CS2336 Discrete Mathematics

Exam 2
December 5, 2016 (2 hours)

Answer all questions. Total marks = 105. Maximum score is 100.

1. It is known that the terms in the sequence $a_0, a_1, a_2, \ldots$ satisfy the following relation:

$$a_0 = 5, \qquad a_1 = 10, \qquad \text{and} \qquad a_n = a_{n-1} + 6a_{n-2} \text{ for } n \geq 2.$$

So, we have $a_2 = 10 + 6 \times 5 = 40$, $a_3 = 40 + 6 \times 10 = 100$.

(15%) Use strong induction to show that $a_n = 4 \times 3^n + (-2)^n$ for all integer $n \geq 0$.

2. There is a square wire frame with a side length of 2 units, a chessboard with a side length of 8 units. See Figure 1.

   (a) (15%) If 33 chess pieces are placed on this board, show that it is always possible to place the wire frame on the board to cover at least 3 chess pieces.

   (b) (5%) If only 32 chess pieces are placed on the board, show that it is possible that no matter where we place the wire frame, it can only cover at most 2 chess pieces at each location.

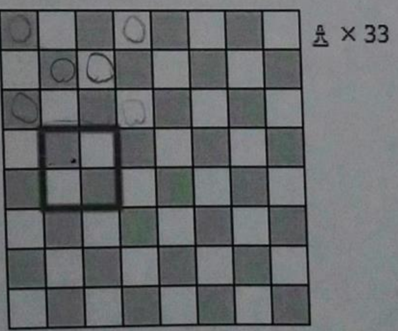

Figure 1: Chessboard and wire frame for Question 2(a)

3. A contiguous sequence of characters in a string X is called a *substring* of X. For instance, `ana` is a substring of `banana`, but `aa` is not a substring of `banana`.

   (15%) Consider all the 5-bit binary strings. How many of them contains `11` but not `101` as its substring?

   For example, `11011` contains both `11` and `101` as its substring, `10001` does not contain `11` and `101` as its substring, while `10011` contains `11` but not `101` as its substrng.

   Hint: Use a tree diagram.

4. Consider the diagram in Figure 2, where each vertex represents a city, and each edge represents a one-way road.

   (a) (5%) How many ways are there to travel from A to B?

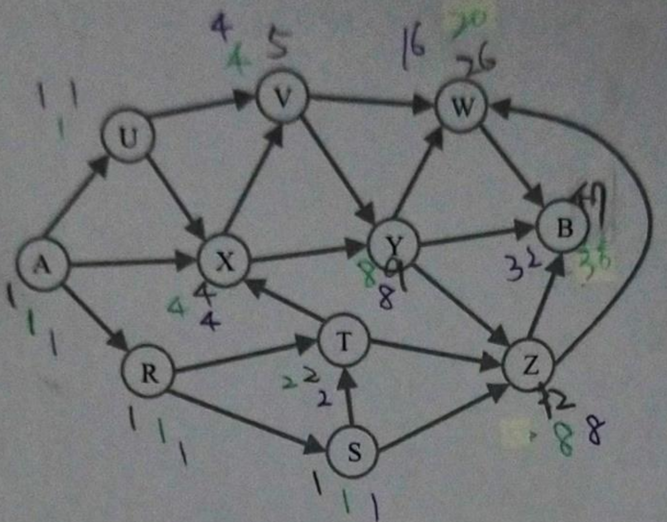

Figure 2: Diagram for Question 4

(b) (5%) How many ways are there to travel from A to B that must pass through X?

(c) (5%) How many ways to travel from A to B that must pass through both X and Y?

Note: For each part of this question, no explanation is needed.

5. (15%) How many ways we can select 4 distinct integers from $\{1, 2, 3, \ldots, 120\}$, so that their sum is divisible by 3?

6. (a) (10%) How many non-negative integral solutions (x, y, z) are there for the inequality

$$x + y + z \leq 100?$$

(b) (5%) How many non-negative integral solutions (x, y, z) are there for the inequality

$$50 \leq x + y + z \leq 100?$$

$x+y+z<50$

Express your answer in a form as simple as possible.

$x+y+z \leq 49.$

7. (10%) Give a combinatorial argument to show that

$$\binom{2n}{2} = 2 \times \binom{n}{2} + n^2.$$

Note: No marks will be given if you are not using a combinatorial argument.