二上 答

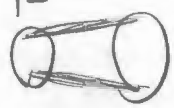
Linear Algebra Exam 2

1. (10%) Find the basis for both N(T) and R(T), then compute the nullity and rank of T, and determine whether T is one-to-one or onto.

   P_2 $P3$

   T : P2(R) → P3(R) defined by T(f(x)) = xf(x) + f'(x)

   $6x^2+5x+5$



   $6x+5x$

2. (5%) T : M2x2(R) → P2(R) by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

   Let $\beta = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ and $\gamma = \{1, x, x^2\}$

   Compute $[T]_\beta^\gamma$.

3. (10%) Let g(x) = 2x. Let T: $P_2(R) \to P_2(R)$ and U: $P_2(R) \to R^3$ be the linear transformations respectively defined by

   β β γ

   $c=1$

   $T(f(x)) = f'(x)g(x)+3f(x)$ and $U(a+bx+cx^2) = (a+2b, c, a-b)$

   Let β and γ be the standard ordered bases of $p_2(R)$ and R^3, respectively.

   (a) Compute $[U]_\beta^\gamma$, $[T]_\beta$

   (b) Let $h(x) = 5-x+x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$.

4. (5%) For each of the following pairs of ordered bases β and β' for $P_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates

   (a) $\beta = \{x^2,x,1\}$ and $\beta' = \{a_2x^2+a_1x+a_0, b_2x^2+b_1x+b_0, c_2x^2+c_1x+c_0\}$

   (b) $\beta = \{2x^2-x,3x^2+1,x^2\}$ and $\beta' = \{1,x,x^2\}$

5. (15%) For each of the following linear transformations T, determine whether T is invertible and justify your answer.

   (a) T: R2 → R3 defined by T(a1, a2) = (a1, a1+a2, a2)

   (b) T: R3 → R3 defined by T(a1, a2, a3) = (a1, a1+a2, a1+a2+a3)

   (c) T: M2x2(R) → M2x2(R) defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-b & d \\ a+c & b \end{pmatrix}$

6. (10%) Let V and W be vector spaces, and let T : V→W be linear. Prove that if T is invertible if and only if T is one-to-one and onto.

7. (15%) Please show the reasons for the statements S1, S2, and S3 below.

Theorem 2.3 (Dimension Theorem). *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. If* V *is finite-dimensional, then*

$$\operatorname{nullity}(\mathsf{T}) + \operatorname{rank}(\mathsf{T}) = \dim(\mathsf{V}).$$

Proof. Suppose that $\dim(\mathsf{V}) = n$, $\dim(\mathsf{N}(\mathsf{T})) = k$, and $\{v_1, v_2, \ldots, v_k\}$ is a basis for $\mathsf{N}(\mathsf{T})$. By the corollary to Theorem 1.11 (p. 51), we may extend $\{v_1, v_2, \ldots, v_k\}$ to a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V. We claim that $S = \{\mathsf{T}(v_{k+1}), \mathsf{T}(v_{k+2}), \ldots, \mathsf{T}(v_n)\}$ is a basis for $\mathsf{R}(\mathsf{T})$.

First we prove that S generates $\mathsf{R}(\mathsf{T})$. Using Theorem 2.2 and the fact that $\mathsf{T}(v_i) = 0$ for $1 \le i \le k$, we have

S1

$$\begin{aligned}\mathsf{R}(\mathsf{T}) &= \underline{\operatorname{span}(\{\mathsf{T}(v_1), \mathsf{T}(v_2), \ldots, \mathsf{T}(v_n)\}}\\ &\underline{= \operatorname{span}(\{\mathsf{T}(v_{k+1}), \mathsf{T}(v_{k+2}), \ldots, \mathsf{T}(v_n)\}} = \operatorname{span}(S).\end{aligned}$$

Now we prove that S is linearly independent. Suppose that

$$\sum_{i=k+1}^{n} b_i \mathsf{T}(v_i) = 0 \quad \text{for } b_{k+1}, b_{k+2}, \ldots, b_n \in F.$$

<u>Using the fact that T is linear, we have</u>

S2

$$\underline{\mathsf{T}\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0.}$$

So

$$\sum_{i=k+1}^{n} b_i v_i \in \mathsf{N}(\mathsf{T}).$$

<u>Hence there exist $c_1, c_2, \ldots, c_k \in F$ such that</u>

S3

$$\underline{\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i} \quad \text{or} \quad \sum_{i=1}^{k} (-c_i) v_i + \sum_{i=k+1}^{n} b_i v_i = 0.$$

8. (5%) Please show the reason for the statement below.

Theorem 2.9. *Let V, W, and Z be vector spaces over the same field F, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}: \mathsf{W} \to \mathsf{Z}$ be linear. Then $\mathsf{UT}: \mathsf{V} \to \mathsf{Z}$ is linear.*

Proof. Let $x, y \in \mathsf{V}$ and $a \in F$. Then

$$\begin{aligned}\mathsf{UT}(ax+y) = \mathsf{U}(\mathsf{T}(ax+y)) &= \underline{\mathsf{U}(a\mathsf{T}(x)+\mathsf{T}(y))} \\ &= \underline{a\mathsf{U}(\mathsf{T}(x)) + \mathsf{U}(\mathsf{T}(y))} = a(\mathsf{UT})(x) + \mathsf{UT}(y).\end{aligned}$$ ∎

9. (5%) Please show the reason for the statement below.

Theorem 2.23

Theorem 2.23. *Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then*

$$[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q.$$

Proof. Let I be the identity transformation on V. Then $\mathsf{T} = \mathsf{IT} = \mathsf{TI}$; hence, by Theorem 2.11 (p. 88),

$$Q[\mathsf{T}]_{\beta'} = [\mathsf{I}]_{\beta'}^{\beta}[\mathsf{T}]_{\beta'}^{\beta'} = [\mathsf{IT}]_{\beta'}^{\beta} = [\mathsf{TI}]_{\beta'}^{\beta} = [\mathsf{T}]_{\beta}^{\beta}[\mathsf{I}]_{\beta'}^{\beta} = [\mathsf{T}]_\beta Q.$$

<u>Therefore $[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q$.</u>

10. (10%) Please show the reasons for the statements S1 and S2 below.

Theorem 2.18 (1/2)

Theorem 2.18. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be linear. Then T is invertible if and only if $[\mathsf{T}]_\beta^\gamma$ is invertible. Furthermore, $[\mathsf{T}^{-1}]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^{-1}$.*

Proof. Suppose that T is invertible. By the lemma, we have $\dim(\mathsf{V}) = \dim(\mathsf{W})$. Let $n = \dim(\mathsf{V})$. So $[\mathsf{T}]_\beta^\gamma$ is an $n \times n$ matrix. Now $\mathsf{T}^{-1}: \mathsf{W} \to \mathsf{V}$ satisfies $\mathsf{TT}^{-1} = \mathsf{I_W}$ and $\mathsf{T}^{-1}\mathsf{T} = \mathsf{I_V}$.

Thus

S1

$$I_n = [\mathsf{I_V}]_\beta = \underline{[\mathsf{T}^{-1}\mathsf{T}]_\beta = [\mathsf{T}^{-1}]_\gamma^\beta[\mathsf{T}]_\beta^\gamma}.$$

Similarly, $[\mathsf{T}]_\beta^\gamma[\mathsf{T}^{-1}]_\gamma^\beta = I_n$. So $[\mathsf{T}]_\beta^\gamma$ is invertible and $\left([\mathsf{T}]_\beta^\gamma\right)^{-1} = [\mathsf{T}^{-1}]_\gamma^\beta$.

Theorem 2.18 (2/2)

Now suppose that $A = [T]_\beta^\gamma$ is invertible. Then there exists an $n \times n$ matrix B such that $AB = BA = I_n$. By Theorem 2.6 (p. 72), there exists $U \in \mathcal{L}(W, V)$ such that

(1,1) (n,1)
(1,n) (n,n).

$$U(w_j) = \sum_{i=1}^{n} B_{ij} v_i \quad \text{for } j = 1, 2, \ldots, n,$$

S2

where $\gamma = \{w_1, w_2, \ldots, w_n\}$ and $\beta = \{v_1, v_2, \ldots, v_n\}$. It follows that $[U]_\gamma^\beta = B$.

11. (10%) Please show the reasons for the statements S1 and S2 below.

Theorem 2.7. Let V and W be vectors spaces over a field F, and let T, U : V →W be linear. For all a ∈ F, aT + U is linear.

Proof. Let x, y ∈ V and c ∈ F. Then

S1

$$\begin{aligned}
(aT + U)(cx + y) &= aT(cx + y) + U(cx + y) \\
&= a[T(cx + y)] + cU(x) + U(y) \\
&= a[cT(x) + T(y)] + cU(x) + U(y) \\
&= acT(x) + cU(x) + aT(y) + U(y) \\
&= c(aT + U)(x) + (aT + U)(y)
\end{aligned}$$

Proof. Let x, y ∈ V and c ∈ F. Then

S2

$$\begin{aligned}
(aT + U)(cx + y) &= aT(cx + y) + U(cx + y) \\
&= a[T(cx + y)] + cU(x) + U(y) \\
&= a[cT(x) + T(y)] + cU(x) + U(y) \\
&= acT(x) + cU(x) + aT(y) + U(y) \\
&= c(aT + U)(x) + (aT + U)(y)
\end{aligned}$$

1.

$N(T)$ is the subspace of all set that make $T(x)=0$. -2

Basis $N(T)$?

$N(T) = \{ f(x) = 0 \ :\ f(x) \in P^2 \}$. nullity $(T) = 0$.

令 $\beta = \{1, x, x^2\}$ is a basis for $P_2(R)$.

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1), T(x), T(x^2)\})$

$= \text{span}(\{x,\ x^2+1,\ x^3+2x\})$ rank $(T) = 3$.

~~EPTO~~

T is 1-1. for $\forall x_1, x_2$, $T(x_1) = T(x_2) \Rightarrow x_1 = x_2$.

and $N(T) = \{0\}$.

but not onto. we can't find $\forall y \in P^3$ $\exists x \in P^2$ $T(x) = y$. ($xf(x) + f(x)$ can't represent any vector in P^3).

2.

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 0x^2$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + 0x + 1x^2$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 + 0x + 0x^2$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 2x + 0x^2$

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3.

(a).

$$U(1) = (1,0,1)$$
$$U(x) = (2,0,-1) \Rightarrow [U]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$
$$U(x^2) = (0,1,0)$$

$$T(1) = 3 = 3 + 0x + 0x^2$$
$$T(x) = 2x + 3x = 0 + 5x + 0x^2 \Rightarrow [T]_\beta = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$
$$T(x^2) = 4x^2 + 3x^2 = 0 + 0x + 7x^2$$

(b)

$$[h(x)]_\beta = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

$$[U(h(x))]_\gamma \underset{\text{(by Thm 2.14)}}{=} [U]_\beta^\gamma \cdot [h(x)]_\beta = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$$

4.

(a) $a_2x^2 + a_1x + a_0 = a_2(x^2) + a_1(x) + a_0(1)$.

$b_2x^2 + b_1x + b_0 = b_2(x^2) + b_1(x) + b_0(1)$.

$c_2x^2 + c_1x + c_0 = c_2(x^2) + c_1(x) + c_0(1)$.

$$A = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix}$$

(b).

$1 = a(2x^2-x) + b(3x^2+1) + cx^2$

$= 0(2x^2-x) + 1(3x^2+1) - 3x^2$

$$\begin{cases} 2a+3b+c=0 \\ -a=0. \\ b=1. \end{cases}$$

$\Rightarrow c=-3$

$x = a(2x^2-x) + b(3x^2+1) + cx^2$

$= -1(2x^2-x) + 0(3x^2+1) + 2x^2$

$$\begin{cases} 2a+3b+c=0 \\ -a=1 \\ b=0 \end{cases}$$

$\Rightarrow c=2$

$x^2 = a(2x^2-x) + b(3x^2+1) + cx^2$

$= 0(2x^2-x) + 0(3x^2+1) + x^2$

$$\begin{cases} 2a+3b+c=1 \\ -a=0 \\ b=0. \end{cases}$$

$\Rightarrow c=1$

$$A: \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

5.

(a) not invertible, They have different dimension

$\because \dim(R2) < \dim(R3). \Rightarrow$ is not on-to.

for example, $(0,1,0)$ in R3 $\Rightarrow \begin{cases} a_1=0 \\ a_1+a_2=1 \\ a_2=0 \end{cases} \Rightarrow$ 無解.

(b) Yes, when $T(a_1,a_2,a_3)=0$. $a_1=a_2=a_3=0$.

$\Rightarrow T$ is 1-1.

$\Leftrightarrow N(T)=\{0\}$.

$\Leftrightarrow \text{rank}(T)=\dim(V), (\dim(V)=\dim(W).)$

$\Leftrightarrow T$ is invertible.

(c) Yes. when $T\begin{pmatrix} a & b \\ c & d \end{pmatrix}=0$. $a=b=c=d=0$.

$\Rightarrow T$ is 1-1

$\Leftrightarrow N(T)=\{0\} \Leftrightarrow \text{rank}(T)=\dim(V), (\dim(V)=\dim(W))$

$\Leftrightarrow T$ is invertible.

6.

Suppose T is invertible, and define $U = T^{-1}$

there exist $y_1, y_2 \in W$.

then.

$$U(T(y_1)) = U(T(y_2))$$

$$\begin{array}{ccc} \| & & \| \\ UT(y_1) & & \| \\ \| & & \| \\ y_1 & = & y_2 \end{array} \quad (1-1)$$

$\forall y \in W$. $\exists x \in V$. $T(x) = y$.

let $x = U(y)$

$T(U(y)) = I_W(y) = y$. (onto).

Suppose T is 1-1 and onto.

$\because$ 1-1

$\forall x \in V$. $UT(x) = U(T(x_1)) = U(y) = x$.

$\Rightarrow UT = I_V$

$\because$ onto.

$\forall y \in W$. $TU(y) = T(U(y_1)) = T(x) = y$.

$\Rightarrow TU = I_W$

$\Rightarrow U = T^{-1}$. $\therefore T$ is invertible.

7、

(S1) if S is a basis for $R(T)$.

it should be 1. generates $R(T)$,

2. linearly independent.

and $R(T) = \text{span}(\{T(v_1), T(v_2) \cdots T(v_n)\})$ (for $T(v_i) = 0$. for $1 \le i \le k$)

$= \text{span}(\{T(v_{k+1}), \cdots T(v_n)\}) = \text{span}(S)$

Next time please write clearly

$\Rightarrow S$ generates $R(T)$.

∵ By theorem 2.2

(S2).

We want to prove S is LI.

$\Rightarrow$. $b_i = 0$. for $k+1 \le i \le n$

$\Rightarrow b_{k+1} v_{k+1} + b_{k+2} v_{k+2} + \cdots + b_n v_n = 0$.

So we use the fact that T is linear, and have

-2

$T(\sum_{i=k+1}^{n} b_i v_i) = 0$. to prove. $b_i = 0$. for. $k+1 \le i \le n$.

(there is a condition that $T(0) = 0$.)

this is the answer.

(S3).

because β is LI.

prove is By S_3

$\Rightarrow a_1 v_1 + a_2 v_2 + \cdots a_k v_k + a_{k+1} v_{k+1} + \cdots a_n v_n = 0$. for $a_1 = a_2 = \cdots = a_n = 0$

$\sum_{i=k+1}^{n} b_i v_i = 0 = \sum_{i=1}^{k} c_i v_i$ (∵ $\{v_1, \cdots v_k\}$ is a basis of $N(T)$. we can find $c_1 \sim c_k \in F$. st. $\sum_{i=1}^{k} c_i v_i = 0$)

$\Rightarrow \sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$

$\Rightarrow -c_1 v_1 - c_2 v_2 - \cdots - c_k v_k + b_{k+1} v_{k+1} + \cdots + b_n v_n = 0$.

$= a_1 v_1 + \cdots + a_k v_k + \cdots + a_n v_n = 0$.

$\Rightarrow$ we know that $b_i = 0$. $k+1 \le i \le n$.

8.

$U(aT(x)+T(y))$

(for U is linear, $U(x+y)=U(x)+U(y)$. $U(cx)=cU(x)$)

$= U(aT(x)) + U(T(y))$

$= aU(T(x)) + U(T(y))$.

9. -1

$Q[T]_{\beta'} = [T]_\beta Q$

$(Q^{-1})Q[T]_{\beta'} = (Q^{-1})[T]_\beta Q$

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

How do you know Q is invertible

10.

(S1)

$I_n = [I_V]_\beta$

(for $I_V = T^{-1}T$)

$= [T^{-1}T]_\beta$

(By Thm 2.17 $\Rightarrow T^{-1}$ is linear)

(By Thm 2.11)

$= [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$

(S2) -5 it want to show that $B=[U]_\gamma^\beta$.

$v_i = A_{ji}$ $(i=1,\cdots,n)$

$U(w_j)=\sum_{i=1}^{n}(B_{ij}\cdot A_{ji})$ for $j=1,\cdots,n$.

為何可以乘上A_{ji}?

$[U]_\gamma^\beta = B$

11.

(S1)

$(aT+U)(cx+y)$, (for. T, U are linearly)

$= aT(cx+y)+U(cx+y)$

(S2) for. U is linearly.

$aT(cx+y)+U(cx+y)$

?

$= a[T(cx+y)] + cU(x)+U(y)$.

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