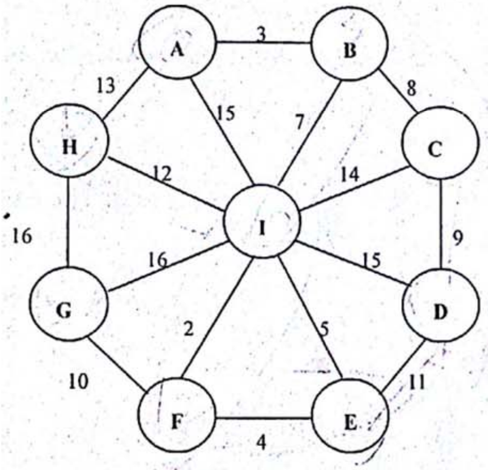
Algorithms Final Examination

Jan. 11, 2012

1. (10%) Give an $O(n^2)$-time algorithm to find the longest monotonically increasing subsequence of a sequence of n numbers.
2. (5%) Give tight asymptotic complexity of $T(n)$, and show your steps:
   $T(n) = 3T(n/3) + O(\lg n)$.
3. (10%) Prove that for any directed graph G, we have $((G^T)^{SCC})^T = G^{SCC}$. That is the transpose of the component graph of G^T is the same as the component graph of G.
4. (10%) Consider the problem of finding the minimum spanning tree in the graph below.



   a. If we use Prim's algorithm, please show the tree growing sequence when the tree starts from vertex A. (Remember to explain Prim's algorithm briefly, don't just show the answer.)

   b. If we use Kruskal's algorithm, please show the sequence of edges added to the minimum spanning tree. (Remember to explain Kruskal's algorithm briefly, don't just show the answer.)
5. (10%) We are given a directed graph $G = (V, E)$ on which each edge $(u, v) \in E$ has an associated value $r(u, v)$, which is a real number in the range $0 \le r(u, v) \le 1$ that represent the reliability of a communication channel from vertex u to vertex v. We interpret $r(u,v)$ as the probability that the channel from u to v will not fail, and we assume that these probabilities are independent. Given an efficient algorithm to find the most reliable path between two given vertices. (You need to show the time complexity of your algorithm.)
6. (20%) Given a directed graph $G = (V, E)$, we want to find the shortest paths of every pair of vertices $u, v \in V$. Please find the algorithm and its corresponding time complexity for the all-pair shortest-paths problem with each of the following

assumptions. (You don't need to detail the algorithm. But you need to describe what algorithm and data structures are used to solve the all-pair shortest-paths problem.)

a) Each edge of the graph G has unit weight.

b) Assume the graph G is acyclic.

c) Assume there is no negative-weight cycle in G.

d) Assume there may exist negative-weight cycle in G.

7. (10%) Given a graph $G = (V, E)$, we want to select the minimum number of vertices such that the two vertices of each edge has at least one vertex selected. Describe an polynomial-time approximation algorithm with approximation ratio two for the problem. What is the time complexity? Prove that your algorithm has a ratio bound two.

8. (25%) For each of the following statements, determine whether it is true or false. If the statement is correct, briefly state why. If the statement is wrong,. explain why. Your answer will be evaluated based on your explanation and not the True/False marking alone.

(1) If we prove that the satisfiability problem can polynomial-time reduce to problem A, then problem A is NP-hard.

(2) If any NP-complete problem can be solved in polynomial time, then NP = P.

(3) The 0/1 Knapsack decision problem is NP-complete.

(4) The "Halting Problem" is a NP-complete problem.

(5) If a problem is NP-complete, then it is unlikely that we can find a polynomial time algorithm to solve it in average cases.

(6) Any NP-complete problem can be solved in polynomial time if there is an algorithm that can solve the 3-CNF satisfiability problem in polynomial time.

(7) If we can find a polynomial-time approximation algorithm with approximation ratio $\rho > 1$ for the general travelling-salesman problem, then P = NP.

(8) Let T be a minimum spanning tree of G. Then, for any pair of vertices of s and t, the shortest path from s to t in G is the path from s to t in T.

(9) The total amortized cost of a sequence of operations gives an upper bound on the actual cost of the sequence.

(10) Give a graph $G = (V, E)$ with cost on edges and a set $S \subseteq V$, let (u, v) be an edge such that (u, v) is the minimum cost edge between any vertex in S and any vertex in V-S. Then the minimum spanning tree of G must include edge (u, v).

Happy New Year!