Test 1 for CS2334 (02)

October 11, 2004

1.(4 pts) A linear system of equations is given below.

$$\begin{array}{rcrcrcr} 3x & + & y & - & z & = & 0 \\ -6x & & & + & 2z & = & -4 \\ 3x & - & 3y & & & = & 9 \end{array}$$

(a) Express this system as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Show the augmented matrix for this system.

(b) Use Gaussian elimination and back substitution to solve this system of equations.

(c) Find $A = LU$, where L is unit lower-Δ and U is upper-Δ.

(d) Find $det(A)$.

2.(3 pts) Given

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

(a) Find $det(M_{21})$, $det(M_{22})$, and $det(M_{23})$.

(b) Find the cofactors A_{21}, A_{22}, and A_{23}.

(c) Compute $det(A)$ from the results of (**b**) and find A^{-1}.

3.(3 pts) Mark $\bigcirc$ if the statement is *true*, otherwise mark $\times$ if the statement is *false*.

(a) If $A, B \in R^{n\times n}$ are nonsingular, then $(A+B)^{-1} \;=\; B^{-1} + A^{-1}$.

(b) The product of unit lower-Δ matrices is also unit lower-Δ.

(c) $det(\alpha A) = \alpha det(A)$, where α is a constant.

(d) $J \in R^{n\times n}$, where $J = [a_{ij}]$ with $a_{i,j} = 1$ if $j - i = 1$ else $a_{ij} = 0$, then J^n is a zero matrix.

Keys for Test 1 of CS2334 (02)

October 11, 2004

1(a)

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -6 & 0 & 2 \\ 3 & -3 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -4 \\ 9 \end{bmatrix}$$

The augmented matrix is $[A \mid \mathbf{b}]$

1(b) $\mathbf{b} = [1, \ -2, \ 1]^t$

1(c)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2(a) $det(M_{21}) = -8$, $det(M_{22}) = -2$, and $det(M_{23}) = 5$

2(b) $A_{21} = 8$, $A_{22} = -2$, and $A_{23} = -5$

2(c) $det(A) = -3$, $A^{-1} = \frac{1}{3}[13, -8, -14; \ -4, 2, 5; \ -7, 5, 8]$.

3. (a) $\times$, (b) $\bigcirc$, (c) $\times$, (d) $\bigcirc$.

Test 2 for CS2334

November 8, 2004

(5 pts) 1. Mark $\bigcirc$ if the statement is *true*, and mark $\times$ otherwise.

(   )**(a)** Let $R^+ = \{x| \ x > 0\}$. For $\forall \ x, y \in R^+$ and $\alpha \in R^+$, the addition $\oplus$ and scalar multiplication $\odot$ are defined as $x \oplus y = xy$ and $\alpha \odot x = x^\alpha$, respectively. Under these definitions, R^+ is a vector space over R^+.

(   )**(b)** For $\forall \ \mathbf{x} = [x_1, x_2]^t, \ \mathbf{y} = [y_1, y_2]^t$ in R^2 and $\alpha \in R$, the vector addition $\oplus$ and scalar multiplication $\odot$ are defined as $\mathbf{x} \oplus \mathbf{y} = [x_1 + y_1 + 1, x_2 + y_2 + 1]^t$ and $\alpha \odot \mathbf{x} = [\alpha x_1, \alpha x_2]^t$, respectively. Under these definitions, R^2 is a vector space over R.

(   )**(c)** If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ span R^n, then they are linearly independent.

(   )**(d)** If $V = span(\mathbf{v}_1, \mathbf{v}_2, \ \cdots, \mathbf{v}_n)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ \cdots, \mathbf{v}_n\}$ is a basis for V.

(   )**(e)** Let $A \in R^{m \times n}$, then $R(A) \subset R^m$ and $Null(A) \subset R^n$.

(   )**(f)** Let $L : \ R^n \ \rightarrow \ R^n$ be a linear transform defined by $L(\mathbf{x}) = A\mathbf{x}$, where $A \in R^{n \times n}$, then $Ker(L) = \mathbf{0}$ iff A is nonsingular.

(   )**(g)** An affine transform is a linear transform.

(15 pts) 2. Answer each of the following questions.

(A) Express $\mathbf{x} = [6, 3, 1]$ as a linear combination of $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [1, 1, 0]$, $\mathbf{w} = [1, 0, 0]$.

(B) Prove or disaprove that $\{[3, 1, -4]^t,\ [2, 5, 6]^t,\ [1, 4, 8]^t\}$ is a basis for R^3.

(C) The rank of A is _____, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

(D) $dim(Null(A)) \ =$ _____, where A is as defined in **(C)**.

(E) Find a basis for $R(A)$, where A is as defined in **(C)**.

(F) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^3$ and let $L : R^3 \rightarrow R^2$ be a linear transformation such that $L(\mathbf{x}) = [1, 0]^t$, $L(\mathbf{y}) = [0, 1]^t$, $L(\mathbf{z}) = [1, -1]^t$. Then $L(2\mathbf{x} - 3\mathbf{y} + 4\mathbf{z}) \ =$ _____

(G) Let $L : R^3 \rightarrow R^2$ be a linear transformation such that $L([1, 0, 0]^t) = [1, 1]^t$, $L([0, 1, 0]^t) = [1, -1]^t$, $L([0, 0, 1]^t) = [1, 0]^t$. Then $Ker(L) \ =$ _____

(H) Let $H \in R^{2\times 2}$ be a Householder matrix defined by $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\mathbf{u} = [1/2, \sqrt{3}/2]^t$. Then $H^tH \ =$ _____

Solutions for Test 2 of CS2334

November 8, 2004

(5 pts) 1. Mark ◯ if the statement is *true*, and mark × otherwise.

(◯)**(a)** Let $V = R^+ = \{x|\ x > 0\}$. For $\forall\ x, y \in V$ and $\alpha \in R^+$, the addition $\oplus$ and scalar multiplication $\odot$ are defined as $x \oplus y = xy$ and $\alpha \odot x = x^\alpha$, respectively. Under these definitions, V is a vector space over R^+.

(×)**(b)** For $\forall\, \mathbf{x} = [x_1, x_2]^t$, $\mathbf{y} = [y_1, y_2]^t$ in R^2 and $\alpha \in R$, the vector addition $\oplus$ and scalar multiplication $\odot$ are defined as $\mathbf{x} \oplus \mathbf{y} = [x_1 + y_1 + 1, x_2 + y_2 + 1]^t$ and $\alpha \odot \mathbf{x} = [\alpha x_1, \alpha x_2]^t$, respectively. Under these definitions, R^2 is a vector space over R.

(◯**(c)** If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ span R^n, then they are linearly independent.

(×)**(d)** If $V = span(\mathbf{v}_1, \mathbf{v}_2, \ \cdots, \mathbf{v}_n)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ \cdots, \mathbf{v}_n\}$ is a basis for V.

(◯)**(e)** Let $A \in R^{m \times n}$, then $R(A) \subset R^m$ and $Null(A) \subset R^n$.

(◯)**(f)** Let $L : R^n \rightarrow R^n$ be a linear transform defined by $L(\mathbf{x}) = A\mathbf{x}$, where $A \in R^{n \times n}$, then $Ker(L) = \mathbf{0}$ iff A is nonsingular.

(×)**(g)** An affine transform is a linear transform.

(15 pts) 2. Answer each of the following questions.

(A) Express $\mathbf{x} = [6, 3, 1]$ as a linear combination of $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [1, 1, 0]$, $\mathbf{w} = [1, 0, 0]$.
[**Answer**] $\mathbf{x} = \mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$

(B) Prove or disaprove that $\{[3, 1, -4]^t,\ [2, 5, 6]^t,\ [1, 4, 8]^t\}$ is a basis for R^3.

[**Answer**] It is a basis since the corresponding determinant is nonzero.

(C) The rank of A is $\underline{\quad \mathbf{2} \quad}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

(D) $dim(Null(A)) = \underline{\quad \mathbf{2} \quad}$, where A is as defined in **(C)**.

(E) Find a basis for $R(A)$, where A is as defined in **(C)**. [**Answer**] $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$

(F) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^3$ and let $L : R^3 \to R^2$ be a linear transformation such that $L(\mathbf{x}) = [1, 0]^t$, $L(\mathbf{y}) = [0, 1]^t$, $L(\mathbf{z}) = [1, -1]^t$. Then $L(2\mathbf{x} - 3\mathbf{y} + 4\mathbf{z}) = \underline{\quad [6, -7]^t \quad}$

(G) Let $L : R^3 \to R^2$ be a linear transformation such that $L([1, 0, 0]^t) = [1, 1]^t$, $L([0, 1, 0]^t) = [1, -1]^t$, $L([0, 0, 1]^t) = [1, 0]^t$. Then $Ker(L) = \underline{\{\alpha[1, 1, -2]^t\}}$

(H) Let $H \in R^{2\times 2}$ be a Householder matrix defined by $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\mathbf{u} = [1/2, \sqrt{3}/2]^t$. Then $H^t H = \underline{\ I_2\ }$

Test 3 for CS2334

December 6, 2004

(10 pts) 1. *Mark* ◯ *if the statement is* true, *and mark* $\times$ *otherwise.*

(a) A set of nonzero orthogonal vectors are linearly independent.

(b) A set of nonzero orthonormal vectors in R^n must be a basis.

(c) Every square matrix can be factored as QR, where Q is orthogonal and R is upper-Δ.

(d) If $\mathbf{x}, \mathbf{y} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 1$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

(e) If U, V, W are vector subspaces of R^n such that $U \perp V$ and $V \perp W$, $U \perp W$.

(f) If $A \in R^{m \times n}$, then AA^t and A^tA have the same rank.

(g) Let $Q_1, Q_2, \cdots, Q_m \in R^{n \times n}$ be orthogonal, then $\prod_{i=1}^{m} Q_i$ is also orthogonal.

(h) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in R^3, then any Gram-Schmidt orthogonalization process constructs the unique orthonormal basis.

(i) A Householder matrix is symmetric, orthogonal, and has determinant 1.

(j) Let $\mathbf{x}, \mathbf{y} \in R^n$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Then $\mathbf{x}$ and $\mathbf{y}$ are orthonormal.

(k) In R^n, if $\mathbf{p}$ is the projection of $\mathbf{b}$ along the line $\mathbf{a}$, then $\mathbf{a}^t(\mathbf{b} - \mathbf{p}) = 0$.

(10 pts) 2. *Choose the best solution in the following questions.*

(a) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in R^n$ be orthonormal vectors, then $\|2\mathbf{u} - 4\mathbf{v} + 4\mathbf{w}\|_2 =$

(1) 4, (2) 5, (3) 6, (4) 7, (5) none.

(b) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

(1) $\frac{\pi}{6}$, (2) $\frac{\pi}{4}$, (3) $\frac{\pi}{3}$, (4) $\frac{\pi}{2}$, (5) none.

(c) Let $V = \{[b, 0, a]^t |\ a, b \in R\} \subset R^3$, then $dim(V^{\perp}) =$?

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

(d) Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t$ is

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

(e) Let $A \in R^{m\times n}$ have rank r, then $dim(Null(A)) + dim(R(A)) =$?

(1) $m - r$, (2) $n - r$, (3) m, (4) n, (5) none.

(f) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is

(1) $[-1, -1]^t$, (2) $[0, 1]^t$, (3) $[1, 0]^t$, (4) $[1, 1]^t$, (5) none.

(g) Let $H \in R^{n\times n}$ be a Householder matrix and define (i) orthogonal, (ii) symmetric, (iii) $\|H\mathbf{x}\|_2 = 1$ for $\mathbf{x} \in R^n$. What statements of (i), (ii), (iii) are true?

(1) (i),(ii) only, (2) (i),(iii) only, (3) (i),(ii),(iii), (4) (ii),(iii) only, (5) (iii) only.

(h) Let $Q \in R^{n\times n}$ be orthogonal, then $det(Q) =$?

(1) 1, (2) -1, (3) n, (4) $\sqrt{n}$, (5) none.

(i) Let $\mathbf{a} = [1, 1, 1]^t$, $\mathbf{b} = [2, 4, 6]^t$, then the projection of $\mathbf{b}$ onto the line $\mathbf{a}$ is

(1) $\mathbf{a}$, (2) $2\mathbf{a}$, (3) $4\mathbf{a}$, (4) $6\mathbf{a}$, (5) none.

(j) Let $f, g \in C[-1, 1]$, and define the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle sin2\pi x, sin2\pi x\rangle =$?

(1) 0, (2) 1, (3) 2π, (4) 4π, (5) none.

(3 pts) 3. Find the point on the line $y = 2x + 1$ that is closest to $[5, 2]^t$.

(3 pts) 4. Let $\mathbf{a}_1 = [1, 1, 0]^t$, $\mathbf{a}_2 = [2, 3, 0]^t$, and $\mathbf{b} = [4, 5, 6]^t$. Find the projection vector of $\mathbf{b}$ onto the plane that is spanned by the vectors $\mathbf{a}_1 = [1, 1, 0]^t$ and $\mathbf{a}_2 = [2, 3, 0]^t$.

(4 pts) 5. (a) Find the best least squares fitting line to the data $[-1, 0]^t$, $[0, 1]^t$, $[1, 3]^t$, $[2, 9]^t$, (b) plot your linear function from (a) along with the data on a coordinate system.

Solutions for Test 3

December 06, 2004

(10 pts) 1. *Mark* ◯ *if the statement is* true, *and mark* $\times$ *otherwise.*

(◯) **(a)** A set of nonzero orthogonal vectors are linearly independent.

($\times$) **(b)** A set of nonzero orthonormal vectors in R^n must be a basis.

(◯) **(c)** Every square matrix can be factored as QR, where Q is orthogonal and R is upper-Δ.

($\times$) **(d)** If $\mathbf{x}, \mathbf{y} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 1$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

($\times$) **(e)** If U, V, W are vector subspaces of R^n such that $U \perp V$ and $V \perp W$, $U \perp W$.

(◯) **(f)** If $A \in R^{m\times n}$, then AA^t and A^tA have the same rank.

(◯) **(g)** Let $Q_1, Q_2, \cdots, Q_m \in R^{n\times n}$ be orthogonal, then $\prod_{i=1}^{m} Q_i$ is also orthogonal.

($\times$) **(h)** Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in R^3, then any Gram-Schmidt orthogonalization process constructs the unique orthonormal basis.

($\times$) **(i)** A Householder matrix is symmetric, orthogonal, and has determinant 1.

($\times$) **(j)** Let $\mathbf{x}, \mathbf{y} \in R^n$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Then $\mathbf{x}$ and $\mathbf{y}$ are orthonormal.

(◯) **(k)** In R^n, if $\mathbf{p}$ is the projection vector of $\mathbf{b}$ along the line $\mathbf{a}$, then $\mathbf{a}^t(\mathbf{b} - \mathbf{p}) = 0$.

(3 pts) 3. Find the point on the line $y = 2x + 1$ that is closest to $[5, 2]^t$. **Ans :** $[1.4, 3.8]^t$

(3 pts) 4. Let $\mathbf{a}_1 = [1, 1, 0]^t$, $\mathbf{a}_2 = [2, 3, 0]^t$, and $\mathbf{b} = [4, 5, 6]^t$. Find the projection vector of $\mathbf{b}$ onto the plane that is spanned by the vectors $\mathbf{a}_1 = [1, 1, 0]^t$ and $\mathbf{a}_2 = [2, 3, 0]^t$. **Ans :** $[4, 5, 0]^t$

(4 pts) 5. (a) Find the best least squares fitting line to the data $[-1, 0]^t$, $[0, 1]^t$, $[1, 3]^t$, $[2, 9]^t$, (b) plot your linear function from (a) along with the data on a coordinate system. $bf Ans$: (a) $\underline{y = 2.9x + 1.8}$.

(10 pts) 2. *Choose the best solution in the following questions.*

[**3** (a)] Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in R^n$ be orthonormal vectors, then $\|2\mathbf{u} - 4\mathbf{v} + 4\mathbf{w}\|_2 =$

(1) 4, (2) 5, (3) 6, (4) 7, (5) none.

[**4** (b)] Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

(1) $\frac{\pi}{6}$, (2) $\frac{\pi}{4}$, (3) $\frac{\pi}{3}$, (4) $\frac{\pi}{2}$, (5) none.

[**1** (c)] Let $V = \{[b, 0, a]^t |\ a, b \in R\} \subset R^3$, then $dim(V^{\perp}) =?$

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

[**1** (d)] Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t$ is

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

[**4** (e)] Let $A \in R^{m \times n}$ have rank r, then $dim(Null(A)) + dim(R(A)) =?$

(1) $m - r$, (2) $n - r$, (3) m, (4) n, (5) none.

[**4** (f)] Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is

(1) $[-1, -1]^t$, (2) $[0, 1]^t$, (3) $[1, 0]^t$, (4) $[1, 1]^t$, (5) none.

[**1** (g)] Let $H \in R^{n \times n}$ be a Householder matrix and define (i) orthogonal, (ii) symmetric, (iii) $\|H\mathbf{x}\|_2 = 1$ for $\mathbf{x} \in R^n$. What statements of (i), (ii), (iii) are true?

(1) (i),(ii) only, (2) (i),(iii) only, (3) (i),(ii),(iii), (4) (ii),(iii) only, (5) (iii) only.

[**5** (h)] Let $Q \in R^{n \times n}$ be orthogonal, then $det(Q) =?$

(1) 1, (2) -1, (3) n, (4) $\sqrt{n}$, (5) none.

[**3** (i)] Let $\mathbf{a} = [1, 1, 1]^t$, $\mathbf{b} = [2, 4, 6]^t$, then the projection of $\mathbf{b}$ onto the line $\mathbf{a}$ is

(1) $\mathbf{a}$, (2) $2\mathbf{a}$, (3) $4\mathbf{a}$, (4) $6\mathbf{a}$, (5) none.

[**2** (j)] Let $f, g \in C[-1, 1]$, and define the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle sin2\pi x, sin2\pi x \rangle =?$

(1) 0, (2) 1, (3) 2π, (4) 4π, (5) none.

Test 4 for CS2334(02)

10:10-11:59 am, January 3, 2005

(15 pts) 1. *Mark* ◯ *if the statement is* true, *and mark* × *otherwise.*

(a) The eigenvalues of a real symmetric matrix must be real.

(b) Let $A \in R^{n \times n}$ be nonsingular, then there exists a unit lower-Δ matrix L and an upper-Δ matrix U such that $A = LU$.

(c) Let $A \in R^{n \times n}$. If $A^t = A$ and $det(A) > 0$, then A is positive definite.

(d) Let $A \in R^{n \times n}$ be positive definite. If λ is an eigenvalue of A with a corresponding eigenvector $\mathbf{v}$, then $\frac{1}{\lambda}$ must be the eigenvalue A^{-1} with a corresponding eigenvector $\mathbf{v}$.

(e) Let $A \in R^{n \times n}$. Then the sum of eigenvalues equals the trace of A.

(f) A is singular iff $0 \in \lambda(A)$.

(g) The determinant of a tridiagonal matrix is the product of its diagonal elements.

(h) Every positive definite matrix has the same set of eigenvalues and singular values.

(i) Every negative definite matrix must be singular.

(j) A real diagonally dominant matrix must be positive definitive.

(k) For a real matrix, the eigenvectors corresponding to distinct eigenvalues are orthogonal.

(l) Let $\mathbf{x}, \mathbf{y} \in R^n$. Then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ iff $\mathbf{x}$ and $\mathbf{y}$ are orthonormal.

(m) Each Markov matrix M has $(\|M\|_1 = 1$.

(n) Every Householder matrix H has $\|H\|_2 = 1$ and $|H| = 1$.

(o) In R^n, if $\mathbf{p}$ is the projection of $\mathbf{b}$ along the line $\mathbf{a}$, then $\mathbf{a}^t(\mathbf{b} - \mathbf{p}) = 0$.

(15 pts) 2. *Choose the best answer in the following questions.*

(a) Let $V = \{[a-b, b-c, 0]^t |\ a, b, c \in R\} \subset R^3$, then $dim(V^\perp) = ?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none.

(b) Define $E(a) = I - a\mathbf{e}_3\mathbf{e}_2^t \in R^{n\times n}$, if $a \neq 0$, then the inverse matrix of $E(a)$ is

(1) $E(a^{-1})$, (2) $E(-a^{-1})$, (3) $E(a)$, (4) $E(-a)$, (5) none.

(c) Let $A \in R^{m\times n}$ have rank k and let $CS(A)$ be the column space of A, then $dim(Null(A)) + dim(CS(A)) = ?$

(1) m, (2) n, (3) $m-k$, (4) $n-k$, (5) none.

(d) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is

(1) $[1,1]^t$, (2) $[-1,-1]^t$, (3) $[0,1]^t$, (4) $[1,0]^t$, (5) none.

(e) Let $Q \in R^{n\times n}$ be orthogonal, then $det(Q) = ?$

(1) 1, (2) 1 or -1, (3) -1, (4) n, (5) none.

(f) Let $A \in R^{n\times n}$ have eigenvalues $0, 2, 4, \cdots, 2(n-1)$. Then $trace(A) = ?$

(1) n^2, (2) $n(n-1)$, (3) $n(n+1)$, (4) n, (5) none.

(g) Let $V = Span([1,1,1]^t) \subset R^3$, then $dim(V^\perp) = ?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none.

(h) Let $A \in R^{m\times n}$ and $\mathbf{b} \in R^m$, then the condition for $A\mathbf{x} = \mathbf{b}$ must have a solution in R^m is

(1) $m \geq n$, (2) $m < n$, (3) $m = n$, (4) $m \neq n$, (5) none.

(i) Let $L \in R^{n\times n}$ be a unit lower triangular matrix, what is $det(L) + trace(L)$?

(1) 1, (2) n, (3) n+1, (4) n^2, (5) none.

(j) Let $A \in R^{m\times n}$ have $Null(A) = Span(\mathbf{e}_1)$ and $m > n$, what is the rank of A?

(1) n, (2) m, (3) $n-1$, (4) $m-1$, (5) none.

(k) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in R^n$ be orthonormal vectors, then $\|2\mathbf{u} - 4\mathbf{v} + 4\mathbf{w}\|_2 =$

(1) 4, (2) 5, (3) 6, (4) 7, (5) none.

(l) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

(1) $\frac{\pi}{6}$, (2) $\frac{\pi}{4}$, (3) $\frac{\pi}{3}$, (4) $\frac{\pi}{2}$, (5) none.

(m) Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t$ is

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

(n) Let $Q \in R^{n\times n}$ be orthogonal, then $\|Q\|_2 =?$

(1) 0, (2) 1, (3) -1, (4) $\sqrt{n}$, (5) none.

(o) Let $A \in R^{n\times n}$ have diagonal elements $1, 3, 5, \cdots, (2n-1)$. The sum of eigenvalues of A is

(1) n, (2) n^2, (3) $n(n-1)$, (4) $n(n+1)$, (5) none.

(5 pts) 3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(a) Find $det(A)$ and A^{-1}.

(b) Find the eigenvalues and corresponding eigenvectors for A.

(c) Find an *orthogonal* matrix U such that U^tAU is diagonal.

(d) Give a singular value decomposition for A.

(5 pts) 4. Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$. For each $\mathbf{x} \in R^n$, the Rayleigh quotient $\rho(\mathbf{x})$ is defined by

$$\rho(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}$$

(a) For $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{u}_i$ with $\sum_{i=1}^n c_i^2 = 1$, prove that $\rho(\mathbf{x}) = \sum_{i=1}^n \lambda_i c_i^2$

(b) Show that $\lambda_n \leq \rho(\mathbf{x}) \leq \lambda_1$

(c) Show that for $\mathbf{x} \neq \mathbf{0}$, $Min\{\rho(\mathbf{x})\} = \lambda_n$ and $Max\{\rho(\mathbf{x})\} = \lambda_1$

(10 pts) 5. Let $B = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(a) Find the eigenvalues and corresponding eigenvectors of B.

(b) Write the spectrum decomposition of B.

(c) Find the singular values of B.

(d) Find e^B.

(e) Find $\|B\|_2$ and $\|B\|_1$.

Solutions for Test 4 of CS2334(02)

10:10-11:59 am, January 3, 2005

(15 pts) 1. *Mark* $\bigcirc$ *if the statement is* true, *and mark* $\times$ *otherwise.*

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
O	X	X	O	O	O	X	O
(i)	(j)	(k)	(l)	(m)	(n)	(o)	
X	X	X	X	O	X	O	

(15 pts) 2. *Choose the best answer in the following questions.*

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
2	4	2	4	2	2	3	5
(i)	(j)	(k)	(l)	(m)	(n)	(o)	
3	3	3	4	1	2	2	

(5 pts) 3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

- **(a)** $det(A) = 3$ and $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
- **(b)** $\lambda(A) = \{3, 1\}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- **(c)** As was given in **(b)**.
- **(d)** $A = UDU^t$, where $D = diag(3, 1)$.

(5 pts) *4. Skip.

(10 pts) 5.

- **(a)** $\lambda(A) = \{4, 2, -1, -5\}$.
- **(c)** $\sigma(A) = \{5, 4, 2, 1\}$.
- **(e)** $\|B\|_2 = 5$, $\|B\|_1 = 5$.