Data Structure for Disjoint Sets

21.1 Disjoint-set operations

***Disjoint set data structure*:**

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1. a data structure maintains a collection $S=\{S_1, S_2, \ldots, S_k\}$ of **disjoint dynamic** sets.

2. Each set is identified by a **representative**, which is some member of the set. In ① some applications, it doesn't matter which member is used as the representative; we only care that if we ask the representative of a set without modifying the set between the requests, we get the same answer. In ② other applications, there may be a representative rule for choosing the representative, such as choosing the smallest member in the set.

3. The following operations should be supported.

   Make-Set(x): create a new set $\{x\}$.
   Union(x, y): unite the two sets containing x, y.
   Find-Set(x): return a pointer to the
   representative of the set containing x.

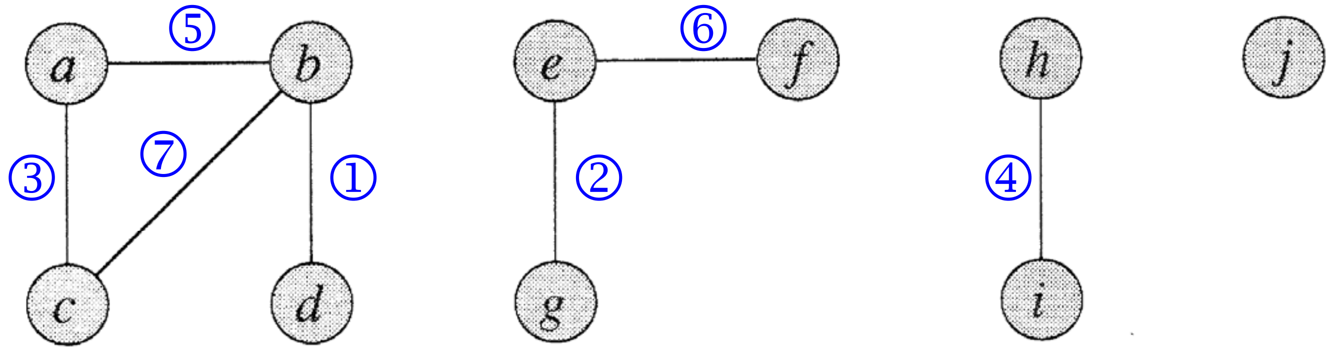
Initially, $S = \emptyset$.

* n: number of *Make-Set* operations
m: total number of *Make-Set*, *Union*, and *Find-Set* operation.

* $m \geq n$ and the number of *Union* operations is at most n-1.

* $m = n + f + u$
* $m \geq n$; $u \leq n-1$

An application of disjoint-set data structures



(a) 4 connected components

Edge processed	Collection of disjoint sets								
initial sets	{a}	{b}	{c}	{d}	{e}	{f}	{g}	{h}	{i}	{j}
1 (b,d)	{a}	{b,d}	{c}		{e}	{f}	{g}	{h}	{i}	{j}
2 (e,g)	{a}	{b,d}	{c}		{e,g}	{f}		{h}	{i}	{j}
3 (a,c)	{a,c}	{b,d}			{e,g}	{f}		{h}	{i}	{j}
4 (h,i)	{a,c}	{b,d}			{e,g}	{f}		{h,i}		{j}
5 (a,b)	{a,b,c,d}				{e,g}	{f}		{h,i}		{j}
6 (e,f)	{a,b,c,d}				{e,f,g}			{h,i}		{j}
★ 7 (b,c)	{a,b,c,d}				{e,f,g}			{h,i}		{j}

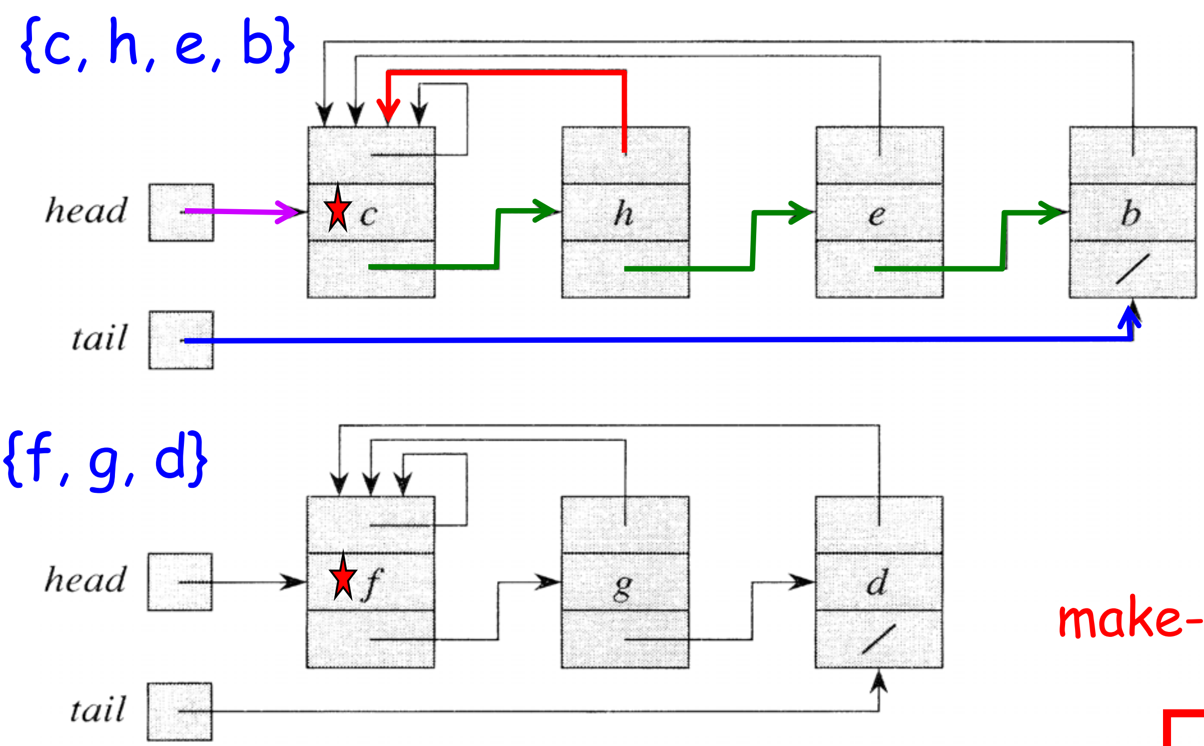
(b)

```
CONNECTED-COMPONENTS(G)
for each vertex v ∈ V[G]
     do MAKE-SET(v)
for each edge (u, v) ∈ E[G]
     do if FIND-SET(u) ≠ FIND-SET(v)
           then UNION(u, v)
```

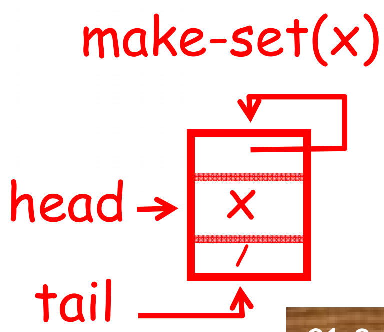
```
SAME-COMPONENT(u, v)
if FIND-SET(u) = FIND-SET(v)
   then return TRUE
   else  return FALSE
```

21.2 Linked-list representation

(First object in a list is the representative.)



(a)



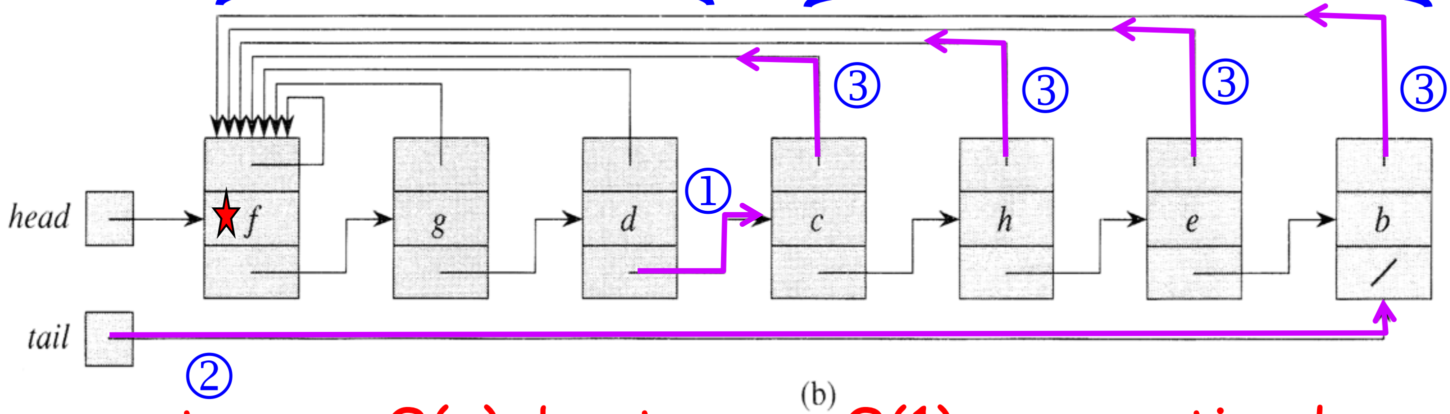
* Make-Set, Find-Set: $O(1)$ time.

Union((1), (2)) $\Rightarrow$ (2) + (1)

A simple implementation of Union(x, y)

(Appending the first list onto the second) O(|(1)|) time

Union(e, d) (2) (1)



worst-case: O(n), best-case: O(1) $\Rightarrow$ amortized

* $O(n^2)$ time for m=2n-1 operations.

* $m = n + u$
* $u = n - 1$

* An example showing $\Omega(n^2)$ for O(n) operations.

Operation		Number of objects updated
MAKE-SET(x_1)		1
MAKE-SET(x_2)		1
⋮		⋮
MAKE-SET(x_n)		1
UNION(x_1, x_2)	1 + 1	1
UNION(x_2, x_3)	1 + 2	2
UNION(x_3, x_4)	1 + 3	3
⋮		⋮
UNION(x_{n-1}, x_n)	1 + (n - 1)	$n - 1$

n Make O(n)

n-1 Union $O(\sum k)$ || $O(n^2)$

* Thus, the *amortized time* of each operation is $O(n)$. (tight)

size

A weighted-union heuristic

1. Each representative stores the length of the list.
2. Append the smaller list onto the longer.

Theorem 21.1: Using the weighted-union heuristic, a sequence of m *Make-Set*, *Union*, and *Find-Set* operations takes $O(m+n\lg n)$ time, where n is the number of *Make-Set* operations.

Make, FIND: $O(m)$
Union: $O(n \lg n)$

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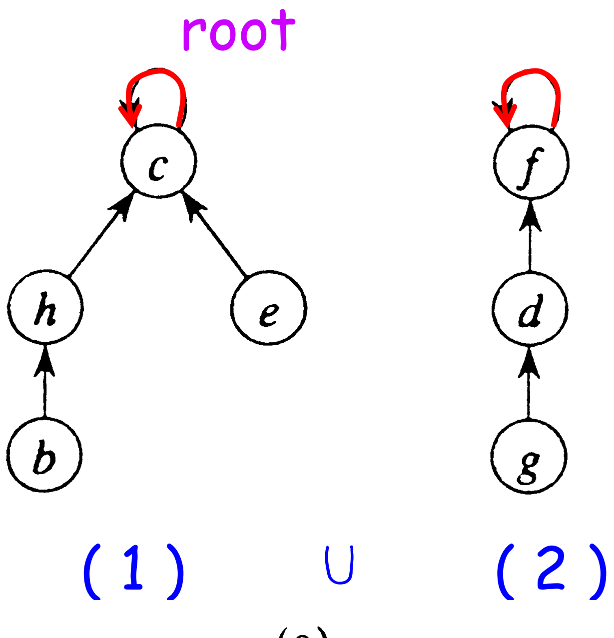
* $m = n + f + u$
* $m \geq n$

Proof: After Make-Set(x) is performed, the list containing x has only one element. At the first time x's representative pointer is updated, the list containing x has at least two elements. Continuing on, we observe that after the k-th time x's representative is updated, the list containing x has at least 2^k elements. Since $k=O(\lg n)$, the time for all *Union* operations is at most $O(n\lg n)$. The time for each *Make-Set* and *Find-Set* operation is $O(1)$. Thus, the theorem holds. Q.E.D.

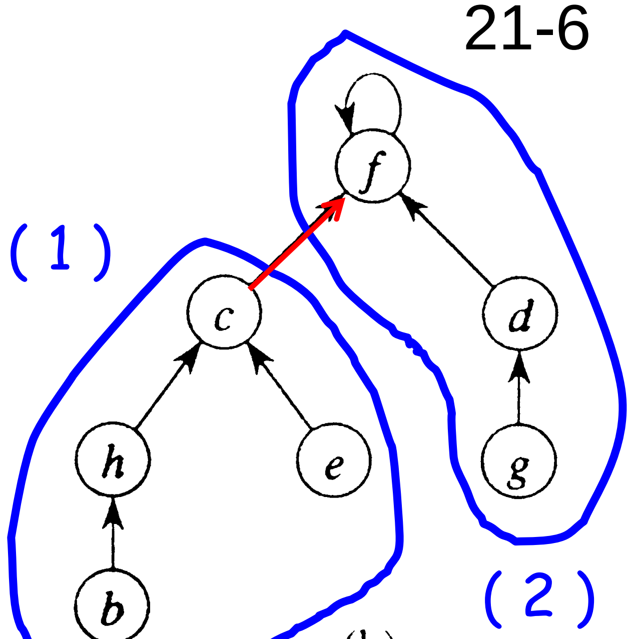
Union: (i) best: $O(1)$ (ii) worst-case: $O(n)$
(iii) amortized: $n \lg n / (n-1) = O(\lg n)$

21.3 Disjoint-set forests

(The root of a tree is the representative.)

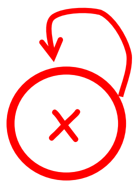


(a)



(b)

Make-Set(x): $O(1)$ time



Find-Set(x): $O(h)$ time, h is the height of the tree containing x. (**Find path: $x \to$ root**)

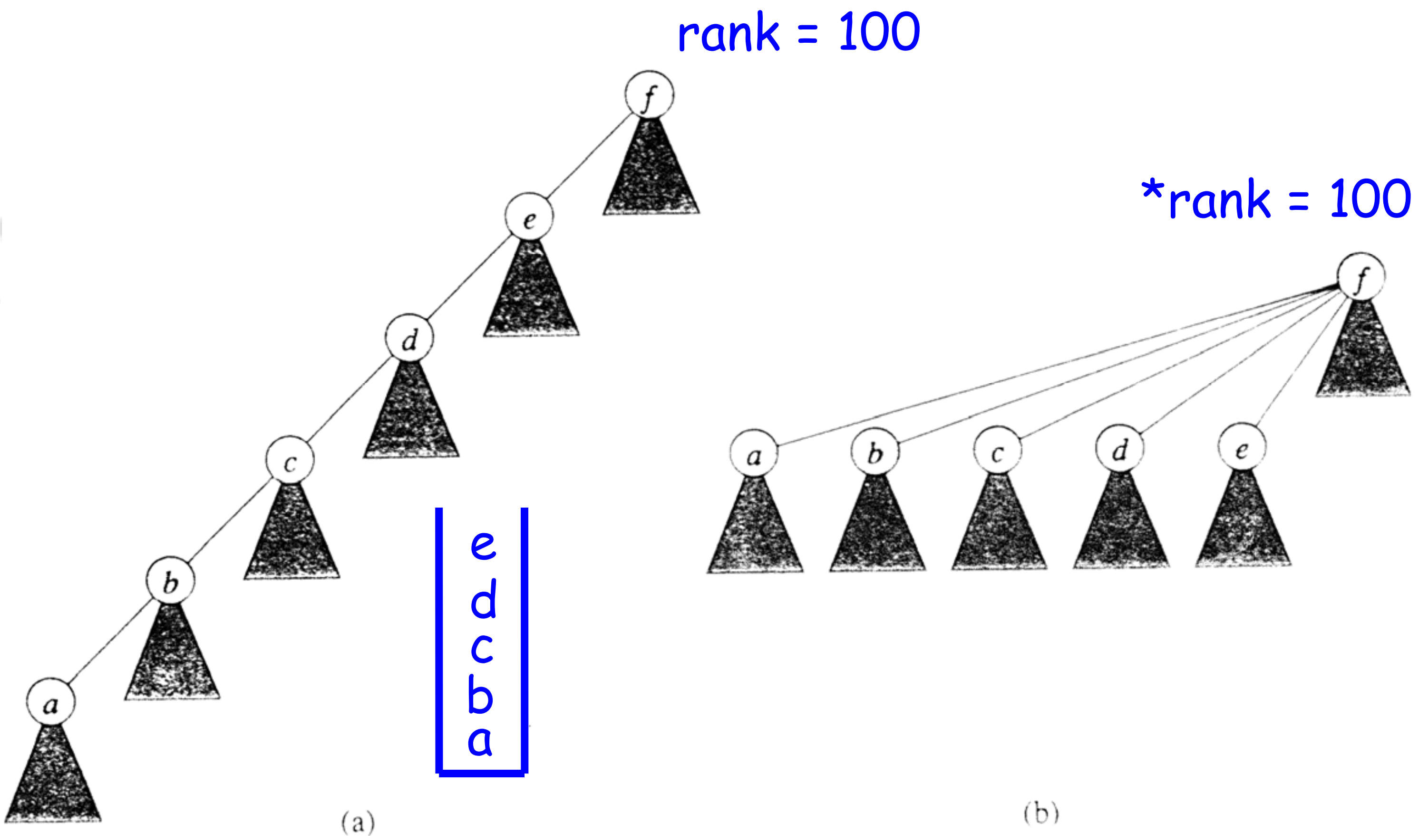
Union(x, y): The root of x points to the root of y. $\to O(h)$ time.

Heuristics to improve the running time

height

1. ***Union by rank***: the root of the smaller tree points to the root of the larger tree (according to heights).
   ***rank*[*x*]:** height of x (number of edges in the longest path between x and a descendant leaf)
   may be an approximation (upper bound)
2. ***Path compression***: During a Find-Set(x) operation, make each node on the find path point to the root. (It will not change any rank.)

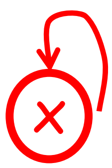
Example: Find-Set(a)



(a) (b)

Pseudo-code for disjoint-set forests

```
MAKE-SET(x)
    p[x] ← x
    rank[x] ← 0
```



rank = 0

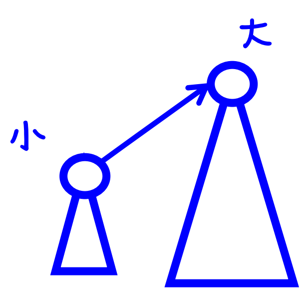
two Find: path compression

```
UNION(x, y)
    LINK(FIND-SET(x), FIND-SET(y))
```

root

```
LINK(x, y)
    if rank[x] > rank[y]
        then p[y] ← x
        else p[x] ← y
            if rank[x] = rank[y]
              then rank[y] ← rank[y] + 1
```

union by rank



FIND-SET(x)
1 **if** $x \neq p[x]$ set x's parent as the root
2 **then** $p[x] \leftarrow$ FIND-SET$(p[x])$ the root
3 **return** $p[x]$ the root

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Effect of the heuristics on the running time

m = n + f + u

1. If only Union-by-rank is used, it can be easily shown that $O(m \lg n)$ time is required.
   ($h \leq \lg n$, Ex 21.4-4)

2. If only path-compression is used, it can be shown (not proved here) that the running time is

$$\Theta(n + f \cdot (1 + \log_{2+f/n} n)),$$

* better than 1

   where n is the number of *Make-Set* operations and f is the number of *Find-Set* operations.

3. When both heuristics are used, the worst-case running time is $O(m\alpha(n))$, where $\alpha(n)$ is the very slowly growing inverse of Ackermann's function. Since $\alpha(n) \leq 4$ for any conceivable application, we can view the running time as linear in m in all practical situations.

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almost linear

Amortized:
⇨ $O(\alpha(n))$ per operation

21-8a

Ackermann's function and its inverse

$g(0) = 2$, $g(1) = 2^2$, $g(2) = 2^{2^2}$

* Let $g(i) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}i$ be a repeated exponentiation.

  (e.g., $g(4) = 2^{2^{2^{2^2}}}$.)

* The function $\lg^* n = \min\{i \geq 0: \lg^{(i)} n \leq 1\}$ is essentially the inverse of $g(i)$. (e.g., $\lg^* 2^{2^{2^{2^2}}} = 5$.)

  $\lg^* g(i) = i + 1$

* The Ackermann's function: for integer $i, j \geq 1$,

$$
\begin{array}{ll}
A(1, j) = 2^j & \text{for } j \geq 1, \\
A(i, 1) = A(i-1, 2) & \text{for } i \geq 2, \\
A(i, j) = A(i-1, A(i, j-1)) & \text{for } i, j \geq 2,
\end{array}
$$

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	
$i = 1$	2^1	2^2	2^3	2^4	2^j
$i = 2$	2^2	2^{2^2}	$2^{2^{2^2}}$	$2^{2^{2^{2^2}}}$	$g(j)$
$i = 3$	2^{2^2}	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$	
$i = 4$	$\gg 10^{80}$				

$\longrightarrow A(4, 1) = A(3, 2) = g(16)$

* Note that $A(2, j) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}j = g(j)$ for all $j \geq 1$.

Thus, $A(i, j) \geq g(j)$ for $i \geq 2$. $A(2, j) = A(1, A(2, j-1))$

$= 2^{A(2, j-1)}$

* The inverse of Ackermann's function:

$$\alpha(n) = \min\{i \geq 1: A(i, 1) > \lg n\}.$$

e.g., $\alpha(4) = 2$, $\alpha(32) = 3$, $\alpha(512) = 3$, $\alpha(2^{10000}) = 4$

* $A(4, 1) = A(3, 2) = g(16) >> 10^{80}$.

* Since $A(4, 1) >> 10^{80}$, we have $\alpha(n) \leq 4$ for all practical cases (unless $\lg n > 10^{80}$).

or $n > 2^{10^{80}}$

* $\lg^* n \leq 5$ for all practical cases (unless $n > 2^{65536}$).

$= g(4) = 2^{2^{2^{2}}} = 2^{65536} \approx 10^{1926}$

* Since $A(i, 1) \geq g(i)$ for $i \geq 4$, $\alpha(n) = O(\lg^* n)$.

$1 \quad \alpha \quad \lg^* \quad \lg\lg \quad \lg\sqrt{n} \quad n \quad n^k \quad 2^n \quad g \quad A$

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Homework: Ex. 21.4-4, Prob. 21-1, 21-3.

21-10a

17-13a