**Addition of n-bit numbers**

$$
\begin{array}{r}
\overbrace{1\,0\,1\,1\,0\,1\,0\,1\,1\,0\,0\,1\,1}^{n} \\
+\,0\,0\,1\,0\,1\,0\,0\,0\,1\,0\,1\,1\,0 \\
\hline
\end{array}
$$

1

0

0

1

. . . .

$\Rightarrow$ O(n) time

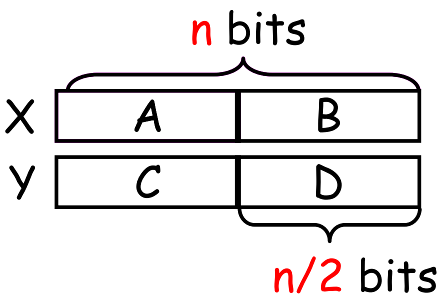
**Multiplication of n-bit numbers**

$$
\begin{array}{r}
\overbrace{1\;0\;1\;1}^{n} \\
\times\;1\;0\;0\;1 \\
\hline
1\;0\;1\;1 \\
0\;0\;0\;0\phantom{\;0} \\
0\;0\;0\;0\phantom{\;0\;0} \\
+\;1\;0\;1\;1\phantom{\;0\;0\;0} \\
\hline
\end{array}
\left.\vphantom{\begin{array}{c}1\\0\\0\\1\end{array}}\right\} n
$$

sum of n numbers

$\Rightarrow$ O($n^2$) time






(1) $X \cdot Y = (A \cdot 2^{n/2} + B) \cdot (C \cdot 2^{n/2} + D)$

$= AC \cdot 2^n + (AD + BC) \cdot 2^{n/2} + BD$

① ② ③ ④

$T(n) = 4\,T(n/2) + O(n) = O(n^2)$

(2) Let $P = AC$, $Q = BD$, $R = (A + B)(C + D)$

$X \cdot Y = P \cdot 2^n + (R - P - Q) \cdot 2^{n/2} + Q$

① ② + ③ ④

$T(n) = 3\,T(n/2) + O(n) = O(n^{\log_2 3})$

$$C \quad = \quad A \quad \times \quad B$$

$$\begin{bmatrix} C_{11} & C_{12} & & C_{1k} \\ C_{21} & C_{22} & & C_{2k} \\ & & \cdots\cdots & \\ C_{k1} & C_{k2} & & C_{kk} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & & A_{1k} \\ A_{21} & A_{22} & & A_{2k} \\ & & \cdots\cdots & \\ A_{k1} & A_{k2} & & A_{kk} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} & & B_{1k} \\ B_{21} & B_{22} & & B_{2k} \\ & & \cdots\cdots & \\ B_{k1} & B_{k2} & & B_{kk} \end{bmatrix}$$

$\frac{n}{k} * \frac{n}{k}$

$$C_{ij} = \sum_{1 \le s \le k} A_{is} B_{sj}$$

$\Rightarrow T(n) = k^3 T(n/k) + O(n^2)$

$\Rightarrow T(n) \overset{?}{=} q\, T(n/k) + O(n^2) \quad (q < k^3)$

$\Rightarrow T(n) = O(n^{\log_k q})$ by Master Thm. ▮

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$T(n) = 2T(\lfloor n/2 \rfloor) + n$ (with $T(1) = 1$)



Claim:

$T(n) = O(n \lg n)$ since

$T(n) \le 3 n \lg n$ for all $n \ge 2$

c $\quad n_0$

Basis: ($\cancel{n = n_0 = 2}$) ($n = 2, 3$)

$T(n_0) = T(2) = 2T(1) + 2 = 4$

$3 n_0 \lg n_0 = 3\ 2 \lg 2 = 6$

$T(2) = 4 \le 6$

$\Rightarrow$ OK!

$3\ 3 \lg 3 \sim 13.5$

$T(3) = 5 \le 13.5$

$\Rightarrow$ OK!

Claim:

$T(n) = O(n \lg n)$ ($\exists$ c and $n_0$ s.t.

$T(n) \le c n \lg n$ for all $n \ge n_0$)

Basis: ($n = n_0$)

$n_0 = 1$? $T(1) = 1 \le c\ 1 \lg 1$? (×)

$n_0 = 2$? $T(2) = 4 \le c\ 2 \lg 2$?

OK for $c \ge 4/(2 \lg 2) = 2$

$n_0 = 3$? $T(3) = 5 \le c\ 3 \lg 3$?

OK for $c \ge 5/(3 \lg 3)$

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$T(n_0) \le c n_0 \lg n_0$

$\Rightarrow$ OK for $\begin{cases} n_0 \ge 2 \\ c \ge T(n_0)/n_0 \lg n_0 \end{cases}$ ①

Induction: ~~$(n > n_0)$~~ (for $n > 3$)

Assume $T(x) \le 3\, x \lg x$

for $x = 2, 3, \ldots, n - 1$.

$x = \lfloor n/2 \rfloor \le n - 1$

$\lfloor n/2 \rfloor \ge 2$

$$
\begin{aligned}
& T(n) \\
= {} & 2T(\lfloor n/2 \rfloor) + n \\
\le {} & 2\,(3 \lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor) + n \\
\le {} & 3\, n \lg (n/2) + n \\
\le {} & 3\, n \lg n - 3\, n \lg 2 + n \\
\le {} & 3\, n \lg n - 3\, n + n \\
\le {} & 3\, n \lg n - 2\, n \\
\le {} & 3\, n \lg n \quad \text{(goal !)}
\end{aligned}
$$

OK !

$\Rightarrow$ Done !

Induction: $(n > n_0)$

Assume $T(x) \le c\, x \lg x$

for $x = n_0, n_0 + 1, \ldots, n - 1$.

⓪ $n_0 \le \lfloor n/2 \rfloor \le n - 1$

$$
\begin{aligned}
& T(n) \\
= {} & 2T(\lfloor n/2 \rfloor) + n \\
\le {} & 2\,(c \lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor) + n \\
\le {} & c\, n \lg (n/2) + n \\
\le {} & c\, n \lg n - c\, n \lg 2 + n \\
\le {} & c\, n \lg n - c\, n + n \\
\le {} & c\, n \lg n - (c-1)\, n \\
\le {} & c\, n \lg n \quad \text{(goal !)}
\end{aligned}
$$

$\Rightarrow$ OK for $c \ge 1$ ②

⓪ $\cap$ ① $\cap$ ② $\neq \emptyset$ $\Rightarrow$ Done !



$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$ (with $T(1) = 1$)

Prove $T(n) = 2n - 1$ (By induction)

Basis: $n = 1$

$T(1) = 1 = 2 * 1 - 1$ OK !

Induction:

Assume $T(x) = 2x - 1$ for $x < n$.

$$
\begin{aligned}
& T(n) \\
= {} & \underline{T(\lfloor n/2 \rfloor)} + \underline{T(\lceil n/2 \rceil)} + 1 \\
= {} & (\underline{2 \times \lfloor n/2 \rfloor - 1}) + (\underline{2 \times \lceil n/2 \rceil - 1}) + 1 \\
= {} & 2n - 1 \quad \text{(goal !)}
\end{aligned}
$$

OK !

$\Rightarrow$ Done !

Prove $T(n) \le 2n$

Basis: $n = 1$

$T(1) = 1 \le 2 * 1$ OK !

Induction:

Assume $T(x) \le 2x$ for $x < n$.

$$
\begin{aligned}
& T(n) \\
= {} & \underline{T(\lfloor n/2 \rfloor)} + \underline{T(\lceil n/2 \rceil)} + 1 \\
\le {} & \underline{2 \times \lfloor n/2 \rfloor} + \underline{2 \times \lceil n/2 \rceil} + 1 \\
\le {} & 2n + 1 \\
\le {} & 2n \quad \text{(goal !)}
\end{aligned}
$$

$\Rightarrow$ Fail ! Why???

Assume that we do not take ceiling for both.

* cost(internal-nodes) of a level is at most n.
(Prove that $n/3 + 2n/3 \le n$ for $\lfloor\rfloor\lfloor\rfloor$, $\lfloor\rfloor\lceil\rceil$, and $\lceil\rceil\lfloor\rfloor$.)

How to compute the number of leaves L?

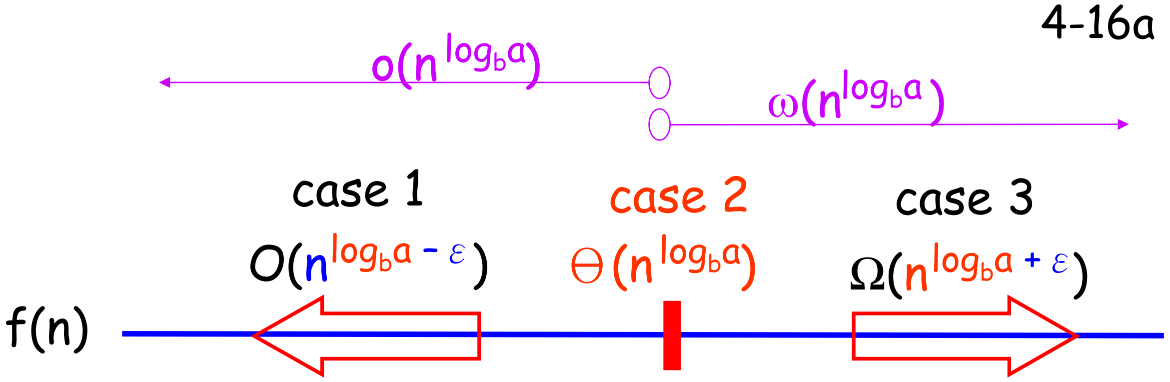
$2^{\lg(3/2)\, n} = n^{\lg(3/2)\, 2} = O(n^{1.xxx})$

* From $L \le 2^{\lg(3/2)\, n}$, we have $L = \omega(n\lg n)$, which is larger than $O(n\lg n)$. （太 悲 觀）

* Avoid the computation of L. (text book)
Prove $O(n\lg n)$ is correct by substitution method.

* Prove by induction that $L \le n$.
(Try it!)






Ex.
$T(n) = 9T(n/3) + f(n)$
$\log_b a = \log_3 9 = 2$

$f(n) = n^2$ -> case 2
$f(n) = n^{1.9}$ -> case 1
$f(n) = n^{2.2}$ -> case 3

$O(n^{2-\varepsilon}) = O(n^2/n^{\varepsilon})$

$\Theta(n^2)$

$\Omega(n^{2+\varepsilon}) = \Omega(n^2 n^{\varepsilon})$

$\underbrace{n^2/\lg n \qquad n^2 \lg n}_{\text{cannot apply M.T.}}$