First Midterm Examination, Advanced Calculus I ,10/31/2006

Total 110 pts.

1. (10%)(i) Let S be an infinite set. State the definition that S is countable.
   (ii) Let Q be the set of rational numbers. Show that Q is countable.
2. (20%) (i) Let $S \subseteq R$. State the definition of $\inf S$.
   (ii) Let $S \subseteq R$ be bounded below and d be a lower bound of S. Show that d$=\inf S \Leftrightarrow$ for any $\varepsilon > 0$ there exists $x \in S$ such that $d+\varepsilon > x$.
   (iii) Let $\{a_k\}$ be a bounded below and monotone decreasing sequence in R.
   Show that $\lim_{k\to\infty} a_k = \inf S, where\ S = \{a_k\}$.
3. (10%) (i) Let $\{a_n\}$ be a sequence in R. State the definitions of $\limsup a_n$ and $\liminf a_n$.
   (ii) Find $\limsup a_n$ and $\liminf a_n$ if a_n is given by
   (a) $(1+\frac{1}{\mathrm{n}})\cos n\pi$ $\qquad$ $(b)(-1)^n n$

4.(10%) (i) State Cauchy-Schwartz Inequality.
   (ii) Let $x, y \in R^n$. Use Cauchy-Schwartz Inequality to prove triangular Inequality $\| x+y \| \le \| x \| + \| y \|$.

5.(15%) (i) Let $A \subseteq R^n$. State the definition that A is an open set in R^n.
   (ii) Show that open ball $B(a,r)=\{x \in R^n : \| x-a \| < r\}$ is an open set.
   (iii) Show that finite intersection of open sets is open.

6.(15%) Determine all accumulation points of the following sets and decide whether the sets are open or closed (or neither).
   (a) Z, the set of integers,
   (b) S=$\{\frac{1}{\mathrm{n}}+\frac{1}{m} : m,n=1,2,3,....\}$
   (c) S=$\{(\mathrm{x,y}):\mathrm{x} \ge 0\}$

7.(10%) Let $A \subseteq R^n$. Show that if for any sequence $\{x_k\} \subseteq A, x_k \to x$ we have $x \in A$, then A is a closed set.

8.(10%) Let $A, B \subseteq R^n$. Show that
   (i) $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$
   $(ii) A \cap Cl(B) \subseteq Cl(A \cap B)$ if A is open

9.(10%) Let $T : R^n \to R^n$ be a contraction map, i.e.,
   $\forall x, y \in R^n, \| Tx - Ty \| \le \theta \| x-y \|$, for some $0 < \theta < 1$.
   Show that there exists a unique fixed point x of $T, i.e. Tx = x$.