BASIC NUMBER THEORY (OCT. 25, 2006)

1.(20) (a) Prove that $\binom{2}{2} + \binom{4}{2} + \binom{6}{2} + \cdots + \binom{2n}{2} = \frac{n(n+1)(4n-1)}{6}, \quad n \geq 2.$

(b) Let p_n be the n-th pentagonal number, show that $p_n = \frac{n(3n-1)}{2}$.

2.(20) (a) For $p > q > 5$ and both p, q are primes. Show that $24 | (p^2 - q^2)$.

(b) For $n \geq 1$, show that $\frac{n(7n^2+5)}{6}$ is an integer.

3.(20) (a) Find all positive integer solutions of $54x + 21y = 906$.

(b) If a, b are relatively prime positive integers. Prove that $ax - by = c$ has infinitely many positive integer solutions for any given integer c.

4.(20) Show that there are infinite many primes of the form $6n + 5$.

5.(20) Let a, b, c be positive integers and $d = \gcd(a, b, c)$. Show that d is the least element of the set $S = \{0 < n = ax + by + cz : x, y, z \in \mathbb{Z}\}$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX