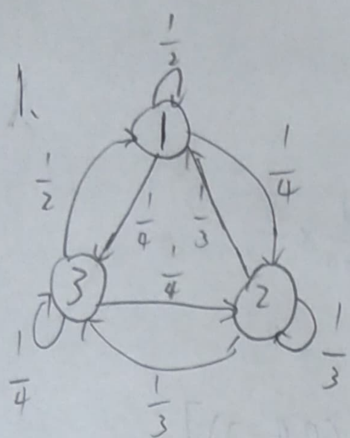
Randomized Algo Assignment 5. 106033233
周聖諺

1.



(a)

$Pr(X_0 = 1) = \frac{1}{2}$, $Pr((X_0 = 1) \cap (X_1 = 2)) = ?$

$$\frac{Pr(A \cap B)}{Pr(B)} = Pr(A|B)$$

$$\Rightarrow Pr((X_1 = 2) \cap (X_0 = 1)) = Pr(X_1 = 2 | X_0 = 1) \cdot Pr(X_0 = 1)$$

$$= \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$ ※

(b)

$Pr(X_0 = 1) = \frac{1}{2}$, $Pr((X_0 = 1) \cap (X_1 = 2) \cap (X_2 = 3))$

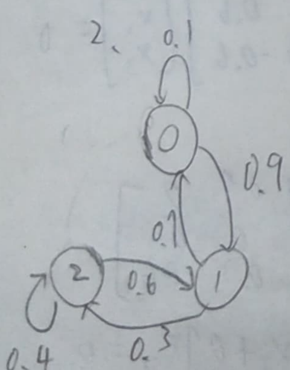
$$\Rightarrow Pr((X_2 = 3) \cap (X_1 = 2) \cap (X_0 = 1))$$

$$= Pr((X_2 = 3) \cap (X_1 = 2) | (X_0 = 1)) \cdot Pr(X_0 = 1)$$

$$= Pr(X_2 = 3 | X_0 = 1 \cap X_1 = 2) \cdot Pr(X_0 = 1) \, Pr(X_1 = 2)$$

$$= \frac{1}{3} \times \frac{1}{4} \times \frac{1}{2} = \frac{1}{24}$$ ※

2.



(a) Aperiodic $\Rightarrow$ For state 2 can go back to state 2 in 1, 2, 3, ... states with $d = 1$

Irreducible $\Rightarrow$ State 0 can arrive 1, 2

State 1 can arrive 2, 0

$\rightarrow$ Strongly Connected ※

(b)

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.1 & 0.1 & 0 \\ 1 & 0.9 & 0 & 0.6 \\ 2 & 0 & 0.3 & 0.4 \end{array}$$

$\Rightarrow$ Eigenvalue Decomposition $\Rightarrow Mu = \lambda u$

$$\begin{bmatrix} 0.1-\lambda & 0.1 & 0 \\ 0.9 & 0-\lambda & 0.6 \\ 0 & 0.3 & 0.4-\lambda \end{bmatrix}$$

$(M - \lambda I)u = 0$

$u \neq 0$

$$\begin{bmatrix} 0.1-\lambda & 0.7 & 0 \\ 0.9 & 0-\lambda & 0.6 \\ 0 & 0.3 & 0.4-\lambda \end{bmatrix}$$

$\frac{54}{100}\times\frac{3}{10}$

$\Rightarrow$ Characterist Polynomial:

$$(0.1-\lambda)[-\lambda(0.4-\lambda)-0.6\times 0.3]-0.7[0.9(0.4-\lambda)]$$

$$=\left(-\frac{1}{10}\lambda+\lambda^2\right)(0.4-\lambda)-\left(\frac{63}{100}\right)(0.4-\lambda)+\frac{18}{100}\left(\lambda-\frac{1}{10}\right)$$

$$=\left(\lambda^2-\frac{\lambda}{10}-\frac{63}{100}\right)(0.4-\lambda)+\frac{18}{100}\left(\lambda-\frac{1}{10}\right)$$

$$=-\lambda^3+\frac{\lambda^2}{2}+\frac{77}{100}\lambda-\frac{27}{100}$$

$$\lambda=1,\ \frac{-\sqrt{133}-5}{20},\ \frac{\sqrt{133}-5}{20}$$

To find stationary distribution we take $\lambda=1$

$$\begin{bmatrix} 0.1 & 0.7 & 0 \\ 0.9 & 0 & 0.6 \\ 0 & 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\Rightarrow\begin{bmatrix} -0.9 & 0.7 & 0 \\ 0.9 & -1 & 0.6 \\ 0 & 0.3 & -0.6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

Gaussian Elimination:

$$\begin{bmatrix} -0.9 & 1 & -0.6 \\ 0.9 & -1 & 0.6 \\ 0 & 0.3 & -0.6 \end{bmatrix}\to\begin{bmatrix} -0.9 & 1 & -0.6 \\ 0 & 0 & 0 \\ 0 & 0.3 & -0.6 \end{bmatrix}\to\begin{bmatrix} -0.9 & 0.7 & 0 \\ 0 & 0 & 0 \\ 0 & 0.3 & -0.6 \end{bmatrix}$$

$$\begin{cases} x_1=\frac{14}{9}t \\ x_2=2t \\ x_3=t \end{cases}\qquad\begin{cases} -0.9x_1+0.7x_2=0 \\ 0.3x_2-0.6x_3=0 \end{cases}$$

$x_2=2x_3$

$7x_2=9x_1$

$14 + 9 + 18 = 41$

$$\frac{14}{9}t+2t+t=1$$

$$\frac{41}{9}t=1\quad t=\frac{9}{41}$$

Stationary Distribution $\Rightarrow\left\langle\frac{14}{41},\frac{18}{41},\frac{9}{41}\right\rangle$ #

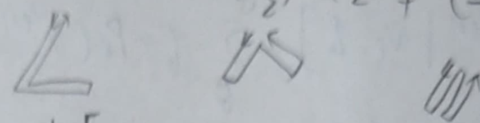
3.

$P_1 = 0$

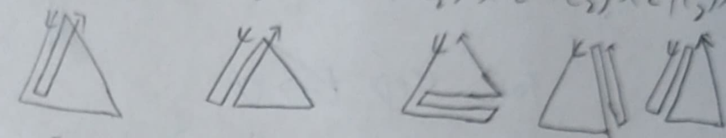
$P_2 = \frac{1}{2}\times\frac{1}{2}\times 2 = \frac{1}{2}$

$P_3 = (\frac{1}{2})^3 \times 2 = \frac{1}{4}$

$P_4 = (\frac{1}{2})^4 \times 2 + (\frac{1}{2})^4 \times 2 + (\frac{1}{2})^4 \times 2 = \frac{3}{8}$



$P_5 = (\frac{1}{2})^5 \times 2 + (\frac{1}{2})^5 \times 2 + (\frac{1}{2})^5 \times 2 + (\frac{1}{2})^5 \times 2 + (\frac{1}{2})^5 \times 2$



$\frac{10}{32}$

$\frac{5}{16}$

$= \frac{5}{16}$ ※

4.

$(1-\frac{1}{P_{k-1}})\times\frac{1}{2} = P_k$, Since for the bugs at state $k-1$ while bugs will arrive starting point at k. The bug will have $\frac{1}{2}$ probability to reach starting point.

$k+(n-k)$

$k-(n-k) = 2k-n$

5. (a) To show $E[2^{W_t}] = E[2^{W_{t+1}}]$, prove $E[W_t] = E[W_{t+1}]$ first

$E[W_{t+1}] = E[E[W_{t+1} | W_t]] = \sum_k E[W_{t+1} | W_t = k]\cdot Pr(W_t = k)$

$= \sum_k (k-\frac{1}{3}) Pr(W_t = k) = E[W_t] - \frac{1}{3}$

$E[W_{t+1} | W_t = k] = \sum_q q\, Pr(W_{t+1} = q | W_t = k) = (k+1)\times Pr(W_{t+1} = k+1 | W_t = k)$

$+ (k-1)\times Pr(W_{t+1} = k-1 | W_t = k) = \frac{1}{3}\times(k+1) + \frac{2}{3}\times(k-1) = k - \frac{1}{3}$