(15 pts) 2. *Choose the best answer in the following questions.*

$x_1 + x_2 = a - c \Rightarrow$ two dim

2. **(a)** Let $V = \{[a-b, b-c, 0]^t \mid a, b, c \in R\} \subset R^3$, then $dim(V^{\perp}) =?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none. $3-2=1$

4 2. **(b)** Define $E(a) = I - a\mathbf{e}_3\mathbf{e}_2^t \in R^{n\times n}$, if $a \neq 0$, then the inverse matrix of $E(a)$ is

(1) $E(a^{-1})$, (2) $E(-a^{-1})$, (3) $E(a)$, (4) $E(-a)$, (5) none.

$n-k$

2. **(c)** Let $A \in R^{m\times n}$ have rank k and let $CS(A)$ be the column space of A, then $dim(Null(A)) + dim(CS(A)) =?$

k

(1) m, (2) n, (3) $m-k$, (4) $n-k$, (5) none.

$A^TAx = A^Tb$

4. **(d)** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, the least squares solution of $A\mathbf{x} = \mathbf{b}$ is

(1) $[1,1]^t$, (2) $[-1,-1]^t$, (3) $[0,1]^t$, (4) $[1,0]^t$, (5) none.

2. **(e)** Let $Q \in R^{n\times n}$ be orthogonal, then $det(Q) =?$

(1) 1, (2) 1 or -1, (3) -1, (4) n, (5) none.

2. **(f)** Let $A \in R^{n\times n}$ have eigenvalues $0, 2, 4, 6 \cdots, 2(n-1)$. Then $trace(A) =?$

(1) n^2, (2) $n(n-1)$, (3) $n(n+1)$, (4) n, (5) none.

3. **(g)** Let $V = Span([1,1,1]^t) \in R^3$, then $dim(V^{\perp}) =?$

(1) 0, (2) 1, (3) 2, (4) 3, (5) none.

5. **(h)** Let $A \in R^{m\times n}$ and $\mathbf{b} \in R^m$, then the condition for $A\mathbf{x} = \mathbf{b}$ must have a solution in R^m is

(1) $m \geq n$, (2) $m < n$, (3) $m = n$, (4) $m \neq n$, (5) none. $b \in R(A)$

3. **(i)** Let $L \in R^{n\times n}$ be a unit lower triangular matrix, what is $det(L) + trace(L)$?

(1) 1, (2) n, (3) n+1, (4) n^2, (5) none. $= 1$

3. **(j)** Let $A \in R^{m\times n}$ have $Null(A) = Span(\mathbf{e}_1)$ and $m > n$, what is the rank of A?

(1) n, (2) m, (3) $n-1$, (4) $m-1$, (5) none. $n - r = 1 \Rightarrow r = n-1$

3. **(k)** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in R^n$ be orthonormal vectors, then $\|2\mathbf{u} - 4\mathbf{v} + 4\mathbf{w}\|_2 =$

(1) 4, (2) 5, (3) 6, (4) 7, (5) none. $4+16+16=36$

4. (l) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y}$ is

(1) $\frac{\pi}{6}$, (2) $\frac{\pi}{4}$, (3) $\frac{\pi}{3}$, (4) $\frac{\pi}{2}$, (5) none.

1. (m) Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t$ is

(1) 1, (2) 2, (3) 3, (4) 4, (5) none.

2. (n) Let $Q \in R^{n\times n}$ be orthogonal, then $\|Q\|_2 =$?

(1) 0, (2) 1, (3) -1, (4) $\sqrt{n}$, (5) none.

2. (o) Let $A \in R^{n\times n}$ have diagonal elements $1, 3, 5, \cdots, (2n-1)$. The sum of eigenvalues of A is

(1) n, (2) n^2, (3) $n(n-1)$, (4) $n(n+1)$, (5) none.

(5 pts) 3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(a) Find $det(A)$ and A^{-1}. $4-1=3$ $\quad \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(b) Find the eigenvalues and corresponding eigenvectors for A.

(c) Find an *orthogonal* matrix U such that U^tAU is diagonal.

(d) Give a singular value decomposition for A. $A = U\Sigma V^t$

(b) $(\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 3$

$2a + b = a$, $a + 2b = b$ $\quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad 2a + b = 3a$, $a + 2b = 3b$ $\quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(C)

$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(d)

$AV = \Sigma U$. $V = U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. $\Sigma = \Lambda$

$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(5 pts) 4. Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$. For each $\mathbf{x} \in R^n$, the Rayleigh quotient $\rho(\mathbf{x})$ is defined by

$$\rho(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}$$

(a) For $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{u}_i$ with $\sum_{i=1}^n c_i^2 = 1$, prove that $\rho(\mathbf{x}) = \sum_{i=1}^n \lambda_i c_i^2$

(b) Show that $\lambda_n \leq \rho(\mathbf{x}) \leq \lambda_1$

(c) Show that for $\mathbf{x} \neq \mathbf{0}$, $Min\{\rho(\mathbf{x})\} = \lambda_n$ and $Max\{\rho(\mathbf{x})\} = \lambda_1$

1

$U^t A U = \Lambda$

$\rho(\vec{x}) = \frac{\vec{x}^t A \vec{x}}{\vec{x}^t \vec{x}} = \langle A\vec{x}, \vec{x}\rangle$ $\quad \vec{x} = \sum_{i=1}^n c_i \vec{u}_i$. with $\sum c_i^2 = 1$.

(a) proof $\rho(\vec{x}) = \sum_{i=1}^n \lambda_i c_i^2$

$\vec{x}^t \vec{x} = (\sum c_i \vec{v}_i)^t (\sum c_i \vec{v}_i) = \sum\sum c_i c_j \langle \vec{u}_i, \vec{u}_j \rangle = \sum_{i=1}^n c_i^2 = 1$

for $\vec{u}_i . \vec{u}_j$ is orthonormal

$\langle A\vec{x}, \vec{x}\rangle = \langle A \sum_{i=1}^n c_i \vec{u}_i, \sum_{i=1}^n c_i \vec{u}_i \rangle = \langle \sum c_i \lambda_i \vec{u}_i, \sum c_i \vec{u}_i \rangle = \sum_{i=1}^n \lambda_i c_i^2 \|\vec{u}\|_2^2$

$\rho(\vec{x}) = \frac{\sum_{i=1}^n \lambda_i c_i^2 \|\vec{u}\|_2^2}{1} = \sum_{i=1}^n \lambda_i c_i^2$

(b) $\rho(\vec{x}) = \sum_{i=1}^n \lambda_i c_i^2 \leq \sum_{i=1}^n \lambda_1 c_i^2 = \lambda_1 \sum_{i=1}^n c_i^2 = \lambda_1$

$\rho(\vec{x}) = \sum_{i=1}^n \lambda_i c_i^2 \geq \sum_{i=1}^n \lambda_n c_i^2 = \lambda_n \sum_{i=1}^n c_i^2 = \lambda_n$

$\Rightarrow \lambda_n \leq \rho(\vec{x}) \leq \lambda_1$

(c) proof $max\{\rho(x)\} = \lambda_1$

let $\vec{x} = \vec{u}_1$. $\langle \vec{x}, \vec{x} \rangle = 1$

$\rho(\vec{x}) = \frac{\langle \lambda_1 \vec{x}, \vec{x} \rangle}{1} = \lambda_1 \Rightarrow \|A\|_2 = \lambda_1$

(10 pts) 5. Let $B = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$(\lambda+3)(\lambda+3)-4 = \lambda^2+6\lambda+5=0$
$\Rightarrow (\lambda+5)(\lambda+1)=0 \quad \lambda_1=-1.\ \lambda_2=-5$

$\lambda^2-6\lambda+\cancel{9}^{8}\cancel{-1} = (\lambda-2)(\lambda-4)=0$
$\Rightarrow \lambda_3=2.\ \lambda_4=4$

(a) Find the eigenvalues and corresponding eigenvectors of B.

(b) Write the spectrum decomposition of B. $B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & \\ 1 & -1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 1 & \end{bmatrix}$

(c) Find the singular values of B. $\cancel{-1.\ -5.\ 2.}\ 4$ $\quad [5, 4, 2, 1]$ $\quad B^TB$

(d) Find e^B. $= Ue^{\Lambda}V$

(e) Find $\|B\|_2$ and $\|B\|_1$. $\quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & 0 \\ 0 & & \ddots \end{bmatrix}$

(a) $-3x_1+2x_2=-x_1$
$2x_1-3x_2=-x_2$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$-3x_1+2x_2=-5x_1 \ .\ 2x_1-3x_2=$

$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

$4\cancel{3}x_3+x_4=\cancel{-x_3}$
$x_3+\cancel{3}^{4}x_4=\cancel{-x_4}$

$\cancel{3}x_3+x_4=\cancel{2x_3}\ 0$
$x_3+3x_4=2x_4$

$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \quad \vec{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

$\cancel{3x_3}+x_4=\cancel{4x}_3 \Rightarrow x_3=x_4$

$\vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}$

(e)
$\|B\|_2 = \max_{1\le i\le n}|\lambda_i| = 5$

$\|B\|_1 = \max_{1\le j\le n}\left\{\sum_{i=1}^{m}|a_{ij}|\right\} = |-3|+|2| = 5$

$B^TB = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -12 & 0 & 0 \\ -12 & 13 & 0 & 0 \\ 0 & 0 & 10 & 6 \\ 0 & 0 & 6 & 10 \end{bmatrix}$

$\lambda^2-26\lambda+25=(\lambda-25)(\lambda-1)=0$
$\lambda_1=25.\ \lambda_4=1$
$\lambda^2-20\lambda+64=0$
$\Rightarrow (\lambda-16)(\lambda-4)=0 \Rightarrow \lambda_3=4.\ \lambda_2=16$

$\lambda_1=25 \quad \sigma_1=5$
$\lambda_2=16 \Rightarrow \sigma_2=4$
$\lambda_3=4 \quad \sigma_3=2$
$\lambda_4=1 \quad \sigma_1=1$