CS2336 Discrete Mathematics

Exam 2
December 8, 2014 (2 hours)

Maximum score is 100.

- **Answer all seven questions (each = 15 marks).**
- **Question 5 and Question 7 are tricky.**

1. (15%) How many integers in $\{1, 2, 3, \ldots, 3600\}$ are divisible by either 4, 6, or 9?

   Remark. Be careful.

2. (15%) Consider all the 5-bit binary strings. How many of them does not contain `010` and `101` as its substring?

   For example, `11011` contains `101` as its substring, `01010` contains both `010` and `101` as its substring, while `11001` does not contain `010` and `101` as its substring.

   Hint: Use a tree diagram.

3. (15%) Let $S(n, k)$ denote the number of ways to distribute n distinguishable balls into k nondistinguishable bins, such that no bin is empty. For example, we have $S(n, 1) = 1$, $S(n, n) = 1$, and $S(n, n + 1) = 0$.

   Use a combinatorial argument to show that for $n \geq k \geq 2$,

   $$S(n, k) = S(n-1, k-1) + k \times S(n-1, k).$$

4. (15%) How many ways we can select 3 distinct integers from $\{1, 2, 3, \ldots, 1000\}$, so that their sum is divisible by 5?

5. (15%) How many non-negative integral solutions (x, y, z) are there for the inequality

   $$3 \leq x + y + z \leq 100?$$

   Express your answer in a form as simple as possible.

6. (15%) Show that the set

   $$\{(i, j) \mid i \text{ and } j \text{ are positive integers}\}$$

   is countable.

7. This question asks you to give examples of uncountable sets that satisfy certain properties. You will get full marks if your sets satisfy the desired properties, and no extra explanation is needed.

   (a) (5%) Give two uncountable sets X and Y such that $X \cap Y$ is countably infinite.

   (b) (10%) Give three uncountable sets A, B, and C such that $A - B$ is countably infinite, $A - C$ is countably infinite, but $B - C$ is uncountable.