CS5314 Randomized Algorithms

Homework 3
Due: May 19, 2020 (before 11:59pm)

1. (a) Determine the moment generating function for the binomial random variable $Bin(n, p)$.
   (b) Let X be a $Bin(n, p)$ random variable and Y be a $Bin(m, p)$ random variable. Suppose that X and Y are independent. Use part (a) to determine the moment generating function of $X + Y$.
   (c) What can we conclude from the form of the moment generating function of $X + Y$?

2. Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \mathrm{E}[X]$. During the class, we have learnt that for any $\delta > 0$,

$$\Pr(X \geq (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu.$$

   In fact, the above inequality holds for the weighted sum of Poisson trials. Precisely, let $a_1, \ldots, a_n$ be real numbers in $[0, 1]$. Let $W = \sum_{i=1}^n a_i X_i$, and $\nu = \mathrm{E}[W]$. Then, for any $\delta > 0$,

$$\Pr(W \geq (1+\delta)\nu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\nu.$$

   (a) Show that the above bound is correct.
   (b) Prove a similar bound for the probability $\Pr(W \leq (1-\delta)\nu)$ for any $0 < \delta < 1$.

3. Let $X_1, \ldots, X_n$ be independent random variables such that

$$\Pr(X_i = 1 - p_i) = p_i \quad \text{and} \quad \Pr(X_i = -p_i) = 1 - p_i.$$

   Let $X = \sum_{i=1}^n X_i$. Prove that

$$\Pr(|X| \geq a) \leq 2e^{-2a^2/n}.$$

   Note: You may assume that the following inequality, which is a special case of **Hoeffding's Lemma**, is correct:

$$p_i e^{\lambda(1-p_i)} + (1-p_i)e^{-\lambda p_i} \leq e^{\lambda^2/8}.$$

4. (No marks) Study Hoeffding's Lemma.[1]

[1] Check this out: `https://en.wikipedia.org/wiki/Hoeffding%27s%5flemma`