Algorithms Middle Examination
Dec. 8, 2017

1. Short questions:

   a. (2%) Consider a modification Quicksort algorithm, such that each time Partition is called, the median of the partitioned array is found (using the SELECT algorithm) and used as a pivot. The worst case running of the algorithm is:

      (1)__

   b. (6%) If a data structure supports an operation *seefoo* such that a sequence of n *seefoo*'s operations takes $\theta(n \log n)$ time to perform in the worst case, then the amortized cost of a seefoo operation is: θ _(2)__, while the actual time of a *seefoo* operation could be as high as: θ ___(3)__.

2. For each of the following statements, determine whether it is true or false. If the statement is correct, briefly state why. If the statement is wrong, explain why. Your answer will be evaluated based on your explanation and not the True/False marking alone.

   a. (4%) We have a dynamic programming (DP) solution to the 0/1 Knapsack problem that runs in O(nW) time, where n is the number of items and W is the maximum weight of items that the thief can put in his knapsack. Therefore, the DP algorithm is a *polynomial* time algorithm.

   b. (2%) The total amortized cost of a sequence of operations gives an upper bound on the actual cost of the sequence.

3. (6%) Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$ by making a change of variables. Your solution should be asymptotically tight. Do not worry about whether values are integral.

4. (5%) What is wrong with the following argument? Your friend has tried to prove 1+2+…+ n = O(n). His proof is by induction. When n = 1, 1 = O(n). Assume 1+2+…+k = O(n). When n = k+1, 1+2+…+k+(k+1) = O(n) + (k+1) = O(n) + O(n) = O(n). So, 1+2+…+n = O(n).

5. (10%) Give an algorithm to find the first kth smallest elements of a set of n distinct integers in O(n + k log k) time.

6. (10%) Consider a variant of the matrix-chain multiplication problem in which the goal is to parenthesize the sequence of matrices so as to maximize, rather than minimize, the number of scalar multiplications. Does this problem exhibit optimal substructure? Please explain in detail.

7. (15%) Peter is an owner of a Japanese sushi restaurant and today he invites you for dinner at his restaurant. In front of you are n sushi dishes that are arranged in a line. All dishes are different and they have different costs. Peter hopes that you can select the dishes you want, starting from the left to the right. However, there is a further restriction: when you select a dish, say A, the next dish you can select must cost higher than A. Design an $O(n^2)$-time algorithm to select the dishes so as to maximize the total costs. (You need to give the algorithm and analysis the time complexity step by step in your algorithm.)

8. (10%) We have a bag which has a weight capacity of x kilograms. We also have n items, each of different weights. Our target is to pack as many items as possible in the bag, without exceeding the weight capacity. Note that, we want to maximize the number of items, not the total weights of the items. Design an algorithm to determine the desired items in $O(n)$ time. (You need to give the algorithm and analysis the time complexity step by step in your algorithm.)

9. (10%) In 1952, David Huffman thinks of a greedy algorithm to obtain the optimal prefix code tree. Let c and c' be chars with least frequencies. Please prove that there is some optimal prefix code tree with c and c' sharing the same parent, and the two leaves are farthest from root.

10. (10%) What is the total cost of executing n of the stack operations PUSH, POP, and MULTIPOP, assuming that the stack begins with S_0 objects and finishes with S_n objects?

11. (10%) In the dynamic table problem, if the table T is full before insertion of an item, we expand T by doubling its size and if table T is below (1/4)-full after deletion of an item, we contract table T by halving its size. We define a potential function Φ as follows:

    (a) If table T is at least half full: $\Phi(T) = 2\,\text{num}(T) - \text{size}(T)$. 2 num(T) - size(T)

    (b) If table T is less than half full: $\Phi(T) = \text{size}(T)/2 - \text{num}(T)$. - 2n(T-1) + size(T-1)

    Initially, the table T is empty. Assume we have the following 12 operations on the table T: *Insert, Insert, Insert, Insert, Delete, Insert, Insert, Delete, Delete, Delete, Delete, Insert*. Please show the amortized cost α_i and actual cost c_i of the above operations, for $1 \le i \le 12$. = 10 - 8 = 2 - 8 + 8