

# CS2336 DISCRETE MATHEMATICS

## Exam 2

December 11, 2017 (2 hours)

**Answer all questions. Total marks = 100. A large portion of marks may be deducted from incomplete proofs or wrong arguments.**

1. Fermat once conjectured that for  $n \geq 0$ , all numbers  $F_n = 2^{2^n} + 1$  are primes. Indeed, the numbers

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$

are all primes. However, in 1732, Euler showed that  $F_5 = 4294967297 = 641 \times 6700417$ , thus disproving Fermat's claim.

(20%) Here, you are asked to show an interesting property of  $F_n$ :

$$\text{For all integer } n \geq 1, F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2.$$

2. A standard chessboard contains  $8 \times 8$  squares. A king controls the squares immediately adjacent to the square that it is placed, in all eight directions. See Figure 1 for an example. A king can attack a piece if it is placed on the squares it controls.
- (a) (5%) If 17 pieces of kings are placed on a chessboard, show that there must be two kings attacking each other.
  - (b) (5%) If only 16 pieces of kings are placed on the board, show that it is possible that no kings are attacking any other.
  - (c) (5%) If 17 pieces of kings are placed on a chessboard, show that we can find five kings such that they are not attacking any other.
  - (d) (5%) If only 16 pieces of kings are placed on a chessboard, show that it is possible that we cannot find five kings such that they are not attacking any other.



Figure 1: A  $8 \times 8$  chessboard and a king, with squares controlled by the king marked

3. A contiguous sequence of characters in a string  $X$  is called a *substring* of  $X$ . For instance, **ana** is a substring of **banana**, but **aa** is not a substring of **banana**.

(10%) Consider all the 5-bit binary strings. How many of them contains 11 but not 101 as its substring?

For example, 11011 contains both 11 and 101 as its substring, 10001 does not contain 11 and 101 as its substring, while 10011 contains 11 but not 101 as its substring.

*Hint:* Use a tree diagram.

4. (10%) Given that  $x \geq 2, y \geq 1$ , and  $z \geq 0$ , how many integral solutions are there for the equation

$$x + y + z = 11?$$

5. Consider the diagram in Figure 2, where each vertex represents a city, and each edge represents a one-way road.

- (a) (5%) How many ways are there to travel from  $A$  to  $B$ ?
- (b) (5%) How many ways are there to travel from  $A$  to  $B$  that must pass through  $X$ ?
- (c) (5%) How many ways are there to travel from  $A$  to  $B$  that must pass through both  $X$  and  $Y$ ?

Note: For each part of this question, no explanation is needed.

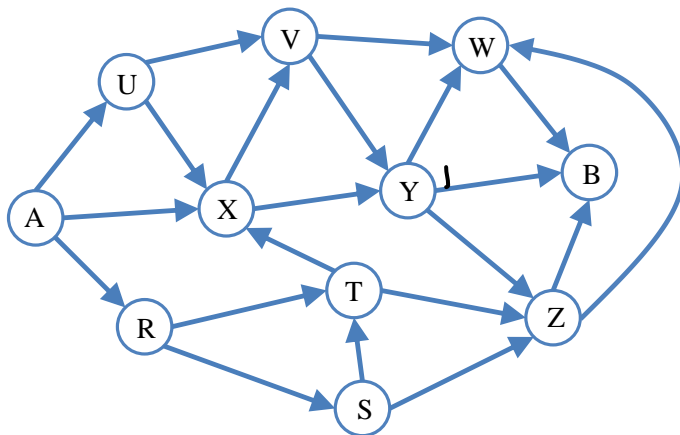


Figure 2: Diagram for Question 5

6. (15%) How many ways we can select 4 distinct integers from  $\{1, 2, 3, \dots, 100\}$ , so that their sum is divisible by 3?

7. (10%) Give a combinatorial argument to show that

$$\binom{2n}{3} = 2 \times \binom{n}{3} + 2n \times \binom{n}{2}.$$

Note: No marks will be given if you are not using a combinatorial argument.