二上 (素)

1. [10%] Let V be the set of $\mathbf{R}^2$ with addition defined by

   $\alpha[(x_1, x_2) + (y_1, y_2)] = (x_1+y_1, x_2+y_2)$

   and scalar multiplication defined by

   $\alpha \cdot (x_1, x_2) = (\alpha x_1, x_2)$ $\quad \alpha(x$

   for $(x_1, x_2), (y_1, y_2) \in V$ and $\alpha \in F$. Is V a vector space with these operations?

2. [10%] Prove that

   (a) $W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : a_1+a_2+\cdots+a_n = 1\}$ is a subspace of F^n or not. (5%)

   (b) $W_2 = \{(b_1, b_2, \cdots, b_n) \in F^n : b_1+b_2+\cdots+b_n = 0\}$ is a subspace of F^n or not. (5%)

3. [10%] In each part, determine whether the given vector is in the span of S.

   (a) (-2, 0, 3) $\qquad S = \{(1, 3, 0), (2, 4, -1)\}$ (2%)

   (b) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ $\qquad S = \{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$ (2%)

   (c) $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ $\qquad S = \{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$ (2%)

   (d) $-x^3+2x^2+3x+3$ $\qquad S = \{x^3+x^2+x+1, x^2+x+1, x+1\}$ (2%)

   (e) x^3+2x^2+4x+4 $\qquad S = \{x^3+x^2+1, x+1, x^2+2x+1\}$ (2%)

4. [15%] Determine whether the following sets are linearly dependent or linearly independent.

   (a) $\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}\}$ in $M_{2\times 2}(R)$ (5%)

   (b) $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$ in $M_{2\times 2}(R)$ (5%)

   (c) $\{x^3-x, 2x^2+4, -2x^3+3x^2+2x+6\}$ in $P_3(R)$. (5%)

5. [15%] The set of all diagonal matrices of nxn is a subspace W of $M_{n\times n}(F)$.

   (a)Find a basis for W. (10%)

   (b)What is the dimension of W? (5%)

6. [20%] Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$.

   (a) Prove if S_1 is linearly dependent, then S_2 is linearly dependent. (10%)

   (b) Prove if S_2 is linearly independent, then S_1 is linearly independent. (10%)

7. [10%] Prove if x, y and z are vectors in a vector space V such that x+z = y+z, then x = y.

8. [10%] Prove that the set S = {(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)} is linearly independent or not.

$a=0$
$b=0$
$c=0$
$-a-b-c+d=0.$

9. [20%] Please show the reasons for the statements S_1 and S_2 below.

Theorem 1.10 (1/2)

Theorem 1.10 (Replacement Theorem). *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

Proof. The proof is by mathematical induction on m. The induction begins with $m = 0$; for in this case $L = \varnothing$, and so taking $H = G$ gives the desired result.

Now suppose that the theorem is true for some integer $m \geq 0$. We prove that the theorem is true for $m + 1$. Let $L = \{v_1, v_2, \ldots, v_{m+1}\}$ be a linearly independent subset of V consisting of $m + 1$ vectors. S_1 By the corollary to Theorem 1.6 (p. 39), $\{v_1, v_2, \ldots, v_m\}$ is linearly independent, and so we may apply the induction hypothesis to conclude that $m \leq n$ and that there is a subset $\{u_1, u_2, \ldots, u_{n-m}\}$ of G such that $\{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_{n-m}\}$ generates V. Thus there exist scalars $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{n-m}$ such that

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m + b_1u_1 + b_2u_2 + \cdots + b_{n-m}u_{n-m} = v_{m+1}. \quad (9)$$

Theorem 1.10 (2/2)

S_2 Note that $n - m > 0$, lest v_{m+1} be a linear combination of $v_1, v_2, \ldots, v_m$, which by Theorem 1.7 (p. 39) contradicts the assumption that L is linearly independent. Hence $n > m$; that is, $n \geq m + 1$. Moreover, some b_i, say b_1, is nonzero, for otherwise we obtain the same contradiction. Solving (9) for u_1 gives

$$u_1 = (-b_1^{-1}a_1)v_1 + (-b_1^{-1}a_2)v_2 + \cdots + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1}$$
$$+ (-b_1^{-1}b_2)u_2 + \cdots + (-b_1^{-1}b_{n-m})u_{n-m}.$$

Let $H = \{u_2, \ldots, u_{n-m}\}$. Then $u_1 \in \text{span}(L \cup H)$, and because $v_1, v_2, \ldots, v_m$, $u_2, \ldots, u_{n-m}$ are clearly in $\text{span}(L \cup H)$, it follows that

$$\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{n-m}\} \subseteq \text{span}(L \cup H).$$

Because $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{n-m}\}$ generates V, Theorem 1.5 (p. 30) implies that $\text{span}(L \cup H) = \mathsf{V}$. Since H is a subset of G that contains $(n - m) - 1 = n - (m + 1)$ vectors, the theorem is true for $m + 1$. This completes the induction. ∎

10. [10%] Please show the reason for the statement S below.

Theorem 1.9. *If a vector space* V *is generated by a finite set* S, *then some subset of* S *is a basis for* V. *Hence* V *has a finite basis.*

Proof. If $S = \varnothing$ or $S = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S that is a basis for V. Otherwise S contains a nonzero vector u_1. By item 2 on page 37, $\{u_1\}$ is a linearly independent set. Continue, if possible, choosing vectors $u_2, \ldots, u_k$ in S such that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent. Since S is a finite set, we must eventually reach a stage at which $\beta = \{u_1, u_2, \ldots, u_k\}$ is a linearly independent subset of S, but adjoining to β any vector in S not in β produces a linearly dependent set. We claim that β is a basis for V. Because β is linearly independent by construction, it suffices to show that β spans V. S By Theorem 1.5 (p. 30) we need to show that $S \subseteq \text{span}(\beta)$. Let $v \in S$. If $v \in \beta$, then clearly $v \in \text{span}(\beta)$. Otherwise, if $v \notin \beta$, then the preceding construction shows that $\beta \cup \{v\}$ is linearly dependent. So $v \in \text{span}(\beta)$ by Theorem 1.7 (p. 39). Thus $S \subseteq \text{span}(\beta)$. ∎

1. No, (VS4) (VS5) fail

$(x_1, x_2) + (-1)(x_1, x_2)$
$= (0, 2x_2) \neq (0, 0)$
$\rightarrow$ VS4 fails

2.

(a) No. Since $\vec{0} \notin W$, -1

-5

(b)
$\because (b_1, b_2, b_3, \cdots, b_n) \in F^n$
① $b_1 + b_2 + b_3 + \cdots + b_n \in F^n$ ✗
② $\{cb_1, cb_2, cb_3 \cdots, cb_n\} \in F^n \ (c \in F)$ ✗
③ by ①. $b_1 + b_2 + \cdots + b_n = 0 \in F^n$

3. +10

(a) Yes

$a(1,3,0)+b(2,4,-1) = (-2,0,3)$

$$\begin{cases} a+2b = -2 \\ 3a+4b = 0 \\ -b = 3 \end{cases}$$

$b = -3$ $(-2,0,3)$ is the linear combination of $(1,3,0)$, $(2,4,-1) \in S$.

$a = 4$ $\Rightarrow (-2,0,3)$ in the span of S.

(b) Yes

$$a\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$\begin{cases} a+c = 1 \\ b+c = 2 \\ -a = -3 \\ b = 4 \end{cases}$$

$a=3$, $b=4$, $c=-2$

(c) No.

$$a\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

$$\begin{cases} a+c = 1 \\ b+c = 5 \\ -a = 5 \\ b = 1 \end{cases} \Rightarrow \text{no solution}$$

(d) Yes.

$a(x^3+x^2+x+1)+b(x^2+x+1)+c(x+1) = -x^3+2x^2+3x+3$

$$\begin{cases} a=-1 \\ a+b=2 \\ a+b+c=3 \\ a+b+c=3 \end{cases} \Rightarrow \begin{array}{l} a=-1 \\ b=3 \\ c=1 \end{array}$$

(e) Yes

$a(x^3+x^2+1)+b(x+1)+c(x^2+2x+1) = x^3+2x^2+4x+4$

$$\begin{cases} a=1 \\ a+c=2 \\ b+2c=4 \\ a+b+c=4 \end{cases} \Rightarrow \begin{array}{l} a=1 \\ c=1 \\ b=2 \end{array}$$

4.

(a) Yes -1

+12 $a\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} + b\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = 0$

$\Rightarrow a=-b$ 、上式有 ∞ 解 and. a.b 可 not all zero

(b) Yes -1

$a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$

$$\begin{cases} a+d=0 \\ b+d=0 \\ b+c=0 \\ a+c=0 \end{cases}$$ $a=-d=b=-c$.

有 ∞ 解, and a.b.c.d 可 not all zero

(c) Yes.

$a(x^3-x)+b(2x^2+4)+c(-2x^3+3x^2+2x+6)=0.$

$$\begin{cases} a-2c=0 \\ 2b+3c=0 \\ -a+2c=0 \\ 4b+6c=0 \end{cases} \Rightarrow \begin{cases} a-2c=0 \\ 2b+3c=0 \end{cases} \Rightarrow \begin{matrix} \infty \text{解}. \\ a,b,c \text{ not all zero.} \end{matrix}$$

5. +10

(a)

$$\{A_{ij} : 1 \le i = j \le n\}$$

where A is the $n\times n$ matrix having 1 in the ith row and jth column.

(b)

n.

6. +10

(a) if $S_1 = \{u_1, u_2, \dots, u_n\}$

$S_2 = \{u_1, u_2, \dots, u_n, \dots, u_{n+m}\}$

$\because S_1$ is linearly dependent

$\therefore a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$ for some nonzero scalars $a_1, a_2, \dots a_n$

then

$a_1u_1 + a_2u_2 + \dots a_nu_n + \dots a_{n+m}u_{n+m} = 0$. for $\underset{\text{scalars}}{a_{n+1}} = a_{n+2} = \dots = a_{n+m} = 0$.

$\Rightarrow S_2$ is linearly dependent $\times$

(b) when S_2 is linearly independent,

if S_1 is linearly dependent $\Rightarrow S_2$ is dependent by (a). contradicts

$\Rightarrow S_1$ is linearly independent. $\times$

7.

By (VS 4), there exists a (unique) vector v in V such that $z+v=0$.

+10

Thus

$$\begin{aligned} x &= x+0 && (\text{VS 3}) \\ &= x+(z+v) \\ &= (x+z)+v && (\text{VS 2}) \\ &= (y+z)+v && \text{by 題意 } x+z=y+z \\ &= y+(z+v) && (\text{VS 2}) \\ &= y+0 \\ &= y && (\text{VS 3}) \end{aligned}$$

$\Rightarrow x=y$ ⊠

8.

+10 To prove S is linearly independent or not,

we must find

$$a_1(1,0,0,-1)+a_2(0,1,0,-1)+a_3(0,0,1,-1)+a_4(0,0,0,1)=(0,0,0,0)$$

whether a_1, a_2, a_3, a_4 are all zero $\left(\begin{cases} \text{all zero} \rightarrow \text{independent} \\ \text{not all zero} \rightarrow \text{dependent} \end{cases}\right)$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ -a_1 - a_2 - a_3 + a_4 = 0 \end{cases} \Rightarrow \text{there is only one solution that } a_1=a_2=a_3=a_4=0.$$

$\Rightarrow S$ is linearly independent.

9.

(a) because $L=\{v_1, v_2, \cdots, v_{m+1}\}$ is linearly independent.

$\because \{v_1, v_2, \cdots, v_m\} \subseteq L \subseteq V$, so by Thm 1.6, $\{v_1, v_2, \cdots, v_m\}$ is linearly independent.

(b) if $n-m=0$, $n=m$. $\Rightarrow a_1v_1+a_2v_2+\cdots a_nv_n = v_{n+1}$

$a_1v_1+a_2v_2+\cdots a_nv_n - v_{n+1} = 0$

$\Rightarrow L$ is linearly dependent $(\rightarrow\!\leftarrow)$

$\Rightarrow n-m\neq 0$.

if $n-m<0$,

$\mathcal{L} \cup G$

$= \{v_1, v_2, \cdots, v_m\}$ can't generates V

10. 10