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Growth of Functions

3-1a

漸近的

3.1 Asymptotic notation

"=" ∈ a set

Θ-notation: $f(n) = \Theta(g(n))$

$g(n)$ is an asymptotically tight bound for $f(n)$.

$\Theta(g(n)) = \{f(n) |$ there exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$$

for all $n \ge n_0\}$ 調小 調大

when n is sufficiently large

f g

Example: Prove that $3n^2 - 6n = \Theta(n^2)$.

Proof: To do so, we have to determine c_1, c_2, and n_0 such that

g f g

$$c_1 n^2 \le 3n^2 - 6n \le c_2 n^2, \qquad \text{(for all } n \ge n_0)$$

dividing which by n^2 yields

$$c_1 \le 3 - 6/n \le c_2.$$

3-1y

Clearly, by choosing c_1=2, c_2=3 and n_0=6 we can verify that $3n^2 - 6n = \Theta(n^2)$. Q.E.D

There are many choices!

- $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$, Ex. $n^2 = \Theta(3n^2 - 6n)$

'$\leq$'

***O*-notation**: $f(n) = O(g(n))$

$g(n)$ is an asymptotically upper bound for $f(n)$.

$O(g(n)) =$ {$f(n)$| there exist positive constants c and n_0 such that

$$0 \leq f(n) \leq cg(n)$$

for all $n \geq n_0$} 調大

- $\Theta(g(n)) \subseteq O(g(n))$
- $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$
- $6n = O(n)$, $6n = O(n^2)$
- "The running time is $O(n^2)$" means "the worst-case running time is $O(n^2)$."

'$\geq$'

***Ω*-notation**: $f(n) = \Omega(g(n))$

$g(n)$ is an asymptotically lower bound for $f(n)$.

$\Omega(g(n)) =$ {$f(n)$| there exists positive constants c and n_0 such that

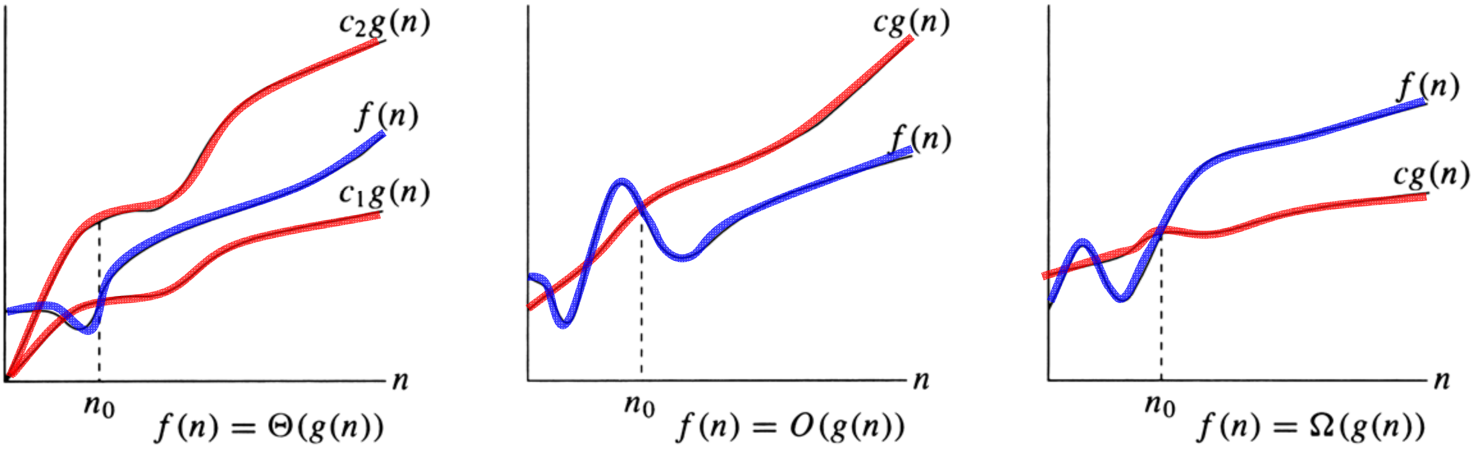
調小

$$0 \leq cg(n) \leq f(n)$$

for all $n \geq n_0$}

- $f(n) = \Theta(g(n))$ iff $(f(n)=O(g(n)))$ & $(f(n)=\Omega(g(n)))$

3-3x



(a) tight bound

(b) upper bound

(c) lower bound

3-3v

$o(g(n)) = O(g(n)) \setminus \Theta(g(n))$???

'<'

***o*-notation**: $f(n) = o(g(n))$ (little-oh of g of n)

$o(g(n))=$ $\{f(n)|$ for any positive constant c, there exists a constant $n_0 > 0$ such that

$$0 \le f(n) < cg(n)$$

for all $n \ge n_0\}$

任意小

- $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.
- $f(n) = o(g(n))$ can also be defined as

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

'>'

***ω*-notation**: $f(n) = \omega(g(n))$ (little-omega of g of n)

$\omega(g(n)) = \{f(n) \mid$ for (any) positive constant c, there exists a constant $n_0 > 0$ such that

$$0 \le cg(n) < f(n) \quad \text{for all } n \ge n_0\}$$

任意大

- $2n^2 = \omega(n)$, but $2n^2 \ne \omega(n^2)$.
- $f(n) = \omega(g(n))$ iff $\lim_{n\to\infty} \dfrac{f(n)}{g(n)} = \infty$.

Comparison of functions

functions:	ω	Ω	Θ	O	o
real numbers:	$>$	$\ge$	$=$	$\le$	$<$

- Transitivity, Reflexivity, Symmetry, Transpose Symmetry

3-4x

- Any two real numbers can be compared. (trichotomy) But, not any two functions can be compared.

  Example: $f(n)=n$ and $g(n)=n^{1+\sin n}$

(a)~(d)

Homework: Problems 3-2, 3-3, 3-4.

3-4y

$\lg^* n = k$ $\quad \underbrace{\lg\lg\ldots\lg}_{k} n \le 1$

Appendix A: Summation formulas

$$\sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$1 + 2 + \dots + n$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) = \Theta(n^2)$$

$x^0 + x^1 + x^2 + \dots + x^n$

$$\sum_{k=0}^{n} x^k = (x^{n+1} - 1)/(x - 1)$$

$\log_e n = \int_1^n \frac{1}{x}\,dx$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \log_e n + O(1) \quad \text{(Harmonic series)}$$

$= O(\lg n)$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x| < 1) \;\Rightarrow\; \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad (|x| < 1)$$

3-5x

(differentiating and then multiplying by x)

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n}$$

$$\lg \prod_{k=1}^{n} a_k = \sum_{k=1}^{n} \lg a_k$$

* $\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$ * $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$

* $\lg_a b = (\lg_c b)/(\lg_c a)$ * $a^{\lg_c b} = b^{\lg_c a}$