二上 (葉)

線性代數 Final Exam

1. [10%] Find the rank of the following matrices.

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

2. [6%] Prove that E^t is an elementary matrix if E is.

3. [6%] Prove that for any m x n matrix A, A is the zero matrix if rank(A) = 0.

4. [10%] Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

5. (10%) Label the following statements as true or false.
   - i) If E is an elementary matrix, then det(E) = ± 1.
   - ii) For any A, B $\in M_{nxn}(F)$, det(AB) = det(A)*det(B).
   - iii) A matrix M $\in M_{nxn}(F)$ has rank n if and only if det(M) $\neq$ 0.
   - iv) For any A $\in M_{nxn}(F)$, det(A^t) = - det(A).
   - v) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

8-i

3+i 5

6. [12%] Provide reasons why each of the following is not an inner product on the given vector spaces.
   i) $\langle (a,b),\ (c,d)\rangle = ac - bd$ on R^2.
   ii) $\langle A, B\rangle$ = tr(A+B) on M_{2x2}(R).

8 $\quad = \begin{bmatrix} -1+i & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 3 2 3

7. [6%] Please show the reason for the statement S below.

Theorem 6.3. *Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in$ span(S), then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2} v_i.$$

Proof. Write $y = \sum_{i=1}^{k} a_i v_i$, where $a_1, a_2, \ldots, a_k \in F$. Then, for $1 \le j \le k$, we have

S

$$\langle y, v_j\rangle = \left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle = \underline{\sum_{i=1}^{k} a_i \langle v_i, v_j\rangle = a_j \langle v_j, v_j\rangle = a_j \|v_j\|^2}.$$

So $a_j = \dfrac{\langle y, v_j\rangle}{\|v_j\|^2}$, and the result follows. ∎

8. [6%] In R^4, let w1 = (1,0,1,0), w2=(1,1,1,1) and w3 = (0,1,1,1). Then {w1, w2, w3} is linearly independent. Use Gram-Schmidt process to compute the orthogonal vectors v1, v2, v3, and then normalize these vectors to obtain an orthonormal set.

9. [4%] For the following inner product spaces V and linear operations T on V, evaluate T* at the given vector in V.

   V = C^2, T(z_1, z_2) = ($2z_1 + iz_2$, $(1-i)z_1$), x = (3-i, 2+2i).

10. [10%] Please show the reason for the statement S.

Theorem 6.8

Theorem 6.8. *Let* V *be a finite-dimensional inner product space over* F, *and let* $\mathsf{g}\colon \mathsf{V} \to F$ *be a linear transformation. Then there exists a unique vector* $y \in \mathsf{V}$ *such that* $\mathsf{g}(x) = \langle x, y \rangle$ *for all* $x \in \mathsf{V}$.

We need to define y and show
(1) $g(x) = \langle x, y \rangle$ for all $x \in V$, and
(2) y is unique.
How to define y ?
Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V.
Let $y = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, where $a_1, a_2, \ldots, a_n \in F$.

Furthermore, for $1 \le j \le n$ we have

S

$$\mathsf{h}(v_j) = \langle v_j, y \rangle = \left\langle v_j, \sum_{i=1}^{n} \overline{\mathsf{g}(v_i)} v_i \right\rangle = \sum_{i=1}^{n} \mathsf{g}(v_i) \langle v_j, v_i \rangle$$
$$= \sum_{i=1}^{n} \mathsf{g}(v_i) \delta_{ji} = \mathsf{g}(v_j).$$

Since g and h both agree on β, we have that $\mathsf{g} = \mathsf{h}$ by the corollary to Theorem 2.6 (p. 73).

11. [5%] Please show the reason for the statement S below.

Theorem 6.9. *Let* V *be a finite-dimensional inner product space, and let* T *be a linear operator on* V. *Then there exists a unique function* $\mathsf{T}^*: \mathsf{V} \to \mathsf{V}$ *such that* $\langle \mathsf{T}(x), y\rangle = \langle x, \mathsf{T}^*(y)\rangle$ *for all* $x, y \in \mathsf{V}$. *Furthermore,* T^* *is linear.*

Proof. Let $y \in \mathsf{V}$. Define $\mathsf{g}: \mathsf{V} \to F$ by $\mathsf{g}(x) = \langle \mathsf{T}(x), y\rangle$ for all $x \in \mathsf{V}$. We first show that g is linear. Let $x_1, x_2 \in \mathsf{V}$ and $c \in F$. Then

S

$$\begin{aligned}\mathsf{g}(cx_1 + x_2) = \langle \mathsf{T}(cx_1 + x_2), y\rangle &= \underline{\langle c\mathsf{T}(x_1) + \mathsf{T}(x_2), y\rangle} \\ &\underline{= c\langle \mathsf{T}(x_1), y\rangle + \langle \mathsf{T}(x_2), y\rangle} = c\mathsf{g}(x_1) + \mathsf{g}(x_2).\end{aligned}$$

Hence g is linear.

We now apply Theorem 6.8 to obtain a unique vector $y' \in \mathsf{V}$ such that $\mathsf{g}(x) = \langle x, y'\rangle$; that is, $\langle \mathsf{T}(x), y\rangle = \langle x, y'\rangle$ for all $x \in \mathsf{V}$. Defining $\mathsf{T}^*: \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}^*(y) = y'$, we have $\langle \mathsf{T}(x), y\rangle = \langle x, \mathsf{T}^*(y)\rangle$.

12. [10%] Please show the reason for the statements S_1, S_2 below.

Lemma. *Let* T *be a linear operator on a finite-dimensional inner product space* V. *If* T *has an eigenvector, then so does* T^*.

Proof. Suppose that v is an eigenvector of T with corresponding eigenvalue λ. Then for any $x \in \mathsf{V}$,

S_1

$$0 = \langle 0, x\rangle = \langle (\mathsf{T} - \lambda\mathsf{I})(v), x\rangle = \underline{\langle v, (\mathsf{T} - \lambda\mathsf{I})^*(x)\rangle = \langle v, (\mathsf{T}^* - \overline{\lambda}\mathsf{I})(x)\rangle},$$

S_2

and hence v is orthogonal to the range of $\mathsf{T}^* - \overline{\lambda}\mathsf{I}$. So $\underline{\mathsf{T}^* - \overline{\lambda}\mathsf{I} \text{ is not onto}}$ and hence is not one-to-one. Thus $\mathsf{T}^* - \overline{\lambda}\mathsf{I}$ has a nonzero null space, and any nonzero vector in this null space is an eigenvector of T^* with corresponding eigenvalue $\overline{\lambda}$. ∎

13. [5%] Please show the reason for the statement S below.

Lemma. Let T be a self-adjoint operator on a finite-dimensional inner product space V. *Then*

(a) *Every eigenvalue of* T *is real.*

(b) *Suppose that* V *is a real inner product space. Then the characteristic polynomial of* T *splits.*

Proof. (a) Suppose that $\mathsf{T}(x) = \lambda x$ for $x \neq \mathit{0}$. Because a self-adjoint operator is also normal, we can apply Theorem 6.15(c) to obtain

$$\lambda x = \mathsf{T}(x) = \mathsf{T}^*(x) = \bar{\lambda} x.$$

So $\lambda = \bar{\lambda}$; that is, λ is real.

S (b) Let $n = \dim(\mathsf{V})$, β be an orthonormal basis for V, and $A = [\mathsf{T}]_\beta$. <u>Then A is self-adjoint.</u> Let T_A be the linear operator on C^n defined by $\mathsf{T}_A(x) = Ax$ for all $x \in \mathsf{C}^n$. Note that T_A is self-adjoint because $[\mathsf{T}_A]_\gamma = A$, where γ is the standard ordered basis for C^n. By the fundamental theorem of algebra, the characteristic polynomial of T_A splits into factors of the form $t - \lambda$. Since each λ is real, the characteristic polynomial splits over R. : T_A has the same characteristic polynomial as A, which has the same characteristic polynomial as T. Therefore the characteristic polynomial of T splits. ∎