Solutions for Exam 1

1. **(a)** $sinh(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

(b) $sinh^{-1}(x) = \ln[x + \sqrt{x^2+1}] \ for \ \infty < x < \infty$

2. $1+\epsilon < e^{\epsilon} = 1 + \epsilon + \frac{\epsilon^2}{2!} + \cdots,$ then

$(1+\epsilon)^n - 1 < e^{n\epsilon} - 1 = \sum_{k=1}^{\infty} \frac{(n\epsilon)^k}{k!} = (n\epsilon)\sum_{k=0}^{\infty} \frac{(n\epsilon)^k}{(k+1)!} \leq (n\epsilon)\sum_{k=0}^{\infty} \frac{(n\epsilon)^k}{2^k} = (n\epsilon)\frac{1}{1-n\epsilon/2} < \frac{0.01}{1-0.01/2} = \frac{0.01}{0.995} < 0.01006$

3. Let $L = [l_{ij}], \quad M = [m_{ij}]$, with $l_{ii} = 1, \ m_{ii} = 1, \quad l_{ij} = 0, \ m_{ij} = 0 \ for \ i < j$, denote $LM = [a_{ij}]$, then

(a)

$$a_{ii} = \sum_{k=1}^{n} l_{ik}m_{ki} = l_{ii}m_{ii} + \sum_{k=1}^{i-1} l_{ik}m_{ki} + \sum_{k=i+1}^{n} l_{ik}m_{ki} = 1 + \sum_{k=1}^{i-1} l_{ik} \times 0 + \sum_{k=i+1}^{n} 0 \times m_{ki} = 1$$

$$a_{ij} = \sum_{k=1}^{n} l_{ik}m_{kj} = \sum_{k=1}^{j-1} l_{ik}m_{kj} + \sum_{k=j}^{n} l_{ik}m_{kj} = \sum_{k=1}^{j-1} l_{ik} \times 0 + \sum_{k=j}^{n} 0 \times m_{kj} = 0 \quad \forall \ i < j$$

(b)

$$L = \prod_{j=1}^{n-1} \prod_{i=j+1}^{n} [I + l_{ij}\mathbf{e}_i\mathbf{e}_j^t]$$

Then

$$L^{-1} = \prod_{j=1}^{n-1} \prod_{i=0}^{j-1} [I - l_{n-i,n-j}\mathbf{e}_{n-i}\mathbf{e}_{n-j}^t]$$

4. Since A is nonsingular, suppose $A = L_1U_1 = L_2U_2$, where L_1, L_2 are unit lower-Δ, and $U_1, \ U_2$ are upper-Δ with nonzero diagonal elements. then $L_1^{-1}L_2 = U_1U_2^{-1}$. $L_1^{-1}L_2$ is unit-Δ and $U_1U_2^{-1}$ is upper-Δ by (**3**), which implies that $L_1^{-1}L_2 = U_1U_2^{-1} = I$. Thus, $L_1 = L_2$ and $U_1 = U_2$.

5. $\|A\| < 1$, $(I+A)(I+A+A^2+\cdots) = I$ implies $\|(I+A)^{-1}\| = \|limit_{n\to\infty}\sum_{k=0}^{n} A^k\|$, then $\|(I+A)^{-1}\| \leq limit_{n\to\infty}\sum_{k=0}^{n} \|A\|^k \leq limit_{n\to\infty}\frac{1-\|A\|^{n+1}}{1-\|A\|}$. Therefore, $\|(I+A)^{-1}\| \leq (1-\|A\|)^{-1}$.

6. **(a)** Show that $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$. Suppose that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{y}$, then it has a solution $\mathbf{x}$ with $\|\mathbf{x}\|_\infty = 1$, w.l.o.g., let $x_k = 1$ such that $|x_j| \le |x_k|\ \forall\ 1 \le j \le n$, from $\sum_{j=1,j\neq k}^{n} a_{kj}x_j = -a_{kk}x_k$, we have $|a_{kk}| \le \sum_{j=1,j\neq k}^{n} |a_{kj}|$ which contradicts the assumption that A is diagonally dominant.

(b)

for $i = 1, n-1$

$\alpha \leftarrow t_{i+1,i}/t_{i,i};$

$t_{i+1,i} \leftarrow \alpha; \quad (l_{i+1,i})$

$t_{i+1,i+1} \leftarrow t_{i+1,i+1} - \alpha * t_{i,i+1}$

endfor

(c) $3(n-1)$ operations

7.

$$\begin{aligned}
\|A\|_1 &= max_{\|\mathbf{x}\|_1=1}\{\|A\mathbf{x}\|_1\} \\
&= max_{\|\mathbf{x}\|_1=1}\{\textstyle\sum_{i=1}^{m} |\sum_{j=1}^{n} a_{ij}x_j|\} \\
&\le max_{\|\mathbf{x}\|_1=1}\{\textstyle\sum_{j=1}^{n}\sum_{i=1}^{m} |a_{ij}||x_j|\} \\
&\le [\textstyle\sum_{i=1}^{m} |a_{ik}|]\,\{\sum_{j=1}^{n} |x_j|\} \quad for \quad some \quad k \\
&= [\textstyle\sum_{i=1}^{m} |a_{ik}|]
\end{aligned}$$

where $\sum_{i=1}^{m} |a_{ik}| = max_{1\le i\le n}\{\sum_{i=1}^{m} |a_{ij}|\}$.

Then

$$\|A\|_1 \le max_{1\le j\le n}\{\sum_{i=1}^{m} |a_{ij}|\}$$

Furthermore, choose $\mathbf{x} = \mathbf{e}_k$, then the equality holds.

8. Since $\mathbf{x}, \mathbf{y}$ are unit vectors, that is, $\langle \mathbf{x}, \mathbf{x}\rangle = 1$ and and $\langle \mathbf{y}, \mathbf{y}\rangle = 1$. Let $\mathbf{v} = \mathbf{x} - \mathbf{y}$, and define $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$. Then, $H = I - 2\mathbf{u}\mathbf{u}^t$ satisfies $H\mathbf{x} = \mathbf{y}$.

9. Let

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \quad and \quad J = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \end{bmatrix}$$

For Jacobi iteration method,

$$A = C - M = 2I - (J^t + J)$$

and

$$B_{Jacobi} = (2I)^{-1}(J^t + J) = \frac{1}{2}(J^t + J)$$

For Gauss-Seidel iteration method,

$$A = (D - L) - M = (2I - J^t) - J = 2(I - \frac{J^t}{2}) - J$$

and

$$B_{GS} = (2I - J^t)^{-1}J = \frac{1}{2}\left[I + \sum_{k=1}^{n-1} \frac{(J^t)^k}{2^k}\right] J$$

$$B_{Jacobi} = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{bmatrix}, \quad B_{GS} = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ 0 & \frac{1}{4} & \frac{1}{2} & & & \\ & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2^{n-1}} & & & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$