First Midterm Examination, Advanced Calculus I ,Practice sheet

Total 110 pts.

1. (10%)(i) Let S be an infinite set. State the definition that S is countable.

   (ii) Let Z be the set of rational numbers. Show that $Z \times Z$ is countable.

2. (20%) (i) Let $S \subseteq R$. State the definition of $\sup S$.

   (ii) Let $S \subseteq R$ be bounded above and d be an upper bound of S. Show that d=$\sup S \Leftrightarrow$ for any $\varepsilon > 0$ there exists $x \in S$ such that $d - \varepsilon < x$.

   (iii) Let $\{a_k\}$ be a bounded above and monotone increasing sequence in R.

   Show that $\lim_{k\to\infty} a_k = \sup S, where\ S = \{a_k\}$.

3. (10%) (i) Let $\{a_n\}$ be a sequence in R. State the definitions of $\limsup a_n$ and $\liminf a_n$.

   (*ii*) Find $\limsup a_n$ and $\liminf a_n$ if a_n is given by

   (a) $n^2 \sin(\frac{1}{2} n\pi)$ $\quad (b) \frac{n}{3} - [\frac{n}{3}]$

4.(10%) A set S in R^n is called convex if for every pair of points x and y in S and every real number θ satisfying $0<\theta<1$, we have $\theta x+(1-\theta)y \in S$. Interpret this statement geometrically in R^2 and R^3. Prove

   (i) every n-ball is convex

   (ii) The interior of a convex set is convex.

   (iii) The closure of a convex set is convex.

5.(15%) (i) Let $A \subseteq R^n$. State the definition that A is an closed set in R^n.

   (ii) Show that $B(a,r) = \{x \in R^n : \| x - a \| \le r\}$ is a closed set.

   (iii) Show that intersection of arbitary collection of closed sets is closed.

6.(15%) Determine all accumulation points of the following sets and decide whether the sets are open or closed (or neither).

   (*a*) Q, the set of rational numbers

   (b) S=$\{(-1)^n + \frac{1}{m} : m, n = 1,2,3,....\}$

   (*c*) S=$\{(x,y): x^2 - y \ge 0\}$

7.(10%) Let $A \subseteq R^n$. Show that if for any sequence $\{x_k\} \subseteq A, x_k \to x$ we have $x \in A$, then A is a closed set.

8.(10%) Let $A, B \subseteq R^n$. Show that

   $(i)\ Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$

   $(ii) A \cap Cl(B) \subseteq Cl(A \cap B)$ if A is open

9.(10%) Let $A \subseteq R^n$. Show that A', the set of accumulation points of A is a closed set.