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Dynamic Programming

Dynamic programming: a tabular (programming) method applied to **optimization problems**.

Divide a problem into several subproblems that are not independent (sharing subproblems). Avoid recomputing the same subproblem by solving every subproblem just once and saving the answer in a table.

Step 1. Characterize the structure of an optimal solution.
Step 2. Recursively define the value of an optimal solution.
Step 3. Compute the value of an optimal solution in a ***bottom-up*** fashion.
Step 4. Construct an optimal solution from the computed information. (sometimes omitted)

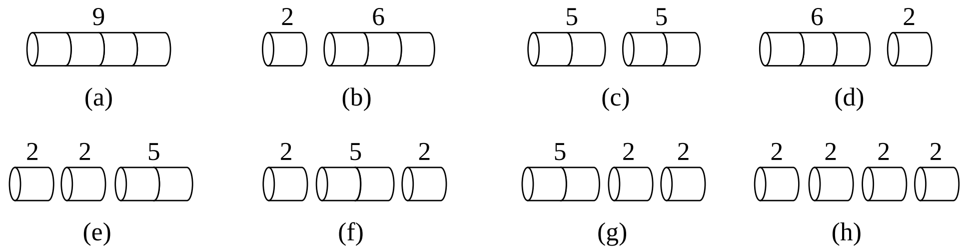
15.1 The rod-cutting problem (1-d DP)

Input:
n, the length of a (steel) rod
$p[i]$, the price of a rod of length i

Output: the maximum revenue r^*

length i	1	2	3	4	5	6	7	8	9	10	11
price $p[i]$	2	5	6	9	11	16	17	20	22	24	25

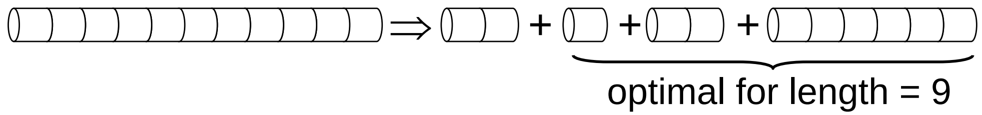
A price table



The 8 possible ways for selling a rod of length 4
((c) is optimal, where r^* = 10)

Step 1. An optimal solution to an instance contains optimal solutions to sub-instances.

Example:
If (2, 1, 2, 6) is optimal for length = 11,
then (1, 2, 6) is optimal for length = 9



Step 2.

Let $r[j]$ be the maximum revenue for length = j. Then

$$r[j]=\begin{cases}0 & \text{if } j=0\\ \max_{1\le i\le j}\{p[i]+r[j-i]\} & \text{if } j>0\,.\end{cases}$$

The maximum revenue r^* is $r[n]$.

Step 3. Compute r and s (for **Step 4**)

BOTTOM-UP-CUT-ROD(*p*, *n*)

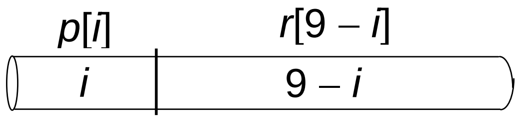
```
let r[0..n] and s[0..n] be new arrays
r[0] ← 0
for j ← 1 to n do    // compute r[j]
    r[j] ← –∞
    for i ← 1 to j do
        if r[j] < p[i] + r[j – i] then
            r[j] ← p[i] + r[j – i]
            s[j] ← i
return r and s
```

- $T(n)=O(n^2)$

Example: (n = 11, j = 9)

length i	1	2	3	4	5	6	7	8	9	10	11
price $p[i]$	2	5	6	9	11	16	17	20	22	24	25

i	0	1	2	3	4	5	6	7	8	9	10	11
$r[i]$	0	2	5	7	10	12	16	18	21	23	26	28
$s[i]$	0	1	2	1	2	1	6	1	2	1	2	1



first cut at i ($1 \le i \le 9$)

$$r[9] = \max\left\{\begin{matrix} p[1]+r[8], & p[2]+r[7], & p[3]+r[6] \\ p[4]+r[5], & p[5]+r[4], & p[6]+r[3] \\ p[7]+r[2], & p[8]+r[1], & p[9]+r[0] \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} 2+21, & 5+18, & 6+16 \\ 9+12, & 11+10, & 16+7 \\ 17+5, & 20+2, & 22+0 \end{matrix}\right\}$$

= 23 (the first cut $s[9]$ = 1, 2, or 6)

Step 4. Using table s, by backtracking we obtain an optimal cutting in $O(n)$ time.

Example: (1, 2, 2, 6) is optimal for n = 11, since $s[11]$ = 1, $s[10]$= 2, $s[8]$ = 2, and $s[6]$ = 6.

16.2 Matrix-chain multiplication (2d DP)

Input: $(p_0, p_1, ..., p_n)$, the dimensions of n matrices $A_1A_2 ...A_n$. (A_i is of size $p_{i-1}\times p_i$)

Output: parenthesize $A_1A_2 ...A_n$ to minimize the number of scalar multiplications.

Example: $(p_0, p_1, p_2, p_3) = (10, 100, 5, 50)$

$((A_1A_2)A_3) \Rightarrow 10\times100\times5 + 10\times5\times50 = 7500$ (√)

$(A_1(A_2A_3)) \Rightarrow 100\times5\times50 + 10\times100\times50 = 75000$ (×)

Step 1. An optimal solution to an instance contains optimal solutions to sub-instances.

Example: if $((A_1(A_2A_3))((A_4(A_5A_6))A_7))$ is an optimal solution to $A_1A_2...A_7$, then

$(A_1(A_2A_3))$ is optimal to $A_1A_2A_3$, and
$((A_4(A_5A_6))A_7)$ is optimal to $A_4A_5A_6A_7$.

Step 2.
Let $m[i, j]$ be the minimum number of scalar multiplications for computing $A_i...A_j$. We have

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i\le k<j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j \end{cases}$$

Step 3. $m[1..n, 1..n]$ $s[1..n, 1..n]$ (for **Step 4**)

Matrix-Chain-Order(*p*)

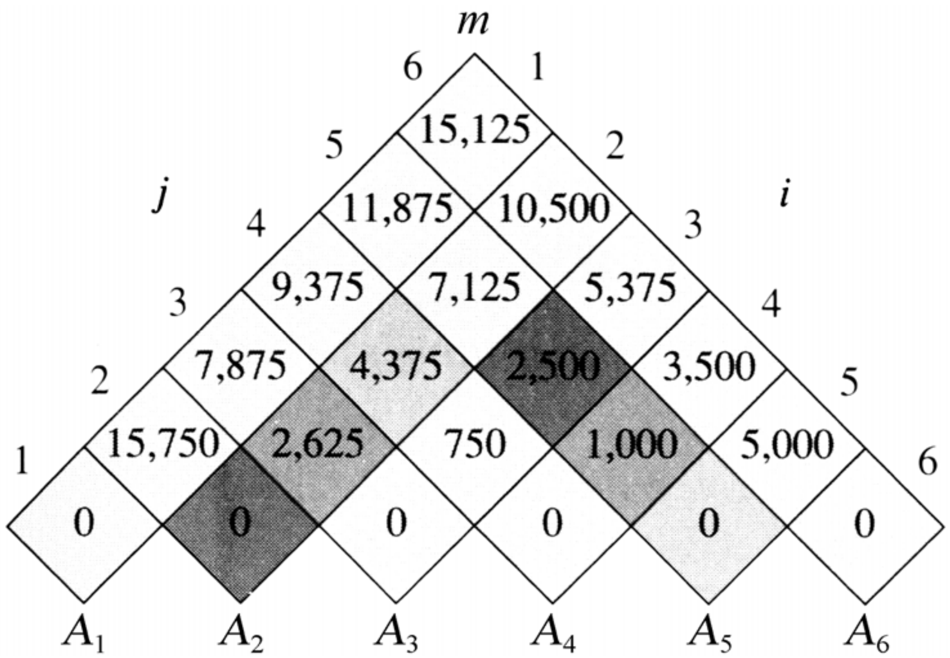
```
for i ← 1 to n do m[i, i] = 0
for l ← 2 to n do
    for i ← 1 to n – l + 1 do
        j ← i + l – 1
        m[i, j] = ∞
        for k ← i to j – 1 do
            q ← m[i, k] + m[k+1, j] + p_{i-1} p_k p_j
            if q < m[i, j] then m[i, j] ← q
                                s[i, j] ← k
return m and s
```

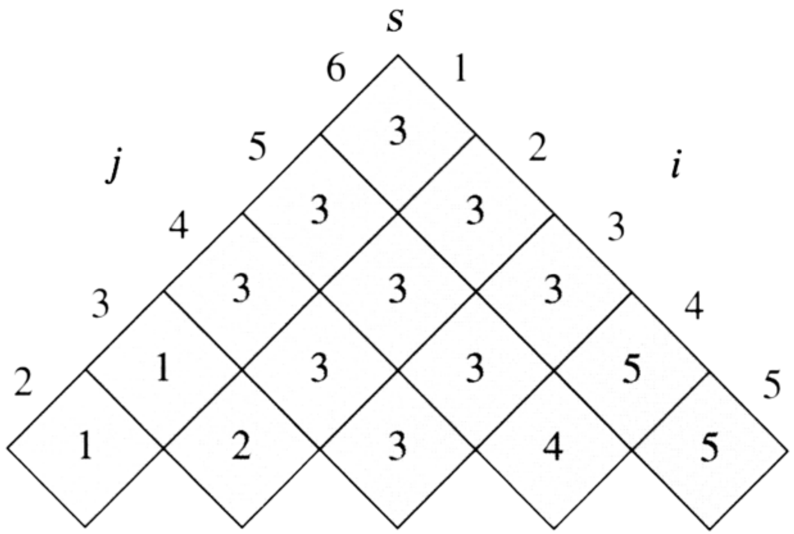
- $T(n)=O(n^3)$

Example: $(p_0, p_1, ..., p_6)=(30,35,15,5,10,20,25)$

$$\begin{cases} m[2,2]+m[3,5]+p_1p_2p_5 = 0+2500+35\times15\times20 = 13000 \\ m[2,3]+m[4,5]+p_1p_3p_5 = 2625+1000+35\times5\times20 = 7125 \\ m[2,4]+m[5,5]+p_1p_4p_5 = 4375+0+35\times10\times20 = 11375 \end{cases}$$

Thus, we have $m[2,5] = 7125$ and $s[2,5] = 3$





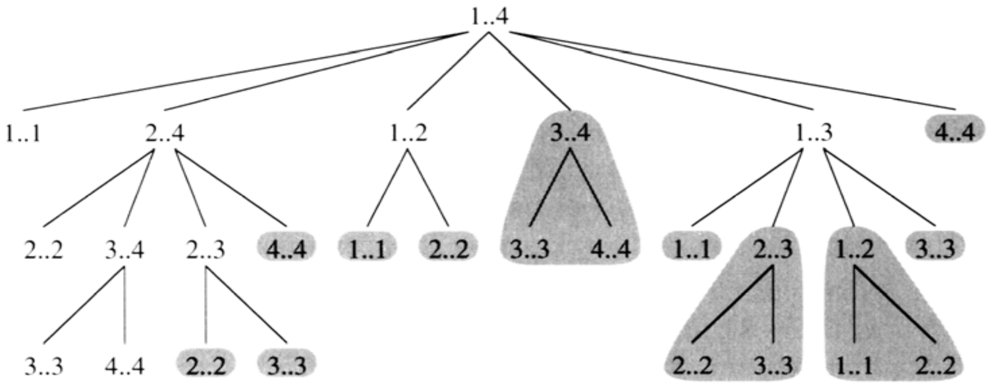
Step 4. Using table s, by backtracking we obtain $((A_1(A_2A_3))((A_4A_5)A_6))$ in $O(n)$ time.

15.3 Elements of dynamic programming

Optimal substructure: an optimal solution to the problem contains optimal solutions to subproblems.

Overlapping subproblems: a recursive algorithm revisits the same subproblem over and over again.

Recursive-Matrix-Chain(*p*, *i*, *j*)
  if $i = j$ **then return** 0
  $m[i, j] = \infty$
  for $k \leftarrow i$ **to** $j - 1$ do
    $q \leftarrow$ Recursive-Matrix-Chain(p, i, k)
      + Recursive-Matrix-Chain(p, k+1, j)
      + $p_{i-1}p_kp_j$
    if $q < m[i, j]$ **then** $m[i, j] \leftarrow q$
  return $m[i, j]$



- $T(n) \geq \sum_{1 \leq k \leq n-1}(T(k) + T(n-k) + 1)$
  $\geq 2\sum_{1 \leq i \leq n-1}T(i) + n$
  $= \Omega(2^n)$ (by substitution method)

Memoization:
  a variation of dynamic programming (top-down)

Memoized-Matrix-Chain(*p*)
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow i$ **to** n **do** $m[i, j] = \infty$
  return Lookup-Chain(m, p, 1, n)

Lookup-Chain(*m*, *p*, *i*, *j*)
  if $m[i, j] < \infty$ **then return** $m[i, j]$
  if $i = j$ **then** $m[i, j] \leftarrow 0$
    else
      for $k \leftarrow i$ **to** $j - 1$ **do**
        $q \leftarrow$ Lookup-Chain(m, p, i, k)
          + Lookup-Chain(m, p, k+1, j)
          + $p_{i-1}p_kp_j$
        if $q < m[i, j]$ **then** $m[i, j] \leftarrow q$
  return $m[i, j]$

- $T(n) = O(n^3)$

* Try to write a memoized recursive algorithm for the rod cutting problem.

15.4 Longest common subsequence (LCS)

Subsequence: Z is a subsequence of X iff Z can be obtained from X by deleting some characters.

Common subsequence:

$X = x_1x_2...x_7$ = abcbdab $\quad Y = y_1y_2...y_6$ = bdcaba

common sequences: ba, bca, bcba, bdab

Longest common subsequence: bcba, bdab

Step 1. Optimal substructure

Example: $X[1..m]$ = a b c b d a <u>b</u> d
$Y[1..n]$ = b d c a <u>b</u> a c

From (b, d, a, <u>b</u>) = $LCS(X, Y)$, we conclude that
(b, d, a) = $LCS(X[1..m–2], Y[1..n–3])$.

Step 2.
Let $Z[1..k] = LCS(X[1..m], Y[1..n])$.
(1) If $x_m = y_n$, then
$x_m = y_n = z_k$ and
$Z[1..k–1] = LCS(X[1..m–1], Y[1..n–1])$.

(2) If $x_m \neq y_n$, then either
$Z[1..k] = LCS(X[1..m–1], Y[1..n])$ or
$Z[1..k] = LCS(X[1..m], Y[1..n–1])$

Let $c[i, j]$ be the length of $LCS(X[1..i], Y[1..j])$

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

Step 3. $c[0..n, 0..n]$, $b[0..n, 0..n]$ (for **Step 4**)

i \ j		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

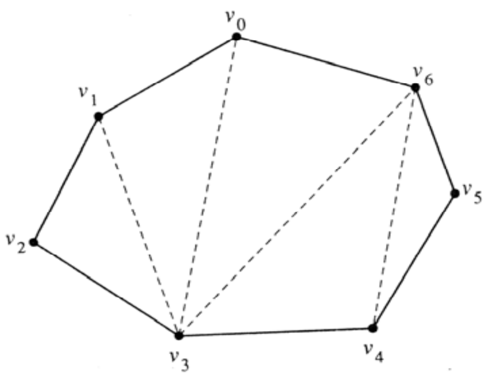
- Time: $O(mn)$ Space: $O(mn)$
- If Step 4 is omitted, c only needs two rows.

Step 4. Using table b, by backtracking we obtain $LCS(X, Y)$ = bcba in $O(m + n)$ time.

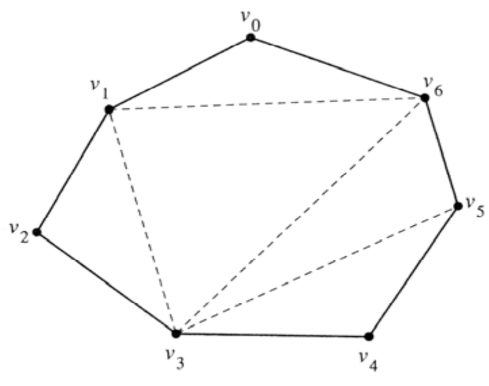
15.5 optimal binary search trees (extra class)

*** Optimal polygon triangulation**

Input: a convex polygon $P = (v_0, v_1, ..., v_{n-1})$
a cost function $w(\Delta v_i v_j v_k)$
Output: an optimal triangulation



(a)



(b)

- Usually, $w(\Delta v_i v_j v_k)$ is $|v_i v_j| + |v_j v_k| + |v_i v_k|$.

Step 2. Let $t[i, j]$ be the weight of an optimal triangulation of polygon $(v_{i-1}, v_i, ..., v_j)$.

$$t[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k \le j-1} \{t[i,k] + t[k+1,j] + w(\Delta v_{i-1} v_k v_j\} & \text{if } i < j \end{cases}$$

Step 3. Similar to Step 3 of matrix chain.

- Time: $O(n^3)$ Space: $O(n^2)$

Homework: Ex. 15.2-2, 15.4-3 15.4-5, Prob. 15-3, 15-4, 15-5, 15-9