**1st Examination**

**Probability**

April 10, 2015

1. (12%) Let $B$ be an event of a sample space $\boldsymbol{S}$ with $P(B) > 0$. For a subset $A$ of $\boldsymbol{S}$, define $Q(A) = P(A|B)$. By Theorem 3.1 we know that $Q$ is a probability function. For $E$ and $F$, events of $\boldsymbol{S}$ [$P(FB) > 0$], show that $Q(E|F) = P(E|FB)$.

2. (20%)

   (a). Show that $\bigcap_{n=1}^{\infty} (1/2 - 1/2n, 1/2 + 1/2n) = \{1/2\}$.

   (b). Using part (a), show that the probability of selecting 1/2 in a random selection of a point from (0, 1) is 0.

3. (20%) There are two stables on a farm, one that houses 20 horses and 13 mules, the other with 25 horses and eight mules. Without any pattern, animals occasionally leave their stables and then return to their stables. Suppose that during a period when all the animals are in their stables, a horse comes out of a stable and then returns. What is the probability that the next animal coming out of the same stable will also be a horse?

4. (12%) Suppose that two points are selected at random and independently from the interval (0, 1). What is the probability that the first one is less than 3/4, and the second one is greater than 1/4?

5. (12%) In Section 3.4 (Bayes' Formula), we have discussed a formula relating probabilities of earlier events, given later events, to the conditional probabilities of later events given the earlier ones. If $P(B) > 0$ and $P(B^c) > 0$, then for any event $A$ of $\boldsymbol{S}$ with $P(A) > 0$, we have $P(B \mid A) = ?$

6. (12%) A box contains seven red and 13 blue balls. Two balls are selected at random and are discarded without their colors being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

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7. (12%) Adam tosses a fair coin $n + 1$ times. Andrew tosses the same coin $n$ times. What is the probability that Adam gets more heads than Andrew?

[Bonus problems: Choose one of them]

8. (15%) An absentminded professor wrote $n$ letters and sealed them in envelopes before writing the addresses on the envelopes. Then he wrote the $n$ addresses on

the envelopes at random. What is the probability that at least one letter was addressed correctly?

9. (15%) Suppose that some individuals in a population produce offspring of the same kind. The offspring of the initial population are called second generation, the offspring of the second generation are called third generation, and so on. If with probability $\exp[-(2n^2+7)/(6n^2)]$ the entire population completely dies out by the $n^{\text{th}}$ generation before producing any offspring, what is the probability that such a population survives forever?

10. (15%) Let $X_0$ be the amount of rain that will fall in the United States on the next Christmas day. For $n > 0$, let $X_n$ be the amount of rain that will fall in the United States on Christmas $n$ years later. Let $N$ be the smallest number of years that elapse before we get a Christmas rainfall greater than $X_0$. Suppose that $P(X_i = X_j) = 0$ if $i \neq j$, the events concerning the amount of rain on Christmas days of different years are all independent, and the $X_n$'s are identically distributed. Find the expected value of $N$.