{\tau_i} M_{ti}(\underline{\mathbf{Z}}_{ti})^2 \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}, \mathbf{v}_0)^2 \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}_0, \mathbf{v}_1) \\
&\qquad + 2 \sum_{s<t}^{\tau_i} L_{si}(\underline{\mathbf{Z}}_{si}) L_t(\underline{\mathbf{Z}}_{ti}) \widetilde{W}_{si}(\underline{\mathbf{Z}}_{si}, \mathbf{v}, \mathbf{v}_0) \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}, \mathbf{v}_0) \widetilde{W}_{ti}(\underline{\mathbf{Z}}_{ti}, \mathbf{v}_0, \mathbf{v}_1) \Big] \Big\}. \qquad (2.27)
\end{aligned}$$

The relation between these two LR estimators $\tilde{\mathcal{L}}_N$ and $\hat{\mathcal{L}}_N$ is discussed in Asmussen and Rubinstein [2], where it is shown that the estimator $\hat{\mathcal{L}}_N$ is typically more accurate than estimator $\tilde{\mathcal{L}}_N$, in the sense that the former has typically a smaller variance.

With the above results, we can estimate the optimal reference parameter $\mathbf{v}_0^*$ of the program (2.24) by modifying the (static) Algorithm 2.1 as follows. Replace (2.1) and (2.6) in Step 2 of Algorithm 2.1 by (2.24) and (2.27), respectively, while all other data remains the same. We shall call such modification of Algorithm 2.1, *Modified Algorithm 2.1* (MA 2.1).

To illustrate that the optimal solution, $\bar{\mathbf{v}}_{0N}^*$, of the stochastic counterpart (2.27), is rather insensitive (robust) to the choice of $\mathbf{v}_1$, extensive simulation experiments were carried out. Our experiments suggest that in typical situations there exists a rather broad set of parameter vectors $\mathbf{v}_1$ to identify (estimate reliably) the optimal reference parameter $\mathbf{v}_0^*$. Consider, for example, an M/M/1 queue with fixed $\lambda = 1$ and $\rho = 0.6$ and let $\{L_t\}$ be the waiting time process. In this case it follows from the analytic results of Asmussen and Rubinstein [3] that the optimal reference parameter (service rate) $v_0^* \approx 1.25$ ($\rho_0^* \approx 0.8$). We found numerically that the stochastic counterpart (2.27) (with v_1 being the service rate in the auxiliary pdf $f(y, v_1)$) estimates the parameter $\rho_0^* \approx 0.8$ rather accurate, provided $v_1 \in (1.06, 2)$ ($\rho_1 = 1/v_1 \in (0.5, 0.95)$).

(ii) *Performance with rare events*

Note first that the regenerative estimator (2.21)–(2.22) is suitable for estimation of rare events: we need to replace L_{ti} by $I_{\{L_{ti} > x\}}$ while all other data remain the same. Unfortunately such a regenerative estimator, which we call the *conventional regenerative* (CR) estimator, is rather limited for rare events estimation, since (see [4]) it is *exponential time* for all $\mathbf{v}_0 \in V$.

To overcome this 'complexity' difficulty we consider the *switching regenerative* (SR) estimators [9,26]. The idea behind SR estimators is to introduce more congestion at the beginning of the cycle by making the corresponding service and arrival parameters of the IS distribution controllable, and then switch to some different parameters (less congested case in order for a system to remain stable), say switch back to the original distribution (parameters). More precisely, we choose $\widetilde{W}_t$ as

$$\widetilde{W}_t(\zeta) = \begin{cases} \prod_{j=1}^{t} W(\mathbf{z}_j, \hat{\mathbf{v}}_{01}) & \text{if } t \leqslant \zeta, \\ \prod_{j=1}^{\zeta} W(\mathbf{z}_j, \hat{\mathbf{v}}_{01}) \prod_{j=\zeta+1}^{t} W(\mathbf{z}_j, \hat{\mathbf{v}}_{02}) & \text{if } \zeta < t \leqslant \tau. \end{cases} \qquad (2.28)$$

Here ζ is a *random variable* (stopping time), defined as $\zeta = \inf\{t : L_t = x\}$, and $\hat{\mathbf{v}}_{01}$ and $\hat{\mathbf{v}}_{02}$ correspond to the parameters in the IS pdf before and after switching. In the particular case where $\hat{\mathbf{v}}_{02} = \mathbf{v}$ (switching back to the original distribution) the LR process (2.28) reduces to

$$\widetilde{W}_t(\zeta) = \begin{cases} \prod_{j=1}^{t} W(\mathbf{z}_j, \hat{\mathbf{v}}_{01}) & \text{if } t \leqslant \zeta, \\ \prod_{j=1}^{\zeta} W(\mathbf{z}_j, \hat{\mathbf{v}}_{01}) & \text{if } \zeta < t \leqslant \tau. \end{cases} \tag{2.29}$$

If not stated otherwise we use below that particular case of (2.28) and we write $\mathbf{v}_0 = \hat{\mathbf{v}}_{01}$, rather than

$$\mathbf{v}_0 = \begin{cases} \hat{\mathbf{v}}_{01} & \text{if } t \leqslant \zeta, \\ \hat{\mathbf{v}}_{02} & \text{if } \zeta < t \leqslant \tau, \end{cases} \tag{2.30}$$

since it is assumed that $\hat{\mathbf{v}}_{02} = \mathbf{v}$ and $\mathbf{v}$ is fixed. For convenience purposes we denote $\mathbf{v}_0$ in (2.30) by $[\hat{\mathbf{v}}_{01}, \hat{\mathbf{v}}_{02}]$.

Consider now the transient regime. In particular assume that we are interested in buffer overflow during a cycle in a G/G/1 queue starting with an empty system. Here a cycle is defined as a duration of time starting with an empty system and ending at the instant the system, for the first time, either becomes empty again or the number of customers in the system, L_t, reaches x. For more details see [20]. It is readily seen that this is a special case of the SR estimator, corresponding to $W(\mathbf{z}_j, \mathbf{v}_{02}) \equiv 0$. Operationally, this means that when level x is reached, one switches to a server with an infinite service rate. The corresponding likelihood ratio process, $\widetilde{W}_t$, (see (2.29)) can be written as

$$\widetilde{W}_t(\zeta) = \begin{cases} \prod_{j=1}^{t} W(\mathbf{Z}_j, \hat{\mathbf{v}}_{01}) & \text{if } t \leqslant \zeta, \\ 0 & \text{if } \zeta < t \leqslant \tau. \end{cases} \tag{2.31}$$

For simple queueing and simple inventory models conditions are presented in [4,9,15] under which the SR estimator (2.21)–(2.28) is polynomial time. In particular, Glasserman and Kou [10] analyze rare events in tandem Jackson queues where the rare event corresponds to the network population starting at 0 and reaching a prescribed large number x before returning to 0. They analyzed an IS estimator based on *interchanging the arrival rate with the smallest service rate, while all other service rates remain unchanged.* They showed that, in certain parameter regions, their estimator has linearly bounded and even purely bounded relative error, while in other regions it is not even asymptotically efficient. A similar phenomenon has been observed in [4], [15] and [18]. They found conditions (for rather general static and some simple queueing models) under which it is sufficient to change the original probability distribution $F(\mathbf{y}, \mathbf{v})$ to $F(\mathbf{y}, \mathbf{v}_{0b})$ rather than to $F(\mathbf{y}, \mathbf{v}_0)$, where $\mathbf{v}_{0b}$ is a well defined *small subset* of $\mathbf{v}_0$. Their IS estimators are associated with the notion of the so-called *bottleneck cut* and *bottleneck queueing module* for static and queueing models, respectively, in the sense that as soon as they are identified, the change of the probability measure is performed for the distributions associated with the parameters of the bottleneck cut (bottleneck queue) alone, rather than with the entire set of input distributions. They also discuss conditions under which the relative error of their IS estimators is bounded by a polynomial. For complex settings, however, the complexity of the SR, even with the optimal choice of the reference parameter $\mathbf{v}_0^* = \hat{\mathbf{v}}_{01}^*$, is still an open problem.

Consider the stochastic counterpart (2.27), where the standard regenerative estimator is replaced by the switching regenerative estimator (2.21), (2.28), and $M_{ti} = L_{ti} - \bar{\ell}(\mathbf{v})$ is replaced by $M_{ti} = I_{\{L_{ti} > x\}} - \bar{\ell}(x)$. Taking this into account, it is readily seen that the (static) Algorithm 2.2 can be modified to dynamic (queueing) systems in the sense that it can estimate consistently the optimal reference parameter vector $\mathbf{v}_0^* \equiv \hat{\mathbf{v}}_{01}^*$. More precisely, the dynamic version of Algorithm 2.2 assumes modifying its Step 2 as follows:

Step 2. Generate a sample $\{\mathbf{Z}_{11}, \ldots, \mathbf{Z}_{\tau_1 1}, \ldots, \mathbf{Z}_{1N}, \ldots, \mathbf{Z}_{\tau_N N}\}$ of N regenerative cycles from the pdf $f\{\mathbf{y}, \bar{\mathbf{v}}_1(0)\}$, and solve the stochastic counterpart

$$\text{Max } y \quad \text{s.t.} \quad N^{-1}\tau^{-1} \sum_{i=1}^{N} \sum_{t=1}^{\tau_i} I_{\{L(\mathbf{Z}_{ti}) > y\}} \geqslant \alpha_0 \tag{2.32}$$

of the program

$$\text{Max } y \quad \text{s.t.} \quad \mathbb{E}\left\{I_{\{L(\mathbf{Z})>y\}}\right\} \geqslant \alpha_0.$$

Denote the solution of (2.32) by $\bar{x}_0$. It is readily seen that, in fact, $\bar{x}_0 = L_{([(1-\alpha_0)N])}$, where $L_{(i)}$ is the i-th order statistics of the sequence $\{L_i \equiv L(\mathbf{Z}_i),\ i = 1, \ldots, N\}$. For $\alpha_0 = 10^{-2}$ this reduces to

$$\bar{x}_0 = L_{([\frac{99}{100}N])}.$$

We call such a modification of Algorithm 2.2, *Modified Algorithm* 2.2 (MA2.2).

As for static models (see Example (2.1)), the optimal reference parameter vectors $\mathbf{v}_0^*$ in the CR and SR estimators might be quite different. To see this consider the following:

Example 2.3. Let $\ell_1(\mathbf{v}, x) = \mathbb{E}_{\mathbf{v}}\{I_{\{L_t>x\}}\}$ and $\ell_2(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L_t\}$ be two associated performance measures. Consider first $\ell_1(\mathbf{v}, x)$, where $\{L_t\}$ is an output process, say the waiting time process, in the M/M/1 queue. It is well known (e.g., [4]) that the optimal $\hat{\mathbf{v}}_{01}^*$ in the RS estimator (2.21)–(2.29) corresponds to the interchange of λ and v, i.e. $\hat{\mathbf{v}}_{01}^* = (v, \lambda)$, and that such an RS estimator is polynomial time. Take for example, $\lambda = 1$ and $v = 2$. We obtain $\hat{\mathbf{v}}_{01}^* = (2, 1)$. Consider next $\ell_2(\mathbf{v})$. For the same $\lambda = 1$ and $v = 2$ ($\rho = 0.5$), we have from [3] that $v_0^* \approx 1.4$ ($\rho_0^* \approx 0.72$). Thus, the optimal reference parameter vectors $\mathbf{v}_0^*$ which minimize the variances of the corresponding CR and SR estimators are quite different.

Note again that the purpose of Modified Algorithms 2.1 and 2.2, if applied to Example 2.3, is to estimate consistently the optimal vector $\mathbf{v}_0^*$, which for $\ell_1(\mathbf{v})$ and $\ell_2(\mathbf{v})$ in Example 2.3 becomes $\mathbf{v}_0^* = \hat{\mathbf{v}}_{01}^* = (2, 1)$ and $v_0^* \approx 1.4$, respectively.

3. Stochastic optimization with rare events

Consider the stochastic program (1.8) for dynamic (queueing) models and assume that the output processes, $\{L_{jt}\}$, $j = 0, 1, \ldots, M$, are regenerative. In this case we may approximate the functions

$$\ell_j(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L_j(\mathbf{Y})\}, \quad j = 0, 1, \ldots, M,$$

by their sample equivalents,

$$\bar{\ell}_{jN}(\mathbf{v}) = \frac{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i} L_{jti}\widetilde{W}_{ti}(\mathbf{v}, \mathbf{v}_0)}{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i} \widetilde{W}_{ti}(\mathbf{v}, \mathbf{v}_0)}, \tag{3.1}$$

and similarly for the case of rare events. In the last case we need only to replace L_{jti} by the indicator $I_{\{L_{jti}>x\}}$ with all other data remaining the same. It is crucial to understand that the likelihood ratio processes $\widetilde{W}_{ti}$ in both cases are quite different: for L_{jti} we use $\widetilde{W}_{ti}$ as per (2.22), while for the indicator $I_{\{L_{jti}>x\}}$ (x is large) we use $\widetilde{W}_{ti}$ as per (2.28). In short, we use the CR estimator (3.1)–(2.22) and the SR estimator (3.1)–(2.28) for L_{jti} and $I_{\{L_{jti}>x\}}$, respectively.

Before turning to the choice of the reference parameter vector $\mathbf{v}_0$ in the stochastic program (1.8)–(3.1) with $\widetilde{W}_t$ as per (2.22) and (2.28), several remarks are in order.

Assume for a moment that the reference parameter vector $\mathbf{v}_0$ is already chosen and *none of the functions* $\ell_j(\mathbf{v})$, $j = 0, 1, \ldots, M$, contains rare events. Then viewing the $\bar{\ell}_{jN}(\mathbf{v})$ as functions of $\mathbf{v}$ rather than estimators for fixed $\mathbf{v}$, one may estimate the optimal solution, $\mathbf{v}^*$, by solving the stochastic program (1.8)–(3.1). Observe that as soon as the sample $\{\mathbf{Z}_{11}, \ldots, \mathbf{Z}_{\tau_1 1}, \ldots, \mathbf{Z}_{1N}, \ldots, \mathbf{Z}_{\tau_N N}\}$ is generated from the IS $g(\mathbf{y}) = f(\mathbf{y}, \mathbf{v}_0)$, the functions $\bar{\ell}_{jN}(\mathbf{v})$, $j = 0, \ldots, m$, become available for all $\mathbf{v} \in V$, and the corresponding gradients, $\nabla\bar{\ell}_{jN}(\mathbf{v})$, and

Hessian matrices, $\nabla^2\overline{\ell}_{jN}(\mathbf{v})$, can be calculated from the same *single simulation* run. Consequently, the optimization problem (1.8), (3.1) can be solved, in principle, by *standard methods of Mathematical Programming*. The resulting optimal value, $\overline{\ell}_{0N}(\bar{\mathbf{v}}_N^*)$, and optimal solution, $\bar{\mathbf{v}}_N^*$, of the program (1.8) provide estimators for the optimal value, $\ell_0(\mathbf{v}^*)$, and the optimal solution, $\mathbf{v}^*$, of the original program, (1.7), respectively. Following is the algorithm for estimating $\mathbf{v}^*$ from the solution of the stochastic counterpart (1.8)–(3.1).

Algorithm 3.1. *Estimating v^* with the stochastic counterpart (1.8)–(3.1) for a fixed v_0.*

Step 1. Generate a random sample, $\{\mathbf{Z}_{11},\dots,\mathbf{Z}_{\tau_1 1},\ \dots,\mathbf{Z}_{1N},\dots,\mathbf{Z}_{\tau_N N}\}$, from the dominating pdf, $g(\mathbf{y}) = f(\mathbf{y},\mathbf{v}_0)$.

Step 2. Generate the output (sample performance) processes $\{L_{jti} : j = 0,\dots,M\}$, and the likelihood ratio (weight) process $\{\widetilde{W}_{ti}(\mathbf{v}) : t = 1,\dots,\tau_i;\ i = 1,\dots,N\}$.

Step 3. Solve the program (1.8)–(3.1) by Mathematical Programming methods.

Step 4. Return the solution, $\bar{\mathbf{v}}_N^*$, of (1.8)–(3.1) as an estimator of $\mathbf{v}^*$.

Convergence conditions of Algorithm 3.1, to the unknown optimal parameter vector, $\mathbf{v}^*$, its asymptotic properties, and supportive simulation results may be found in [26].

Assume that the parameter set V in program (1.7) is given by

$$V = \{\boldsymbol{v} : \mathbf{0} \leqslant \boldsymbol{\rho}^- \leqslant \boldsymbol{\rho}(\boldsymbol{v}) \leqslant \boldsymbol{\rho}^+ < \mathbf{1},\quad \boldsymbol{\rho} = (\rho_1,\dots,\rho_r)\}, \tag{3.2}$$

where $\rho_k = \rho_k(\boldsymbol{v})$, $k = 1,\dots,r$, is the traffic intensity at the k-th queue, $\boldsymbol{\rho}^-$ and $\boldsymbol{\rho}^+$ are fixed, $\mathbf{1}$ is a vector of r one's, r is the number of nodes in the network, and inequalities between the vectors are component wise.

Note that there are two types of errors involved in a numerical solution of the program (1.7) by simulation based methods. One error comes from the fact that the approximating program (1.8) is solved imprecisely by an employed deterministic algorithm. The magnitude of that error depends on a chosen stopping criterion and is controlled by an associated precision value. The other error is of stochastic nature. It appears since the deterministic optimization is performed on the approximating program (1.8) which involves random functions as well (see [29]). That type of error is usually measured in terms of appropriate variances (mean square errors) which in turn are functions of the sample size N and of the reference parameter vector $\mathbf{v}_0$. We will be concerned with the stochastic error, and in particular with the error associated with the choice of $\mathbf{v}_0$ while solving the program (1.7), say while optimizing non-Markovian queueing networks consisting of r nodes.

With these remarks we turn to the choice of optimal ('good') parameter vector(s) $\mathbf{v}_0$ for the program (1.8)–(3.1). We consider separately the case where (i) *none* of the functions $\ell_j(\mathbf{v})$, $j = 0,1,\dots,M$, contain rare events; (ii) *all* functions $\ell_j(\mathbf{v})$, $j = 0,1,\dots,M$ contain rare events; (iii) *some* of the functions $\ell_j(\mathbf{v})$, $j = 0,1,\dots,M$, contain rare events.

(i) *None of the functions $\ell_j(\boldsymbol{v})$ contain rare events*

The following result is from [26]:

Solving program (1.7) with the stochastic counterpart (1.8)–(3.1), take, as a 'good' vector of reference parameter for $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_0(\mathbf{v}_0)$, either $\boldsymbol{\rho}^+$, or even a little larger than $\boldsymbol{\rho}^+$.

Remark 3.1. We take below $\boldsymbol{\rho}_0 = \boldsymbol{\rho}^+$. Note that the above statement means that a 'good' reference parameter $\mathbf{v}_0$ should correspond to the *highest traffic intensity vector* $\boldsymbol{\rho}^+$ among all traffic intensities defined in the constraint (3.2). Since the highest traffic intensity $\boldsymbol{\rho}^+$ corresponds to the 'noisiest' (most variable) system, the statement in turn means that choosing a 'good' $\mathbf{v}_0$, one can *ignore* all configurations (regimes) but the 'noisiest' one. It turns out that the 'noisiest' configuration is the 'most informative' one in the sense that taking $\boldsymbol{\rho}_0 = \boldsymbol{\rho}^+$, we have a good control over the entire system with the decision vector set $\mathbf{v} \in V$ given in (3.2).

Note that Remark 3.1 can be readily reformulated in terms of the original parameter vector $\mathbf{v}_0$, rather than in terms of $\boldsymbol{\rho}_0$. If, for example, $\mathbf{v}$ is the parameter vector of the interarrival rates in a queueing network, then the 'noisiest' configuration corresponds to the one where the values $\boldsymbol{\rho}_0$, $\boldsymbol{\rho}^-$ and $\boldsymbol{\rho}^+$ in (3.2) are replaced by $\boldsymbol{v}_0$, $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$, respectively. If, further, $\mathbf{v}$ is the parameter vector of the service rates, then we replace in (3.2), $\boldsymbol{\rho}^+$ by $\boldsymbol{v}^-$ and $\boldsymbol{\rho}^-$ by $\boldsymbol{v}^+$.

Remark 3.2. It is shown in [26] that for typical performance measures (e.g., for waiting time process in queueing models), there exist $\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}(\hat{v}, k)$, such that if the reference traffic intensity vector, $\boldsymbol{\rho}_0$, satisfies

$$\boldsymbol{\rho} \leqslant \boldsymbol{\rho}_0 \leqslant \hat{\boldsymbol{\rho}}, \tag{3.3}$$

then variance reduction is obtained in the sense of

$$\mathrm{Var}_{\boldsymbol{\rho}_0}\{\nabla^k \bar{\ell}_N(\boldsymbol{\rho}, \boldsymbol{\rho}_0)\} \leqslant \mathrm{Var}_{\boldsymbol{\rho}}\{\nabla^k \bar{\ell}_N(\boldsymbol{\rho}, \boldsymbol{\rho})\}, \quad k = 0, 1, \tag{3.4}$$

with strict equality holding for $\boldsymbol{\rho} = \hat{\boldsymbol{\rho}}$. Here $\bar{\ell}_N(\boldsymbol{\rho}, \boldsymbol{\rho})$ corresponds to the Crude Monte Carlo (CMC). In other words, for the score function (SF) estimator, $\nabla^k \bar{\ell}_N(\boldsymbol{\rho}_0)$, to perform *'well'* (in the sense of (3.4)), the reference traffic intensity, $\boldsymbol{\rho}_0$, must belong to the set $(\boldsymbol{\rho}, \hat{\boldsymbol{\rho}})$. Of course, it may happen that

$$\boldsymbol{\rho}^- = \min_{\mathbf{v} \in V} \boldsymbol{\rho}(\mathbf{v}_k) < \hat{\boldsymbol{\rho}},$$

in which case the variance is decreased in the sense of (3.4) whenever $\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{v})$ is in $[\hat{\boldsymbol{\rho}}, \boldsymbol{\rho}^+]$, and is increased otherwise. In the latter case, the variance increase is typically moderate when $\hat{\boldsymbol{\rho}} - \boldsymbol{\rho}$ is not too large and the cycle length τ tends to be small (see [26] for more details). If this is not the case, then the set V should be partitioned into smaller subsets and a different $\mathbf{v}_0$ should be chosen over each subset, or a recursive algorithm of stochastic approximation type could be used.

(ii) *All functions $\ell_j(\mathbf{v})$ contain rare events*

In this case we estimate the optimal solution $\mathbf{v}^*$ from the stochastic program (1.8)–(3.1),(3.2) while using the SR estimator (3.1)–(2.28) with the reference parameter vector $\mathbf{v}_0 = [\hat{\mathbf{v}}_{01}, \hat{\mathbf{v}}_{02}]$. As in case (i), for the standard performance functions, we suggest choosing (estimating) the optimal reference parameter vector $\mathbf{v}_0^* = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$ as in Modified Algorithm (2.2), but with $\boldsymbol{\rho}$ replaced by $\boldsymbol{\rho}^+$. That is, our estimator of the optimal reference parameter vector $\mathbf{v}_0^* = \mathbf{v}_0^*(\boldsymbol{\rho}^+) = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*](\boldsymbol{\rho}^+)$ will correspond to a configuration with the highest traffic intensity $\boldsymbol{\rho}^+$ defined in (3.2). Notice that as in Modified Algorithm (2.2), we estimate part $\hat{\mathbf{v}}_{01}^*$ from the program (2.27) (but with $\boldsymbol{\rho}$ replaced by $\boldsymbol{\rho}^+$), and take part $\hat{\mathbf{v}}_{02}^*$ as in the SR estimator (2.21)–(2.29), but with $\boldsymbol{\rho}$ replaced by $\boldsymbol{\rho}^+$ The above considerations lead to the following two-stage procedure for estimating the optimal solution $\mathbf{v}^*$ using the stochastic counterpart (1.8)–(3.1),(3.2).

Algorithm 3.2. *Two-stage procedure for estimating the optimal solution $\mathbf{v}^*$.*

Step 1. Apply the Modified Algorithm 2.2 to estimate the optimal reference parameter vector $\mathbf{v}_0^* = \mathbf{v}_0^*(\boldsymbol{\rho}^+) = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*](\boldsymbol{\rho}^+)$, which corresponds to the highest traffic intensity $\boldsymbol{\rho}^+$ defined in (3.2).

Step 2. Solve the program (1.8)–(3.1),(3.2) using the derived estimator $\bar{\mathbf{v}}_{0N}^* = \bar{\mathbf{v}}_{0N}^*(\rho^+)$ and deliver the solution as an estimate of $\mathbf{v}^*$.

Still one would be interested to know how reliable, in fact, is such an estimator of $\mathbf{v}^*$, when using the reference parameter vector $\bar{\mathbf{v}}_{0N}^*(\boldsymbol{\rho}^+)$ of Algorithm 3.2. The answer follows from the result of [15], where it is shown that, for fixed $\mathbf{v}$, the RS estimator (2.21)–(2.28) is *robust* in the sense that perturbing the optimal reference vector $\mathbf{v}_0^*$ by 20–30%, one loses only 2–3 orders of magnitude of variance reduction compared to the orders of ten under the optimal value $\mathbf{v}_0^*$. That is, perturbing $\mathbf{v}_0^*$ by 20–30% one still gets dramatic variance

reduction compared with the Crude Monte Carlo method. It is important to note that since $\boldsymbol{\rho}^+$ corresponds to the highest traffic intensity, the perturbations in $\boldsymbol{\rho}^+$ must be toward lower traffic intensities, which exactly correspond to (3.2). This implies (see also supporting numerical results below) that Algorithm 3.2 should work reasonably well if $(\boldsymbol{\rho}^+ - \boldsymbol{\rho}^-)\boldsymbol{\rho}^+ \leqslant 0.3$, component-wise. The above consideration must be taken carefully into account while defining the 'trust optimization region' and, in particular, the constraint (3.2).

(iii) *Some of the functions $\ell_j(v)$ contain rare events*

In this case, Step 1 of Algorithm 3.2 can be modified as follows: Take the highest traffic intensity $\boldsymbol{\rho}^+$ as a reference parameter vector $\boldsymbol{\rho}_0$ for the standard performance functions and for the rest (performance functions with rare events) take the parameter vector $\mathbf{v}_0^* = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$ from Algorithm 3.2. We call such an algorithm, *Modified Algorithm 3.2.*

To see again that the reference vectors $\mathbf{v}_0^*$ might be quite different in CS and SR estimators, while solving the program (1.8)–(3.1),(3.2), consider the following example:

Example 3.1. Let $\ell_1(\mathbf{v}) = \mathbb{E}_{\mathbf{v}}\{L_t\}$ and $\ell_2(\mathbf{v}, x) = \mathbb{E}_{\mathbf{v}}\{I_{\{L_t > x\}}\}$ be two performance functions in the program (1.7)–(3.2), where $\{L_t\}$ is the waiting time process at the M/M/1 queue. Assume that the parameters ρ^+ and ρ^- in (3.2) are $\rho^+ = 0.7$ and $\rho^- = 0.5$, respectively, i.e. $0.5 \leqslant \rho \leqslant 0.7$. Assume finally that $\lambda = 1$ and the service rate v is the decision variable. For the part $\ell_1(\mathbf{v})$, we take $\rho_0 = \rho^+ = 0.7$ ($v_0 = 1/0.7 \approx 1.4$), which corresponds to the highest traffic intensity. For the part $\ell_2(\mathbf{v})$, we take $\mathbf{v}_0^* = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$, where $\hat{\mathbf{v}}_{01}^* = (v_0, \lambda) = (1.4, 1)$ and $\hat{\mathbf{v}}_{02}^* = (\lambda, v_0) = (1, 1.4)$. Note that in $\hat{\mathbf{v}}_{01}^*$ we interchange λ with $v_0 = v_0(\rho^+)$ while in $\hat{\mathbf{v}}_{02}^*$ we switch back.

Since again the reference parameter vectors $\mathbf{v}_0$ in the CR and SR estimators might be quite different we could use *two different simulations*: one simulation for the standard performance functions and another for performance functions containing rare events. Our numerical results (see below) suggest, in fact, that Algorithm 3.2 based on a *single simulation* with SR estimators is sufficient, provided $\mathbb{E}_{\mathbf{v}}\{L_t\} < 10^{-6}$. More precisely, simulating the entire system only once, with the reference parameters $\mathbf{v}_0^* = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$ corresponding to performance functions containing rare events, we derived estimators of the optimal solution $\mathbf{v}^*$ of the program (1.7)–(3.2) which are as good (accurate) as estimators based on two independent simulations mentioned above. In other words, ignoring the standard performance functions, while making a decision about the optimal reference parameter vector $\mathbf{v}_0^*$ does not influence the accuracy of the estimator of the optimal decision vector $\mathbf{v}^*$. Our argument for this is in the spirit of Remark 3.2, namely taking $\mathbf{v}_0^* = \mathbf{v}_0^*(\boldsymbol{\rho}^+) = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$ as reference parameter, for both the standard performance functions and performance functions with rare events, we have a good control over the entire system with the decision vector set $\mathbf{v} \in V$ given in (3.2). This important (still empirical) evidence requires solid mathematical and statistical grounds, and thus further investigation.

4. Numerical results

We present separate numerical results concerning (i) estimating the optimal reference parameter $\mathbf{v}_0^*$ and (ii) estimating the optimal solution $\mathbf{v}^*$ of the constrained program (P_0).

4.1. Estimating the optimal parameter vector v_0^*

Below we present numerical results for a static model and for dynamic models. Our first example concerns a coherent reliability model while the second concerns a queueing model.

Consider the double-bridge reliability model containing 10 elements and depicted in Fig. 1.

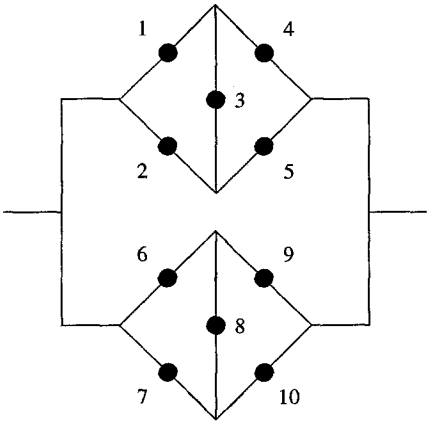

Fig. 1. A double-bridge structure.

Table 1
Performance of the LR and CMC estimators for the failure probability $\ell = P\{L > x\} = 10^{-4}$ in the double-bridge structure

			$\bar{\ell}_N(\mathbf{v}, \bar{\mathbf{v}}_0^*)$		$\bar{\ell}_{N_{\text{CMC}}}(\mathbf{v})$	
i	v_i	$\bar{v}_{0i}^*$	PE	CW	PE	CW
1	1.0	0.30	$1.1 \cdot 10^{-4}$	0.21	$9.7 \cdot 10^{-5}$	0.73
2	1.0	0.29				
3	1.0	0.89				
4	2.0	1.90				
5	2.0	1.89				
6	1.0	0.30				
7	1.0	0.30				
8	1.0	0.88				
9	2.0	1.88				
10	2.0	1.76				

Table 1 displays point estimators (PE), $\bar{\ell}_N(\mathbf{v}, \bar{\mathbf{v}}_0^*)$, and the 95% relative width of the confidence interval (CW) for the failure probability $\ell = P\{L > x\} = 10^{-4}$ for exponentially distributed component life. A sample size of $N = 10^5$ from $\boldsymbol{Y} \sim f(\mathbf{y}, \mathbf{v}_0)$ was used to estimate the optimal vector of reference parameters, $\mathbf{v}_0^*$, according to Algorithm 2.2, and an additional sample size of $N = 10^5$ was used to estimate the rare event probability $\ell(\mathbf{v}, x) = \mathbb{E}_{\mathbf{v}}\{I_{L>x}\}$. We used two iterations of Algorithm 2.2, starting at $x_0 = 1$, which under $\mathbf{v}_0 = \mathbf{v}$ corresponds to $P_{\mathbf{v}}\{L > x_0\} \approx 10^{-2}$.

It is readily seen that the LR estimators, $\bar{\ell}_N(\mathbf{v}, \mathbf{v}_0^*)$, outperform their CMC counterpart, $\bar{\ell}_{N_{\text{CMC}}}(\mathbf{v})$. The efficiency of the estimator $\bar{\ell}_N(\mathbf{v}, \mathbf{v}_0^*)$ relative to their CMC counterpart $\bar{\ell}_{N_{\text{CMC}}}(\mathbf{v})$ was found to increase with the rarity of the event $\{L > x\}$.

Consider now two M/M/1 queues in tandem with routing as depicted in Fig. 2. Here v_1 denotes the customer interarrival rate, v_2, v_3 denote the service rate of the first and the second queue, respectively, and v_4 is the routing probability.

Our goal is to estimate the optimal reference parameter vector $\mathbf{v}_0^* = (v_{01}^*, \ldots, v_{04}^*)$ using Modified Algorithm 2.2 and the associated (small) probability $\ell(x) = P(L_t > x)$ of excessive backlog (buffer overflow) in the *transient* regime. Recall that a cycle is defined as a duration starting with an empty system and ending at the instant the system, for the first time, either becomes empty again or the number of customers in the system, L_t, reaches x.

Tables 2 and 3 present the results of two simulation experiments with the model. Both experiments were conducted for $x = 20$. In the first case $\ell(x) = 3.269 \times 10^{-4}$, while in the second $\ell(x) = 2.390 \times 10^{-8}$. The relevant statistics were obtained on the basis of 20 experiments, each containing 1500 cycles in the first case and 2000 in the second.

For both experiments we used two iterations of Modified Algorithm 2.2, each starting at $x_0 = 7.0$ and $\mathbf{v}_0 = \mathbf{v}$; a total of 2×10^4 cycles to find $\hat{v}_0^*$ and the same number of cycles to calculate $\bar{\ell}_N(\hat{v}_0^*, x)$. Our results are

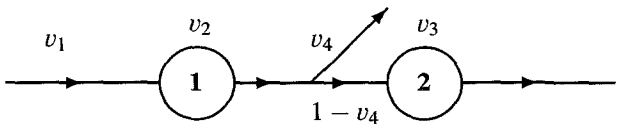

Fig. 2. Two M/M/1 queues in tandem with routing.

Table 2
Performance of Algorithm 2.2 for two M/M/1 queues in tandem with routing for probability $\ell(x) = 3.269 \times 10^{-4}$

				Parekh and Walrand		Algorithm 2.2	
i	v_i	v_{0i}^*	$\hat{v}_{0i}^*$	$\bar{\ell}_N(\mathbf{v}_0^*, x)$	$\hat{\sigma}$	$\bar{\ell}_N(\hat{v}_0^*, x)$	$\hat{\sigma}$
1	0.200	0.300	0.310	$3.194 \cdot 10^{-4}$	$1.370 \cdot 10^{-5}$	$3.286 \cdot 10^{-4}$	$1.218 \cdot 10^{-5}$
2	0.300	0.200	0.205				
3	0.500	0.500	0.479				
4	0.100	0.100	0.099				

Table 3
Performance of Algorithm 2.2 for two M/M/1 queues in tandem with routing for probability $\ell(x) = 2.390 \times 10^{-8}$

				Parekh and Walrand		Algorithm 2.2	
i	v_i	v_{0i}^*	$\hat{v}_{0i}^*$	$\bar{\ell}_N(\mathbf{v}_0^*, x)$	$\hat{\sigma}$	$\bar{\ell}_N(\hat{v}_0^*, x)$	$\hat{\sigma}$
1	0.100	0.220	0.228	$2.405 \cdot 10^{-8}$	$5.110 \cdot 10^{-10}$	$2.378 \cdot 10^{-8}$	$6.391 \cdot 10^{-10}$
2	0.700	0.700	0.687				
3	0.200	0.080	0.082				
4	0.200	0.090	0.086				

compared with those of Parekh and Walrand in [20], where the optimal reference parameter $\mathbf{v}_0^*$ was calculated analytically (using large deviation theory) and then a total of 2×10^4 cycles were used to estimate $\ell_N(\mathbf{v}_0^*, x)$. The true parameters v_i of the considered model are given in the first column of the tables. The optimal values of the reference parameters v_{0i}^*, $i = 1, 2, 3, 4$, calculated analytically in [20] and these $\hat{v}_{0i}^*$, $i = 1, 2, 3, 4$, estimated according to Algorithm 2.2 are given in the second and the third columns, respectively.

It readily follows from Tables 2 and 3 that the accuracies (variances) of estimators obtained according to Modified Algorithm 2.2 are close to that of Parekh and Walrand in [20]. At this end it is crucial to note that the latter approach is limited to some particular settings, like simple Markovian models, while the former is rather general.

4.2. *Estimating the optimal solution of the program* (P_0)

We consider here only the following unconstrained program:

$$\min_{\mathbf{v} \in V} \ell(\mathbf{v}) = \min_{\mathbf{v} \in V} \left[c \left(\mathbb{E}_{\mathbf{v}}\{L_1\} \right)^{1/4} + b\, \mathbb{E}_{\mathbf{v}}\{I_{\{L_2 > x\}}\} + d||\mathbf{v}|| \right], \quad \mathbf{v} \in V, \tag{4.1}$$

where L_1 and L_2 are some steady-state random variables, say the steady-state random variables of the sojourn time and queue length processes, respectively, in a GI/G/1 queue, c is the cost of a waiting customer, b is the unit cost associated with the probability $P\{L_2 > x\}$ (x is large), and d is the cost associated with the service rates.

Note as before that, under some mild regularity conditions (see, e.g., Rubinstein and Shapiro [26]), $\ell(\mathbf{v})$ is a strictly convex differentiable function with respect to $\mathbf{v}$. Thus, $\mathbf{v}^*$ is a unique minimizer of (2.6) over the convex region V. This allows to estimate the optimal solution $\mathbf{v}^*$ of the program (4.1) from the following stochastic counterpart:

$$\nabla \bar{\ell}_N(\mathbf{v}) = \mathbf{0}, \quad \mathbf{v} \in V.$$

Note also that the estimator of $\nabla \ell_1(\mathbf{v}) = \nabla \mathbb{E}_{\mathbf{v}}\{L_1\}$ can be written as

$$\nabla\bar{\ell}_{1N}(\mathbf{v}) = \frac{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i} L_{ti1}\nabla\widetilde{W}_{ti}}{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i}\widetilde{W}_{ti}(\mathbf{v}_0)} - \frac{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i} L_{ti1}\widetilde{W}_{ti}(\mathbf{v}_0)}{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i}\widetilde{W}_{ti}(\mathbf{v}_0)} \cdot \frac{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i}\nabla\widetilde{W}_{ti}(\mathbf{v}_0)}{\sum_{i=1}^{N}\sum_{t=1}^{\tau_i}\widetilde{W}_{ti}(\mathbf{v}_0)}, \quad \mathbf{v}\in V, \tag{4.2}$$

and similarly the estimator of $\nabla\ell_2(\mathbf{v}) = \nabla\mathbb{E}_{\mathbf{v}}\{I_{\{L_2>x\}}\}$.

Example 4.1. *The* M/M/1 *queue.* Let λ and v be the interarrival and service rates, respectively, and let v be the decision parameter. Assume that L_1 and L_2 are the steady-state random variables presenting the sojourn time and the queue length, respectively, and consider the program (4.1) with $v \in V$ corresponding to

$$0.2 = \rho^- \leqslant \rho(v) \leqslant \rho^+ = 0.8.$$

Taking into account that $\mathbb{E}_v\{L_1\} = 1/(v-\lambda)$ and $\mathbb{E}_v\{I_{\{L_2>x\}}\} = (\lambda/v)^x$ it is readily seen that the optimal v^* that minimizes the performance measure given in (4.1) is the solution of the following equation:

$$-\frac{c}{4(v-\lambda)^{5/4}} - \frac{bx}{v}\left(\frac{\lambda}{v}\right)^x + d = 0.$$

Table 4 displays the theoretical curves, $\ell(\rho)$ and $\nabla\ell(\rho)$, and their stochastic analogs ($\bar{\ell}_N(\rho)$ and $\nabla\bar{\ell}_N(\rho)$) as functions of $\rho = \rho(v)$ ($\rho = \lambda/v$) for different sample sizes N. The parameters chosen were $\lambda = 1$, $b = 5\times10^5$, $c = 1$, $d = 5$, and $x = 20$ (these correspond to $v^* \approx 2$, $\rho^* = \rho(v^*) \approx 0.5$ and $\mathbb{E}_v\{I_{\{L_2>x\}}\} = (\lambda/v)^x \approx 10^{-6}$). In this experiment we used two different simulations: one for the part $\mathbb{E}_v\{L_1\}$ and another for the part $\mathbb{E}_v\{I_{\{L_2>x\}}\}$ (involving rare events). More precisely, we used the conventional regenerative and switching regenerative estimators (3.1)–(2.22) and (3.1)–(2.28), respectively. According to Modified Algorithm (3.2) the corresponding reference parameters (for $\mathbb{E}_v\{L_1\}$ and $\mathbb{E}_v\{I_{\{L_2>x\}}\}$) were set to $v_0 = 1.25$ and $\mathbf{v}_0^* = [\hat{\mathbf{v}}_{01}^*, \hat{\mathbf{v}}_{02}^*]$, where $\hat{\mathbf{v}}_{01}^* = (v_0, \lambda) = (1.25, 1)$ and $\hat{\mathbf{v}}_{02}^* = (\lambda, v_0) = (1, 1.25)$, respectively. This corresponds to $\rho_0^* = \rho^+ = 0.8$ for part $\mathbb{E}_v\{L_1\}$, and to the interchange of λ and v_0^* (in $\hat{\mathbf{v}}_{01}^*$) and then switching back (in $\hat{\mathbf{v}}_{02}^*$) for part $\mathbb{E}_v\{I_{\{L_2>x\}}\}$.

Table 4 clearly indicates that the estimated curves, $\bar{\ell}_N(\rho)$ and $\nabla\bar{\ell}_N(\rho)$, *converge* to their respective theoretical curves, $\ell(\rho)$ and $\nabla\ell(\rho)$, for $\rho \leqslant \rho_0 = 0.8$, and *diverge* from $\ell(\rho)$ and $\nabla\ell(\rho)$, respectively, for $\rho > \rho_0 = 0.8$. These results are in agreement with Remark 3.2.

Table 4
The theoretical values, $\ell(\rho)$ and $\nabla\ell(\rho)$, and the corresponding estimated values ($\bar{\ell}_N(\rho)$, $\nabla\bar{\ell}_N(\rho)$), as functions of ρ, for $b = 5\times10^5$, $c = 1$ and $d = 5$, $\lambda = 1$, and reference traffic intensity $\rho_0 = \rho^+ = 0.8$

	5000		50000		500000		Theoretical value	
ρ/N	$\bar{\ell}_N$	$\nabla\bar{\ell}_N$	$\bar{\ell}_N$	$\nabla\bar{\ell}_N$	$\bar{\ell}_N$	$\nabla\bar{\ell}_N$	ℓ	$\nabla\ell$
0.1	50.1	4.98	50.6	4.98	50.6	4.98	50.6	4.98
0.2	25.4	4.96	25.6	4.96	25.7	4.96	25.7	4.96
0.3	17.2	4.91	17.3	4.91	17.5	4.91	17.5	4.91
0.4	13.0	4.81	13.2	4.79	13.3	4.81	13.4	4.81
0.5	6.41	0.11	6.44	0.06	6.47	0.00	6.48	0.00
0.6	26.9	−196	27.1	−206	27.7	−214	27.7	−216
0.7	382	$-5\cdot10^4$	396	$-5\cdot10^4$	409	$-6\cdot10^4$	407	$-6\cdot10^4$
0.8	$5.6\cdot10^3$	$-8\cdot10^4$	$5.7\cdot10^3$	$-8.4\cdot10^4$	$6\cdot10^3$	$-9\cdot10^4$	$6\cdot10^3$	$-9\cdot10^4$
0.85	$1.2\cdot10^4$	$-9\cdot10^4$	$1.4\cdot10^4$	$-2\cdot10^5$	$1.3\cdot10^4$	$-2.1\cdot10^5$	$2\cdot10^4$	$-3\cdot10^5$
0.88	$2.2\cdot10^4$	$-3\cdot10^5$	$2.2\cdot10^4$	$-5\cdot10^5$	$2.3\cdot10^4$	$-7\cdot10^5$	$4\cdot10^4$	$-7\cdot10^5$

Table 5
Performance of the stochastic counterpart (4.2), for the M/M/1 queue with reference traffic intensity $\rho_0 = 0.8$

ρ^*	b	α	$\bar{v}_N^*(b)$	v^*	95% CI
0.88	1.453	−0.091	0.675	1.136	0.07, 1.27
0.85	4.271	−0.058	0.978	1.176	0.78, 1.18
0.8	19.44	0.000	1.261	1.255	1.15, 1.37
0.7	383.1	0.143	1.433	1.429	1.33, 1.54
0.6	$1.0 \cdot 10^4$	0.333	1.654	1.667	1.58, 1.73
0.5	$5.0 \cdot 10^5$	0.600	1.971	2.000	1.91, 2.03
0.4	$5.5 \cdot 10^7$	1.000	2.467	2.500	2.42, 2.51
0.3	$2.3 \cdot 10^{10}$	1.667	3.324	3.333	3.25, 3.39
0.2	$1.2 \cdot 10^{14}$	3.000	4.947	5.000	4.74, 5.15
0.1	$2.5 \cdot 10^{20}$	7.000	9.741	10.00	9.42, 10.06

Table 5 displays theoretical values of v^*, the point estimators, $\bar{v}_N^* = \bar{v}_N^*(b)$, and the 95% confidence intervals of v^* (denoted by 95% CI) as functions of b, $\rho^* = \lambda/v^*$ and $\alpha = [\rho_0 - \rho^*]/\rho^*$. The M/M/1 queue was then simulated for $N = 10\,000$ cycles (approximately 50 000 customers). Note that *all* estimators $\bar{v}_N^*(b)$ were obtained *simultaneously from a single simulation run* (with $\rho_0 = 0.8$), by solving the system of equations (4.2) for different values of b.

It is readily seen that the estimator $\bar{v}_N^*$ performs reasonably well for $\rho \in (0.3, 0.8)$. Poor performance of the estimator $\bar{v}_N^*$ may be noted for the following cases:

(a) For $\rho > 0.8$, this is a consequence of violating the requirement of Remark 3.2 which calls for ρ in the range $\rho \leqslant \rho_0 = 0.8$; in fact, we have $0.8 \geqslant \rho \geqslant \rho_0 = 0.8$.

(b) For $\rho < 0.2$ ($\alpha = [\rho_0 - \rho^*]/\rho^* > 3$), this is a consequence of the fact that ρ is perturbed from ρ^* by more than 300%.

5. Concluding remarks and further research

We showed how to optimize complex computer simulation models involving rare events from using a specific setup of the IS distribution, namely we assumed that the IS comes from the same parametric family as the original (true) one. More specifically, we presented a two-stage Algorithm 3.2 suited for optimization with the stochastic counterpart (1.8)–(3.1),(3.2): at the first stage we identified (estimated) the optimal parameter vector $\mathbf{v}_0^*$ of the IS distribution, while at the second stage we estimated the optimal solution $\mathbf{v}^*$ of the constrained program (1.7)–(3.2). Particular emphasis was placed on estimation of rare events and on integration of the associated performance function into stochastic optimization programs.

This, in fact, was the first attempt to solve such important problems. The whole area is still in its infancy and no doubt much more must be and can be done. As for further research we suggest to:

(i) consider optimization of computer simulation models involving rare events with respect to structural decision variables and in particular to exploit the capabilities of the 'push-out' method;

(ii) combine recursive and non-recursive procedures, like stochastic approximation and stochastic counterpart;

(iii) investigate in more detail the complexity properties of estimators of $\mathbf{v}_0^*$ and $\mathbf{v}^*$ based on both *single* and *double* simulation runs, as discussed in the paper; and

(iv) apply the methodology for optimal decision making with rare events in such important fields as queueing, inventories, reliability and insurance risk.

Acknowledgements

We would like to thank Arkadii Nemirovskii, Sergey Porotsky and Dmitrii Lieber from the Technion, Vladimir Katkovnik from UNISA, South Africa, Alexander Shapiro from the Georgia Institute of Technology and two anonymous referees for several valuable suggestions on the earlier draft of the paper.

References

[1] Asmussen, S., *Applied Probability and Queues*, Wiley, New York, 1987.
[2] Asmussen, S., and Rubinstein, R.Y., "The efficiency and heavy traffic properties of the score function method in sensitivity analysis of queueing models", *Advances of Applied Probability* 24/1 (1992) 172–201.
[3] Asmussen, S., and Rubinstein, R.Y., "Response surface estimation and sensitivity analysis via efficient change of measure", *Stochastic Models* 9/3 (1993) 313–339.
[4] Asmussen, S., and Rubinstein, R.Y., "Complexity properties of steady-state rare events simulation in queueing models", in: J. Dshalalow (ed.), *Advances in Queueing: Theory, Methods and Open Problems*, CRC Press, Boca Raton, FL, 1995, 429–462.
[5] Asmussen, S., and Rubinstein, R.Y., "Sensitivity analysis of insurance risk models", Manuscript, University of Lund, Sweden, 1996.
[6] Bucklew, J.A., Ney, P., and Sadowsky, J.S., "Monte Carlo simulation and large deviations theory for uniformly recurrent Markov chains", *Journal of Applied Probability* 27 (1990) 44–59.
[7] Ermoliev, Y., "Stochastic quasi-gradient methods and their application to systems optimization", *Stochastics* 4 (1983) 1–37.
[8] Ermoliev, Y.M., and Gaivoronski, A.A., "Stochastic programming techniques for optimization of discrete event systems", *Annals of Operations Research* 39 (1992) 1–41.
[9] Glasserman, P., and Liu, T.-W., "Rare-event simulation for multistage production-inventory systems", *Management Science*, to appear.
[10] Glasserman, P., and Kou, S.-G., "Analysis of importance sampling estimator for tandem queues", *ACM TOMACS* 5, 22–42.
[11] Frater, M.R., Lennon, T.M., and Anderson, B.D.O., "Optimally efficient estimation of the statistics of rare events in queueing networks", *IEEE Transactions on Automatic Control* 36 (1991) 1395–1405.
[12] Goyal, A., Shahabuddin, P., Heidelberger, P., Nicola, V.F., and Glynn, P.W., "A unified framework for simulating Markovian models of highly dependable systems", *IEEE Transactions on Computers* C-41 (1992) 36–51.
[13] Heidelberger, P., "Fast simulation of rare events in queueing and reliability models", *ACM Transactions on Modeling and Computer Simulation* 5 (1995) 43–85.
[14] Kovalenko, I., "Approximations of queues via small parameter method", in: J. Dshalalow (ed.), *Advances in Queueing: Theory, Methods and Open Problems*, CRC Press, Boca Raton, FL, 1995, 481–509.
[15] Kriman, V., and Rubinstein, R.Y., "Polynomial and exponential time algorithms for estimation of rare events in queueing models", in: J. Dshalalow (ed.), *Frontiers in Queueing: Models and Applications in Science and Engineering*, CRC Press, Boca Raton, FL, to appear.
[16] L'Ecuyer, P., Giroux, N., and Glynn, P.W., "Stochastic optimization by simulation: Numerical experiments with M/M/1 queue in steady-state", *Management Science* 40 (1994) 1245–1261.
[17] L'Ecuyer, P., and Glynn, P.W., "Stochastic optimization by simulation: Convergence proofs for the GI/G/1 queue in steady-state", *Management Science* 40 (1994) 1562–1578.
[18] Lieber, D., Rubinstein, R.Y., and Elmakis, D., "Quick estimation of rare events in stochastic networks", Manuscript, Technion, Haifa, Israel, 1993.
[19] Ljung, P., Pflug, G., and Walk, H., *Stochastic Approximation and Optimization of Stochastic Systems*, Birkhauser, Basel, 1992.
[20] Parekh, S., and Walrand, J., "A quick simulation method for excessive backlogs in networks of queues", *IEEE Transactions on Automatic Control* 34 (1989) 54–66.
[21] Pflug, G.Ch., "Optimization of simulated discrete event processes", *Annals of Operations Research* 39 (1992) 173–194.
[22] Pflug, G.Ch., and Rubinstein, R.Y., "Simulation based optimization of the (s, S)-policy inventory model by the push-out method", Manuscript, Technion, Haifa, Israel, 1996.
[23] Plambeck, E.L., Fu, B.R., Robinson, S.M., and Suri, R., "Sample-path optimization of convex stochastic performance functions", *Mathematical Programming B* 75 (1996) 137–176.
[24] Polyak, B.T., and Juditsky, A.B., "Acceleration of stochastic approximation by averaging", *SIAM Journal on Control and Optimization* 30 (1992) 838–855.
[25] Rubinstein, R.Y., "Sensitivity analysis of discrete event systems by the 'push-out' method", *Annals of Operations Research* 39 (1992) 229–250.
[26] Rubinstein, R.Y., and Shapiro, A., *Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization via the Score Function Method*, Wiley, New York, 1993.
[27] Sadowsky, J.S., "On the optimality and stability of exponential twisting in Monte Carlo simulation", *IEEE Transactions on Information Theory* 39 (1993) 119–128.

[28] Shapiro, A., "Asymptotic behavior of optimal solutions in stochastic programming", *Mathematics of Operations Research* 18 (1993) 829–845.
[29] Shapiro, A., "Simulation based optimization-convergence analysis and statistical inference", *Stochastic Models*, to appear.
[30] Shapiro, A., and Wardi, Y., "Nondifferentiability of the steady-state function in discrete event dynamic systems", *IEEE Transactions on Automatic Control* 39 (1994) 1707–1711.
[31] Shapiro, A., and Wardi, Y., "Convergence analysis of stochastic algorithms", *Mathematics of Operations Research*, to appear.
[32] Shapiro, A., and Wardi, Y., "Convergence analysis of gradient descent stochastic algorithms", *Journal of Optimization Theory and Applications*, to appear.
[33] Shwartz, A., and Weiss, A., *Large Deviation for Performance Analysis: Queues, Communication and Computers*, Chapman & Hall, London, 1995.