Probability & Statistics

2004.11.12

1. (15%) At a certain gas station, 50% of the customers use regular unleaded gas, 30% use extra unleaded gas, and 20% use premium unleaded gas. Of those customers using regular gas, only 60% fill their tanks. Of those customers using extra gas, 50% fill their tanks, whereas of those using premium, 40% fill their tanks.
   (a) What is the probability that the next customer will request extra unleaded gas and fill the tank?
   (b) What is the probability that the next customer fills the tank?
   (c) If the next customer fills the tank, what is the probability that regular gas is requested?
2. (15%) A CD player has a magazine that holds four CDs. The machine is capable of randomly selecting a CD at random and then selecting a song randomly from that CD. Suppose that three CDs are albums by Paul McCartney and one is by Billy Joel. The player selects songs until a song by Billy Joel is played after which the machine is turned off. Give the probability that the machine is turned off after
   (a) The fifth song.
   (b) At least three songs have been played.
   (c) Suppose now that the songs continue to be played until the second song by Billy Joel has been played. Find the probability that at most three songs are played.
3. (10%) Suppose that 500 points are selected independently and at random from the unit square $\{(x, y): 0 \le x < 1, 0 \le y < 1\}$. Let W equal the number of points that fall in $A = \{(x, y): x^2 + y^2 < 1\}$.
   (a) Give the mean and variance of W.
   (b) What are the expected value and variance of $\dfrac{W}{100} - 5$?
4. (10%) Let the random variable X have the p.m.f. $f(x) = \dfrac{(|x|+1)^2}{9}$, $x = -1, 0, 1$. Compute $E(X^2)$ and $E(X^2 - X + 2 - \dfrac{1}{X})$.
5. (10%) An urn contains 5 balls numbered from 1 through 5. A random sample of 5 balls is selected from the urn, one at a time. A match occurs if ball numbered i is selected on the ith draw. Find the probability of at least one match if the sampling is done
   (a) With replacement.
   (b) Without replacement.
6. (10%) Let the moments of X be defined by $E(X^r) = 0.3$, $r = 1, 2, 3, \ldots, \infty$. Find the p.m.f of X.
7. (10%) The random variable X has the p.m.f. $f(x) = \dfrac{1}{x(x+1)}$, $x = 1, 2, 3, \ldots, \infty$. What is the expected value of X?
8. (10%) There are N distinct types of coupons, and each time one is obtained it will, independently of past choices, be of type i with probability P_i, $i = 1, \ldots, N$. Let T denote the number one need select to obtain at least one of each type. Compute $P(T=n)$.
9. (10%) Let X be such that $P(X=1) = p = 1 - P(X=-1)$. Find $c \neq 1$ such that $E(c^X) = 1$.
10. (10%) Let X be a random variable having expected value μ and variance σ^2. Find the excepted value and variance of $Y = (X-\mu)/\sigma$.