記分					
1	10	2	10		
3	10	4	10		
5	10	6	10		
7	10	8	10		
9		10			
11		12			
13		14			
15		16			
17		18			
19		20			
總分	100				

國立清華大學試卷

所系 資工

科目 工數

學號 104062229

姓名 毛翊宣

日期 12/2

1. $y''' - y'' = 2\cos x - 5e^x$

$m^3 - m^2 = 0$

$m^2(m-1) = 0$

$m = 0, 1$

$y_h = C_1 + C_2 x + C_3 e^x$

$y_p = A\cos x + B\sin x + Cxe^x$

$y_p' = -A\sin x + B\cos x + Ce^x + Cxe^x$

$y_p'' = -A\cos x - B\sin x + 2Ce^x + Cxe^x$

$y_p''' = A\sin x - B\cos x + 3Ce^x + Cxe^x$

$A\sin x - B\cos x + 3Ce^x + Cxe^x + A\cos x + B\sin x - 2Ce^x - Cxe^x = 2\cos x - 5e^x$

$(A+B)\sin x + (A-B)\cos x + Ce^x = 2\cos x - 5e^x$

$A+B = 0$

$A-B = 2$

$A = 1,\ B = -1,\ C = -5$

$y_p = \cos x - \sin x - 5xe^x$

$y = y_h + y_p = C_1 + C_2 x + C_3 e^x + \cos x - \sin x - 5xe^x$ #.

2. 令 $x = e^t$, $t = \ln x$, $dt = \frac{1}{x} dx$

$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 7\frac{dy}{dt} + 13y = 0$

$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0$

$m^2 + 6m + 13 = 0$

$m = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$

$y = C_1 e^{-3t}\cos 2t + C_2 e^{-3t}\sin 2t$

$y = C_1 x^{-3}\cos(2\ln x) + C_2 x^{-3}\sin(2\ln x)$ #.

3. $(2x+1)^2 y'' - 6(2x+1)y' + 16y = 20$

令 $u = 2x+1$, $du = 2dx$

$y' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 2\frac{dy}{du}$ $\quad y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\frac{dy}{du}\right) = 4\frac{d^2y}{du^2}$

$4u^2\frac{d^2y}{du^2} - 12u\frac{dy}{du} + 16y = 20$

$u^2\frac{d^2y}{du^2} - 3u\frac{dy}{du} + 4y = 5$

令 $u = e^t$ $\quad t = \ln u$, $dt = \frac{1}{u}du$

$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 3\frac{dy}{dt} + 4y = 5$

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 5$

$y_h = C_1 e^{2t} + C_2 t e^{2t}$

令 $y_p = A$, $y_p' = 0$, $y_p'' = 0$

$0 - 0 + 4A = 5$

$A = \frac{5}{4}$

$y = C_1 e^{2t} + C_2 t e^{2t} + \frac{5}{4}$

$y = C_1 u^2 + C_2 u^2 \ln u + \frac{5}{4}$

$y = C_1(2x+1)^2 + C_2(2x+1)^2 \ln|2x+1| + \frac{5}{4}$ #

4. y_3 is a solution of $y'' + P(x)y' + Q(x)y = 0$

By Albert's identity

$W(y_1, y_2) = y_1y_2' - y_1'y_2 = k_{12}e^{-\int P(x)dx}$

$W(y_1, y_3) = y_1y_3' - y_1'y_3 = k_{13}e^{-\int P(x)dx}$ — ①

$W(y_2, y_3) = y_2y_3' - y_2'y_3 = k_{23}e^{-\int P(x)dx}$ — ②

① × y_2 − ② × y_1

$y_1y_2y_3' - y_1'y_3y_2 - y_1y_2y_3' + y_1y_2'y_3 = (y_2k_{13} - y_1k_{23})e^{-\int P(x)dx}$

$(y_1y_2' - y_1'y_2)y_3 = (y_2k_{13} - y_1k_{23})e^{-\int P(x)dx}$

$y_3k_{12}e^{-\int P(x)dx} = (y_2k_{13} - y_1k_{23})e^{-\int P(x)dx}$

$y_3 = \underbrace{\frac{k_{13}}{k_{12}}}_{= C_2}y_2 - \underbrace{\frac{k_{23}}{k_{12}}}_{= C_1}y_1$

$y_3 = C_1y_1 + C_2y_2$ ⇒ every solution of this ODE is a linear combination of y_1 and y_2.

5、 $(x^2-x)y''-2xy'+2y=0$

$$y''-\frac{2}{x-1}y'+\frac{2}{x^2-x}y=0$$

令 $u=x$, $u'=1$ $u''=0$

$$0-\frac{2}{x-1}+\frac{2x}{x^2-x}=0$$

$\Rightarrow u=x$ is a solution of $y''-\frac{2}{x-1}y'+\frac{2}{x^2-x}y=0$

令 $y=u(x)v(x)$

$$v''+\frac{2+\frac{-2}{x-1}x}{x}v'=0$$

令 $t=v'$, $t'=v''$

$$t'+\left(\frac{2}{x}+\frac{-2}{x-1}\right)t=0$$

$$t\cdot e^{\int\left(\frac{2}{x}+\frac{-2}{x-1}\right)dx}=C_1$$

$$t\cdot\frac{x^2}{(x-1)^2}=C_1$$

$$t=C_1\cdot\frac{(x-1)^2}{x^2}$$

$$v=\int C_1\frac{x^2-2x+1}{x^2}dx$$

$$v=C_1\int 1-\frac{2}{x}+\frac{1}{x^2}dx$$

$$v=C_1\left(x-2\ln x-\frac{1}{x}\right)+C_2$$

$$y=C_1(x^2-2x\ln x-1)+C_2x$$ #

6、 $y''-4y'+53y=0$

$$m=\frac{4\pm\sqrt{-196}}{2}=\frac{4\pm 14i}{2}=2\pm 7i$$

$$y=C_1e^{2x}\cos 7x+C_2e^{2x}\sin 7x$$

$$y'=2C_1e^{2x}\cos 7x-7C_1e^{2x}\sin 7x+2C_2e^{2x}\sin 7x+7C_2e^{2x}\cos 7x$$

$$y(\pi)=-C_1e^{2\pi}=-3 \qquad C_1=\frac{3}{e^{2\pi}}$$

$$y'(\pi)=-2C_1e^{2\pi}-7C_2e^{2\pi}=3$$

$$-6-7C_2e^{2\pi}=3$$

$$-7C_2e^{2\pi}=9$$

$$C_2=\frac{-9}{7e^{2\pi}}$$

$$y=3e^{2x-2\pi}\cos 7x-\frac{9}{7}e^{2x-2\pi}\sin 7x$$ #

7. $y''+y=\tan x$

$m=\pm i$

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$y_h = C_1\cos x + C_2 \sin x$

$$y_p = \cos x \int \frac{-\tan x \cdot \sin x}{1}dx + \sin x \int \frac{\tan x \cdot \cos x}{1}dx$$

$$= \cos x \int -\frac{\sin^2 x}{\cos x}dx + \sin x \int \sin x\, dx$$

$$= -\cos x \int \frac{1-\cos^2 x}{\cos x}dx + \sin x\,(-\cos x)$$

$$= -\cos x \int (\sec x - \cos x)dx - \sin x \cos x$$

$$= -\cos x\,(\ln|\sec x+\tan x| - \sin x) - \sin x\cos x$$

$$= -\cos x \ln|\sec x + \tan x| + \cos x \sin x - \cos x \sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$ #

8.

① 令 $y_h = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ 且 $m_1 \neq m_2$, $m_1 = m$.

$$\begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = (m_2 - m_1)e^{(m_1+m_2)x}$$

$$y_p = e^{m_1 x}\int \frac{-Ce^{mx}\cdot e^{m_2 x}}{(m_2-m_1)e^{(m_1+m_2)x}}dx + e^{m_2 x}\int \frac{Ce^{mx}\cdot e^{m_1 x}}{(m_2-m_1)e^{(m_1+m_2)x}}dx$$

$$= e^{m_1 x}\int \frac{-C}{m_2-m_1}dx + e^{m_2 x}\int \frac{Ce^{(m_1-m_2)x}}{(m_2-m_1)}dx$$

$$= -\frac{C}{m_2-m_1}xe^{m_1 x} - \frac{C}{(m_2-m_1)^2}e^{m_2 x}\cdot e^{(m_1-m_2)x}$$

$$= -\frac{C}{m_2-m_1}\underline{\underline{x}}\,e^{m_1 x} - \frac{C}{(m_2-m_1)^2}e^{m_1 x}$$

↳ x^1, 1 is the least positive integer s.t. $xe^{mx} \neq e^{mx}$

② 令 $y_h = C_1 e^{mx} + C_2 x e^{m_2 x}$, $m_1 = m_2 = m$

$$\begin{vmatrix} e^{mx} & xe^{mx} \\ me^{mx} & e^{mx}+mxe^{mx} \end{vmatrix} = e^{2mx}$$

$$y_p = e^{mx}\int \frac{-Ce^{mx}\, xe^{mx}}{e^{2mx}}dx + xe^{mx}\int \frac{Ce^{mx}, e^{mx}}{e^{2mx}}dx$$

$$= e^{mx}\int -Cx\,dx + xe^{mx}\int C\,dx$$

$$= -\frac{1}{2}Cx^2e^{mx} + Cx^2e^{mx}$$

$$= \frac{1}{2}C\underline{\underline{x^2}}\,e^{mx}$$

↳ x^2, 2 is the least positive integer s.t. $x^2e^{mx} \neq e^{mx}$ and $x^2e^{mx} \neq xe^{mx}$