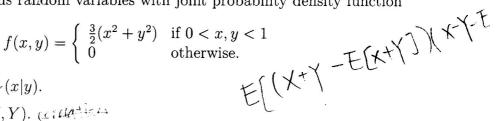
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## Probability (CS 3332) – Spring 2010

Final Exam (June 18, 2010)

1. Let X and Y be continuous random variables with joint probability density function



- (a) (10 points) Find  $f_{X|Y}(x|y)$ .
- (b) (10 points) Find  $\rho(X,Y)$ .
- 2. (20 points) Let  $X_1, X_2, \ldots$  be a sequence of independent and identically distributed continuous random variables. Let  $N \geq 2$  be such that

$$X_1 \ge X_2 \ge \ldots \ge X_{N-1} < X_N$$

That is, N is the point at which the sequence stops decreasing. Show that E[N] = e. Hint: First find  $P(N \ge n)$  by arguing that all orderings of  $X_1, X_2, \ldots, X_n$  are equally likely. Then apply the identity  $E[N] = \sum_{n=1}^{\infty} P(N \ge n)$ .

3. (15 points) Show that X and Y are identically distributed and not necessarily independent, then

$$Cov(X + Y, X - Y) = 0$$

4. (10 points) An urn initially contains b black and w white balls. At each stage, we add r black balls and then withdraw, at random, r balls from the b+w+r balls in the urn. Show by induction that

$$E[\text{number of white balls after stage } t] = \left(\frac{b+w}{b+w+r}\right)^{t} w$$

- 5. An urn contains a white ball and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let  $M_n$  denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.
  - (a) (10 points) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a+b}\right)M_n + 1$$

(b) (5 points) Use part (a) to show that

$$M_n = a + b - b \left(1 - \frac{1}{a+b}\right)^n$$

- (c) (5 points) What is the probability that the (n+1)st ball drawn is white?
- 6. (15 points) The joint density function of X and Y is given by

$$f(x,y) = \frac{1}{\sqrt{2\pi}}e^{-y}e^{-(x-y)^2/2}$$
  $0 < y < \infty, -\infty < x < \infty$ 

Compute the joint moment generating function of X and Y.