CS2336 Discrete Mathematics

Exam 1
November 3, 2014 (2 hours)

Maximum score is 100.

- **Questions 1 to 6 are regular questions (Total = 90 marks).**
- **Question 7 is tricky (15 marks).**

$\neg Q \vee Q$

1. (10%) Draw the truth table for the expression $\neg(P \wedge Q) \wedge (P \rightarrow Q)$, and hence find an equivalent expression that is as short as possible. $(\neg P \vee \neg Q) \wedge (\neg P \vee Q)$

2. (15%) Use logical equivalences and rules of inferences to show that the following argument is valid. Refer to the last page for some common equivalences and rules.

   - Premises: $(P \vee Q) \rightarrow R$, $\neg R$, $S \rightarrow P$
   - Conclusion: $\neg S$

3. (30%) Recall that a real number is *rational* if it can be expressed as A/B, such that such that A and B are both integers, and $B \neq 0$. Given a rational number x, we say x is an *odd-type* rational number if it can be represented as A/B where A is an integer and B is an odd integer; else, we say x is a *non-odd-type* rational number.

   For instance, any integer K is odd-type because $K = K/1$; other odd-type examples include $\pm 1/3, \pm 2/3, \pm 1/5, \pm 2/5$, etc.

   $B = \overline{2k+1}$ $\quad \overline{2M+1}$

   (a) (15%) Prove that if x and y are odd-type rational numbers, then their sum $x + y$ must also be an odd-type rational number.

   (b) (15%) Prove that if x is an odd-type rational number and y is a non-odd-type rational number, then their sum $x + y$ must be a non-odd-type rational number.

4. (10%) Find two irrational numbers x and y whose sum $x + y$ and whose product $x \times y$ are both rational.

   Remark. You will get full marks if your x and y satisfy the above criteria. You do not need to show your x and y are irrational.

5. (10%) Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

   Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

   Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k + 1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

   $-8+9$

6. (15%) The nth Fibonacci number F_n, for integer $n \geq 0$, is defined as follows:

$$F_0 = 0, \ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for any integer } n \geq 2.$$

For instance, the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Show that $F_n \cdot F_{n+2} = F_{n+1} \cdot F_{n+1} + (-1)^{n+1}$ for any integer $n \geq 0$.

7. (15%) A party has n people ($n \geq 2$). At the end, people shake hands randomly with the others. It is known that between any two persons, they shake hands at most once (that is, they either do not shake hands, or shake hands exactly once).

Assume that each person keeps a *shaking-count*, which is the total number of times she shakes hands with the others. So the shaking-count of any person is always an integer between 0 and $n-1$.

(a) (5%) If someone has 0 as her shaking-count, argue that the shaking-count of any person is between 0 and $n-2$.

(b) (0%) If nobody has 0 as her shaking-count, argue that the shaking-count of any person is between 1 and $n-1$.

(c) (10%) Show that there exist two persons such that their shaking-counts are the same.

n-1 people

0

0 0 0 0 0 0

$F_K (F_K + F_{K+1}) - F_{K+1} \cdot F_{K+1}$

$$F_{K+1} \cdot F_{K+3} = F_{K+1} \cdot (F_{K+1} + F_{K+2})$$
$$= (F_{K+1})^2 + F_{K+1} \cdot F_{K+2}$$
$$= \left[F_K \cdot F_{K+2} + (-1)^{K+1}\right] + F_{K+1} \cdot F_{K+2}$$