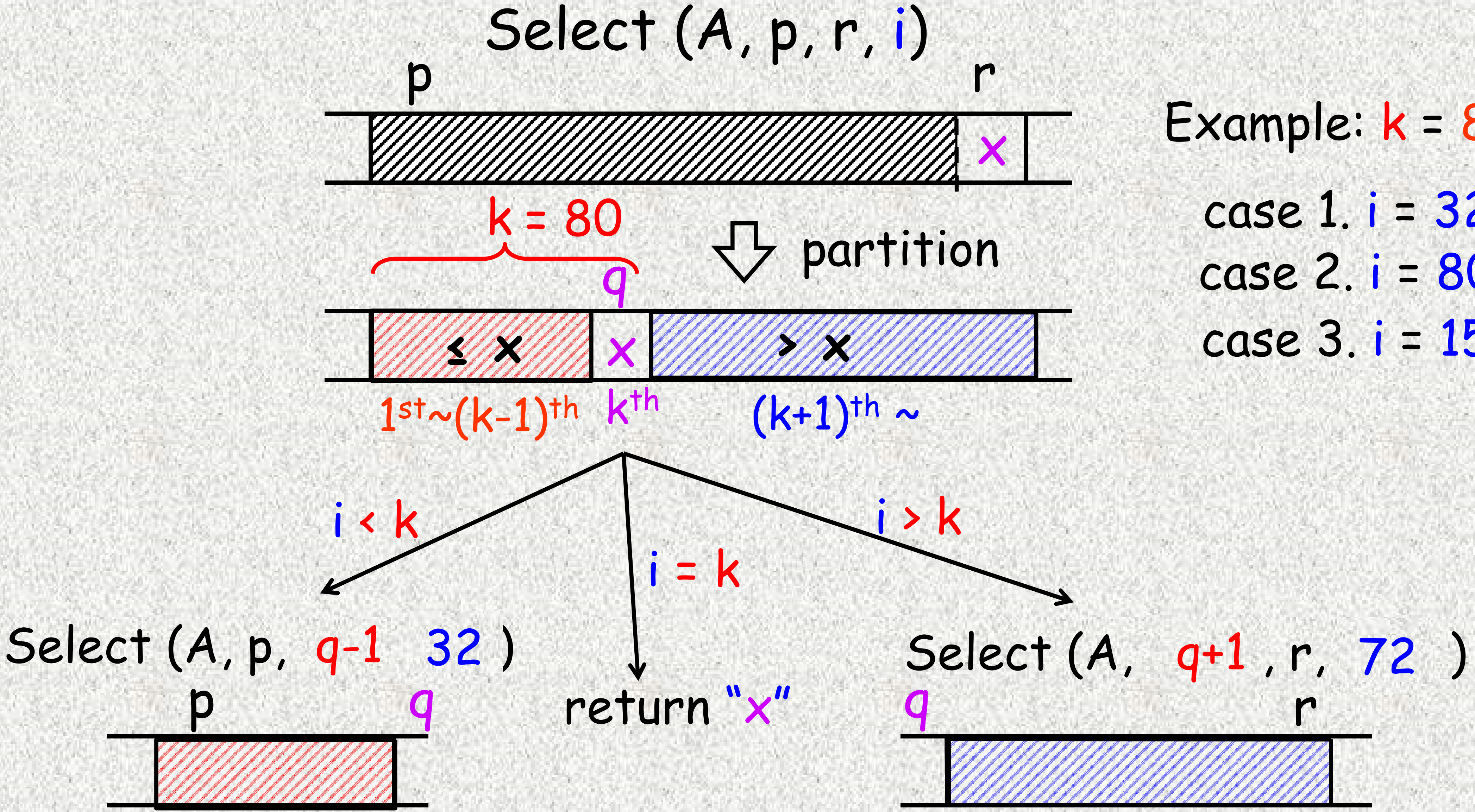
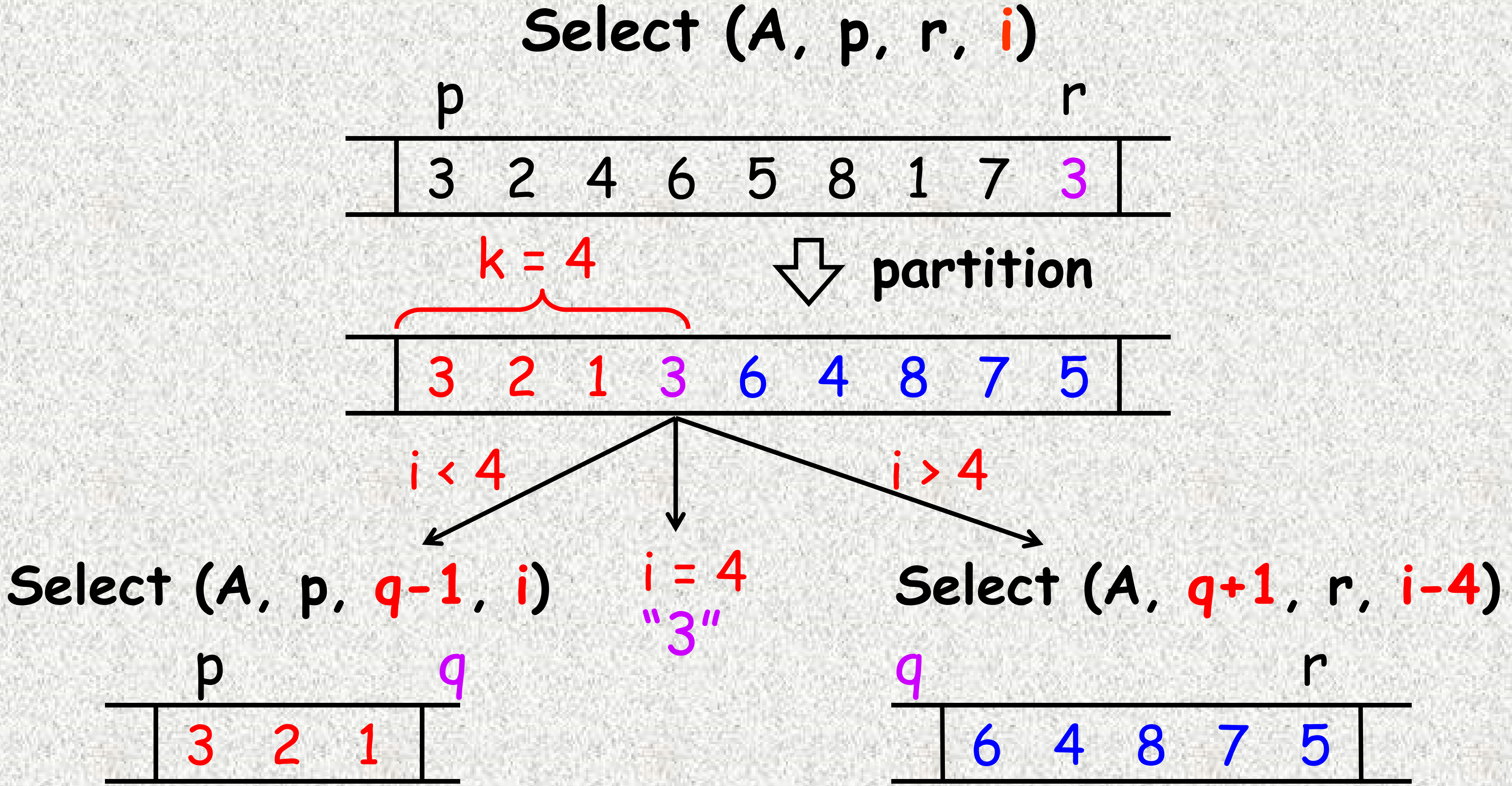
min $\boxed{1}$

8	7	9	3	4	5	1	9

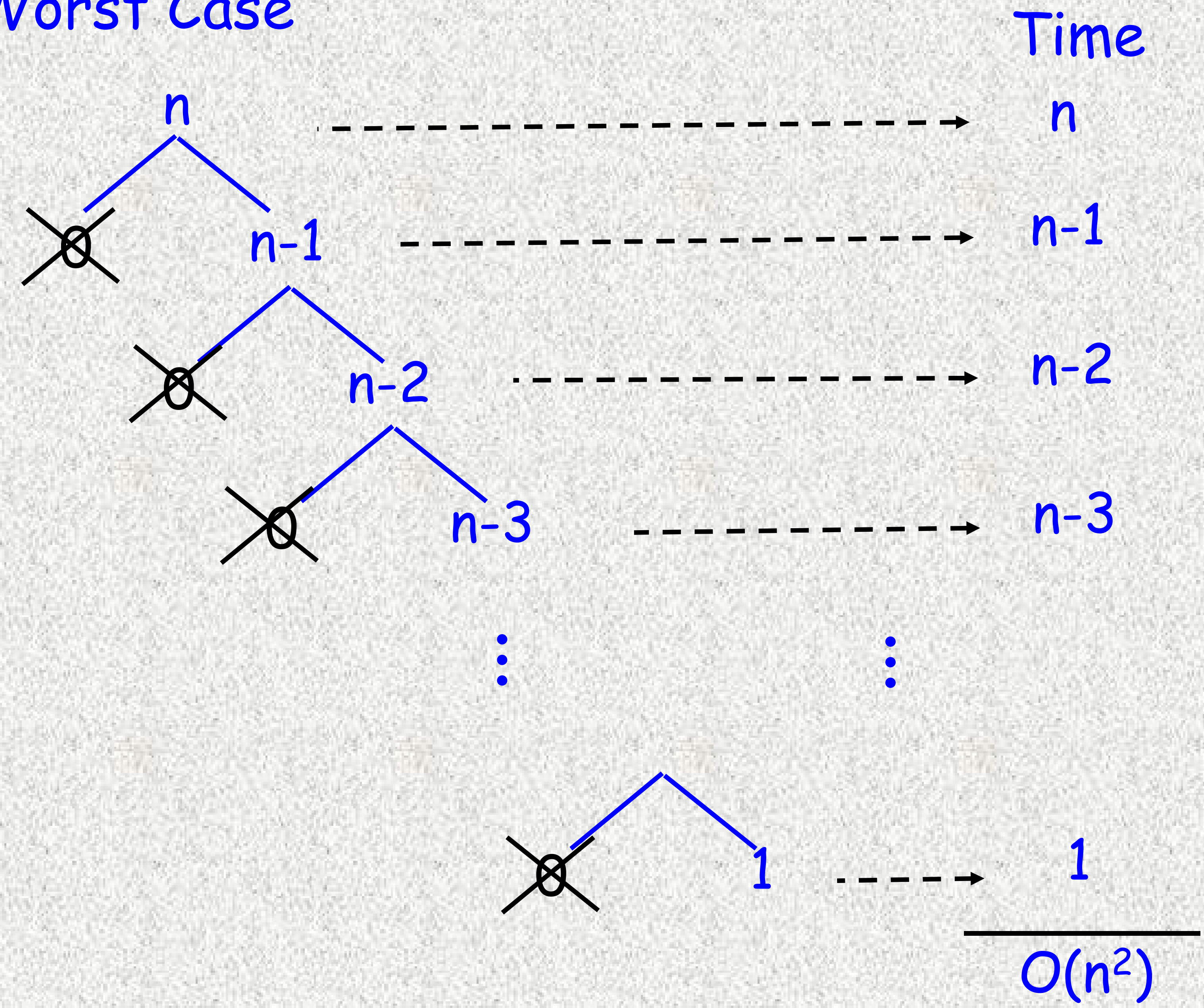


Example: $k = 80$

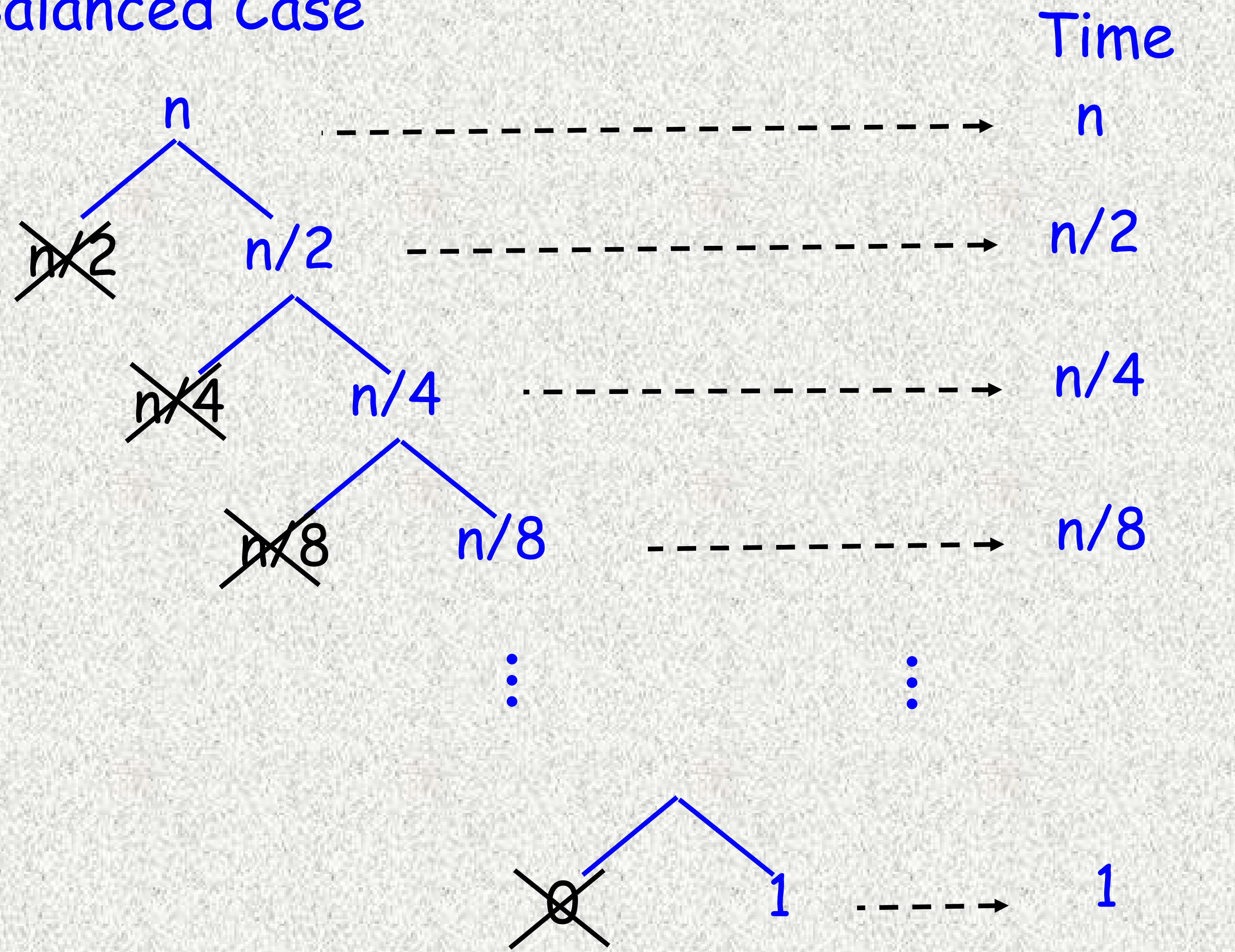
case 1. $i = 32$
case 2. $i = 80$
case 3. $i = 152$


Select (A, p, r, i)
p
r
3 2 4 6 5 8 1 7 3
k = 4
partition
3 2 1 3 6 4 8 7 5
i < 4
i = 4
"3"
i > 4
Select (A, p, q-1, i)
p
q
3 2 1
Select (A, q+1, r, i-4)
q
r
6 4 8 7 5

Selection: Worst Case



Selection: Balanced Case

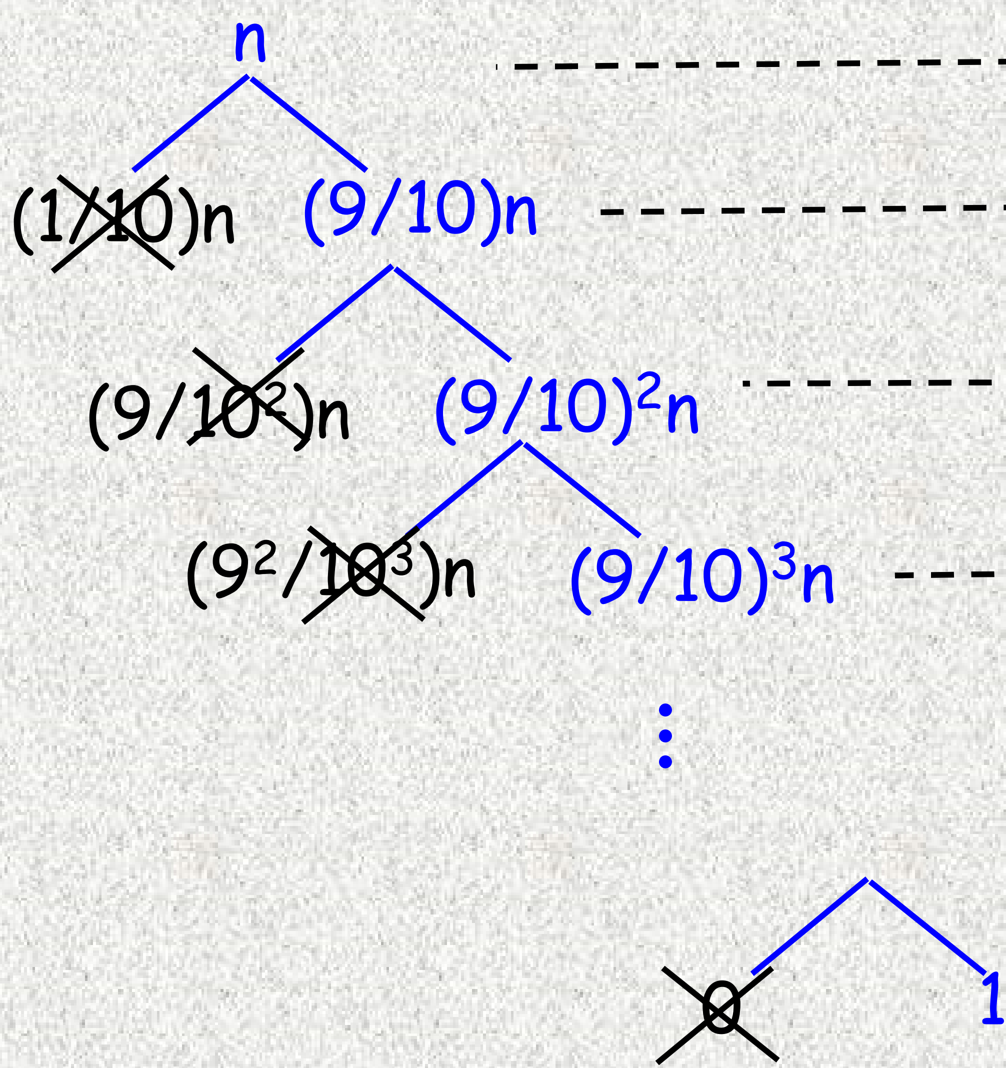


$r = 0.5$

$O(n) \approx 2n$

等比會收斂

Selection: Assume 1/10 candidates are removed



Time

n

$(9/10)n$

$(9/10)^2n$

$(9/10)^3n$

⋮

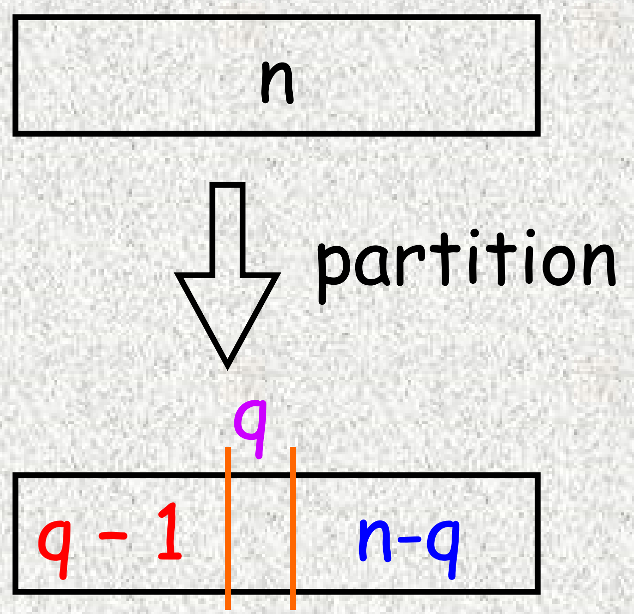
1

$r = 0.9$

$O(n) \approx 10n$

等 比 會 收 斂

Quick Sort: Average Case



0	1	n-1
1	2	n-2
2	3	n-3
⋮	⋮	⋮
n-1	n	0

$\cancel{0},1,2\ldots,n-1 \qquad n-1,\ldots,2,1,\cancel{0}$

$$E(n) = (n-1) + \underbrace{\frac{1}{n}}_{\text{機率}} \sum_{q=1}^{n} \big(\underbrace{E(q-1) + E(n-q)}_{\text{兩邊都要做}} \big)$$

$$= (n-1) + \frac{2}{n}\big(E(1)+E(2)+\ldots+E(n-1)\big)$$

$$= (n-1) + \frac{2}{n}\sum_{k=1}^{n-1} E(k)$$

$$= O(n \lg n)$$ (substitution method or Knuth's approach)

Quicksort: Average Case

$$E(n) = n\text{-}1 + \frac{1}{n}\sum_{q=1}^{n}\{\ E(q-1) + E(n-q)\ \} \qquad = n\text{-}1 + \frac{2}{n}\sum_{k=1}^{n-1} E(k)$$

兩 邊 都 要 做

substitution method or Knuth's approach $\Rightarrow$ $O(n \lg n)$

Selection: Average Case

永 遠 都 做 大 的 那 一 邊 (多 算 沒 關 係)

$$E(n) = n\text{-}1 + \frac{1}{n}\sum_{q=1}^{n}\{\ E(\max\{q-1, n-q\})\ \}$$

substitution method $\Rightarrow$ $O(n)$

只 做 一 邊，機 率 按 比 例

$$E(n) = n\text{-}1 + \frac{1}{n}\sum_{q=1}^{n}\left\{\ \frac{q-1}{n-1}E(q-1) + \frac{n-q}{n-1}E(n-q)\ \right\}$$

Knuth's approach $\Rightarrow$ $O(n)$

Selection: Average Case

只做一邊，機率按比例

$$E(n) = n-1 + \frac{1}{n}\sum_{q=1}^{n}\left\{\frac{q-1}{n-1}E(q-1) + \frac{n-q}{n-1}E(n-q)\right\}$$

$$E(n) = n-1 + \frac{1}{n(n-1)}\{0E(0) + 1E(1) + 2E(2) + \ldots + (n-1)E(n-1) +$$

$$(n-1)E(n-1) + (n-2)E(n-2) + \ldots + 0E(0)\}$$

$$E(n) = n-1 + \frac{2}{n(n-1)}\sum_{k=1}^{n-1} kE(k)$$

Knuth's approach

$$E(n) = n+1 + \frac{2}{n(n-1)}\sum_{k=1}^{n-1} kE(k) \qquad (1)$$

不換也可以，但推導比較比較不簡潔漂亮

$$E(n) = n+1 + \frac{2}{n(n-1)} \sum_{k=1}^{n-1} kE(k) \quad \text{——— (1)}$$

$$n(n-1)E(n) = (n+1)n(n-1) + 2\sum_{k=1}^{n-1} kE(k) \quad \text{——— (2): (1)} \times n(n-1)$$

$$(n-1)(n-2)E(n-1) = n(n-1)(n-2) + 2\sum_{k=1}^{n-2} kE(k) \quad \text{——— (3): (2) with } n = n-1$$

$$n(n-1)E(n) = n(n-1)(3) + 2(n-1)E(n-1) + (n-1)(n-2)E(n-1)$$

$$n(n-1)E(n) = 3n(n-1) + n(n-1)E(n-1)$$

$$E(n) = 3 + E(n-1) = 3n = O(n) \quad \text{(by iteration method)}$$

1. $\underline{4\ 1\ 6\ 3\ 1}\ \underline{2\ 5\ 7\ 8\ 7}\ \underline{6\ 5\ 9\ 1\ 7}\ \underline{1\ 3\ 5\ 7\ 6}\ \underline{9\ 2\ 4\ 3\ 5}$ $(r = 5)$

2.
$$
\begin{array}{ccccc}
 & 1 & & 2 & & 1 & & 1 & & 2 \\
 & 1 & & 5 & & 5 & & 3 & & 3 \\
m_1 & 3 & m_2 & 7 & m_3 & 6 & m_4 & 5 & m_5 & 4 \\
 & 4 & & 7 & & 7 & & 6 & & 5 \\
 & 6 & & 8 & & 9 & & 7 & & 9
\end{array}
$$

$M = \{3, 7, 6, 5, 4\}$

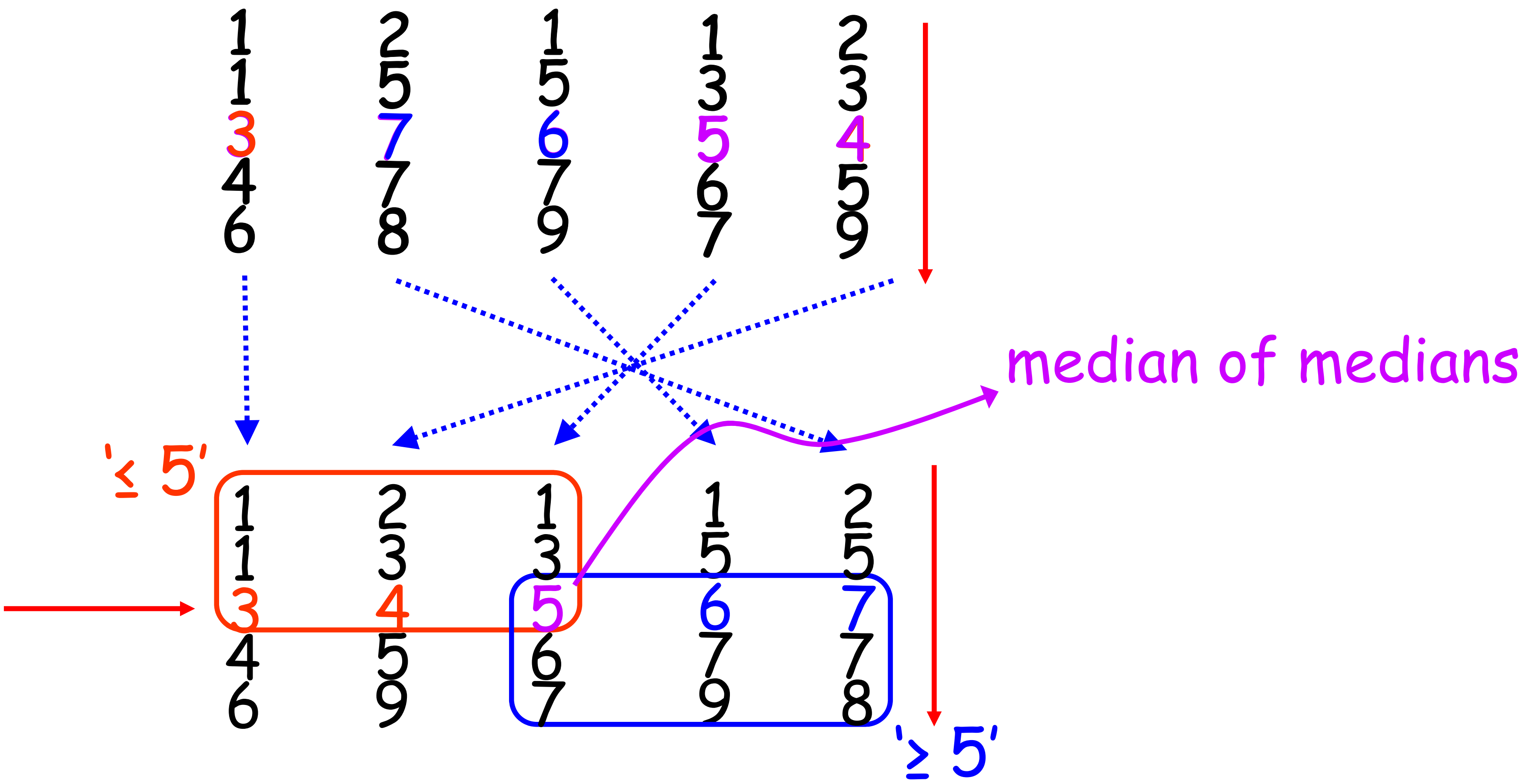
3. $m = \text{Select}(M, \lceil |M|/2 \rceil)$ $= 5$ (median of medians)

4. $S_1 = \{4\ \ 1\ \ 3\ \ 1\ \ 2\ \ 1\ \ 1\ \ 3\ \ 2\ \ 4\ \ 3\}$
$S_2 = \{5\ \ 5\ \ 5\ \ 5\}$
$S_3 = \{6\ \ 7\ \ 8\ \ 7\ \ 6\ \ 9\ \ 7\ \ 7\ \ 6\ \ 9\}$

$\boxed{|S_1| = 11,\ |S_2| = 4,\ |S_3| = 10}$

5.

case 1. $i = 7$	case 2. $i = 13$	case 3. $i = 22$
Select(S_1, 7)	return $m = 5$	Select(S_3, $22 - 15$)



⇒ '≤ 5' 超過 $\frac{1}{4}$ ⇒ '> 5' 最多 $\frac{3}{4}$ ⇒ $|S_3| \le \frac{3}{4}n$

⇒ '≥ 5' 超過 $\frac{1}{4}$ ⇒ '< 5' 最多 $\frac{3}{4}$ ⇒ $|S_1| \le \frac{3}{4}n$

1. 4 1 6 3 1 2 5 7 8 7 6 5 9 1 7 1 3 5 7 6 9 2 4 3 5 (r = 5)

2.

	m_1	m_2	m_3	m_4	m_5
	1	2	1	1	2
	1	5	5	3	3
	3	7	6	5	4
	4	7	7	6	5
	6	8	9	7	9

$M = \{3, 7, 6, 5, 4\}$

3. $m = \text{Select}(M, |M|/2) = 5$ (median of medians)

4. $S_1 = \{4\ 1\ 3\ 1\ 2\ 1\ 1\ 3\ 2\ 4\ 3\}$
$S_2 = \{5\ 5\ 5\ 5\}$
$S_3 = \{6\ 7\ 8\ 7\ 6\ 9\ 7\ 7\ 6\ 9\}$

$\boxed{|S_1| = 11,\ |S_2| = 4,\ |S_3| = 10}$

5. $\boxed{\text{case 1. } i = 7}$ $\boxed{\text{case 2. } i = 13}$ $\boxed{\text{case 3. } i = 22}$

Select(S_1, 7)

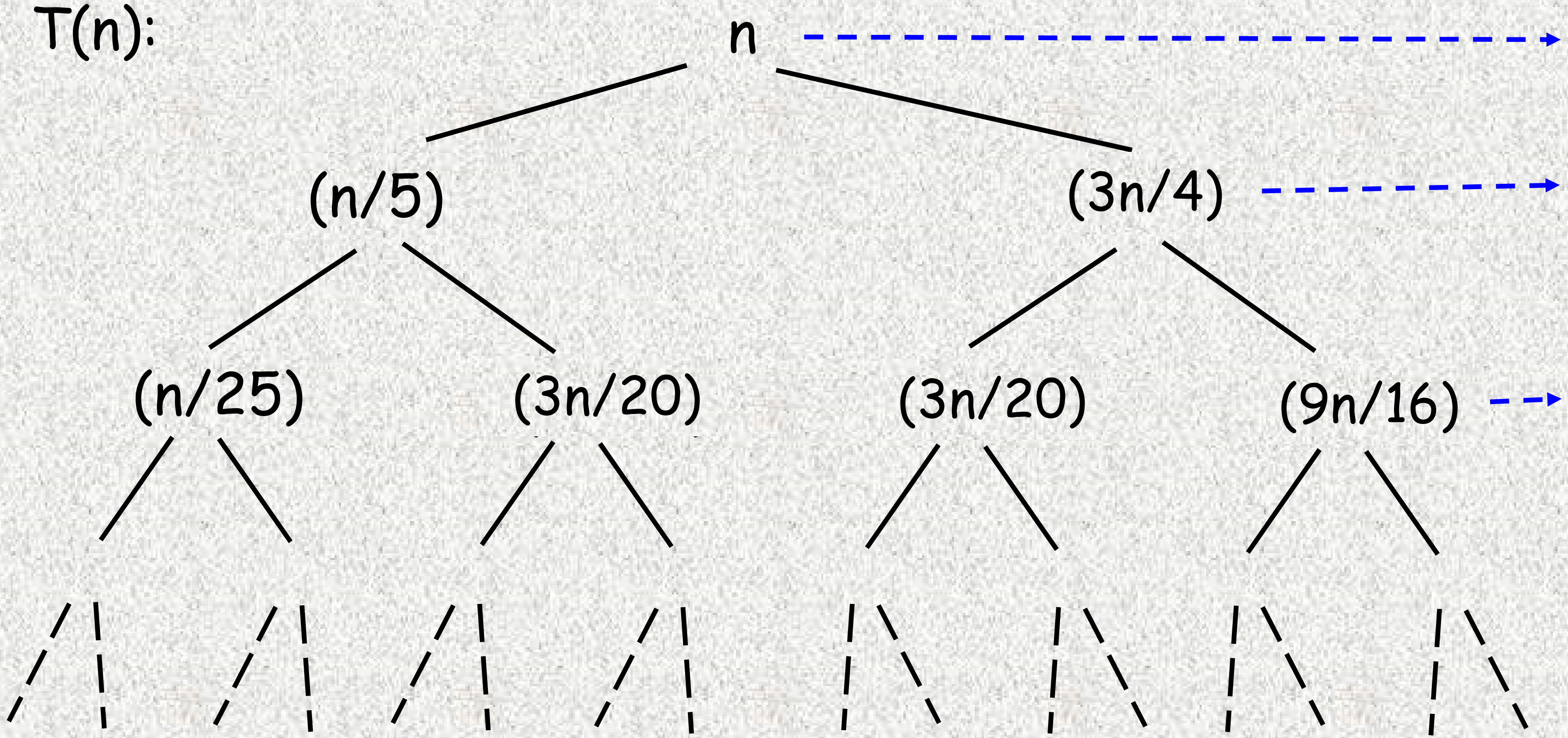
return $m = 5$

Select(S_3, 22 – 15)

$T(n) = T(n/5) + T(3n/4) + n$

Why $T(n) = O(n)$?

T(n):



$$\le n$$

$$\le \frac{19}{20} n$$

$$\le \left(\frac{19}{20}\right)^2 n$$

$$\vdots$$

$$\le \cdots$$

$$O(n)$$

等比會收斂

Divide & Conquer

D & C

1. partition the input into same subproblems

2. recursively solve the subproblems

3. combine subsolutions

(Ex. merge-Sort)

$$T(n) = \sum T(n_i) + C(n)$$

Partition

1. break the problem into independent subproblems

2. solve the subproblems

(Ex. quick-sort)

$$T(n) = P(n) + \sum T_i(n_i)$$

Prune & Search

repeatedly remove invalid candidates

(Ex. selection)

(Ex. binary search)

$$T(n) = r(n) + T(n')$$