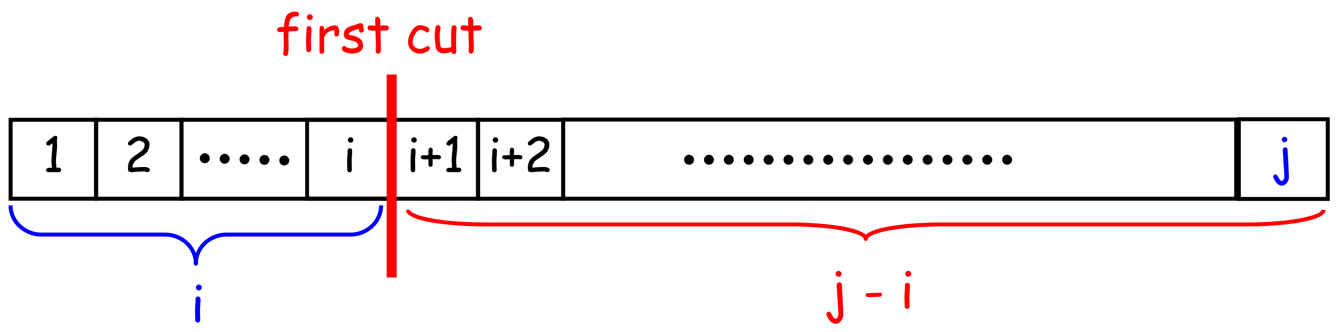


$$r[j] = \underset{1 \le i \le j}{\text{MAX}} \left\{ \underset{\text{left-part}}{p[i]} + \underset{\text{right-part}}{r[j-i]} \right\}$$

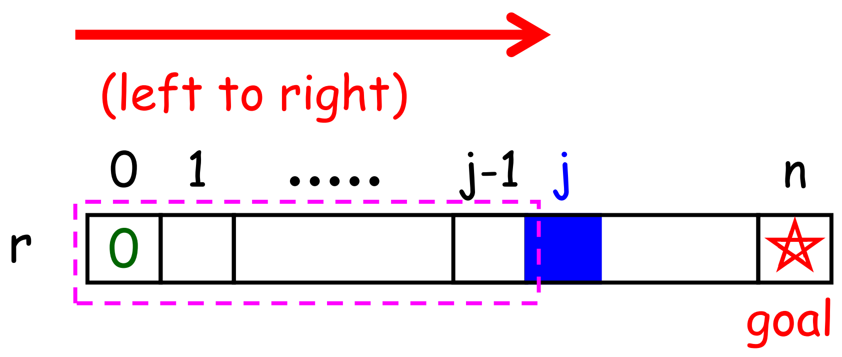
Step 3

$$r[j] = \underset{1 \le i \le j}{\text{MAX}} \left\{ p[i] + r[j-i] \right\}$$

(i) Draw a table

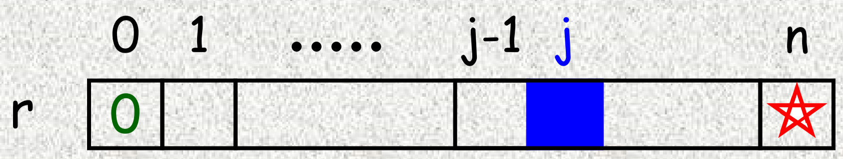
(ii) Observe the dependency

(iii) Find a good order



* r[j] needs r[0], r[1], ..., r[j-1]
(ex. r[9] needs r[0], r[1], ..., r[8])

Time complexity



$$r[j] = \max_{1 \le i \le j} \{ p[i] + r[j] \}$$

$$\text{Time:} \quad \sum_{j=1}^{n} O(j) = \sum_{j=1}^{n} O(n) = n \times O(n) = O(n^2)$$

table size

$= O(1+2+\ldots+n)$

$= O(n(n+1)/2)$

$= O(n^2)$

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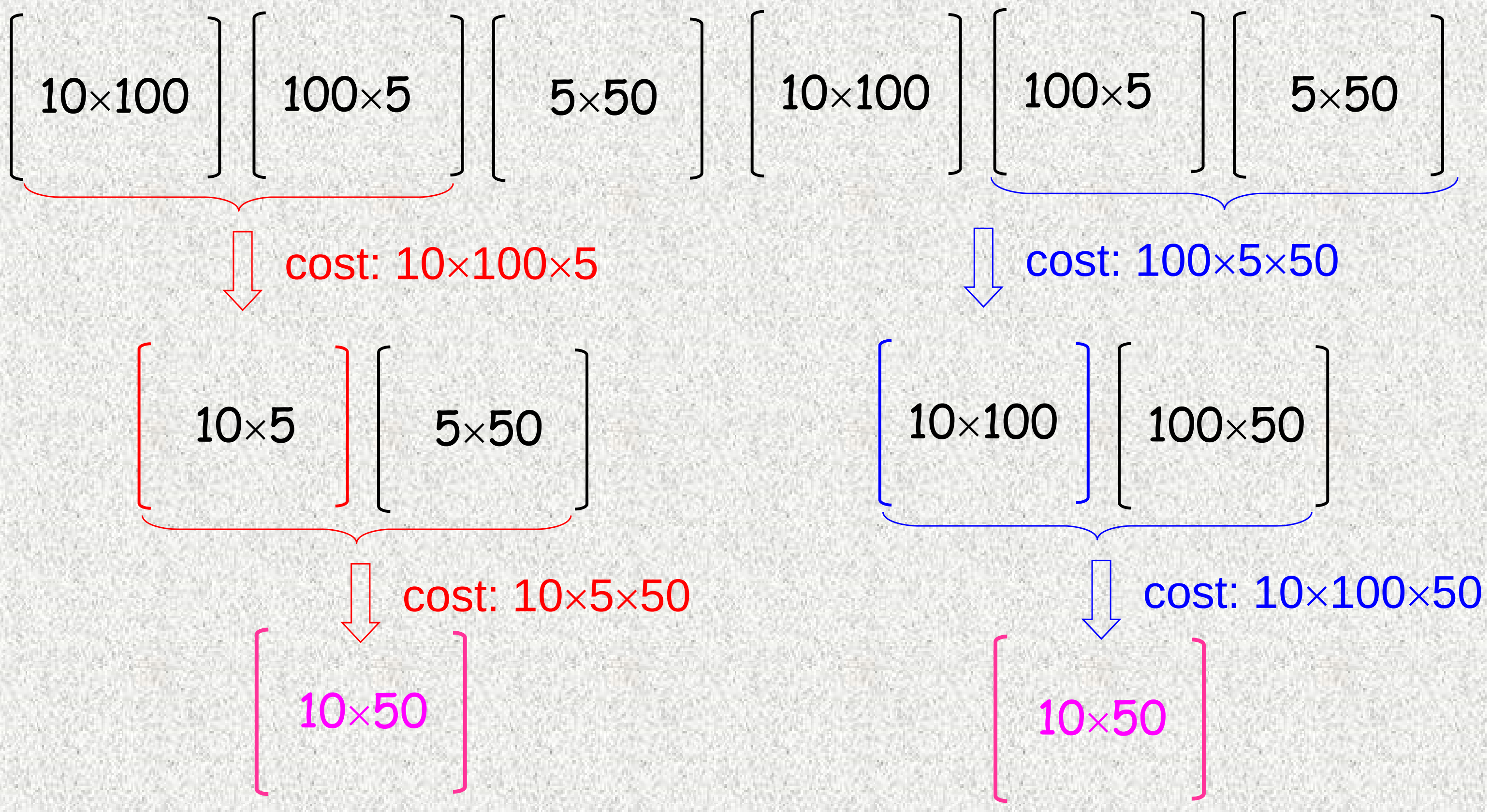
$$Z_{a\times c} = X_{a\times b} \times Y_{b\times c}$$

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & & \\ & z_{ij} & \\ & & \end{bmatrix} \end{array} = \underbrace{\begin{array}{cc} & \\ & \begin{bmatrix} x_{11} & x_{12} & & x_{1b} \\ x_{21} & x_{22} & & x_{2b} \\ i & & \cdots\cdots & \\ x_{a1} & x_{a2} & & x_{ab} \end{bmatrix} \end{array}}_{b} \times \begin{array}{c} \begin{array}{cccc} & & j & \end{array} \\ \begin{bmatrix} y_{11} & y_{12} & & y_{1c} \\ y_{21} & y_{22} & & y_{2c} \\ & & \cdots\cdots & \\ y_{b1} & y_{k2} & & y_{bc} \end{bmatrix} \end{array}$$

cost of computing z_{ij} : b multiplications

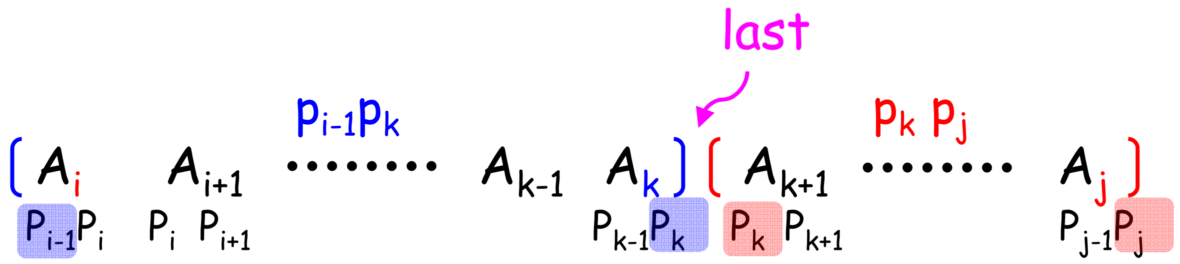
cost of computing Z: abc multiplications

Note: size of Z is $a\times c$



* total cost: 7500

* total cost: 75000


last
p_{i-1}p_k
p_k p_j
(A_i A_{i+1} .... A_{k-1} A_k) (A_{k+1} .... A_j)
P_{i-1}P_i P_i P_{i+1} P_{k-1}P_k P_k P_{k+1} P_{j-1}P_j

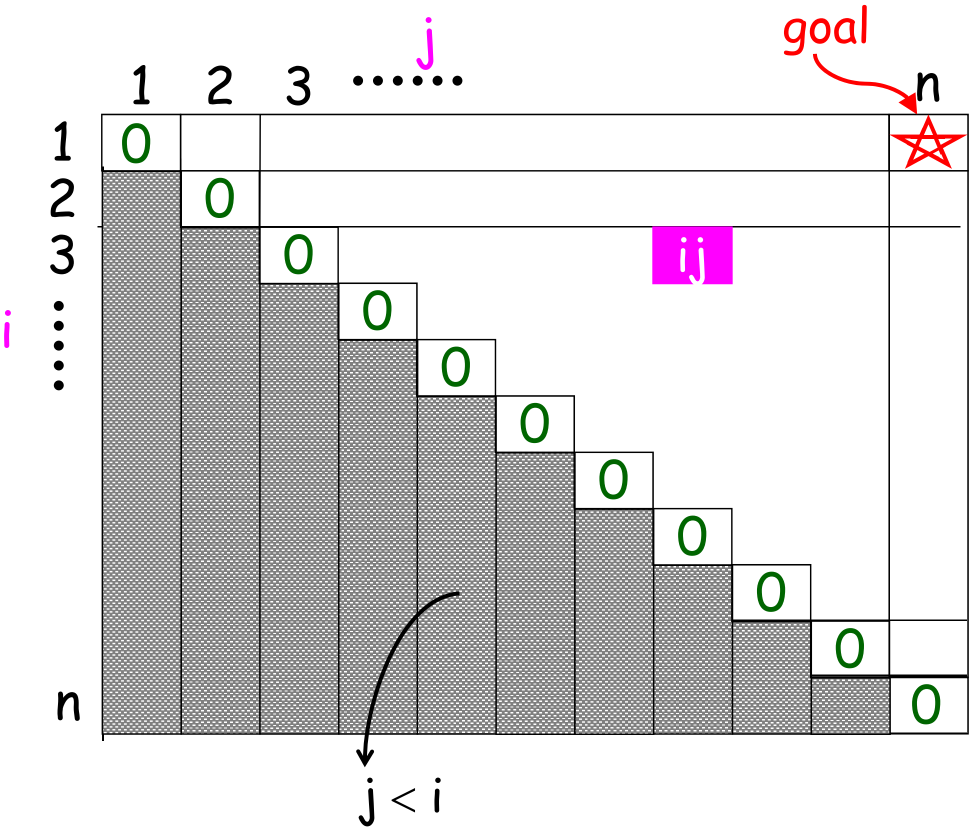
$$m_{ij} = \underset{i \le k \le j-1}{\text{MIN}} \left\{ \underbrace{m_{ik}}_{\text{left-part}} + \underbrace{m_{k+1j}}_{\text{right-part}} + \underbrace{p_{i-1}p_k p_j}_{\text{last}} \right\}$$

Step 3

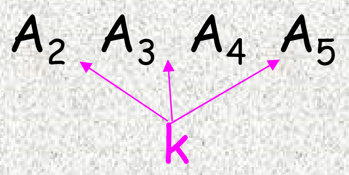
$$m_{ij} = \text{MIN}\{m_{ik} + m_{k+1j} + p_{i-1}p_kp_j\}$$

(i) Draw a table

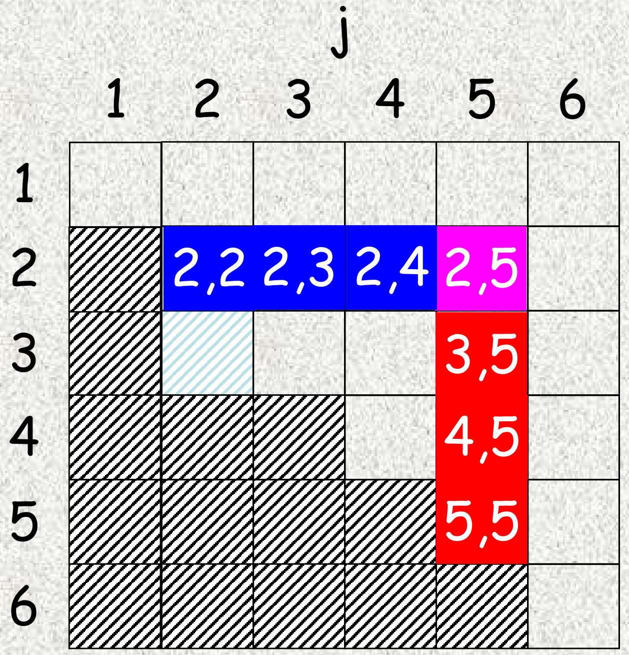
(ii) Observe the dependency



Compute m[2, 5] (See page 15-7)



$$\text{MIN}\begin{cases} m[2,2] + m[3,5] + p_1p_2p_5 & k=2\\ m[2,3] + m[4,5] + p_1p_3p_5 & k=3\\ m[2,4] + m[5,5] + p_1p_4p_5 & k=4\end{cases}\Bigg\} i$$



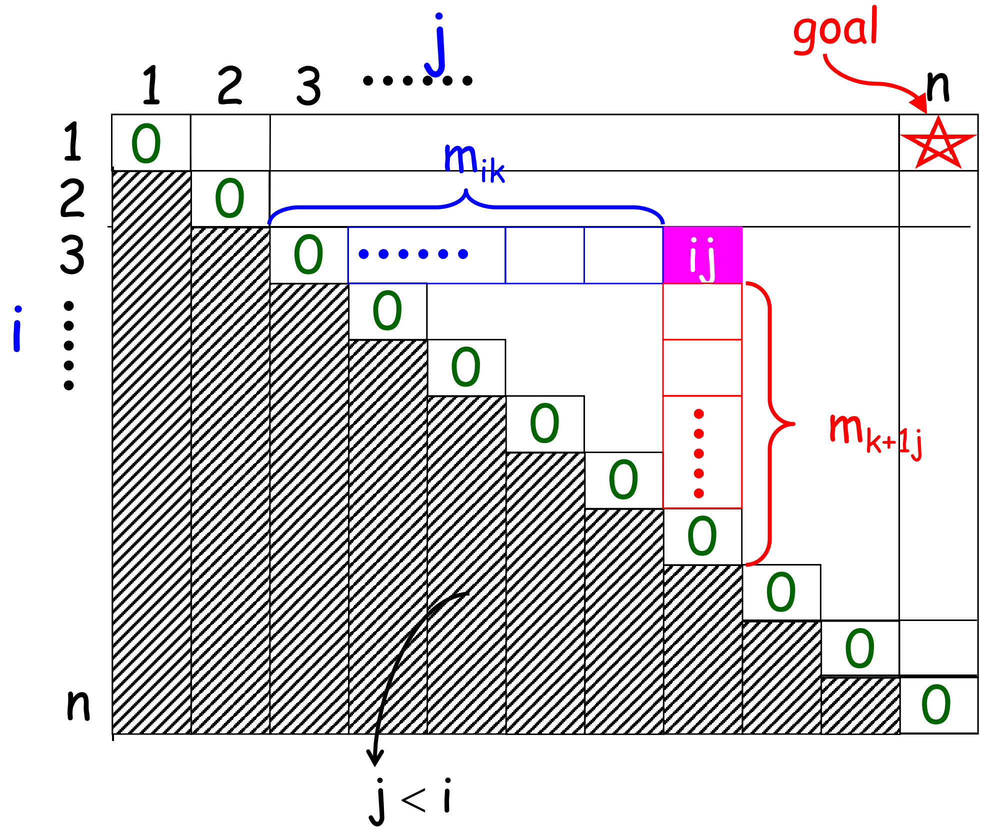
Step 3

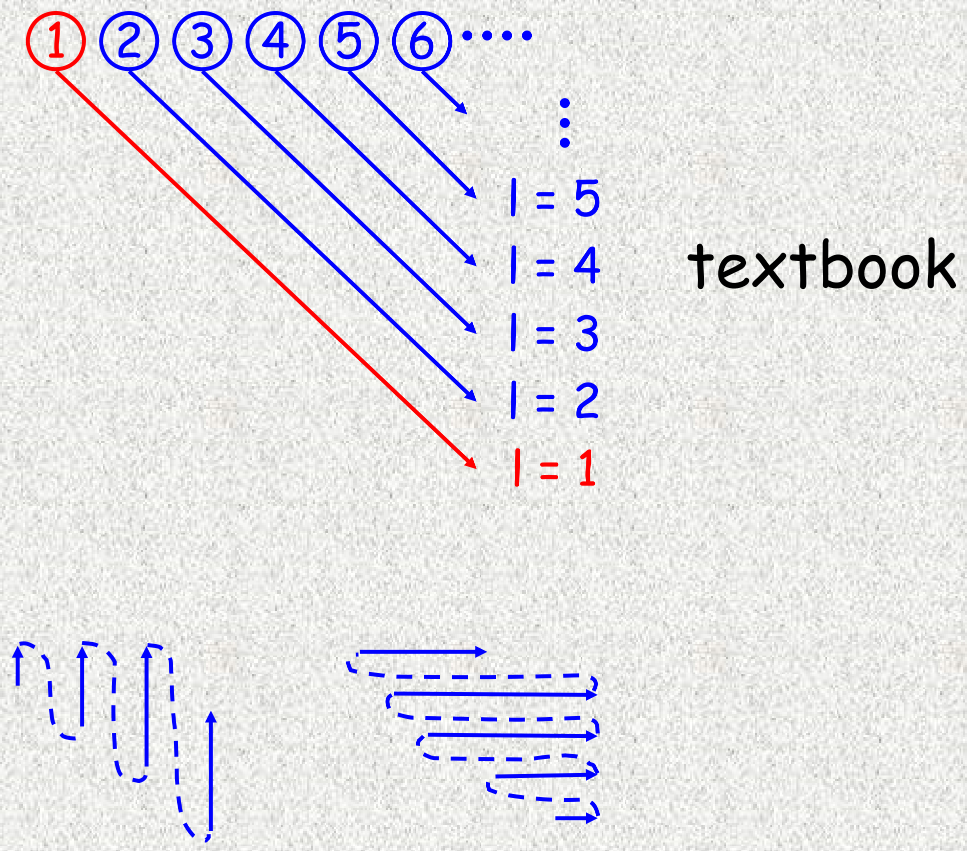
$m_{ij} = \text{MIN}\,\{m_{ik} + m_{k+1j} + p_{i-1}p_kp_j\}$

(i) Draw a table

(ii) Observe the dependency

(iii) Find a good order



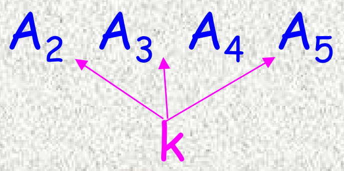


Compute m[2, 5]

(See page 15-7)

p0	p1	p2	p3	p4	p5	p6
30	35	15	5	10	20	25

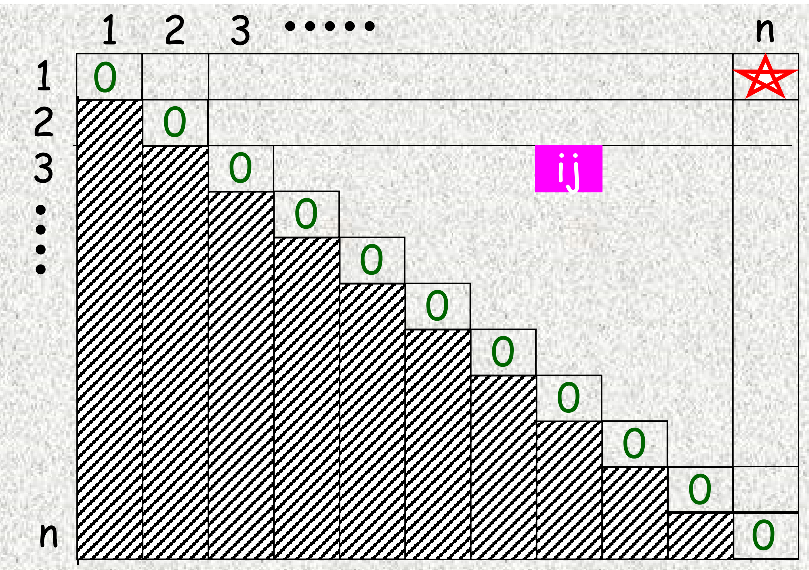
	1	2	3	4	5	6
1	0	15750	7875	9375		
2	▨	0	2625	4375	7125	
3	▨	▨	0	750	2500	
4	▨	▨	▨	0	1000	3500
5	▨	▨	▨	▨	0	5000
6	▨	▨	▨	▨	▨	0



$$\text{MIN}\begin{Bmatrix} 0 + 2500 + 35\times15\times20 & (k = 2) \\ 2625 + 1000 + 35\times5\times20 & (k = 3) \\ 4375 + 0 + 35\times10\times20 & (k = 4) \end{Bmatrix} = \text{MIN}\begin{Bmatrix} 13000 \\ 7125 \\ 11375 \end{Bmatrix}$$

s[2,5]

Time complexity



$$m_{ij} = \underset{i \le k \le j-1}{\text{MIN}} \{m_{ik} + m_{k+1j} + p_{i-1}p_kp_j\}$$

$$\text{Time: } \sum_{i,j} O(j-i) = \sum_{i,j} O(n) = O(n^2) \times O(n) = O(n^3)$$

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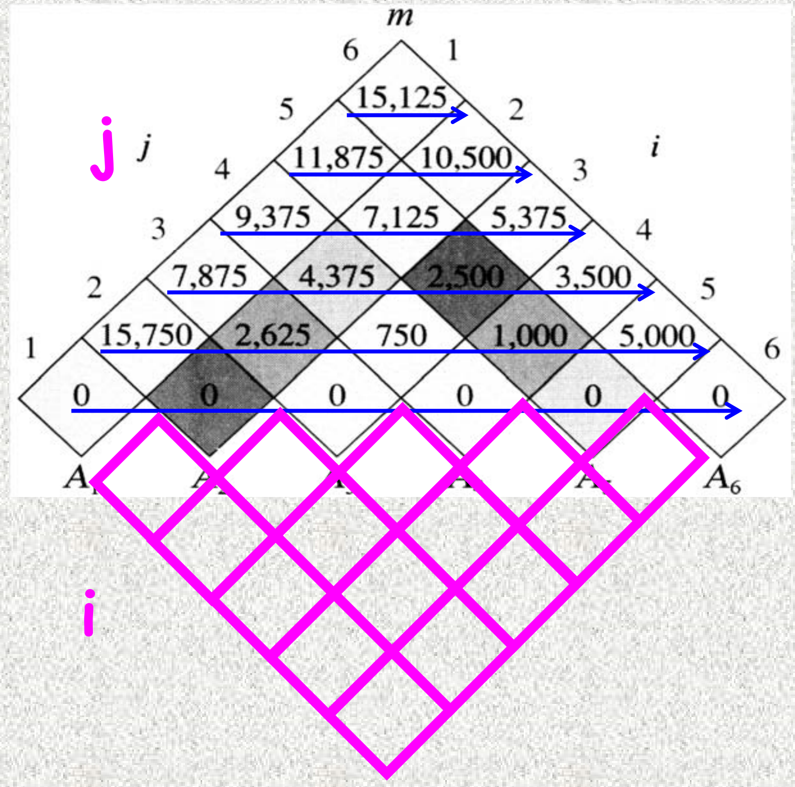
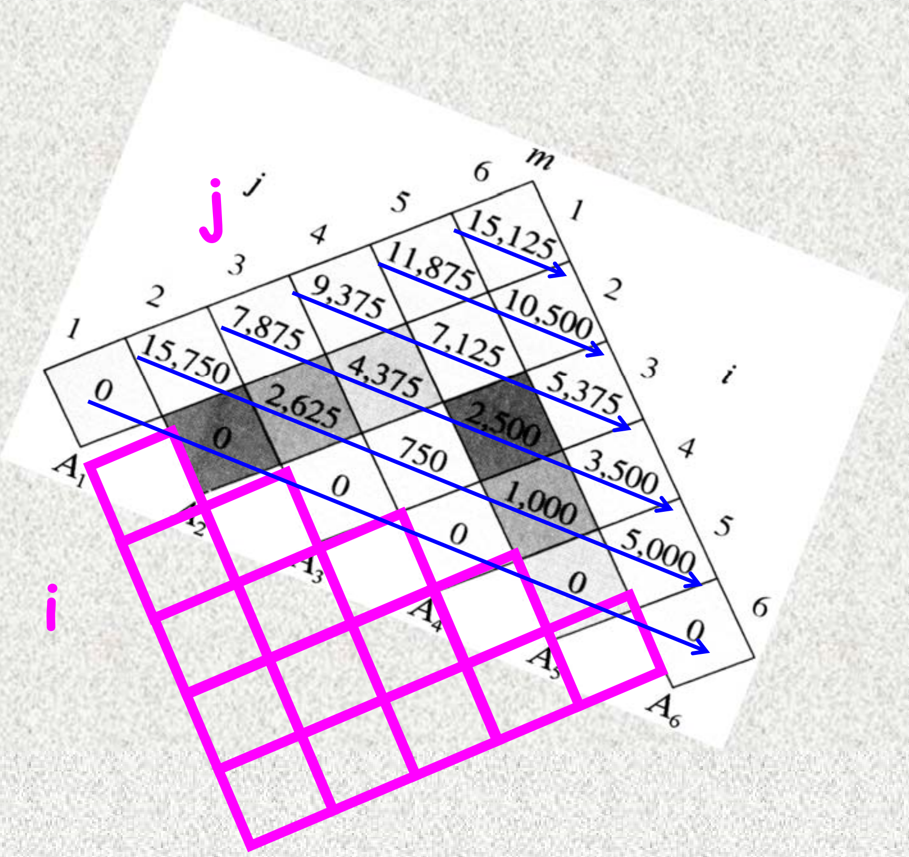
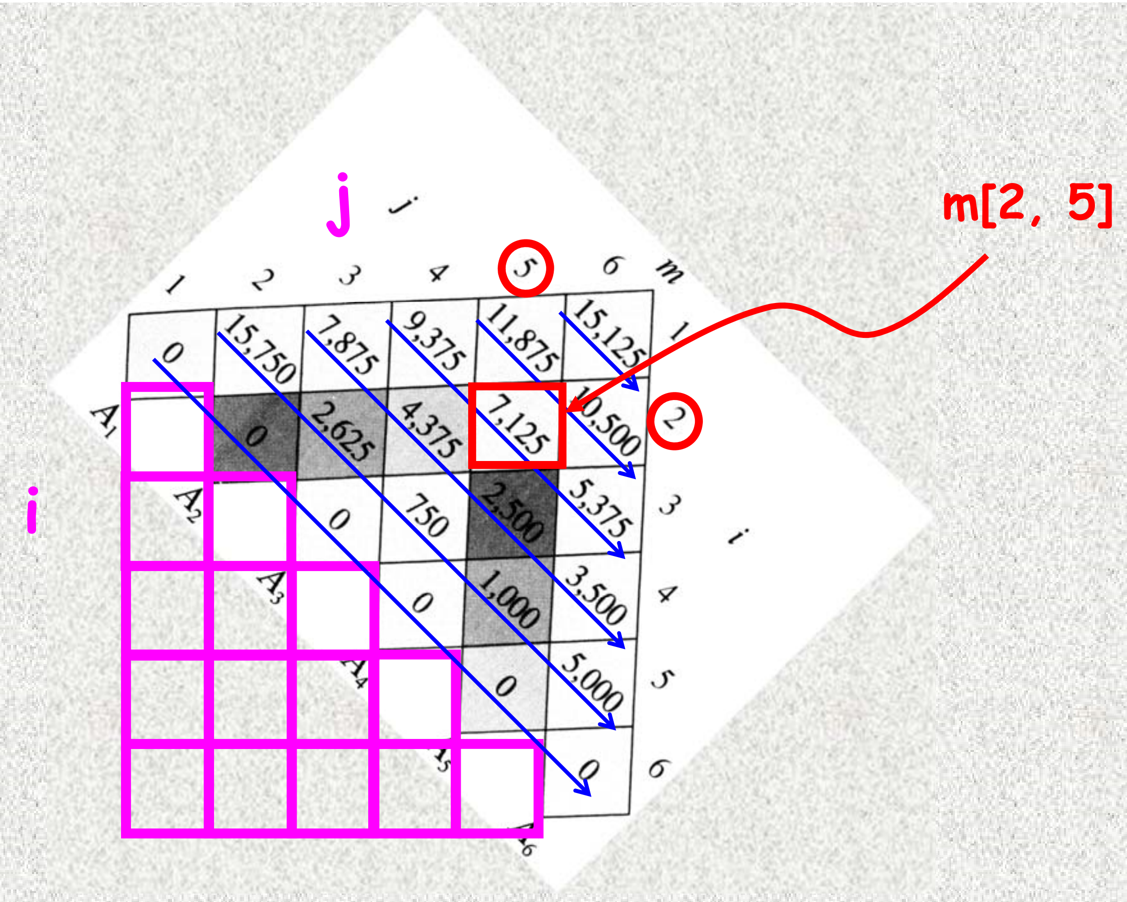
table size



m
j
i
1
2
3
4
5
6
15,125
11,875
10,500
9,375
7,125
5,375
7,875
4,375
2,500
3,500
15,750
2,625
750
1,000
5,000
0
0
0
0
0
0
A1
A2
A3
A4
A5
A6
j
i



m
j
i
1
2
3
4
5
6
15,125
11,875
10,500
9,375
7,125
5,375
7,875
4,375
2,500
3,500
15,750
2,625
750
1,000
5,000
0
0
0
0
0
0
A1
A2
A3
A4
A5
A6
j
i

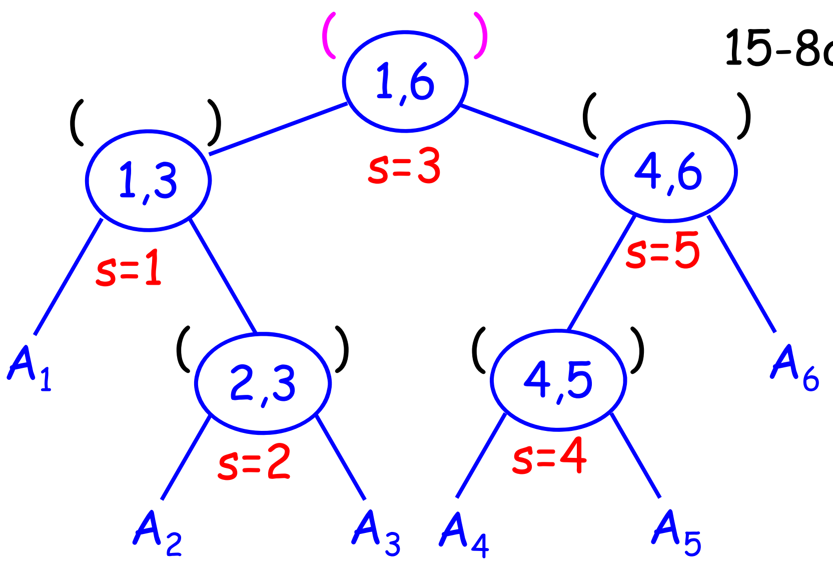


15-7y

15-8a

	1	2	3	4	5	6
1	▨	1	1	3	3	3
2	▨	▨	2	3	3	3
3	▨	▨	▨	3	3	3
4	▨	▨	▨	▨	4	5
5	▨	▨	▨	▨	▨	5
6	▨	▨	▨	▨	▨	▨

table s



$(\,(A_1(A_2A_3))\,((A_4A_5)A_6)\,)$

a recursive procedure { one matrix: print A_i
otherwise: ① (↙② ↘③) ④

Fibonacci numbers

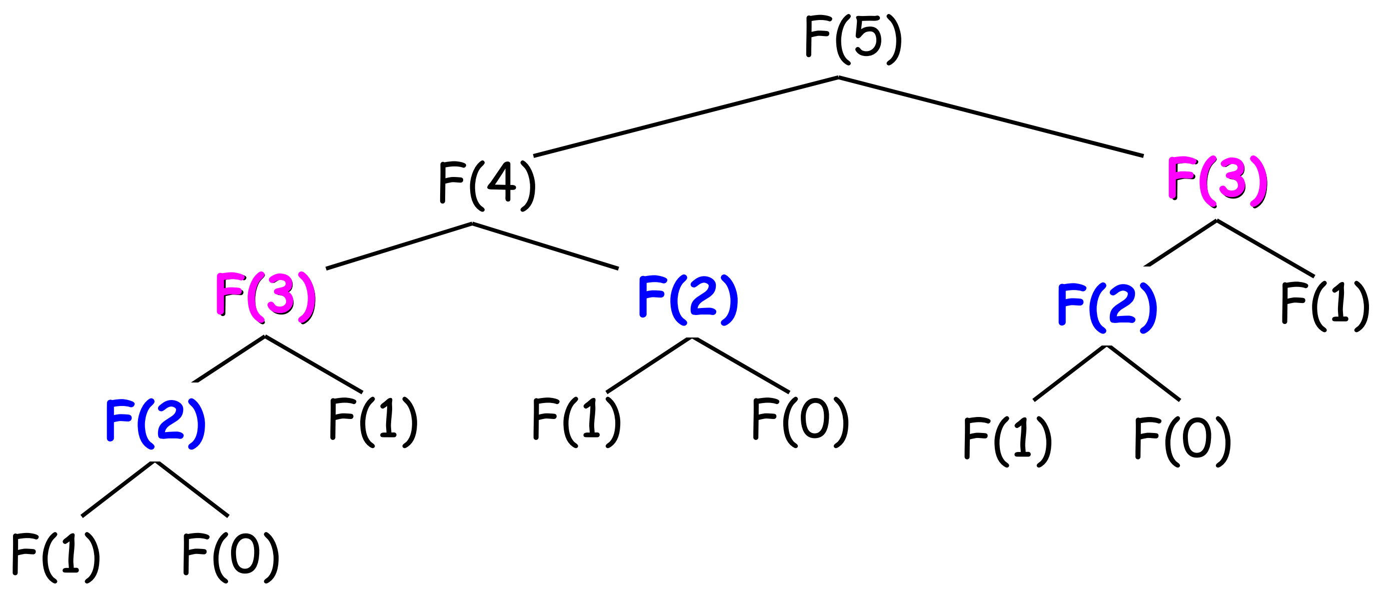
$$\begin{cases} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

(i) Recursive

(top-down, $O(2^n)$)

```
function F(n)
begin
    if $n \leq 1$ then return 1
    else
        return F(n-1)+F(n-2);
end;
```

Top-down: $O(2^n)$



Fibonacci numbers

$$\begin{cases} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

(i) Recursive
(top-down, $O(2^n)$)

```
function F(n)
begin
    if $n \le 1$ then return 1
    else
        return F(n-1)+F(n-2);
end;
```

(ii) Tabular
(DP: bottom-up, $O(n)$)

```
F: array [0..n] of integer;

F[0] := 1; F[1] := 1;
for i:=2 to n do
    F[i] := F[i-1] + F[i-2];
return F[n];
```

Memoization : $F_n = F_{n-1} + F_{n-2}$

(top-down DP!)

	0	1	2	3	4	5
F	∞	∞	∞	∞	∞	∞

```
Mem-F(n)
    F: array [0..n] of integer;
    for i=0 to n do F[i] := $\infty$
    return Lookup-F(n);

Lookup-F(i)
    if F[i] $\ne \infty$ then return F[i]
    ~~else~~
    if $i \le 1$ then ~~return~~ F[i] := 1
    else
        ~~return~~ F[i] := Lookup-F(i-1) +
                           Lookup-F(i-2);
    return F[i];
```

avoid recomputing

compute and save

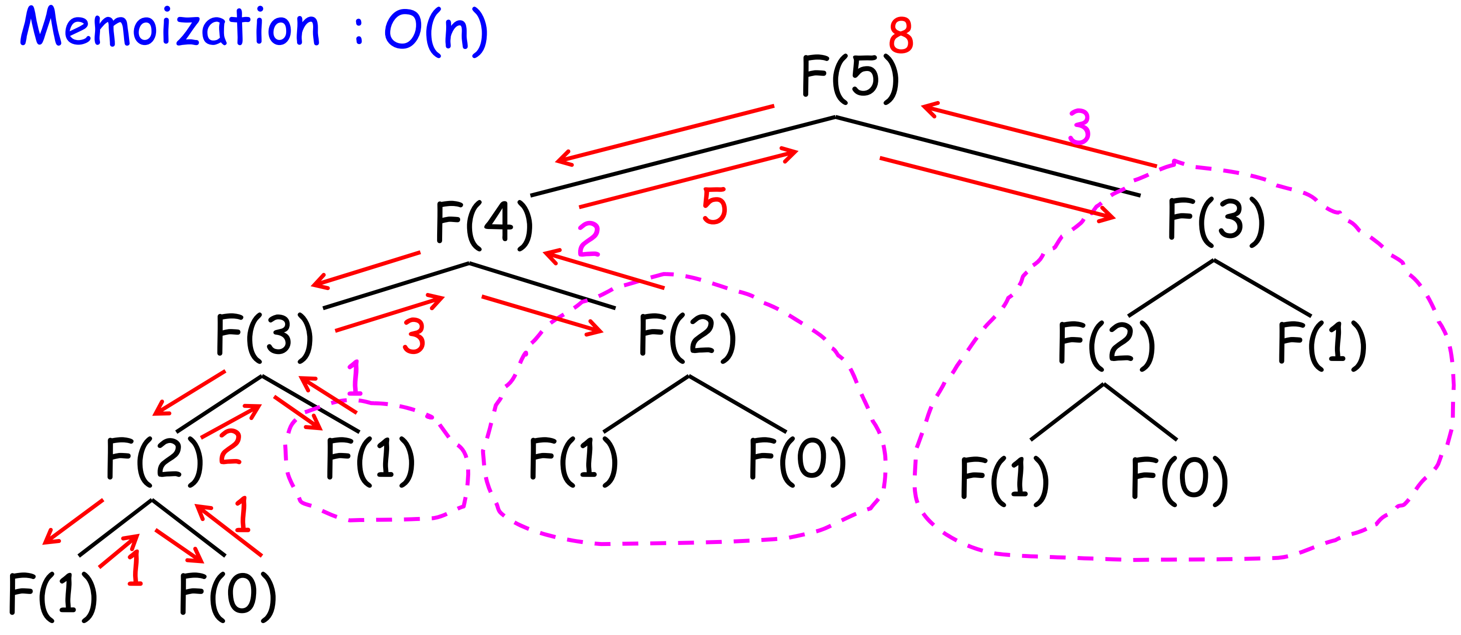
save for latter usage

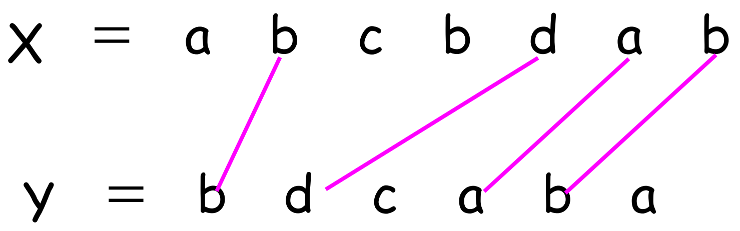
Top-down: $\cancel{O(2^n)}$

+

Memoization : O(n)

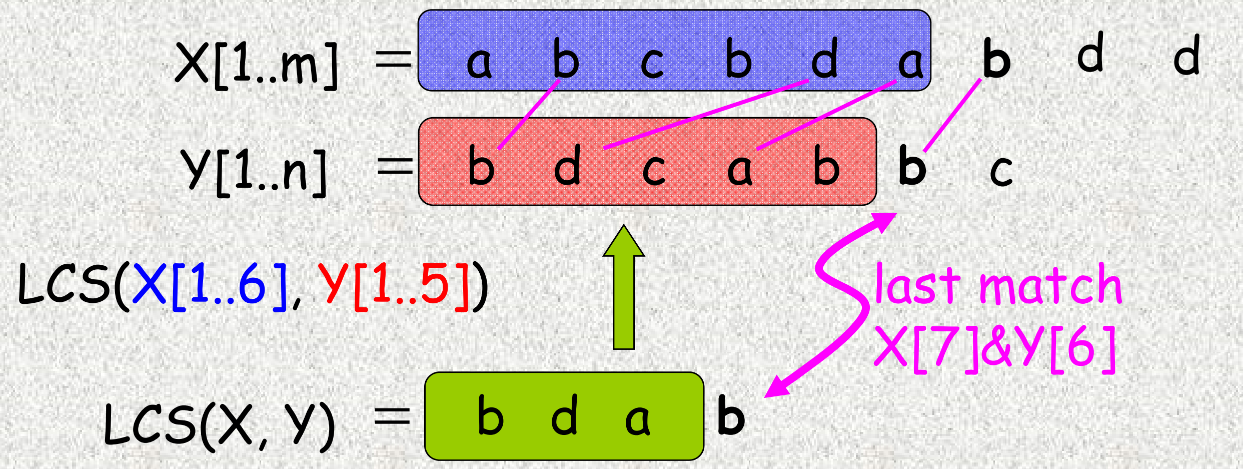
0	1	2	3	4	5
1	1	2	3	5	8





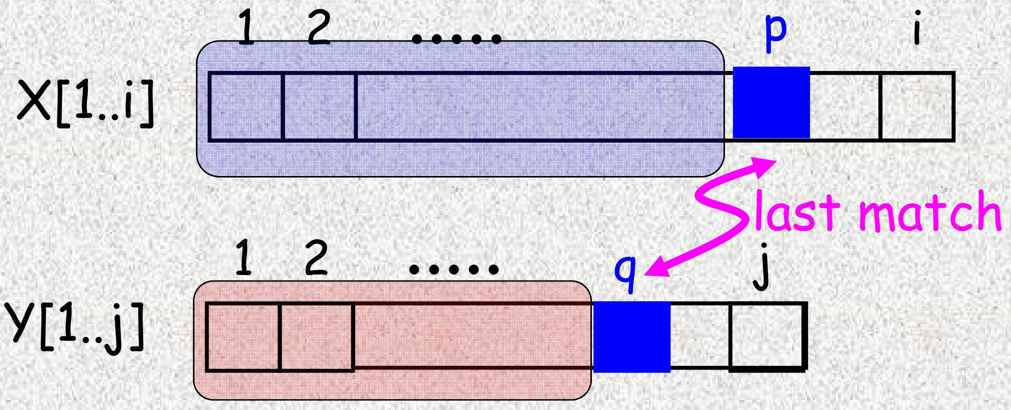
LCS $\Longrightarrow$ max # of non-crossing matching

Optimal substructure



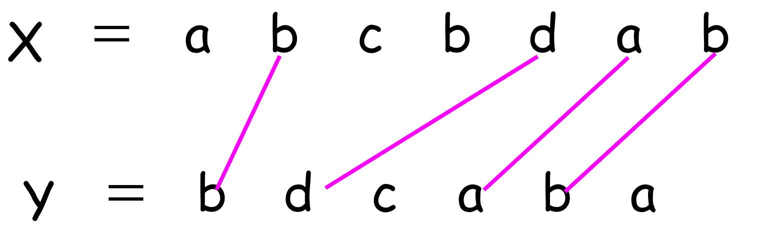
A naive DP

Let c[i,j] = len of LCS(X[1..i], Y[1..j])



$$c[i,j] = \max_{\substack{1 \le p \le i \\ 1 \le q \le j \\ X[p]=Y[q]}} \left\{ c[p-1,q-1] + 1 \right\}$$

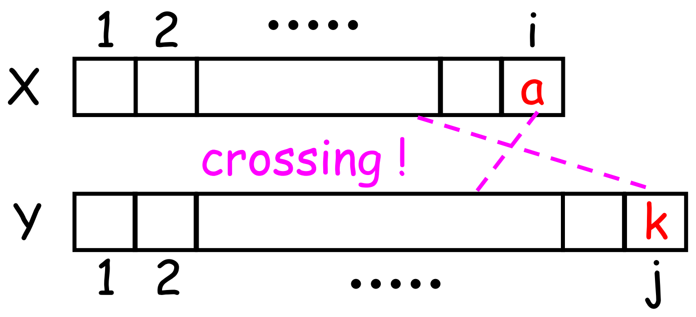
Time: $O(n^4)$



LCS ⇨ max # of non-crossing matching

Case 1. $x_i = y_j$ (discussed latter)

Case 2. $x_i \neq y_j$

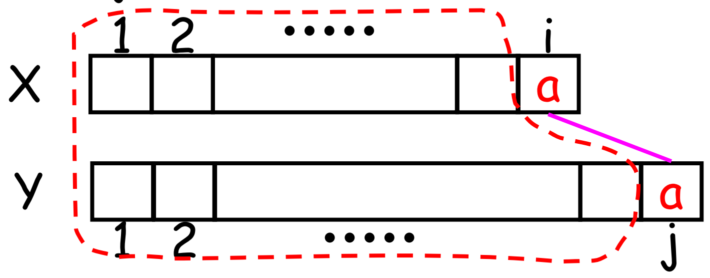


At least one of x_i and y_j can be discarded !

Let c[i,j] = LCS(X[1..i], Y[1..j])

$$c[i, j] = \text{MAX}\left\{\begin{array}{l} \overset{\text{discard } y_j}{c[i, j-1]} \\ \underset{\text{discard } x_i}{c[i-1, j]} \end{array}\right\}$$

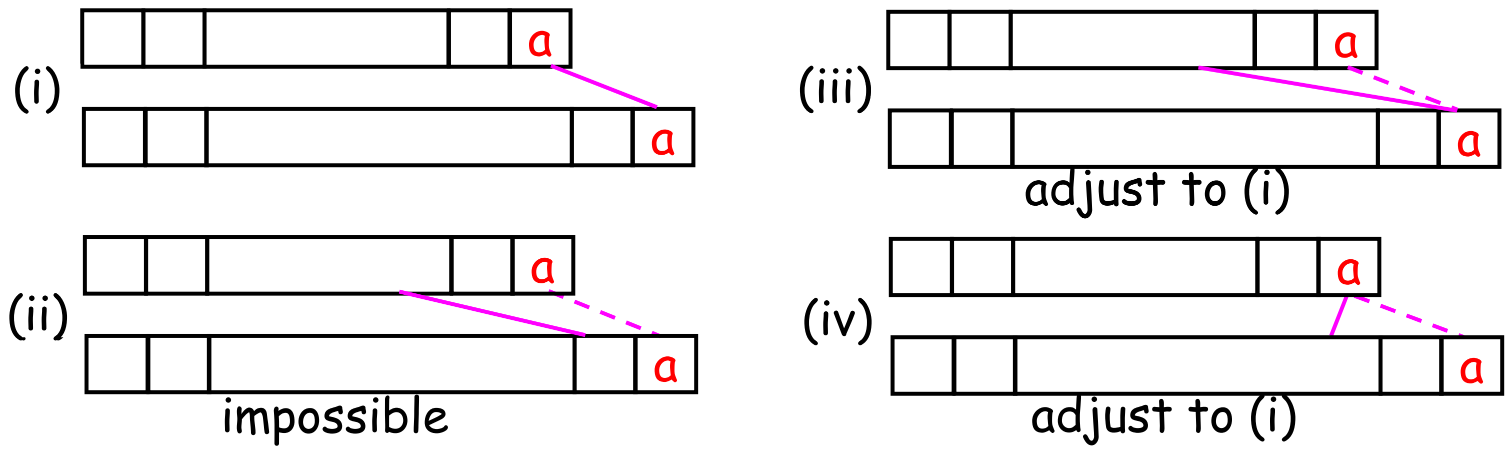
Case 1. $x_i = y_j$



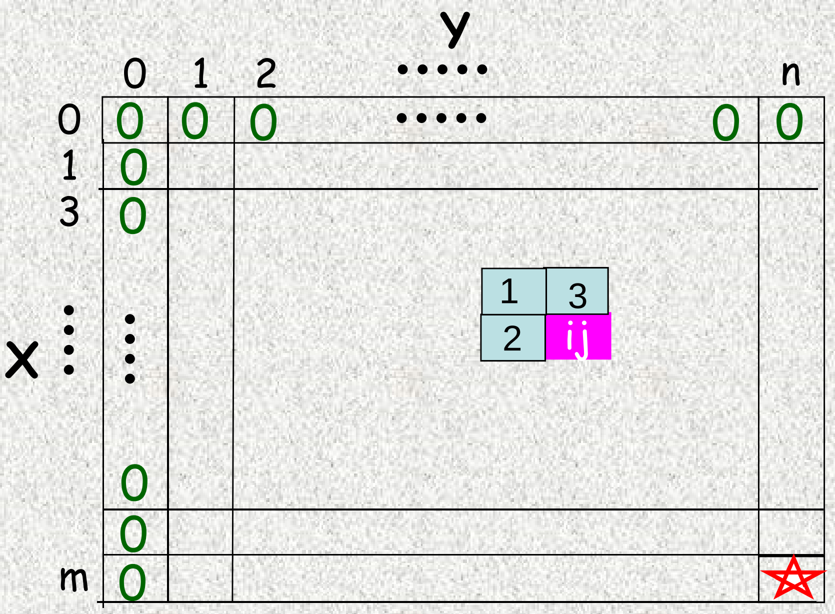
Match the tails !

$$c[i, j] = c[i-1, j-1] + 1$$

------------ last matching of an optimal solution: 4 cases ----------



Dependency and Time complexity



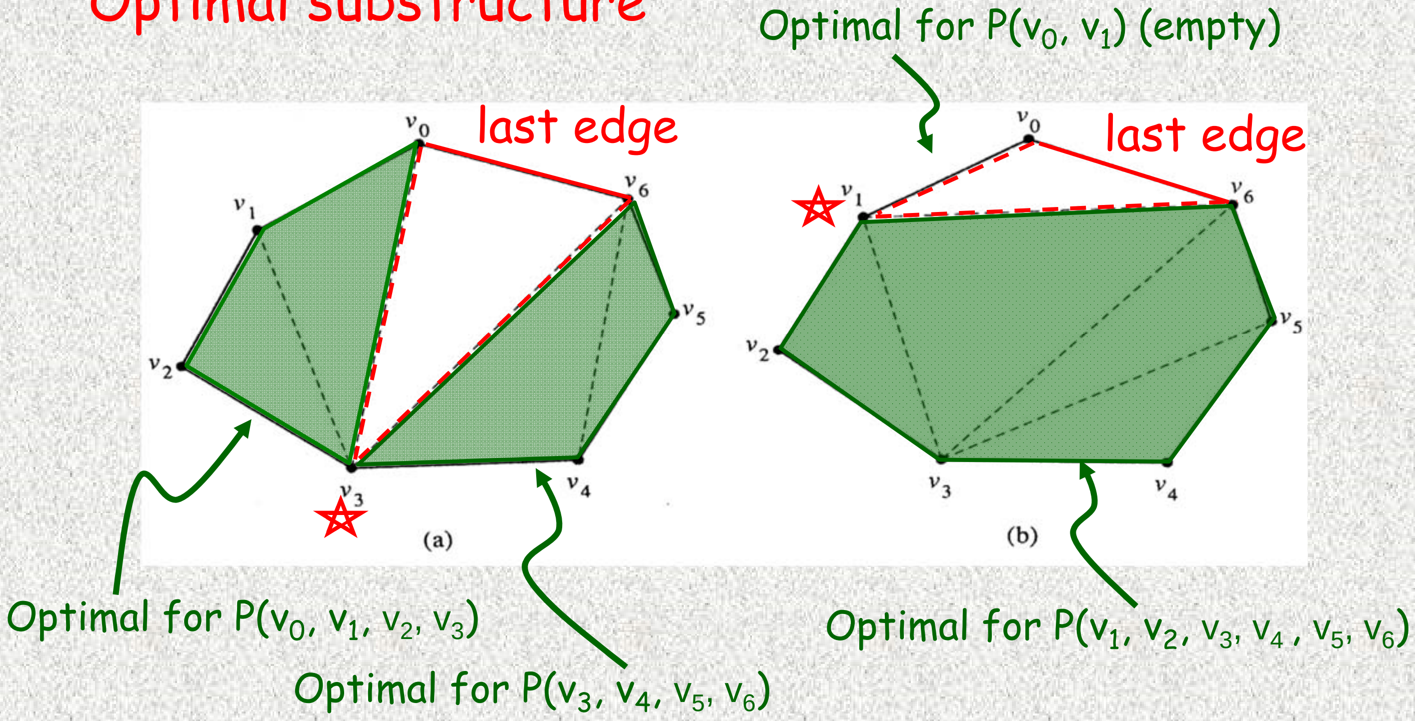
Time:

$$\sum_{i,j} O(1) = \underset{\text{table size}}{O(mn)} \times O(1) = O(mn)$$

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ \overset{1}{c[i-1, j-1]} + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{\underset{2}{c[i, j-1]}, \underset{3}{c[i-1, j]}\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

$(0 \leq i \leq m, 0 \leq j \leq n)$

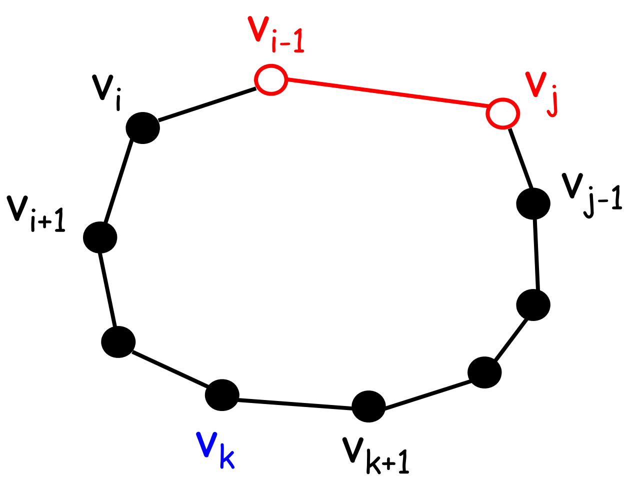
Optimal substructure



Optimal for P(v0, v1) (empty)
last edge
last edge
v0
v1
v2
v3
v4
v5
v6
(a)
(b)
Optimal for P(v0, v1, v2, v3)
Optimal for P(v3, v4, v5, v6)
Optimal for P(v1, v2, v3, v4 , v5, v6)

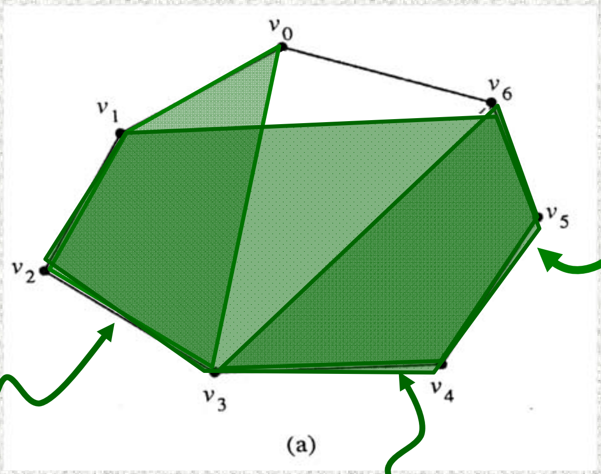
$t[i,j]$: optimal triangulation of $P(v_{i-1}, v_i, \ldots, v_j)$

☆ 包 含 v_{i-1}



$t[i,j]$: optimal triangulation of $P(v_{i-1}, v_i, \ldots, v_j)$

Optimal cost for the whole input: $t[1, 6]$ (or $t[1, n-1]$)



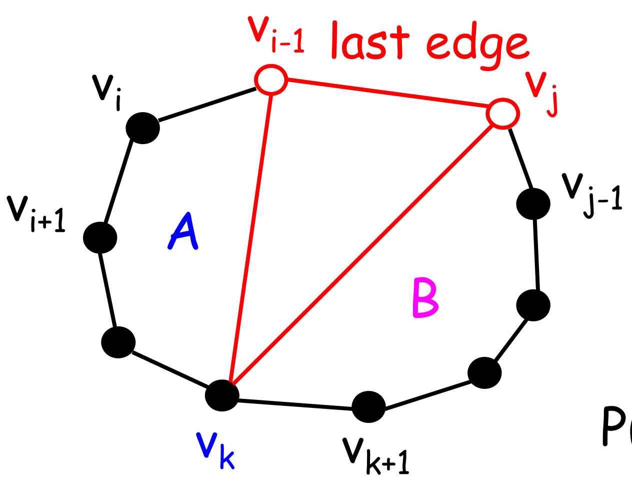
$t[2, 6]$: optimal cost for $P(v_1, v_2, v_3, v_4, v_5, v_6)$

$t[1, 3]$: optimal cost for $P(v_0, v_1, v_2, v_3)$

$t[4, 6]$: optimal cost for $P(v_3, v_4, v_5, v_6)$

t[i,j] : optimal triangulation of $P(v_{i-1}, v_i, \ldots, v_j)$

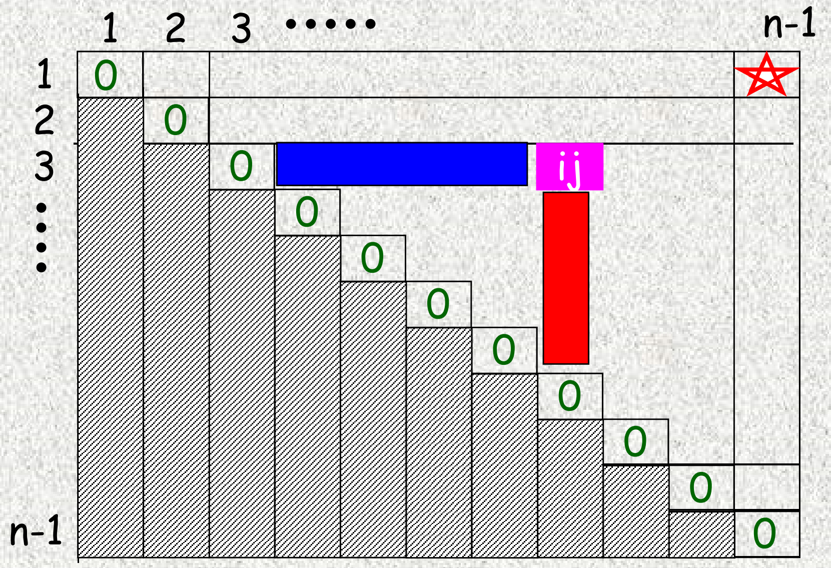
☆ 包 含 v_{i-1}



$$\overset{A}{P(v_{i-1},\ldots,v_k)} + \Delta v_{i-1}v_kv_j + \overset{B}{P(v_k,\ldots, v_j)}$$

$$t[i,j] = \min_{i \le k \le j-1} \{ t[i, k] + w(\Delta v_{i-1}v_kv_j) + t[k+1, j] \}$$

Time complexity



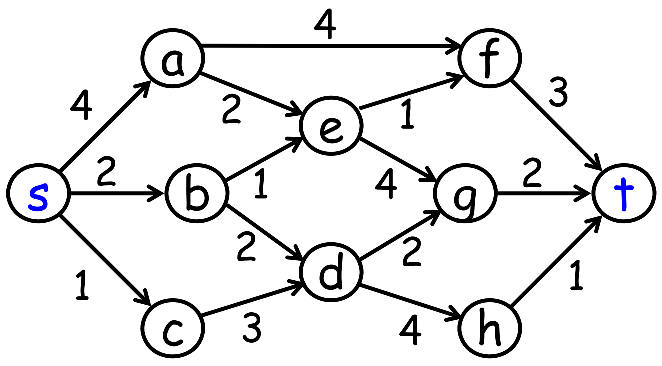
$$t_{ij} = \min_{i \le k \le j-1} \{t_{ik} + t_{k+1j} + w(v_{i-1}v_kv_j)\}$$

Similar to Matrix-Chain

$$m_{ij} = \min_{i \le k \le j-1} \{m_{ik} + m_{k+1j} + p_{i-1}p_kp_j\}$$

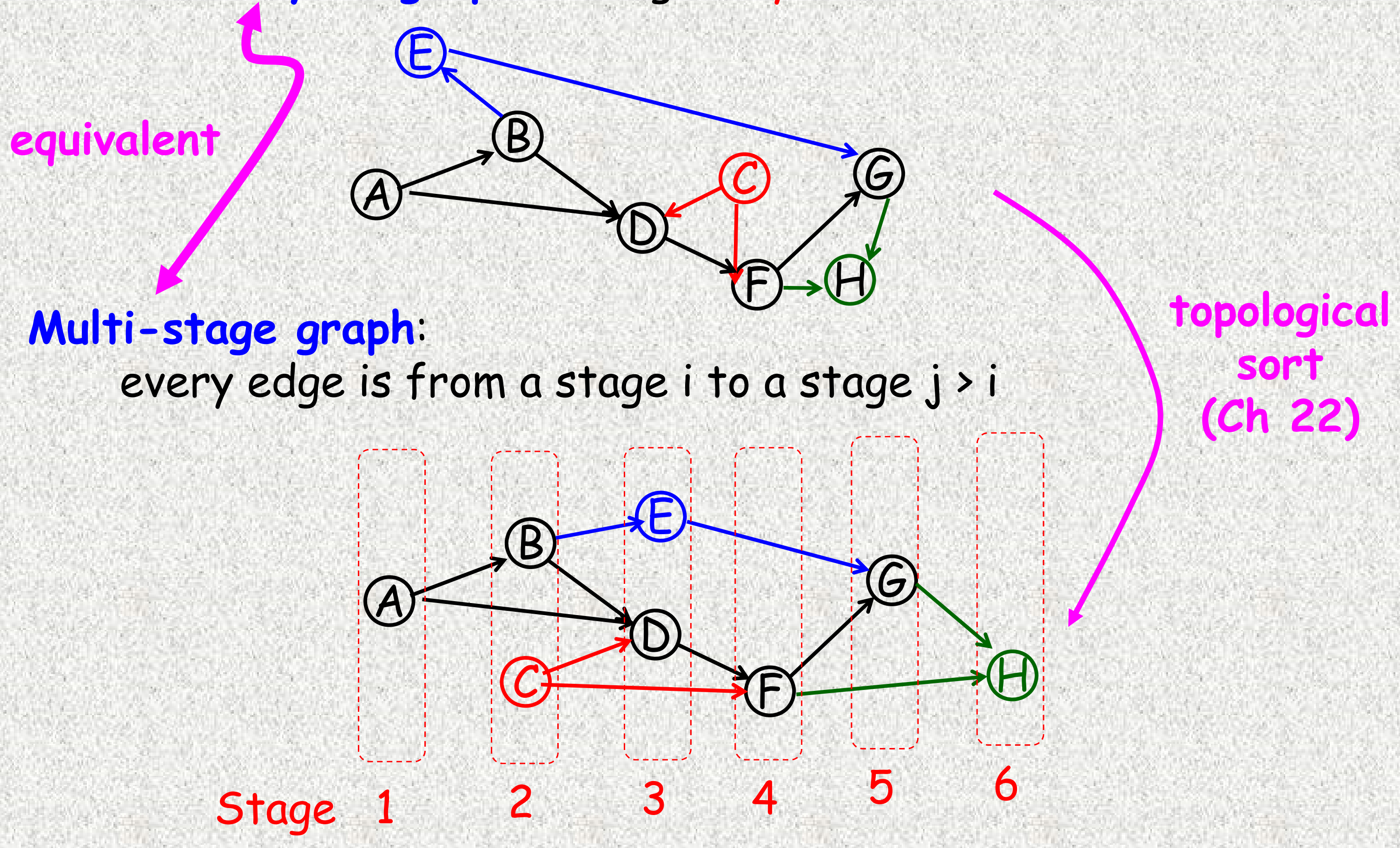
$$\text{Time: } \sum_{i,j} O(j-i) = \sum_{i,j} O(n) = \underbrace{O(n^2)}_{\text{table size}} \times O(n) = O(n^3)$$

Directed Acyclic graph (multi-stage graph)



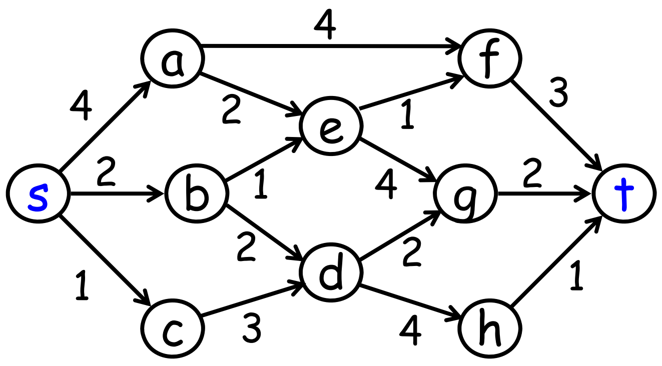
Given a DAG G = (V, E), find a shortest path from s to t

Directed Acyclic graph: having no cycles



Multi-stage graph:
every edge is from a stage i to a stage j > i

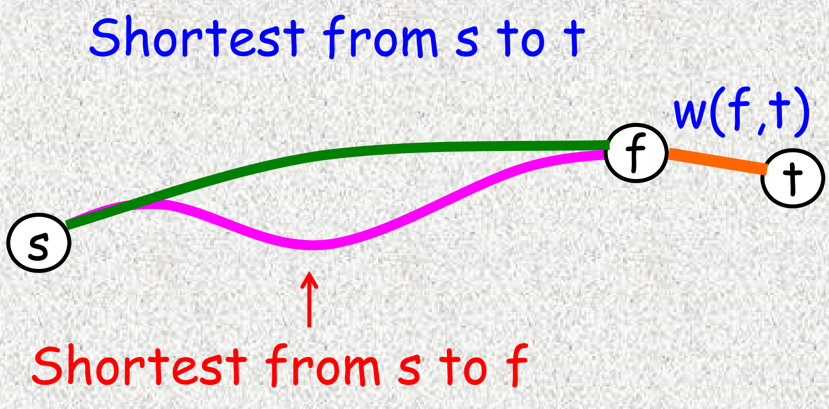
Directed Acyclic graph (multi-stage graph)



Given a DAG G = (V, E), find a shortest path from s to t

* d(v) : shortest distance from s to v

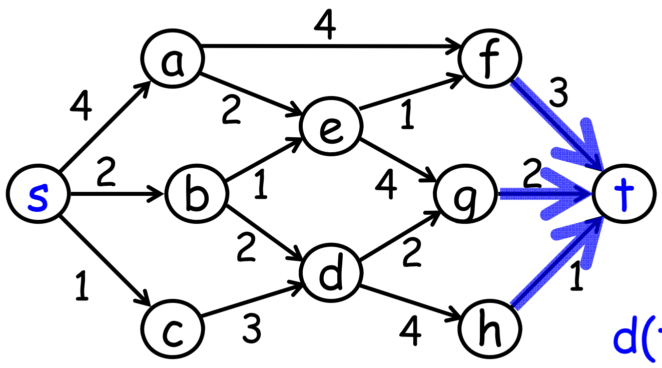
Optimal substructure



$\Rightarrow$ d(t) = d(f) + w(f,t)

Directed Acyclic graph (multi-stage graph)

15supp-a



Given a DAG G = (V,E), find a shortest path from s to t

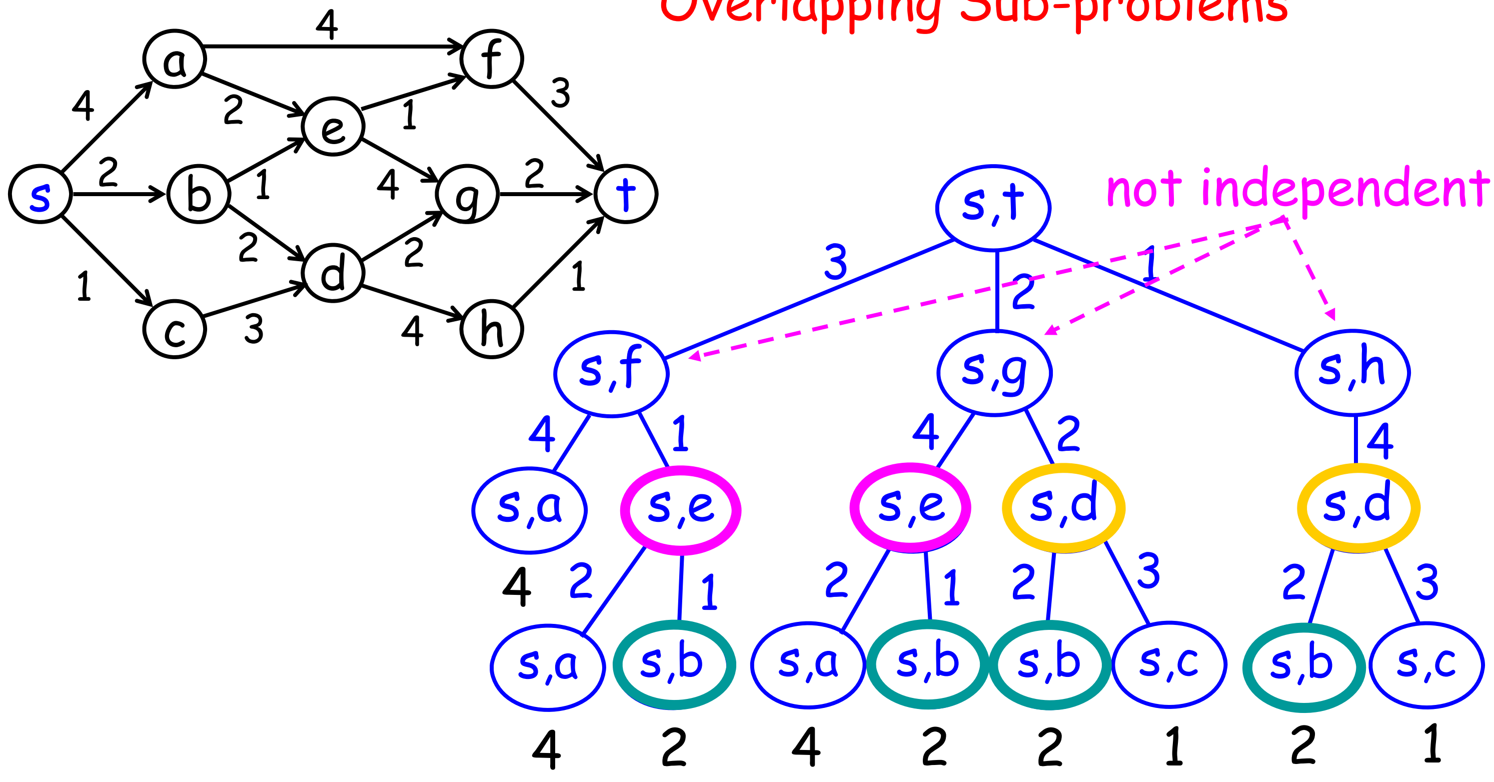
d(t) = MIN{ d(f)+3, d(g)+2, d(h)+1 }

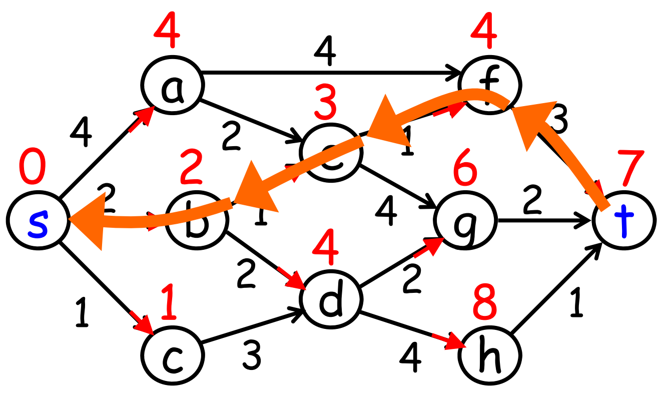
* d(v) : shortest distance from s to v

$$\begin{cases} d(s) = 0 \\ d(v) = \min_{\langle x,v\rangle \in E} \{ d(x) + w(x,v) \} \end{cases}$$

15supp-b

Overlapping Sub-problems





$$\begin{cases} d(s) = 0 \\ d(v) = \underset{\langle x,v\rangle \in E}{\text{MIN}} \{ d(x) + w(x,v) \} \end{cases}$$

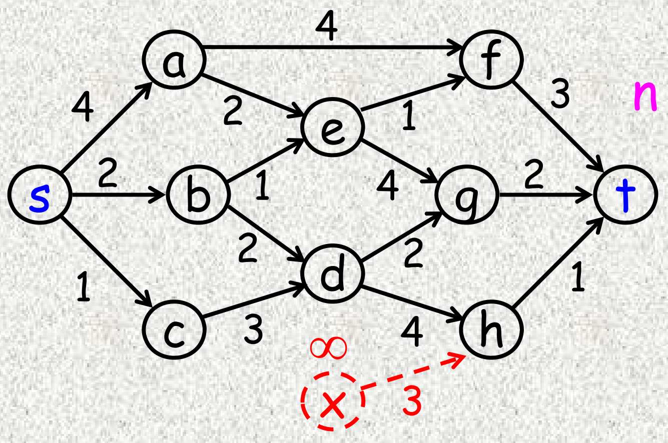
* shortest distance d(t): DP (bottom-up, left-to-right)
- a graph-shaped table
- find an order: topological sort or top-down DP

* shortest s-t path: backtracking (a simple recursive procedure)

* O(V + E)

* A longest path (critical path) ⇨ MIN → MAX

Time complexity



n = |V|, m = |E| (or n = V, m = E)

$$d(v) = \underset{\langle x,v\rangle \in E}{\text{MIN}} \{ d(x) + w(x,v) \}$$

table size

$$\text{Time: } \sum_v O(\text{in\_deg}(v)) = \sum_v O(n) = O(n) \times O(n) = O(V^2)$$

$$\text{Time: } \sum_v O(\text{in\_deg}(v)) = O(n + \text{total-in-degree})$$

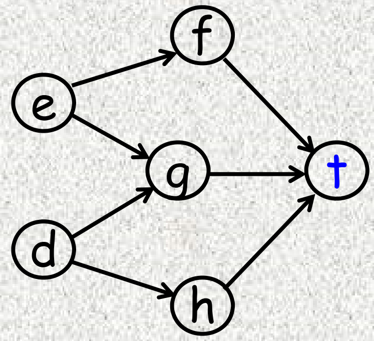
max{1, in_deg(v)}

why???

total-in-degree

in = 1; out = 1

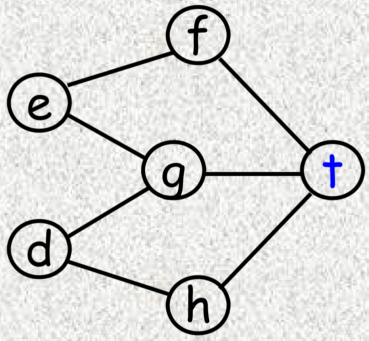
directed graph



total-in-degree = E

in = 2; out = 2

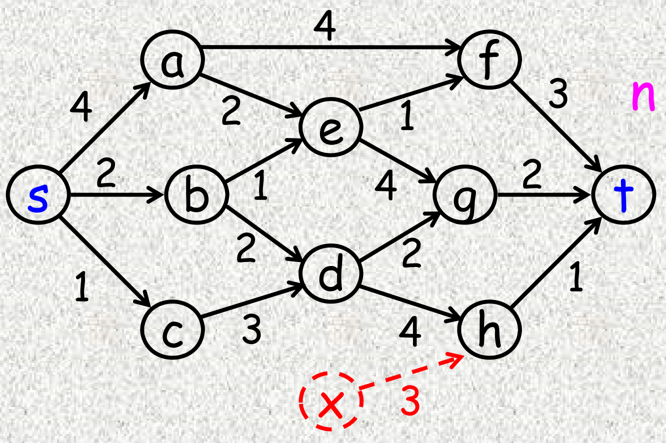
undirected graph



total-in-degree = 2E

Time complexity

directed: total-in-degree = E



n = |V|, m = |E| (or n = V, m = E)

$$d(v) = \underset{\langle x,v\rangle \in E}{\mathrm{MIN}} \{ d(x) + w(x,v) \}$$

table size

$$\text{Time: } \sum_{v} O(\text{in\_deg}(v)) = \sum_{v} O(n) = O(n) \times O(n) = O(V^2)$$

細算有好處

$$\text{Time: } \sum_{v} O(\text{in\_deg}(v)) = O(n + \text{total-in-degree})$$

max{1, in_deg(v)}

why???

$$= O(n + E) = O(V + E)$$

A simple exercise

S = {1, 3, 5, 10} : a set of stamps

F(n): minimum # of stamps having a total of n

$n = 1 \Rightarrow \{1\}$

$n = 2 \Rightarrow \{1, 1\}$

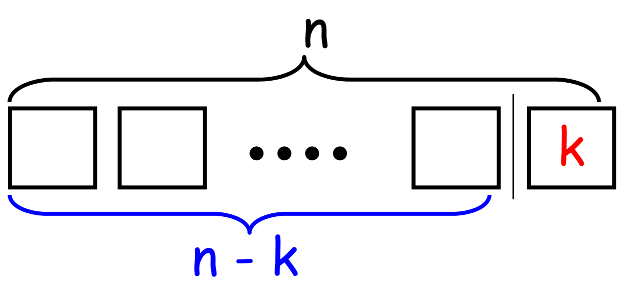
$n = 3 \Rightarrow \{3\}$

$n = 4 \Rightarrow \{1, 3\}$

$n = 9 \Rightarrow \{1, 3, 5\}$

optimal substructure

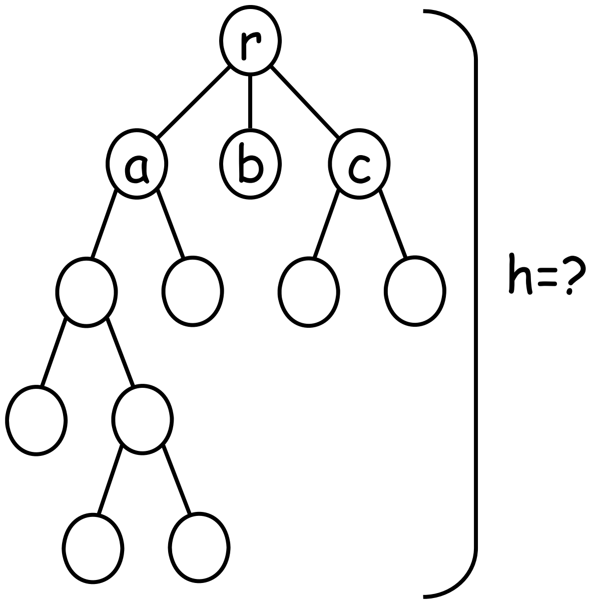
$\Rightarrow F(9) = F(4) + 1$



$$F(n) = \min_{k \in S,\, k \le n} \{ F(n-k) + 1 \}$$

* F(0) = 0

Dynamic Programming ?



* find a deepest leaf
  - an optimization problem

* h(r) = max{ h(a), h(b), h(c) } + 1

$$* \; h(v) = \max_{c \in CHILD(v)} \{ h(c) \} + 1$$

(h(v) = 0 if v is a leaf)

optimal substructure

* table is tree-shaped

* no overlapping sub-problems

not need to avoid recomputing by saving answers