Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.10

No. 1

Circular membranes are important parts of drums, pumps, microphones, telephones, and other devices. This accounts for their great importance in engineering. Therefore, Polar coordinates are used for this purpose.

No. 2

If $u = u(r)$ and we set $u' = v$, then $\nabla^2 u = u'' + u'/r = v' + v/r = 0$. Hence $v'/v = -1/r$, $\ln v = -\ln r + \tilde{c} = \ln(c_1/r)$, $v = c_1/r$. By integration, $u = c_1 \ln r + c_2$.

No. 3

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\nabla^2 u = \frac{(r u_r)_r}{r} + \frac{u_{\theta\theta}}{r^2}, \quad \text{where } u_r = \frac{\partial u}{\partial r}, u_{\theta\theta} = \frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{r u_{rr}}{r} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

故得証、

No. 4

Team Project. **(a)** $r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = x^2 - y^2$, $r^2 \sin 2\theta = 2xy$, etc.

(c) $u = \dfrac{400}{\pi}\left(r\sin\theta + \dfrac{1}{3}r^3 \sin 3\theta + \dfrac{1}{5}r^5 \sin 5\theta + \cdots\right)$

(d) The form of the series results as in (b), and the formulas for the coefficients follow from

$$u_r(R, \theta) = \sum_{n=1}^{\infty} nR^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = f(\theta).$$

(f) $u = -(r + 4/r)(\sin\theta)/3$ by separating variables

No. 5

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_n = \frac{1}{\pi \cdot n R^{n-1}} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$B_n = \frac{1}{\pi \cdot n R^{n-1}} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$f(\theta) = u(1,\theta) = \begin{cases} 220 & ; \ -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \\ 0 & ; \ \text{otherwise} \end{cases} \qquad ; \ R = 1.$$

$$A_n = \frac{1}{\pi \cdot n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 220 \cos n\theta \, d\theta$$

$$A_0 = \frac{1}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 220 \, d\theta = 110.$$

$$A_n = \frac{1}{\pi \cdot n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 220 \cos n\theta \, d\theta = \frac{220}{n\pi} (-1)^{n+1} \left[1 - (-1)^n \right]$$

$$B_n = \frac{1}{\pi \cdot n} \int_{-\frac{1}{2}\pi}^{\frac{\pi}{2}} 220 \sin n\theta \, d\theta = 0$$

$$u(r,\theta) = 110 + \sum_{n=1}^{\infty} r^n \cdot \left(\frac{220}{n\pi} (-1)^{n+1} \left[1 - (-1)^n \right] \right) \cos n\theta$$

$$= 110 + \frac{440}{\pi} \left(r\cos\theta - \frac{1}{3} r^3 \cos 3\theta + \frac{1}{5} r^5 \cos 5\theta - + \cdots \right)$$

No. 6

$$f(\theta) = u(1, \theta) = 400\cos^3\theta \qquad ; \quad R = 1.$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 400\cos^3\theta \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(300\cos\theta + 100\cos3\theta)d\theta = 0$$

$$A_n = \frac{1}{\pi \cdot n}\int_{-\pi}^{\pi} 400\cos^3\theta \cdot \cos n\theta \, d\theta = \frac{1}{n\pi}\int_{-\pi}^{\pi}(300\cos\theta + 100\cos3\theta) \cdot \cos n\theta \, d\theta$$

$$= \begin{cases} \frac{300}{n} & ; n = 1 \\ \frac{100}{n} & ; n = 3 \\ 0 & ; n \neq 1, 3 \end{cases}$$

$$B_n = \frac{1}{\pi \cdot n}\int_{-\pi}^{\pi} 400\cos^3\theta \sin n\theta \, d\theta = 0$$

$$\therefore u(r, \theta) = \sum_{n=1}^{\infty} r^n A_n \cos n\theta$$

$$= 300 \cdot r\cos\theta + \frac{300}{3} r^3 \cos3\theta$$

No. 7

$$f(\theta) = u(1, \theta) = 110|\theta| \quad ; \ -\pi < \theta < \pi, \quad ; R = 1$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 110|\theta| \, d\theta = \frac{2}{2\pi}\int_{0}^{\pi} 110\theta \, d\theta = 55\pi$$

$$A_n = \frac{1}{\pi \cdot n}\int_{-\pi}^{\pi} 110|\theta| \cos n\theta \, d\theta = \frac{2}{n\pi}\int_{0}^{\pi} 110\,\theta\cos n\theta \, d\theta$$

$$= \frac{220}{n^2\pi}\left[(-1)^n - 1\right]$$

$$B_n = \frac{1}{\pi \cdot n}\int_{-\pi}^{\pi} 110|\theta| \sin n\theta \, d\theta = \frac{2}{\pi n}\int_{0}^{\pi} 110\theta \sin n\theta \, d\theta = 0$$

$$\therefore u(r, \theta) = 55\pi + \sum_{n=1}^{\infty} r^n \frac{220}{n^2\pi}\left[(-1)^n - 1\right]\cos n\theta$$

$$= 55\pi - \frac{440}{\pi}\left(r\cos\theta + \frac{1}{3^2}r^3\cos3\theta + \frac{1}{5^2}r^5\cos5\theta + \cdots\right)$$

$$f(\theta) = u(1,\theta) = \begin{cases} \theta & ; \ -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \theta \, d\theta = 0$$

$$A_n = \frac{1}{n\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \theta \cos n\theta \, d\theta = 0$$

$$B_n = \frac{1}{n\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \theta \sin n\theta \, d\theta = \frac{1}{n^3\pi} \sin \frac{n}{2}\theta$$

$$\therefore u(r,\theta) = \sum_{n=1}^{\infty} r^n \left(\frac{1}{n^3\pi} \sin \frac{n}{2}\theta\right) \sin n\theta$$

$$= \frac{1}{\pi}\left(r\sin\theta - \frac{1}{3^3} r^3 \sin 3\theta + \frac{1}{5^3} r^5 \sin 5\theta - + \cdots\right)$$

No. 8

$$-2\frac{r\sin(\theta)}{\pi} - 1/2\, r^2 \sin(2\theta) + 2/9\frac{r^3 \sin(3\theta)}{\pi} + 1/4\, r^4 \sin(4\theta) + \cdots$$

Except for the presence of the variable *r*, this is just another important application of Fourier series, and we concentrate on a few simple practically important types of boundary values. Of course, earlier problems on Fourier series can now be modified by introducing the powers of *r* and considered from the present point of view.

No. 9

問答或證明題，不解

No.10

To get $u = 0$ on the x-axis, the idea is to extend the given potential from $0 < \theta < \pi$ skew-symmetrically to the whole boundary circle $r = 1$; that is,

$$u(1, \theta) = \begin{cases} 110\theta(\pi - \theta) & \text{if } 0 < \theta < \pi \quad \text{(given)} \\ 110\theta(\pi + \theta) & \text{if } -\pi < \theta < 0. \end{cases}$$

Then you obtain (valid in the whole disk and thus in the semidisk)

$$u(r, \theta) = \frac{880}{\pi}\left(r \sin \theta + \frac{1}{3^3} r^3 \sin 3\theta + \frac{1}{5^3} r^5 \sin 5\theta + \cdots\right).$$

No.11

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (\cos n\theta + B_n \sin n\theta)$$

$$A_n = \frac{1}{\pi \cdot n R^{n-1}} \int_0^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$B_n = \frac{1}{\pi \cdot n R^{n-1}} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$f(\theta) = \begin{cases} u_0 & ; \ 0 < \theta < \pi \qquad R = a \\ 0 & ; \ \text{otherwise} \\ -u_0 & ; \ 0 < \theta < \pi \qquad R = -a \end{cases}$$

$A_n \cancel{= \frac{\quad}{\pi \cdot n a^{n-1}}}$

$$u(r, \theta) = \frac{4u_0}{\pi}\left(\frac{r}{a} \sin \theta + \frac{1}{3a^3} r^3 \sin 3\theta + \frac{1}{5a^5} r^5 \sin 5\theta + \cdots\cdots\right)$$

No.12

CAS Project. **(b)** Error 0.04863 ($m = 1$), 0.02229, 0.01435, 0.01056, 0.00835, 0.00691, 0.00589, 0.00513, 0.00454, 0.00408 ($m = 10$)

(c) The approximation of the partial sums is poorest for $r = 0$.

(d) The radii of the nodal circles are

u_2: $\alpha_1/\alpha_2 = 0.43565$	Comparison	$0.435/0.500 = 0.87$
u_3: $\alpha_1/\alpha_3 = 0.27789$		$0.278/0.333 = 0.83$
$\alpha_2/\alpha_3 = 0.63788$		$0.638/0.667 = 0.96$
u_4: $\alpha_1/\alpha_4 = 0.2039$		$0.204/0.250 = 0.82$
$\alpha_2/\alpha_4 = 0.4681$		$0.468/0.500 = 0.94$
$\alpha_3/\alpha_4 = 0.7339$		$0.734/0.750 = 0.98.$

We see that the larger radii are better approximations of the values of the nodes of the string than the smaller ones. The smallest quotient does not seem to improve (to get closer to 1); on the contrary, e.g., for u_6 it is 0.80. The other ratios seem to approach 1 and so does the sum of all of them divided by $m - 1$.

No.13

$$u_{tt} = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy})$$

$$c^2 = \frac{T}{\rho}$$

$$\lambda_m = c k_m = c\alpha_m / R$$

frequency : $\frac{\lambda_m}{2\pi}$.

$T = 2T_0$. $c = \sqrt{2} c_0$. $\lambda_m = \sqrt{2} \lambda_{m0}$.

so frequency increase by a factor $\sqrt{2}$

No.14

1 : smaller drum ; 2 : larger drum.

$$C_1^2 = \frac{T_1}{\rho_1} = \frac{T_2}{\rho_2} = C_2^2$$

$$\lambda_{m1} = C_1 k_{m1} = \frac{C_1 \alpha_m}{R_1} . \quad \lambda_{m2} = C_2 k_{m2} = \frac{C_2 \alpha_m}{R_2}.$$

$\because R_1 < R_2, \quad \lambda_{m1} > \lambda_{m2}.$

$\therefore$ A smaller drum have a higher fundenmental frequency than a larger one.

No.15

$$f_1 = \frac{\lambda_1}{2\pi} = \frac{1}{2\pi}\frac{C\alpha_1}{R} = \frac{\sqrt{\frac{T}{\rho}}\cdot(2.4048)}{2\pi R}$$

$$T = 6.826\,\rho R^2 f_1^2$$

No.16

The reason is that $f(0) = 1$. The partial sums equal

1.10801	0.96823	1.01371	0.99272	1.00436	$\cdots$

the last value having 3-digit accuracy. Musically the values indicate substantial contributions of overtones to the overall sound.

No.17

No,

$$u_m(r,t) = (A_m \cos\lambda_m t + B_m \sin\lambda_m t) J_0(k_m r)$$

$$\lambda_m = \frac{c\alpha_m}{R} . \quad C. R : \text{fixed}$$

λ_m are different for different m

No.18

$$u(r,t)=\sum_{m=1}^{\infty}(A_m\cos\lambda_m t+B_m\sin\lambda_m t)J_0\left(\frac{\alpha_m}{R}r\right)$$

$$u(r,0)=f(r)\quad,\quad u_t(r,0)=g(r)$$

$$u(r,0)=\sum_{m=1}^{\infty}A_mJ_0\left(\frac{\alpha_m}{R}r\right)=f(r)$$

$$A_m=\frac{2}{R^2J_1^2(\alpha_m)}\int_0^R rf(r)J_0\left(\frac{\alpha_m}{R}r\right)dr$$

$$u_t(r,t)=\sum_{m=1}^{\infty}(-A_m\lambda_m\sin\lambda_m t+B_m\lambda_m\cos\lambda_m t)J_0\left(\frac{\alpha_m}{R}r\right)$$

$$u_t(r,0)=\sum_{m=1}^{\infty}B_m\lambda_mJ_0\left(\frac{\alpha_m}{R}r\right)=g(r)$$

$$B_m\lambda_m=\frac{2}{R^2J_1^2(\alpha_m)}\int_0^R rg(r)J_0\left(\frac{\alpha_m}{R}r\right)dr$$

$$B_m=\frac{2}{R^2\lambda_mJ_1^2(\alpha_m)}\int_0^R rg(r)J_0\left(\frac{\alpha_m}{R}r\right)dr$$

$$=\frac{2}{(c\alpha_mR)J_1^2(\alpha_m)}\int_0^R rg(r)J_0\left(\frac{\alpha_m}{R}r\right)dr$$

No.19

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$$

Let $u = F(r, \theta)G(t)$

$$F(r,\theta)\ddot{G}(t) = c^2\left(F_{rr}G + \frac{1}{r}F_r G + \frac{1}{r^2}F_{\theta\theta}G\right)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta}}{F} = -k^2$$

$\ddot{G} + c^2k^2G = 0$ Let $\lambda = ck$

$$\underline{\ddot{G} + \lambda^2 G = 0}$$

$$\underline{F_{rr} + \frac{1}{r}F_r + \frac{1}{r^2}F_{\theta\theta} + k^2F = 0}$$

Let $F(r,\theta) = W(r)Q(\theta)$

$$W''Q + \frac{1}{r}W'Q + \frac{1}{r^2}WQ'' + k^2WQ = 0$$

$$-\frac{Q''}{Q} = \frac{r^2W'' + rW' + k^2r^2W}{W} = n^2$$

$$\underline{Q'' + n^2 Q = 0}$$

$$\underline{r^2W'' + rW' + (k^2r^2 - n^2)W = 0}$$

No.20

$$Q'' + n^2 Q = 0$$

$$Q_n = A_n \cos n\theta + B_n \sin n\theta$$

$$r^2 W'' + r W' + (k^2 r^2 - n^2) W = 0. \quad \text{Bessel's equation}$$

$$W_n = J_n(kr), \quad n = 0, 1, \cdots$$

No.21

$$u(R, \theta, t) = W(R)\, Q(\theta)\, G(t) = 0$$

$$W(R) = 0$$

$$W_n(R) = J_n(kR) = 0$$

$$kR = \alpha_{mn}$$

$$k_{mn} = \frac{\alpha_{mn}}{R}$$

where $s = \alpha_{mn}$ is the mth positive zero of $J_n(s)$.

No.22

On Notation. n is standard for Legendre polynomials and for Bessel functions of integer order. Hence we needed another letter for numbering the zeros of $J_1, J_2, \cdots$, and we took m. Hence, for example, the positive zeros of J_2 are numbered α_{21}, α_{22},

$\alpha_{23}, \cdots$. (In the 9th Edition we used the probably less advantageous opposite order $\alpha_{12}, \alpha_{22}, \alpha_{32}, \cdots$.)

For consistency, we should have numbered the positive zeros of J_0 by $\alpha_{01}, \alpha_{02}, \alpha_{03}, \cdots$, but this would make formulas unnecessarily clumsy, and we wrote $\alpha_1, \alpha_2, \cdots$, in particular since the "problem" occurred only at the very end, in the last problems of Sec. 12.10.

No.23

$$u_{nm} = (A_{nm} \cos ck_{nm}t + B_{nm} \sin ck_{nm}t) J_n(k_{nm}r) \cos n\theta$$

$$u_{nm}^{*} = (A_{nm}^{*} \cos ck_{nm}t + B_{nm}^{*} \sin ck_{nm}t) J_n(k_{nm}r) \sin n\theta$$

$$(u_{nm})_t = (-ck_{nm}A_{nm} \sin ck_{nm}t + ck_{nm}B_{nm} \cos ck_{nm}t) J_n(k_{nm}r) \cos n\theta$$

$$u_t(r, \theta, 0) = ck_{nm}B_{nm} J_n(k_{nm}r) \cos n\theta = 0$$

$$\Rightarrow B_{nm} = 0$$

$$(u_{nm}^{*})_t = (-ck_{nm}A_{nm}^{*} \sin ck_{nm}t + ck_{nm}B_{nm}^{*} \cos ck_{nm}t) J_n(k_{nm}r) \sin n\theta$$

$$u_t^{*}(r, \theta, 0) = ck_{nm}B_{nm}^{*} J_n(k_{nm}r) \sin n\theta = 0$$

$$\Rightarrow B_{nm}^{*} = 0$$

No.24

$$u_{0m} = (A_{0m} \cos ck_{0m}t + B_{0m} \sin ck_{0m}t) J_0(k_{0m}r)$$

$$= (A_m \cos ck_m t + B_m \sin ck_m t) J_0(k_m r)$$

$$= u_m$$

$$u_{0m}^{*} = (A_{0m}^{*} \cos ck_{0m}t + B_{0m}^{*} \sin ck_{0m}t) J_0(k_{0m}r) \sin 0 = 0$$

No.25

$$u_{11} = (A_{11} \cos ck_{11} t + B_{11} \sin ck_{11} t) J_1(k_{11} r) \cos\theta$$

$$f = \frac{\lambda_{nm}}{2\pi} = \frac{c\alpha_{nm}}{2\pi R}$$

$c^2 = 1$, $R = 1$. $\alpha_{11} = 3.82954$

$$f_{11} = \frac{\alpha_{11}}{2\pi} = 0.6098$$