Part C FOURIER ANALYSIS. PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

CHAPTER 11 Fourier Analysis

SECTION 11.1. Fourier Series, page 474

Purpose. To derive the Euler formulas (6) for the coefficients of a Fourier series (5) of a given function of period 2π, using as the key property the orthogonality of the trigonometric system.

Main Content, Important Concepts

Periodic function

Trigonometric system, its orthogonality (Theorem 1)

Fourier series (5) with Fourier coefficients (6)

Representation by a Fourier series (Theorem 2)

Comment on Notation

If we write $a_0/2$ instead of a_0 in (1), we must do the same in (6.0) and see that (6.0) then becomes (6a) with $n = 0$. This is merely a small notational convenience (but may be a source of confusion to poorer students).

Comment on Fourier Series

Whereas their theory is quite involved, practical applications are simple, once the student has become used to evaluating integrals in (6) that depend on n.

Figure 260 should help students understand why and how a series of continuous terms can have a discontinuous sum.

Comment on the History of Fourier Series

Fourier series were already used in special problems by Daniel Bernoulli (1700–1782) in 1748 (vibrating string, Sec. 12.3) and Euler (Sec. 2.5) in 1754. Fourier's book of 1822 became the source of many mathematical methods in classical mathematical physics. Furthermore, the surprising fact that Fourier series, whose terms are *continuous* functions, may represent *discontinuous* functions led to a reflection on, and generalization of, the concept of a function in general. Hence the book is a landmark in both pure and applied mathematics. [That surprising fact also led to a controversy between Euler and D. Bernoulli over the question of whether the two types of solution of the vibrating string problem (Secs. 12.3 and 12.4) are identical; for details, see E. T. Bell, *The Development of Mathematics,* New York: McGraw-Hill, 1940, p. 482.] A mathematical theory of Fourier series was started by Peter Gustav Lejeune Dirichlet (1805–1859) of Berlin in 1829. The concept of the Riemann integral also resulted from work on Fourier series. Later on, these series became the model case in the theory of orthogonal functions (Sec. 5.7). An English translation of Fourier's book was published by Dover Publications in 1955.

Further Comments on Text and Problems

Figure 260, showing the rectangular periodic wave and partial sums of its Fourier series, is typical and should give the student an intuitive feel for convergence of Fourier series, notably of their behavior near discontinuities. The latter are of great practical importance, as will appear as we proceed.

The essential property of the trigonometric system is its orthogonality. Other orthogonal systems will follow later in connection with families of solutions of linear ODEs (Legendre polynomials, Bessel functions, etc.).

The integrals needed for Fourier coefficients are those of calculus, but their dependence on n will create a new situation, and the student will need some time and particular attention to become familiar with them.

Problem 25 suggests that students look critically at the speed of convergence, which will depend on the power of $1/n$ in the Fourier coefficients, which in turn depends on continuity of $f(x)$ to be developed.

The figures in Probs. 16–21 should help students to get familiar with "piecewise given" functions, as they will occur in practical work on potential problems, heat flow, and so on.

SOLUTION OF PROBLEM SET 11.1, page 482

2. $2\pi/n, 2\pi/n, k, k, k/n, k/n$

4. $f(x + p) = f(x)$ implies

$$f(ax + p) = f(a[x + (p/a)]) = f(ax) \text{ or } g[x + (p/a)] = g(x),$$

where

$$g(x) = f(ax).$$

Thus $g(x)$ has the period p/a. This proves the first statement. The other statement follows by setting $a = 1/b$.

12. $\pi - 8\dfrac{\cos(x)}{\pi} - \dfrac{8}{9}\dfrac{\cos(3x)}{\pi} - \dfrac{8}{25}\dfrac{\cos(5x)}{\pi} + \cdots$

13. $4\dfrac{\cos(x)}{\pi} + 4/9\dfrac{\cos(3x)}{\pi} + \dfrac{4}{25}\dfrac{\cos(5x)}{\pi} + 6\sin(x) + \cdots -2\sin(2x)$

$+ 2\sin(3x) - \sin(4x) + 6/5\sin(5x) - 2/3\sin(6x) + \cdots$

14. $\frac{1}{3}\pi^2 - 4(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - + \cdots)$

16. $\dfrac{2}{\pi}\sin x + \dfrac{1}{2}\sin 2x - \dfrac{2}{9\pi}\sin 3x - \dfrac{1}{4}\sin 4x + \dfrac{2}{25\pi}\sin 5x + \dfrac{1}{6}\cos 6x + \cdots$

18. $\dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin x + \dfrac{1}{3}\sin 3x + \dfrac{1}{5}\sin 5x + \cdots\right)$. $f(x) - \frac{1}{2}$ is odd.

20. $\dfrac{1}{\pi}\left[(2 + \pi)\sin x + \dfrac{1}{9}(-2 + 3\pi)\sin 3x + \dfrac{1}{25}(2 + 5\pi)\sin 5x + \cdots\right] - \dfrac{1}{2}\sin 2x$

$-\dfrac{1}{4}\sin 4x - \dfrac{1}{6}\sin 6x - \cdots$

22. CAS Experiment. Experimental approach to Fourier series. This should help the student obtain a feel for the kind of series to expect in practice, and for the kind and quality of convergence, depending on continuity properties of the sum of the series.

(a) The 2π-periodic function $f(x) = x(-\pi < x < \pi)$ has discontinuities at $\pm\pi$. The instructor will notice the **Gibbs phenomenon** (see Sec. 11.7) at the points of discontinuity.

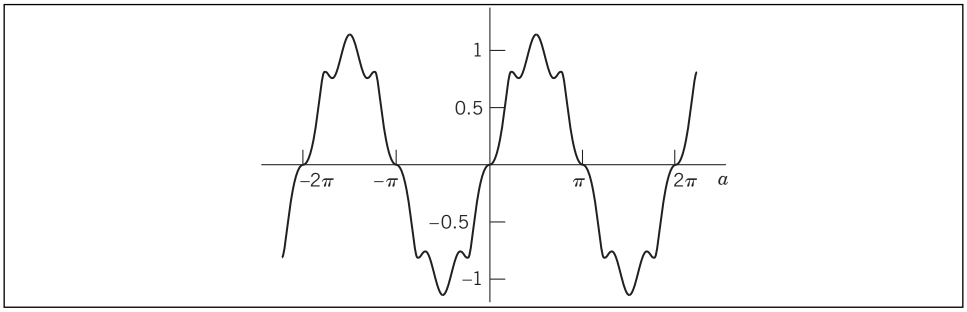
(b) $f(x) = 1 + x/\pi$ if $-\pi < x < 0$ and $1 - x/\pi$ if $0 < x < \pi$, is continuous throughout, and the accuracy is much better than in (a).

(c) $f(x) = \pi^2 - x^2$ has about the same continuity as (b), and the approximation is good.

The coefficients in (a) involve $1/n$, whereas those in (b) and (c) involve $1/n^2$. This is typical. See also CAS Experiment 25.

24. CAS Experiment. The student should recognize the importance of the interval in connection with orthogonality, which is the basic concept in the derivation of the Euler formulas.

For instance, for $\sin 3x \sin 4x$ the integral equals $\sin a - \frac{1}{7}\sin 7a$, and the graph suggests orthogonality for $a = \pi$, as expected.



Section 11.1. Integral in Problem 24 as a function of *a*

SECTION 11.2. Arbitrary Period. Even and Odd Functions, Half-Range Expansions, page 483

Purpose. The three topics considered in this section are listed in the title. The three main points are as follows.

The transition from period 2π to period $2L$ amounts to a linear transformation in x. From (5), (6) in Sec. 11.1 it produces (5), (6) in this section.

For even functions the Fourier series reduces to a cosine series (hence a series without sine terms) (5*) with coefficients (6*).

For odd functions the Fourier series reduces to a sine series (5**) with coefficients (6**).

For period 2π the corresponding simpler formulas are separately listed in the *Summary*.

Typical illustrations of all this are shown in Examples 1–5.

The third and last topic is half-range expansions, typically illustrated in Example 6 and Fig. 272. This will be applied to physical problems of vibrations and heat conduction in the next chapter.

In the problem set we start with some general questions on even and odd functions (Probs. 1–7), followed by Fourier series developments for various periods (Probs. 8–17) and by some general problems (Probs. 18–22).

Finally, half-range expansions are needed in Probs. 23–30, cosine series as well as sine series in each case, that is, functions $f(x)$ given on an interval $0 \leqq x \leqq L$ are to be represented on $-L \leqq x \leqq L$ as a cosine series as well as a sine series, of a function $\tilde{f}(x)$ of period $2L$, obtained by extending the given function from $0 \leqq x \leqq L$ to $-L \leqq x \leqq L$ as an even or odd function, respectively.

SOLUTIONS TO PROBLEM SET 11.2, page 490

1. Neither, even, odd, odd, neither

2. Even, even, neither, odd, even

4. Odd for sums and for products of an odd number $2k + 1$ of factors,

$$f(-x) = f_1(-x) \cdots f_{2k+1}(-x) = (-1)^{2k+1} f_1(x) \cdots f_{2k+1}(x) = -f(x).$$

Even for products of an even number of factors.

6. Odd. This is important in connection with the integrand in the Euler formulas for the Fourier coefficients. It implies the simplification of the Fourier series of an odd function to a Fourier sine series and of the Fourier series of an even function to a Fourier cosine series.

8. Even, $L = 1$. Cf. Prob. 12 in Sec. 11.1. The Fourier series is

$$1/2 + 4\frac{\cos(\pi x)}{\pi^2} + 4/9\frac{\cos(3\pi x)}{\pi^2} + \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} + \cdots$$

9. Odd, $L = 2$. $-\frac{4}{\pi}(\sin(1/2\pi x) + 1/3\sin(3/2\pi x) + 1/5\sin(5/2\pi x) + \cdots)$

10. Odd, $L = 4$. Cf. also Prob. 21 in Sec. 11.1. The series is

$$\frac{8}{\pi}\left(\sin\frac{\pi x}{4} + \frac{1}{2}\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{4} + \cdots\right)$$

11. Even, $L = 1$, $-\frac{4}{\pi^2}(\cos\pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - \cdots)$

12. Even. The series is

$$\frac{2}{3} + \frac{4}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} - \frac{1}{16}\cos 2\pi x + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right)$$

14. Even, $L = \frac{1}{2}$, full-wave rectification of a cosine current. The series is

$$\frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\pi x - \frac{1}{3\cdot 5}\cos 4\pi x + \frac{1}{5\cdot 7}\cos 6\pi x - + \cdots\right).$$

15. Odd, $L = \pi$. $-\frac{4}{\pi}(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x + \cdots)$

16. Odd, $L = 1$, $f(x) = -x^2$ if $-1 < x < 0$, $f(x) = x^2$ if $0 < x < 1$, series

$$\frac{2}{\pi^3}\left((\pi^2 - 4)\sin\pi x + \frac{1}{27}(9\pi^2 - 4)\sin 3\pi x + \frac{1}{125}(25\pi^2 - 4)\sin 5\pi x + \cdots\right)$$
$$- \frac{1}{\pi}\left(\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \cdots\right)$$

17. Even, $L = 1$. $1/2 - 4\frac{\cos(\pi x)}{\pi^2} - 4/9\frac{\cos(3\pi x)}{\pi^2} - \frac{4}{25}\frac{\cos(5\pi x)}{\pi^2} + \cdots$

18. $b_n = 0$, $a_0 = \frac{V_0}{\pi}$,

$$a_n = 100V_0\int_{-1/200}^{1/200}\cos 100\pi t\cos 100n\pi t\,dt$$
$$= 50V_0\int_{-1/200}^{1/200}\cos 100(n+1)\pi t\,dt + 50V_0\int_{-1/200}^{1/200}\cos 100(n-1)\pi t\,dt;$$

hence the series is

$$\frac{V_0}{\pi} + \frac{V_0}{2}\cos 100\pi t$$
$$+ \frac{2V_0}{\pi}\left(\frac{1}{1 \cdot 3}\cos 200\pi t - \frac{1}{3 \cdot 5}\cos 400\pi t + \frac{1}{5 \cdot 7}\cos 600\pi t - + \cdots\right)$$

20. Set $x = -1$. Then $f(-1) = 1 - 1/3 = 2/3 = \frac{4}{\pi^2}\,(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\cdots)$. Hence, the results.

22. Note that the functions in Prob. 8 and Prob. 17 are related as follows. The function in Prob. 8 is

$$f_8 = \begin{cases} -x & -1 < x \text{ and } x < 0 \\ x & 0 < x \text{ and } x < 1 \end{cases}$$

and the function in Prob. 17 is

$$f_{17} = \begin{cases} -x & -1 < x \text{ and } x < 0 \\ x & 0 < x \text{ and } x < 1 \end{cases}$$

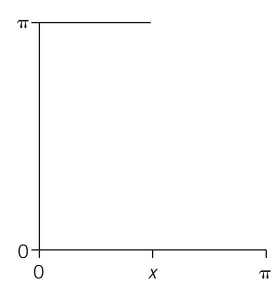
It can readily be verified that $f_{17} = 1 - f_8$ and hence the result.

24. $L = 4$, **(a)** $\frac{1}{2} - \frac{2}{\pi}\left(\cos\frac{\pi x}{4} - \frac{1}{3}\cos\frac{3\pi x}{4} + \frac{1}{5}\cos\frac{5\pi x}{4} - + \cdots\right)$

(b) $$\frac{2}{\pi}\left(\sin\frac{\pi x}{4} - \sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} - \frac{1}{3}\sin\frac{3\pi x}{2}\right.$$
$$\left. + \frac{1}{7}\sin\frac{7\pi x}{4} + \frac{1}{9}\sin\frac{9\pi x}{4} - \frac{1}{5}\sin\frac{5\pi x}{2} + \frac{1}{11}\sin\frac{11\pi x}{4} + \cdots\right)$$

25. (a) $1/2\,\pi + 2\cos(x) - 2/3\,\cos(3x) + 2/5\cos(5x) + \cdots$

(b) $1/2\,\pi + 2\sin(x) + 2\,\sin(2x) + 2/3\sin(3x) + 2/5\sin(5x) + 2/3\sin(6x) + \cdots$



26. $L = \pi$, **(a)** $$\frac{3\pi}{8} - \frac{2}{\pi}\left(\cos x + \frac{1}{2}\cos 2x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \frac{1}{18}\cos 6x\right.$$
$$\left. + \frac{1}{49}\cos 7x + \frac{1}{81}\cos 9x + \frac{1}{50}\cos 10x + \frac{1}{121}\cos 11x + \cdots\right)$$

(b) $$\left(1 + \frac{2}{\pi}\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3} - \frac{2}{9\pi}\right)\sin 3x - \frac{1}{4}\sin 4x$$
$$+ \left(\frac{1}{5} + \frac{2}{25\pi}\right)\sin 5x - \frac{1}{6}\sin 6x + \left(\frac{1}{7} - \frac{2}{49\pi}\right)\sin 7x + \cdots$$

The student should be invited to find the two functions that the sum of the series represents. This can be done by graphing $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + - \cdots = x/2$ and then $(2/\pi)(\sin x - 3^{-2}\sin 3x + - \cdots) = f_2$, where

$$f_2(x) = \begin{cases} x/2 & \text{if} \quad -\pi/2 < x < \pi/2 \\ \pi/2 - x/2 & \text{if} \quad \pi/2 < x < 3\pi/2. \end{cases}$$

The first of these functions is discountinuous, the coefficients being proportional to $1/n$, whereas f_2 is continuous, its Fourier coefficients being proportional to $1/n^2$, so that they go to zero much faster than the others.

28. **(a)** $\dfrac{L}{2} - \dfrac{4L}{\pi^2}\left(\cos\dfrac{\pi x}{L} + \dfrac{1}{9}\cos\dfrac{3\pi x}{L} + \dfrac{1}{25}\cos\dfrac{5\pi x}{L} + \cdots\right)$

(b) $\dfrac{2L}{\pi}\left(\sin\dfrac{\pi x}{L} - \dfrac{1}{2}\sin\dfrac{2\pi x}{L} + \dfrac{1}{3}\sin\dfrac{3\pi x}{L} - \dfrac{1}{4}\sin\dfrac{4\pi x}{L} + - \cdots\right)$

30. Shift by π. This cosine series is obtained. The sine series needs to be multiplied by -1. The result is that the coefficients of odd multiples of π remain the same, whereas the coefficients of even multiples of π are multiplied by -1.

SECTION 11.3. Forced Oscillations, page 492

Purpose. To show that mechanical or electrical systems with periodic but nonsinusoidal input may respond predominantly to one of the infinitely many terms in the Fourier series of the input, giving an unexpected output; see Fig. 277, where the output frequency is essentially five times that of the input.

Further Comments on Text and Problems

Example 1 in the text is sufficient to show the general idea and is typical of the problems.

Problems 2 and 20 are particularly important. In fact, students should decrease the damping term in (4), letting it approach zero. This will give an additional better understanding of the present situation.

SOLUTIONS TO PROBLEM SET 11.3, page 494

2. For $k = 49$ (and $c = 0.05$ as before) the amplitudes are $C_1 = 0.0265$, $C_3 = 0.0035$, $C_5 = 0.0021$, $C_7 = 0.0742$, $C_9 = 0.0005$, $C_{11} = 0.0001$.

An increase of the damping constant $c\,(>0)$ increases D_n for all n, hence it decreases all amplitudes C_n.

4. $r'(t)$ is given by the sine series in Example 1 with $k = -1$. The new C_n is n times the old. Hence C_5 is now so large that the output is practically a cosine vibration having five times the input frequency.

6. $y = C_2 \sin(\omega t) + C_1 \cos(\omega t) + \dfrac{(-\beta^2 + \omega^2)\cos(\alpha t) + (\omega^2 - \alpha^2)\cos(\beta t)}{\omega^4 + (-\beta^2 - \alpha^2)\omega^2 + \alpha^2\beta^2}$

7. $y = C_1 \sin\omega t + C_2 \cos\omega t + a(\omega)\cos t,\ a(\omega) = 1/(\omega^2 - 1)$

$= -1.19, -2.78, -5.26, 4.76, 2.27, 0.0417$

8. The given $r(t)$ is

$$r(t) = \frac{\pi}{4}\cos t \text{ if } -\frac{\pi}{2} < t < \frac{\pi}{2} \text{ and } -\frac{\pi}{4}\cos t \text{ if } \frac{\pi}{2} < t < \frac{3\pi}{2}.$$

The corresponding Fourier series, a Fourier cosine series, is

$$r(t) = \frac{1}{2} + \frac{1}{1\cdot 3}\cos 2t - \frac{1}{3\cdot 5}\cos 4t + \frac{1}{5\cdot 7}\cos 6t \; - + \cdots .$$

Substituting this into the above ODE and solving it gives the *answer*

$$y = C_1\cos\omega t + C_2\sin\omega t + \frac{1}{2\omega^2} + \frac{1}{1\cdot 3\,(\omega^2-4)}\cos 2t$$
$$-\frac{1}{3\cdot 5(\omega^2-16)}\cos 4t + - \cdots .$$

10. $y = C_1\cos\omega t + C_2\sin\omega t + \dfrac{1}{2\omega^2} - \dfrac{1}{1\cdot 3\,(\omega^2-4)}\cos 2t$
$$-\frac{1}{3\cdot 5(\omega^2-16)}\cos 4t - \frac{1}{5\cdot 7(\omega^2-36)}\cos 6t - \cdots$$

11. $y = C_1\cos\omega t + C_2\sin\omega t - \dfrac{4}{\pi}\left(\dfrac{\sin t}{\omega^2-1} + \dfrac{1}{3}\dfrac{\sin 3t}{\omega^2-9} + \dfrac{1}{5}\dfrac{\sin 5t}{\omega^2-25} + \cdots\right)$

14. The Fourier series is a Fourier sine series, as given and derived in Example 1 of Sec. 11.1 ($k = 1$) with coefficients

$$b_n = 4/(n\pi) \qquad (n \text{ odd}).$$

Hence the ODE must be solved with the right side

$$r_n(t) = (4/n\pi)\sin nt \qquad (n \text{ odd}).$$

The steady-state solution of this ODE is

$$y = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} (A_n\cos nt + B_n\sin nt)$$

where

$$A_n = -4nc/(n\pi D_n), \qquad B_n = (4 - 4n^2)/(n\pi D_n)$$

with $D_n = (1-n^2)^2 + n^2c^2$ as in Prob. 13.

So we have $b_1 = 4/\pi$, $b_2 = 0$, $b_3 = 4/(3\pi)$, etc. and no cosine terms in the Fourier series on the right side of the ODE. In the solution the damping constant appears with the cosine terms (and in D_n), causing a phase shift, which is zero if $c = 0$. Also, increasing c increases D_n, hence it decreases the amplitudes; this is physically understandable.

16. For the right side we have the Fourier sine series

$$\frac{4}{\pi}\left(\sin t - \frac{1}{9}\sin 3t + \frac{1}{25}\sin 5t - + \cdots\right)$$

with the coefficients $b_n = 4/(n^2\pi)$ if $n = 1, 5, 9, \cdots$, and $b_n = -4/(n^2\pi)$ if $n = 3, 7, 11, \cdots$. Substitution of this series into the ODE gives

$$y = A_1\cos t + B_1\sin t + A_3\cos 3t + B_3\sin 3t + \cdots$$

with coefficients

$$A_n = -ncb_n/D_n, \qquad B_n = (1 - n^2)b_n/D_n, \qquad D_n = (1 - n^2)^2 + n^2c^2.$$

The damping constant c appears in the cosine terms, causing a phase shift, which is zero if $c = 0$. Also, c increases D_n, hence it decreases the amplitudes, which is physically understandable.

18. The ODE in Probs. 17–19 is the same, except for the changing right sides, whose Fourier series we use term-by-term, as in the text. The solution of the ODE is of the general form

$$I = A_0 + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt)$$

with coefficients obtained by substitution

$$A_n = -\frac{n^2 - 10}{D_n}a_n, \quad B_n = \frac{10n}{D_n}a_n, \quad D_n = (n^2 - 10)^2 + 100n^2,$$

in particular, $A_0 = a_0/10$, the ODE being

$$I'' + 10I' + 10I = a_n \cos nt.$$

For the present problem we have the Fourier series

$$100 + 100\pi - \frac{800}{\pi}\left(\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \cdots\right).$$

Hence $A_0 = 10 + 10\pi$ and all the other A_n and B_n with n even are zero. The formula for the a_n is $-800/(\pi n^2)$ where n is odd. Numerically evaluating the terms, we obtain the solution (the current in the *RLC*-circuit)

$$\begin{aligned} I = 41.416 &- 12.662 \cos t - 14.069 \sin t - 0.031 \cos 3t \\ &- 0.942 \sin 3t + 0.056 \cos 5t - 0.187 \sin 5t + \cdots. \end{aligned}$$

20. $C_n = \sqrt{A_n^2 + B_n^2} = 4/(n^2\pi\sqrt{D_n})$, $D_n = (n^2 - k)^2 + n^2c^2$ with A_n and B_n obtained as solutions of

$$\begin{aligned} (k - n^2)A_n + \qquad ncB_n &= 4/n^2\pi \\ -ncA_n + (k - n^2)B_n &= 0. \end{aligned}$$

SECTION 11.4. Approximation by Trigonometric Polynomials, page 495

Purpose. We show how to find "best" approximations of a given function by trigonometric polynomials of a given degree N.

Important Concepts

Trigonometric polynomial

Square error, its minimum (6)

Bessel's inequality, Parseval's identity

Short Courses. This section can be omitted.

Comment on Quality of Approximation

This quality can be measured in many ways. Particularly important are (i) the absolute value of the maximum deviation over a given interval, and (ii) the mean square error considered here. See Ref. [GenRef7] in App. 1.

Further Comments

The ideas in this section play a basic role in more advanced applied and abstract courses.

CAS Experiment 10 will give the student a feel for the size of the error of the present approximation and its size as a function of the number of terms considered, that is, for the rapidity of its decrease.

For other types of interpolation and approximation and for numeric work, see Secs. 19.3, 19.4, and 20.5.

SOLUTIONS OF PROBLEM SET 11.4, page 498

2. $a_n = 0$ since $f(x) = x$ is odd. Calculation of the b_n gives the approximating trigonometric polynomial

$$F = 2\left(\sin x - \frac{1}{2}\sin 2x + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx\right).$$

From this, the minimum square error is obtained as shown in Example 1 of the text; note that the present function and that in Example 1 differ just by an additive constant π.

4. $F = 2/3\,\pi^2 + 4\cos(x) - \cos(2x) + 4/9\cos(3x) - 1/4\cos(4x)$
$+ \frac{4}{25}\cos(5x) - 1/9\cos(6x) + \cdots$
$E^* = 4.13, 0.99, 0.39, 0.17, 0.11$

5. $F = 4\sin(x) + 4/3\sin(3x) + 4/5\sin(5x) + \cdots,\ b_n = \frac{2}{n}((-1)^{1+n} + 1)$
$E^* = 62.012, 60.739, 60.739, 60.420, 60.420$

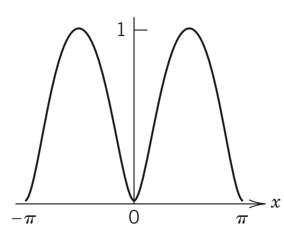
6. The function in Prob. 3 is continuous, the function in Prob. 5 is not; indeed, it is the derivative of the function in Prob. 3, and differentiation produces a factor n in each term.

8. The approximating trigonometric polynomial of minimum square error is

$$F = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos 2x + \frac{1}{3\cdot 5}\cos 4x + \frac{1}{5\cdot 7}\cos 6x + \cdots\right),$$

From this we obtain $E^* = 0.5951,\quad 0.0292,\quad 0.0292,\quad 0.0066,\quad 0.0066$, etc.

These values are small; indeed, a graph shows that the four terms given are such that this partial sum of F^* approximates $f(x)$ rather accurately.



Section 11.4. Problem 8

10. CAS Experiment. Factors are the continuity or discontinuity and the speed with which the coefficients go to zero, $1/n$, $1/n^2$.

For $f(x)$ given on $-\pi < x < \pi$ some data are as follows (f, decrease of the coefficients, continuity or not, smallest N such that $E^* < 0.1$).

$f = x, 1/n$, discontinuous, $N = 126$

$f = x^2, 1/n^2$, continuous, $N = 5$

$f = x^3, 1/n^2$, discontinuous, $E^* = 6.105$ for $N = 200$

$f = x^4, 1/n$, continuous, $N = 40$

$f = x^6, 1/n$, continuous. For $N = 200$ we still have $E^* = 0.1769$.

For f in Prob. 9 we have $1/n^2$, continuity, $N = 2$.

The functions

$$f(x) = \begin{cases} (x + \frac{1}{2}\pi)^{2k} - (\frac{1}{2}\pi)^{2k} & \text{if} \quad -\pi < x < 0 \\ -(-x - \frac{1}{2}\pi)^{2k} + (\frac{1}{2}\pi)^{2k} & \text{if} \quad 0 < x < \pi \end{cases}$$

with $k = 1, 2, 3, 4$ have coefficients proportional to $1/n^2$ and $E^* < 0.1$ when $N \geqq 1$ $(k = 1)$, $5\,(k = 2)$, $11\,(k = 3)$, $21\,(k = 4)$.

These data indicate that the whole situation is more complex than one would at first assume. So the student may need your help and guidance.

14. The Fourier series is

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x$$

and Parseval's identity (8) gives

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^4 x \, dx.$$

Section 11.5. Sturm—Liouville Problems. Orthogonal Functions, page 498

Purpose. Discussion of eigenvalue problems for ordinary second-order ODEs (1) under boundary conditions (2).

Main Content, Important Concepts

Sturm–Liouville equations, Sturm–Liouville problem

Reality of eigenvalues

Orthogonality of eigenfunctions

Orthogonality of Legendre polynomials and Bessel functions

Short Courses. Omit this section.

Comment on Importance

This theory owes its significance to two factors. On the one hand, boundary value problems involving practically important ODEs (Legendre's, Bessel's, etc.) can be cast into Sturm–Liouville form, so that here we have a general theory with several important particular cases. On the other hand, the theory gives important general results on the spectral theory of those problems.

Comment on Existence of Eigenvalues

This theory is difficult. Quite generally, in problems where we can have *infinitely many* eigenvalues, the existence problem becomes nontrivial, in contrast with matrix eigenvalue problems (Chap. 8), where existence is trivial, a consequence of the fact that a polynomial equation $f(x) = 0$ (f not constant) has at least one solution and at most n numerically different ones (where n is the degree of the polynomial).

Move of This Theory from Chap. 5 to Chap. 11

Sturm–Liouville theory is motivated to a large extent by the use of orthogonality in connection with Fourier series and by generalizations from $y'' + \lambda y = 0$ (in the separation of the vibrating string PDE, the wave PDE) to more general linear ODEs (Legendre and Bessel above all).

The Sturm–Liouville material is now close to one of its main applications in PDEs (Chap. 12). More importantly, orthogonality seems more complicated to grasp than other theories and needs digestion time and more maturity than the average student probably has in Part A on ODEs. At least, this is my observation resulting from teaching these matters many times to engineers, physicists, and mathematicians, representing groups of various interests and maturity in applied mathematics.

SOLUTIONS TO PROBLEM SET 11.5, page 503

2. If y_m is a solution of (1), so is z_m because (1) is linear and homogeneous; here, $\lambda = \lambda_m$ is the eigenvalue corresponding to y_m. Also, multiplying (2) with $y = y_m$ by c, we see that z_m also satisfies the boundary conditions. This proves the assertion.

4. $a = -\pi$, $b = \pi$, $c = \pi$, $k = 0$. Problems 2 to 6 are most useful in applications; they concern situations that appear rather frequently.

6. Perform the differentiations in (1), divide by p, and compare; that is,

$$py'' + p'y' + (q + \lambda r)y = 0, \qquad y'' + \frac{p'}{p}\,y' + \left(\frac{q}{p} + \lambda\,\frac{r}{p}\right)y = 0.$$

Hence $f = p'/p$, $p = \exp\left(\int f\,dx\right)$, $q/p = g$, $q = gp$, $r/p = h$, $r - hp$. A reason for performing this transformation may be the discovery of the weight function needed for determining the orthogonality. We see that

$$r(x) = h(x)p(x) = h(x)\exp\left(\int f(x)\,dx\right).$$

This problem shows that a Sturm–Liouville equation is rather general; more precisely, it is equivalent to a general second-order homogeneous linear ODE whose coefficient of the unknown function $y(x)$ contains a parameter λ, the coefficient being of the form $Q + \lambda R$.

7. $\lambda_m = (m\pi/5)^2$, $m = 1, 2, \cdots$; $y_m = \sin(m\pi x/5)$

8. $\lambda = (m\pi/L)^2$, $m = 1, 2, \cdots$; $y_m = \sin(m\pi x/L)$

10. We need

$$y = A\cos kx + B\sin kx$$
$$y' = -Ak\sin kx + Bk\cos kx.$$

From the boundary conditions we obtain

$$y(0) = A = y(1) = A\cos k + B\sin k$$
$$y'(0) = Bk = y'(1) = -Ak\sin k + Bk\cos k.$$

Ordering gives

$$(1 - \cos k)A - (\sin k)B = 0$$
$$(k \sin k)A + k(1 - \cos k)B = 0.$$

By eliminating A and then requiring $B \neq 0$ (to have $y \neq 0$, an eigenfunction) or simply by noting that for this homogeneous system to have a nontrivial solution A, B, the determinant of its coefficients must be zero; that is,

$$k(1 - \cos k)^2 + k \sin^2 k = k(2 - 2\cos k) = 0;$$

hence $\cos k = 1$, $k = 2m\pi$, so that the eigenvalues and eigenfunctions are

$$\lambda_m = (2m\pi)^2, \quad m = 0, 1, \cdots;$$

$$y_0 = 1, \quad y_m(x) = \cos(2m\pi x), \sin(2m\pi x), \quad m = 1, 2, \cdots.$$

12. A general solution $y = e^{2x}(A \cos kx + B \sin kx)$, $k = \sqrt{\lambda}$, of this ODE with constant coefficients is obtained as usual. The Sturm–Liouville form of the ODE is obtained by using the formulas in Prob. 6,

$$(e^{-4x}y')' + e^{-4x}(k^2 + 1)y = 0$$

From this and the boundary conditions we expect the eigenfunctions to be orthogonal on $0 \leq x \leq 1$ with respect to the weight function e^{-4x}. Now, from that general solution and $y(0) = A = 0$, we see that we are left with $y = e^{2x} \sin kx$. From the second boundary condition $y(1) = 0$ we now obtain

$$y(1) = e^2 \sin k = 0, k = m\pi, \ m = 1, 2, \cdots.$$

Hence the eigenvalues and eigenfunctions are

$$\lambda_m = (m\pi)^2, y_m = e^{2x} \sin m\pi x.$$

13. $p = e^{6x}$, $q = 9e^{6x}$, $r = e^{6x}$, $\lambda_m = m^2$, $y_m = e^{-3x} \sin mx$, $m = 1, 2, \cdots$

14. Team Project. (a) We integrate over x from -1 to 1, hence over θ defined by $x = \cos\theta$ from π to 0. Using $(1 - x^2)^{-1/2}\,dx = -d\theta$, we thus obtain

$$\int_{-1}^{1} \cos(m \arccos x)\cos(n \arccos x)(1 - x^2)^{-1/2}dx$$
$$= \int_0^\pi \cos m\theta \cos n\theta \, d\theta = \frac{1}{2}\int_0^\pi (\cos(m+n)\theta + \cos(m-n)\theta)\, d\theta,$$

which is zero for integer $m \neq n$.

(b) Following the hint, we calculate $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$:

$$\int_0^\infty e^{-x}x^k L_n(x)\,dx = \frac{1}{n!}\int_0^\infty x^k \frac{d^n}{dx^n}(x^n e^{-x})\,dx = -\frac{k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx$$
$$= \cdots = (-1)^k \frac{k}{n!}\int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x})\,dx = 0.$$

SECTION 11.6. Orthogonal Series. Generalized Fourier Series, page 504

Purpose. To show how families (sequences) of orthogonal functions, as they arise in eigenvalue problems and elsewhere, are used in series for representing other functions, and to show how orthogonality becomes crucial in simplifying the determination of the coefficients of such a series by integration.

Main Content, Important Concepts

Standard notation (y_m, y_n)

Orthogonal expansion (1), eigenfunction expansion

Fourier constants (2)

Fourier–Legendre series (Example 1)

Fourier–Bessel series (Example 2)

Orthogonality of Bessel functions (Theorem 1)

Mean square convergence

Completeness, also called totality (Theorem 2)

Comments on Text

Formula (2) for the Fourier constants (the coefficients of an orthogonal series) generalizes (6) in Sec. 11.2 for the Fourier coefficients of a Fourier series.

Note that for Fourier–Bessel series (9) with coefficients (10) you obtain infinitely many orthogonal families, each consisting of infinitely many Bessel functions.

In many applications of these series (and other orthogonal series), it turns out that one needs only a relatively small number of terms for obtaining a reasonable accuracy.

Completeness of orthogonal families of functions guarantees that the set of given functions to be developed is sufficiently large to be of practical (and theoretical!) interest.

In practical work, numeric methods may be needed for obtaining values of Fourier constants.

SOLUTIONS TO PROBLEM SET 11.6, page 509

1. $8(P_1(x) - P_2(x) + P_4(x))$

2. $\frac{2}{3}P_2(x) + 2P_1(x) + \frac{4}{3}P_0(x)$. This is probably most simply obtained by the method of undetermined coefficients, beginning with the highest power, x^2 and $P_2(x)$. The point of these problems is to make the student aware that these developments look totally different from the usual expansions in terms of powers of x.

3. $\frac{6}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x)$

4. $P_0(x)$, $P_1(x)$, $\frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$, $\frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$, $\frac{1}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x)$

6. The series may contain all even powers, not just powers x^{4m}.

8. $m_0 = 5$. The size of m_0, that is, the rapidity of convergence seems to depend on the variability of $f(x)$. A discontinuous derivative (e.g., as for $|\sin x|$ occurring in connection with rectifiers) makes it virtually impossible to reach the goal. Let alone when $f(x)$ itself is discontinuous. In the present case the series is

$$f(x) = 0.95493P_1(x) - 1.15824P_3(x) + 0.21929P_5(x) - + \cdots.$$

Rounding seems to have considerable influence in Prob. 8–13.

10. $f(x) = 0.7468P_0(x) - 0.4460P_2(x) + 0.0739P_4(x) - \cdots, m_0 = 4$

12. $f(x) = 0.6116P_0(x) - 0.7032P_2(x) + 0.0999P_4(x) + \cdots, m_0 = 4$. Compare with Prob. 13!

14. Team Project. (b) A Maclaurin series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ has the coefficients $a_n = f^{(n)}(0)/n!$ We thus obtain

$$f^{(n)}(0) = \frac{d^n}{dt^n}\left(e^{tx-t^2/2}\right)\Big|_{t=0} = e^{x^2/2}\frac{d^n}{dt^n}\left(e^{-(x-t)^2/2}\right)\Big|_{t=0}.$$

If we set $x - t = z$, this becomes

$$f^{(n)}(0) = e^{x^2/2}(-1)^n \frac{d^n}{dz^n}\left(e^{-z^2/2}\right)\Big|_{z=x} = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) = He_n(x).$$

(c) $G_x = \sum a_n'(x)t^n = \sum He_n'(x)t^n/n! = tG = \sum He_{n-1}(x)t^n/(n-1)!$, etc.

(d) We write $e^{-x^2/2} = v$, $v^{(n)} = d^n v/dx^n$, etc., and use (21). By integrations by parts, for $n > m$,

$$\begin{aligned}\int_{-\infty}^{\infty} vHe_m He_n\,dx &= (-1)^n \int_{-\infty}^{\infty} He_m v^{(n)}\,dx = (-1)^{n-1}\int_{-\infty}^{\infty} He_m' v^{(n-1)}\,dx \\ &= (-1)^{n-1} m \int_{-\infty}^{\infty} He_{m-1} v^{(n-1)}\,dx = \cdots \\ &= (-1)^{n-m} m! \int_{-\infty}^{\infty} He_0 v^{(n-m)}\,dx = 0.\end{aligned}$$

(e) $nHe_n = nxHe_{n-1} - nHe_{n-1}'$ from (22) with $n - 1$ instead of n. In this equation, the first term on the right equals xHe_n' by (21). The last terms equals $-He_n''$, as follows by differentiation of (21).

SECTION 11.7. Fourier Integral, page 510

Purpose. Beginning in this section, we show how ideas from Fourier series can be extended to nonperiodic functions defined on the real line, leading to integrals instead of series.

Main Content, Important Concepts

Fourier integral (5)

Existence Theorem 1

Fourier cosine integral, Fourier sine integral, (10)–(13)

Application to integration

Short Courses. This section can be omitted.

Comments on Text and Problems

The simplest example on Fourier series can also serve here as an introduction (Example 1 and Fig. 280) to motivate the present extension to Fourier *integrals* as well as the result in Theorem 1, which we must leave without proof (a reference is given in the text); so the situation is somewhat similar to that on Fourier series near the beginning of the chapter.

It is interesting that we shall now be able to prove and understand the Gibbs phenomenon in terms of the sine integral and Dirichlet's discontinuous factor in Example 2. See, in particular, Fig. 283.

Fourier cosine and Fourier sine integral in (10) and (11) are analogs of Fourier cosine and Fourier sine series.

Example 3 shows a basic application.

The evaluation of integrals by the present method is shown in Probs. 1–6 for the Fourier integral itself and in Probs. 7–12 and 16–20 for Fourier cosine and sine integrals, respectively.

CAS Experiments 13 and 15 should help the student in gaining additional insight beyond the present formalism.

Project 14 is a first step into transform theory similar to that of the Laplace transform in Chap. 6, the latter being of much greater importance to the engineer.

SOLUTIONS TO PROBLEM SET 11.7, page 517

1. $f(x) = \pi e^{-x} (x > 0)$ gives $A = \int_0^\infty e^{-x/2} \cos wv\, dv = \frac{1}{2(1/4 + w^2)}$, $B = \frac{w}{(1/4 + w^2)}$, (see Example 3) etc.

2. Use (11) and $f(x) = (\pi/2) \sin x \quad (0 \leqq x \leqq \pi)$ to get, with the help of (11) in App. 3.1,

$$B(w) = \int_0^\pi \sin v \sin wv\, dv = \frac{\sin \pi w}{1 - w^2}.$$

3. Use (11); $B = \frac{2}{\pi} \int_0^\infty \frac{\pi}{4} \sin wv dv = \frac{1 - \cos w\pi}{2w}$

4. Use (10). Also use (11) in App. A3.1. Take $f(x) = \frac{1}{2}\pi \cos x$. Then

$$A(w) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\pi}{2} \cos v \cos wv\, dv = 2 \frac{\cos \frac{1}{2}\pi w}{1 - w^2}.$$

5. $B(w) = \dfrac{2}{\pi} \displaystyle\int_0^1 - \frac{\pi x}{2} \sin wv\, dv = \frac{w \cos w - \sin w}{w^2}$

6. Take $f(x) = \pi e^{-x} \cos x \quad (x > 0)$. Then (11) in this section and (11) in App. A3.1 give

$$B(w) = \frac{2}{\pi} \int_0^\infty \pi e^{-v} \cos v \sin wv\, dv$$

$$= \int_0^\infty e^{-v} \sin (w + 1)v\, dv + \int_0^\infty e^{-v} \sin (w - 1)v\, dv.$$

Integrate by parts, obtaining

$$B(w) = \frac{w + 1}{1 + (w + 1)^2} + \frac{w - 1}{1 + (w - 1)^2} = \frac{2w^3}{w^4 + 4}.$$

Now use the first formula in (11) to obtain the result.

7. $\dfrac{2}{\pi} \displaystyle\int_0^\infty \frac{1 - \cos w}{w^2} \cos xw\, dw$

8. $A = \dfrac{2}{\pi}\displaystyle\int_0^1 v^2 \cos wv\, dv = \dfrac{2}{\pi w}\left(\sin w - \dfrac{2}{w^2}\sin w + \dfrac{2}{w}\cos w\right)$, so that the *answer* is

$$\frac{2}{\pi}\int_0^\infty \left[\left(1 - \frac{2}{w^2}\right)\sin w + \frac{2}{w}\cos w\right]\frac{\cos xw}{w}\, dw.$$

Although many students will do the actual integration by their CAS, problems of the present type have the merit of illustrating the ideas of integral representations and transforms, a rather deep and versatile creation, and the techniques involved, such as the proper choice of integration variables and integration limits. Moreover, graphics will help in understanding the transformation process and its properties, for instance, with the help of Prob. 18 or similar experiments.

10. $\dfrac{4}{\pi}\displaystyle\int_0^\infty \dfrac{\sin aw - aw\cos aw}{w^3}\cos xw\, dw$

12. $A = \dfrac{2}{\pi}\displaystyle\int_0^a e^{-v}\cos wv\, dv = \dfrac{2}{\pi}\left(\dfrac{1 - e^{-a}(\cos wa - w\sin wa)}{1 + w^2}\right)$, so that the integral representation is

$$\frac{2}{\pi}\int_0^\infty \frac{1 - e^{-a}(\cos wa - w\sin wa)}{1 + w^2}\cos xw\, dw.$$

14. Project. (a) Formula (a1): Setting $wa = p$, we have from (10)

$$f(ax) = \int_0^\infty A(w)\cos axw\, dw = \int_0^\infty A\left(\frac{p}{a}\right)\cos xp\,\frac{dp}{a}.$$

If we again write w instead of p, we obtain (a1).

Formula (a2): From (11) with $f(v)$ replaced by $vf(v)$ we have

$$B^*(w) = \frac{2}{\pi}\int_0^\infty vf(v)\sin wv\, dv = -\frac{dA}{dw}$$

where the last equality follows from (10).

Formula (a3) follows by differentiating (10) twice with respect to w,

$$\frac{d^2A}{dw^2} = -\frac{2}{\pi}\int_0^\infty f^*(v)\cos wv\, dv, \qquad f^*(v) = v^2 f(v).$$

(b) In Prob. 7 we have

$$A = \frac{2}{\pi}w^{-1}\sin w.$$

Hence by differentiating twice we obtain

$$A'' = \frac{2}{\pi}(2w^{-3}\sin w - 2\,w^{-2}\cos w - w^{-1}\sin w).$$

By (a3) we now get the result, as before,

$$x^2 f(x) = \frac{2}{\pi}\int_0^\infty \left[\left(-\frac{2}{w^3}+\frac{1}{w}\right)\sin w + \frac{2}{w^2}\cos w\right]\cos xw\, dw.$$

(c) $A(w) = (2\sin w)/(\pi w)$; see Prob. 7. By differentiation,

$$B^*(w) = -\frac{dA}{dw} = -\frac{2}{\pi}\left(\frac{\cos w}{w} - \frac{\sin w}{w^2}\right).$$

This agrees with the result obtained by using (11). Note well that here we are dealing with a relation between the *two* Fourier transforms under consideration.
(d) The derivation of the following formulas is similar to that of (a1)–(a3).

(d1) $\quad f(bx) = \dfrac{1}{b}\displaystyle\int_0^\infty B\left(\frac{w}{b}\right)\sin xw\, dw \qquad (b > 0)$

(d2) $\quad xf(x) = \displaystyle\int_0^\infty C^*(w)\cos xw\, dw, \qquad C^*(w) = \frac{dB}{dw}, \quad B \text{ as in (11)}$

(d3) $\quad x^2 f(x) = \displaystyle\int_0^\infty D^*(w)\sin xw\, dw, \qquad D^*(w) = -\frac{d^2B}{dw^2}.$

16. From (11) we obtain

$$B(w) = \frac{2}{\pi}\int_0^a v\sin wv\, dv = \frac{2}{\pi w^2}(\sin aw - aw\cos aw)$$

so that the *answer* is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin aw - aw\cos aw}{w^2}\sin xw\, dw.$$

17. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos w - 1}{w}\sin xw\, dw$

18. $B(w) = \dfrac{2}{\pi}\displaystyle\int_0^\pi \cos v \sin wv\, dv = \frac{2}{\pi}\frac{w(1+\cos \pi w)}{w^2-1}$ gives the integral representation

$$\frac{2}{\pi}\int_0^\infty \frac{w(1+\cos\pi w)}{w^2-1}\sin xw\, dw.$$

Note that the Fourier sine series of the odd periodic extension of $f(x)$ of period 2π has the Fourier coefficients $b_n = (2/\pi)n(1+\cos n\pi)/(n^2-1)$. Compare this with $B(w)$.

20. $\dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{w - (w\cos w + \sin w)/e}{1+w^2}\sin xw\, dw$

SECTION 11.8. Fourier Cosine and Sine Transforms, page 518

Purpose. Fourier cosine and sine transforms are obtained immediately from Fourier cosine and sine integrals, respectively, and we investigate some of their properties.

Content

Fourier cosine and sine transforms

Transforms of derivatives (8), (9)

Comment on Purpose of Transforms

Just as the Laplace transform (Chap. 6), these transforms are designed for solving differential equations. We show this for PDEs in Sec. 12.7.

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.8, page 522

1. $\hat{f}_c(w) = \dfrac{\sqrt{2}(-1 + 2\sin(w)w + \cos(2w))}{\sqrt{\pi}w^2}$

2. $f(x) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f_c(w) \cos wx \, dw = \begin{cases} 1 & \text{if } 0 < x < 1 \\ -1 & \text{if } 1 < x < 2. \end{cases}$

If you got your answer by a CAS (e.g., by Maple) in a somewhat unusual form, plot it to see that it is correct.

3. $\hat{f}_c(w) = \dfrac{\sqrt{2}(1 - \cos(2w))}{\sqrt{\pi}w^2}$

4. This is standard integral of calculus. If you want to do it by your CAS, you may have trouble for general a (>0); however, you should still be able to see what the limit is and your CAS should be able to do evaluation for any fixed number a. For instance, this is the situation for Maple.

5. $\hat{f}_c(w) = 2\dfrac{\sqrt{2}(\sin(w) - \cos(w)w)}{\sqrt{\pi}w^3}$

6. We have $f''(x) = 2 = g(x)$ if $0 < x < 1$, $\quad f''(x) = 0$ if $x > 1$.
By integration,

$$g_c(w) = 2\sqrt{\frac{2}{\pi}}\frac{\sin w}{w}.$$

Hence (5a) with $f'(0) = 0$ would give

$$-2\sqrt{\frac{2}{\pi}}\frac{\sin w}{w^3},$$

just one of the three terms shown in the answer to Prob. 5. This should show the student the importance of the continuity assumptions in the present and similar cases.

8. The defining integral (1a) has no limit,

$$\int_0^\infty k \cos wx \, dx = k \lim_{x\to\infty} \frac{\sin wx}{w} \qquad (w \text{ fixed!}).$$

Similarly for (2a).

9. $\dfrac{e^{-a}\sqrt{2}w}{\sqrt{\pi}(w^2+1)}$

10. Use $f''(x) = a^2 f(x)$ to obtain from (5b)

$$\mathcal{F}_s(f''(x)) = a^2\mathcal{F}_s(f(x)) = -w^2\mathcal{F}_s(f(x)) + \sqrt{\frac{2}{\pi}}\,w,$$

hence by collecting terms

$$(a^2 + w^2)\mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}}\,w$$

and so on.

11. $\dfrac{\sqrt{2}\,(w^2 - 2w\sin(w) - 2\cos(w) + 2)}{\sqrt{\pi}w^3}$

12. $\mathcal{F}_s(xe^{-x^2/2}) = -\mathcal{F}_s((e^{-x^2/2})') = w\mathcal{F}_c(e^{-x^2/2}) = we^{-w^2/2}$ from formula 4 in Table I (see Sec. 11.10 of text).

14. Formula 4 in Table II (see Sec. 11.10 of text) with $a = \frac{1}{2}$ gives

$$\mathcal{F}_s(x^{-1/2}) = \sqrt{\frac{2}{\pi}}\;w^{-1/2}\,\Gamma\left(\frac{1}{2}\right)\sin\,\frac{\pi}{4} = \frac{1}{\sqrt{\pi}}\,w^{-1/2}\,\Gamma\left(\frac{1}{2}\right).$$

On the other hand, by formula 2 in Table II,

$$\mathcal{F}_s(x^{-1/2}) = w^{-1/2}.$$

Comparison proves $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

SECTION 11.9 Fourier Transform. Discrete and Fast Fourier Transforms, page 522

Purpose. Derivation of the Fourier transform from the complex form of the Fourier integral; explanation of its physical meaning and its basic propertites.

Main Content, Important Concepts

Complex Fourier integral (4)

Fourier transform (6), its inverse (7)

Spectral representation, spectral density

Transforms of derivatives (9), (10)

Convolution $f * g$

Comments on Content

The complex Fourier integral is relatively easily obtained from the real Fourier integral in Sec. 11.7, and the definition of the Fourier transform is then immediate.

Note that convolution $f * g$ differs from that in Chapter 6, and so does the formula (12) in the convolution theorem (we now have a factor $\sqrt{2\pi}$).

Short Courses. This section can be omitted.

SOLUTIONS TO PROBLEM SET 11.9, page 533

2. This involves a transformation of exponential functions into a sine, as mentioned in Prob. 1. Integration of the defining integral gives

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{(1-w)ix}\,dx = \frac{-i}{\sqrt{2\pi}\,(2-w)}\,(e^{(1-w)i} - e^{-(1-w)i})$$

$$= \frac{-i}{\sqrt{2\pi}\,(1-w)}\,2i\sin(1-w)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin(w-1)}{w-1}.$$

4. By integration of the defining integral we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(k-iw)x}\,dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{e^{(k-iw)x}}{k-iw}\bigg|_{-\infty}^{0}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{k-iw}.$$

6. $\sqrt{2/\pi}(1 + w^2)$, as obtained by integration and simplification, namely

$$\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x(1-iw)}\,dx + \int_{0}^{\infty} e^{x(-1-iw)}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-iw} + \frac{1}{1+iw}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2}{1+w^2} = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+w^2}.$$

7. $-1/2\dfrac{\sqrt{2}(-1 - aw^2 + \mathrm{e}^{-iwa} + i\mathrm{e}^{-iwa}wa)}{\sqrt{\pi}w^2}$

8. By integration by parts we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{0} xe^{-x-iwx}\,dx = \frac{1}{\sqrt{2\pi}}\left(\frac{xe^{-(1+iw)x}}{-(1+iw)}\bigg|_{-1}^{0} + \frac{1}{1+iw}\int_{-1}^{0} e^{-(1+iw)x}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(-(-1)\,\frac{e^{1+iw}}{-(1+iw)} + \frac{e^{-(1+iw)x}}{-(1+iw)(1+iw)}\bigg|_{-1}^{0}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{-e^{1+iw}}{1+iw} - \frac{1-e^{1+iw}}{(1+iw)^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}(-w+i)^2}\,(1 + e^{1+iw}(-1 - i(-w+i)))$$

$$= \frac{1}{\sqrt{2\pi}(-w+i)^2}\,(1 + iwe^{1+iw}).$$

Problems 2 to 11 should help the student get a feel for integrating complex exponential functions and for their transformation into cosine and sine, as needed in this context. Here, it is taken for granted that complex exponential functions can be handled in the same fashion as real ones, which will be justified in Part D on complex analysis. The problems show that the technicalities are rather formidable for someone who faces these exponential functions for the first time. This is so for relatively simple $f(x)$, and since a CAS will give all the results without difficulty, it would make little sense to deal with more complicated $f(x)$, which would involve increased technical difficulties but no new ideas.

Furthermore, the present problem illustrates the following situation. Maple gave a result quite different and much more complicated than that obtained by the usual formal integration. In such a case, consider the difference between the result obtained and that expected and try to see whether you can reduce this difference to zero, thereby confirming what you had expected.

Problem 9 is of a similar type, that is, the difference between the result expected and that obtained can be reduced to zero.

10. By integration by parts,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} xe^{-iwx}dx &= \frac{1}{\sqrt{2\pi}}\left(\frac{xe^{-iwx}}{-iw}\bigg|_{-1}^{1} - \frac{1}{-iw}\int_{-1}^{1} e^{-iwx}\,dx\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-iw}}{-iw} + \frac{e^{iw}}{-iw} - \frac{1}{(-iw)^2}\, e^{-iwx}\bigg|_{-1}^{1}\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{2\cos w}{-iw} + \frac{1}{w^2}\,(e^{-iw} - e^{iw})\right)\\ &= \sqrt{\frac{2}{\pi}}\left(\frac{i\cos w}{w} - \frac{i\sin w}{w^2}\right)\\ &= \sqrt{\frac{2}{\pi}}\cdot\frac{i}{w^2}\,(w\cos w - \sin w).\end{aligned}$$

11. $\dfrac{4\,i(\sin(1/2w))^2}{w}$

12. Let $f(x) = xe^{-x} \quad (x > 0)$ and $g(x) = e^{-x} \quad (x > 0)$. Then $f' = g - f$ and by (9)

$$iw\,\mathcal{F}(f) = \mathcal{F}(f') = \mathcal{F}(g) - \mathcal{F}(f).$$

From this and formula 5 in Table III (see Sec. 11.10 of text) with $a = 1$,

$$(1 + iw)\,\mathcal{F}(f) = \mathcal{F}(g) = \frac{1}{\sqrt{2\pi}\,(1 + iw)}.$$

Now divide by $1 + iw$.

14. Formula 8 with $-b$ instead of b and b instead of c is

$$\frac{i}{\sqrt{2\pi}}\,\frac{e^{ib(a-w)} - e^{-ib(a-w)}}{a - w}.$$

Multiply numerator and denominator by -1, use $\sin(-\alpha) = -\sin\alpha$ and the formula for the sine Prob. 1. This gives

$$\frac{i}{\sqrt{2\pi}}\,\frac{e^{ib(a-w)} - e^{-ib(a-w)}}{w-a} = \frac{i}{\sqrt{2\pi}}\,\frac{2i \sin b(a-w)}{w-a} = \sqrt{\frac{2}{\pi}}\,\frac{\sin b(w-a)}{w-a}.$$

16. Team Project. **(a)** Use $t = x - a$ as a new variable of integration.

(b) Use $c = 3b$. Then (a) gives

$$e^{2ibw}\,\mathscr{F}(f(x)) = \frac{e^{ibw} - e^{-ibw}}{iw\sqrt{2\pi}} = \frac{2i \sin bw}{iw\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin bw}{w}.$$

(c) Replace w with $w - a$. This gives a new factor e^{iax}.

(d) We see that $\hat{f}(w)$ in formula 7 is obtained from $\hat{f}(w)$ in formula 1 by replacing w with $w - a$. Hence by (c), $f(x)$ in formula 1 times e^{iax} should given $f(x)$ in formula 7, which is true. Similarly for formulas 2 and 8.

20. $$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}\begin{bmatrix} 14i \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

22. $$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} f_1 + f_2 \\ f_1 - f_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

24. The powers of w are

$$1 \quad \frac{1-i}{\sqrt{2}} \quad -i \quad \frac{-1-i}{\sqrt{2}} \quad -1 \quad \frac{-1+i}{\sqrt{2}} \quad i \quad \frac{1+i}{\sqrt{2}}.$$

This is the second row of $\mathbf{F}$. The third row is

$$1 \quad -i \quad -1 \quad i \quad 1 \quad -i \quad -1 \quad i,$$

and so on.

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11. $3/2 + 6\dfrac{\sin(1/3\pi x)}{\pi} + 2\dfrac{\sin(\pi x)}{\pi} + 6/5\dfrac{\sin(5/3\pi x)}{\pi} + \cdots$

12. $f(x) - 3/2$ is an odd function.

14. The even function

$$f_{\text{even}}(x) = \begin{cases} -x/2 & \text{if} \quad -1 < x < 0 \\ x/2 & \text{if} \quad 0 < x < 1 \end{cases}$$

and the odd function

$$f_{\text{odd}}(x) = x/2 \quad (-1 < x < 1),$$

respectively.

16. $8\,\dfrac{\cos(1/2\pi x)}{\pi^2} + \dfrac{8}{9}\,\dfrac{\cos(3/2\pi x)}{\pi^2} + \dfrac{8}{25}\,\dfrac{\cos(5/2\pi x)}{\pi^2} + \cdots$

18. The function is $f(x) = \begin{cases} 1 & -2 < x < 0 \\ -1 & 0 < x < 2 \end{cases}$. The series is

$$-4\,\frac{\sin(1/2\pi x)}{\pi} - 4/3\,\frac{\sin(3/2\pi x)}{\pi} - 4/5\,\frac{\sin(5/2\pi x)}{\pi}\cdots$$

19. $-4\,\dfrac{\cos(\pi x)}{\pi^2} - 4/9\,\dfrac{\cos(3\pi x)}{\pi^2} - \dfrac{4}{25}\,\dfrac{\cos(5\pi x)}{\pi^2} + \cdots,$

$$-\frac{\sin(2\pi x)}{\pi} - 1/2\,\frac{\sin(4\pi x)}{\pi} - 1/3\,\frac{\sin(6\pi x)}{\pi} + \cdots$$

20. $\pi^2 - 12\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - + \cdots\right)$

22. $y = C_1 \cos \omega t + C_2 \sin \omega t + \dfrac{\pi}{2\omega^2} - \dfrac{4}{\pi}\left(\dfrac{\cos t}{\pi^2 - 1} + \dfrac{1}{9}\cdot\dfrac{\cos 3t}{\pi^2 - 9}\right.$

$$\left. + \frac{1}{25}\cdot\frac{\cos 5t}{\pi^2 - 25} + \cdots\right)$$

24. By a factor k^2, as can be seen directly from the formula in Sec. 11.4.

26. $\dfrac{\sqrt{2}\,\sqrt{\pi}(-w + \sin(w))}{w^2}$

28. $k[(ibw + 1)e^{-ibw} - (iaw + 1)e^{-iaw}]/(w^2\sqrt{2\pi})$

30. $\dfrac{\sqrt{2}}{\sqrt{\pi}(2iw + 1)}$