CHAPTER 16 Laurent Series. Residue Integration

This is another powerful and elegant integration method that has no analog in calculus. It uses Laurent series (roughly, series of positive and negative powers of z), more precisely, it uses just a single term of such a series (the term in $1/(z - z_0)$, whose coefficient is called the **residue** of the sum of the series that converges near z_0).

SECTION 16.1. Laurent Series, page 708

Purpose. To define Laurent series, to investigate their convergence in an annulus (a ring, in contrast to Taylor series, which converge in a disk), to discuss examples.

Major Content, Important Concepts

Laurent series

Convergence (Theorem 1)

Principal part of a Laurent series

Techniques of development (Examples 1–5)

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2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} z^{2n-1} = \pi z^{-1} - \frac{1}{6}\pi^3 z + \frac{1}{120}\pi^5 z^3 - + \cdots, \quad 0 < |z| < \infty$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} z^{2n-1} = 2z^{-1} + \frac{8}{6} z + \frac{32}{120} z^3 + \cdots, \quad 0 < |z| < \infty$

5. We may proceed as follows:

$$\frac{z^2 - 3i}{(z-3)^2} = \frac{[3 + (z-3)]^2 - 3i}{(z-3)^2} = \frac{9 - 3i}{(z-3)^2} + \frac{6}{z-3} + 1$$

8. Using the MacLaurin series of the exponential function, we obtain

$$\begin{aligned} \frac{e^{az}}{z-b} &= \frac{e^{ab + a(z-b)}}{z-b} \\ &= e^{ab} \sum_{n=0}^{\infty} \frac{a^n}{n!} (z-b)^{n-1} \\ &= e^{ab}((z-b)^{-1} + a + \tfrac{1}{2}a^2 (z-b) + \tfrac{1}{6}a^3 (z-b)^2 + \cdots) \end{aligned}$$

convergent for $0 < |z - b| < \infty$. If $a = 0$, the series reduces to its first term, which is the principal part.

11. $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n!} (z-1)^{n-4}, \quad a_n = \sinh 1 \quad (n \text{ even}), \quad a_n = \cosh 1 \ (n \text{ odd}), \quad |z - 1| > 0$

SECTION 16.2. Singularities and Zeros. Infinity, page 715

Purpose. *Singularities* just appeared in connection with the convergence of Taylor and Laurent series in the last sections, and since we now have the instrument for their classification and discussion (i.e., Laurent series), this seems the right time for doing so. We also consider *zeros*, whose discussion is somewhat related.

Main Content, Important Concepts

Principal part of a Laurent series convergent near a singularity

Pole, behavior (Theorem 1)

Isolated essential singularity, behavior (Theorem 2)

Zeros are isolated (Theorem 3)

Relation between poles and zeros (Theorem 4)

Point ∞, extended complex plane, behavior at ∞

Riemann sphere

SOLUTIONS TO PROBLEM SET 16.2, page 719

1. $z^4 - 81$ has simple zeros at ± 3 and $\pm 3i$. Hence the given function has third-order zeros at these points.

4. $\frac{1}{2}\pi \pm 2n\pi, n = 0, 1, \cdots$, third order

6. $\pm\sqrt{8}$, third order, $(\pm 1 \pm i)\sqrt{n\pi}$, simple, because

$$e^{z^2} = 1, \qquad z^2 = \text{Ln } 1 \pm 2n\pi i = \pm 2n\pi i;$$

hence

$$z = \sqrt{\pm 2n\pi i} = \pm\frac{1 \pm i}{\sqrt{2}}\sqrt{2n\pi}.$$

7. Third-order pole at $z = i$, essential singularity at $z = \infty$.

10. Essential singularity at $z = 1$, third-order pole at $z = \infty$.

12. Expressing $\cos z$ and $\sin z$ in terms of exponential functions, we have

$$\frac{1}{2}(e^{iz} + e^{-iz}) - \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2}((1 + i)e^{iz} + (1 - i)e^{-iz}) = 0.$$

Hence

$$e^{2iz} = -\frac{1 - i}{1 + i} = i$$

and

$$2iz = \text{Ln } i \pm 2n\pi i.$$

This gives simple poles at $z = \frac{1}{4}\pi \pm n\pi, n = 0, 1, \cdots$.

Note that these are the points of intersection of the curves of $\cos x$ and $\sin x$. Since they have different tangent directions at these points, the corresponding zeros must be simple, giving simple poles. Our analysis has shown that these are the only points at which the complex functions $\cos z$ and $\sin z$ are equal, so that their difference gives those zeros, but no further zeros, hence no further poles.

SECTION 16.3. Residue Integration Method, page 719

Purpose. To explain and apply this most elegant integration method.

Main Content, Important Concepts

Formulas for the residues at poles (3)–(5)

Residue theorem (several singularities inside the contour)

Comment

The extension from the case of a single singularity to several singularities (residue theorem) is immediate.

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3. 0 at 0 because the Laurent series of $(\cos z)/z^4$ has only even powers.

5. $\pm\frac{1}{4}\pi$ at ∓ 1

7. $z^2 - iz + 2 = (z - 2i)(z + i)$ shows that the given function has two simple poles at $2i$ and $-i$ with residues

$$\left.\frac{z^4}{z+i}\right|_{z=2i} = \frac{16}{3i} = -\frac{16}{3}\, i \text{ at } z = 2i$$

and

$$\left.\frac{z^4}{z-2i}\right|_{z=-i} = \frac{1}{-3i} = \frac{1}{3}\, i.$$

This calculation used (3), and (4) gives the same results, of course, with about the same amount of work.

10. $z^4 - 2z^3 = z^3(z - 2)$. Third-order pole at 0, residue

$$\frac{1}{2!}\left.\left(\frac{z+1}{z-2}\right)''\right|_{z=0} = \frac{1}{2}\left.\left(\frac{-3}{(z-2)^2}\right)'\right|_{z=0} = \frac{6}{2(-8)} = -\frac{3}{8}.$$

Simple pole at 2, residue

$$\left.\frac{z+1}{4z^3 - 6z^2}\right|_{z=2} = \frac{3}{8}.$$

Both poles lie inside the contour. Hence the *answer* is 0.

12. $(z^2 + 1)^3 = (z + i)^3\,(z - i)^3$. Third-order poles at $\pm i$ with residues

$$\begin{aligned}[(z+i)^{-3}]''|_{z=i} &= [-3(z+i)^{-4}]'|_{z=i} \\ &= 12(z+i)^{-5}|_{z=i} \\ &= 12(2i)^{-5} \\ &= -3i/8\end{aligned}$$

and

$$\begin{aligned}[(z-i)^{-3}]''|_{z=-i} &= [-3(z-i)^{-4}]'|_{z=-i} \\ &= 12(z-i)^{-5}|_{z=-i} \\ &= 12(-2i)^{-5} \\ &= 3i/8.\end{aligned}$$

Answer: 0

14. Simple poles at $\pm\frac{1}{2}$. By (4) the residues are obtained from

$$\frac{z^2 \sin z}{8z} = \frac{1}{8} z \sin z.$$

The *answer* is

$$2\pi i \left[\tfrac{1}{8} \cdot \tfrac{1}{2} \cdot \sin \tfrac{1}{2} + \tfrac{1}{8}\left(-\tfrac{1}{2}\right) \sin\left(-\tfrac{1}{2}\right)\right] = \tfrac{1}{4}\pi i \sin \tfrac{1}{2}.$$

SECTION 16.4. Residue Integration of Real Integrals, page 725

Purpose 1. To show that certain classes of *real* integrals over finite or infinite intervals of integration can also be evaluated by residue integration.

Comment on Content

Since residue integration requires a closed path, one must have methods for producing such a path. We see that for the finite intervals in the text, this is done by (2), perhaps preceded by a translation and change of scale if another interval is given. (This is not shown in the text.) In the case of an infinite interval, we start from a finite one, close it by some curve in complex (here, a semicircle; Fig. 374), blow it up, and make assumptions on the integrand such that we can prove (once and for all) that the value of the integral over the complex curve added goes to zero.

Purpose 2. Extension of the second of the two methods just mentioned to integrals of practical interest in connection with Fourier integral representations (Sec. 11.7) and to discuss the case of singularities on the real axis.

Main Content

Integrals involving cos and sin (1), their transformation (2)

Improper integral (4), Cauchy principal value

Fourier integrals

Poles on the real axis (Theorem 1), Cauchy principal value

SOLUTIONS TO PROBLEM SET 16.4, page 733

1. Note that in each of the present problems we are dealing with a *whole class* of integrals obtainable by replacing numerical coefficients with parameters a, b, etc., as follows.

The present integral is of the form

$$\begin{aligned}\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} &= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \\ &= \frac{1}{2}\oint_C \frac{dz}{iz\left[a + \frac{1}{2}b(z + 1/z)\right]} \\ &= \frac{1}{ib}\oint_C \frac{dz}{z^2 + 2az/b + 1} \qquad (a > b > 0).\end{aligned}$$

The zeros of the denominator,

$$z_1 = -\frac{a}{b} + K, \qquad z_2 = -\frac{a}{b} - K, \qquad K^2 = \frac{a^2}{b^2} - 1$$

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give poles at z_1 (inside the unit circle C because $a > b > 0$) and z_2 (outside C) with the residue at z_1 given by

$$\operatorname*{Res}_{z=z_1} \frac{1}{z^2 + 2az/b + 1} = \frac{1}{z - z_2}\bigg|_{z_1} = \frac{1}{2K}$$

and the integral from 0 to π equals

$$2\pi i \cdot \frac{1}{2ibK} = \frac{\pi}{bK} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

and $2\pi/bK$ if we integrate from 0 to 2π. In our case $a = \pi, b = 3$ and the *answer* is $\pi\sqrt{\pi^2 - 9}$.

Note that for $a = b$ the integrand is infinite at $\theta = \pi$, and the value of the integral approaches ∞ as $a \to b > 0$ from above.

2. The integral equals

$$\frac{1}{i}\oint_C \frac{1 + 2(z + z^{-1})}{z[17 - 4(z + z^{-1})]}\,dz = \frac{1}{-4i}\oint_C \frac{2z^2 + z + 2}{z(z - z_1)(z - z_2)}\,dz$$

where $z_1 = \frac{1}{4}$ (inside the unit circle) and $z_2 = 4$ (outside) give simple poles. The residue at $z = 0$ is $2/(z_1z_2) = 2$, and at $z = \frac{1}{4}$ it is

$$\frac{\frac{2}{16} + \frac{1}{4} + 2}{(\frac{1}{4})(\frac{1}{4} - 4)} = -\frac{38}{15}.$$

This gives the *answer*

$$\frac{2\pi i}{-4i}\left(2 - \frac{38}{15}\right) = \frac{4\pi}{15}.$$

8. The integral is of the form

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{a - b\sin\theta} &= \oint_C \frac{dz}{iz[a - (b/2i)(z - 1/z)]} \\
&= \frac{2}{-b}\oint_C \frac{dz}{z^2 - (2ai/b)z - 1} \\
&= \frac{-2}{b}\oint_C \frac{dz}{(z - z_1)(z - z_2)} \qquad (a > b > 0)
\end{aligned}$$

where the zeros of the denominator (the poles of the integrand) are

$$z_1 = \frac{ai}{b} + iK, \qquad z_2 = \frac{ai}{b} - iK, \qquad K^2 = \frac{a^2}{b^2} - 1$$

and from (3), Sec. 16.3, we obtain the residue of the pole inside the unit circle

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - z_1)(z - z_2)} = \frac{1}{z_2 - z_1} = -\frac{1}{2iK}.$$

Hence the integral equals

$$2\pi i(-2/b)(-1/(2Ki)) = 2\pi/(bK) = 2\pi/\sqrt{a^2 - b^2}.$$

In our case, $a = 8$, $b = 2$ gives the *answer* $2\pi/\sqrt{60} = \pi/\sqrt{15}$.

Note that the present formula $\pi/\sqrt{a^2 - b^2}$ also occurred in the solution to Online Prob. 1. The reason is that $-\sin\theta = \cos\theta^*$, where $\theta^* = \theta + \frac{1}{2}\pi$ and the integrand is periodic with period 2π.

10. Second-order pole at $z_1 = 1 + 2i$ in the upper half-plane (and at $z_2 = 1 - 2i$ in the lower) with residue

$$\left[\frac{1}{(z - 1 + 2i)^2}\right]'_{z=z_1} = \frac{-2}{(z_1 - z_2)^3} = \frac{-2}{(4i)^3} = \frac{1}{32i}.$$

Answer: $2\pi i(1/32i) = \pi/16$.

13. Second-order poles at $z_1 = i$ and $z_2 = -i$ (in the lower half-plane). By (5) Sec. 15.3, we get the residue

$$\left[\frac{e^{2iz}}{(z+i)^2}\right]'_{z=i} = \frac{e^{2iz}}{(z+i)^3}[2i(z+i) - 2]\bigg|_{z=i} = \frac{e^{-2}}{(2i)^3}(-6) = \frac{-3e^{-2}i}{4}.$$

Multiplying the imaginary part $-3e^{-2}/4$ by -2π gives the *answer*

$$3\pi e^{-2}/2 = 0.6378.$$

15. $z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4)$ shows that we have two simple poles at $z_1 = i$ and $z_2 = 2i$ in the upper half-plane (and two at $-i$ and $-2i$ in the lower). By (4), Sec. 16.3, the sum of the residues of $e^{4iz}/(z^4 + 5z^2 + 4)$ at z_1 and z_2 is

$$\frac{e^{-4}}{4i^3 + 10i} + \frac{e^{-8}}{4(2i)^3 + 20i} = -\frac{i}{6}e^{-4} + \frac{i}{12}e^{-8}.$$

From the first formula in (10) we thus obtain the *answer*

$$\pi(2e^{-4} - e^{-8})/6.$$

17. Equivalently to sequences, you may use a parameter p, obtaining

$$\int_{-\infty}^{-p} \frac{dx}{x^2 - ix} + \int_{p}^{\infty} \frac{dx}{x^2 - ix} = \pi - 2\arctan p.$$

Now let $p \to 0$.

SOLUTIONS TO CHAPTER 16 REVIEW QUESTIONS AND PROBLEMS, page 733

2. Essential singularity at $z = 0$; this point lies inside the contour. The residue $z = 2$ can be seen from the Laurent series

$$e^{2/z} = 1 + \frac{2}{z} + \frac{2^2}{2!z^2} + \cdots,$$

which converges for $|z| > 0$. This gives the *answer* $4\pi i$.

4. Simple poles at $\pm 2i$. $z = 2i$ lies inside C, and the *answer* is

$$2\pi i \operatorname*{Res}_{z=2i} \frac{5z^3}{z^2 + 4} = 2\pi i \cdot \frac{-40i}{2 \cdot 2i} = -20\pi i.$$

6. $z^3 - 9z = z(z + 3)(z - 3)$. Simple poles at $-3, 0, 3$ inside C with residues

$$\frac{15z + 9}{3z^2 - 9} = -2, -1, 3,$$

respectively. *Answer:* 0.

8. Simple poles at $z = 0$ inside C and $\pm\frac{1}{4}, \pm\frac{1}{2}, \cdots$ all outside C. The residue $\frac{1}{4}$ at $z = 0$ is obtained from the Laurent series

$$\cot 4z = \frac{1}{4z} + \cdots$$

or from the formula

$$\left.\frac{\cos 4z}{(\sin 4z)'}\right|_{z=0} = \frac{1}{4}.$$

The *answer* is $\pi i/2$.

10. The integral equals

$$\oint_C \frac{(z - 1/z)/(2i)}{iz[3 + \frac{1}{2}(z + 1/z)]}\, dz = -\oint_C \frac{z^2 - 1}{z[z^2 + 6z + 1]}\, dz.$$

At the simple pole at $z = 0$ the residue is -1. At the simple pole at $-3 + \sqrt{8}$ (inside the unit circle) the residue is

$$\frac{(-3 + \sqrt{8})^2 - 1}{3(-3 + \sqrt{8})^2 + 12(-3 + \sqrt{8}) + 1} = 1.$$

Answer: 0. Simpler: integrate from $-\pi$ to π and note that the integrand is odd.

12. $1 + 4z^4 = 0$ gives simple poles of $f(z) = 1/(1 + 4z^4)$ at $\pm\frac{1}{2} + \frac{1}{2}i$ in the upper half-plane and at $\pm\frac{1}{2} - \frac{1}{2}i$ in the lower half-plane. The residues of the first two poles are (use the formula for the residue of a simple pole)

$$\left.\frac{1}{16z^3}\right|_{\pm\frac{1}{2}+\frac{1}{2}i} = \mp\frac{1}{8} - \frac{1}{8}i.$$

The sum of these two residues is $-i/4$. *Answer:* $2\pi i(-\frac{1}{4}i) = \frac{1}{2}\pi$.

14. $x^2 - 4ix = x(x - 4i)$. Simple poles of $1/(z^2 - 4iz)$ at 0 and $4i$. The residues are

$$\operatorname{Res} \frac{1}{z^2 - 4iz} = \frac{1}{2z - 4i} = \frac{i}{4}, -\frac{i}{4}.$$

This gives the *answer*

$$\begin{aligned} \pi i \cdot \tfrac{1}{4}i + 2\pi i(-\tfrac{1}{4}i) &= -\tfrac{1}{4}\pi + \tfrac{1}{2}\pi \\ &= \tfrac{1}{4}\pi. \end{aligned}$$