CHAPTER 3 Higher Order Linear ODEs

Chapters 1 and 2 demonstrate and illustrate that first- and second-order ODEs are, by far, the most important ones in usual engineering and physical applications, and the linear ODEs of orders 1 and 2 foreshadow much of the general theory. This implies that the material of the present chapter is less important than that of the previous chapters. For this reason the present chapter was created from the previous edition by rearranging and extending material previously contained in Chap. 2. This rearrangement parallels the arrangement in Chap. 2, in order to facilitate comparisons.

Root Finding

For higher order ODEs you may need Newton's method or some other method from Sec. 19.2 (which is independent of other sections in numerics) in work on a calculator or with your CAS (which may give you a root-finding method directly).

Linear Algebra

The typical student may have taken an elementary linear algebra course simultaneously with a course on calculus and will know much more than is needed in Chaps. 2 and 3. Thus Chaps. 7 and 8 need not be taken before Chap. 3.

In particular, although the Wronskian becomes useful in Chap. 3 (whereas for $n = 2$ one hardly needs it), a very modest knowledge of determinants will suffice. (For $n = 2$ and 3, determinants are treated in a reference section, Sec. 7.6.)

SECTION 3.1. Homogeneous Linear ODEs, page 105

Purpose. Extension of the basic concepts and theory in Secs. 2.1 and 2.6 to homogeneous linear ODEs of any order n. This shows that practically all the essential facts carry over without change. Linear independence, now more involved than for $n = 2$, causes the Wronskian to become indispensable (whereas for $n = 2$ it played a marginal role).

Main Content, Important Concepts

Superposition principle for the homogeneous ODE (2) (Theorem 1)

General solution (3), basis, particular solution

Linear independence and dependence of functions (see (4) and Examples 1 and 2)

Linear combination

General solution of (2) with continuous coefficients exists.

Existence and uniqueness of solution of initial value problem (2), (5) (Theorem 2, Example 4)

Linear independence of solutions, Wronskian (6) (Theorem 3, Example 5)

Existence of general solution (Theorem 4)

General solution includes all solutions of (2) (Theorem 5)

Comment on Order of Material

In Chap. 2 we first gained practical experience and skill and presented the theory of the homogeneous linear ODE at the end of the discussion, in Sec. 2.6. In this chapter, with all the experience gained on second-order ODEs, it is more logical to present the whole theory at the beginning and the solution methods (for linear ODEs with constant coefficients) afterward. Similarly, the same logic applies to the nonhomogeneous linear ODE, for which Sec. 3.3 contains the theory as well as the solution methods.

Problem Set 3.1

Problems 1–6 should emphasize the importance of the Wronskian for higher-order ODEs.

Team Project 7 is another demonstration of the importance of linearity in connection with ODEs.

Problems 8–16 invite students to deepen their understanding of linear independence and dependence from a practical point of view, with Probs. 8–15 involving special functions and Prob. 16 concerning the basic general theorems.

SOLUTIONS TO PROBLEM SET 3.1, page 111

2. $W = \frac{3}{2}e^{\frac{1}{2}x}$

5. $W = -5e^{-4x}$

8. Functions $y_1 = 0, y_2, \cdots, y_n$ are always linearly dependent because (4) holds with any $k_1 \neq 0$ and the other k_j's all 0.

9. Linearly independent

10. Linearly independent. *First proof.* Divide (4) by e^{2x} to obtain $c_1 + c_2x + c_3x^2 = 0$. Set $x = 0$ to get $c_1 = 0$. Etc.

Second proof. The functions are solutions of the ODE

$$y''' - 6y'' + 12y' - 8y = 0.$$

Their Wronskian equals

$$2e^{6x}.$$

11. Linearly independent

12. Linearly dependent because

$$\cos 2x = \cos^2 x - \sin^2 x.$$

This problem is typical of the use of functional relations for proofs of linear dependence; for instance, $\ln x^2 = 2 \ln x$, etc.

14. Linearly dependent because $\cos^2 x + \sin^2 x = 1$

15. Linearly dependent

16. Team Project. (a) (1) No. If $y_1 \equiv 0$, then (4) holds with any $k_1 \neq 0$ and the other k_j all zero.

(2) Yes. If S were linearly dependent on I, then (4) would hold with a $k_j \neq 0$ on I, hence also on J, contradicting the assumption.

(3) Not necessarily. For instance, x^2 and $x|x|$ are linearly dependent on the interval $0 < x < 1$, but linearly independent on $-1 < x < 1$.

(4) Not necessarily. See the answer to (3).

(5) Yes. See the answer to (2).

(6) Yes. By assumption, $k_1y_1 + \cdots + k_py_p = 0$ with $k_1, \cdots, k_p$ not all zero (this refers to the functions in S), and for T we can add the further functions with coefficients all zero; then the condition for linear dependence of T is satisfied.

(b) We can use the Wronskian for testing linear independence only if we know that the given functions are solutions of a homogeneous linear ODE with continuous coefficients. Other means of testing are the use of functional relations, e.g., $\ln x^2 = 2 \ln x$ or trigonometric identities, or the evaluation of the given functions at several values of x, to see whether we can discover proportionality.

SECTION 3.2. Homogeneous Linear ODEs with Constant Coefficients, page 111

Purpose. Extension of the algebraic solution method for constant-coefficient ODEs from $n = 2$ (Sec. 2.2) to any n, and discussion of the increased number of possible cases:

Real different roots (Theorems 1 and 2, Example 1)

Complex simple roots (Example 2)

Real multiple roots ((7), Example 3)

Complex multiple roots (11)

Combinations of the preceding four basic cases

Comment on Numerics

In practical cases, one may have to use Newton's method or another method for computing (approximate values of) roots in Sec. 19.2.

Problem Set 3.2

Problems 1–13 concern general solutions and IVPs.

Reduction of order when solutions are known (for instance, are obtained by inspection) was considered for second-order ODEs in Sec. 2.1 and is extended to third-order ODEs in the present Project 14 as well as in the open-ended CAS Experiment 15, where the difficulty seems to be in finding ODEs simple enough so that they can be successfully handled as requested.

SOLUTIONS TO PROBLEM SET 3.2, page 116

1. $y = c_1 + c_2 \sin 3x + c_3 \cos 3x$

2. From the characteristic equation

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda^2 + 1)^2 = 0$$

we conclude that the corresponding real general solution is

$$y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x.$$

3. $y = c_1 + c_2 x + c_3 \sin 4x + c_4 \cos 4x$

4. From the characteristic equation

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda + 1)(\lambda - 1)^2$$

we conclude that a general solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x)e^x.$$

6. $y = c_1 + c_2 \sin x + c_3 \cos x + c_4 x \sin x + c_5 x \cos x$

7. $y = \frac{3}{2} + \frac{9}{4} e^{-\frac{3}{2}x} \sin \frac{8x}{5} + \frac{9}{2} e^{-\frac{3}{2}x} \cos \frac{8x}{5}$

8. $y = 10e^{-4x} (\cos 1.5x - \sin 1.5x) + 0.05e^{0.5x}$

10. $y = e^x(\frac{5}{16} \cos x - \frac{3}{16} \sin x) + e^{-x}(\frac{3}{16} \cos x - \frac{23}{16} \sin x)$

12. $y = 1 + 3e^{-2x} - e^{-x}$

13. $y = 5.2e^{0.25x} + 2.15e^{-0.6x} + 0.25 \sin 2x + 0.0625 \cos 2x$

14. Project. (a) Divide the characteristic equation by $\lambda - \lambda_1$ if $e^{\lambda_1 x}$ is known.

(b) The idea is the same as in Sec. 2.1.

(c) We first produce the standard form because this is the form under which the equation for z was derived. Division by x^3 gives

$$y''' - \frac{3}{x}y'' + \left(\frac{6}{x^2} - 1\right)y' - \left(\frac{6}{x^3} - \frac{1}{x}\right)y = 0.$$

With $y_1 = x, y_1' = 1, y_1'' = 0$, and the coefficients p_1 and p_2 from the standard equation, we obtain

$$xz'' + \left[3 + \left(-\frac{3}{x}\right)x\right]z' + \left[2\left(-\frac{3}{x}\right)\cdot 1 + \left(\frac{6}{x^2} - 1\right)x\right]z = 0.$$

Simplification gives

$$xz'' + \left(-\frac{6}{x} + \frac{6}{x} - x\right)z = x(z'' - z) = 0.$$

Hence

$$z = c_1 e^x + \tilde{c}_2 e^{-x}.$$

By integration we get the *answer*

$$y_2 = x\int z\,dx = (c_1 e^x + c_2 e^{-x} + c_3)x.$$

SECTION 3.3. Nonhomogeneous Linear ODEs, page 116

Purpose. To show that the transition from $n = 2$ (Sec. 2.7) to general n introduces no new ideas but generalizes all results and practical aspects in a straightforward fashion; this refers to existence, uniqueness, and the need for a particular solution y_p to get a general solution in the form

$$y = y_h + y_p.$$

Comment on Elastic Beams

This is an important standard example of a fourth-order ODE that needs no lengthy preparations and explanations.

Vibrating beams follow in Problem Set 12.3. This leads to PDEs, since time t comes in as an additional variable.

Comment on Variation of Parameters

This method looks perhaps more complicated than it is; also the integrals may often be difficult to evaluate, and handling the higher order determinants that occur may require some more skill than the average student will have at this time. Thus it may be good to discuss this matter only rather briefly.

Problem Set 3.3 concerns:

Determination of general solutions (Probs. 1–7) and solving IVPs (Probs. 8–13), with y_p obtained by the method of undetermined coefficients, where Prob. 6 shows an ODE reducible to second order.

CAS Experiment 14 invites students to explore the extent to which the method of undetermined coefficients can be used, and Writing Report 15 should establish this method as a special case of variation of parameters for the ODE indicated.

SOLUTIONS TO PROBLEM SET 3.3, page 122

1. $y = \frac{1}{6}x^3e^x + x + 4 + c_1e^x + c_2xe^x + c_3x^2e^x$

2. $y = c_1e^{-2x} + c_2e^{-x} + c_3e^x + 2x^3 - 3x^2 + 15x - 8$

3. $-\frac{7}{160}e^{-2x} + \frac{7}{160}e^{2x} + c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$

4. The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 5\lambda - 39 = (\lambda^2 + 6\lambda + 13)(\lambda - 3) = [(\lambda + 3)^2 + 4](\lambda - 3) = 0.$$

Hence a general solution of the homogeneous ODE is

$$y_h = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x) + c_3e^{3x}.$$

Using the method of undetermined coefficients, substitute $y_p = a \cos x + b \sin x$, obtaining

$$y_p = 7 \cos x + \sin x.$$

5. $y = c_2x \ln x + c_3x + \frac{1}{36}\,\frac{9c_1x-4}{x^2}$

6. Set $y' = z$ to obtain

$$z'' + 4z = \sin x.$$

By undetermined coefficients,

$$z = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{3} \sin x.$$

By integration,

$$y = \int z\, dx = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{3} \cos x + c_3.$$

7. $y = \cos x + \sin x + c_1e^x + c_2xe^x + c_3x^2e^x$

8. A general solution of the homogeneous ODE is obtained from the characteristic equation

$$(\lambda^2 - 1)(\lambda^2 - 4) = \lambda^4 - 5\lambda^2 + 4 = 0$$

in the form

$$y_h = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x},$$

A particular solution of the nonhomogeneous ODE is obtained by undetermined coefficients, namely, $e^{-3x}/4$. From this and the initial conditions we obtain the answer

$$y = \tfrac{1}{4}e^x + \tfrac{3}{2}e^{-x} - e^{-2x} + \tfrac{1}{4}e^{-3x}$$

10. $y_1 = x, y_2 = x \ln x, y_3 = x (\ln x)^2,$ *Answer:* $x^2 + x \ln x + \frac{11}{2} x (\ln x)^2$

11. $y = \frac{2}{5}e^{-2x} \cos x + \frac{2}{5}e^{-2x} \sin x$

12. The characteristic equation of the homogeneous ODE

$$\lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$$

has the roots -3, 2, and 3. Hence a general solution of the homogeneous ODE is

$$y_h = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{3x}.$$

This also shows that the function on the right is a solution of the homogeneous ODE. Hence the Modification Rule applies, and the particular solution obtained is

$$y_p = -0.2xe^{2x}.$$

A general solution of the nonhomogeneous ODE is

$$y = y_h + y_p = c_1 e^{-3x} + c_2 e^{2x} + c_3 e^{3x} - 0.2xe^{2x}.$$

From this and the initial conditions we obtain

$$y = (4.5 - 0.2x)e^{2x}.$$

14. CAS EXPERIMENT. The first equation has as a general solution

$$y = (c_1 + c_2 x + c_3 x^2)e^x + \tfrac{8}{105} e^x x^{7/2},$$

so in cases such as this, one could try

$$y_p = x^{1/2}(a_0 + a_1 x + a_2 x^2 + a_3 x^3)e^x.$$

However, the equation alone does not show much, so another idea is needed. One could modify the right side systematically and see how the solution changes. The solution of the second suggested equation shows that the equation is not accessible by undetermined coefficients; its solution is

$$y = c_1 x^{-1} + c_2 x + c_3 x^2 + \tfrac{1}{8} x^3 \ln x - \tfrac{7}{32} x^3.$$

And one could perhaps modify this equation, too, in an attempt to obtain a form of solution that might be suitable for undetermined coefficients.

SOLUTIONS TO CHAPTER 3 REVIEW QUESTIONS AND PROBLEMS, page 122

6. The characteristic equation is

$$\lambda^4 - 3\lambda^2 - 4 = (\lambda^2 - 1)(\lambda^2 + 4) = 0.$$

Hence a general solution is

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x.$$

7. $y = c_1 + c_2 e^x \sin 2x + c_3 e^x \cos 2x$

8. y_p is obtained by the method of undetermined coefficients. A general solution of the nonhomogeneous ODE is

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{4x} - 5e^{2x}.$$

9. $y = \frac{1}{5} e^x - \frac{1}{5} e^{-x} + c_1 \cos 3x + c_2 e^{-3x} + c_3 e^{3x} + c_4 \sin 3x$

10. The auxiliary equation is

$$m(m - 1)(m - 2) + 3m(m - 1) - 2m = m(m^2 - 3) = 0.$$

Hence a general solution is

$$y = c_1 + c_2 x^{\sqrt{3}} + c_3 e^{-\sqrt{3}}.$$

12. The characteristic equation is

$$\lambda^3 - \lambda = \lambda(\lambda + 1)(\lambda - 1) = 0.$$

A particular solution is obtained by undetermined coefficients. We thus obtain as a general solution of the nonhomogeneous ODE

$$y = c_1 + c_2 \cosh x + c_3 \sinh x - \tfrac{125}{36} \cosh 0.8x.$$

13. $y = -24 - 12x - 2x^2 + c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

14. The characteristic equation is quadratic in λ^2, the roots being ± 2 and ± 3. A particular solution of the nonhomogeneous ODE is obtained by undetermined coefficients, namely, $e^{-x}/2$. Hence a general solution of the nonhomogeneous ODE is

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{-3x} + c_4 e^{3x} + \tfrac{1}{2} e^{-x}.$$

15. $y = -5 + c_1 + c_2 \sqrt{x} + c_3 x^{\frac{1}{4}}$

16. $y = \frac{9}{49} e^{-\frac{4}{3}x} - \frac{9}{49} e^x + \frac{3}{7} x e^x$

18. The characteristic equation is quadratic in λ^2, the roots being $-5, -1, 1, 5$. The right side requires a quadratic polynomial whose coefficients can be determined by substitution. Finally, the initial conditions are used to determine the four arbitrary constants in the general solution of the nonhomogeneous ODE thus obtained. The *answer* is

$$y = 5e^{-x} + e^{-5x} + 6.16 + 4x + 2x^2.$$

19. $y = -\frac{1}{3} e^{-4x}$

20. Two of four possible terms resulting from the homogeneous ODE are not visible in the answer. The student should recognize that all or some or none of the solutions of a basis of the homogeneous ODE may be present in the final answer; this will depend on the initial conditions.

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$; hence a general solution of the homogeneous equation is

$$y_h = (c_1 + c_2 x + c_3 x^2) e^{-x}.$$

By the method of undetermined coefficients and from the initial conditions we get the *answer*

$$y = (1 + x^2) e^{-x} - 2 \cos x - 2 \sin x.$$