Final Exam

2016/1/3

1. (18 points) There are **3** mistakes in the following proof. Pick up **3** correct answers.

Let $F\colon [1,\infty) \to \mathbb{R}$ be defined by $F(x) = \int_1^x \sqrt{1+\frac{1}{3}\sin^6 t}\,dt$, $x \in [1,\infty)$. Then prove that $\exists\ a \in [1,\infty)$ such that $F(a) = 2$.

Proof: Since $\sin^6 t \geq 0$, $\forall\ t \in [1,\infty)$, $\sqrt{1+\frac{1}{3}\sin^6 t} \geq 1$, $\forall\ t \in [1,\infty)$. So $F(x) = \int_1^x \sqrt{1+\frac{1}{3}\sin^6 t}\,dt \geq \int_1^x 1dt = x-1$, $\forall\ x \in [1,\infty)$. So $F(4) \geq 4-1 = 3 > 2$. Let $f(t) = \sqrt{1+\frac{1}{3}\sin^6 t}$, $t \in \mathbb{R}$. Then by the chain rule, $f'(t) = (1+\frac{1}{3}\sin^6 t)^{-\frac{1}{2}}\sin^5 t\cos t$, $\forall\ t \in \mathbb{R}$. Then since f is differentiable on $\mathbb{R}$, by Theorem 6.1.4, f is continuous on $[1,\infty)$, hence by Fundemental theorem of calculus I, F is continuous on $[1,\infty)$. Then since $F(1) = \int_1^1 \sqrt{1+\frac{1}{3}\sin^6 t}\,dt = 0$, by Mean value theorem, $\exists\ a \in (1,2)$ such that $F(a) = 2$. ∎

2. (12 points) There are **2** mistakes in the following proof. Pick up **2** correct answers.

Prove that there exists a positive number δ such that for $x \in [-\delta, 0]$ and $n \in \mathbb{N}$, we have

$$e^x \leq 1 - (\frac{1}{2n}x^{2n} + \frac{1}{2n-2}x^{2n-2} + \frac{1}{2n-4}x^{2n-4} + \cdots + \frac{1}{2}x^2).$$

Proof: Let $f(x) = e^x + p(x) - 1$, where $p(x) = \frac{1}{2n}x^{2n} + \frac{1}{2n-2}x^{2n-2} + \frac{1}{2n-4}x^{2n-4} + \cdots + \frac{1}{2}x^2$, $n \in \mathbb{N}$. Then since $\lim_{x\to 0^-} e^x = 1$, $\forall\ \delta_1 > 0$, $\exists\ \frac{1}{4} > 0$ such that if $-\delta_1 < x < 0$, then $|e^x - 1| < \frac{1}{4}$, hence $\frac{3}{4} < e^x < \frac{5}{4}$. Moreover, since $\lim_{x\to 0^-} p'(x) = 0$, where $p'(x) = x^{2n-1} + x^{2n-3} + x^{2n-5} + \cdots + x$, for $\frac{1}{4} > 0$, $\exists\ \delta_2 > 0$

such that if $-\delta_2 < x < 0$, then $-\frac{1}{4} < p'(x) < \frac{1}{4}$. So take $\delta = \min\{\delta_1, \delta_2\} > 0$. So if $-\delta < x < 0$, then $-\delta_1 < x < 0$ and $-\delta_2 < x < 0$, hence $\frac{3}{4} < e^x < \frac{5}{4}$ and $-\frac{1}{4} < p'(x) < \frac{1}{4}$. So $f'(x) = e^x + p'(x) > 0$, $\forall\, x \in (-\delta, 0)$. Then since $f(x) = e^x + p(x) - 1$ is continuous on $[-\delta, 0]$, by Theorem 3.5.3, f is concave up on $[-\delta, 0]$. Finally since $f(0) = e^0 + p(0) - 1 = 0$, $f(x) = e^x + p(x) - 1 \leq 0$, $\forall\, x \in [-\delta, 0]$. ■

3. (18 points) There are **3** mistakes in the following proof. Pick up **3** correct answers.

If f is differentiable on $(0, 2)$, f is continuous on $[0, 2]$, the range of f contains 0, and $|f'(x)| \leq \frac{1}{3}|f(x)|$, $\forall\, x \in (0, 2)$, then prove f is constant on $[0, 2]$.

Proof: Since f is continuous on $[0, 2]$, by the ~~first derivative~~ test, $\exists\, x_1, x_2 \in [0, 2]$ such that $f(x_1)$ is the absolute maximum of f and $f(x_2)$ is the absolute minimum of f. Then we discuss in the following two cases.

Case 1: $f(x_1) = f(x_2)$.

Since $f(x_1)$ is the absolute maximum and $f(x_2)$ is the absolute minimum, $f(x_2) \leq f(x) \leq f(x_1)$, $\forall\, x \in [0, 2]$. Then since $f(x_1) = f(x_2)$, $\exists\, x \in [0, 2]$ such that $f(x_2) = f(x) = f(x_1)$. So f is constant on $[0, 2]$.

Case 1: $f(x_1) \neq f(x_2)$.

First suppose $x_1 < x_2$. Then since f is differentiable on $(0, 2)$ and is continuous on $[0, 2]$, f is differentiable on (x_1, x_2) and is continuous on $[x_1, x_2]$. So by Mean-value Theorem, $\exists\, c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Then since $f(x_1) \neq f(x_2)$, $f'(c) \neq 0$. So we have $\left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| = |f'(c)| \leq \frac{1}{3}|f(c)|$. Then since $f'(c) \neq 0$, $|f(c)| \geq 3|f'(c)| > 0$. So

$$|f(x_2) - f(x_1)| \leq \frac{1}{3}|f(c)| \cdot \overset{2}{|x_2 - x_1|} \leq \frac{1}{3}|f(c)| < |f(c)|.$$

Then since the range of f contains 0, $f(x_1) \geq 0$ and $f(x_2) \leq 0$. So $|f(x_2) - f(x_1)| = |f(x_2)| + |f(x_1)| < |f(c)|$. This is a contradiction. On the other hand, if $x_1 > x_2$, by the analogous argument, this also leads to a contradicton. ■

4. (7 points) Find the volumn by revolving about the x-axis the region bounded by the graphs $y = 4x + 6$ and $y = x^3 - x^2 + 2x + 6$.

$u = \ln x$

$du = \frac{1}{x}\,dx$

5. (8 points) Let $S_n = \frac{1}{n}\frac{\ln(1+\frac{1}{n})}{1+\frac{1}{n}} + \frac{1}{n}\frac{\ln(1+\frac{2}{n})}{1+\frac{2}{n}} + \frac{1}{n}\frac{\ln(1+\frac{3}{n})}{1+\frac{3}{n}} + \cdots + \frac{1}{n}\frac{\ln(1+\frac{n}{n})}{1+\frac{n}{n}}$. Find $\lim_{n\to\infty} S_n$.

$= \int_1 \frac{b-a}{n} f($

$\int_1^2 \frac{\ln x}{x}\,dx = \int_1 u\,du = \frac{1}{2}u^2\Big|_1^2$

$= \frac{1}{2}\ln x^2\Big|_1^2$

$= \ln x\Big|_1^2$

$= \ln 2 - \ln 1$

6. (8 points) Compute $\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{3}} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx$.

$x(1-x^2)$

7. (7 points) Let $f(x) = \ln\left|\tan^3(x^4+1)\right|$. Find $f'(x)$.

8. (7 points) Let $F(x) = \int_0^{\sin^3(x^5+1)} \frac{1}{t^3+1}\,dt$. Find $F'(x)$.

9. (7 points) Find the area bounded by the graph $y = (\frac{x}{2}-1)^2 + 2$, $x \in [-1, 7]$, and the graphs $y = 2x + 3$, and $y = -x + 12$.

$-\frac{2}{7} + \frac{1}{3} - \frac{1}{5} = \frac{-30+35-21}{105} = \frac{-16}{105}$

$F(0) = 0$ F

10. (8 points) Let $F(x) = \int_0^x (\sin^2 t - \frac{1}{4})\cos t\,dt$, $x \in [0, \pi]$. Find the absolute maximum and the points of inflection of F.

$(x^3-x^2+2x+6)(x^3-x^2+2x+6)$

$= x^6 - 2x^5 + 3x^4 + 8x^3 - 8x^2 + 24x + 36$

$-\frac{2}{7} + \frac{2}{3} - \frac{9}{5} = \frac{-30+70-12}{105}$

$= \frac{18}{105}$

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$16x^2+48x+36$

$\begin{cases} 4x+6 \\ x^3-x^2+2x+6 \end{cases}$

$\int_0^2 \pi\left((4x+6)^2 - (x^3-x^2+2x+6)^2\right)dx + \int_{-1}^0 \pi\left((x^3-x^2+2x+6)^2 - (4x+6)^2\right)dx$

$\int_0^2 (-x^6+2x^5-3x^4-8x^3+24x^2+24x)\,dx + \int_{-1}^0 (x^6-2x^5+3x^4+8x^3-24x^2-24x)$

Final Exam Answers B

1. 3 answers

(A) $f'(t) = \frac{1}{2}(1 + \frac{1}{3}\sin^6 t)^{-\frac{1}{2}}(1 + 2\sin^5 t\cos t)$,
(B) $f'(t) = (1 + \frac{1}{3}\sin^6 t)^{-\frac{1}{2}}\cos^5 t$,
(C) $f'(t) = (1 + \frac{1}{3}\sin^6 t)^{-\frac{1}{2}}\sin^5(\cos t)$,
(D) $f'(t) = \frac{1}{2}(2\sin^5(\cos t))^{-\frac{1}{2}}$,
(E) $\exists\ a \in [1,2]$ such that $F(a) = 2$,
(F) $\forall\ a \in (1,4)$, $F(a) = 2$,
(G) $\exists\ a \in (1,5)$ such that $F(a) = 2$,
(H) $\exists\ a \in (3,4)$ such that $F(a) = 2$,
(I) by Theorem 2.1.6,
(J) by Theorem 3.2.5,
(K) by the first derivative test,
(L) by Theorem 3.2.4,
(M) by Fundemental theorem of calculus II,
(N) by Theorem 3.5.3,
(O) by Theorem 1.4.8,
(P) by Theorem 1.4.6,
(Q) by Intermediate value theorem,
(R) by Extreme value theorem,

2. 2 answers

(A) by Theorem 3.2.4, f is increasing on $(-\delta, 0)$.
(B) by Theorem 3.2.4, f is decreasing on $(-\delta, 0)$.
(C) by Theorem 3.2.5, f is increasing on $[-\delta, 0]$.
(D) by Theorem 3.2.5, f is decreasing on $[-\delta, 0]$.
(E) by Theorem 3.5.3, f is concave up on $(-\delta, 0)$.
(F) by Theorem 3.5.3, f is concave down on $[-\delta, 0]$.
(G) So take $\delta = \frac{\delta_1 + \delta_2}{2} > 0$.
(H) So take $\delta = \max\{\delta_1, \delta_2\} > 0$.
(I) So $f'(x) = e^x + p'(x) = 0$,
(J) So $f'(x) = e^x + p'(x) < 0$,
(K) Finally since $f(-\delta) = e^{-\delta} + p(-\delta) - 1 < 0$,

(**L**) Finally since $f(-\delta) = e^{-\delta} + p(-\delta) - 1 > 0$,
(**M**) Finally since $f(0) = e^0 + p(0) - 1 < 0$,
(**N**) Finally since $f(0) = e^0 + p(0) - 1 > 0$,
(**O**) $\exists\ \frac{1}{4} > 0,\ \forall\ \delta_2 > 0$, if $-\delta_2 < x < 0$,
(**P**) for $\frac{1}{4} > 0,\ \exists\ \delta_1 > 0$ such that if $-\delta_1 < x < 0$,
(**Q**) $\forall\ \delta_2 > 0,\ \exists\ \frac{1}{4} > 0$ such that if $-\delta_2 < x < 0$,
(**R**) $\exists\ \frac{1}{4} > 0,\ \forall\ \delta_1 > 0$, if $-\delta_1 < x < 0$,

3. 3 answers
(**A**) by Fundemental theorem of calculus I,
(**B**) by Theorem 3.2.4,
(**C**) by Theorem 3.2.5,
(**D**) by Theorem 6.1.4,
(**E**) by the chain rule,
(**F**) by Theorem 3.5.3,
(**G**) by Theorem 1.4.8,
(**H**) by Theorem 1.4.6,
(**I**) by Intermediate value theorem,
(**J**) by Extreme value theorem,
(**K**) **So** $|f(x_2) - f(x_1)| \le \frac{1}{3}|f(c)| \cdot |x_2 - x_1| < \frac{1}{3}|f(c)| \le |f(c)|$.
(**L**) **So** $|f(x_2) - f(x_1)| \le \frac{1}{3}|f(c)| \cdot |x_2 - x_1| < \frac{2}{3}|f(c)| \le |f(c)|$.
(**M**) **So** $|f(x_2) - f(x_1)| \le \frac{1}{3}|f(c)| \cdot |x_2 - x_1| \le \frac{2}{3}|f(c)| \le |f(c)|$.
(**N**) **So** $|f(x_2) - f(x_1)| \le \frac{1}{3}|f(c)| \cdot |x_2 - x_1| \le \frac{2}{3}|f(c)| < |f(c)|$.
(**O**) Then since $f(x_1) \neq f(x_2)$, $f(c) \neq 0$.
(**P**) Then since $f(x_1) \neq f(x_2)$, $f'(c) > 0$.
(**Q**) Then since $f(x_1) = f(x_2)$, $f(x_2) = f(x) = f(x_1)$, $\forall\ x \in [0, 2]$.
(**R**) Then since $f(x_1) = f(x_2)$, this is a contradiction.
(**S**) $\forall\ c \in (x_1, x_2)$, $f'(c) > 0$.
(**T**) $\exists\ c \in (x_1, x_2)$ such that $f'(c) = 0$.

B

4 - 10.

(a1) $\frac{15x^4\sin^2(x^5+1)\cos(x^5+1)}{\sin^9(x^5+1)+1}$, **(a2)** $\frac{15x^4\sin^2(x^5+1)\cos(x^5+1)}{\sin^3(x^5+1)+1}$, **(a3)** $\frac{15x^4\sin^2(x^5+1)\cos^2(x^5+1)}{\sin^9(x^5+1)+1}$,

(a4) $\frac{1}{\sin^9(x^5+1)+1}$, **(a5)** $\frac{15x^4\sin^2(x^5+1)}{\sin^3(x^5+1)+1}$, **(a6)** $\frac{1145}{21}\pi$, **(a7)** $\frac{333}{7}\pi$, **(a8)** $\frac{162}{35}\pi$,

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(a9) $\frac{1072}{21}\pi$, **(a10)** $\frac{37}{12}\pi$, **(a11)** 36, **(a12)** $\frac{21}{2}$, **(a13)** $\frac{45}{2}$, **(a14)** $\frac{45}{4}$, **(a15)** $\frac{99}{4}$,

(a16) $\frac{12x^3}{\tan(x^4+1)}$, **(a17)** $\left|\frac{1}{\tan(x^4+1)}\right|$, **(a18)** $\left|\frac{12x^3\sec(x^4+1)}{\tan(x^4+1)}\right|$, **(a19)** $12x^3\sec(x^4+1)$,

(a20) $\frac{12x^3\sec^2(x^4+1)}{\tan(x^4+1)}$, **(a21)** $(\ln 2)^2$, **(a22)** $\frac{3}{2}$, **(a23)** $-1+\frac{1}{2}\ln 2$, **(a24)** $\ln 2$,

(a25) $\frac{1}{2}(\ln 2)^2$, **(a26)** 10, **(a27)** -10, **(a28)** $-\frac{45}{8}$, **(a29)** $-\frac{51}{8}$, **(a30)** $\frac{51}{8}$,

(a31) 0 is the absolute maximum. $x=\frac{\pi}{6}$ and $x=\frac{5\pi}{6}$ are the points of inflection.
(a32) $\frac{1}{12}$ is the absolute maximum. $x=\frac{\pi}{3}$ and $x=\frac{2\pi}{3}$ are the points of inflection.
(a33) 0 is the absolute maximum. There are no any points of inflection.
(a34) $\frac{7}{12}$ is the absolute maximum. $x=\frac{\pi}{3}$ and $x=\frac{2\pi}{3}$ are the points of inflection.
(a35) $\frac{1}{4}$ is the absolute maximum. $x=\frac{\pi}{6}$ and $x=\frac{5\pi}{6}$ are the points of inflection.

Final Exam References

Theorem 1.4.6. If f and g are continuous at c, then
(a) $f \pm g$ is continuous at c (sum rule),
(b) αf is continuous at c, $\alpha \in \mathbb{R}$ (constant multiple),
(c) fg is continuous at c (product rule),
(d) $\frac{f}{g}$ is continuous at c, where $g(c) \neq 0$ (quotient rule).

Theorem 1.4.8. If g is continuous at c, and f is continuous at $g(c)$, then the composite function $h = f \circ g$ is continuous at c.

Theorem 2.1.6. If f is differentiable at c, then f is continuous at c.

Theorem 3.2.4. If f is differentiable on an open interval I, then
(1) if $f'(x) > 0$, $\forall\, x \in I$, then f is increasing on I,
(2) if $f'(x) < 0$, $\forall\, x \in I$, then f is decreasing on I,
(3) if $f'(x) = 0$, $\forall\, x \in I$, then f is constant on I.

Theorem 3.2.5. If f is differentiable on the interior of an interval I, and is continuous on I, then
(1) if $f'(x) > 0$ for all x in the interior of I, then f is increasing on I,
(2) if $f'(x) < 0$ for all x in the interior of I, then f is decreasing on I,
(3) if $f'(x) = 0$ for all x in the interior of I, then f is constant on I.

Theorem 3.5.3. If f is twice differentiable on an open interval I, then
(1) if $f''(x) > 0$, $\forall\, x \in I$, then f' is increasing on I, so f is concave up on I,
(2) if $f''(x) < 0$, $\forall\, x \in I$, then f' is decreasing on I, so f is concave down on I.

Theorem 6.1.4. If f is 1-1 and onto, f is differentiable at a, $f'(a) \neq 0$, and $f(a) = b$, then f^{-1} is differentiable at b, and $(f^{-1})'(b) = \frac{1}{f'(a)}$.