Problems for Exam 3

Name: ______________ **StuID:**______________

1. For each of the following statements, mark "O" if it is *true* and mark "×" if it is *false*.

(   )**(a)** There exists a unique $x_0 \in [0, 1]$ such that $sin(x_0) - 2x_0 = 0$.

(   )**(b)** A cubic spline interpolant S for a given set of points $[x_i, y_i]^t$, $0 \le i \le n$, with $x_0 < x_1 < \cdots < x_n$, can be regarded as a continuous function defined on $[x_0, x_n]$ whose derivative exists everywhere on (x_0, x_n).

(   )**(c)** Let $A \in R^{n\times n}$, if A is nonsingular and $A = LDU$, where L is unit lower-Δ, U is unit upper-Δ, and D is diagonal, then $\lambda(A) \; = \; \lambda(D)$.

(   )**(d)** Let $A \in R^{n\times n}$, and $\mathbf{b} \in R^n$, then $A\mathbf{x} = \mathbf{b}$ can always be solved by the Gauss-Seidel iterative procedure if A is positive definite.

(   )**(e)** Let $A \in R^{n\times n}$ be an orthogonal matrix, if λ is the eigenvalue of A, then $\lambda = 1 \; or \; -1$.

(   )**(f)** Every nonsingular matrix can be decomposed into LU, where L is unit lower-Δ and U is upper-Δ.

(   )**(g)** Let $Q \in R^{n\times n}$ be orthogonal, then $\|Q\|_2 = 1$ and $det(Q) = -1$.

(   )**(h)** Every positive definite matrix can be diagonalized so it must have distinct eigenvalues.

(   )**(i)** Let $A \in R^{n\times n}$ be a symmetric matrix, then A can be diagonalized by applying $n(n-1)/2$ Givens' rotations.

(   )**(j)** A singular value for a real symmetric matrix must be an eigenvalue.

2. Let $f(x) = ln(1+x), \quad x \in [0,1]$.

(a) Show that $f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \quad for \quad k = 1, 2, \cdots$.

(b) Show that the Taylor polynomial of degree n for $f(x) = ln(1+x)$ about $x = 0$ is

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n}$$

(c) Show that the error term $f(x) - P_n(x)$ is

$$E_n(x) = \frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}, \quad c \in [0,1]$$

(d) Evaluate $P_3(0.5)$ with the precision upto 10^{-4}.

3. Using the nodes $x_0 = 2, \ x_1 = 2.5, \ and \ x_2 = 4$ to find a quadratic interpolating polynomial $P_2(x)$ for $f(x) = \frac{1}{x}$.

(a) Write $P_2(x)$ in Lagrange form

(b) Write $P_2(x)$ in Newton form

(c) If $P_2(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$, what are $\alpha_2, \alpha_1, \alpha_0$, respectively?

4. Let $f(x) = \sqrt{x}$ for $x \geq 0$. Do numerical integration for $\int_0^4 f(x)dx$, based on the five nodes $x = 0, 1, 2, 3, 4$, to the precision within 10^{-4} by using

(a) trapezoidal rule

(b) Simpson's $\frac{1}{3}$ rule

- Use $\sqrt{2} = 1.4142, \ \sqrt{3} = 1.7321, \ \sqrt{5} = 2.2361$.

5. For $x \in [-1,1]$, define the Chebyshev polynomial of degree n by

$$T_n(x) = \cos(n\cos^{-1}x) \;\; \forall \;\; n \geq 0$$

Denote the inner product by $\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx$

(a) Write down $T_n(x)$ in the polynomial format for $0 \leq n \leq 4$.

(b) Show that $\langle T_n, T_m \rangle = 0$ if $n \neq m$.

(c) Compute $\langle T_0, T_0 \rangle$.

(d) Compute $\langle T_n, T_n \rangle$ for $n \geq 1$.

(e) Find all of the roots of $T_n(x) = 0$.

6. Define $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$.

(a) Find $\Gamma(5)$.

(b) Find $\Gamma(\frac{1}{2})$.

(c) Find $\Gamma(n)$, where n is a positive integer.

(d) Find $\Gamma(\frac{2n+1}{2})$, where n is a positive integer.

7. Define $f(x) = \sqrt{x+6} \;\; for \; x \in [0,4]$.

(a) Show that f is contractive.

(b) What is the fixed point for f?

8. Let $f : [a,b] \rightarrow [a,b]$ and given $n+1$ distinct points $a = x_0 < x_1 < \cdots < x_n = b$. In no more than **50** words, discribe the difference among a cubic spline interpolation, Bezier curve fit, and a cubic B-spline fit to the above data.