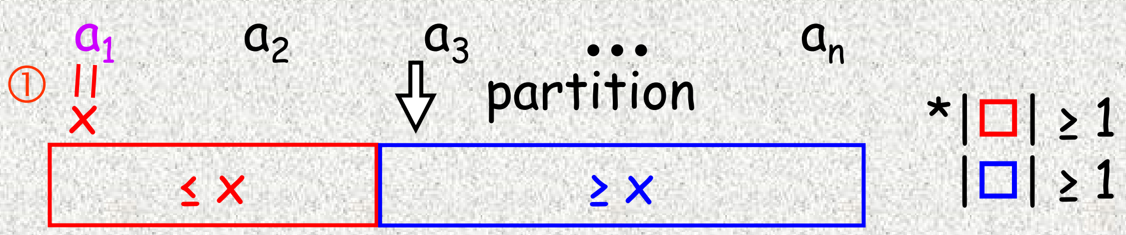
Proof of the worst-case of quicksort in 7-4

Q: Is "basis" unnecessary?

Q: Which one, "basis" or "induction", first?

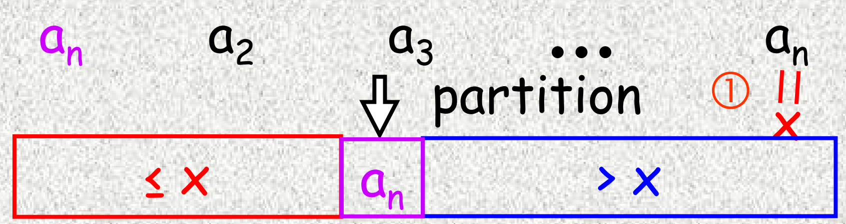
Q: How to handle b in "basis"?

1st Edition



$T(n) = \max_{k \in [1,n-1]} \{T(k) + T((n-1)-k)\} + bn$ (with $T(1) = 1$)

3rd Edition



$T(n) = \max_{k \in [0,n-1]} \{T(k) + T((n-1)-k)\} + bn$

(with $T(0)=T(1) = 1$)

$T(n) = \max_{k\in[0,n-1]} \{T(k) + T((n-1)-k)\} + bn$, where b is a constant

(with $T(0)=T(1) = 1$)

Claim: $\exists$ c and n_0 s.t. $T(n) \le cn^2$ for all $n \ge n_0$

Basis: ($n = n_0$)

$n_0 = 0$? $T(0) = 1 \le c0^2$? (×)

$n_0 = 1$? $T(1) = 1 \le c1^2$ OK for $c \ge 1$

$n_0 = 2$? $T(2) = 1+2b \le c2^2$? OK for $c \ge (1+2b)/4$

$n_0 = 3$? $T(3) = \max\{T0+T2, 2T1, T2+T0\} + 3b$

$= 2+5b \le c3^2$? OK for $c \ge (2+5b)/9$

$T(n_0) \le cn^2$

$\Rightarrow$ OK for $\begin{cases} n_0 \ge 1 \\ c \ge T(n_0)/n_0^2 \end{cases}$ ①

$T(n) = \max_{k\in[0,n-1]} \{T(k) + T((n-1)-k)\} + bn$, where b is a constant

(with $T(0)=T(1) = 1$)

Claim: $\exists$ c and n_0 s.t. $T(n) \le cn^2$ for all $n \ge n_0$

Induction: ($n > n_0$) Assume $T(x) \le c\,x^2$ for $x = n_0, n_0+1, \ldots, n-1$.

$T(n) = \max_{k\in[0,n-1]} \{T(k) + T((n-1)-k)\} + bn$

*this recur. always needs $T(x)$ for all $x = 0, 1, 2, \ldots, n_0-1$

*try $n_0=1$ directly -- we need $c \ge 1$ for ①

$\le \max\{T(0)+T(n-1), \max_{k\in[1,n-2]}\{T(k) + T((n-1)-k)\}\} + bn$

$\le \max\{1+c(n-1)^2, \max_{k\in[1,n-2]}\{c\,k^2 + c\,((n-1)-k)^2\} + bn$

$\le c\,(n-1)^2 + 1 + bn$

(since $c(n-1)^2 = \max_{k\in[0,n-1]}\{c\,k^2 + c\,((n-1)-k)^2\}$)

$\le c\,n^2 - 2cn + c + 1 + bn$

$\le c\,n^2$ (goal !) $\Rightarrow$ OK for $c \ge \max\{1, b\}$ ②

① $\cap$ ② $\ne \emptyset$ (take $n_0 = 1$, $c \ge \max\{1, b\}$) $\Rightarrow$ Done !