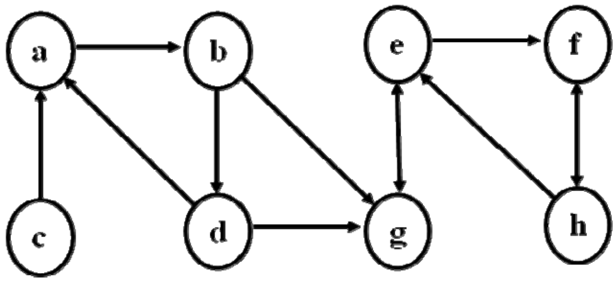
# Algorithms Final Examination
## Jan. 16, 2013

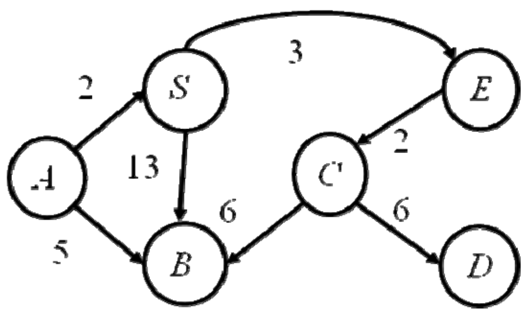
1. (5%) Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$. (Please show your computational steps in detail.)

2. (10%) Please find the strongly connected components of the following directed graph. (Please show your algorithm time complexity and the computational steps in detail.)




3. (10%) Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Kruskal's algorithm run? What if the edge weights are integers in the range from 1 to $W$ for some constant $W$?

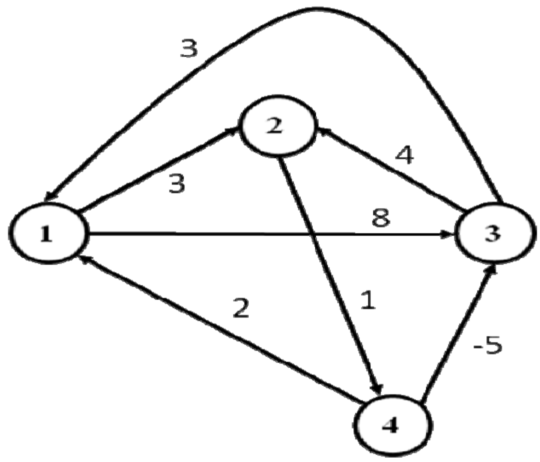
4. (10%) Please find the one-to-all shortest paths from the node $S$ in the following graph. Show the major steps and state clearly the time complexity of your algorithm.




5. (10%) Give a simple example of a directed graph with negative-weight edges for which Dijkstra's algorithm produces incorrect answers to solve the single-source shortest paths problem.

6. (10%) Please find the all-to-all shortest paths cost for the following directed graph. Show the major steps and state clearly the time complexity

of your algorithm.

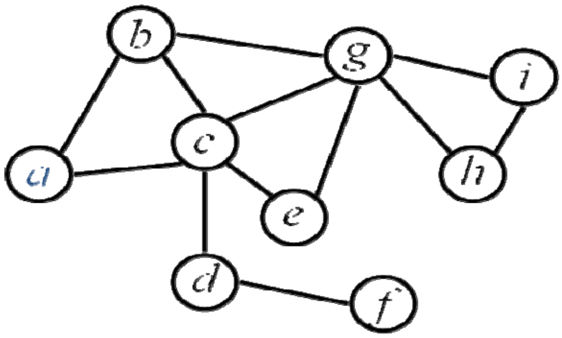



7. (10%) Given a weighted directed graph $G = (V, E)$ with weight function $w$: $E \rightarrow R$, let $h$: $V \rightarrow R$ be any function mapping vertices to real numbers. For each edge $(u,v) \in E$, define $\hat{w}(u,v) = w(u,v) + h(u) - h(v)$. Let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be any path from vertex $v_0$ to $v_k$. Please prove that $p$ is a shortest from $v_0$ to $v_k$ with weight function $w$ if and only if it is a shortest with weight function $\hat{w}$. Furthermore, $G$ has a negative-weight cycle using weight function $w$ if and only if $G$ has a negative-weight cycle using weight function $\hat{w}$.

8. (10%) Assume we have a graph $G = (V, E)$. Please answer the following question.
   a. The time complexity of using Dijkstra's algorithm to solve all-to-all shortest path problem with the binary heap implementation and Fibonacci heap implementation are ___(1)___ and ___(2)___, respectively.
   b. The time complexity of Prim's algorithm with binary heap implementation to find a minimum spanning tree of a graph $G = (V, E)$ is ___(3)___.
   c. The time complexity of using Johnson algorithm to solve all-to-all shortest path problem is ___(4)___.
   d. The time complexity of using Bellman-Ford algorithm to solve all-to-all shortest path problem is ___(5)___.

9. (10%) Please give an approximation algorithm to find the maximum cut of the following graph. Show the optimal solution is at most two times of your

solution.




10. (30%) For each of the following statements, determine whether it is true or false. If the statement is correct, briefly state why. If the statement is wrong, explains why. Your answer will be evaluated based on your explanation and not the True/False marking alone.
    a. We can solve the 0/1 knapsack problem by dynamic-programming method that runs in $O(nW)$ time, where $n$ is the number of items and $W$ is the maximum weight of items that the thief can put in his knapsack. Therefore, the dynamic-programming method is a polynomial-time algorithm.
    b. If we prove that the satisfiability problem can polynomial-time reduce to problem $A$, then problem $A$ is NP-hard.
    c. If we can prove that problem $A$ can polynomial-time reduce to satisfiability problem, then problem $A$ is NP-complete.
    d. If any NP-complete problem can be solved in polynomial time, then NP = P.
    e. The "Halting Problem" is a NP-hard problem.
    f. Any NP-complete problem can be solved in polynomial time if there is an algorithm that can solve the 3-CNF satisfiability problem in polynomial time.
    g. Let $T$ be a minimum spanning tree of $G$. Then, for any pair of vertices of $s$ and $t$, the shortest path from $s$ to $t$ in $G$ is the path from $s$ to $t$ in $T$.
    h. Give a graph $G = (V, E)$ with cost on edges and a set $S \subseteq V$, let $(u, v)$ be an edge such that $(u, v)$ is the minimum cost edge between any vertex in $S$ and any vertex in $V$-$S$. Then the minimum spanning tree of $G$ must include edge $(u, v)$.
    i. If we can find a polynomial-time approximation algorithm with approximation ratio $\rho > 1$ for the general travelling-salesman problem, then P = NP.
    j. If NP = P then NP-hard problem can be solved in polynomial time.