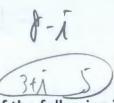


線性代數 Final Exam

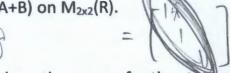
- 1. [10%] Find the rank of the following matrices.
- (a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$
 - 2. [6%] Prove that E^t is an elementary matrix if E is.
 - 3. [6%] Prove that for any m x n matrix A, A is the zero matrix if rank(A) = 0.
 - 4. [10%] Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.
 - 5. (10%) Label the following statements as true or false.
 - i) If E is an elementary matrix, then $det(E) = \pm 1$.
 - ii) For any A, B \in M_{nxn} (F), det(AB) = det(A)*det(B).
 - iii) A matrix $M \in M_{nxn}(F)$ has rank n if and only if $det(M) \neq 0$.
 - iv) For any $A \in M_{nxn}(F)$, $det(A^t) = det(A)$.
 - v) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

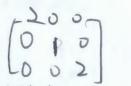


[12%] Provide reasons why each of the following is not an inner product on the given vector spaces.

i) $\langle (a,b), (c,d) \rangle = ac - bd \text{ on } \mathbb{R}^2$

ii) $\langle A, B \rangle = \text{tr}(A+B)$ on $M_{2\times 2}(R)$.





7. [6%] Please show the reason for the statement S below.

Theorem 6.3. Let V be an inner product space and $S = \{v_1, v_2, \dots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \text{span}(S)$, then

$$y = \sum_{i=1}^k \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Proof. Write $y = \sum_{i=1}^{n} a_i v_i$, where $a_1, a_2, \ldots, a_k \in F$. Then, for $1 \leq j \leq k$.

we have

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^k a_i v_i, v_j \right\rangle = \sum_{i=1}^k a_i \left\langle v_i, v_j \right\rangle = a_j \left\langle v_j, v_j \right\rangle = a_j \|v_j\|^2.$$

So $a_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2}$, and the result follows.

- 8. [6%] In R⁴, let w1 = (1,0,1,0), w2=(1,1,1,1) and w3 = (0,1,1,1). Then {w1, w2, w3} is linearly independent. Use Gram-Schmidt process to compute the orthogonal vectors v1, v2, v3, and then normalize these vectors to obtain an orthonormal set.
- 9. [4%] For the following inner product spaces V and linear operations T on V, evaluate T* at the given vector in V.

$$V = C^2$$
, $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$, $x = (3-i, 2+2i)$.

10. [10%] Please show the reason for the statement S.

Theorem 6.8

Theorem 6.8. Let V be a finite-dimensional inner product space over F, and let $g: V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

Furthermore, for $1 \le j \le n$ we have

$$\begin{split} \mathsf{h}(v_j) &= \langle v_j, y \rangle = \underline{\left\langle v_j, \sum_{i=1}^n \overline{\mathsf{g}(v_i)} v_i \right\rangle} = \sum_{i=1}^n \mathsf{g}(v_i) \left\langle v_j, v_i \right\rangle \\ &= \sum_{i=1}^n \mathsf{g}(v_i) \delta_{ji} = \mathsf{g}(v_j). \end{split}$$

Since g and h both agree on β , we have that g = h by the corollary to Theorem 2.6 (p. 73).

11. [5%] Please show the reason for the statement S below.

Theorem 6.9. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^*: V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. Furthermore, T^* is linear.

Proof. Let $y \in V$. Define $g: V \to F$ by $g(x) = \langle T(x), y \rangle$ for all $x \in V$. We first show that g is linear. Let $x_1, x_2 \in V$ and $c \in F$. Then

$$g(cx_1 + x_2) = \langle \mathsf{T}(cx_1 + x_2), y \rangle = \langle c\mathsf{T}(x_1) + \mathsf{T}(x_2), y \rangle = c \langle \mathsf{T}(x_1), y \rangle + \langle \mathsf{T}(x_2), y \rangle = c\mathsf{g}(x_1) + \mathsf{g}(x_2).$$

Hence g is linear.

We now apply Theorem 6.8 to obtain a unique vector $y' \in V$ such that $g(x) = \langle x, y' \rangle$; that is, $\langle T(x), y \rangle = \langle x, y' \rangle$ for all $x \in V$. Defining $T^* : V \to V$ by $T^*(y) = y'$, we have $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$.

12. [10%] Please show the reason for the statements S_1 , S_2 below.

Lemma. Let T be a linear operator on a finite-dimensional inner product space V. If T has an eigenvector, then so does T*.

Proof. Suppose that v is an eigenvector of T with corresponding eigenvalue λ . Then for any $x \in V$,

$$0 = \langle \theta, x \rangle = \langle (\mathsf{T} - \lambda \mathsf{I})(v), x \rangle = \underbrace{\langle v, (\mathsf{T} - \lambda \mathsf{I})^*(x) \rangle}_{} = \underbrace{\langle v, (\mathsf{T}^* - \overline{\lambda} \mathsf{I})(x) \rangle}_{},$$

and hence v is orthogonal to the range of $\mathsf{T}^* - \bar{\lambda}\mathsf{l}$. So $\underline{\mathsf{T}^* - \bar{\lambda}\mathsf{l}}$ is not onto and hence is not one-to-one. Thus $\mathsf{T}^* - \bar{\lambda}\mathsf{l}$ has a nonzero null space, and any nonzero vector in this null space is an eigenvector of T^* with corresponding eigenvalue $\bar{\lambda}$.

13. [5%] Please show the reason for the statement S below.

Lemma. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Then

- (a) Every eigenvalue of T is real.
- (b) Suppose that V is a real inner product space. Then the characteristic polynomial of T splits.

Proof. (a) Suppose that $T(x) = \lambda x$ for $x \neq 0$. Because a self-adjoint operator is also normal, we can apply Theorem 6.15(c) to obtain

$$\lambda x = \mathsf{T}(x) = \mathsf{T}^*(x) = \overline{\lambda} x.$$

So $\lambda = \overline{\lambda}$; that is, λ is real.

S(b) Let $n = \dim(V)$, β be an orthonormal basis for V, and $A = [T]_{\beta}$. Then A is self-adjoint. Let T_A be the linear operator on C^n defined by $T_A(x) = Ax$ for all $x \in C^n$. Note that T_A is self-adjoint because $[T_A]_{\gamma} = A$, where γ is the standard ordered basis for C^n .

By the fundamental theorem of algebra, the characteristic polynomial of T_A splits into factors of the form $t - \lambda$. Since each λ is real, the characteristic polynomial splits over R. T_A has the same characteristic polynomial as A, which has the same characteristic polynomial as T. Therefore the characteristic polynomial of T splits.