------------------------------- 9 -------------------------------

Medians and Order Statistics

***i*-th order statistic of a set of *n* elements**: *i*-th *smallest element*.

minimum: $i = 1$
median: $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$
maximum: $i = n$

The selection problem:

(For simplicity, assume that all elements in the input set are distinct.)

Input: $A[1..n]$ and i
Output: the i-th order statistic of A

9.1 Minimum (Maximum)

```
Minimum(A)
   min ← A[1]
   for i ← 2 to n do
      if A[i]<min then min ← A[i]
   return min
```

- $T(n)=O(n)$ (exactly n-1 comparisons)
- n-1 is optimal (every element loses at least once)

- Simultaneous minimum and maximum:

  Step 1: Perform $\lfloor n/2 \rfloor$ disjoint pairwise comparison.
  Step 2: Find minimum among the set containing the smaller elements.
  Step 3: Find maximum among the set containing the larger elements.

  $T(n)$: at most $3\lfloor n/2 \rfloor$ comparisons.

9.2 Selection in expected linear time

(divide & conquer, or prune-and-search)

```
Randomized-Select(A, p, r, i)
   if p=r then return A[p]
   q ← Randomized-Partition(A, p, r)
   k ← q-p+1
   if i = k then return A[q]
   elseif i < k
      then return Randomized-Select(A, p, q–1, i)
   else return Randomized-Select(A, q+1, r, i-k)
```

- Worst case: $T(n) = O(n) + T(n-1) = O(n^2)$.

- Average case:

$$E(n) = O(n) + \frac{1}{n}\sum_{1 \le k \le n} E(\max\{k-1, n-k\})$$
$$= O(n) + \frac{2}{n}\sum_{\lfloor n/2 \rfloor \le k \le n-1} E(k)$$
$$= O(n)$$

(Prove it using substitution method by yourself.)

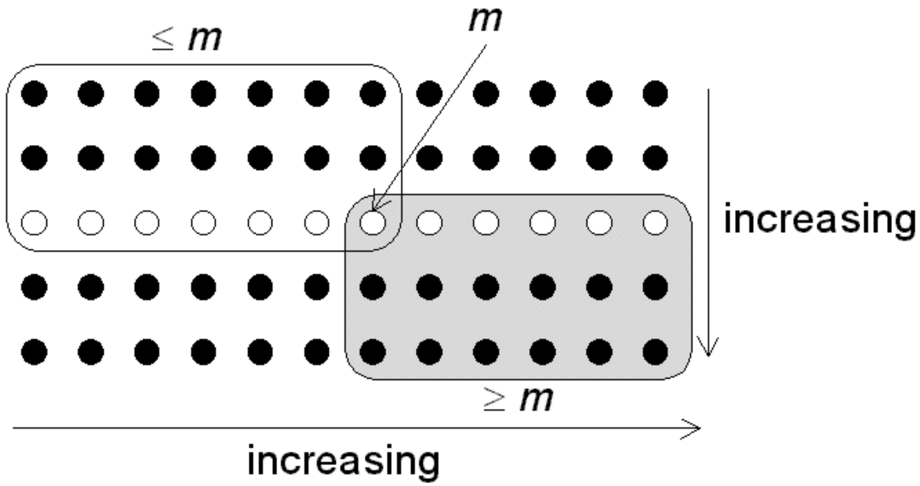
9.3 Selection in worst case linear time
(D&C, or prune and search)

Select(S, i)

1. Divide S into n/r subsequences of r integers ($r \ge 5$, odd, constant).
2. Sort every group. Let m_k be the median of the k-th subsequence.
3. Find the median m of m_k's.
4. Partition S into three subsequences:
   $S_1=\{x|x<m\}$, $S_2=\{x|x=m\}$, $S_3=\{x|x>m\}$.
   Then, the i-th smallest element of S is located in one of the three subsequences.

5. if $i \le |S_1|$ then Select(S_1, i)
   elseif $i \le (|S_1|+|S_2|)$ then return m
   else Select(S_3, i-($|S_1|+|S_2|$)).

- Analysis (Assume r=5)



At least one fourth of S is discard. ($|S_1| \le 3|S|/4$ and $|S_3| \le 3|S|/4$)

Step	1.	2.	3.	4.	5.
Time	--	$O(n)$	$T(n/5)$	$O(n)$	$T(3n/4)$

$$T(n) \le T(n/5)+T(3n/4)+\Theta(n)$$
$$= O(n) \text{ (Prove it by substitution method.)}$$

Homework: Ex. 9.1-1, 9.3-3, 9.3-6, 9.3-7, 9.3-8, 9.3-9, Pro. 9-1, 9-2, 9-3. Read Ch10 ~ 12.