------------------------------ 17 ------------------------------

Amortized Analysis

Amortized Analysis:

Let $op_1, op_2, \ldots, op_n$ be a sequence of n operations.

In an *amortized analysis*, the average time of each op_i in the worst case is computed.

That's $T(n)/n$ is computed, where $T(n)$ is the **worst case** time required for all of the n operations.

17.1 *The aggregate method*:

(First, compute $T(n)$. Then, compute $T(n)/n$.)

Example: n stack operations on a stack S

Initially, S is empty.

Each op_i is Push, Pop, or Multi-Pop(S,k).

Multi-Pop(S, k)

1. **while** not Stack-Empty(S) and $k > 0$
2. **do** Pop(S)
3. $k \leftarrow k-1$

- Push, Pop: $O(1)$; Multi-Pop: $O(\min\{|S|, k\})$.

Analysis (I):

1. The worst-case of an op_i is $O(n)$.
2. The worst-case of all op_i is $T(n)=O(n^2)$.
3. The average cost of each op_i is $T(n)/n=O(n)$.

---> correct but not tight.

Analysis (II):

1. At most n objects are pushed into S.
2. Each object in S can be poped at most once.
3. The total number of Push's and Pop's is $O(n)$.
4. $T(n)=O(n)$ and the average cost of each op_i is $T(n)/n=O(1)$.

Example: Incrementing a binary counter of k-bit

Initially, all bits in $A[0..k-1]$ are 0.

Each op_i is to increase A by one.

Analysis (I):

1. The worst-case of an op_i takes $O(k)$ time.
2. The worst-case of all op_i is $T(n)=O(nk)$.
3. The average cost of an op_i is $T(n)/n=O(k)$.

Counter value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	Total cost
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1
2	0	0	0	0	0	0	1	0	3
3	0	0	0	0	0	0	1	1	4
4	0	0	0	0	0	1	0	0	7
5	0	0	0	0	0	1	0	1	8
6	0	0	0	0	0	1	1	0	10
7	0	0	0	0	0	1	1	1	11
8	0	0	0	0	1	0	0	0	15
9	0	0	0	0	1	0	0	1	16
10	0	0	0	0	1	0	1	0	18
11	0	0	0	0	1	0	1	1	19
12	0	0	0	0	1	1	0	0	22
13	0	0	0	0	1	1	0	1	23
14	0	0	0	0	1	1	1	0	25
15	0	0	0	0	1	1	1	1	26
16	0	0	0	1	0	0	0	0	31

```
INCREMENT(A)
i ← 0
while i < length[A] and A[i] = 1
      do A[i] ← 0
         i ← i + 1
if i < length[A]
   then A[i] ← 1
```

Analysis (II):

1. $A[0]$ flips each time.
2. $A[1]$ flips every other time.
3. $A[i]$ flips $\lfloor n/2^i \rfloor$ times in total.
4. $T(n) = n + \lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \ldots \le 2n$

5. Average cost of each op_i is $T(n)/n = O(1)$

Emphasize again:

amortized analysis $\neq$ average-case analysis
(expected cost)
(probabilistic cost)

17.2 *The accounting method*:

1. Assign different charges to different operations. The charge of an operation is called the ***amortized cost*** of the operation.

2. If (amortized cost > actual cost), the value (amortized cost – actual cost) is assigned to specific objects in the data structure as ***credit***.

3. Credit can be used later to pay for operations with (amortized cost < actual cost).

Operation	Amortized cost	Actual cost	Total credit
op_1	40	10	30
op_2	40	20	50
op_3	40	70	20

$$\Sigma a_i \quad = \quad \Sigma t_i \quad + \quad C$$

4. The credit after each operation should never become negative, which guarantees
   $T(n) \leq$ total amortized cost.

Example: n stack operations on an empty stack S

Analysis:

1. Actual costs:

	Push:	1
	Pop:	1
	Multi-Pop:	$\min\{\|S\|, k\}$

2. Assign amortized cost:

	Push:	2
	Pop:	0
	Multi-Pop:	0

3. *Push an object* -- we pay 2 dollars, 1 for the actual cost and 1 for credit on the object.
   Pop an object -- we pay by the credit on the object.
   The total credit on S -- $|S|$, which is never negative.

4. $T(n) \leq$ total amortized cost $\leq 2n$ (at most n push). Thus, the average cost of each op_i is $O(1)$.

Example: Incrementing a binary counter $A[0..k\text{-}1]$

Analysis:

1. Actual costs:

	Set a bit to 1:	1
	Reset a bit to 0:	1

2. Assign amortized cost:

	Set a bit to 1:	2
	Reset a bit to 0:	0

3. *Set* a bit to 1 -- pay 2 dollars, 1 for the actual cost and 1 for credit on the bit.
   Reset a bit to 0 -- pay by the credit on the bit.

   The total credit on A -- the number of 1's in A, which is never negative.

4. $T(n) \leq$ total amortized cost $\leq 2n$ (INCREMENT(A) sets at most one bit at Line 6).
   Thus, the average cost of each op_i is $O(1)$.

17.3 *The potential method*:

-- Similar to accounting method. But potential is associated with the *whole data structure* instead of with *specific objects*.

1. Let $D_0 \rightarrow_{op_1} D_1 \rightarrow_{op_2} D_2 \rightarrow \ldots \rightarrow_{op_n} D_n$.

2. Let t_i be the actual cost of op_i.

3. A ***potential function*** Φ (fai) maps each D_i to its potential $\Phi(D_i)$ (a real number).

4. The potential cost a_i of op_i is defined by

$$a_i = t_i + \Phi(D_i) - \Phi(D_{i-1}).$$
$$(\text{or } \Phi(D_i)=\Phi(D_{i-1})+(a_i - t_i))$$

5. Total amortized cost of k operations is

$$\sum_{1\le i\le k} a_i = \sum_{1\le i\le k}\{t_i + \Phi(D_i) - \Phi(D_{i-1})\}$$
$$= \sum_{1\le i\le k} t_i + \Phi(D_k) - \Phi(D_0)$$

6. To guarantee $T(n)=O(\sum_{1\le i\le n} a_i)$, $\Phi(D_k)$-$\Phi(D_0)$ should never be negative.
(Usually, we define $\Phi(D_0)=0$ and then prove $\Phi(D_i)\ge 0$ for all i.)

Example: n stack operations on a stack S

Analysis:

1. Let $\Phi(S)=|S|$.

2. Clearly, we have $\Phi(D_0)=0$ and $\Phi(D_i)\ge 0$ for all i.

3. The amortized cost of a Push operation is
$$\begin{aligned} a_i &= t_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= 1 + (|S|+1) - |S| \\ &= 2. \end{aligned}$$
The amortized cost of a Pop operation is
$$\begin{aligned} a_i &= 1 + (|S|-1) - |S| \\ &= 0. \end{aligned}$$
The amortized cost of a Multi-Pop operation is
$$a_i = k' + (|S|-k') - |S| = 0. \quad (k'=\min\{|S|, k\})$$

4. Each a_i is a constant. Thus,

$$T(n) = O(\sum_{1\le i\le n} a_i) = O(n).$$

Example: Incrementing a binary counter $A[0..k-1]$

Analysis:

1. Let $\Phi(A)$ be the number of 1's in A.

2. Clearly, we have $\Phi(D_0)=0$ and $\Phi(D_i)\geq 0$ for all i.

3. The amortized cost of an increment operation is

$$\begin{aligned} a_i &= t_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= k' + (|A|-(k'-1)+1) - |A| \\ &= 2. \qquad \text{(reset } k'\text{-1 bits to 0)} \end{aligned}$$

4. Each a_i is a constant. Thus,

$$T(n) = O\left(\sum_{1\leq i\leq n} a_i\right) = O(n).$$

17.4 Dynamic table

- Initially, T is a table of size 0.
- Perform a sequence of n operations on T, each of which is either *Insert* or *Delete*.

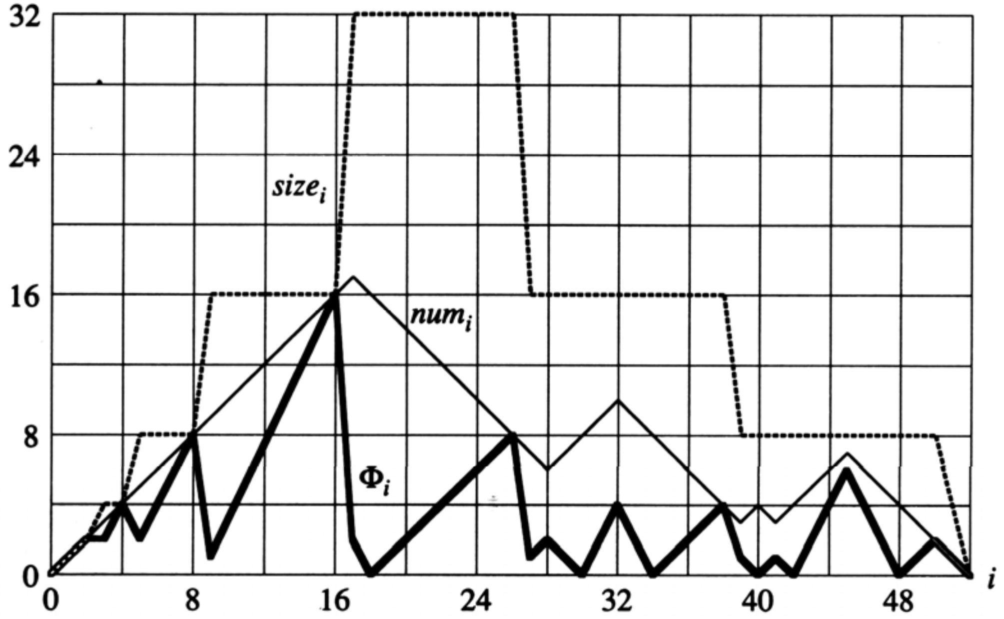
$$\text{Insert} \begin{cases} 1 \text{ step} & \text{if } T \text{ has a free slot or } |T| = 0 \\ |T|+1 \text{ steps} & \text{Otherwise} \\ & \text{(1. } T \text{ is expanded to size } 2|T| \\ & \text{ 2. Copy elements in } T \text{ to a new space} \\ & \text{ 3. Perform Insert)} \end{cases}$$

$$\text{Delete} \begin{cases} 1 \text{ step} & \text{if } T \text{ is at least } 1/4 \text{ full} \\ & \text{after deleting one item} \\ |T|/4 \text{ steps} & \text{Otherwise} \\ & \text{(1. Perform Delete.} \\ & \text{ 2. } T \text{ is contracted to } |T|/2 \\ & \text{ 3. Copy elements in } T \text{ to a new space)} \end{cases}$$

Analysis based on potential method:

1. Let $\Phi(T) = \begin{cases} 2num[T] - size[T] & \text{if } \alpha \geq 1/2 \\ size[T]/2 - num[T] & \text{if } \alpha < 1/2 \end{cases}$.

-- $num[T]$: the number of items in T
-- $size[T]$: the size of T
-- $\alpha = num[T]/size[T]$: the *load factor* of T
(If $num[T]=size[T]= 0$, define $\alpha=1$.)



2. Clearly, $\Phi(D_0)=0$ and $\Phi(D_i)\geq 0$ for all i.

3. The amortized cost of an Insert operation:

case 1a: $\alpha_{i-1}\geq 1/2$ and T is not expanded

$$\begin{aligned} a_i &= t_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= 1 + 2 \\ &= 3. \end{aligned}$$

case 1b: $\alpha_{i-1}\geq 1/2$ and T is expanded

$$\begin{aligned} a_i &= (size(D_{i-1})+1) + (2) - (size(D_{i-1})) \\ &= 3. \end{aligned}$$

case 2a: $\alpha_{i-1}<1/2$ and $\alpha_i<1/2$
(T is not expanded)

$$\begin{aligned} a_i &= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\ &= 0. \end{aligned}$$

case 2b: $\alpha_{i-1}<1/2$ and $\alpha_i\geq 1/2$
(T is not expanded)

$$\begin{aligned} a_i &= 1 + (2num_i - size_i) - (size_{i-1}/2 - num_{i-1}) \\ &< 3. \end{aligned}$$

4. The amortized cost of a Delete operation:

case 1a: $\alpha_{i-1}<1/2$ and T is not contracted

$$\begin{aligned} a_i &= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\ &= 2. \end{aligned}$$

case 1b: $\alpha_{i-1}<1/2$ and T is contracted

$$\begin{aligned} a_i &= (num_i+1)+(size_i/2 - num_i)-(size_{i-1}/2 - num_{i-1}) \\ &= 1. \end{aligned}$$

case 2: $\alpha_{i-1}\geq 1/2$
a_i is also bounded by a constant.
(See Exercise 17.4-2.)

5. Each a_i is bounded by a constant. Thus,

$$T(n) \leq \sum_{1 \leq i \leq n} a_i \leq c \times n = O(n)$$

6. Amortized cost of each op_i is $T(n)/n = O(1)$

Homework: Ex. 17.1-3, 17.2-2, 17.3-2, 17.4-2, Prob. 17-2. Read Ch18.