1. Let number of infected people be y

$$\Rightarrow \frac{dy}{dt} = C\cdot y(200-y) = \underbrace{-Cy^2}_{P} + \underbrace{200Cy}_{Q} + \underbrace{200C}_{R}$$

$$\Rightarrow s(t) = \frac{1}{Ct} \Rightarrow s' = \frac{-1}{Ct^2},\quad Cs(200-s) = \frac{-1}{Ct^2} + \frac{200}{t}$$

$$\Rightarrow \text{令 } y = s + \frac{1}{z} \Rightarrow z' + (2Ps + Q)z = -R$$

$$\Rightarrow z =$$

2.

$$\Rightarrow \frac{dQ}{dt} = 0.3\times r - \frac{Q}{120}\times r \Rightarrow Q' = 0.3r - \frac{r}{120}Q$$

$$\Rightarrow Q' + \frac{r}{120}Q = 0.3r \Rightarrow I = e^{\int \frac{r}{120}dt} = e^{\frac{r}{120}t}$$

$$\Rightarrow Q'e^{\frac{r}{120}t} + \frac{r}{120}Qe^{\frac{r}{120}t} = 0.3re^{\frac{r}{120}t}$$

$$\Rightarrow \frac{d}{dx}\left(Q\cdot e^{\frac{r}{120}t}\right) \Rightarrow \int dt \Rightarrow Q\cdot e^{\frac{rt}{120}} = 0.3\times\frac{120}{\cancel{r}}\cancel{r}e^{\frac{rt}{120}} + C = 36\cdot e^{\frac{rt}{120}} + C$$

$$\Rightarrow Q(t) = 36 + C\cdot e^{\frac{-rt}{120}},\ \text{since } Q(0) = Q_0 \Rightarrow 36 + C = Q_0$$

$$\Rightarrow C = Q_0 - 36$$

$$\Rightarrow Q(t) = 36 + (Q_0 - 36)\cdot e^{\frac{-rt}{120}} \quad \#$$

when $t \to \infty \Rightarrow Q(t) = 36$ #

3.

(proof)

since y_1, y_2 are two solutions

$\Rightarrow y_1'' + py_1' + y_1 = 0 \Rightarrow y_1''y_2 + py_1'y_2 + y_1y_2 = 0$

$y_2'' + py_2' + y_2 = 0 \qquad y_2''y_1 + py_2'y_1 + y_1y_2 = 0$

$\Rightarrow (y_1''y_2 - y_2''y_1) + (y_1'y_2 - y_2'y_1)p = 0$ --- ①

where $W' = (y_1'y_2 - y_2'y_1)' = (y_1''y_2 - y_2''y_1) + (y_1'y_2' - y_1'y_2')$

$= (y_1''y_2 - y_2''y_1)$

$\Rightarrow$ ① 改為 $W' + p(x)W = 0 \Rightarrow I = e^{\int p(x)dx}$

$\Rightarrow \frac{d}{dx}(W \cdot e^{\int p(x)dx}) = 0 \Rightarrow W \cdot e^{\int p(x)dx} = c_1$

$\Rightarrow W = c_1 \cdot e^{\int -p(x)dx}$ #

4.

$\Rightarrow x^2y\,p^2 + (xy^2 - x^2y - 2x^3)p + (2x^3 - 2x^2y) = 0, \quad p \equiv y'$

$\Rightarrow (xy)p^2 + (y^2 - xy - 2x^2)p + (2x^2 - 2xy) = 0$

$\Rightarrow (xp + (y - x))(yp + (-2x)) = 0$ ✓

$\Rightarrow p = \frac{-y+x}{x}$ or $\frac{+2x}{y} \Rightarrow y' = \frac{-y+x}{x} = \frac{-y}{x} + 1$

or $y' = \frac{+2x}{y}$

I. calculate : $y' = \frac{-y}{x} + 1 \Rightarrow$ 令 $\frac{y}{x} = u \Rightarrow dy = udx + xdu$

$\Rightarrow \frac{x}{dx}du + \cancel{u} = \cancel{-u+1} \Rightarrow -du = \frac{dx}{x} \Rightarrow -u = \cancel{\ln|x| - c_1}$

$\Rightarrow \cancel{y = -x\ln|x| + c_1x}$

-3

II calculate $\frac{dy}{dx} = \frac{-2x}{y}$ (red: $+2x$) $\Rightarrow y\,dy = -2x\,dx \Rightarrow \int \Rightarrow \frac{y^2}{2} = -x^2 + C_2$

-1

$$\Rightarrow \left(\frac{y^2}{2} + x^2 - C_2\right)\left(y + x\ln|x| - C_1 x\right) = 0$$

※

5. $y'' + y = (x-1)\cos x = x\cos x - \cos x$

I. find $y_h \Rightarrow m^2 + m = 0$? $\Rightarrow m = 0$ or -1

+3

$\Rightarrow \boxed{y_h = C_1 + C_2 e^{-x}}$ ✗

~~II. find y_p ⇒ 令 $y_p = (Ax+B)(C\cos x +$~~

~~$\Rightarrow y_p' = A(C\cos x + D\sin x)$~~

~~$\Rightarrow y_p'' = A(-C\sin x + D\cos x) + (Ax+B)(-C\cos$~~

~~$\Rightarrow y_p'' + y_p = (AC - AC)x\cos x$~~

$\Rightarrow 2A + 2B = 1$, $2A - 2B = 0$ $\Rightarrow A = \frac{1}{4}$, $B = \frac{1}{4}$

$\Rightarrow AC + 2A + BC = \frac{C}{2} + \frac{1}{2} = 0$

$\Rightarrow C = -1$

$\Downarrow$

$$y = y_h + y_p = C_1 + C_2 e^{-x} + (x^2 - x)\left(\frac{1}{4}\cos x + \frac{1}{4}\sin x\right)$$

※

II find y_p ⇒ 令 $y_p = (x+C)(A\cos x + B\sin x)$

$\Rightarrow y_p' = (x+C)(-A\sin x + B\cos x) + (A\cos x + B\sin x)$

$\Rightarrow y_p'' = (x+C)(-A\cos x - B\sin x) + (-A\sin x + B\cos x) + (-A\sin x + B\cos x)$

$\Rightarrow y_p'' + y_p = -2A\sin x + 2B\cos x = (x-1)\cos x \Rightarrow A = 0,\ B = \frac{x-1}{2}$ 不合)

⇒令 $y_p = (x^2 + Cx)(A\cos x + B\sin x)$

$\Rightarrow y_p' = (x^2 + Cx)(-A\sin x + B\cos x) + (2x + C)(A\cos x + B\sin x)$

正負不合

$\Rightarrow y_p'' = (x^2 + Cx)(-A\cos x - B\sin x) + (2x + C + 2)(A\cos x + B\sin x) + (2x + C)(-A\sin x + B\cos x)$

$\Rightarrow y_p'' + y_p = (2A + 2B)x\cos x + (AC + 2A + BC)\cos x + (BC + 2B - AC)\sin x + (2B - 2A)x\sin x$

6. $x^2y'' - 2xy' + 2y = x^2 + 2$

令 $x = e^t$ ($\log x = t$) $\Rightarrow x^2y'' = y''(t) - y'(t)$

$xy' = y'(t)$

$\Rightarrow y''(t) - 3y'(t) + 2y(t) = e^{2t} + 2$

I: find $y_h \Rightarrow m^2 - 3m + 2 = 0 = (m-2)(m-1) \Rightarrow y_h = c_1 e^{2t} + c_2 e^t$

II. 令 $y_p = Ate^{2t} + B$

$\Rightarrow y_p' = 2Ate^{2t} + Ae^{2t}$

$\Rightarrow y_p'' = 4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t}$

$\Rightarrow (4A - 6A + 2A)te^{2t} + \cdots = e^{2t} + 2 \Rightarrow$ ~~$0Ae^{2t} + \cdots = e^{2t} + 2$~~

$+ (4A - 3A)e^{2t} + 2B$ ~~$\Rightarrow$ 令~~

~~$\Rightarrow$ 令 $y_p = Ax^2e^{2t} + B$~~

~~$\Rightarrow y_p' = 2Ax^2e^{2t} + 2Axe^{2t}$~~

~~$\Rightarrow y_p'' = 4Ax^2e^{2t} + 4Axe^{2t} + 2Ae^{2t}$~~

~~$\Rightarrow (4A - 6A + 2A)x^2e^{2t} + (4A - 6A$~~

$A = 1$

$B = 1$

$\Rightarrow y_p = te^{2t} + 1$

$\Rightarrow y = y_h + y_p = c_1 e^{2t} + c_2 e^t + te^{2t} + 1$

$= c_1 x^2 + c_2 x + (\ln x)x^2 + 1$ ※

7. $x^2y'' - 4xy' + 6y = 2(\ln x)^2$

令 $x = e^t$ ($\ln x = t$)

$\Rightarrow y''(t) - 5y'(t) + 6y(t) = 2t^2$

I. find y_h: $m^2 - 5m + 6 = 0$

$\Rightarrow m = 3, 2$

$\Rightarrow y_h = c_1 e^{3t} + c_2 e^{2t}$

II 令 $y_p = At^2+Bt+C$

$\Rightarrow y_p' = 2At+B$

$\Rightarrow y_p'' = 2A$

$\Rightarrow 2A - 10At - 5B + 6At^2 + 6Bt + 6C = 2t^2$

$= 6At^2 + (6B-10A)t + (2A-5B+6C)$

$\Rightarrow A = \frac{1}{3} \Rightarrow 6B - \frac{10}{3} = 0 \Rightarrow B = \frac{5}{9}$

$\Rightarrow \left(\frac{2}{3} - \frac{25}{9} + 6C\right) = 0 \Rightarrow 6C = \frac{25-6}{9} = \frac{19}{9} \Rightarrow C = \frac{19}{54}$

$\Rightarrow y_p = \frac{t^2}{3} + \frac{5}{9}t + \frac{19}{54} \Rightarrow y = y_h + y_p = C_1e^{3t} + C_2e^{2t} + \left(\frac{1}{3}\right)t^2 + \frac{5}{9}(t) + \frac{19}{54}$

$= C_1x^3 + C_2x^2 + \frac{1}{3}(\ln x)^2 + \frac{5}{9}(\ln x) + \frac{19}{54}$ #

8.

$2xy'' + (1-4x)y' + (2x-1)y = e^x$

$\Rightarrow y'' + \underbrace{\left(\frac{1}{2x}-2\right)}_{P} y' + \underbrace{\left(1-\frac{1}{2x}\right)}_{Q} y = \underbrace{\frac{e^x}{2x}}_{R}$

已知 $u = e^x$ 為 homogeneous solution $\Rightarrow$ 令 $\boxed{y = u \cdot v}$

$\Rightarrow v'' + \left(\frac{2e^x + \left(\frac{1}{2x}-2\right)e^x}{e^x}\right)v' = \frac{\frac{e^x}{2x}}{e^x} \Rightarrow v'' + \left(\frac{1}{2x}\right)v' = \frac{1}{2x}$

$\Rightarrow I = e^{\int \frac{1}{2x}dx} = e^{\ln|x|^{\frac{1}{2}}} = x^{\frac{1}{2}} \Rightarrow x^{\frac{1}{2}}v'' + \left(\frac{1}{2}\right)x^{-\frac{1}{2}}v' = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{d}{dx}\left(x^{\frac{1}{2}}v'\right)$

$\Rightarrow \int dx \Rightarrow \sqrt{x}\,v' = \frac{1}{2} \times 2 \cdot \sqrt{x} + C_1$

$\Rightarrow v' = 1 + C_1x^{-\frac{1}{2}} \Rightarrow \int dx \Rightarrow v = x + 2C_1x^{\frac{1}{2}} + C_2$

$\Rightarrow y = v \cdot u = e^x\left(x + 2C_1x^{\frac{1}{2}} + C_2\right)$ #

9.

$$2y''-4y'+2y=\frac{4e^x}{2x-1}$$

$$\Rightarrow y''-2y'+y=\frac{2e^x}{2x-1} \quad R$$

I. find $y_h \Rightarrow m^2-2m+1=0 \Rightarrow m=1$ (重根)

$$\Rightarrow y_h=c_1\underbrace{e^x}_{y_1}+c_2\underbrace{xe^x}_{y_2}$$

II. find $y_p \Rightarrow$ 令 $y_p=\phi_1 y_1+\phi_2 y_2 \Rightarrow \begin{bmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{bmatrix}\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix}=\begin{bmatrix} 0 \\ \frac{2e^x}{2x-1} \end{bmatrix}$

$e^{2x}(1+x)-e^{2x}\cdot x$

$$\Rightarrow \phi_1'=\frac{\left(\frac{-2x}{2x-1}\right)e^{2x}}{e^{2x}}=\left(\frac{-2x}{2x-1}\right)=-1+\frac{-1}{2x-1}$$

$$\Rightarrow \phi_2'=\frac{\frac{2}{2x-1}\cdot e^{2x}}{e^{2x}}=\frac{2}{2x-1}$$

$$\Rightarrow \phi_1=\int(-1)\,dx-\int\frac{\overset{\frac{u}{2}}{dx}}{\underset{u}{2x-1}}=-x-\frac{1}{2}\ln|2x-1|$$

$$\phi_2=\int\frac{2}{\underset{u}{2x-1}}\overset{du}{dx}=\ln|2x-1|$$

$$\Rightarrow y_p=\phi_1 y_1+\phi_2 y_2=e^x\left(-x-\frac{1}{2}\ln|2x-1|\right)+xe^x\left(\ln|2x-1|\right)$$

$$\Rightarrow y=y_h+y_p=\underbrace{c_1e^x+c_2xe^x+(-1)xe^x}_{\downarrow}-\frac{e^x}{2}\ln|2x-1|+xe^x\left(\ln|2x-1|\right)$$

$$=c_1e^x+c_2'xe^x-\frac{e^x}{2}\ln|2x-1|+xe^x\left(\ln|2x-1|\right)$$