[CS 2336]Discrete Mathematics: Autumn 2009

Final exam (close book)
Examination Date: Jan. 11, 2010
Time: 13:10-15:00

1. Can a bipartite graph contain a cycle of odd length? Explain your reasons. (5%)

2. Let $G = (V, E)$ be an undirected connected loop-free graph. Suppose further that G is planar and determines 11 regions. If, for some planar embedding of G, each region has at least eleven edges in its boundary, prove that $|V| \geq 52$. (5%)

3. Let P be the union of the vertices in the partial semipaths which start from a but cannot reach ~~z~~. Find P for each transport network shown in the Fig.1. (6%)

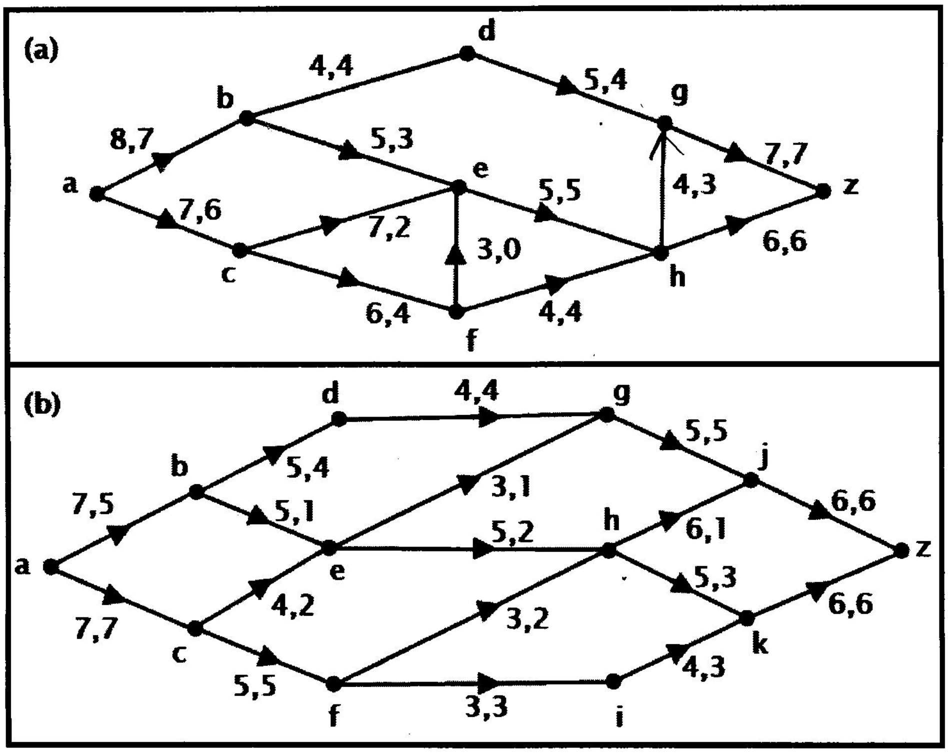

Fig.1

4. Let $G = (V, E)$ be a loop-free weighted connected graph. We want to prove that any spanning tree for G that is obtained by Kruskal's algorithm is optimal. The following steps are first half of the proof.

   1) Let $|V| = n$, and let T be a spanning tree for G obtained by Kruskal's algorithm.
   2) The edges in T are labeled $e_1, e_2, ..., e_{n-1}$, according to the order in which they are generated by the algorithm.
   3) For each optimal tree T' of G, define $d(T') = k$ if k is the smallest positive integer such that T and T' both contain $e_1, e_2, ..., e_{k-1}$, but $e_k \notin T'$.
   4) Let T_1 be an optimal tree for which $d(T_1) = r$ is maximal.
   5) If $r = n$, then $T = T_1$ and the result follows.
   6) Otherwise, $r \leq n-1$ and adding edge e_r(of T) to T_1 produces the cycle C, where there exists an edge e_r' of C that is in T_1 but not in T.
   7) Start with tree T_1. Adding e_r to T_1 and deleting e_r', we obtain a connected graph with n vertices and $n-1$ edges.
   8) This graph is a spanning tree, T_2. The weights of T_1 and T_2 satisfy $wt(T_2) = wt(T_1) + wt(e_r) - wt(e_r')$.
   9) Following the selection of $e_1, e_2, ..., e_{r-1}$ in Kruskal's algorithm, the edge e_r is chosen so that $wt(e_r)$ is minimal and no cycle results when e_r is added to the subgraph H of G determined by $e_1, e_2, ..., e_{r-1}$.
   10) Since e_r' produces no cycle when added to the subgraph H, by the minimality of $wt(e_r)$ it follows that $wt(e_r') \geq wt(e_r)$.
   11) Hence $wt(e_r) - wt(e_r') \leq 0$, so $wt(T_2) \leq wt(T_1)$.

$\vdots$

   a) Please explain why the result follows in step 5. **(3%)**
   b) In step 6, please explain why there exists an edge e_r' of C that is in T_1 but not in T. **(3%)**
   c) In step 10, please explain why $wt(e_r') \geq wt(e_r)$. **(3%)**
   d) Please complete the proof by yourself. **(5%)**
   e) In the steps, only the case,$r \leq n$, is considered. Please explain why the case, $r > n$, is not considered. **(3%)**

5. For $n \geq 3$, let $G_n = (V, E)$ be the undirected graph obtained from the complete graph K_n upon deletion of one edge. Determine the chromatic polynomials $P(G_n, \lambda)$ and chromatic number $\chi(G_n)$. **(10%)**

$\lambda \cdot \lambda-1 \cdot \lambda-2$

$\lambda \cdot 1 \cdot \lambda-1$

6. Answer following questions.

   a) For $n \geq 3$, how many different Hamilton cycles are there in the complete graph K_n? **(3%)**

   b) How many edge-disjoint Hamilton cycles are there in K_{21}. **(3%)**

   c) How many different Hamilton path are there for $K_{n,n}$, $n \geq 1$? **(2%)**

7. For $n \geq 1$, let t_n count the number of spanning trees for the fan on n+1 vertices. The fan for n=4 is shown in Fig.2

   a) Show that $t_{n+1} = t_n + \sum_{i=0}^{n} t_i$, where $n \geq 1$ and $t_0 = 1$. **(6%)**

   b) For $n \geq 2$, show that $t_{n+1} = 3t_n - t_{n-1}$. **(6%)**

1 1+1

2 2+2+1

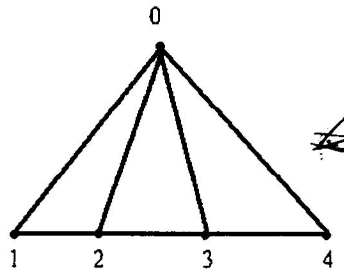

Fig.2

26·3=78

8. Please answer the following questions:

   26

   a) A complete ternary (or 3-ary) tree $T = (V, E)$ has 26 internal vertices. How many edges does T have? How many leaves does T have? **(4%)**

   78
   -26
   52

   b) How many internal vertices does a complete 7-ary tree with 1219 leaves have? **(3%)**

   79

9. Let $T = (V, E)$ be a binary tree with $|V| = n > 3$.

   a) What are the smallest and the largest numbers of articulation points that T can have? Describe the trees for each of these cases. **(6%)**

   b) How many biconnected components does T have in each of the cases in part (a)? **(4%)**

10. Let f be a flow in a transport network $N = (V, E)$ and let $(P, \overline{P})$ be a cut, where $val(f) = c(P, \overline{P})$. Please prove that f is a maximum flow for the network N and $(P, \overline{P})$ is a minimum cut [that is, $(P, \overline{P})$ has minimum capacity in N]. **(5%)**

$7 \cdot n + 1 - 1219 = n$

n=203

l=1219

V=1422

$7 \cdot n - 1218 = n$

$6n = 1218$

$n = 203$

18
43
9
9
61
71
$4\overline{)86}$
21.5

11. For a transport network $N = (V, E)$, let f be a flow in N and let $(P, \overline{P})$ be a cut. Please prove that $val(f) = c(P, \overline{P})$ if and only if

    a) $f(e) = c(e)$ for each edge $e = (x, y)$, where $x \in P$ and $y \in \overline{P}$.

    b) $f(e) = 0$ for each edge $e = (v, w)$, where $v \in \overline{P}$ and $w \in P$. 3%

    (10%)

12. Find a maximum flow and the value of maximum flow for the network shown in Fig.3. The capacities on the undirected edges indicate that the capacity is the same in either direction. [For an undirected edge a flow can go in only one direction at a time.] **(5%)**

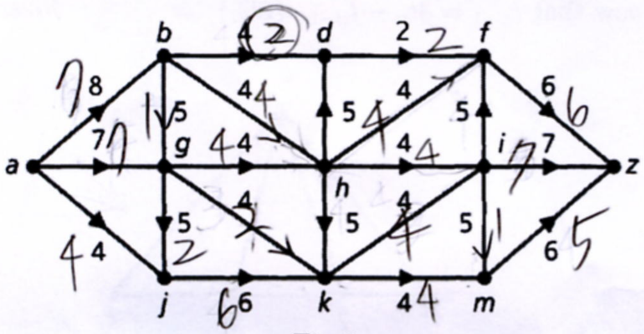

Fig.3