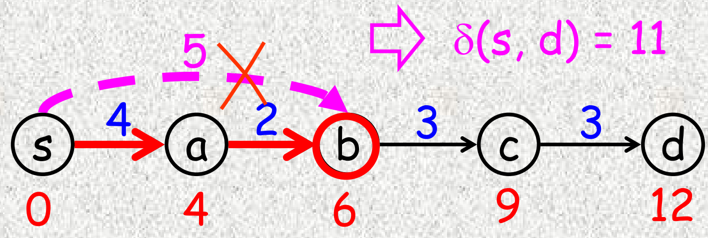
Main Idea ----- 1

$\Rightarrow$ $\delta(s, d) = 11$

If



$\pi(d) = c$

is a shortest path from s to d

Then

(i) all subpaths are shortest optimal substructure !

(ii) After $\delta(s, \pi(v))$ is known,
we can get $\delta(s, v)$ by Relax($\pi(v)$, v, w)

e.g. After $\delta(s, c) = 9$ is known,
we have $\delta(s, d) = 9 + w(c, d) = 12$

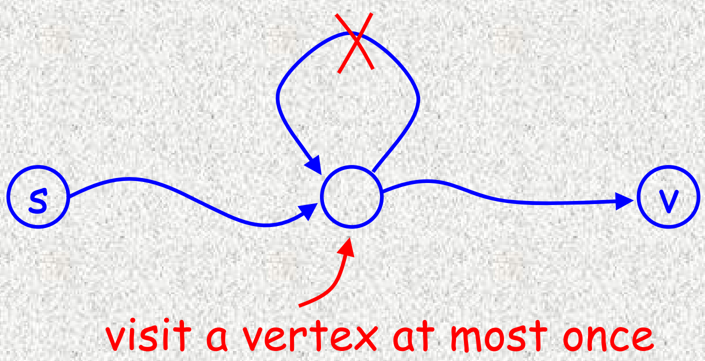
Relax(c, d, w)

Main Idea ----- 2

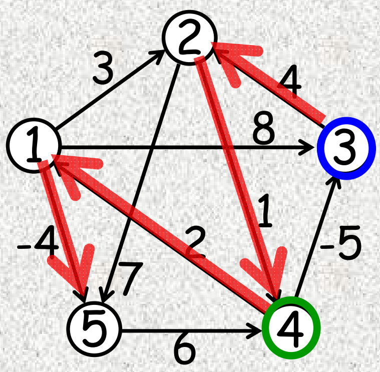
If G contains no negative cycles,

(i) every shortest path is a simple path

(ii) every shortest path has at most n – 1 edges



(For ease of discussion, assume that there are no 0-cycles)



(in adjacency matrix A)

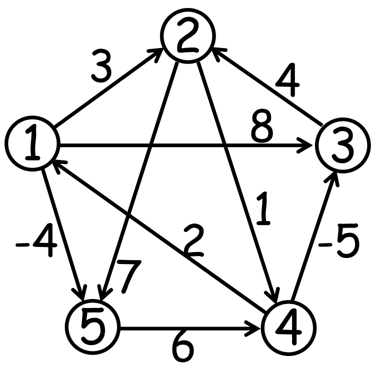
$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & & & & \\ 7 & 4 & 0 & 5 & 3 \\ & & & & \\ & & & & \end{array}\right) \end{array}$$

single source with s = 3

$$\pi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & & & & \\ 4 & 3 & \text{NIL} & 2 & 1 \\ & & & & \\ & & & & \end{array}\right) \end{array}$$

shortest path tree with root 3

$d_{ij}^{(m)}$: shortest distance from i to j using at most m edges



$d_{25}^{(1)} = 7$ $\quad$ $d_{35}^{(1)} = \infty$

$d_{25}^{(2)} = 7$ $\quad$ $d_{35}^{(2)} = 11$

變小，多一根 edge 有用

$d_{25}^{(3)} = -1$ $\quad$ $d_{35}^{(3)} = 11$

不變，多一根 edge 沒用

$d_{25}^{(4)} = -1$ $\quad$ $d_{35}^{(4)} = 3$

$\cancel{d_{25}}^{(5)} = -1$ $\quad$ $\cancel{d_{35}}^{(5)} = 3$

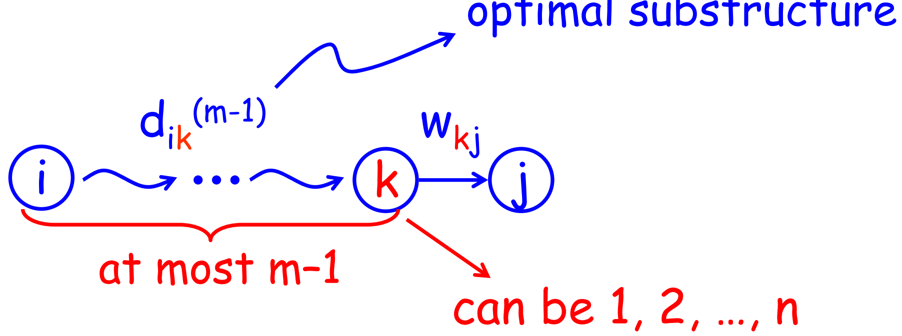
no negative cycles

➔ at most n−1 edges

➔ $d_{ij} = d_{ij}^{(4)} = d_{ij}^{(5)} = d_{ij}^{(6)} = \ldots$

$(d_{25} = d_{25}^{(4)} = -1)$

$d_{ij}^{(m)}$: at most m steps (edges)



$$d_{ij}^{(m)} = \underset{1 \le k \le n}{\mathrm{MIN}}\{\, d_{ik}^{(m-1)} + w_{kj} \,\}$$

$$* \ \left[\, D^{(m)} \,\right] \Leftarrow \left[\, D^{(m-1)} \,\right], \left[\, W \,\right]$$ ($D^{(m)}$ 可 由 $D^{(m-1)}$, W 得 到)

Matrix multiplication

$$C = A \times B$$

$$c_{ij} = \sum_k \{\, a_{ik} \times b_{kj} \,\}$$

$(op_1, op_2) = (\times, +)$

$$\begin{array}{c} i \\ \\ \\ \end{array}\begin{bmatrix} & \boxed{8} & \\ & & \\ & & \end{bmatrix} = \begin{array}{c} i \\ \\ \\ \end{array}\begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix} \times \begin{bmatrix} & 1 \\ & 2 \\ & 1 \end{bmatrix}$$

$$\qquad\qquad j \qquad\qquad\qquad\qquad\qquad\qquad\qquad j$$

Boolean matrix multiplication

$$C = A \times B$$

$$c_{ij} = \underset{k}{\mathrm{or}} \{\, a_{ik} \,\&\, b_{kj} \,\}$$

$(op_1, op_2) = (\&, \mathrm{or})$

$$\begin{array}{c} i \\ \\ \\ \end{array}\begin{bmatrix} & \boxed{1} & \\ & & \\ & & \end{bmatrix} = \begin{array}{c} i \\ \\ \\ \end{array}\begin{bmatrix} 0 & 1 & 1 \\ & & \\ & & \end{bmatrix} \times \begin{bmatrix} & 1 \\ & 0 \\ & 1 \end{bmatrix}$$

$$\qquad\qquad j \qquad\qquad\qquad\qquad\qquad\qquad\qquad j$$

Matrix multiplication with (op_1, op_2)

$$C = A \otimes B$$

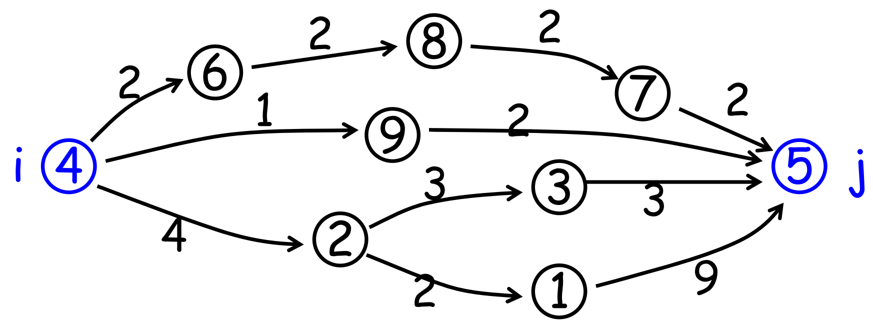
$$c_{ij} = \underset{k}{op_2}\{\, \underset{\text{row } i}{a_{ik}} \ op_1 \ \underset{\text{column } j}{b_{kj}} \,\}$$

$$d_{ij}^{(m)} = \min_{k} \{ d_{ik}^{(m-1)} + w_{kj} \}$$

$$\Rightarrow \quad \begin{array}{c} D^{(m)} \\ i \begin{bmatrix} & & \boxed{4} \\ & & \\ & & \end{bmatrix} \\ \hspace{4em} j \end{array} \longleftarrow \begin{array}{c} D^{(m-1)} \\ i \begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix} \end{array} , \begin{array}{c} W \\ \begin{bmatrix} & 4 \\ & 2 \\ & 3 \end{bmatrix} \\ \hspace{3em} j \end{array}$$

$$\Rightarrow \quad D^{(m)} = D^{(m-1)} \otimes W$$

(multiplication with (op1, op2) = (+, min))



$d_{45}^{(0)} = \infty$

$d_{45}^{(1)} = \infty$

$d_{45}^{(2)} = 15$

$d_{45}^{(3)} = 10$

$d_{45}^{(4)} = 10$

$d_{45}^{(5)} = 10$

$d_{45}^{(6)} = 10$

$d_{45}^{(7)} = 10$

$d_{45}^{(8)} = 8$

$d_{45}^{(9)} = 3$

* $d_{45}^{(3)} < d_{45}^{(2)}$ ⇨ 有 經 過 3
* $d_{45}^{(4)} = d_{45}^{(3)}$ ⇨ 不 必 經 過 4

$d_{ij}^{(k)}$: shortest distance from i to j
via only {1, 2, 3, …, k}

$$\underbrace{i \xrightarrow{d_{ik}^{(k-1)}} k}_{\text{only }\{1,\,2,\,\ldots,\,k-1\}} \underbrace{\xrightarrow{d_{kj}^{(k-1)}} j}_{\text{only }\{1,\,2,\,\ldots,\,k-1\}}$$

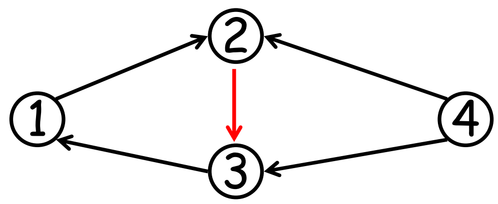
optimal substructure

simple path
(no negative cycles)

$$d_{ij}^{(k)} = \min\{\, d_{ij}^{(k-1)},\; d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \,\}$$

不必經過 k　　　經過 k

* $D^{(k)}$ 可由 $D^{(k-1)}$ 得到

G: 

Adjacency Matrix

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

Transitive Closure

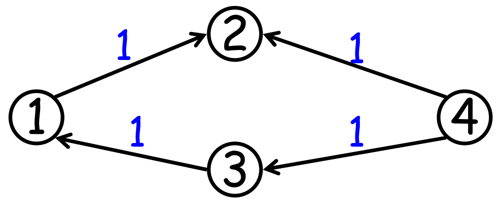
$$A^* = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

↳ null path

* Using A^* ➔ strongly connected components

i,j are in the same component
iff $A^*[i, j] = A^*[j, i] = 1$

G:



Method 1.

1. Assign $w(e) = 1$ for each edge $e \in E$

$$W = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ 1 & \infty & 0 & \infty \\ \infty & 1 & 1 & 0 \end{pmatrix}$$

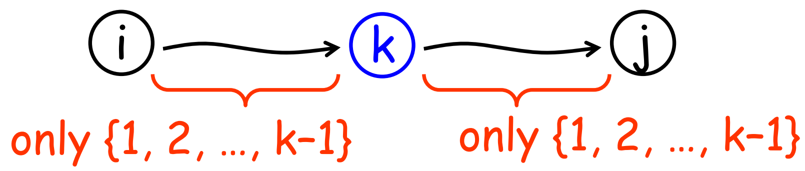
2. Perform an all-pair shortest paths algorithm

$$D = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ 1 & 2 & 0 & \infty \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

3. $D_{ij} \neq \infty \leftrightarrow a^*_{ij} = 1$

$$A^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Method 2: Modify the second Algo.

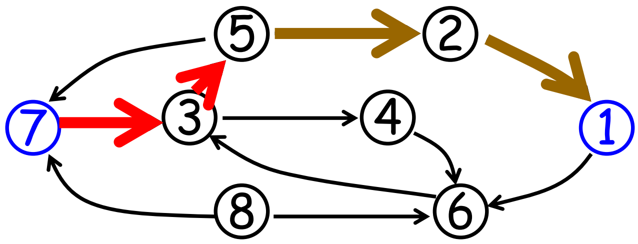


$d_{ij}^{(k)}$: shortest distance, via only $\{1, 2, \ldots, k\}$

$$d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

$$t_{ij}^{(k)} = \begin{cases} 1\text{: reachable, via only } \{1, 2, \ldots, k\} \\ 0 \end{cases}$$

$$t_{ij}^{(k)} = \text{OR}\ \{t_{ij}^{(k-1)}, t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\}$$
$$= t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$$



$t_{71}^{(0)} = 0$ $t_{71}^{(4)} = 0$ $t_{71}^{(8)} = 1$

$t_{71}^{(1)} = 0$ $\boxed{t_{71}^{(5)} = 1}$

$t_{71}^{(2)} = 0$ $t_{71}^{(6)} = 1$

$t_{71}^{(3)} = 0$ $t_{71}^{(7)} = 1$

$t_{75}^{(4)} = 1$ and $t_{51}^{(4)} = 1$

* $t_{ij}^{(k)} = 1$ $\begin{cases} \text{case 1. } t_{ij}^{(k-1)} = 1 \text{ (不必經過 k)} \\ \text{case 2. } t_{ik}^{(k-1)} = 1 \text{ and } t_{kj}^{(k-1)} = 1 \text{ (經過 k)} \end{cases}$

⇨ $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$

* $T^{(k)}$ 可由 $T^{(k-1)}$ 得到

Shortest Paths Algorithms - Review

Main Ideas

Optimal substructure: (1) $\underset{\text{ok}}{\pi(v)} \xrightarrow[\text{relax}]{} \underset{\text{ok}}{v}$ (2) DP

No negative cycles: simple path (at most n-1 edges)

Single-Source (relax)

Bellman-Ford (no negative cycles, can detect) O(VE)

$\underset{\text{ok}}{U_0} \overset{= \{s\}}{\to} \underset{\text{ok}}{U_1} \to \underset{\text{ok}}{U_2} \to \underset{\text{ok}}{U_3} \to \ldots \to \underset{\text{ok}}{U_{n-1}}$

Dijkstra (no negative edges) O(Vlg V+E)

$\underset{\text{ok}}{\text{rank}(1)} \overset{= \{s\}}{\to} \underset{\text{ok}}{\text{rank}(2)} \to \underset{\text{ok}}{\text{rank}(3)} \to \ldots \to \underset{\text{ok}}{\text{rank}(n)}$

Two important special cases

Single-Source on un-weighted graph O(V+E)

BFS

Single-Source on a DAG: shortest/longest O(V+E)

(1) Bellman-Ford: one phase – left to right

(2) classical: DP

25-8x

25-8b

All-Pairs (DP) check no negative cycles first

Matrix Multiplication (no negative cycles) $O(V^3 \lg V)$

$d_{ij}^{(m)}$: shortest distance using at most m edges

$D^{(m)} = D^{(m-1)} \otimes W = W^m$ (op1, op2) = (+, Min)

$D = D^{(m)}$ for $m \geq n-1$ (no negative cycles)

$\underset{W}{\overset{\parallel}{D^{(1)}}} \to D^{(2)} \to D^{(4)} \to D^{(8)} \to \ldots \to D^{(n)}$ (lg (n-1) times)

Floyd-Warshall (no negative cycles) $O(V^3)$

$d_{ij}^{(k)}$: shortest distance via only {1, 2, ..., k}

$D = D^{(n)}$

$\underset{W}{\overset{\parallel}{D^{(0)}}} \to D^{(1)} \to D^{(2)} \to D^{(3)} \to \ldots \to D^{(n)}$