(1)

Clearly $\mathrm{R}(T^2) \subseteq \mathrm{R(T)}$ and $rank(T) = rank(T^2)$

By theorem 2.2 : Let V and W be vector spaces, and let T : V -> W be linear. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V, then R(T) = span(T(β)) = span ({T(v_1), T(v_2), …T(v_n)}).
We can get a basis of V which is also a basis of R(T).
So dim(V) = rank(T).

By theorem 2.3 : nullity (T) + rank(T) = dim(V).
So the nullity(T) = 0.
Then N(T) = {0}

(3)

Prove $\mathrm{T(f(x) + f(y)) = T(f(x)) + T(f(y))}$, and $\mathrm{T(c \cdot f(x)) = c \cdot T(f(x))}$
Then, T(f(x)) is linear

(4)

$\mathrm{T}(1, 0, -1) = (1, 2) = -5(1, -1) + 3(2, -1)$
$\mathrm{T}(1, 2, 1) = (3, 0) = -3(1, -1) + 3(2, -1)$
$\mathrm{T}(-1, 1, 1) = (0, -2) = 4(1, -1) + (-2)(2, -1)$

$$[\boldsymbol{T}]_{\boldsymbol{\beta}}^{\boldsymbol{\gamma}} = \begin{bmatrix} \mathbf{-5} & \mathbf{-3} & \mathbf{4} \\ \mathbf{3} & \mathbf{3} & \mathbf{-2} \end{bmatrix}$$