Midterm Examination II

Probability May 13, 2015

s + δt

1. (15%) Explain the following terms briefly.
   (a) Stationarity (b) Orderliness
   (c) Memoryless property (d) The law of unconscious statistician
   (e) De Moivre-Laplace Theorem

2. (15%) Binomal → Poisson

   (a) (9%) Poisson introduced a procedure to obtain the formula that approximates the binomial probability mass function when the number of trials is large ($n \to \infty$), the probability of success is small ($p \to 0$), and the average number of successes remains a fixed quantity of moderate value ($np = \lambda$ for some constant λ). That is the binomial probability mass function with parameters (n, p),

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i},$$

b
4

   can be approximated by

$$P(X = i) \to \frac{e^{-\lambda}\lambda^i}{i!}.$$

   Briefly explain the procedure.

   (b) (6%) The time of occurrence of the n^{th} event of a Poisson process has a gamma (or n-Erlang) distribution. Let X be a gamma random variable with parameters (n, λ), the distribution function $F(X)$ of X =?

3. (10%)
   Suppose that, on the Richter scale, earthquakes of magnitude 5.5 or higher have probability 0.015 of damaging certain types of bridges. Suppose that such intense earthquakes occur at a Poisson rate of 1.5 (λ, ×6 = 9) per ten years. If a bridge of this type is constructed to last at least 60 years, what is the probability that it will be undamaged by earthquakes for that period of time?

$$\sum_{i=0}^{\infty} (1-0.015)^9 \cdot \frac{e^{-9}\cdot 9^i}{i!}$$

4. (10%)
   To estimate the number of trout in a lake, we caught 50 trout, tagged and returned them. Later we caught 50 trout and found that four of them were tagged. From this experiment estimate n, the total number of trout in the lake. (Hint: Let p_n be the probability of four tagged trout among the 50 trout caught. Find the value of n that maximizes p_n. This value n is called the *maximum likelihood estimate*.)

5. (10%)

A point is selected at random on a line segment of length l. What is the probability that none of the two segments is smaller than $l/3$?

6. (10%)

To examine the accuracy of an algorithm that selects random numbers from the set $\{1, 2, \ldots, 40\}$, 100,000 numbers are selected and there are 3500 ones. Given that the expected number of ones is 2500, is it fair to say that this algorithm is not accurate?

7. (10%) The grades for a certain exam are normally distributed with mean 67 and variance 64. What percent of students get A($\geq$ 90), B(80 – 90), C(70 – 80), D(60 – 70), and F(< 60)?

表

8. (10%)

A man invites his fiancée to an elegant hotel for a Sunday brunch. They decide to meet in the lobby of the hotel between 11:30 A.M. and 12 noon. If they arrive at random times during this period, what is the probability that the first to arrive has to wait at least 12 minutes?

9. (10%)

偶数

What is the probability of an *even* number of successes in n independent Bernoulli trials?

Hint: conditioning on the first trial and using the law of total probability.

[Bonus problems: Choose one of them]

10. (15%) If random events occur in time in a way that the conditions *stationarity*, *independent increments*, and *orderliness* are always satisfied, $N(0) = 0$ and, for all $t > 0$, $0 < P(N(t) = 0) < 1$, then there exists a positive number λ such

$$P(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}.$$

That is, for all $t > 0$, $N(t)$ is a Poisson random variable with parameter λt. Hence $E(N(t)) = \lambda t$ and therefore $\lambda = E(N(1))$. You just have to point out the motivating argument for the $\boxed{\text{validity}}$.

11. (15%)

$\frac{1}{3}$

Bertrand's paradox: What is the probability that a random chord of a circle is longer than a side of an equilateral triangle inscribed into the circle?

We can interpret the expression random chord in three different ways and solve the problem in each case. Describe any two of the three interpretations and show the corresponding solutions.

6 - 9 - 10