Qsort(A, 1, 8)

1	2	3	4	5	6	7	8
2	8	7	1	3	5	6	4

partition

1	2	3	4	5	6	7	8
2	1	3	4	7	5	6	8

Qsort(A, 1, 3)

Qsort(A, 5, 8)

Done!

h^* : max height

prove $h^* \le 185$

An Example: 185 cm

$h^* \ge 185$?

or

$\cancel{h^* \le 185}$?

$T(n)$: worst-case

prove $T(n) = O(n^2)$

An Example: n^2

at least (lower bound)

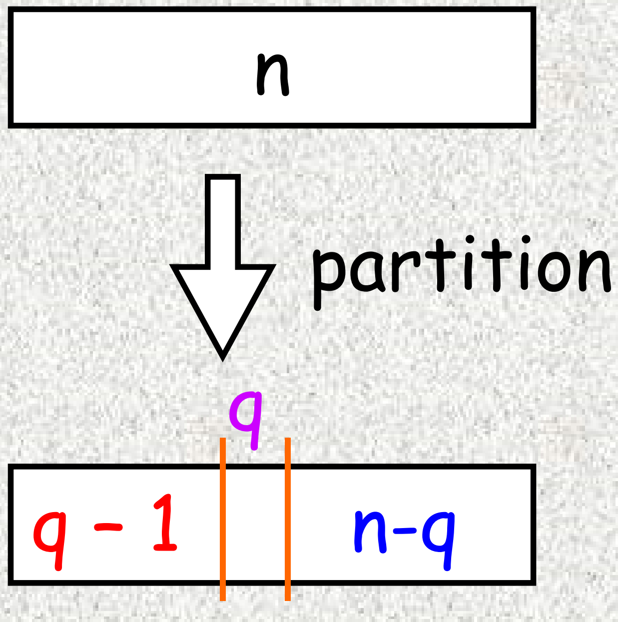
$T(n) = \Omega(n^2)$?

at most (upper bound)

or

$\cancel{T(n) = O(n^2)}$?

Quick Sort: Worst Case



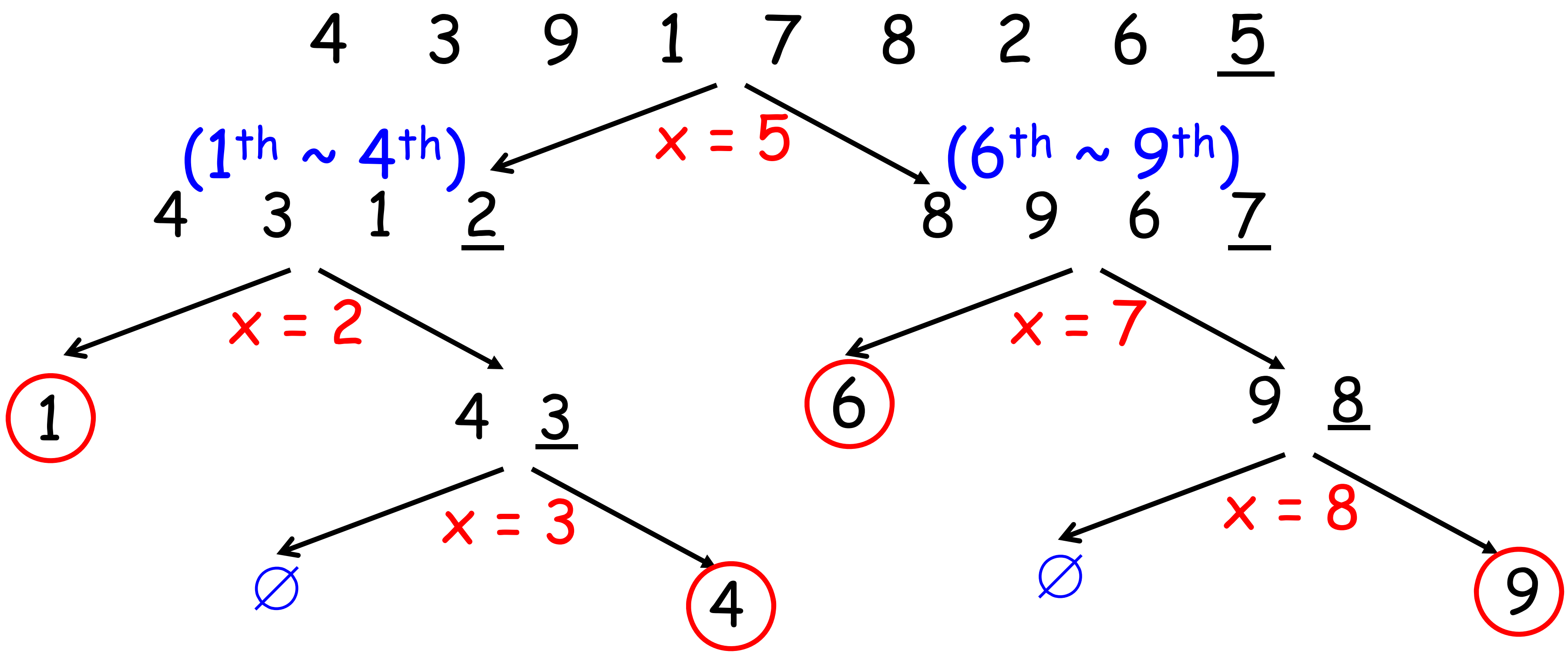
0	1	n-1
1	2	n-2
2	3	n-3
⋮	⋮	⋮
n-1	n	0

$$T(n) = (n - 1) + \max_{1 \le q \le n} \{ T(q-1) + T(n-q) \}$$

$$= \underline{(n - 1)} + \max_{0 \le k \le n-1} \{ T(k) + T(n-k-1) \}$$

n, bn, $\Theta(n)$, $O(n)$

$A = 4^{th}\ 3^{rd}\ 9^{th}\ 1^{st}\ 7^{th}\ 8^{th}\ 2^{nd}\ 6^{th}\ 5^{th}$



*Every number serves as a pivot at most once!

*Each call is on "consecutive" numbers!

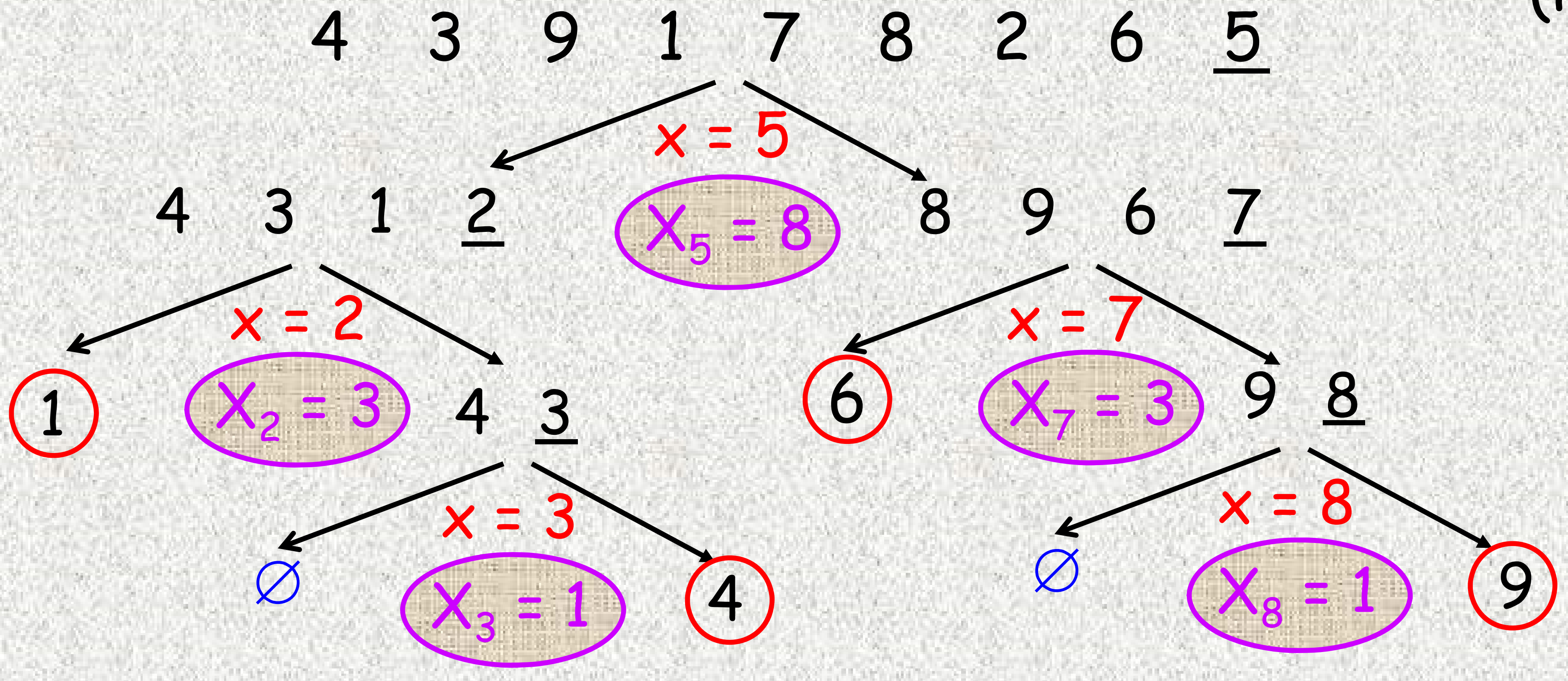
at most n internal nodes

at most n+1 leaves

$\Rightarrow$ at most 2n+1 calls to Quicksort

$A = 4^{th}\ 3^{rd}\ 9^{th}\ 1^{st}\ 7^{th}\ 8^{th}\ 2^{nd}\ 6^{th}\ 5^{th}$

(n = 9)

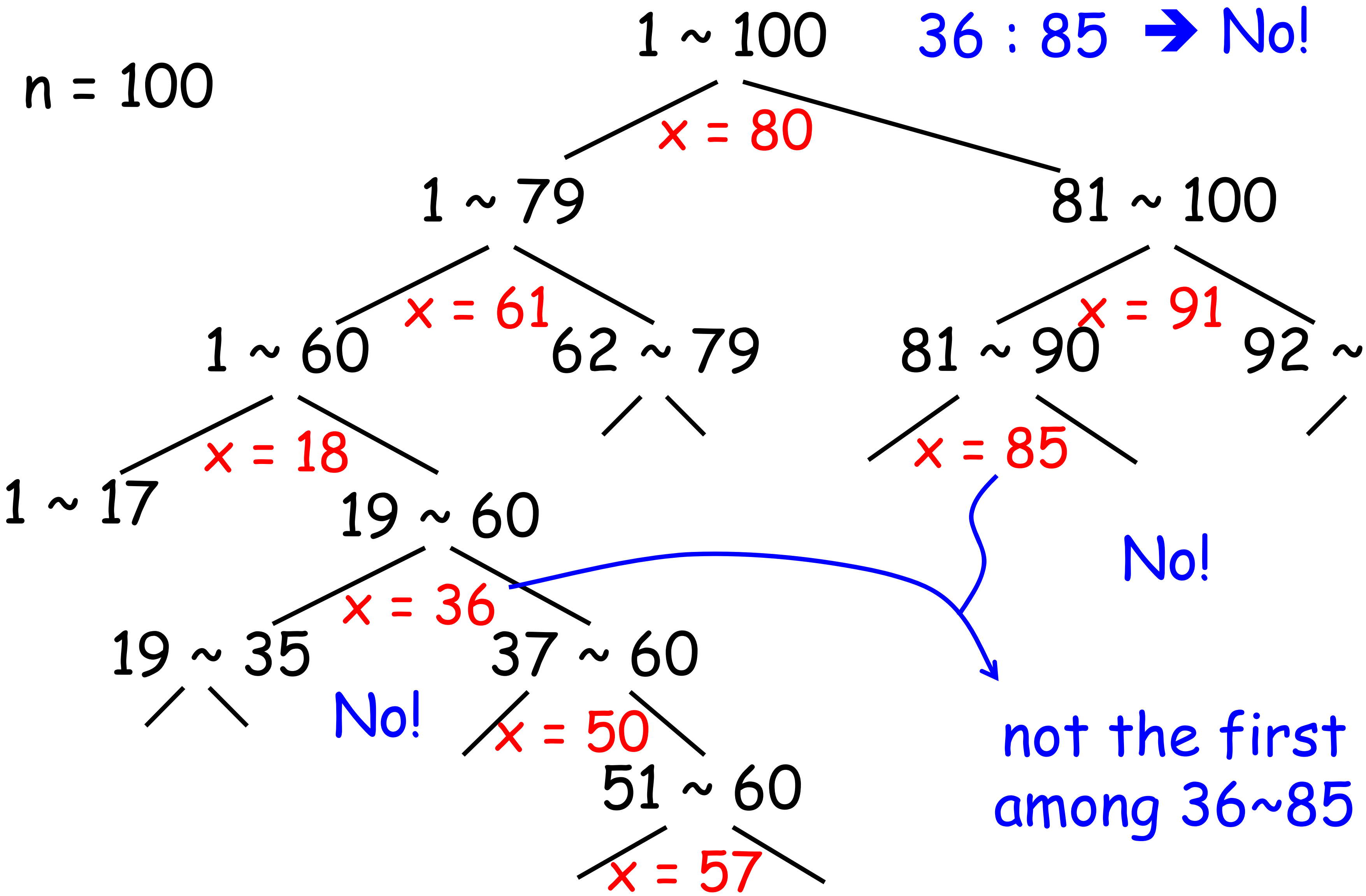


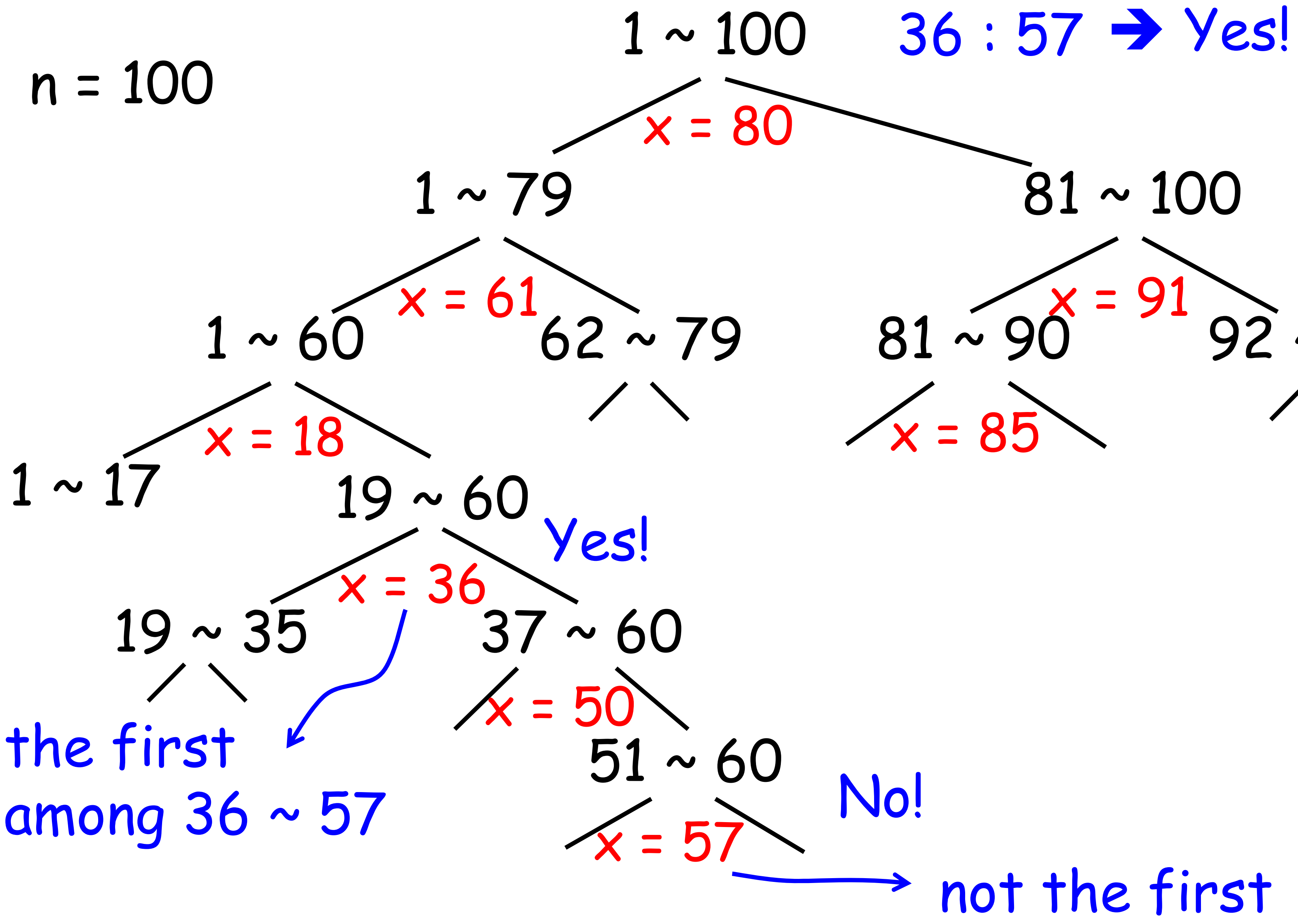
* X: total # of comparisons of all calls to Partition

(In this example, $X = 8 + 3 + 1 + 3 + 1 + (6 \times 0) = 16$)

* $T(n) = O(n + X)$

→ at most $O(n)$ calls to Quicksort

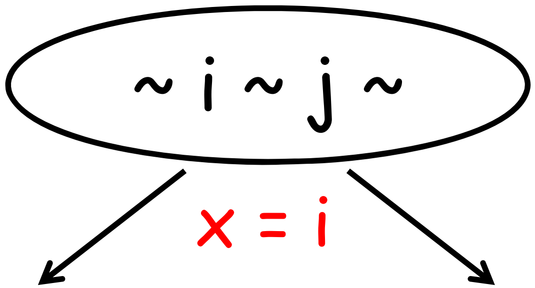

n = 100
1 ~ 100
36 : 85 → No!
x = 80
1 ~ 79
81 ~ 100
x = 61
x = 91
1 ~ 60
62 ~ 79
81 ~ 90
92 ~ 100
x = 18
x = 85
1 ~ 17
19 ~ 60
No!
x = 36
19 ~ 35
37 ~ 60
No!
x = 50
51 ~ 60
x = 57
not the first
among 36~85



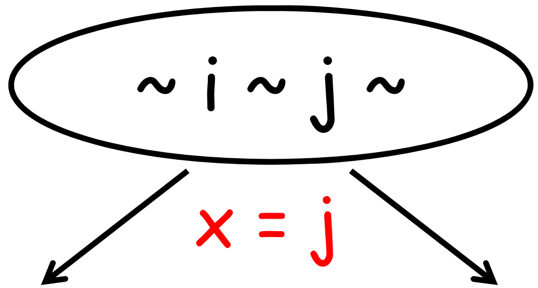
n = 100
1 ~ 100
36 : 57 → Yes!
x = 80
1 ~ 79
81 ~ 100
x = 61
x = 91
1 ~ 60
62 ~ 79
81 ~ 90
92 ~ 100
x = 18
x = 85
1 ~ 17
19 ~ 60
Yes!
x = 36
19 ~ 35
37 ~ 60
x = 50
the first
among 36 ~ 57
51 ~ 60
No!
x = 57
not the first

$i^{th} : j^{th}$

(1)



(2)

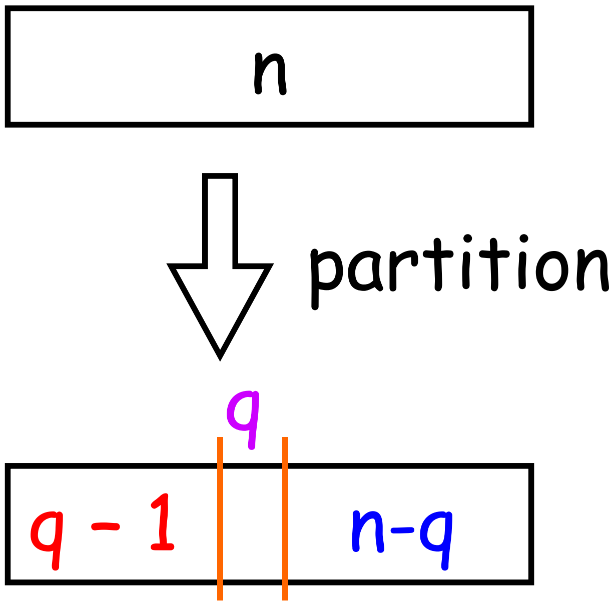


$i : j$ if and only if

no one in $[i+1, j-1]$ becomes a pivot before i and j

➔ probability $= \dfrac{|\{i, j\}|}{|\{i, i+1, \ldots, j\}|} = \dfrac{2}{j - i + 1}$

Quick Sort: Average Case



0	1	n-1
1	2	n-2
2	3	n-3
⋮	⋮	⋮
n-1	n	0

$$E(n) = (n-1) + \begin{cases} 1/n * (\cancel{E(0)} + E(n-1)) \\ 1/n * (E(1) + E(n-2)) \\ 1/n * (E(2) + E(n-3)) \\ \qquad \vdots \qquad\qquad \vdots \\ 1/n * (E(n-1) + \cancel{E(0)}) \end{cases}$$

$$= (n-1) + \frac{1}{n}\sum_{q=1}^{n}\left(E(q-1) + E(n-q) \right)$$

$$= \underbrace{(n-1)}_{n,\ bn,\ \Theta(n)} + \frac{2}{n}\sum_{k=1}^{n-1} E(k)$$

Classic approach:

$$E(n) = \left\{ \begin{array}{c} \Theta(n) \\ bn \\ n \\ n-1 \end{array} \right\} + \frac{2}{n} \sum_{k=1}^{n-1} E(k)$$

guess $E(n) = O(n \lg n)$
prove by substitution method
(very hard to understand!!!)

Knuth's approach:

$$E(n) = n+1 + \frac{2}{n} \sum_{k=1}^{n-1} E(k)$$

Classic approach:

$$E(n) = \left\{ \begin{matrix} \Theta(n) \\ bn \\ n \\ n-1 \end{matrix} \right\} + \frac{2}{n}\sum_{k=1}^{n-1} E(k)$$

guess $E(n) = O(n \lg n)$
prove by substitution method
(very hard to understand!!!)

Knuth's approach:

$$E(n) = n+1 + \frac{2}{n}\sum_{k=1}^{n-1} E(k) \longrightarrow E(n) = \frac{n+1}{n} E(n-1) + 2$$

compute $E(n) \approx 2n \lg n$
by iteration method!

Randomized-variable approach:

$$E(n) = O(n + X) = O(n + \sum_{i,j} Pro(i : j))$$

assume independent

$a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n$

① $a_1 = x$

partition

$\le x$ | $\ge x$

② *a_1 may be either □ or □

*guarantee $|\square| \ge 1$

$|\square| \ge 1$