8 R
6 B

R_1R_2
R_1B_2
B_1R_2
B_1B_2

$\frac{\binom{8}{1}\binom{6}{1}}{\binom{14}{2}} \times \frac{9}{10} \times \frac{1}{6}$

10R
6B

Probability (CS 3332)

Mid-term Exam 2 (May 11, 2016)

Preliminary problems

1. (15%) An urn contains 9 red and 1 blue balls. A second urn contains 1 red and 5 blue balls. One ball is removed from each urn at random and without replacement, and all the remaining balls are put into a third urn. If we draw two balls randomly from the third urn, what is the probability that one of them is red and the other one is blue?

A

2. (15%) The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \leq x < 1 \\ 1/2 & 1 \leq x < 2 \\ \frac{x}{12} + \frac{1}{2} & 2 \leq x < 3 \\ 1 & x \geq 3. \end{cases}$$

Compute the following quantities: (a) $\Pr(X < 2)$; (b) $\Pr(X = 2)$; (c) $\Pr(X = 5/2)$.

$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k}$

3. (15%) What is the probability of an even number of successes in n independent Bernoulli trials? *Hint*: Let r_n be the probability of an even number of successes in n Bernoulli trials. By conditioning on the first trial and using the law of total probability, show that for $n \geq 1$,

$$r_n = p(1 - r_{n-1}) + (1 - p)r_{n-1}.$$

Then prove that $r_n = [1 + (1 - 2p)^n]/2$.

$P\left(\frac{X - 39.8}{2.05} \geq \frac{40 - 39.8}{2.05}\right)$

4. (15%) Suppose that a Scottish soldier's chest size is normally distributed with mean 39.8 and standard deviation 2.05 inches, respectively. What is the probability that 20 randomly selected Scottish soldiers, five have a chest of at least 40 inches? *Hint*: Use the provided tables of normal distributions.

Challenging problems

5. (10%) Suppose that independent Bernoulli trials with parameter p are performed successively. Let N be the number of trials needed to get x successes, and X be the number of successes in the first n trials. Show that

neg bino

$$\Pr(N = n) = \frac{x}{n} \Pr(X = x).$$

$P(N(t) = i) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$

6. (10%) Suppose that male customers arrive at a post office according to a Poisson process with rate λ_1. Denote the number of male customers who arrive at the post office between time zero and time t by $N_1(t)$. The point process $\{N_1(t), t \geq 0\}$ is said to be a Poisson process if it satisfies the following three assumptions

4
0.2
2.05

- Stationarity: For all $n \geq 0$, and for any two equal time intervals T_1 and T_2, the probability of n events in T_1 is equal to the probability of n events in T_2.
- Independent increments: For all $n \geq 0$, and for any time interval $(t, t+s)$ is independent of how many events have occurred earlier or how they have occurred. In particular, suppose that the times $0 \leq t_1 < t_2 < \ldots < t_k$ are given. For $1 \leq i < k-1$, let A_i be the event that n_i events of the process occur in $[t_i, t_{i+1})$. The independent increments mean that $\{A_1, A_2, \ldots, A_{k-1}\}$ is an independent set of events.
- Orderliness: This condition is mathematically expressed by $\lim_{h\to 0} \Pr(N_1(h) > 1)/h = 0$ and $\lim_{h \to 0} \Pr(N_1(h) = 1)/h = \lambda_1$.

$\lim_{h\to 0} \frac{P(N_1(h)=1)}{h} = \lambda_1$ $\quad \lim_{h\to 0} \frac{P(N_1(h)>1)}{h} = 0$

Suppose that female customers arrive at the post office also according to a Poisson process with rate λ_2. Denote this Poisson process by $\{N_2(t), t \geq 0\}$. Assume that the two Poisson processes are independent. Now consider the superposed process $\{N_1(t) + N_2(t), t \geq 0\}$, which is a point process. In this superposed process, each event corresponds to either a male customer arrival time or a female customer arrival time. Show that the superposed process satisfies the three assumptions above and thus, is a Poisson process. What is the rate of the superposed Poisson process?

7. (20%) Let the probability density function of random variable X be

$h(x) = Y = X\sqrt{X} = (X)^{\frac{3}{2}}$

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow h^{-1}(Y) = X = Y^{\frac{2}{3}}$

- Find the probability density function of $Y = X\sqrt{X}$.
- Find the expectation $E(Y)$.

$\Rightarrow$ from $f_Y(y) = f_X(h^{-1}(y)) \cdot |(h^{-1})'(y)|$

$\frac{d}{d\theta}\theta = \frac{d}{dx}\sin^{-1}(x)$

$\frac{e^{-x}}{\sqrt{x}}$

$\int x^{\frac{3}{2}} e^{-x} dx \Rightarrow$

$E(h(x)) = \int_{-\infty}^{\infty} h(x) f(x)$

$\int \frac{dx}{1+x^2} = \tan^{-1}(x)$

$= x^{\frac{3}{2}}(-e^{-x}) + \frac{3}{2}\int x^{\frac{1}{2}} e^{-x} dx$

$\int \underbrace{x^{\frac{3}{2}}}_{u} \underbrace{e^{-x} dx}_{dv}$