CS2336 Discrete Mathematics

Exam 3
January 12, 2015 (2 hours)

Total marks is 105. Maximum score is 100.

- **Answer all seven questions.**
- **Question 6 is tricky. Question 2 and Question 7(c) need some thinking.**

1. (10%) Prove that the function $f(x) = 2^x$ is not bijective when the domain and codomain are both $\mathbb{R}$.

2. (15%) Construct a relation on a set $\{a, b, c\}$ which is not reflexive, not symmetric, not antisymmetric, but transitive. (*Double check your answer for transitivity!*)

3. (10%) Is the following relation a equivalence relation? Write down your proof.

$$\{\ (a, b) \mid a, b \in \mathbb{N},\ \gcd(a, b) > 1\ \},$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

4. (15%) The *girth* of a graph is the length of the smallest cycle in the graph; if there is no cycle, the girth is defined as ∞. For example, the girth of the graph in Figure 1 is 5.

Let G be a finite, connected, simple, planar graph with V vertices and E edges. Suppose that the girth of G is k. Show that $E \times (k - 2) \leq k \times (V - 2)$.

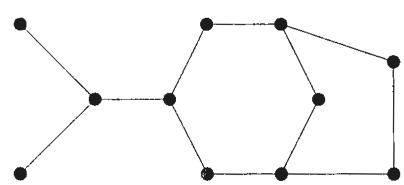

Figure 1: A graph with girth 5

Hint: Find an upper bound on F (the number of regions in a planar depiction of G), and then apply Euler formula.

5. (20%) For each graph in Figure 2, prove or disprove that the graph is planar. (No marks if no proof is written down.)

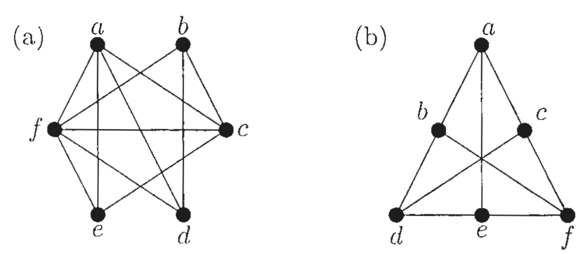

Figure 2: Planar or non-planar?

6. (20%) Show that the two graphs G_1 and G_2 in Figure 3 are non-isomorphic. (No marks if no proof is written down.)

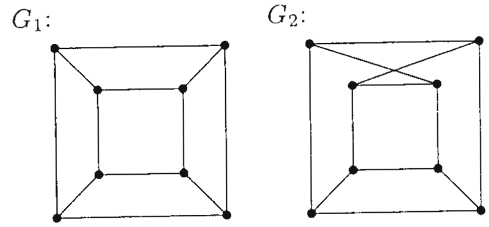

Figure 3: Non-isomorphic graphs.

Hint: Try all graph properties you have learnt. Some leads to a very simple proof.

7. Prove or disprove the following:

   (a) (5%) There exists a graph G isomorphic to $\overline{G}$ such that $G \cup \overline{G}$ equals to K_4.

   (b) (5%) There exists a graph H isomorphic to $\overline{H}$ such that $H \cup \overline{H}$ equals to K_5.

   (c) (5%) There exists a graph F isomorphic to $\overline{F}$ such that $F \cup \overline{F}$ equals to K_6.