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Test 3 for CS2334 (01)

December 15, 2008

ID .
Name :

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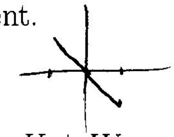
(30%) **1.** *Mark* $\bigcirc$ *if the statement is* true, *and mark* $\times$ *otherwise.*

(X) **(a)** Let $\mathbf{x}, \mathbf{y} \in R^n$ be nonzero vectors and the vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is equal to the vector projection of $\mathbf{y}$ onto $\mathbf{x}$, then $\mathbf{x}$ and $\mathbf{y}$ must be linearly dependent.

✓ (X) **(b)** Let $U = span([1, -1]^t)$ and $V = span([-1, 1]^t)$, then $R^2 = U \oplus V$.



(X) **(c)** If U, V, W are vector subspaces of R^3 such that $U \perp V$ and $V \perp W$, then $U \perp W$.

(O) **(d)** If $Null(A) = \{\mathbf{0}\}$, the ~~the~~ system $A\mathbf{x} = \mathbf{b}$ will have a unique least squares solution.

(O) **(e)** The product of orthogonal matrices in $R^{n\times n}$ must be orthogonal.

(O) **(f)** A set of nonzero orthogonal vectors are linearly independent.

(X) **(g)** A set of nonzero orthonormal vectors in R^n must be a basis.

(O) **(h)** Every square matrix can be factored as QR, where Q is orthogonal and R is upper-Δ.

(X) **(i)** If $\mathbf{x}, \mathbf{y} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 1$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(O) **(j)** If $A \in R^{m\times n}$, then AA^t and A^tA have the same rank.

(O) **(k)** Let $H_1, H_2, \cdots, H_m \in R^{n\times n}$ be Householder matrices, then $\Pi_{i=1}^m H_i$ is an orthogonal matrix.

(X) **(l)** Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in R^3, then any Gram-Schmidt orthogonalization process constructs the unique orthonormal basis.

(X) **(m)** Let H be a Householder matrix, then H is symmetric, orthogonal, and $det(H) = 1$.

(O) **(n)** Let $\mathbf{x}, \mathbf{y} \in R^n$ be such that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, then $\mathbf{x}$ and $\mathbf{y}$ are orthonormal.

(O) **(o)** In R^n, if $\mathbf{p}$ is the projection of $\mathbf{b}$ along the line $\mathbf{a}$, then $\langle \mathbf{a}, \mathbf{b} - \mathbf{p} \rangle = 0$.

(30%) **2.** Fill the following blanks.

$\sqrt{4+16+16}=6$

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(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in R^n$ be orthonormal vectors, then $\|2\mathbf{u} - 4\mathbf{v} + 4\mathbf{w}\|_2 =$ 6

(b) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ $=$ $\frac{\pi}{2}$

(c) Let $V = \{[x, y, z]^t | \, x - y + z = 0\} \subset R^3$, then $V^\perp$ $=$ $\{[\alpha, -\alpha, \alpha]^t \mid \alpha \in R\}$

(1, -1, 1)

(d) Let $\mathbf{u} = [1, 3, -2, 4]^t$, then the rank of $\mathbf{uu}^t$ $=$ 1

uu^t 2

(e) Let $A \in R^{m\times n}$ have r independent column vectors, then $dim(Null(A))+dim(R(A))$ $=$ ~~m~~ n

~~n+r~~ ~~m~~ r

n−r

腦殘啊！

(f) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ $=$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(g) Define a Householder matrix $H = I - 2\mathbf{uu}^t$, where $\mathbf{u} = [0.6, 0.8, 0]^t$, and let $\mathbf{x} = [2, -1, 2]^t$, then $\|H\mathbf{x}\|_2 =$ 3

$\sqrt{4+1+4}=3$

(h) Let $\mathbf{x} = [2, -1, 2]^t$, then $\|\mathbf{x}\|_1 \cdot \|\mathbf{x}\|_2 \cdot \|\mathbf{x}\|_\infty$ $=$ 30

5 3 2

(i) Let $\mathbf{a} = [1, 1, 1]^t$, $\mathbf{b} = [1, 3, 8]^t$, then the projection of $\mathbf{b}$ onto the line $\mathbf{a}$ $=$ $\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$

(j) The distance from the point $[1, 1, 1]^t$ to the plane $2x + y + 2z + 7 = 0$ *is* 4

$\frac{|2+1+2+7|}{\sqrt{4+1+4}} = \frac{12}{3} = 4$

(10%) **3.** Sketch the set of points $\mathbf{x} = [x_1, x_2]^t \in R^2$ such that

(*a*) $\|\mathbf{x}\|_2 = 1$, (*b*) $\|\mathbf{x}\|_1 = 1$, (*c*) $\|\mathbf{x}\|_\infty = 1$.

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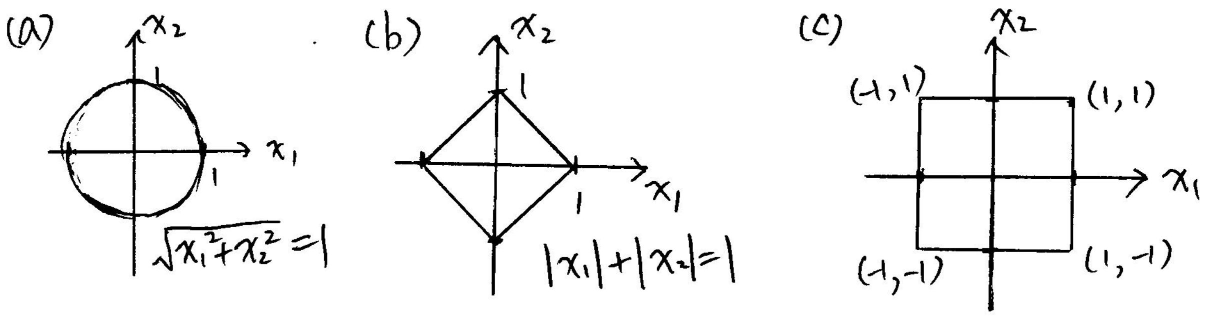

(10%) **4.** Find the point on the line $y = 2x + 1$ that is closest to $[2, 3]^t$.

$L: 2x - y + 1 = 0 \Rightarrow \vec{n} = (2, -1)$

$L_\perp: (2t + 2, -t + 3)$

$4t + 4 + t - 3 + 1 = 0 \Rightarrow 5t = -2 \Rightarrow t = -\frac{2}{5}$

$-\frac{4}{5} + \frac{10}{5} = \frac{6}{5}$, $\frac{2}{5} + \frac{15}{5} = \frac{17}{5}$

$P = \left(\frac{6}{5}, \frac{17}{5}\right)^t$ *

(10%) **5.** Let $\mathbf{a}_1 = [1, 1, 0]^t$, $\mathbf{a}_2 = [2, 3, 0]^t$, and $\mathbf{b} = [4, 5, 6]^t$. Find the projection vector of $\mathbf{b}$ onto the plane that is spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$. $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ find least square solution of $Ax = b$

$x_1 + 2x_2 = 4$
$x_1 + 3x_2 = 5 \Rightarrow x_1 = 2$
$0x_1 + 0x_2 = 6 \quad x_2 = 1$

$p = Ax = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$ *

(10%) **6.** Let $\mathbf{x} = [2, -1, 2]^t$, $\mathbf{y} = [-2, 2, -1]^t$, find a Householder matrix H such that $H\mathbf{x} = \mathbf{y}$.

$16 + 18 = 34$

$H = I - 2\vec{u}\vec{u}^t$

$H\vec{x} = \vec{y}$

$(I - 2\vec{u}\vec{u}^t)\vec{x} = \vec{y}$ $\quad \vec{u}^t\vec{u} = 1$

$\vec{x} - 2\vec{u}\underbrace{\boxed{\vec{u}^t\vec{x}}}_{k} = \vec{y} \Rightarrow 2\vec{u}\vec{u}^t\vec{x} = \vec{x} - \vec{y}$ $\quad u = (x - y)/2k$

$\vec{u}\vec{u}^t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} / 2k$

$\cancel{\vec{u}^t}\vec{u}\vec{u}^t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \vec{u}^t \cdot \frac{1}{2}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}$ $\quad u = \frac{1}{\sqrt{34}}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}$

$\cancel{2u_1} - u_2 + 2u_3 = \cancel{2u_1} - \frac{3}{2}u_2 + \frac{3}{2}u_3$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \frac{1}{2}u_2 = -\frac{1}{2}u_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \cancel{2} \times \frac{1}{\cancel{34}_{17}}\begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}[4 \ -3 \ 3]$

$\vec{u}\vec{u}^t = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\left[0 \ \frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow H = I - 2\vec{u}\vec{u}^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{17}\begin{bmatrix} 16 & -12 & 12 \\ -12 & 9 & -9 \\ 12 & -9 & 9 \end{bmatrix}$

$\frac{2 - 12 - 24}{17} = \frac{-34}{17} = -2$