張延工科 midterm

Subject: Close-book midterm of CS3334-01 (Engineering Mathematics), Nov. 26, 2004

Seven hints and rules for a fair examination:

一、Close books, close everything, and turn off all personal electronics including the cell phone's sound and vibration; calculators can be left on.

二、Please raise your hand with patience when you want to talk to the teacher or a TA.

三、Please try to understand and answer the questions properly.

四、Please express your "thought process" succinctly to avoid jump-to-conclusion answer; please make each of your final answers stands out clearly and explicitly to avoid grading errors.

五、Between 10:10AM and 12:00noon, you are welcome and allowed to talk to the teaching assistants. No chatting or any other form of communications between classmates will be allowed.

六、A "model" is not the same as an "initial value problem".

七、Ten credits for each question.

$\begin{pmatrix} 3x+4y \\ x \end{pmatrix} = \begin{matrix} \lambda x \\ \lambda y \end{matrix}$

$(3-\lambda)(-\lambda)+4$

1. A linear system is modeled as:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the ten questions below, if you write down the correct answers without any mathematic explanation/process, you can only get very few partial credits.

(a) Compute the two eigenvalues for the above;

(b) For the two eigenvalues, compute the associated eigenvectors, respectively;

(c) Compute the general solution;

(d) If the initial condition is $(x(t), y(t))|_{t=0} = (5, 0)$, solve the IVP by generating a solution.

2. A linear system is modeled as:

$$\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda^2+4\lambda+4-12$

$\lambda^2+4\lambda-8$

(a) Compute the two eigenvalues that are conjugated complex numbers;

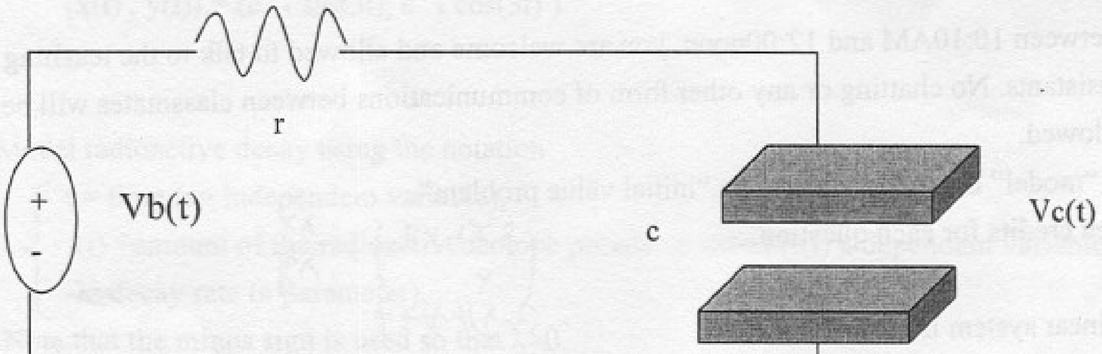
(b) Selecting one of the two eigenvalues, compute its eigenvector;

(c) Compute the general solution that has real-numbered parameters;

(d) If the initial condition is $(x(t), y(t))|_{t=0} = (0.5, 0.5)$, solve the initial-value problem (IVP) by generating a solution that has only real numbers.

$(-2+3i)\begin{pmatrix} i \\ 1 \end{pmatrix}$

3. Using $Qc(t) = c * Vc(t)$, $dQ(t)/dt = I(t)$, $V_r(t) = r * I_r(t)$, and the following simple circuit:



(a) Due to Kirchhoff's Voltage Law ($Vb = V_r + Vc$), explain and prove that the above circuit can be modeled by the following ordinary differential equation (ODE):

$$dVc(t)/dt = (Vb(t) - Vc(t)) / (r*c).$$

(b) If Vb(t) is a constant direct current battery at 1.0 volt and Vc(t=0) = 0, please write down the initial-value problem (IVP) that consists of an ODE and the initial condition.

(c) Explain and prove that the analytic solution of IVP in (b) is $Vc(t) = 1.0 - e^{-t/(r*c)}$

4. For the function:

$$d^2y/dt^2 + (k/m)y = 0$$

describing the motion of a simple harmonic oscillator.

(a) Consider the function $y(t) = \cos \beta t$. Under **what conditions on** β is y(t) a solution?

(b) If v = dy/dt, what initial condition (t = 0) in the yv-plane corresponds to this solution?

(c) In terms of k and m, **what is the period** of this solution?

5. Perform Euler's method with the given step size $\Delta t=0.4$ on the following IVP over the time interval from 0 to 2:
$dy/dt = y^2 - (2 * y) + 1, y(0)=2.$

6. We consider the system:

$$\begin{cases} dx(t)/dt = -2x(t) - y(t) \\ dy(t)/dt = 2x(t) - 5y(t). \end{cases}$$

For the given function $Y(t) = (x(t),y(t))$, check to see if $Y(t)$ is a solution to the system.

(a) $(x(t), y(t)) = (e^{-3t} - 2 * e^{-4t},\ \ e^{-3t} - 4 * e^{-4t})$

(b) $(x(t), y(t)) = (2 * e^{-3t} + e^{-4t},\ \ 2 * e^{-3t} + 2 * e^{-4t})$.

7. We consider the system:

$$\begin{cases} dx(t)/dt = 2x(t) + y(t) \\ dy(t)/dt = -y(t). \end{cases}$$

(a) Derive the general solution.
(b) If the initial condition is $(x(0), y(0)) = (-1, 3)$, determine the solution for the IVP.

8. A cup of hot chocolate is initially 150°F and is left in a room with an ambient temperature of 50°F. Suppose that at time t=0 it is cooling at a rate of 20°F per minute.
   (a) Assume that Newton's law of cooling applies: The rate of cooling is proportional to the difference between the current temperature and the ambient temperature. Write an initial-value problem that models the temperature of the hot chocolate.
   (b) How long does it take the hot chocolate to cool to a temperature of 110°F?

9. We consider the system:

$$\begin{cases} dx(t)/dt = -x(t) + 3 * y(t) \\ dy(t)/dt = -3 * x(t) - y(t). \end{cases}$$

For the given function Y(t) = (x(t),y(t)), check to see if Y(t) is a solution to the system.
$(x(t), y(t)) = (e^{-t} \cdot \sin(3t), e^{-t} \cdot \cos(3t))$

10. Model radioactive decay using the notation

$$\begin{cases} t = \text{time (an independent variable)}, \\ r(t) = \text{amount of the radioactive isotope present at time } t \text{ } (r(t) \text{ a dependent variable}), \\ -\lambda = \text{decay rate (a parameter)}. \end{cases}$$

Note that the minus sign is used so that $\lambda>0$.

a) Using this notation, write a model for the decay of a particular radioactive isotope.
b) If the amount of the isotope present at t=0 is r_0, state the corresponding initial-value problem for the model in part (a).
c) The half-life of a radioactive isotope is the amount of time it takes for a quantity of radioactive material to decay to one-half of its original amount, and the half-life of Carbon 14(C-14) is 400 years. Determine the decay-rate parameter λ for C-14 (note: please explain the unit of your answer λ).