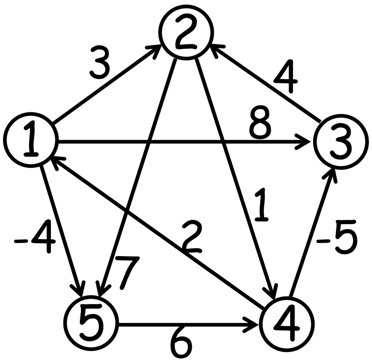
$d_{ij}^{(m)}$: shortest distance from i to j using at most m edges



$d_{25}^{(1)} = 7$ $d_{35}^{(1)} = \infty$

$d_{25}^{(2)} = 7$ $d_{35}^{(2)} = 11$ ← 變小，多一根 edge 有用

$d_{25}^{(3)} = -1$ $d_{35}^{(3)} = 11$ ← 不變，多一根 edge 沒用

$d_{25}^{(4)} = -1$ $d_{35}^{(4)} = 3$

$\cancel{d_{25}^{(5)}} = -1$ $\cancel{d_{35}^{(5)}} = 3$

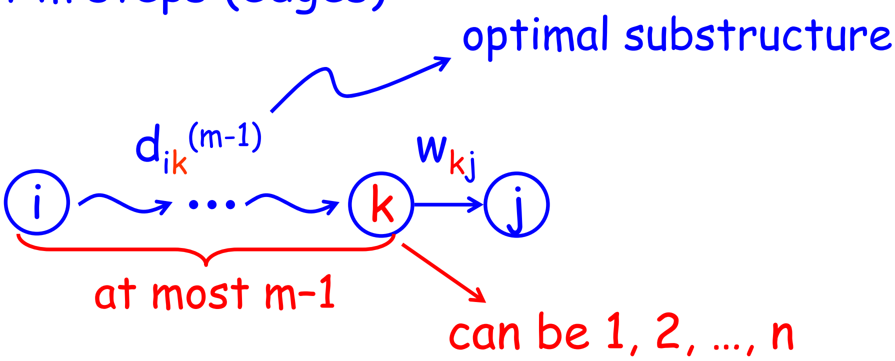
no negative cycles

➔ at most n−1 edges

➔ $d_{ij} = d_{ij}^{(4)} = d_{ij}^{(5)} = d_{ij}^{(6)} =$

$(d_{25} = d_{25}^{(4)} = -1)$

$d_{ij}^{(m)}$: at most m steps (edges)



$$d_{ij}^{(m)} = \min_{1 \le k \le n} \{ d_{ik}^{(m-1)} + w_{kj} \}$$

$* \left[D^{(m)} \right] \Leftarrow \left[D^{(m-1)} \right], \left[W \right]$ ($D^{(m)}$ 可由 $D^{(m-1)}$, W 得到)

Matrix multiplication

$$C = A \times B$$

$$c_{ij} = \sum_k \{ a_{ik} \times b_{kj} \}$$

$$(op_1, op_2) = (\times, +)$$

$$\begin{array}{c c} i & \begin{pmatrix} & \boxed{8} & \\ & & \\ & & \end{pmatrix} \\ & j \end{array} = \begin{array}{c c} i & \begin{pmatrix} 1 & 2 & 3 \\ & & \\ & & \end{pmatrix} \end{array} \times \begin{array}{c} \begin{pmatrix} & 1 \\ & 2 \\ & 1 \end{pmatrix} \\ j \end{array}$$

Boolean matrix multiplication

$$C = A \times B$$

$$c_{ij} = \underset{k}{\mathrm{or}} \{ a_{ik} \,\&\, b_{kj} \}$$

$$(op_1, op_2) = (\&, \mathrm{or})$$

$$\begin{array}{c c} i & \begin{pmatrix} & \boxed{1} & \\ & & \\ & & \end{pmatrix} \\ & j \end{array} = \begin{array}{c c} i & \begin{pmatrix} 0 & 1 & 1 \\ & & \\ & & \end{pmatrix} \end{array} \times \begin{array}{c} \begin{pmatrix} & 1 \\ & 0 \\ & 1 \end{pmatrix} \\ j \end{array}$$

Matrix multiplication with (op_1, op_2)

$$C = A \otimes B$$

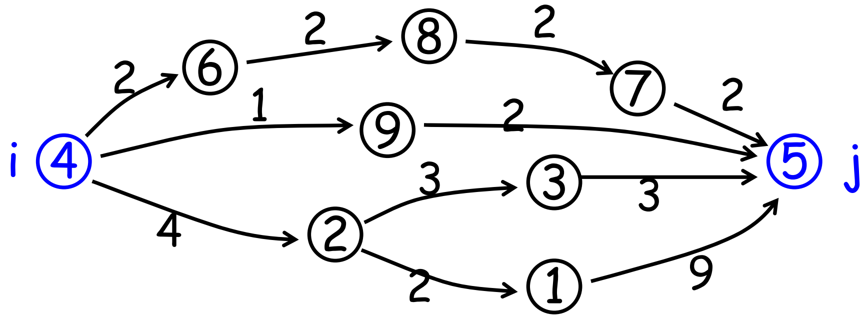
$$c_{ij} = \underset{k}{op_2} \{ \underbrace{a_{ik}}_{\text{row } i} \; op_1 \; \underbrace{b_{kj}}_{\text{column } j} \}$$

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + w_{kj} \}$$

$$\Rightarrow \quad \begin{array}{c c} & D^{(m)} \\ i & \begin{pmatrix} & \boxed{4} & \\ & & \\ & & \end{pmatrix} \\ & j \end{array} \Leftarrow \begin{array}{c c} & D^{(m-1)} \\ i & \begin{pmatrix} 1 & 2 & 3 \\ & & \\ & & \end{pmatrix} \end{array} , \begin{array}{c} W \\ \begin{pmatrix} & 4 \\ & 2 \\ & 3 \end{pmatrix} \\ j \end{array}$$

$$\Rightarrow \quad D^{(m)} = D^{(m-1)} \otimes W$$

(multiplication with $(op1, op2) = (+, \min)$)



$d_{45}^{(0)} = \infty$

$d_{45}^{(1)} = \infty$

$d_{45}^{(2)} = 15$

$d_{45}^{(3)} = 10$

$d_{45}^{(4)} = 10$

$d_{45}^{(5)} = 10$

$d_{45}^{(6)} = 10$

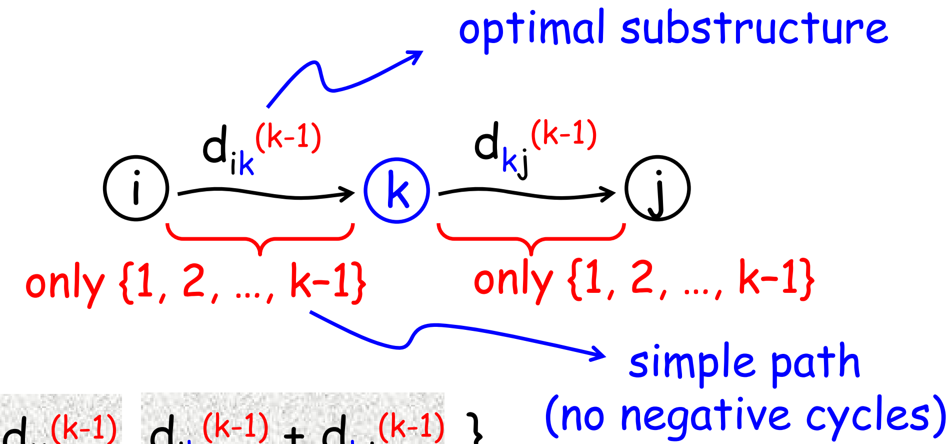
$d_{45}^{(7)} = 10$

$d_{45}^{(8)} = 8$

$d_{45}^{(9)} = 3$

* $d_{45}^{(3)} < d_{45}^{(2)} \Rightarrow$ 有經過 3
* $d_{45}^{(4)} = d_{45}^{(3)} \Rightarrow$ 不必經過 4

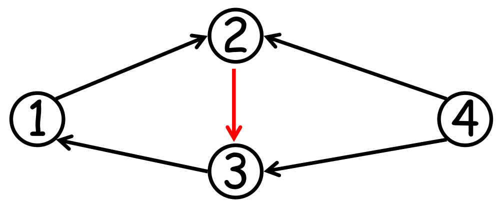
$d_{ij}^{(k)}$: shortest distance from i to j via only {1, 2, 3, …, k}



$$d_{ij}^{(k)} = \min\{\, \underbrace{d_{ij}^{(k-1)}}_{\text{不必經過 } k},\ \underbrace{d_{ik}^{(k-1)} + d_{kj}^{(k-1)}}_{\text{經過 } k} \,\}$$

* $D^{(k)}$ 可由 $D^{(k-1)}$ 得到

G:



Adjacency Matrix

$$A \quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

Transitive Closure

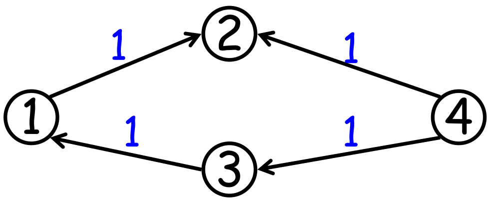
$$A^* \quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

null path

* Using A^* ➔ strongly connected components

i,j are in the same component
iff $A^*[i, j] = A^*[j, i] = 1$

G:



Method 1.

1. Assign w(e) = 1
   for each edge $e \in E$

$$W = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ 1 & \infty & 0 & \infty \\ \infty & 1 & 1 & 0 \end{pmatrix}$$

2. Perform an all-pair shortest paths algorithm

$$D = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ 1 & 2 & 0 & \infty \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

3. $D_{ij} \neq \infty \leftrightarrow a^*_{ij} = 1$

$$A^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Method 2: Modify the second Algo.

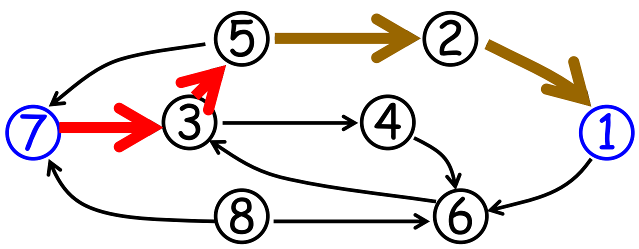
$$\underbrace{(i) \longrightarrow}_{\text{only } \{1,\,2,\,\dots,\,k-1\}} (k) \underbrace{\longrightarrow (j)}_{\text{only } \{1,\,2,\,\dots,\,k-1\}}$$

$d_{ij}^{(k)}$: shortest distance, via only $\{1, 2, \dots, k\}$

$$d_{ij}^{(k)} = \min \{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\}$$

$$t_{ij}^{(k)} = \begin{cases} 1\text{: reachable, via only } \{1, 2, \dots, k\} \\ 0 \end{cases}$$

$$t_{ij}^{(k)} = \text{OR } \{t_{ij}^{(k-1)}, t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}\}$$
$$= t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$$



$t_{71}^{(0)} = 0$ $\quad t_{71}^{(4)} = 0$ $\quad t_{71}^{(8)} = 1$

$t_{71}^{(1)} = 0$ $\quad \boxed{t_{71}^{(5)} = 1}$

$t_{71}^{(2)} = 0$ $\quad t_{71}^{(6)} = 1$

$t_{71}^{(3)} = 0$ $\quad t_{71}^{(7)} = 1$

$t_{75}^{(4)} = 1$ and $t_{51}^{(4)} = 1$

$$* \; t_{ij}^{(k)} = 1 \begin{cases} \text{case 1. } t_{ij}^{(k-1)} = 1 & (\text{不必經過 } k) \\ \text{case 2. } t_{ik}^{(k-1)} = 1 \text{ and } t_{kj}^{(k-1)} = 1 & (\text{經過 } k) \end{cases}$$

$\Rightarrow$ $t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)})$

* $T^{(k)}$ 可由 $T^{(k-1)}$ 得到

Shortest Paths Algorithms - Review

Main Ideas

Optimal substructure: (1) $\pi(v) \xrightarrow{\text{relax}} v$, (2) DP
ok ok

No negative cycles: simple path (at most n-1 edges)

Single-Source (relax)

Bellman-Ford (no negative cycles, can detect) O(VE)

$\underset{\text{ok}}{U_0} \overset{=\{s\}}{} \rightarrow \underset{\text{ok}}{U_1} \rightarrow \underset{\text{ok}}{U_2} \rightarrow \underset{\text{ok}}{U_3} \rightarrow \dots \rightarrow \underset{\text{ok}}{U_{n-1}}$

Dijkstra (no negative edges) O(Vlg V+E)

$\underset{\text{ok}}{\text{rank}(1)} \overset{=\{s\}}{} \rightarrow \underset{\text{ok}}{\text{rank}(2)} \rightarrow \underset{\text{ok}}{\text{rank}(3)} \rightarrow \dots \rightarrow \underset{\text{ok}}{\text{rank}(n)}$

All-Pairs (DP)

check no negative cycles first

Matrix Multiplication (no negative cycles) $O(V^3 \lg V)$

$d_{ij}^{(m)}$: shortest distance using at most m edges

$D^{(m)} = D^{(m-1)} \otimes W = W^m$ (op1, op2) = (+, Min)

$D = D^{(m)}$ for $m \geq n-1$ (no negative cycles)

$\underset{W}{\underset{\|}{D^{(1)}}} \rightarrow D^{(2)} \rightarrow D^{(4)} \rightarrow D^{(8)} \rightarrow \dots \rightarrow D^{(n)}$ (lg (n-1) times)

Floyd-Warshall (no negative cycles) $O(V^3)$

$d_{ij}^{(k)}$: shortest distance via only {1, 2, ..., k}

$D = D^{(n)}$

$\underset{W}{\underset{\|}{D^{(0)}}} \rightarrow D^{(1)} \rightarrow D^{(2)} \rightarrow D^{(3)} \rightarrow \dots \rightarrow D^{(n)}$