Algebra

November 6, 10:10 am–1:00 pm, 2006

Do carry out your computations step by step. Also do give complete proofs of the statements you made in your answers.

(1)(15%) (i) Write the following permutation as a product of disjoint cycles:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 5 & 2 \end{bmatrix}.$$

(ii) What is the sign of this permutation σ?

(iii) Compute $[1\ 6]\sigma[1\ 6]$ inside the group S_6.

(2)(10%) Compute the multiplicative inverse of the polynomial $X^2 + 1$ modulo the polynomial $X^5 - 2$. Both polynomials are considered as polynomial with rational coefficients.

(3)(15%) Give an example of a group G for which the map $x \mapsto x^{-1}$ on G is NOT a group homomorphism.

(4)(15%) (i) Let R be a commutative ring. Let $I \subset R$ be an ideal. Write down the definition of the ideal $I^2 = I \cdot I$. Inductively one can then define the ideal I^n for $n \geq 1$.

(ii) Let $R = F[X]$ be the ring of polynomials in one variable X with coefficients from a field F. Let $I = XR$ the principal ideal generated by X in R. Show that $\bigcap_{n=1}^{\infty} I^n = \{0\}$.

(5)(15%) Let R be a commutative ring with proper ideals I, J such that $I \subset J$. Let $f_1 : R \longrightarrow R/I$ and $f_2 : R \longrightarrow R/J$ be the canonical homomorphisms onto the factor rings. Write down a homomorphism $f : R/I \longrightarrow R/J$ satisfying $f \circ f_1 = f_2$.

(6)(15%) (i)Let m, n be relatively prime positive integers. Let G be the cyclic group $\mathbb{Z}/d\mathbb{Z}$ with $d = mn$. Show that there is a subgroup $H_1 \subset G$ of order m, and also a subgroup $H_2 \subset G$ of order n such that G is the direct sum of H_1 and H_2.

(ii) State the Chinese remainder theorem, then explain the connection of this theorem with question (i).

(7)(15%) Let G be a finite group of odd order. Suppose that $H \subset G$ is a subgroup of index 3. Prove that H is a normal subgroup.