NP-Completeness: 數學家眼中的 CS

數學家的特性:

1. 很聰明但太多話說不清楚
   - 太聰明:腦袋會轉彎,簡單事想的太複雜,也說的太複雜
   - 數學家:其實是聽的人程度太差

2. 目標遠大,心中想的是全世界
   - 喜歡一次解決一大堆(甚至全世界)問題,而不是一個問題

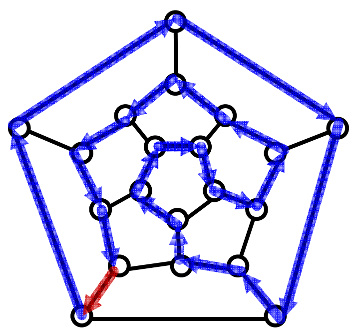
想像自己是數學家才有辦法理解!

數學家的目標:

寫一個程式,一次解決全世界的所有問題
(目標很好,但心很壞,想讓 CS 的都失業)

The Hamiltonian cycle problem

Input: G = (V, E)



a Hamiltonian cycle H

A nondeterministic algorithm

Step 1: Guess a cycle H

Step 2: Verify whether H is a Hamiltonian cycle

(i) all edges exist?
(ii) visit each vertex exactly once?

What dose "solve" mean?

⇨ an algorithm that runs in polynomial time (P)

What is the target?

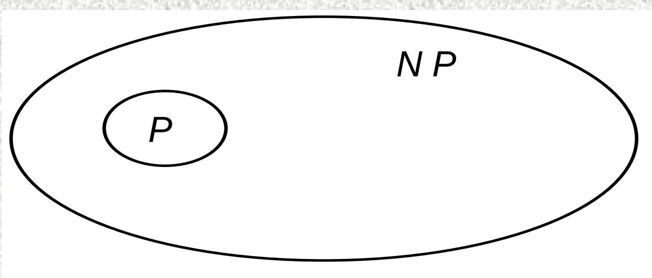
Set of all problems
- ~~un-computable~~
- computable
  - ~~problems whose answers can not be verified in $O(n^k)$ time~~
  - problems whose answers can be verified in $O(n^k)$ time ✓ (NP)

NP-Completeness (數學家眼中的 CS) – Review

目標： 寫一個程式，一次解決全世界的所有問題，為 CS 的人帶來光明(解救所有 programmers)

何謂解決(定義標準)： polynomial (easy , can be solved)
- P (problems "can be solved")

何謂全世界(決定對手)： NP (problems "can" be verified)



– A ∈ NP 表示 A 是一個對手
– A ∈ P 表示 A 被解決了
– 何謂解決全世界： prove NP = P

CS: I have an $O(n^4)$ algo for A
=> Math: I prove A ∈ P

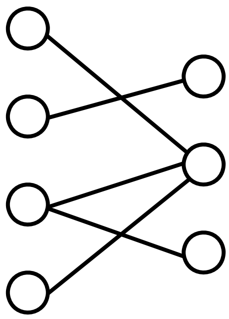
Reduction:

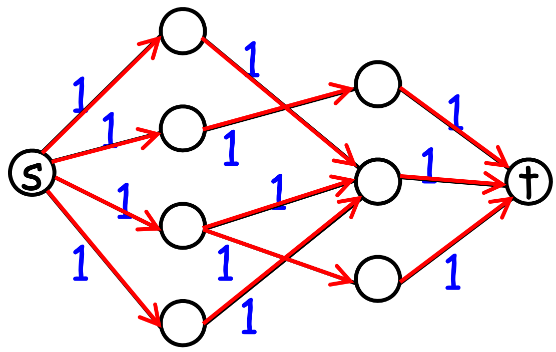
Problem A $\xrightarrow[\text{reduction}]{O(T)}$ Problem B

Example:

max bipartite matching $\xrightarrow[\text{reduction}]{O(V + E)}$ maximum flow

$G = (V, E)$





Partition Problem:

$S = (a_1, a_2, \ldots, a_n)$ $\Rightarrow$ S_1 and S_2 such that $Sum(S_1) = Sum(S_2)$

3-Partition Problem:

$S = (a_1, a_2, \ldots, a_n)$ $\Rightarrow$ S_1, S_2, S_3 such that $Sum(S_1) = Sum(S_2) = Sum(S_3)$

Partition problem $\xrightarrow[\text{reduction}]{O(n)}$ 3-partition problem

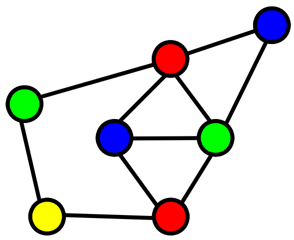
$S = (a_1, a_2, \ldots, a_n)$ $\qquad$ $S' = (a_1, a_2, \ldots, a_n, Sum(S)/2)$

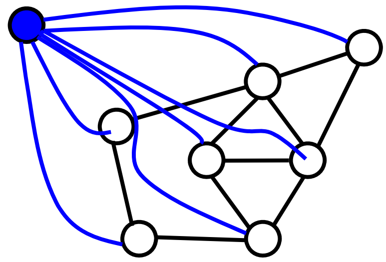
$S = (3, 5, 7, 9)$ $\qquad$ $S' = (3, 5, 7, 9, 24/2)$

$= (3, 5, 7, 9, 12)$

Coloring: Given G, assign color to each node such that adjacent nodes have different colors.

3-coloring $\xrightarrow[\text{reduction}]{O(n)}$ 4-coloring





3-coloring: Determine whether a given G can be colored by using {0, 1, 2}.

4-coloring: Determine whether a given G can be colored by using {0, 1, 2, 3}.

Reduction:

Problem A $\xrightarrow[\text{reduction}]{O(T)}$ Problem B

* usually: easy $\Rightarrow$ hard (or, as hard as)

A can be solved by solving B

B is more difficult

* may: hard $\Rightarrow$ easy

Example:

embed the difficulty

Find(A, x) $\xrightarrow[\text{sort } A]{O(n \lg n)}$ Binary Search(S, x)

O(n) O(lg n)

(i) If $A \Rightarrow B$ then $A: O(n^a + n^b)$

A: ? $O(n^a)$ $O(n^b)$

imply a solution of A

(ii) If $A \Rightarrow B$ then B: ?

$O(n^a)$ $O(n^b)$?

imply nothing for B

(iii) If $A \Rightarrow B$ then

$\Omega(n^a)$ $O(n^b)$?

may imply difficulty of B

case 1. $\Omega(n^{10})$ $O(n^5)$

case 2. $\Omega(n^{10})$ $O(n^{11})$

case 1: $b < a \Rightarrow$ B: $\Omega(n^a)$

/* e.g. B: $O(n^9)$ then A: $O(n^5) + O(n^9) = \cancel{O}(n^9)$

case 2: $b \geq a \Rightarrow$ B: ?

/* may: hard $\Rightarrow$ easy, or easy $\Rightarrow$ hard

Problem A $\Rightarrow$ Problem B

$O(T_1)$ $O(T_2)$

imply a solution of A

* A: $O(T_1 + T_2)$

* Assume that T_1 is polynomial

➡ $A \in P$ if T_2 is polynomial

➡ $A \in P$ if $B \in P$ 解掉 B 就解掉 A

* 數學家心中的 reduction 都是 polynomial

- 數學家：A can be reduced to B

⇨ A can be reduced to B in $O(n^k)$ time

⇨ $A \in P$ if $B \in P$ (解掉 B 就解掉 A)

⇨ B's hardness $\geq$ A's

Polynomial Reduction is transitive

$A \xrightarrow[p]{T_1} B$

$B \xrightarrow[p]{T_2} C$

➡ $A \xrightarrow[p]{T_1 + T_2} C$

in sense of hardness

$B \geq_{hard} A$

$C \geq_{hard} B$

➡ $C \geq_{hard} A$

(解掉 C 就解掉 A 和 B)

Problem A $\xrightarrow{O(T_1)}$ Problem B $O(T_2)$

$\boxed{\text{imply a solution of } A}$

* A: $O(T_1 + T_2)$
* Assume that T_1 is polynomial

數學家心中的 reduction 都是 polynomial

⇨ $A \in P$ if T_2 is polynomial

⇨ $A \in P$ if $B \in P$

解掉 B 就解掉 A

* If $A \xRightarrow{O(n^k)} B$, then $A \in P$ if $B \in P$ (i.e., A can be solved by solving B.)

解掉 B 就解掉全世界 (NP)

* If all $NP \xRightarrow{O(n^k)} B$, then all $NP \in P$ if $B \in P$ (i.e., all NP can be solved by only solving B.) (i.e., $NP = P$ if $B \in P$.)

假設真的存在

What is the goal?

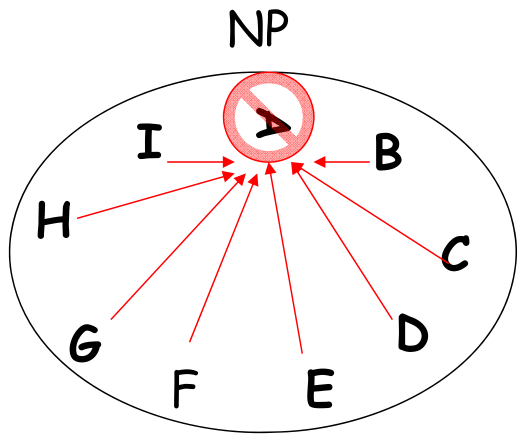
⇨ solve all problems in NP (in polynomial time) at a time
(i.e., prove NP = P)

$A \in NP$ 表示 A 是一個對手
$A \in P$ 表示 A 被解決了

How can "all problems" be solved at a time?

⇨ Idea: reduction

(1) find a problem A in NP such that all problems in NP can be reduced to A in polynomial time
(2) solve A in polynomial time

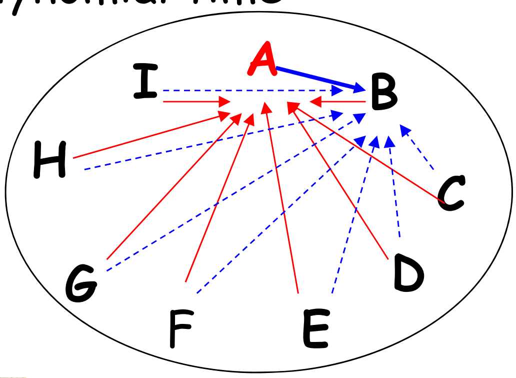


such a problem A is NP-C
(if exists)

A is NP-C:
(1) A is in NP
(2) all NP problems can be reduced to A in polynomial time

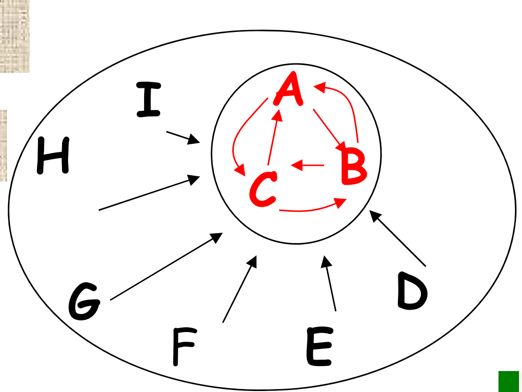
Assume that there exists an NP-C A.

If A can be reduced to B in polynomial time
(1) all NP problems can also be reduced to B
(2) B is NP-C
(3) B is as hard as A



All NP-C problems are of the same difficulty.
(They can be reduced to each other.)

all NP $\Rightarrow_p$ B $\cong$ an NP-C $\Rightarrow_p$ B $\cong$ all NP-C $\Rightarrow_p$ B

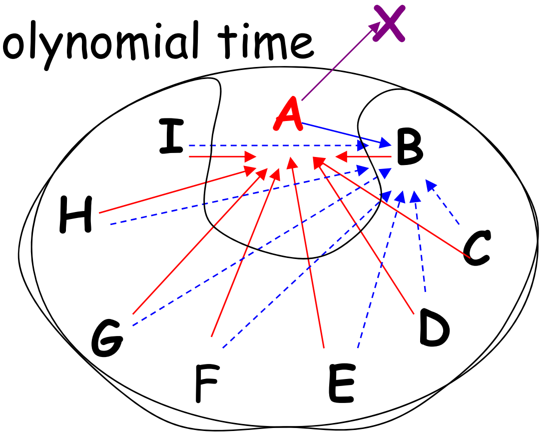


If an NP-C is solved

⇨ all NP (including all NP-C) are solved,

A is NP-C:
(1) A is in NP
(2) all NP problems can be reduced to A in polynomial time

A is NP-H: only (2)



Assume that there exists an NP-C A.

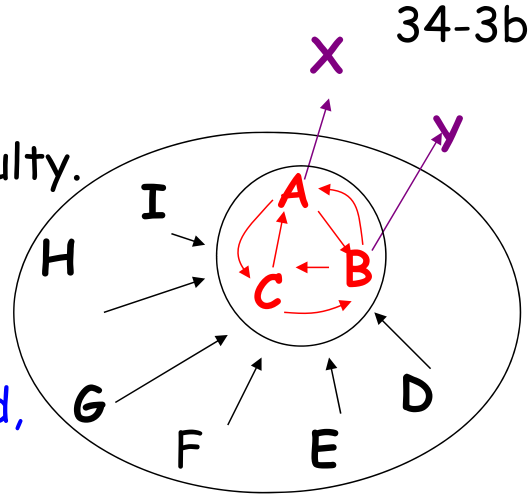
If A can be reduced to B in polynomial time
(1) all NP problems can also be reduced to B
(2) B is NP-C
(3) B is as hard as A

If A can be reduced to an X $\notin$ NP in polynomial time
(1) all NP problems can be reduced to X
(2) X is NP-H, but not NP-C
(3) X is harder than A

NP-H: A, B, C, X, Y
NP-C: A, B, C

All NP-C problems are of the same difficulty.
But, all NP-H problems are not.

If an NP-H or NP-C is solved
⇨ all NP (including all NP-C) are solved, but not all NP-H



Goal: Solve all problems in NP at a time.

How: Solve a problem in NP-C?

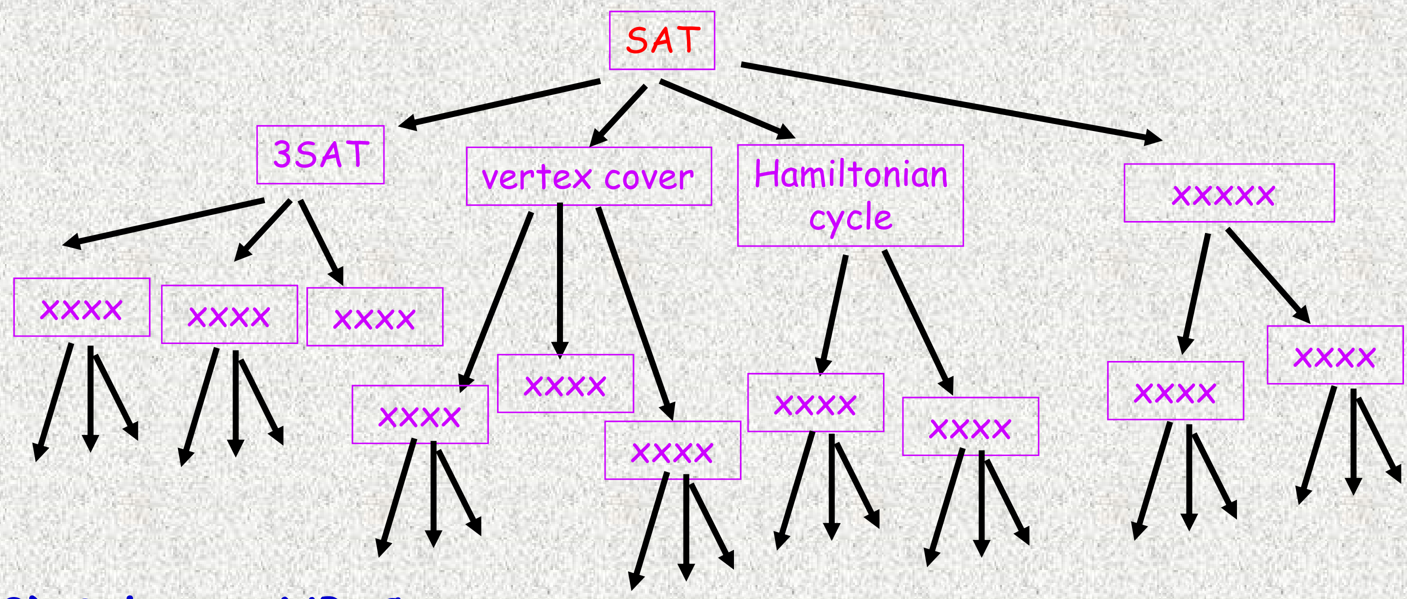
QUESTION: Dose NP-C exist?

Note that it is impossible to solve a problem in NP-H, but not in NP-C. Why?

Two Efforts

(1) Find problems in NP-C

- 1st: SAT /* all NP-C $\Rightarrow_p$ SAT (turing award)



(2) Solve an NP-C

- ????!!!!

How to prove a problem A is in NP-C (NP-H)?

① Show that A is in NP.
give an $O(n^k)$-time nondeterministic algo for A

② Show that all in NP $\xrightarrow{O(n^k)}$ A.
(a) Find a problem $Y \in$ NP-C
(b) Show $Y \xrightarrow{O(n^k)} A$

all NP $\xrightarrow{O(n^k)} Y \xrightarrow{O(n^k)} A$

$O(n^k)$

Example:
Prove 3-partition $\in$ NP-C.

① Show that A is in NP.
(i) guess S_1, S_2, S_3
(ii) check $S_1 \cup S_2 \cup S_3 = S$ and $\mathrm{Sum}(S_1) = \mathrm{Sum}(S_2) = \mathrm{Sum}(S_3)$

$O(n \lg n)$ time

② Show that all in NP $\xrightarrow{O(n^k)}$ A.
(a) It is known 2-partition $\in$ NP-C
(b) 2-partition $\xrightarrow{O(n)}$ 3-partition

(omit ① for NP-H)

NP-Completeness (數學家眼中的 CS) – Review

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為 CS 的人帶來光明 (解救所有 programmers)

何謂解決 (定義標準)： polynomial (easy , can be solved)
- P (problems "can be solved")

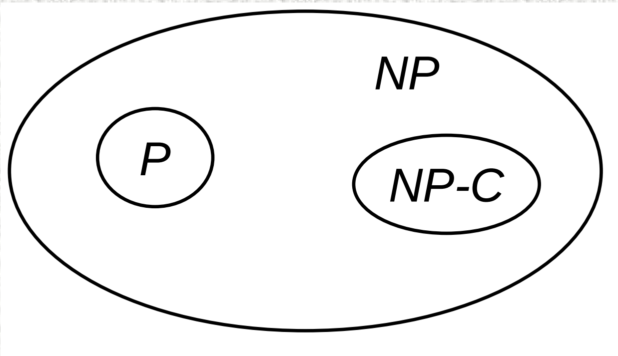
何謂全世界 (決定對手)： NP (problems "can" be verified)
- 解決全世界： prove NP = P

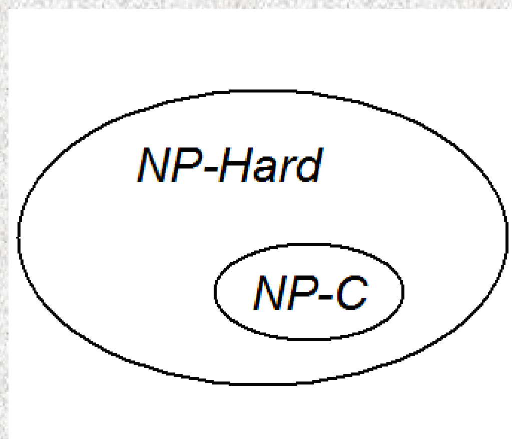
如何下手：擒賊先擒王 (直接解決敵軍中的大將軍)
- reduction：找出對手中的大魔王 NP-C
- 解決大魔王

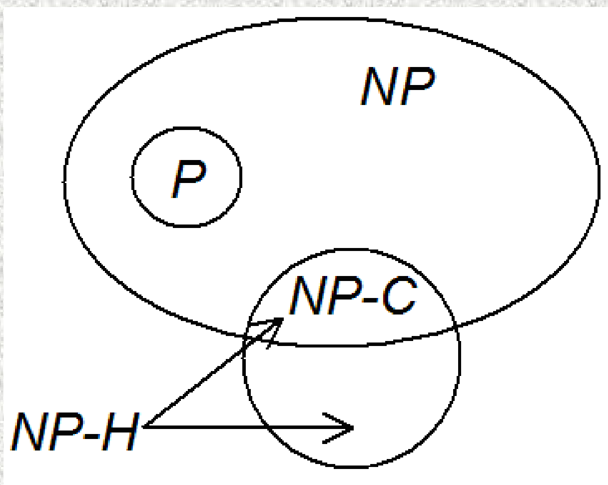
問題： 大魔王真的存在嗎？ (yes, 1st - SAT, Turing award)

剩下的問題：大魔王可以被打敗嗎 (can an N-PC be solved) ?
- NP = P or NP ≠ P? (unknown, but believe NP ≠ P)

結果：只帶來悲慘的消息 (沒帶來光明，反而帶來黑暗)
- 這世界到處都是不知如何對付的大魔王







(a) NP-C: No one knows how to solve these problems. (Y/N) 34-4b
(No algorithms exist for these problems.)

(b) If we consider NP as an army, then
NP-C: ? NP-H but not NP-C: ?

If an NP-C surrenders, then
all NP too? all NP-C too? all NP-H too?

If an NP-H surrenders, then
all NP too? all NP-C too? all NP-H too?

(1) How to prove a problem is P?

(2) How to prove a problem is NP?

(3) How to prove a problem is NP-C?

(4) How to prove a problem is NP-H?

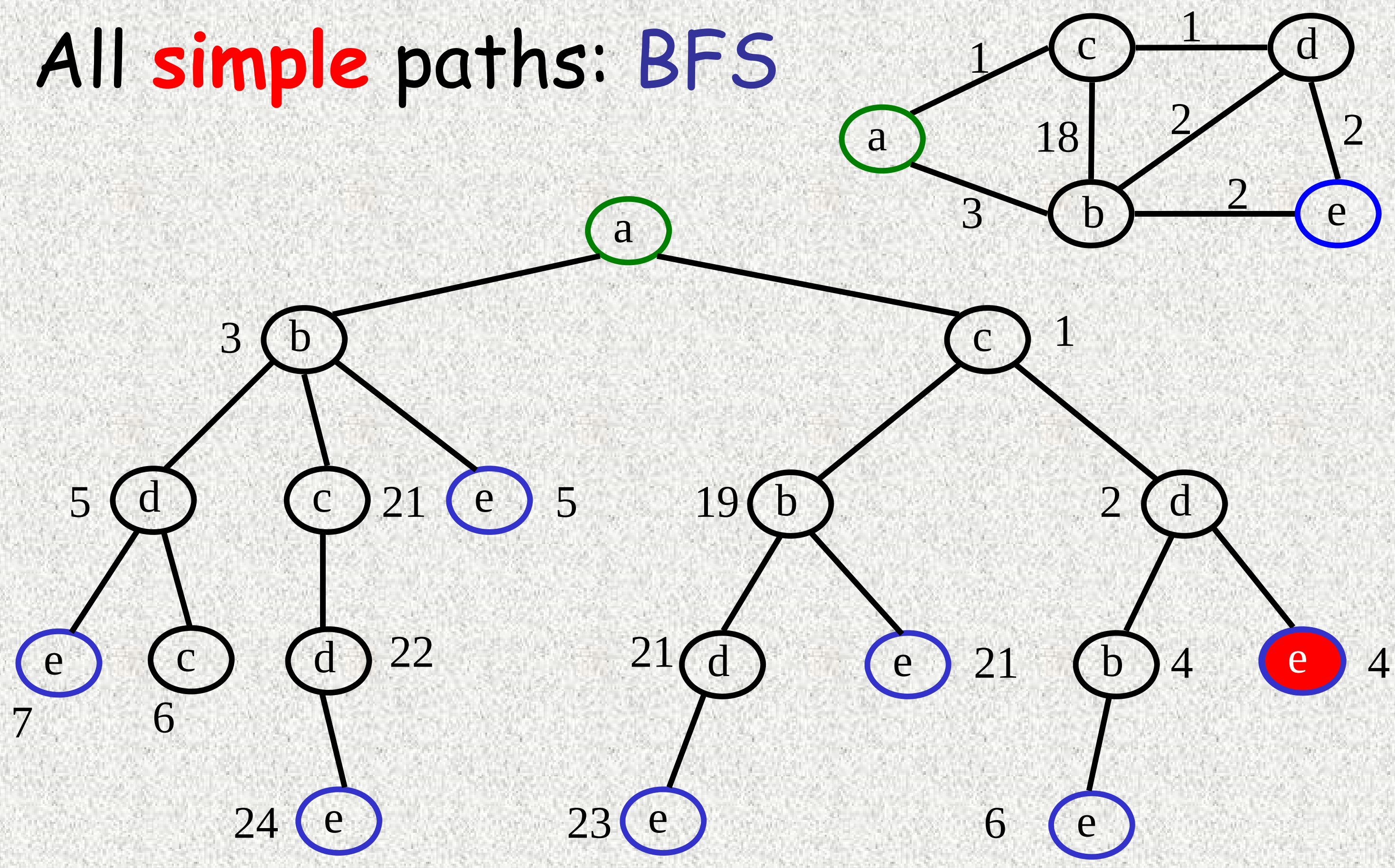
(5) How to prove NP = P?

(6) How to prove NP $\neq$ P?

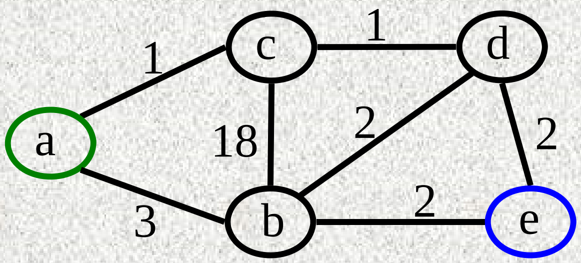
(7) NP = P or NP $\neq$ P ?

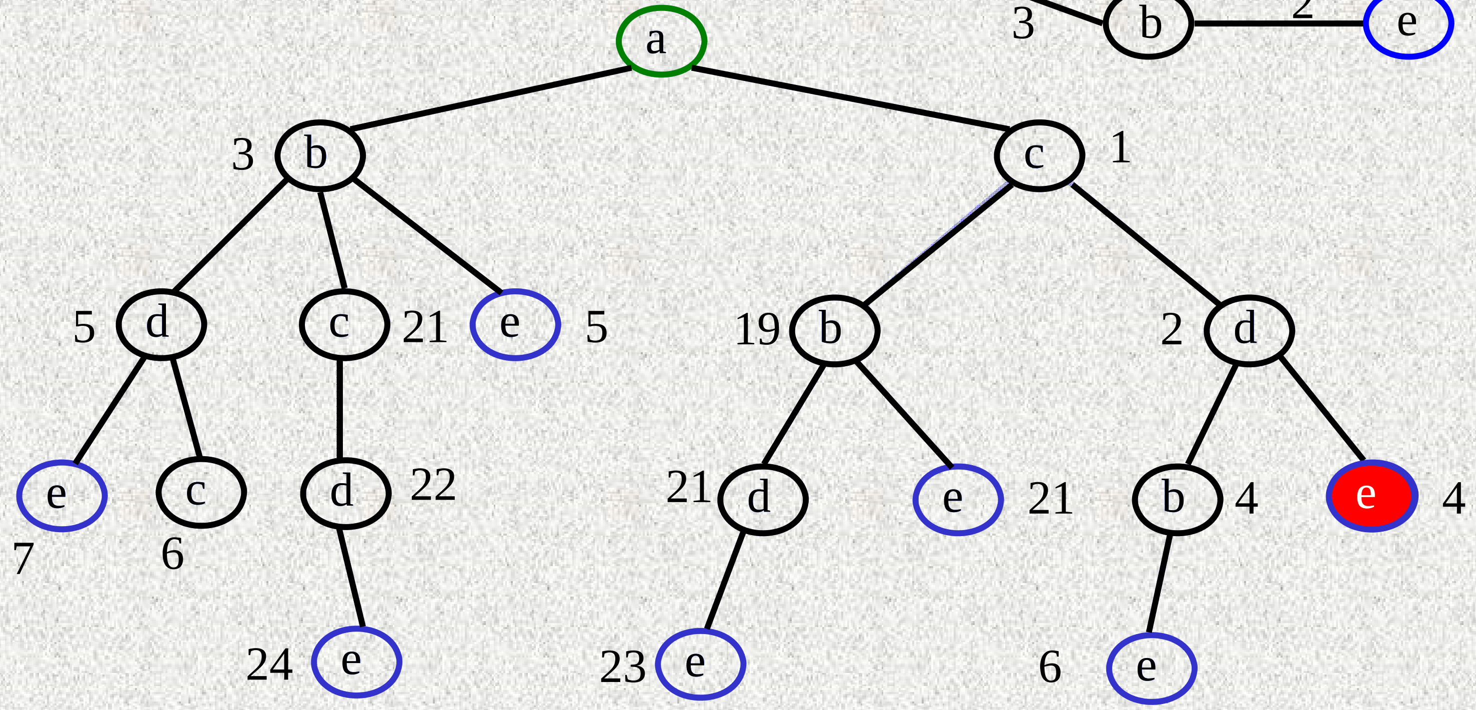
(8) What do we learn ?

All simple paths: BFS



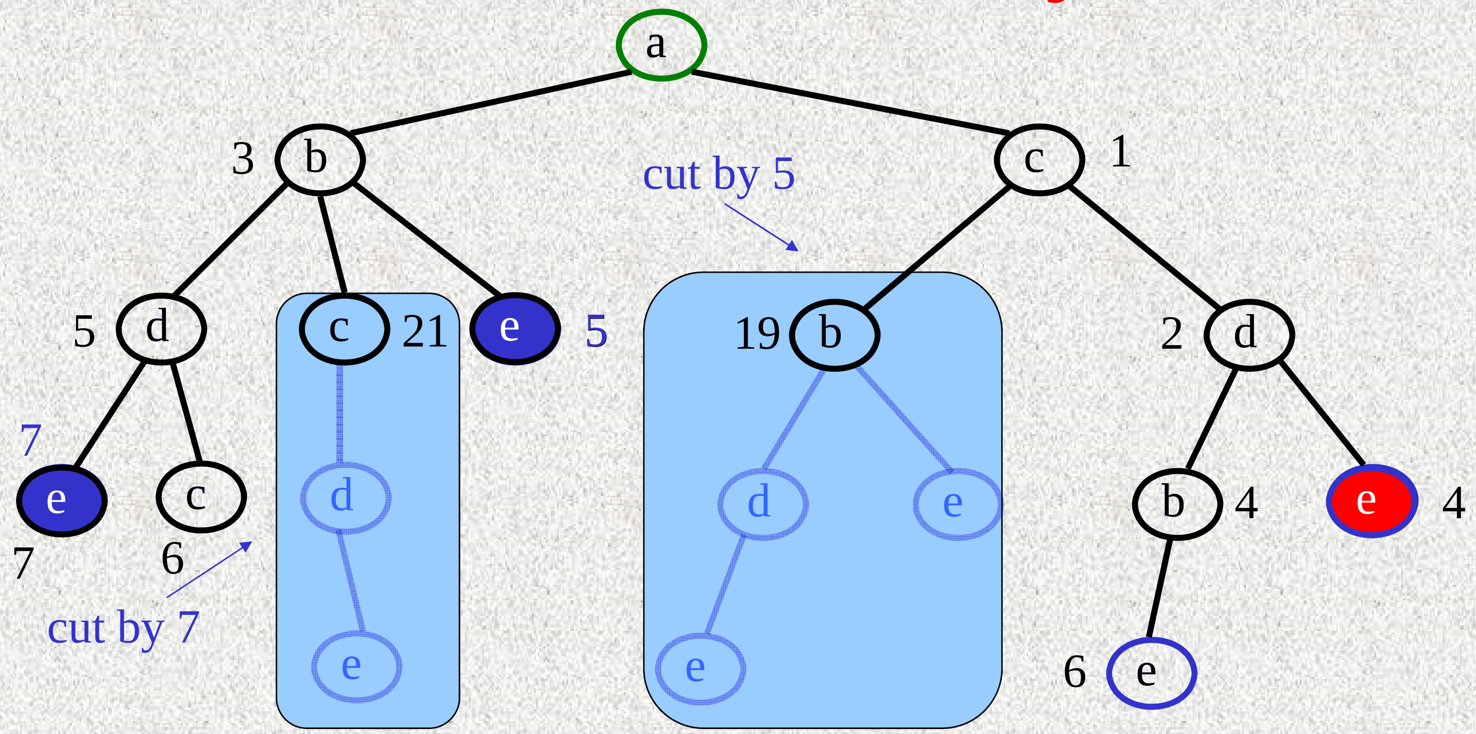
All simple paths: DFS





Branch-and-bound

- Branch-and-bound: Brute-force + intelligent cuts



Optimization Problems

P
- * trivial
- * greedy
- * DP

NP-hard
- * brute-force ($n \leq 20$) ⇒ exact solution
- * B & B ($n \leq 30$~100)
- * $n > 100$???

- * approximation ⇒ near-optimal solution
- * heuristic