Engineering Math Final Examination

Class:____________
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1. (8%) Given the Gamma Function, $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$, show that $\Gamma(n+1)=n!$ for positive integers n. (hint: $\Gamma(1)=1$)

first show $\frac{\Gamma(n+1)}{\Gamma(n)} = \frac{\int_0^\infty x^n e^{-x}dx}{\int_0^\infty x^{n-1}e^{-x}dx}$

$te^{-t}dt = dv$

$\int \underbrace{t}_{u}\underbrace{e^{-t}dt}_{dv} = t(-e^{-t}) + \int e^{-t}dt$

$= -te^{-t} - e^{-t} - e^{-t}$

2. (8%) Assume that $\mathcal{L}\{f(t)\}=F(s)$ and $F(s)$ is differentiable for 2 times. Show that $\mathcal{L}\{t^2 f(t)\}=\frac{d^2}{ds^2}F(s)$.

$\int_s^\infty \underbrace{f(t)}_{u} \underbrace{e^{-t}dt}_{dv}$

$\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt$

first show $\mathcal{L}\{tf(t)\} = (-1)\frac{d}{ds}F(s)$

$= \int_0^\infty \left(\frac{d}{ds}e^{-st}\right)f(t)\,dt$

$\left[f(t)(-te^{-t}-e^{-t})\right]\Big|_s^\infty + \int (te^{-t}+e^{-t})f$

$-\int_s^\infty$

$\mathcal{L}\{f(t)\} - \int f(t)\quad dt$

e^{-ts}

$\int t^a\,dt = \frac{a!}{s^{a+1}}$

$= \int_0^\infty (-t)f(t)$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

3. (6%) Find $\mathcal{L}\left\{\int_0^t e^{-3x}\cos 2x\,dx\right\}$.

$= \frac{1}{s}F(s)$

$\mathcal{L}\{tf(t)\} = \int_0^\infty \underbrace{tf(t)}_{u}\underbrace{e^{-st}dt}_{dv}$

$= \frac{tf(t)}{-s}e^{-st} + \int[f(t)+tf'(t)]$

$\mathcal{L}\{t\} = \int_0^\infty \underbrace{t}_{u}\underbrace{e^{-t}dt}_{dv} = -te^{-t} + \int e^{-t}dt$

$= \frac{1}{s^2}$

$= -te^{-t} - e^{-t}$

$e^{-t(a+1)}$

$= e^{-t}(-t-1)$

$\int \cancel{e^{at}e^{-t}}\,dt$

$\mathcal{L}\{e^{t}\}$

$= \int e^{t}e^{-t}$

$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s}$ $\quad \mathcal{L}\{\int_0^t$

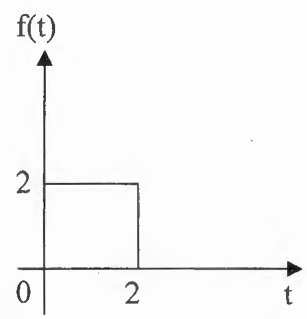
4. (10%) Given $\mathcal{L}\left\{\cos\sqrt[3]{t}\right\}=A$, find $\mathcal{L}\left\{\dfrac{\sin\sqrt[3]{t}}{-\sqrt[3]{t^2}}\right\}$. $= \mathcal{L}\left\{\dfrac{\sin t^{\frac{1}{3}}}{-t^{\frac{2}{3}}}\right\}$

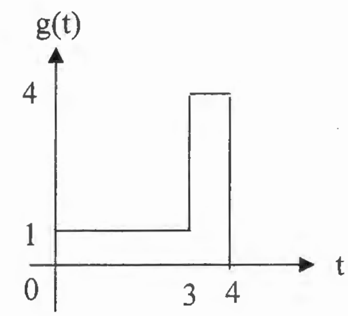
$\mathcal{L}\{\cos t^{\frac{1}{3}}\} = A$ $\qquad = (-1)\mathcal{L}\left\{\dfrac{t^{\frac{1}{3}}\sin t^{\frac{1}{3}}}{(t^{\frac{1}{3}})^3}\right\}$

$\Rightarrow \mathcal{L}\{\int_0^t \cos\tau^{\frac{1}{3}}\,d\tau\}$

$\sin\tau$

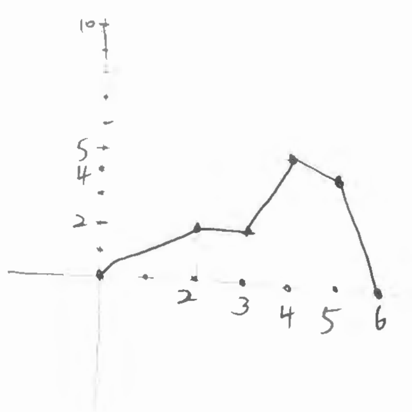
5. (15%) $f(t)$ and $g(t)$ are shown below, find $f(t)*g(t)$ by using the Laplace Transform method. Also show the waveform result of $f(t)*g(t)$. $\int_0^t f(\tau)g(t-\tau)\,d\tau$





$\int_0^\infty t e^{-st}$

$\cancel{2u(t-2)}$



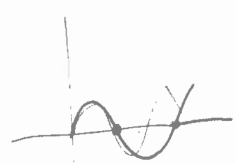
$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$

$\Rightarrow \mathcal{L}\{e^t\} = \int_0^\infty e^{t(1-s)}\,dt$

$= \frac{1}{1-s}\left(e^{t(1-s)}\right)\Big|_0^\infty$

$2\left[t + 3(t-3)u(t-3)\right. -4($

$= \frac{0-1}{1-s}$, for $s>1$



6. (6%) Draw the figures of u(t-π)sin(t), u(t+π)sin(t), and u(t-π)sin(t-π) for -2π≤ t ≤2π. Please label all the necessary values.

$\cos(t^{\frac{2}{3}})^{\frac{1}{2}}$ $\qquad \mathcal{L}\left\{\dfrac{\sin t^{\frac{1}{3}}}{t^{\frac{2}{3}}}\right.$

$\mathcal{L}\{t\}=\frac{2}{s^2}$ $\int te^{-t}dt = -te^{-t}+\int e^{-t}dt = -te^{-t}-e^{-t}$

$= \left(\frac{1-t}{e^t}\right)\Big|_s^\infty$

$\int_s^\infty f(t)e^{-t}dt$

$=$

7. (7%) Assume that $\mathcal{L}\{f(t)\}=F(s)$ and a > 0, show that $\mathcal{L}\{f(at)\}=\frac{1}{a}F(\frac{s}{a})$.

$\int_s^\infty f(t)e^{-t}dt$ $\int_0^\infty f(at)e^{-st}dt$

$\mathcal{L}\{\cancel{f(t)}\} = \int_s^\infty \cancel{f(t)}e^{-ts}dt$

$= \left[t\left(\frac{-1}{s}\right)e^{-st}\right]\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-ts}dt$

$= 0 + \frac{s}{s}e^{-s} + \frac{1}{s^2}\left(e^{-ts}\right)\Big|_0^\infty = 1+\frac{1}{s^2}$

8. (10%) Find the Maclaurin series of the general solution of $y''-2y'+x^2y=x$ with $a_0=y(0)$, $a_1=y'(0)$. Also show the recurrence relation and the first 5 terms (from a_0 to a_4) of solution.

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

$\sin t = \frac{1}{s^2+1}$

$\to \frac{1}{a}\frac{1}{\left(\frac{s}{a}\right)^2+1} = \frac{a}{s^2+a^2}$

$\frac{2\left(\frac{-1}{3}\right)}{\left(\frac{2}{3}\right)^3}$ $= \frac{-\cancel{2}\times 3^{3^2}}{\cancel{4}8\times\cancel{3}}$

$\frac{2}{3}$ $\frac{-\frac{2}{4}}{\frac{4}{9}} = \frac{}{9}$ $=\frac{-9}{2}$

9. (15%) $3x'-y=2t$, $x'+y'-y=0$, $x(0)=y(0)=0$, solve x(t), y(t) by using the Laplace Transform method.

$$\begin{bmatrix} 3s & -1 \\ s & s-1 \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{2}{s^2} \\ 0 \end{bmatrix}$$

$Y(s) = \frac{-\frac{2}{s}}{3s^2-3s+s}$ $\frac{-2}{s(3s-2)}$ $\frac{-2}{s(\ldots)}$

$\frac{\frac{2(s-1)}{s^2}}{s(3s-2)}$

$\frac{-1}{2}e^{\frac{2}{3}t}+t+\frac{1}{2}$

$\frac{-3}{2}e^{\frac{2}{3}t}+3t+\frac{3}{2}$

$t(1)e^{\frac{2}{3}t}, 1 + \frac{3}{2}e^{\frac{2}{3}t}$ ...

10. (15%) Find the Fourier series of the ~~non-periodic~~ function

$$f(x)=\begin{cases} x & -\pi \le x < 0 \\ -x & 0 \le x < \pi \end{cases}$$

$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$

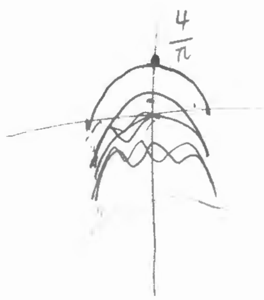
$\left(\frac{\sin nx}{n}\right)$

$\frac{\pi}{2} + \frac{4}{\pi} \cos x$

$\int_{0}^{\pi} -x\,dx \qquad \int_{-\pi}^{0} x$

$\left(\frac{x^2}{2}\right)\Big|_{-\pi}^{0}$

$\left(\frac{x^2}{2}\right)\Big|_{0}^{\pi} = \frac{\pi^2}{2}$



HAVE A NICE VACATION !!!