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Growth of Functions

3.1 Asymptotic notation

***Θ*-notation:** $f(n) = \Theta(g(n))$

$g(n)$ is an asymptotically tight bound for $f(n)$.

$\Theta(g(n)) = \{f(n)|$ there exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$$

for all $n \ge n_0\}$

Example: Prove that $3n^2 - 6n = \Theta(n^2)$.

Proof: To do so, we have to determine c_1, c_2, and n_0 such that

$$c_1 n^2 \le 3n^2 - 6n \le c_2 n^2, \qquad \text{(for all } n \ge n_0)$$

dividing which by n^2 yields

$$c_1 \le 3 - 6/n \le c_2.$$

Clearly, by choosing $c_1=2$, $c_2=3$ and $n_0=6$ we can verify that $3n^2 - 6n = \Theta(n^2)$. Q.E.D

- $f(n) = \Theta(g(n))$ iff $g(n)= \Theta(f(n))$, Ex. $n^2=\Theta(3n^2-6n)$

***O*-notation:** $f(n) = O(g(n))$

$g(n)$ is an asymptotically upper bound for $f(n)$.

$O(g(n))=$ $\{f(n)|$ there exist positive constants c and n_0 such that

$$0 \le f(n) \le cg(n)$$

for all $n \ge n_0\}$

- $\Theta(g(n)) \subseteq O(g(n))$
- $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$
- $6n = O(n)$, $6n = O(n^2)$
- "The running time is $O(n^2)$" means "the worst-case running time is $O(n^2)$."

***Ω*-notation:** $f(n) = \Omega(g(n))$

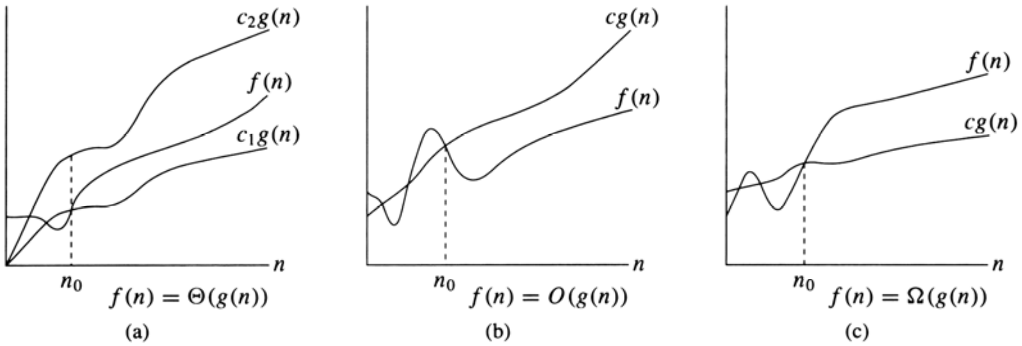
$g(n)$ is an asymptotically lower bound for $f(n)$.

$\Omega(g(n))=$ $\{f(n)|$ there exists positive constants c and n_0 such that

$$0 \le cg(n) \le f(n)$$

for all $n \ge n_0\}$

- $f(n) = \Theta(g(n))$ iff ($f(n)=O(g(n))$) & ($f(n)=\Omega(g(n))$)



tight bound upper bound lower bound

***o*-notation**: $f(n) = o(g(n))$ (little-oh of g of n)

$o(g(n))=$ $\{f(n)|$ for any positive constant c, there exists a constant $n_0 > 0$ such that

$$0 \le f(n) < cg(n)$$

for all $n \ge n_0\}$

- $2n = o(n^2)$, but $2n^2 \ne o(n^2)$.
- $f(n) = o(g(n))$ can also be defined as

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

***ω*-notation**: $f(n) = \omega(g(n))$ (little-omega of g of n)

$\omega(g(n)) = \{f(n)|$ for any positive constant c, there exists a constant $n_0 > 0$ such that

$$0 \le cg(n) < f(n) \quad \text{for all } n \ge n_0\}$$

- $2n^2 = \omega(n)$, but $2n^2 \ne \omega(n^2)$.
- $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Comparison of functions

- functions: ω Ω Θ O o
  real numbers: $>$ $\ge$ $=$ $\le$ $<$

- Transitivity, Reflexivity, Symmetry, Transpose Symmetry

- Any two real numbers can be compared. (trichotomy) But, not any two functions can be compared.

  Example: $f(n)=n$ and $g(n)=n^{1+\sin n}$

Homework: Problems 3-2, 3-3, 3-4.

Appendix A: Summation formulas

$$\sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n}a_k + \sum_{k=1}^{n}b_k$$

$$\sum_{k=1}^{n}k = \frac{1}{2}n(n+1) = \Theta(n^2) \qquad \sum_{k=0}^{n}x^k = (x^{n+1}-1)/(x-1)$$

$$H_n = \sum_{k=1}^{n}\frac{1}{k} = \log_e n + O(1)$$ (Harmonic series)

$$\sum_{k=0}^{\infty}x^k = \frac{1}{1-x} \quad (|x|<1) \qquad \sum_{k=0}^{\infty}kx^k = \frac{x}{(1-x)^2} \quad (|x|<1)$$

$$\sum_{k=1}^{n-1}\frac{1}{k(k+1)} = \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n}$$

$$\lg\prod_{k=1}^{n}a_k = \sum_{k=1}^{n}\lg a_k$$