Ordinary Differential Equation Examination 2 (Two hours)

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Problems designed by Professor S. S. Cheng

1a (10 pts). State the Existence and Uniqueness theorem for analytic solutions of differential systems of the form

$$x'(t) = A(t)x(t),$$

under the side condition

$$x(t_0) = x_0.$$

1b (20 pts). Use the above Theorem to show the Newton binomial expansion formula:

$$(1+x)^\alpha = \sum_{n=0}^{\infty} C_n^{(\alpha)} x^n, \; x \in R, |x| < 1,$$

where we recall that $C_n^{(\alpha)}$ is the extended binomial coefficient defined by $C_0^{(\alpha)} = 1$ for $\alpha \in \mathbf{C}$, and $C_n^{(\alpha)} = \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$ for $n \in \{1, 2, ..., \}$ and $\alpha \in \mathbf{C}$. (Hint: First find the differential equation satisfied by $(1+x)^\alpha$. Second, solve this equation under appropriate algebraic condition).

1c (20 points) Find the analytic solution of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

of the differential equation

$$f''(x) = x f(x)$$

subject to the condition

$$f(0) = 0, \; f'(0) = 1.$$

Find the radius of convergence of the solution. Is it true that $0 = a_0 = a_3 = a_6 = \cdots$? and $0 = a_2 = a_5 = a_8 = \cdots$?

2 (20 pts). Consider the differential system

$$\begin{aligned} x'(t) &= x(t) - x(t)y(t), \\ y'(t) &= x(t)y(t), \end{aligned}$$

subject to the conditions $x(0) = 10$ and $y(0) = 2$. Use Euler method to construct approximate solution over the time interval $[0, 4]$ using the partition $0 < 1 < 2 < 3 < 4$. Write the **explicit formula of the Euler curve** and plot the Euler curve in the phase plane.

3 (30 pts). Consider the differential system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where $\alpha \in R$ and $\alpha^2 - 1 \neq 0$. By finding the eigenvalues and eigenvectors of the coefficient matrix, find all possible solutions. Then according to whether $\alpha < -1$, $|\alpha| < 1$ and $\alpha > 1$, plot (**as precisely as possible**) the corresponding orbits of these solutions in the phase plane. (Do not forget the trivial solution!)

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5. (Additional points). In problem 1c, find all other solutions of $f''(x) = x f(x)$ and their domain of convergence.

Solutions

1a. **THEOREM**. Let $A(t)$ be analytic at $t = t_0$ and have the expansion

$$A(t) = \sum_{k=0}^{\infty} A_k (t - t_0)^k, \ t \in (t_0 - \rho, t_0 + \rho)$$

and let x_0 be a given vector. Then the sequence $\{x_k\}_{k=0}^{\infty}$ defined by x_0 and

$$x_{k+1} = \frac{1}{k+1} \{A_0 x_k + A_1 x_{k-1} + \cdots + A_k x_0\}$$

will yield an unique analytic solution

$$x(t) = \sum_{k=0}^{\infty} x_k (t - t_0)^k, \ t \in (t_0 - \rho, t_0 + \rho)$$

1b. Observe that $f(x) = (1 + x)^{\alpha}$ for $x \in (-1, 1)$ satisfies

$$(1 + x) f'(x) = \alpha f(x), \ |x| < 1.$$

Assume that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is an analytic solution of the above equation. Then $a_0 = f(0)$ and

$$(n + 1) a_{n+1} + n a_n = \alpha a_n, \ n \in \mathbf{N}.$$

The above recurrence is easily solved and

$$a_n = \frac{1}{n!} \alpha (\alpha - 1) \cdots (\alpha - n + 1), \ n \in \mathbf{Z}^+.$$

(The convergence of solution is eithe by Theorem or by checking (by means of the root test) that $\sum_{n=0}^{\infty} C_n^{(\alpha)} x^n < \infty$.)

1c. Airy's equation is $f''(x) = x f(x)$. Substitution of solution $f(x) = \sum_{i=0}^{\infty} a_i x^i$ to Airy's equation about the 'ordinary' point $x_0 = 0$ gives

$$2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ 2a_2 &= 0, \end{aligned}$$

and

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \quad n \in \mathbf{N}.$$

Hence

$$\begin{aligned} 0 &= a_0 = a_3 = a_6 = \cdots \\ 0 &= a_2 = a_5 = a_8 = \cdots \end{aligned}$$

and

$$\begin{aligned} a_4 &= \frac{1}{4\cdot 3}a_1 = \frac{1}{4\cdot 3}, \\ a_7 &= \frac{1}{7\cdot 6}\frac{1}{4\cdot 3} \\ a_{10} &= \frac{1}{10\cdot 9}\frac{1}{7\cdot 6}\frac{1}{4\cdot 3}, \end{aligned}$$

etc. Thus

$$f(x) = \sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{1}{(3i+1)3i}x^{3k}.$$

It converges everywhere by 1a.

2. See Exercise 7.5.5. Here you need to write down the the Euler curve equations and plot the corresponding parametric curves in the plane.

3. The matrix

$$\begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix}$$

has eigenvalues $\lambda_2 = \alpha + 1$ and $\lambda_1 = \alpha - 1$ with corresponding eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Hence all solution are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{(\alpha+1)t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{(\alpha-1)t} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{(\alpha+1)t} \\ c_2 e^{(\alpha-1)t} \end{pmatrix}.$$

Since $\alpha^2 - 1 \neq 0$, there are three cases, $\alpha < -1$, $|\alpha| < 1$ and $\alpha > 1$.

Case 1. $\alpha > 1$. Then $\lambda_2 > \lambda_1 > 0$. Hence we may plot the orbits (see text) of

$$\begin{pmatrix} c_1 e^{(\alpha+1)t} \\ c_2 e^{(\alpha-1)t} \end{pmatrix}.$$

Then the matrix rotates the orbits $\pi/4$ counterclockwise.

Case 2. $|\alpha| < 1$. Then $\lambda_2 > 0 > \lambda_1$. Hence we may plot the orbits of

$$\begin{pmatrix} c_1 e^{(\alpha+1)t} \\ c_2 e^{(\alpha-1)t} \end{pmatrix}$$

again, and then the rotated orbits.

Case 3. $\alpha < -1$, then $\lambda_1 < \lambda_2 < 0$. Hence we may plot the orbits of

$$\begin{pmatrix} c_1 e^{(\alpha+1)t} \\ c_2 e^{(\alpha-1)t} \end{pmatrix}$$

again, and then the rotated orbits.

5. As in 1c. The solution that satisfies $f(0) = 1$ and $f'(0) = 0$ is

$$f(x) = \sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{1}{3i(3i-1)}x^{3k}.$$

Hence the general solution is

$$\alpha f(x) = \sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{1}{(3i+1)3i}x^{3k} + \beta\sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{1}{3i(3i-1)}x^{3k}.$$