Ordinary Differential Equation Examination 1 (Two hours)
May 15, 2007
Problems designed by Professor S. S. Cheng

1 (15 pts) Find all (complex valued) solutions of the following first order differential system

$$x'(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} x(t),\ t \in R$$

by finding the eigenvalues and eigenvectors of the coefficient matrix. Then find the unique solution that also satisfies $x(0) = (1, 0, 1)^{\dagger}$.

2(i)(10 pts) Find a fundamental matrix of the following first order differential system

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x(t),\ t \in R.$$

2(ii)(15 pts) Then find the fundamental matrix (solution) $\Phi(t|0)$ normalized at $t = 0$ and also the inverse $\Phi^{-1}(t)$.

2(iii)(10 pts) Find a particular solution of the first order differential system

$$x'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ t \in R.$$

and then all solutions.

3 (20 pts). Let $A = (a_{ij})_{n\times n}$ be a real matrix. Let $u = u(t)$ and $v = v(t)$ be two solutions of the differential system

$$x'(t) = Ax(t),\ t \in R \tag{1}$$

Show that if $u(t)$ and $v(t)$ are equal at $t = 0$, then they are equal at every t.

4 (30 pts) List as many as possible the properties of fundamental matrices of the differential system

$$x'(t) = Ax(t),\ A \in R^{n\times n}, t \in R.$$

Explain very carefully why these properties are correct.

Solutions:

1. The eigenvectors and eigenvalues of the coefficient matrix are $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix}1\\2\\4\end{pmatrix}\right\} \leftrightarrow 2, \left\{\begin{pmatrix}1\\3\\9\end{pmatrix}\right\} \leftrightarrow 3$. Thus every solution is of the form

$$c_1 e^t \begin{bmatrix}1\\1\\1\end{bmatrix} + c_2 e^{2t} \begin{bmatrix}1\\2\\4\end{bmatrix} + c_3 e^{3t} \begin{bmatrix}1\\3\\9\end{bmatrix}.$$

Furthermore,

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + 4c_2 + 9c_3 &= 1 \end{aligned}$$

we see that $c_1 = 7/2, c_2 = -4, c_3 = 3/2$.

2. Since the coefficient matrix is a companion matrix, the corresponding third order scalar differential equation is

$$y'' - 3y' + 2y = 0.$$

Its characteristic polynomial is

$$P(\lambda) = (\lambda - 1)(\lambda - 2).$$

Hence independent solutions are e^t and e^{2t}. A fundamental matrix is then

$$\Phi(t) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}.$$

Since

$$\Phi(0) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \Phi^{-1}(0) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

hence

$$\Phi(t|0) = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{bmatrix}$$

$$\Phi^{-1}(t) = \Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0) = \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}.$$

A particular solution is

$$\begin{aligned} \int_0^t \Phi(t)\Phi^{-1}(s) \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds &= \Phi(t) \int_0^t \begin{bmatrix} 2e^{-s} & -e^{-s} \\ -e^{-2s} & e^{-2s} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds \\ &= \Phi(t) \int_0^t \begin{bmatrix} 2e^{-s} \\ -e^{-2s} \end{bmatrix} ds = \Phi(t) \begin{bmatrix} -2e^{-t} + 2 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{bmatrix}. \end{aligned}$$

3. Uniqueness **THEOREM**. If $x(t)$ and $y(t)$ are solutions in $S(I, \mathbf{C}^n)$ of (1) and they satisfy $x(t_0) = y(t_0)$ for some $t_0 \in I$, then $x(t) = y(t)$ for all $t \in I$.

Proof. There are several proofs for this theorem. The one that follows is based on the technique of 'integrating factor". First note that we need to prove that if $u(t)$ is a solution of (1) satisfying $u(t_0) = 0$, then $u(t) = 0$ for $t \in I$. If $A = 0$, then the conclusion is obvious. Assume that $A \neq 0$. We write

$$u'(t) = Au(t),\ t \in I,$$

so that

$$u(t) = \int_{t_0}^t Au(s)ds,\ t \in I.$$

Hence,

$$\|u(t)\|_2 = \left\| \int_{t_0}^t Au(s)ds \right\|_2 \leq \int_{t_0}^t \|Au(s)\|_2 \, ds,\ t > t_0,$$

which implies

$$\|u(t)\|_2 \leq \int_{t_0}^t \|A\| \, \|u(s)\| \, ds,\ t > t_0.$$

Let

$$\|A\| = K > 0,$$

and

$$f(t) = \|u(t)\|_2 \, ,\ t \geq t_0.$$

Then $f(t) \geq 0$ for $t \geq t_0$, f is differentiable for $t \geq t_0$ and

$$f(t) \leq K \int_{t_0}^t f(s)ds,\ t > t_0.$$

Hence

$$\frac{d}{dt}\left\{K\int_{t_0}^{t} f(s)ds\right\} = Kf(t) \leq K\left\{K\int_{t_0}^{t} f(s)ds\right\},\ t > t_0.$$

If we denote

$$G(t) = K\int_{t_0}^{t} f(s)ds,\ t \geq t_0,$$

then

$$\begin{aligned} G'(t) - KG(t) &\leq 0,\ t > t_0, \\ G(t_0) &= 0, \\ G(t) &\geq 0,\ t > t_0. \end{aligned}$$

Thus by multiplying the positive integrating factor $\exp\left\{-K(t-t_0)\right\}$, we see that

$$0 \geq \exp\left\{-K(t-t_0)\right\} G'(t) - KG(t)\exp\left\{-K(t-t_0)\right\} = \frac{d}{dt}\left\{G(t)\exp\left(-K(t-t_0)\right)\right\},\ t > t_0.$$

Finally we see that

$$0 \leq G(t)\exp\left(-K(t-t_0)\right) \leq G(0)\exp\left(-K(t_0-t_0)\right) = G(0) = 0,\ t > t_0.$$

This shows that $G(t) = 0$ for $t \geq t_0$ and hence

$$0 \leq \|u(t)\|_2 = f(t) \leq G(t) = 0,\ t \geq t_0.$$

The proof of the case $t < t_0$ is similar.

4. There are some important properties of fundamental matrices Φ.

(i) For any $c \in \mathfrak{F}^n$, $x(t) = \Phi(t)c$ is a solution of (1).

(ii) For any $t \in \mathbf{R}$, $\det \Phi(t) \neq 0$. To see this, let $\Phi(t) = \left(x^{(1)}(t), x^{(2)}(t), ..., x^{(n)}(t)\right)$. If $\det \Phi(t_0) = 0$, then the set $\left\{x^{(1)}(t_0), ..., x^{(n)}(t_0)\right\}$ of vectors is linearly dependent, or there exist $(c_1, c_2, ..., c_n)^\dagger \neq 0$ such that

$$c_1 x^{(1)}(t_0) + c_2 x^{(2)}(t_0) + \cdots + c_n x^{(n)}(t_0) = 0,$$

or,

$$\Phi(t_0)c = 0.$$

By uniqueness, the function

$$y(t) = \Phi(t)c \equiv 0, \; t \in \mathbf{R}.$$

In other words, the set $\left\{x^{(1)}(t), ..., x^{(n)}(t)\right\}$ of solutions is dependent, which is contrary to the definition of a fundamental matrix.

(iii) If B is a nonsingular matrix, then $\Phi(t)B$ is also a fundamental matrix. Indeed, we may write B as $(b^{\langle 1\rangle}, ..., b^{\langle n\rangle})$, then $\Phi(t)B = \left(\Phi(t)b^{\langle 1\rangle}, ..., \Phi(t)b^{\langle n\rangle}\right)$ and each $\Phi(t)b^{\langle i\rangle}$ is a solution. Furthermore, $\Phi(t)Bc = 0 \Rightarrow Bc = 0 \Rightarrow c = 0$.

(iv) $\Phi(t|s) = \Phi(t)\Phi^{-1}(s)$ (by uniqueness again). The unique solution of $x'(t) = Ax(t)$ for $t \in \mathbf{R}$ under the condition $x(t_0) = c$ is $x(t) = \Phi(t|t_0)c$.

(v) (Transition Property) For any $t_1, t_2 \in \mathbf{R}$,

$$\Phi(t|t_1) = \Phi(t|t_2)\Phi(t_2|t_1).$$

Indeed, to show two matrices P and Q with the same dimension are equal, it suffices to show $Pc = Qc$ for any c. Now, $u(t) = \Phi(t|t_1)c$ is a solution of (1) and satisfies $u(t_2) = \Phi(t_2|t_1)c$, while $v(t) = \Phi(t|t_2)\Phi(t_2|t_1)c$ is also a solution of (1) and satisfies

$$v(t_2) = \Phi(t_2|t_2)\Phi(t_2|t_1)c = I\Phi(t_2|t_1)c = \Phi(t_2|t_1)c.$$

Thus $u(t) \equiv v(t)$ as required.

(vi) (Inversion Property) For any $s, t \in \mathbf{R}$, $\Phi^{-1}(t|s) = \Phi(s|t)$. First, since $\Phi(t|s)$ is nonsingular for each $t \in I$, thus it make sense to talk about $\Phi^{-1}(t|s)$. Now, by (v),

$$\Phi(t|s)\Phi(s|t) = \Phi(t|t) = I,$$

hence $\Phi^{-1}(t|s) = \Phi(s|t)$. We remark that the same conclusion can be reached by the uniqueness argument.

(vii) (Decomposition Property) A nonsingular matrix $B \in \mathbf{R}^{n\times n}$ exists such that

$$\Phi(t|s) = B(t)B^{-1}(s).$$

Indeed, pick $t_0 \in \mathbf{R}$, then

$$\Phi(t|s) = \Phi(t|t_0)\Phi(t_0|s) = \Phi(t|t_0)\Phi^{-1}(s|t_0).$$

Now we can pick $B(t) = \Phi(t|t_0)$. We remark that it does no harm to take $t_0 = 0$ in which case $\Phi(t|0)$ and $\Phi^{-1}(s|0)$ are easier to construct.

(viii) (Translation Invariance) Suppose $a, b, t+a, t+b \in \mathbf{R}$, then

$$\Phi(t+b|b) = \Phi(t+a|a).$$

Indeed,

$$\Phi(0+b|b)c = c = \Phi(0+a|a)c.$$

(ix) For $s, t \in \mathbf{R}$, we have

$$\Phi(s+t|t_0) = \Phi(s+t_0|t_0)\Phi(t|t_0).$$

In particular,

$$\Phi(s+t|0) = \Phi(s|0)\Phi(t|0).$$

Indeed, from the transition property and the translation invariance property,

$$\Phi(s+t|t_0) = \Phi(s+t|t)\Phi(t|t_0) = \Phi(s+t_0|t_0)\Phi(t|t_0).$$

(x) From (ix),

$$\Phi^{-1}(t|0) = \Phi(-t|0),$$

which can also be shown by uniqueness argument. More generally,

$$\Phi^{-1}(t_0+t|t_0) = \Phi(t_0-t|t_0).$$

(xi) We have

$$\Phi^{-1}(t) = \Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0).$$

Indeed, we only need to show that

$$I = \Phi(t)\Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0).$$

But

$$\Phi(t)\Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0) = \Phi(t|0)\Phi(-t|0) = I$$

by (x). This sometimes offer an easier method for computing $\Phi^{-1}(t)$. For instance, the differential system

$$x'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} x(t)$$

has a fundamental matrix (solution)

$$\Phi(t)=\begin{bmatrix} e^t & te^t & e^{2t} \\ e^t & te^t+e^t & 2e^{2t} \\ e^t & te^t+2e^t & 4e^{2t} \end{bmatrix}.$$

Since

$$\Phi(0)=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix},\ \Phi^{-1}(0)=\begin{bmatrix} 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix},$$

thus

$$\begin{aligned}\Phi^{-1}(t) &= \Phi^{-1}(0)\Phi(-t)\Phi^{-1}(0) \\ &= \begin{bmatrix} 2te^{-t} & 2e^{-t}-3te^{-t} & -e^{-t}+te^{-t} \\ -2e^{-t} & 3e^{-t} & -e^{-t} \\ e^{-2t} & -2e^{-2t} & e^{-2t} \end{bmatrix}.\end{aligned}$$

(xii) We have

$$\Phi'(t)=A\Phi(t),\ t\in\mathbf{R}.$$

Example. Consider the matrix differential system of the form

$$X'(t)=AX(t),\ t\in I, \tag{2}$$

and $X(t)$ is a matrix function. A solution $X=X(t)$ is a matrix function in $C^{(1)}(I,\mathfrak{F}^{n\times n})$ which renders this equation into an identity. Since we can write

$$X(t)=\left(x^{(1)}(t),...,x^{(n)}(t)\right),$$

thus the above matrix system is equivalent to

$$\frac{d}{dt}x^{(i)}(t)=Ax^{(i)}(t),\ i=1,2,...,n.$$

Thus if $x^{(1)}(t),...,x^{(n)}(t)$ are n vector solutions of (1), then $\left(x^{(1)}(t),...,x^{(n)}(t)\right)$ is a matrix solutions of (2). Conversely, if $X(t)$ is a matrix solution of (2), then $X(t)e^{(i)}$ is a vector solution of (1). In particular, a fundamental matrix function of (1) is a matrix solution of (2). Furthermore, the differential system with 'additional given initial value' problem

$$\begin{aligned} X'(t) &= AX(t),\ t\in I, \\ X(t_0) &= B,\ t_0\in I, \end{aligned}$$

where $B \in \mathfrak{F}^n$, has a unique solution $X(t) = \Phi(t|t_0)B$ in $S(I, \mathfrak{F}^{n\times n})$. Indeed, the fact that $X(t)$ is a solution is clear (xii) and $X(t_0) = \Phi(t_0|t_0)B = B$. If $Y = Y(t)$ is another solution, then $Y(t)c$ and $\Phi(t|t_0)Bc$ are solutions of (1) and $Y(t_0)c = Bc = \Phi(t_0|t_0)Bc$. By uniqueness, $Y(t)c = \Phi(t|t_0)Bc$ for every c. By taking $c = e^{(i)}$, we see that $Y(t) = \Phi(t|t_0)B$.

(xiii) We have

$$\begin{aligned}\frac{d}{dt}\left(\Phi^\dagger(t)\right)^{-1} &= -\left(\Phi^\dagger(t)\right)^{-1}\frac{d\Phi^\dagger(t)}{dt}\left(\Phi^\dagger(t)\right)^{-1} = -\left(\Phi^\dagger(t)\right)^{-1}(A\Phi(t))^\dagger\left(\Phi^\dagger(t)\right)^{-1} \\ &= -\left(\Phi^\dagger(t)\right)^{-1}\Phi^\dagger(t)A^\dagger\left(\Phi^\dagger(t)\right)^{-1} = -A^\dagger\left(\Phi^\dagger(t)\right)^{-1}.\end{aligned}$$

Thus $\left(\Phi^\dagger(t)\right)^{-1}$ is a fundamental matrix of

$$y'(t) = -A^\dagger y(t).$$

Example. The differential system

$$y'(t) = -A^\dagger y(t),\ t \in I, \tag{3}$$

where $y = y(t)$ is a vector function, is called the adjoint system of (1). Note that

$$\frac{d}{dt}y^\dagger(t) = (y'(t))^\dagger = \left(-A^\dagger y(t)\right)^\dagger = -y^\dagger(t)A.$$

Thus

$$\begin{aligned}\frac{d}{dt}\left(y^\dagger(t)x(t)\right) &= \left(\frac{d}{dt}y^\dagger(t)\right)x(t) + y^\dagger(t)x'(t) = -y^\dagger(t)Ax(t) + y^\dagger(t)x'(t) \\ &= y^\dagger(t)\left\{x'(t) - Ax(x)\right\}.\end{aligned}$$

As a consequence, if $x(t)$ is a solution of (1), we will have

$$y^\dagger(t)x(t) = \alpha,\ t \in I,$$

for some constant α. Similarly, the matrix differential system

$$Y'(t) = -A^\dagger Y(t)$$

is called the adjoint system of (2). By similar reasoning, we see further that

$$\frac{d}{dt}\left(Y^\dagger(t)X(t)\right) = Y^\dagger(t)\left(X'(t) - AX(t)\right).$$

Hence if $\Phi(t)$ is a fundamental matrix for (1) and $\Psi(t)$ a fundamental matrix for (3), then

$$\Psi^\dagger(t)\Phi(t) = B,\, t \in I,$$

for some constant matrix B. Since Ψ and Φ are nonsingular, B is also nonsingular, and hence

$$\Psi(t) = \left(B\Phi^{-1}(t)\right)^\dagger = \left(\Phi^\dagger(t)\right)^{-1} B^\dagger,$$

and

$$\left(\Phi^\dagger(t)\right)^{-1} = \Psi(t)\left(B^\dagger\right)^{-1} = \left(\Phi^\dagger(t)\right)^{-1} B^\dagger \left(B^\dagger\right)^{-1}$$

are fundamental matrices for (3). The last statement has already been seen in (xiii).