CS2336 Discrete Mathematics

Exam 1
Halloween, 2016 (2 hours)

Answer all questions. Total marks = 100.

1. (10%) Consider the following compound proposition:

$$[\neg q \wedge (p \vee q)] \leftrightarrow (p \wedge q).$$

In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.

2. (15%) Use logical equivalences and rules of inferences to show that the following arguments are valid. Refer to the last page for some common equivalences and rules.
   - Premises: $\forall x(P(x) \rightarrow Q(x))$, $\exists x(P(x) \vee R(x))$
   - Conclusion: $\exists x(Q(x) \vee R(x))$

3. (30%) Let x be an integer.
   (a) (15%) Show that if x is of the form $3M+1$, where M is an integer, then x^2+x+1 is divisible by 3.
   (b) (15%) Show that if x^2+x+1 is divisible by 3, then x is of the form $3M+1$, where M is an integer.

4. (15%) A real number is *rational* if it can be expressed as p/q, for some integer p and some integer $q \neq 0$; otherwise, it is *irrational*. Let x be a rational number and y be an irrational number. Show that $x+y$ must be irrational.

5. (10%) Lagrange's four-square theorem states that for any integer $n \geq 0$, we can find integers $p, q, r, s \geq 0$ (possibly the same) such that $n = p^2+q^2+r^2+s^2$. For instance, for $n=5$, we can write

$$5 = 0^2 + 0^2 + 1^2 + 2^2.$$

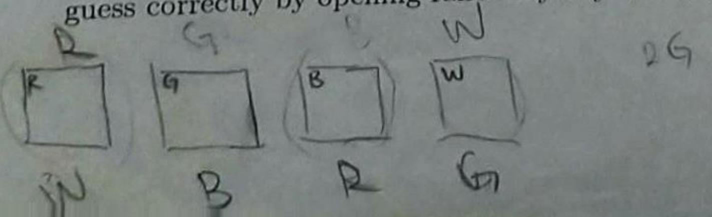
Show that Lagrange's four-square theorem holds when $n=55$ and $n=56$.

6. (20%) Peter has four boxes. One is labeled R, which contains all red balls, another one is labeled G, which contains all green balls, another one is labeled B, which contains all blue balls, and the last one is labeled W, which contains all white balls.

On Halloween, Peter's little brother, Patrick, asked Peter for candies, but Peter refuses to do so. Patrick then plays a trick by secretly exchanging the labels on the boxes, so that now all the boxes are with wrong labels.

Patrick tells Peter about this, and would help Peter to correct the labels only if Peter can guess correctly which box is which. To make the guessing easier, Peter is allowed to open one of the boxes, see what is inside, and then open another box, and see what is inside.

   (a) (10%) What should Peter do to guarantee he can guess all the boxes correctly? (Explain your answer.)
   (b) (10%) Does it matter which two boxes Peter open? That is, is it true that Peter can guess correctly by opening randomly any two of the boxes? (Explain your answer.)



1. Identity Laws:	$p \wedge T_0 \equiv p$	$p \vee F_0 \equiv p$
2. Domination Laws:	$p \wedge F_0 \equiv F_0$	$p \vee T_0 \equiv T_0$
3. Idempotent Laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
4. Double Negation Law:	$\neg(\neg p) \equiv p$	
5. Commutative Laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
6. Associative Laws:	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
7. Distributive Laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8. De Morgan's Laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption Laws:	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
10. Negation Laws:	$p \wedge \neg p \equiv F_0$	$p \vee \neg p \equiv T_0$
11. De Morgan's Laws with Quantifiers:	$\neg \forall x P(x) \equiv \exists x \neg P(x)$	$\neg \exists x P(x) \equiv \forall x \neg P(x)$
12. Conditional Statement Equivalences:	$p \rightarrow q \equiv \neg p \vee q$	$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Figure 1: Some useful logical equivalences

1. Modus Ponens:
   Premises: $p, \quad p \rightarrow q$ — Conclusion: q
2. Modus Tollens:
   Premises: $\neg q, \quad p \rightarrow q$ — Conclusion: $\neg p$
3. Hypothetical Syllogism:
   Premises: $p \rightarrow q, \quad q \rightarrow r$ — Conclusion: $p \rightarrow r$
4. Disjunctive Syllogism:
   Premises: $\neg p, \quad p \vee q$ — Conclusion: q
5. Addition:
   Premise: p — Conclusion: $p \vee q$
6. Simplification:
   Premise: $p \wedge q$ — Conclusion: p
7. Conjunction:
   Premises: $p, \quad q$ — Conclusion: $p \wedge q$
8. Resolution:
   Premises: $p \vee q, \quad \neg p \vee r$ — Conclusion: $q \vee r$
9. Universal Instantiation:
   Premise: $\forall x P(x)$ — Conclusion: $P(c)$, for any c
10. Universal Generalization:
    Premise: $P(c)$, for any c — Conclusion: $\forall x P(x)$
11. Existential Instantiation:
    Premise: $\exists x P(x)$ — Conclusion: $P(c)$, for some c
12. Existential Generalization:
    Premise: $P(c)$, for some c — Conclusion: $\exists x P(x)$

Figure 2: Some useful rules of inference