A Review of Variation Bayesian Gaussian Mixture Model

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Introduction

The disadvantage of K-Means

- Hard to **decide the number of clusters**
- **Poor** performance on **unbalanced dataset**

Variational Bayesian Expectation Maximization(VBEM)

- Only need to choose the **maximum number of clusters, self-adapt to the best**
- **Better** performance on unbalanced dataset

EM In General Form

In naive EM, the goal is to **maximize likelihood** $\mathcal{L}(Y;\theta)$

$$\arg\max_{\theta} \mathcal{L}(Y;\theta) = \arg\max_{\theta}\ log \int_Z p(Y,Z;\theta)dZ$$

where Y **is observing data,** Z **is latent variable,** θ **is the parameter**

With ELBO, we can derive the **lower bound** $\mathcal{L}(\theta,\gamma)$ **of likelihood** $\mathcal{L}(Y;\theta)$

$$\mathcal{L}(\theta,\gamma) = \mathbb{E}_q[log(\frac{p(Y,Z;\theta)}{q(Z;\gamma)})]$$

$$= \int_Z q(Z;\gamma)log\ \frac{p(Y,Z;\theta)}{q(Z;\gamma)}dZ$$

$$= log\ p(Y;\theta) - KL[q(Z;\gamma)||p(Z|Y)]$$

$$= \mathcal{L}(Y;\theta) - KL[q(Z;\gamma)||p(Z|Y)]$$

EM In General Form

Thus

$$\mathcal{L}(\theta, \gamma) = \mathcal{L}(Y; \theta) - KL[q(Z; \gamma)||p(Z|Y)]$$

Since the KL-divergence always ≥ 0

$$\arg\max_{\theta} \mathcal{L}(Y; \theta) \geq \arg\max_{\theta, \gamma} \mathcal{L}(\theta, \gamma)$$

With KKT and Lagrange multiplier, the optimization problem can be written as

$$\arg\max_{\theta, \gamma} \mathcal{L}(\theta, \gamma) = \arg\max_{\theta, \gamma} \mathcal{L}(Y; \theta) - \beta KL[q(Z; \gamma)||p(Z|Y)]$$

EM In General Form

Pseudo Code

Iterate until θ converge

- E Step at k-th iteration

  $$\gamma_{k+1} = \arg\max_{\gamma_k} \mathcal{L}(\theta_k, \gamma_k)$$

- M Step at k-th iteration

  $$\theta_{k+1} = \arg\max_{\theta_k} \mathcal{L}(\theta_k, \gamma_{k+1})$$

Variational Bayesian Expectation Maximization(VBEM)

In VBEM, we consider an **additional prior** $p(\theta; \lambda)$

$$log\, p(Y) = log \int_{Z,\theta} p(Y, Z, \theta; \lambda) dZ d\theta$$

$$= log\, \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[\frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

$$\geq \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[log\, \frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

Thus, we get the ELBO $\mathcal{L}(\phi^Z, \phi^\theta)$

$$\mathcal{L}(\phi^Z, \phi^\theta) = \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[log\, \frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

Variational Bayesian Expectation Maximization(VBEM)

According to the general form of EM

$$\arg\max_{\gamma_k} \mathcal{L}(\theta_k, \gamma_k)$$

$$\arg\max_{\theta_k} \mathcal{L}(\theta_k, \gamma_k)$$

We can derive

$$\frac{d}{d\phi^Z}\mathcal{L}(\phi^Z, \phi^\theta) = 0, \quad ln\, q(Z; \phi^Z) \propto \mathbb{E}_{q(\theta;\phi^\theta)}[log\, p(Y, Z, \theta)]$$

$$\frac{d}{d\phi^\theta}\mathcal{L}(\phi^Z, \phi^\theta) = 0, \quad ln\, q(\theta; \phi^\theta) \propto \mathbb{E}_{q(Z;\phi^Z)}[log\, p(Y, Z, \theta)]$$

Variational Bayesian Expectation Maximization(VBEM)

Pseudo Code

Iterate until $\mathcal{L}(\phi^Z, \phi^\theta)$ converge

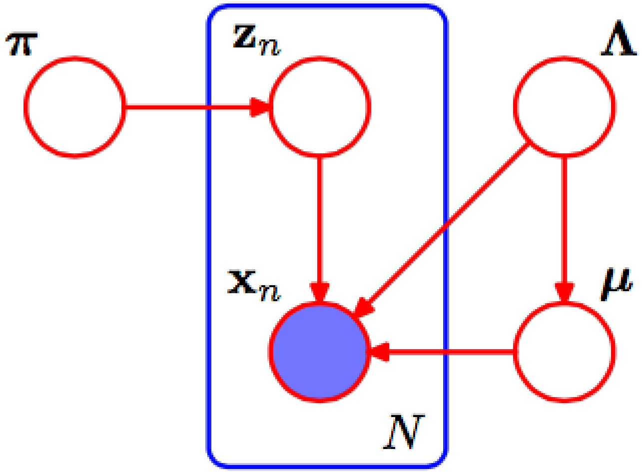
- E Step: Update the variational distribution on Z

  $q(Z; \phi^Z) \propto e^{(\mathbb{E}_{q(\theta;\phi^\theta)}[log\, p(Y,Z,\theta)])}$

- M Step: Update the variational distribution on θ

  $q(\theta; \phi^\theta) \propto e^{(\mathbb{E}_{q(Z;\phi^Z)}[log\, p(Y,Z,\theta)])}$

Variational Bayesian Gaussian Mixture Model(VB-GMM)



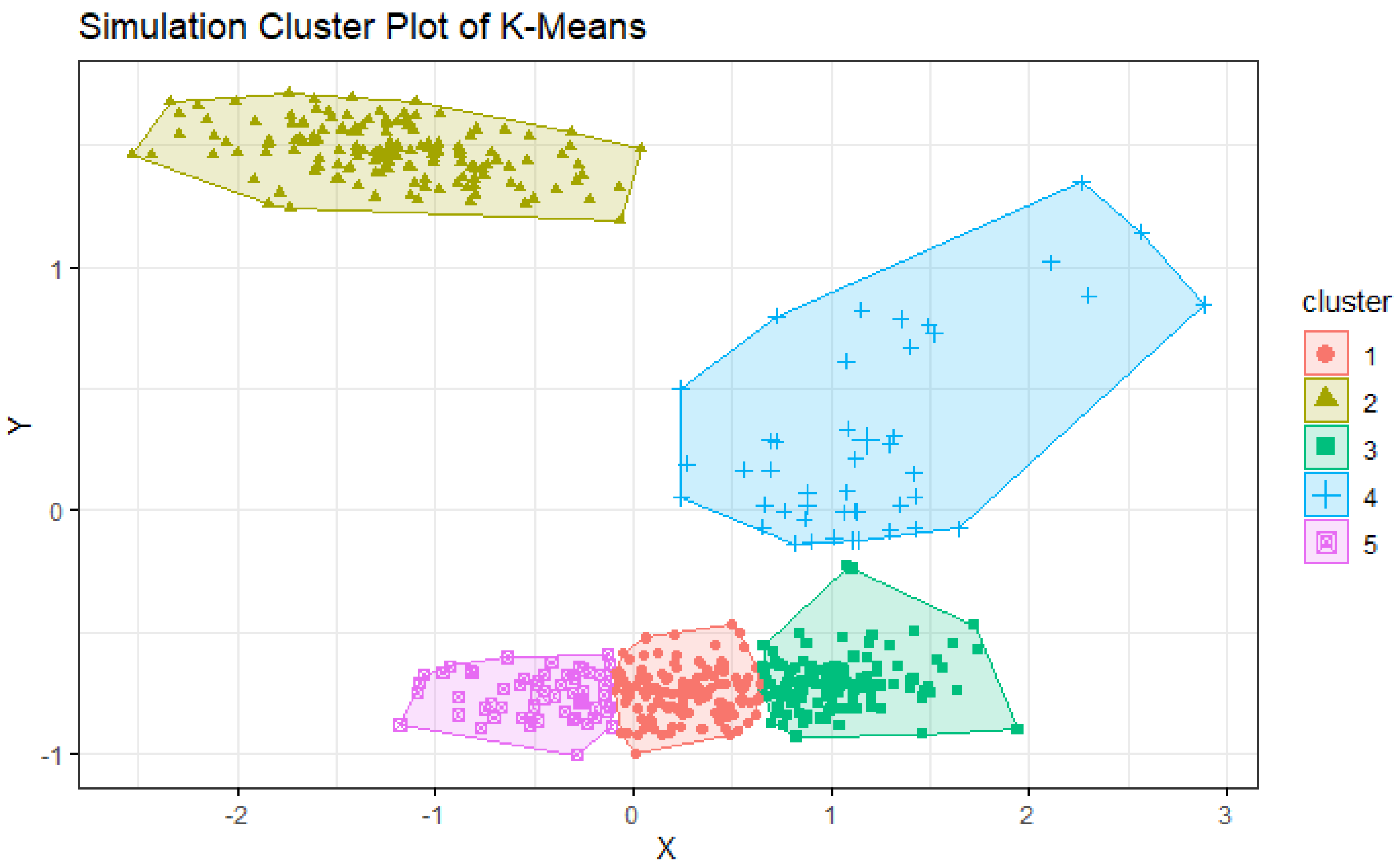
$$p(X, Z, \pi, \mu, \Lambda) = p(X|Z, \pi, \mu, \Lambda)p(Z|\pi)p(\pi)p(\mu|\Lambda)p(\Lambda)$$

- $p(X|Z, \pi, \mu, \Lambda)$ denotes the **Gaussian Mixture Model**
- $p(Z|\pi)$ denotes the **Latent Variables**
- $p(\pi)$ denotes the **Dirichlet Prior Distribution Over The Latent Variables** Z
- $p(\mu|\Lambda)p(\Lambda)$ denotes the **Gaussian-Wishart Prior Distribution Over GMM** X

Ignore The Complex Derivation

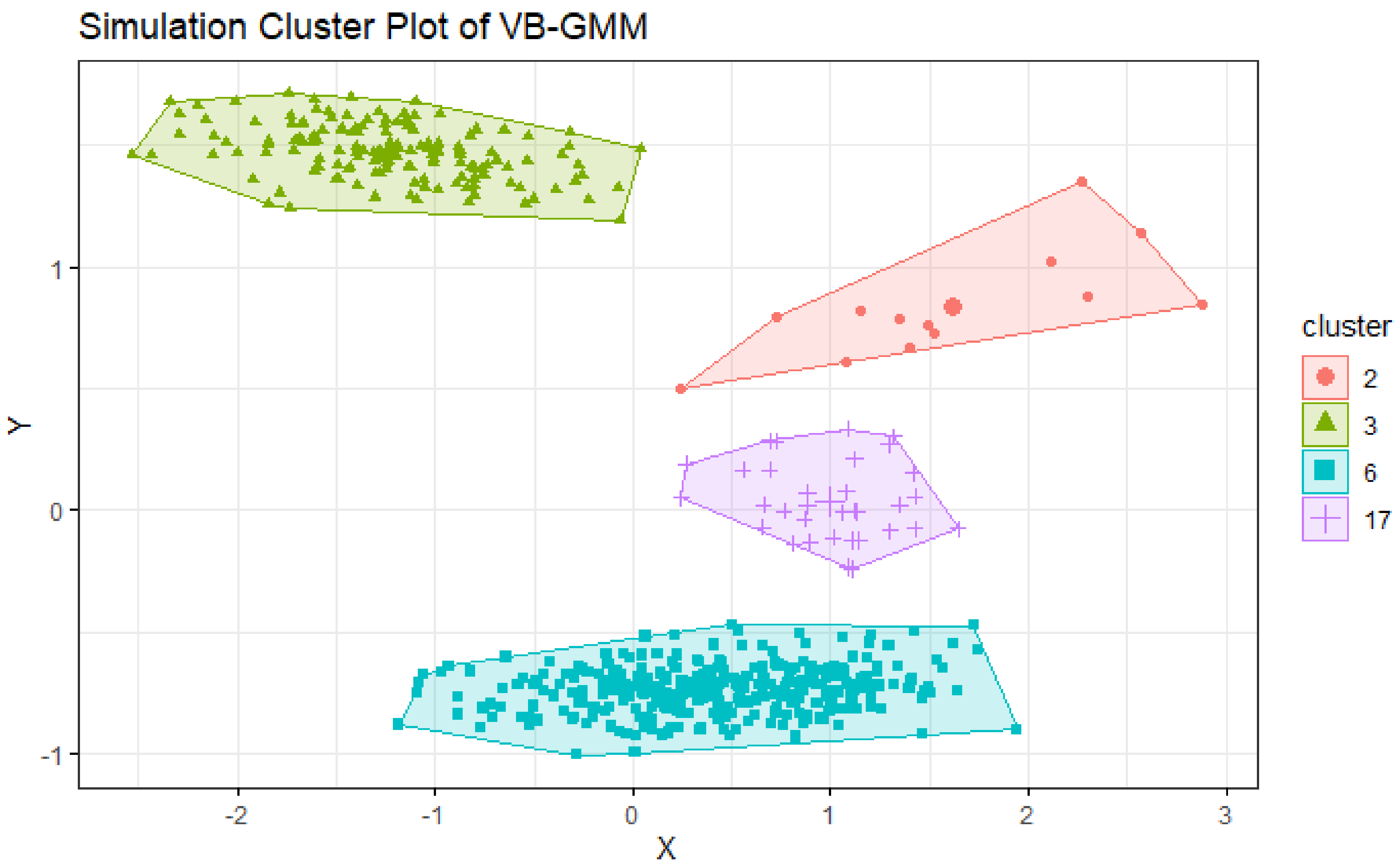
K-means Cluster Plot

Fit On 5 Modal Simulation Dataset Generated by GMM



VB-GMM Cluster Plot

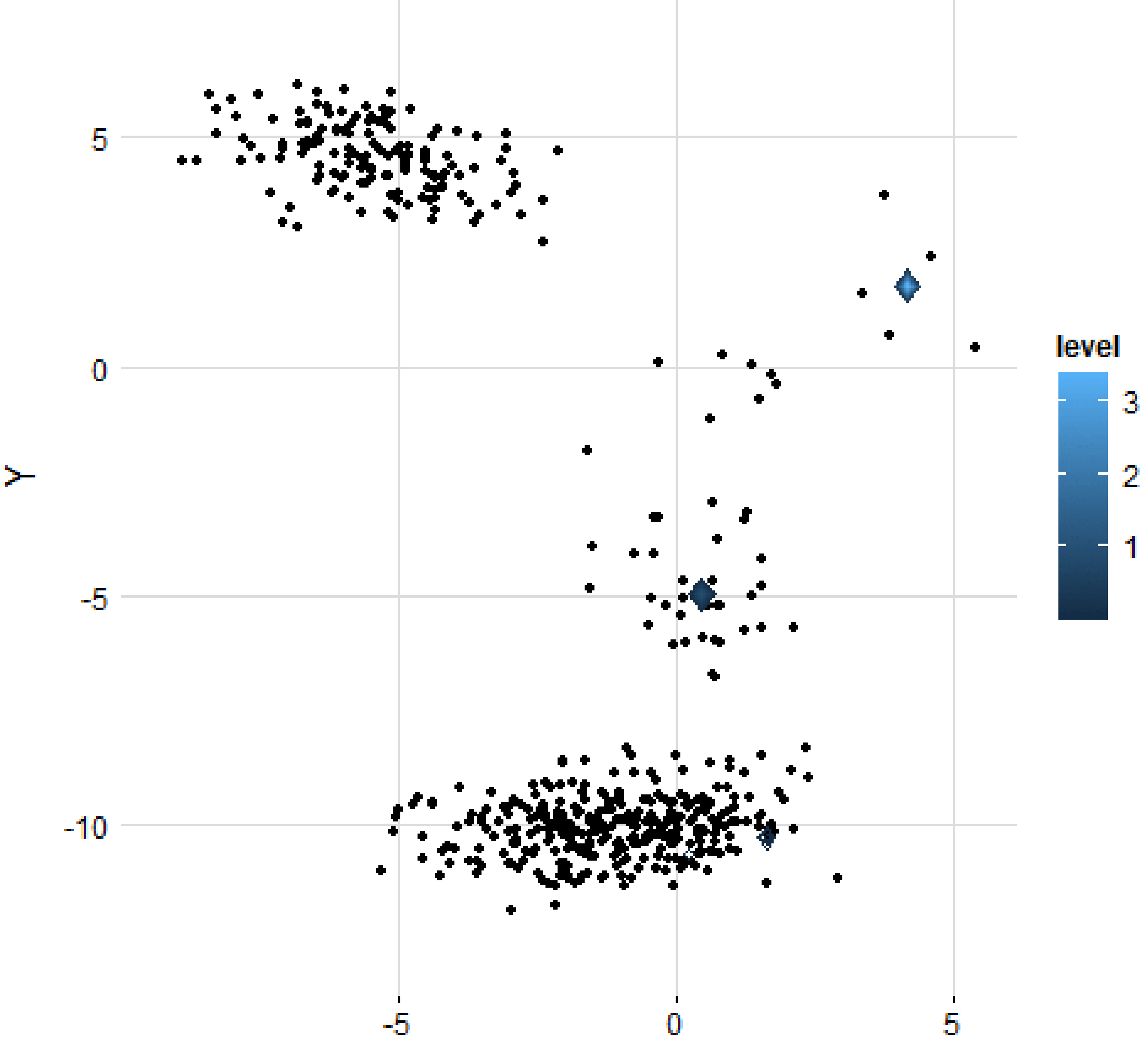
Fit On 5 Modal Simulation Dataset Generated by GMM



VB-GMM Contour Plot Animation

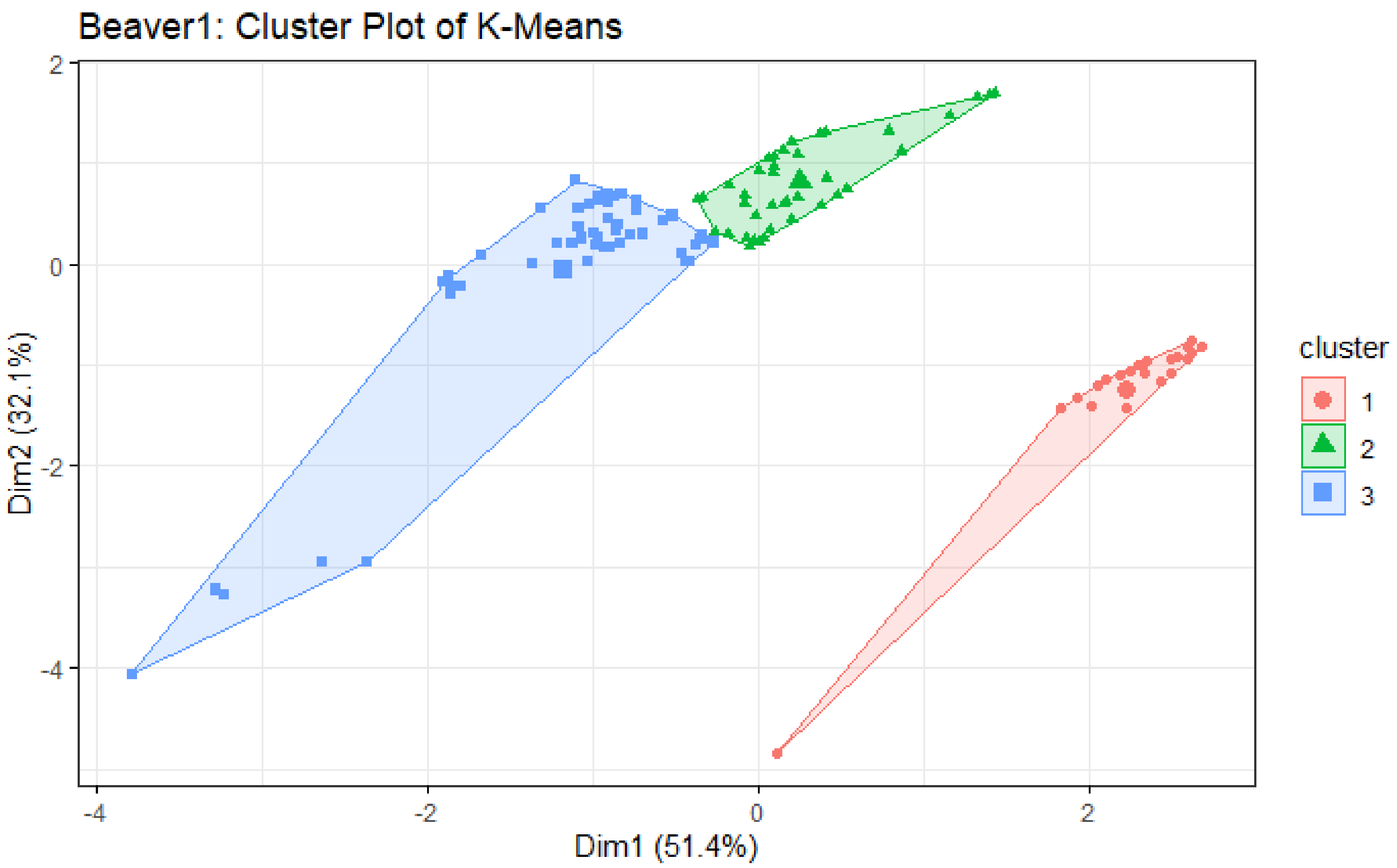
Fit On 5 Modal Simulation dataset

Link



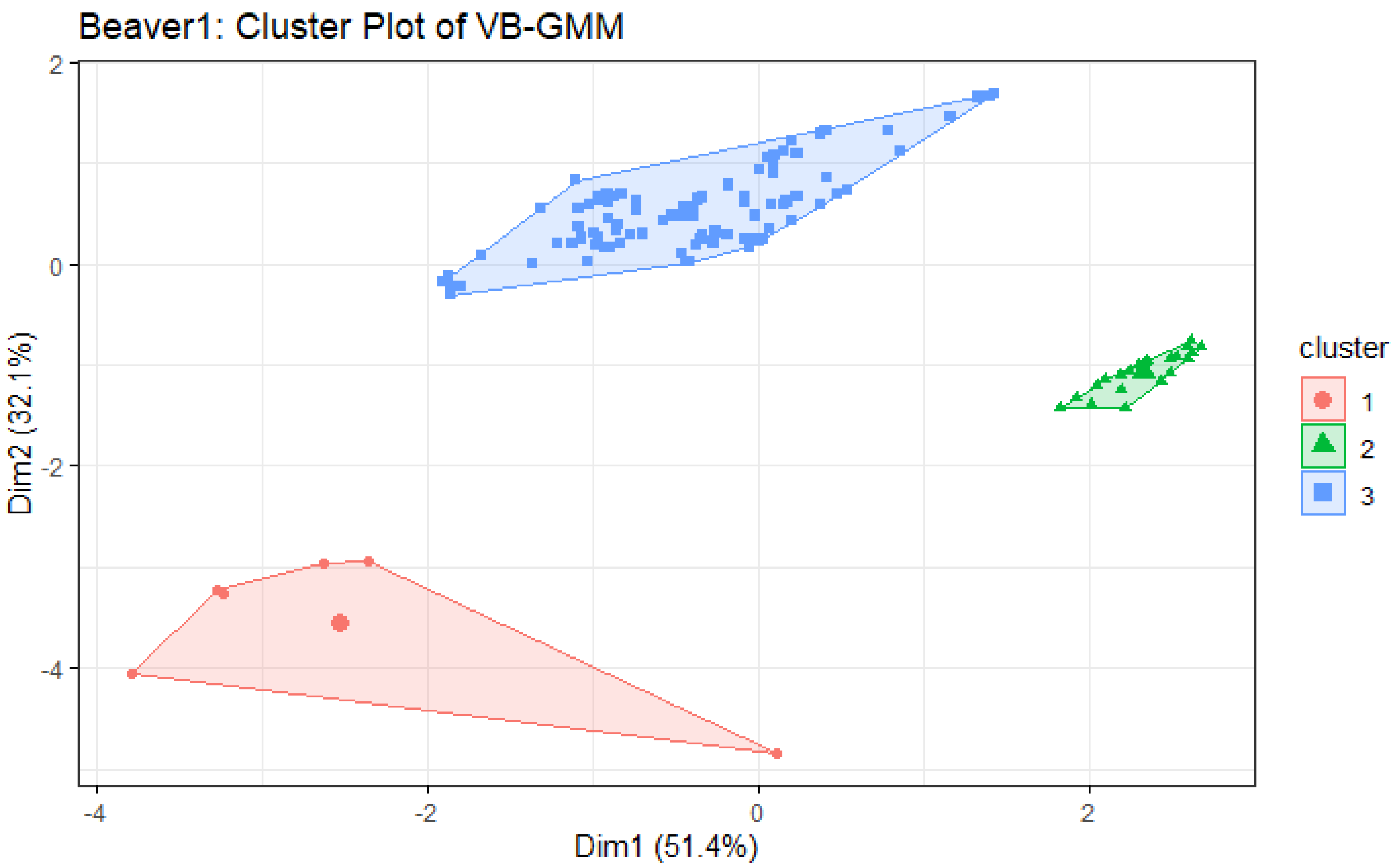
K-means Cluster Plot

Fit On Beaver1 dataset



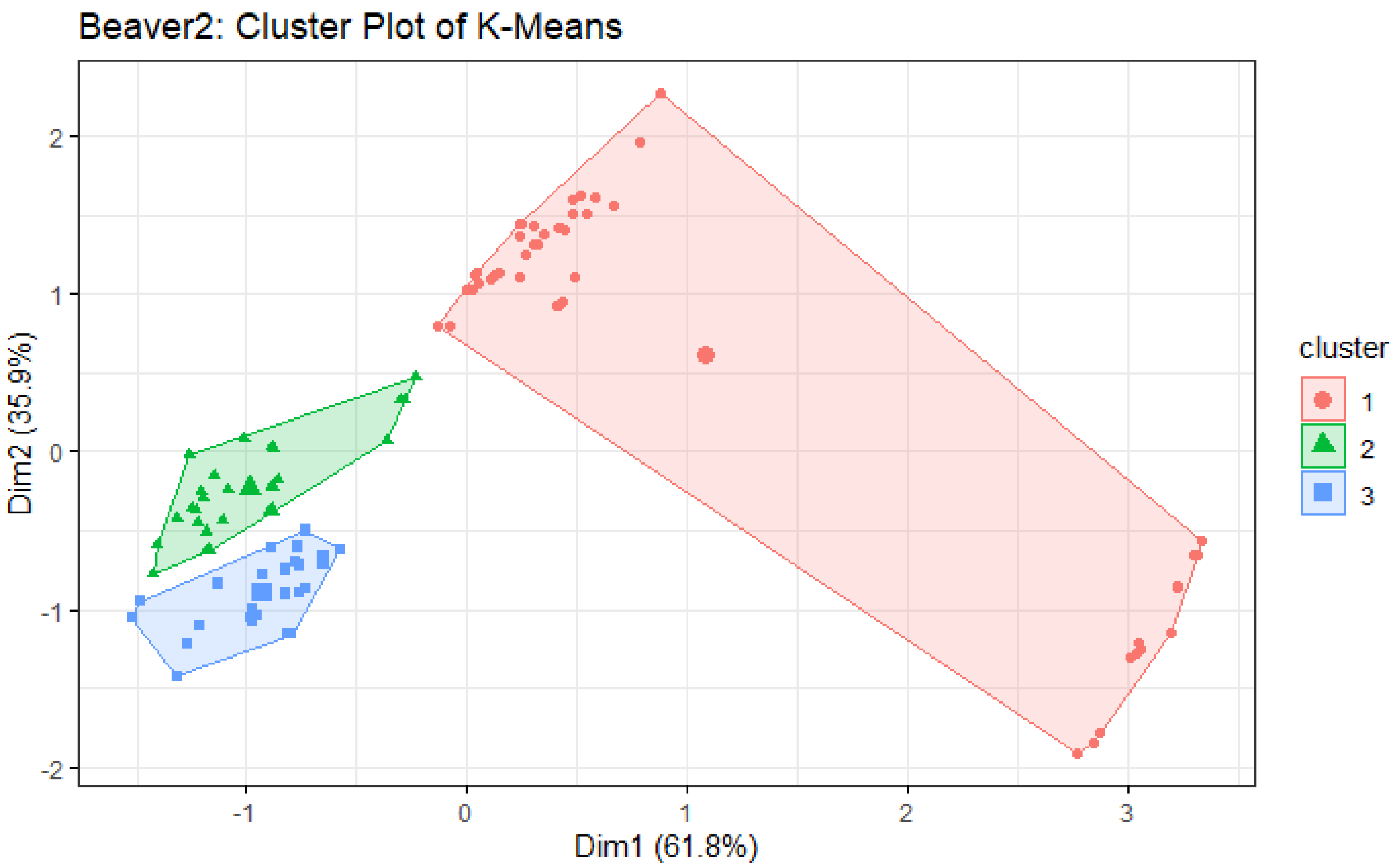
VB-GMM Cluster Plot

Fit On Beaver1 dataset



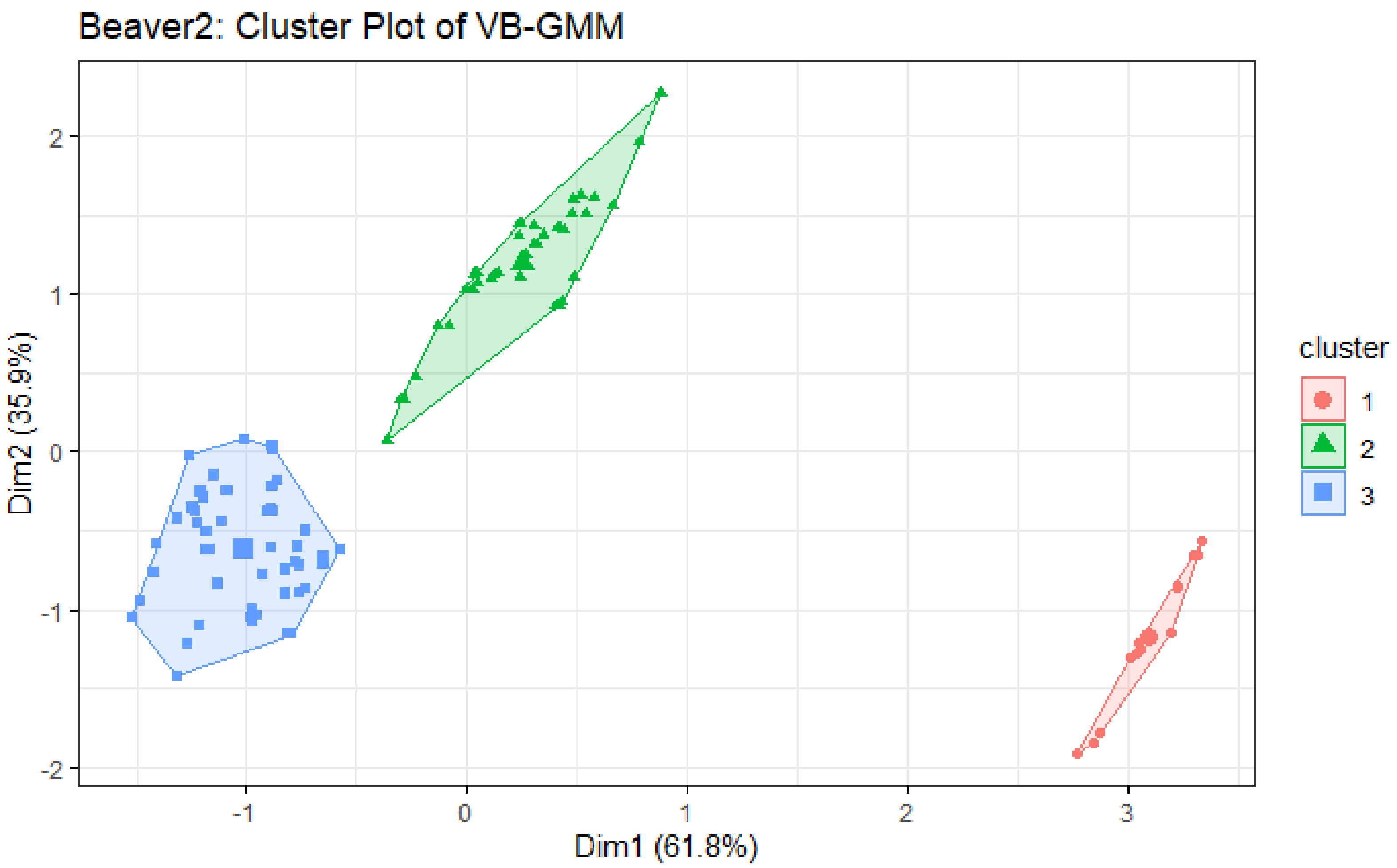
K-means Cluster Plot

Fit On Beaver2 dataset



VB-GMM Cluster Plot

Fit On Beaver2 dataset



Thanks For Listening