$E[X^2 - Y^2] - E[X+Y]E[$

Probability (CS 3332) – Spring 2010

Final Exam (June 18, 2010)

1. Let X and Y be continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2) & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$E[(X+Y - E[X+Y])(X-Y-E$

   (a) (10 points) Find $f_{X|Y}(x|y)$.

   (b) (10 points) Find $\rho(X, Y)$. correlation

2. (20 points) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \ldots \geq X_{N-1} < X_N$$

   That is, N is the point at which the sequence stops decreasing. Show that $E[N] = e$. *Hint:* First find $P(N \geq n)$ by arguing that all orderings of $X_1, X_2, \ldots, X_n$ are equally likely. Then apply the identity $E[N] = \sum_{n=1}^{\infty} P(N \geq n)$.

3. (15 points) Show that X and Y are identically distributed and not necessarily independent, then

$$\mathrm{Cov}(X + Y, X - Y) = 0$$

4. (10 points) An urn initially contains b black and w white balls. At each stage, we add r black balls and then withdraw, at random, r balls from the $b + w + r$ balls in the urn. Show by induction that

$$E[\text{number of white balls after stage } t] = \left(\frac{b + w}{b + w + r}\right)^t w$$

5. An urn contains a white ball and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.

   (a) (10 points) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a + b}\right) M_n + 1$$

   (b) (5 points) Use part (a) to show that

$$M_n = a + b - b\left(1 - \frac{1}{a + b}\right)^n$$

(c) (5 points) What is the probability that the $(n+1)$st ball drawn is white?

6. (15 points) The joint density function of X and Y is given by

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-y} e^{-(x-y)^2/2} \quad 0 < y < \infty, -\infty < x < \infty$$

Compute the joint moment generating function of X and Y.