CHAPTER 14 Complex Integration

Comment

The introduction to the chapter mentions two reasons for the importance of complex integration. Another practical reason is the extensive use of complex integral representations in the higher theory of special functions; see, for instance, Ref. [GenRef10] listed in App. 1.

SECTION 14.1. Line Integral in the Complex Plane, page 643

Purpose. To discuss the definition, existence, and general properties of complex line integrals. Complex integration is rich in methods, some of them very elegant. In this section we discuss the first two methods, integration by the use of path and (under suitable assumptions given in Theorem 1!) by indefinite integration.

Main Content, Important Concepts

Definition of the complex line integral

Existence

Basic properties

Indefinite integration (Theorem 1)

Integration by the use of path (Theorem 2)

Integral of $1/z$ around the unit circle (basic!)

ML-inequality (13) (needed often in our work)

Comment on Content

Indefinite integration will be justified in Sec. 14.2, after we have obtained Cauchy's integral theorem. We discuss this method here for two reasons: (i) to get going a little faster and, more importantly, (ii) to answer the students' natural question suggested by calculus, that is, whether the method works and under what condition—that it does not work unconditionally can be seen from Example 7!

SOLUTIONS TO PROBLEM SET 14.1, page 651

1. Straight segment from (1, 0) to (6, 1.5).
2. Straight segment from (3, 1) sloping downward to (6, -2).
3. Parabola $y = 4x^2$ from (1, 4) to (2, 16).
4. Parabola $y = 1 - x^2$ $(-1 \leqq x \leqq 1)$, opening downward.
5. Circle through (0, 0), center (2, -2), radius $\sqrt{5}$, oriented clockwise.
6. Circle of radius 1 and center (1, 1), oriented clockwise.
7. Quarter circle in the first quadrant, center at (1, 0), radius 2.
8. Quartercircle from (5, 0) to (0, -5) in the fourth quadrant, center 0.
9. Cubic $y = 1 - x^3$, $(-2 \le x \le 2)$.

10. Ellipse with coordinate axes as axes, semiaxes of lengths 2 (in x-direction) and 1.

11. $z(t) = t + (3 + t)i$

12. First along the x-axis and then along the parallel to the y-axis gives

$$C_1: \quad z_1 = t_1 \quad (0 \leqq t_1 \leqq 2), \qquad C_2: \quad z_2 = 2 + it_2, \quad (0 \leqq t_2 \leqq 1).$$

First along the y-axis and then parallel to the x-axis gives a second path consisting of C_3 and C_4,

$$C_3: \quad z_3 = it_3 \quad (0 \leqq t_3 \leqq 1), \qquad C_4: \quad z_4 = t_4 + i \quad (0 \leqq t_4 \leqq 2).$$

13. $z(t) = 1 - i + 4e^{it}, (0 \le t \le \pi)$

14. $z(t) = e^{-it} \quad (0 \leqq t \leqq 2\pi)$
Emphasize here, and elsewhere, the advantage of parametric representations that they represent the entire curve, whereas $y = y(x)$ gives only the upper half, and $y'(x) \to \infty$ as $x \to \pm 1$, i.e., vertical tangents. Also, essential for our present purpose is the fact that a parametric representation gives an *orientation* of the curve.

15. $z(t) = \cosh t + i2 \sinh t, (-\infty < t < \infty)$

16. $z(t) = 3 \cos t + 2i \sin t \quad (0 \leqq t \leqq 2\pi)$

17. Circle $z(t) = -a + ib + re^{-it}, (0 \le t \le 2\pi)$

18. Hyperbola $z(t) = t + i/t \quad (1 \leqq t \leqq 5)$

19. $z(t) = t + (1 - \frac{1}{2}t^2)i, (-2 \le t \le 2)$

20. Ellipse $z(t) = 2 + \sqrt{5} \cos t + (-1 + 2 \sin t)i \quad (0 \leqq t \leqq 2\pi)$

21. $z(t) = (1 + i)t, (1 \le t \le 5)$, Re $z = t, z'(t) = 1 + i$; $12 + 12i$

22. We have

$$\text{Re}\, z = t \qquad \text{and} \qquad z' = 1 + (t - 1)i$$

so that

$$\int_C \text{Re}\, z\, dz = \int_1^3 t(1 + (t - 1)i)\, dt = 4 + \tfrac{14}{3}i.$$

Compare with Prob. 21. Both paths have the same endpoints, so that the comparison illustrates path dependence, as expected because the integrand is not analytic.

23. $e^{\pi i} - e^{\pi/2i} = -1 - i$

24. First method,

$$\int_C \cos 2z\, dz = \tfrac{1}{2} \sin 2z \Big|_{-\pi i}^{\pi i} = \sin 2\pi i = i \sinh 2\pi = 267.7i$$

26. By linearity we can integrate the two terms separately. The integral of z is zero by Theorem 1. The integral of $1/z$ equals $2\pi i$; see Example 5 in the text.

28. $-5 \cdot 2\pi - 0$ by Example 6.

30. Re $z^2 = x^2 - y^2$. (1) Upward, $z(t) = it, \dot{z}(t) = i, \displaystyle\int_0^1 -t^2 i\, dt = -\frac{1}{3}i$

(2) To the right, $z(t) = t + i, \dot{z}(t) = 1, \displaystyle\int_0^1 (t^2 - 1)\, dt = -\frac{1}{3} - 1$

(3) Down, $z(t) = 1 + it$, t goes from 1 to 0, $\dot{z}(t) = i$, $\displaystyle\int_1^0 (1 - t^2)i\,dt = i\left(-1 + \frac{1}{3}\right)$

(4) To the left, $z(t) = t$, t goes from 1 to 0, $\dot{z}(t) = 1$, $\displaystyle\int_1^0 t^2\,dt = -\frac{1}{3}$.
Answer: $-1 - i$

34. Team Experiment. **(b)** (i) $12.8i$, (ii) $\frac{1}{2}(e^{2+4i} - 1)$

(c) The integral of Re z equals $\frac{1}{2}\pi^2 - 2ai$. The integral of z equals $\frac{1}{2}\pi^2$. The integral of Re (z^2) equals $\pi^3/3 - \pi a^2/2 - 2a\pi i$. The integral of z^2 equals $\pi^3/3$.

(d) The integrals of the four functions in (c) have, for the present paths, the values $\frac{1}{2}a\pi i$, 0, $(4a^2 - 2)i/3$, and $-2i/3$, respectively.

Parts (c) and (d) may also help in motivating further discussions on path independence and the principle of deformation of path.

SECTION 14.2. Cauchy s Integral Theorem, page 652

Purpose. To discuss and prove the most important theorem in this chapter, Cauchy's integral theorem, which is basic by itself and has various basic consequences to be discussed in the remaining sections of the chapter.

Main Content, Important Concepts

Simply connected domain

Cauchy's integral theorem, Cauchy's proof

(Goursat's proof in App. 4)

Independence of path

Principle of deformation of path

Existence of indefinite integral

Extension of Cauchy's theorem to multiply connected domains

SOLUTIONS TO PROBLEM SET 14.2, page 659

2. **(a)** $z = 0$ outside C. **(b)** $z = 0, \pm 2i$ outside C.

4. No, it would contradict the deformation principle.

6. **(a)** $z(t) = (1 + i)t \quad (0 \leqq t \leqq 1)$, $\quad \dot{z}(t) = 1 + i$, hence

$$\int_C e^z\,dz = \int_0^1 e^{(1+i)t}(1 + i)\,dt = e^{(1+i)t}\Big|_0^1 = e^{1+i} - 1.$$

In **(b)** we can choose

$$z_1(t_1) = t_1 \quad (0 \leqq t_1 \leqq 1), \quad \dot{z}_1 = 1$$
$$z_2(t_2) = 1 + it_2 \quad (0 \leqq t_2 \leqq 1), \quad \dot{z}_2 = i$$

and obtain the corresponding integrals

$$\int_0^1 e^{t_1}\,dt_1 + \int_0^1 e^{1+it_2}\,i\,dt_2 = e - 1 + e^{1+i} - e$$
$$= e^{1+i} - 1.$$

8. Team Experiment. (b) (i) $\dfrac{2z + 3i}{z^2 + \frac{1}{4}} = \dfrac{4}{z - i/2} - \dfrac{2}{z + i/2}$. From this, the principle of deformation of path, and (3) we obtain the *answer*

$$4 \cdot 2\pi i - 2 \cdot 2\pi i = 4\pi i.$$

(ii) Similarly,

$$\frac{z + 1}{z^2 + 2z} = \frac{\frac{1}{2}}{z} + \frac{\frac{1}{2}}{z + 2}.$$

Now $z = -2$ lies outside the unit circle. Hence the *answer* is $\frac{1}{2} \cdot 2\pi i = \pi i$.

(c) The integral of z, Im z, z^2, Re z^2, Im z^2 equals $\frac{1}{2}$, $a/6$, $\frac{1}{3}$, $\frac{1}{3} - a^2/30 - ia/6$, $a/6 - ia^2/30$, respectively. Note that the integral of Re z^2 plus i times the integral of Im z^2 must equal $\frac{1}{3}$. Of course, the student should feel free to experiment with any functions whatsoever.

9. 0, yes

10. 0, yes. The points $\frac{1}{4}z = (2n + 1)\pi/2$, thus $z = (4n + 2)\pi$, at which $\cos \frac{1}{4}z = 0$, lie outside the unit circle.

11. $\dfrac{\pi}{2}i$, *no*

12. $\displaystyle\int_0^{2\pi} e^{-3it}\, ie^{it}\, dt = -\tfrac{1}{2}e^{-2it}\Big|_0^{2\pi} = 0$, no

13. 0, Yes

14. $\displaystyle\int_0^{2\pi} e^{-it}\, ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i$

15. $i\pi$, no

16. $\dfrac{1}{\pi z - 1} = \dfrac{1/\pi}{z - 1/\pi}$. *Answer* $\dfrac{1}{\pi}2\pi i = 2i$ by deformation and (11) in Sec. 14.1.

18. $\dfrac{1}{5z - 1} = \dfrac{\frac{1}{5}}{z - \frac{1}{5}}$, so that the integral equals $\frac{1}{5} \cdot 2\pi i = \frac{2}{5}\pi i$ by the principle of deformation of path. The theorem does not apply.

20. Sketch C to see what is going on. Ln $(1 - z)$ is not analytic at $z = 1$ and on the portion $x > 1$ of the real axis. Since this lies outside the contour, the integral is 0 by Cauchy's theorem, which applies.

21. $2\pi i$

22. $\displaystyle\int_{-1}^{1} x\, dx = 0$, $z(t) = e^{it} \quad (0 \leqq t \leqq \pi)$; hence the integral over the semicircle is

$$\int_0^{\pi} (\cos t) ie^{it}\, dt = i\int_0^{\pi} \tfrac{1}{2}(e^{it} + e^{-it})e^{it}\, dt = i[0 + \tfrac{1}{2}\pi] = \tfrac{1}{2}\pi i.$$

24. We obtain

$$\frac{1}{z^2-1} = \frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1}.$$

For the first fraction, integration around the left loop gives 0 and around the right loop $(\frac{1}{2})2\pi i = \pi i$. For the second fraction, with the minus sign, clockwise integration around the left loop gives $(-1)(-\frac{1}{2})2\pi i = \pi i$ and for the integration around the right loop 0. *Answer* $2\pi i$.

26. 0 because the points $\pm 4n\pi i$ at which $\sinh \frac{1}{2}z = 0$ all lie outside the contour of integration, so that Cauchy's theorem applies.

28. 0 because the points $z = \pm\pi, \pm 2\pi, \cdots$ as well as $\pm\frac{3}{2}, \pm\frac{3}{2}i$ lie outside the contour C.

29. 0

30. The integrand equals

$$\frac{2z}{z^2+4} + \frac{1}{z^2} = \frac{1}{z+2i} + \frac{1}{z-2i} + \frac{1}{z}.$$

The integrals of the three terms on the right equal $-2\pi i, -2\pi i, 0$, respectively, so that the *answer* is $-4\pi i$, the minus sign arising because we integrate clockwise.

SECTION 14.3. Cauchy s Integral Formula, page 660

Purpose. To prove, discuss, and apply Cauchy's integral formula, the second major consequence of Cauchy's integral theorem (the first being the justification of indefinite integration).

Comment on Examples

The student has to find out how to write the integrand as a product $f(z)$ times $1/(z - z_0)$, and the examples (particularly Example 3) and problems are designed to give help in that technique.

SOLUTIONS TO PROBLEM SET 14.3, page 663

1. $2\pi i z^2/(z-1)|_{z=-1} = -\pi i$

2. The integrand is not analytic at ± 1. $z = 1$ lies inside the contour, $z = -1$ outside. Hence, by Cauchy's formula, the integral has the value

$$2\pi i \cdot \frac{z^2}{z+1}\bigg|_{z=1} = 2\pi i \cdot \frac{1}{2} = \pi i.$$

3. 0

4. $z = -1$ lies inside C, $z = 1$ outside. Hence Cauchy's formula gives for

$$\frac{z^2}{z^2-1} = \frac{z^2}{z-1} \cdot \frac{1}{z+1} \quad \text{the value} \quad 2\pi i \cdot \frac{z^2}{z-1}\bigg|_{z=-1} = 2\pi i \cdot \left(-\frac{1}{2}\right) = -\pi i.$$

Note that by the deformation principle this must be the same value as in Prob. 1.

5. $2\pi i(\cos 2z)/4|_{z=0} = \frac{1}{2}\pi i$

6. The given function

$$\frac{e^{2z}}{\pi z - i} = \frac{e^{2z}/\pi}{z - i/\pi}$$

is not analytic at $z = i/\pi$ inside the unit circle. By Cauchy's formula the integral equals

$$\frac{2\pi i}{\pi} e^{2i/\pi} = 2i\left(\cos\frac{2}{\pi} + i \sin\frac{2}{\pi}\right) = -1.189 + 1.608i.$$

7. $2\pi i(i/4)^2/4 = -\dfrac{1}{32}\pi i$

8. From

$$\frac{z \sin z}{2z - 1} = \frac{(z \sin z)/2}{z - \frac{1}{2}}$$

we obtain by Cauchy's formula

$$2\pi i \cdot \tfrac{1}{2} z \sin z\big|_{z=1/2} = \tfrac{1}{2}\pi i \sin(1/2) = 0.7530i$$

10. Team Project. (a) Eq. (2) is

$$\oint_C \frac{z^3 - 6}{z - \frac{1}{2}i}\,dz = \left[\left(\frac{i}{2}\right)^3 - 6\right]\oint_C \frac{dz}{z - \frac{1}{2}i} + \oint_C \frac{z^3 - (\frac{1}{2}i)^3}{z - \frac{1}{2}i}\,dz$$

$$= \left(-\frac{i}{8} - 6\right)2\pi i + \oint_C \left(z^2 + \frac{1}{2}iz - \frac{1}{4}\right)dz = \frac{1}{4}\pi - 12\pi i$$

because the last integral is zero by Cauchy's integral theorem. The result agrees with that in Example 2, except for a factor 2.

(b) Using (12) in App. A3.1, we obtain (2) in the form

$$\oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\,dz = (\sin\tfrac{1}{2}\pi)\oint_C \frac{dz}{z - \frac{1}{2}\pi} + \oint_C \frac{\sin z - \sin\frac{1}{2}\pi}{z - \frac{1}{2}\pi}\,dz$$

$$= 2\pi i + \oint_C \frac{2\sin(\frac{1}{2}z - \frac{1}{4}\pi)\cos(\frac{1}{2}z + \frac{1}{4}\pi)}{z - \frac{1}{2}\pi}\,dz.$$

As ρ in Fig. 357 approaches 0, the integrand approaches 0. Indeed, the expression

$$\frac{2\sin(\frac{1}{2}z - \frac{1}{4}\pi)}{z - \frac{1}{2}\pi}$$

approaches 1 (calculus!), whereas the cosine factor approaches $\cos\frac{1}{2}\pi$, which is 0.

12. The integrand is

$$\frac{z}{z^2 + 4z + 3} = \frac{z}{(z + 1)(z + 3)}.$$

$z = -1$ lies inside the contour, $z = -3$ outside. Accordingly, Cauchy's integral formula gives

$$2\pi i \frac{z}{z + 3}\bigg|_{z=-1} = 2\pi i \frac{-1}{2} = -\pi i.$$

14. $z = 0$ lies inside the contour. The solutions of $e^z - 2i = 0$ lie outside because $e^z = 2i, z = \ln 2i \pm 2n\pi i$ and $|\ln 2i| > \ln 2 > 0.6$. We thus obtain the answer

$$2\pi i \left. \frac{e^z}{e^z - 2i} \right|_{z=0} = \frac{2\pi i}{1 - 2i} = -0.8\pi + 0.4\pi i = -2.51 + 1.26i.$$

16. $2\pi i \tan i = -2\pi \tanh 1 = -4.785$

18. $4z^2 - 8iz = 4z(z - 2i) = 0$ at $z = 2i$ is in the "annulus" in the figure and $z = 0$ is not. Hence

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\, dz = 2\pi i \cdot \frac{1}{4} \cdot \left. \frac{\sin z}{z} \right|_{z=2i} = \frac{\pi i}{4} \sinh 2 = 2.849i$$

20. For $z_1 = z_2$ use Example 6 in Sec. 14.1 with $m = -2$. For $z_1 \neq z_2$ use partial fractions

$$\frac{1}{(z - z_1)(z - z_2)} = \frac{1}{(z_1 - z_2)(z - z_1)} - \frac{1}{(z_1 - z_2)(z - z_2)}.$$

SECTION 14.4. Derivatives of Analytic Functions, page 664

Purpose. To discuss and apply the most important consequence of Cauchy's integral formula, the theorem on the existence and form of the derivatives of an analytic function.

Main Content

Formulas for the derivatives of an analytic function (1)

Cauchy's inequality

Liouville's theorem

Morera's theorem (inverse of Cauchy's theorem)

Comments on Content

Technically the application of the formulas for derivatives in integration is practically the same as that in the last section.

The basic importance of (1) in giving the existence of all derivatives of an analytic function is emphasized in the text.

SOLUTIONS TO PROBLEM SET 14.4, page 667

1. $(2\pi i/3!)(-8 \cos 0) = -\frac{8}{3}\pi i$

2. We obtain

$$\frac{z^6}{(2z - 1)^6} = \frac{z^6}{2^6(z - \frac{1}{2})^6}.$$

Hence the solution is

$$\frac{2\pi i}{5!} \cdot \frac{1}{2^6} \cdot \left. (z^6)^{(5)} \right|_{z=1/2} = \frac{2\pi i}{5!} \cdot \frac{1}{2^6} \cdot 6!z \left. \right|_{z=1/2} = \frac{3\pi i}{32}.$$

3. $(2\pi/(n-1)!)e^0$

4. We have to differentiate twice, obtaining

$$\begin{aligned}\frac{2\pi i}{2!}(e^z \cos z)''\Big|_{z=\pi/4} &\\ &= \pi i e^z(\cos z - \sin z)' \\ &= \pi i e^z(\cos z - \sin z - \sin z - \cos z) \\ &= -2e^z \sin z.\end{aligned}$$

Evaluation at $z = \pi/4$ gives $-\sqrt{2}e^{\pi/4} = -3.102$.

5. $\dfrac{2\pi i}{3!}(\sinh 2z)''' = \dfrac{8}{3}\pi i \cosh 1 = 12.93i$

6. $z = \frac{1}{2}$ lies within the contour, $z = 2i$ lies outside. Accordingly, differentiating once, we obtain

$$\begin{aligned}2\pi i\left[\frac{1}{(z-2i)^2}\right]' &= 2\pi i(-2)(z-2i)^{-3}|_{z=i/2} \\ &= -4\pi i(-3i/2)^{-3} \\ &= -\frac{32}{27}\pi.\end{aligned}$$

8. We have to differentiate twice, so that (1″) gives

$$\begin{aligned}\frac{2\pi i}{2!}(z^3 + \sin z)'' &= \pi i(6z - \sin z)|_{z=i} \\ &= \pi i(6i - \sin i) \\ &= -6\pi + \pi \sinh 1 \\ &= -15.158.\end{aligned}$$

10. We have to differentiate once and obtain from (1′)

$$\begin{aligned}2\pi i\left(\frac{4z^3-6}{z}\right)'\Big|_{z=1+i} &= 2\pi i\left(8z + \frac{6}{z^2}\right)\Big|_{z=1+i} \\ &= 2\pi i(8 + 8i + 6/2i) \\ &= 2\pi i(8 + 5i) \\ &= \pi(-10 + 16i) \\ &= -31.42 + 50.27i.\end{aligned}$$

11.

$$\begin{aligned}\frac{2\pi i}{4}((1+z)\cos z)'|_{z=1/2} &= \frac{\pi i}{2}(\cos z - (1+z)\sin z) \\ &= \frac{1}{2}\pi i\left(\cos\frac{1}{2} - \frac{3}{8}\sin\frac{1}{2}\right) \\ &= 0.2488i\end{aligned}$$

12. We have to differentiate once and obtain from (1), with the minus sign because we integrate clockwise,

$$\begin{aligned} -2\pi i\left(\frac{\exp(z^2)}{z}\right)'\bigg|_{z=2i} &= -2\pi i\left(2\exp(z^2) - \frac{1}{z^2}\exp(z^2)\right)\bigg|_{z=2i} \\ &= -2\pi i e^{-4}(2 + \tfrac{1}{4}) \\ &= -\tfrac{9}{2}\pi e^{-4} i \\ &= -0.2589i. \end{aligned}$$

13. $2\pi i \cdot \dfrac{1}{z}\bigg|_{z=4} = \dfrac{1}{2}\pi i = 1.571i$

14. Differentiating once, we obtain from (1), since $z = -1$ lies inside the contour and $z = 2$ outside,

$$\begin{aligned} 2\pi i\left(\frac{\mathrm{Ln}(z+3)}{z-2}\right)'\bigg|_{z=-1} &= 2\pi i\left(\frac{1}{(z+3)(z-2)} - \frac{\mathrm{Ln}(z+3)}{(z-2)^2}\right)\bigg|_{z=-1} \\ &= 2\pi i\left(\frac{1}{-6} - \frac{\mathrm{Ln}\,2}{9}\right) \\ &= -1.531i. \end{aligned}$$

16. $z = 0$ lies outside the "annulus" bounded by the two circles of C. The point $z = 2i$ lies inside the larger circle but outside the smaller. Hence the integral equals (differentiate once)

$$2\pi i\left(\frac{2e^{2z}}{z} - \frac{e^{2z}}{z^2}\right)\bigg|_{z=2i} = 2\pi i\left(\frac{2e^{4i}}{2i} - \frac{e^{4i}}{-4}\right) = \pi e^{4i}\left(2 + \frac{1}{2}i\right).$$

18. 0 when n is a negative integer or 0, by Cauchy's integral theorem. Also 0 when $n = 1, 3, 5, \cdots$ because then we have in (1) an even number of differentiations, which reproduces $\sinh z$, which is 0 at $z = 0$. When $n = 2, 4, \cdots$, We have to differentiate an odd number of times, producing $\cosh z$, which is 1 at $z = 0$, so that by (1) the answer is $2\pi i/(n-1)!$

20. Team Project. (a) If no such z existed, we would have $|f(z)| \leqq M$ for every $|z|$, which means that the entire function $f(z)$ would be bounded, hence a constant by Liouville's theorem.

(b) Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n = z^n\left(c_n + \dfrac{c_{n-1}}{z} + \cdots + \dfrac{c_0}{z^n}\right)$, $c_n \neq 0$, $n > 0$. Set $|z| = r$. Then

$$|f(z)| > r^n\left(|c_n| - \frac{|c_{n-1}|}{r} - \cdots - \frac{|c_0|}{r^n}\right)$$

and $|f(z)| > \frac{1}{2}r^n|c_n|$ for sufficiently large r. From this the result follows.
(c) $|e^z| > M$ for real $z = x$ with $x > R = \ln M$. On the other hand, $|e^z| = 1$ for any pure imaginary $z = iy$ because $|e^{iy}| = 1$ for any real y (Sec. 13.5).

(d) If $f(z) \neq 0$ for all z, then $g = 1/f$ would be analytic for all z. Hence by (a) there would be values of z exterior to every circle $|z| = R$ at which, say, $|g(z)| > 1$ and thus $|f(z)| < 1$. This contradicts (b). Hence $f(z) \neq 0$ for all z cannot hold.

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18. Not in general.

20. The *ML*-inequality gives such a bound.

21. $-\frac{1}{2}\sinh(\pi^2/4) = -2.928$

22. The integral of z is zero. $|z|$ is not analytic, so we must use a representation $z(t) = e^{-it}(0 \leqq t \leqq 2\pi)$ of the path, obtaining

$$\oint_C |z|\, dz = \int_0^{2\pi} 1 \cdot (-ie^{-it})\, dt = e^{-it}\Big|_{t=0}^{2\pi} = 0.$$

23. $2\pi i \cdot \frac{1}{3!}(e^{-z})^{(3)}\Big|_{z=0} = -\frac{1}{3}\pi i e^0 = -1.047i$

24. $z(t) = t + i2t^2, \quad (0 \le t \le 2), \quad \dot{z} = 1 + 4it$, $\operatorname{Re} z(t) = t$, hence

$$\int_0^2 t(1 + 4it)\, dt = 2 + \frac{32}{3}i$$

26. $z^2 + \bar{z}^2 = 2(x^2 - y^2)$. From 0 to 2 we have $z = x = t\ (0 \leqq t \leqq 2)$, $dz = dt$, and thus

$$\int_0^2 2t^2\, dt = \frac{2}{3}t^3\Big|_0^2 = \frac{16}{3}.$$

From 2 to $2 + i$ we have $z = 2 + it\ (0 \leqq t \leqq 2)$, $dz = i\, dt$, and thus

$$\int_0^2 2(2^2 - t^2)i\, dt = \frac{32}{3}i.$$

28. $2\pi i(\operatorname{Ln} z)'\big|_{z=2i} = \frac{2\pi i}{z}\Big|_{z=2i} = \pi$

29. $8\pi i$

30. $\sin z\big|_0^{\pi/2-i} = \sin(\pi/2 - i) = \cosh 1 = 1.543$