Final Examination

Probability

June 17, 2015

1. (18%) Which of the following statements are FALSE? Give a brief explanation if your answer is "FALSE".

   (a) For random variables X and Y, $E(XY)^2 \geq E(X^2)E(Y^2)$.

   (b) Two dependent random variables may be uncorrelated.

   X (c) If $X_1, X_2, \ldots, X_n$ are independent normal random variables all with the same mean μ, and the same variance σ^2, then the sample mean is N($\cancel{\mu/n}$, $\sigma^2 n$).
   $N(\mu, \frac{\sigma^2}{n})$

   (d) $E(X|Y)$ is a constant, where X and Y are two random variables.

   (e) Let $X_1, X_2, \ldots, X_n$ be independent standard normal random variables. Then $X = X_1^2 + X_2^2 + \cdots + X_n^2$ is gamma with parameters $(n/2, 1/2)$.

   (f) Let $X_1, X_2, \ldots, X_n$ be independent geometric random variables each with parameter p, then $X_1 + X_2 + \ldots + X_n$ is negative binomial with parameters (n, np).

2. (12%) A coin is tossed n times ($n > 4$). What is the expected number of exactly three consecutive heads?

   Hint: Let E_1 be the event that the first three outcomes are heads and the fourth outcome is tails. For $2 \leq i \leq n-3$, let E_i be the event that the outcome $(i-1)$ is tails, the outcomes i, $(i+1)$, and $(i+2)$ are heads, and the outcome $(i+3)$ is tails. Let E_{n-2} be the event that the outcome $(n-3)$ is tails, and the last three outcomes are heads. Let

   $$X_i = \begin{cases} 1 & if\ E_i\ occurs \\ 0 & otherwise \end{cases}.$$

   Then calculate the expected value of an appropriate sum of X_i's.

3. (12%)

   (a) (6%) Find $M_X(t)$, the moment-generating function of a Poisson random variable X with parameter λ.

   (b) (6%) Use $M_X(t)$ to find $E(X)$ and $Var(X)$.

4. (12%) The distributions of the grades of the students of probability and calculus at a certain university are $N(65, 418)$ and $N(72, 448)$, respectively. Dr. Olwell teaches a calculus section with 28 and a probability section with 22 students. What is the probability that the difference between the averages of the final grades of these two classes is at least 2?

5. (10%) Let X be a random number from the interval (0, 1) and $Y = X^2$. The probability density function of X is

$$f(x) = \begin{cases} 1 & if\ 0 < x < 1\ occurs \\ 0 & otherwise \end{cases}$$

$\rho(X, Y) = ?$

6. (12%) Let B be an event associated with an experiment and X be a discrete random variable with possible set of values A. Show that $P(B) =$

$$\sum_{x \in A} P(B \mid X = x) P(X = x)$$

7. (12%) Suppose that, with equal probabilities, the value of a specific stock on any trading day increases 30% or decreases 25%, independent of the fluctuations of the stock value on the past and future trading days. Let

$$r_i = \begin{cases} 0.30 & \text{with probability } 1/2 \\ -0.25 & \text{with probability } 1/2. \end{cases}$$

Then r_i is the rate of return on the i^{th} trading day. Schiller, a very conservative investor, invests A dollars in this stock thinking that $E(r_i) = 0.025$ implies that, on average, every day the value of his investment increases by 2.5% of its value on the previous day. However, Schiller is mistaken, and if he holds his shares of this stock long enough, eventually he will lose the entire value of his investment in this stock.

Using the central limit theorem, find n, the number of trading days after which, with probability 0.99, the value of the stock decreases to 10% of its original value.

Hint: Let $Y_i = 1 + r_i$, and using: $ln0.10 = -2.303$, $E(ln\ Y_i) = -0.127$, $\sigma_{lnYi} = 0.244$.

8. (12%) A system consists of n components whose lifetimes form an independent sequence of random variables. Suppose that the system functions as long as at least one of its components functions. Let $F_1, F_2, \ldots, F_n$ be the distribution functions of the lifetimes of the components of the system. In terms of $F_1, F_2, \ldots, F_n$, find the survival function of the lifetime of the system.

9. (12%) In n independent Bernoulli trials, each with probability of success p, let X be the number of successes and Y the number of failures. Calculate $E(XY)$ and $\mathrm{Cov}(X, Y)$. Are X and Y uncorrelated?

z_0	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936

Definition *A discrete random variable X with possible values 0, 1, 2, 3, ... is called* ***Poisson*** *with parameter λ, $\lambda > 0$, if*

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}, \qquad i = 0, 1, 2, 3, \ldots.$$