Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.6

No. 1

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ; \quad c^2 = \frac{k}{\sigma \rho}$$

$$u_n(x,t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$\lambda_n = \frac{cn\pi}{L} = \frac{\sqrt{\frac{k}{\sigma\rho}}\, n\pi}{L}$$

$$u_n(x,t) = B_n \sin\frac{n\pi x}{L} e^{-\frac{k}{\sigma\rho}\left(\frac{n\pi}{L}\right)^2 t}.$$

No. 2

$$u_1(x,t) = B_1 \sin\frac{\pi x}{L} e^{-\lambda_1^2 t}.$$

$$u_1(x,20) = \frac{1}{2} u_1(x,0)$$

$$B_1 \sin\frac{\pi x}{L} e^{-\lambda_1^2 \cdot 20} = \frac{1}{2} B_1 \sin\frac{\pi x}{L}$$

$$e^{-\lambda_1^2 \cdot 20} = \frac{1}{2}$$

$$\lambda_1^2 = \frac{1}{20}\ln 2 = 0.03466$$

$$\lambda_1 = \frac{c\pi}{L} = 0.18616.$$

No. 3

$$u_n(x,t) = B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

Let $B_n = 1$, $c = 1$, $L = \pi$.

$$\lambda_n = \frac{cn\pi}{L} = n.$$

$$u_n(x,t) = \sin nx \, e^{-n^2 t}.$$

$$u_1(x,t) = \sin x \, e^{-t}$$

$$u_2(x,t) = \sin 2x \, e^{-2^2 t}$$

$$u_3(x,t) = \sin 3x \, e^{-3^2 t}$$

No. 4

問答或證明題，不解

No. 5

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t} \qquad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

$$L = 10. \quad c = \frac{K}{\sigma\rho} = \frac{1.04}{10.6 \times 0.056} = 1.752$$

$$u(x,0) = f(x) = \sin 0.1\pi x = \sum_{n=1}^{\infty} B_n \sin(0.1n\pi x)$$

$$B_n = \frac{2}{10}\int_0^{10} \sin(0.1\pi x)\sin(0.1n\pi x)\,dx$$

$$= \begin{cases} 1, & n = 1 \\ 0, & n \neq 1. \end{cases}$$

$$\therefore u(x,t) = \sin(0.1n\pi x)\, e^{-\frac{(1.752)^2\pi^2}{100} t}.$$

No. 6

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin(0.1n\pi x)\, e^{-\frac{(1.752)^2 n^2\pi^2}{100}t}$$

$$u(x,0)=f(x)=4-0.8|x-5| = \sum_{n=1}^{\infty} B_n \sin(0.1n\pi x)$$

$$B_n=\frac{2}{10}\int_0^{10}[4-0.8|x-5|]\sin(0.1n\pi x)\,dx$$

$$=\frac{2}{10}\left[\int_0^5 (0.8x)\sin(0.1n\pi x)\,dx+\int_5^{10}(8-0.8x)\sin(0.1n\pi x)\,dx\right]$$

$$=16\left(\frac{2\sin 0.5n\pi}{n^2\pi^2}-\frac{\sin n\pi}{n^2\pi^2}\right)$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty} 16\left(\frac{2\sin 0.5n\pi}{n^2\pi^2}-\frac{\sin n\pi}{n^2\pi^2}\right)\sin(0.1n\pi x)\, e^{-\frac{(1.752)^2 n^2\pi^2}{100}t}$$

No. 7

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin(0.1n\pi x)\, e^{-\frac{(1.752)^2 n^2\pi^2}{100}t}$$

$$u(x,0)=f(x)=x(10-x)=\sum_{n=1}^{\infty} B_n \sin(0.1n\pi x)$$

$$B_n=\frac{2}{10}\int_0^{10} x(10-x)\sin(0.1n\pi x)\,dx$$

$$=\frac{400}{n^3\pi^3}[1-(-1)^n]$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}\frac{400}{n^3\pi^3}[1-(-1)^n]\sin(0.1n\pi x)\, e^{-\frac{(1.752)^2 n^2\pi^2}{100}t}$$

$$=\frac{800}{\pi^3}\left(\sin 0.1\pi x\, e^{-\frac{(1.752)^2\pi^2}{100}t}+\frac{1}{3^3}\sin 0.3\pi x\, e^{-\frac{(1.752)^2 3^2\pi^2}{100}t}+\cdots\right.$$

No. 8

$u_I = U_1 + (U_2 - U_1)x/L$. This is the solution of (1) with $\partial u/\partial t = 0$ satisfying the boundary conditions.

No. 9

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = U_1 \quad , \quad u(L,t) = U_2$$

Let $\hat{u} = u - U_1$

$$\frac{\partial \hat{u}}{\partial t} = c^2 \frac{\partial^2 \hat{u}}{\partial x^2} \quad , \quad \hat{u}(0,t) = 0 \, , \quad \hat{u}(L,t) = U_2 - U_1$$

$$\hat{u} = F(x)G(t).$$

$$F(x) = A\cos px + B\sin px$$

$$F(0) = 0 \Rightarrow A = 0.$$

$$F(L) = 0 \Rightarrow B\sin pL = 0 \, . \; B \neq 0 \quad pL = n\pi \, . \; p = \frac{n\pi}{L} \quad (n = 1, 2, \cdots)$$

Let $B = 1$

$$F_n(x) = \sin\frac{n\pi}{L}x$$

$$G_n(t) = B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$\hat{u} = \sum_{n=1}^{\infty} F_n G_n = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$u = \hat{u} + U_1 = U_1 + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$I.C: \quad u(x, 0) = f(x) = U_1 + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L [f(x) - U_1] \sin\frac{n\pi x}{L} dx$$

No.10

$u(x, 0) = f(x) = 100$, $U_1 = 100$, $U_2 = 0$, $u_I = 100 - 10x$. Hence

$$\begin{aligned} B_n &= \frac{2}{10}\int_0^{10} [100 - (100 - 10x)] \sin\frac{n\pi x}{10}\,dx \\ &= \frac{2}{10}\int_0^{10} 10x \sin\frac{n\pi x}{10}\,dx \\ &= -\frac{200}{n\pi}\cos n\pi \\ &= \frac{(-1)^{n+1}}{n}\cdot 63.66. \end{aligned}$$

This gives the solution

$$u(x, t) = 100 - 10x + 63.66\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{10}\,e^{-1.752(n\pi/10)^2 t}.$$

For $x = 5$ this becomes

$$u(5, t) = 50 + 63.66\left[e^{-1.729t} - \tfrac{1}{3}e^{-1.556t} + \tfrac{1}{5}e^{-4.323t} - + \cdots\right].$$

Obviously, the sum of the first few terms is a good approximation of the true value at any $t > 0$. We find:

t	1	2	3	10	50
$u(5, t)$	99	94	88	61	50

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No.11

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u_x(0, t) = 0$, $u_x(L, t) = 0$

$u(x, t) = F(x)G(t)$.

$u_x(0, t) = F'(0)G(t) = 0$

$u_x(L, t) = F'(L)G(t) = 0$

$\therefore$ $F(x) = A\cos px + B\sin px$

$F'(x) = -Ap\sin px + Bp\cos px$

$$\begin{cases} F'(0) = Bp = 0 \\ F'(L) = -Ap\sin pL = 0 \end{cases} \Rightarrow \begin{cases} B = 0. \\ \sin pL = 0 \end{cases}, A \neq 0.$$

$p = p_n = \frac{n\pi}{L}$ $\quad (n = 0, 1, 2, \cdots)$

Let $A = 1$.

$F_n(x) = \cos\frac{n\pi x}{L}$. $\quad n = 0, 1, 2 \cdots$

$G_n(t) = e^{-\lambda_n^2 t}$. $\quad \lambda_n = cp = \frac{cn\pi}{L}$ $\quad (n = 0, 1, 2, \cdots)$

$\therefore$ $u_n(x, t) = F_n(x)G_n(t) = A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2 t}$ $\quad (n = 0, 1, \cdots)$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2 t}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}e^{-\lambda_n^2 t} \quad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

$$u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} = f(x)$$

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $\quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, $\quad (n = 1, 2, \cdots$

No.12

$L = \pi, \quad c = 1. \quad \lambda_n = \frac{cn\pi}{L} = n.$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(\frac{cn\pi}{L})^2 t}$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx \, e^{-n^2 t} \qquad u(x,0) = f(x) = x$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x \, dx = \frac{\pi}{2}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1] \cos\frac{n\pi x}{L} e^{-(\frac{cn\pi}{L})^2 t}$$

No.13

$u(x,0) = f(x) = 1.$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} 1 \cdot dx = 1$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \cos nx \, dx = \frac{2}{n\pi}(\sin n\pi - \sin 0) = 0$$

$\therefore u(x,t) = 1.$

No.14

$u(x, t) = \cos 4x \, e^{-16t}$ (Due to orthogonality all the terms except for $n = 4$ vanish. When $n = 4$, the integral evaluates to 1).

No.15

$$f(x) = 1 - \frac{x}{\pi}$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} \left(1 - \frac{x}{\pi}\right) dx = \frac{1}{2}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \cos nx \, dx = \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$\therefore \ u(x, t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right] \cos nx \, e^{-n^2 t}.$$

$$u(x, t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos x \, e^{-t} + \frac{1}{3^2}\cos 3x \, e^{-9t^2} + \frac{1}{5^2}\cos 5x \, e^{-25t^2} + \cdots\right)$$

No.16

$c^2 v_{xx} = v_t$, $v(0, t) = 0$, $v(\pi, t) = 0$, $v(x, 0) = f(x) + Hx(x - \pi)/(2c^2)$, so that, as in (9) and (10),

$$u(x, t) = -\frac{Hx(x - \pi)}{2c^2} + \sum_{n=1}^{\infty} B_n \sin nx \, e^{-c^2 n^2 t}$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi}\left(f(x) + \frac{Hx(x - \pi)}{2c^2}\right)\sin nx \, dx.$$

No.17

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}e^{-\lambda_n^2 t} \qquad \left(\lambda_n=\frac{cn\pi}{L}\right)$$

$$\cancel{u(0,t)=\phi(t)=-ku_x(0,t)}$$

$$\cancel{\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}\sum_{n=1}^{\infty}B_n e^{-\lambda_n^2 t}=-k\frac{n\pi}{L}-k\sum_{n=1}^{\infty}B_n\frac{n\pi}{L}e^{-\lambda_n^2 t}}$$

$$\cancel{\sum_{n=1}^{\infty}B_n\sin\frac{}{L}}$$

$$\phi(t)=-ku_x(0,t)=-k\sum_{n=1}^{\infty}B_n\cdot\frac{n\pi}{L}\cdot e^{-\lambda_n^2 t}$$

$$=-\frac{k\pi}{L}\sum_{n=1}^{\infty}nB_n e^{-\lambda_n^2 t}$$

No.18

$$u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

$a=20$,

$$u(x,40)=110=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi}{20}x\sinh\left(\frac{n\pi}{20}\cdot 40\right)$$

$$110=\sum_{n=1}^{\infty}A_n^*\sinh(2n\pi)\sin\frac{n\pi}{20}x$$

$$A_n^*\sinh(2n\pi)=\frac{2}{20}\int_0^{20}110\sin\frac{n\pi}{20}x\,dx$$

$$=\frac{220}{n\pi}\left[(-1)^n-1\right]$$

$$A^*=\frac{220\left[(-1)^n-1\right]}{n\pi\sinh(2n\pi)}$$

$$u(x,y)=\sum_{n=1}^{\infty}\frac{220\left[(-1)^n-1\right]}{n\pi\sinh(2n\pi)}\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}$$

No.19

$$u(x, y) = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$a = 2,$

$$u(x, 2) = 1000 \sin \frac{1}{2}\pi x = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi}{2} x \sinh \frac{n\pi \cdot 2}{2}$$

$$1000 \sin \frac{1}{2}\pi x = \sum_{n=1}^{\infty} A_n^* \sinh(n\pi) \sin \frac{n\pi}{2} x$$

$$A_n^* \sinh(n\pi) = \frac{2}{2} \int_0^2 \left(1000 \sin \frac{1}{2}\pi x\right) \sin \frac{n\pi}{2} x \, dx$$

$$A_n^* = \frac{1}{\sinh(n\pi)} \int_0^2 \left(1000 \sin \frac{1}{2}\pi x\right) \sin \frac{n\pi}{2} x \, dx$$

$$= \begin{cases} \dfrac{1000}{\sinh \pi}, & n = 1 \\ 0, & n \neq 0 \end{cases}$$

$$\therefore \; u(x, y) = \frac{1000}{\sinh \pi} \sin \frac{\pi x}{2} \sinh \frac{\pi y}{2}$$

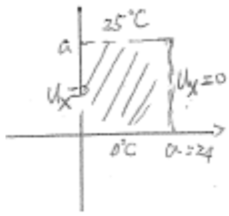
No.20

CAS Project. **(a)** $u = 80 (\sin \pi x \sinh \pi y)/\sinh 2\pi$

(b) $u_y(x, 0, t) = 0, \quad u_y(x, 2, t) = 0, \quad u = 0$

No.21

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$u(x, 0) = 0, \quad u(24, y) = 25$$

$$u_x(0, y) = 0, \quad u_x(24, y) = 0$$

$$u = F(x) G(y)$$

$$\frac{1}{F} \cdot \frac{d^2 F}{dx^2} = -\frac{1}{G} \frac{d^2 G}{dy^2} = -k$$

$$\frac{d^2 F}{dx^2} + kF = 0$$

$$F_x(0) = 0, \quad F_x(24) = 0$$

$$\Rightarrow \quad k = \left(\frac{(2n-1)\pi}{24}\right)^2 \qquad (n = 1, 2, \cdots\cdots)$$

$$F(x) = F_n(x) = \sin\left(\frac{(2n-1)\pi}{24}\right) x$$

$$\frac{d^2 G}{dy^2} - \left(\frac{(2n-1)\pi}{24}\right)^2 G = 0$$

$$G(y) = G_n(y) = A_n e^{\frac{(2n-1)\pi}{24} y} + B_n e^{-\frac{(2n-1)\pi}{24} y}$$

$$G_n(0) = 0 = A_n + B_n \qquad \Longrightarrow \qquad B_n = -A_n$$

$$G_n(y) = A_n\left(e^{\frac{(2n-1)\pi}{24}y} - e^{-\frac{(2n-1)\pi}{24}y}\right) = 2A_n \sinh\left(\frac{(2n-1)\pi}{24}y\right)$$

$$u_n(x,y) = F_n(x)G_n(y) = A_n^* \sin\left(\frac{(2n-1)\pi}{24}\right)x \sinh\left(\frac{(2n-1)\pi}{24}y\right).$$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{(2n-1)\pi}{24}x\right) \sinh\left(\frac{(2n-1)\pi}{24}y\right)$$

$$u(x,24) = 25 = \sum_{n=1}^{\infty}\left(A_n^* \sinh(2n-1)\pi\right) \sin\left(\frac{(2n-1)\pi}{24}x\right)$$

$$A_n^* \sinh(2n-1)\pi = \frac{2}{24}\int_0^{24} 25 \sin\left(\frac{(2n-1)\pi}{24}x\right) dx$$

$$= \frac{100}{\pi}\,\frac{1}{(2n-1)}$$

$$A^* = \frac{100}{\pi}\,\frac{1}{(2n-1)\sinh(2n-1)\pi}$$

$$\therefore\ u = \sum_{n=1}^{\infty} \frac{100}{\pi}\,\frac{1}{(2n-1)\sinh(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{24}\right) \sinh\left(\frac{(2n-1)\pi y}{24}\right).$$

No.22

$u = u_{\mathrm{I}} + u_{\mathrm{II}}$, where

$$u_{\mathrm{I}} = \frac{4U_1}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{24}\,\frac{\sinh\left[(2n-1)\pi y/24\right]}{\sinh(2n-1)\pi}$$

$$u_{\mathrm{II}} = \frac{4U_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{24}\,\frac{\sinh\left[(2n-1)\pi(1-y/24)\right]}{\sinh(2n-1)\pi}.$$

No.23

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u_y(x, 0) = 0$, $u_y(x, 24) = 0$

$u(0, y) = 0$, $u(24, y) = f(y)$

$$u = F(x)G(y)$$

$$-\frac{1}{F}\frac{d^2F}{dx^2} = \frac{1}{G}\frac{d^2G}{dy^2} = -k$$

$$\frac{d^2G}{dy^2} + kG = 0$$

$$G_y(0)=0, \quad G_y(24)=0 \quad \Rightarrow k=\left(\frac{n\pi}{24}\right)^2 \qquad (n=0,1,2,\cdots)$$

$$G(y)=G_n(y)=\cos\frac{n\pi}{24}y \qquad (n=0,1,2\cdots)$$

$$\frac{d^2F_n}{dx^2}-\left(\frac{n\pi}{24}\right)^2F_n=0$$

$$F(x)=F_n(x)=A_n e^{\frac{n\pi}{24}x}+B_n e^{\frac{-n\pi}{24}x}, \qquad F_0(x)=A_0x$$

$$F_n(0)=0=A_n+B_n \Rightarrow B_n=-A_n$$

$$F_n(x)=A_n\left(e^{\frac{n\pi}{24}x}-e^{-\frac{n\pi}{24}x}\right)=2A_n\sinh\left(\frac{n\pi}{24}x\right)$$

$$u_n(x,y)=F_n(x)G_n(x)=A_n^*\sinh\left(\frac{n\pi}{24}x\right)\cos\left(\frac{n\pi}{24}y\right)+A_0x \qquad (n=0,1,2\cdots)$$

$$u(x,y)=\sum_{n=0}^{\infty}A_n^*\sinh\left(\frac{n\pi}{24}x\right)\cos\left(\frac{n\pi}{24}y\right)=A_0x+\sum_{n=1}^{\infty}A_n^*\sinh\left(\frac{n\pi x}{24}\right)\cos\left(\frac{n\pi y}{24}\right)$$

$$u(24,y)=f(y)=24A_0+\sum_{n=1}^{\infty}A_n^*\sinh(n\pi)\cos\left(\frac{n\pi}{24}y\right)$$

$$A_0=\frac{1}{24^2}\int_0^{24}f(y)\,dy, \qquad A_n^*\sinh(n\pi)=\frac{2}{24}\int_0^{24}f(y)\cos\frac{n\pi y}{24}\,dy$$

$$A_n^*=\frac{1}{12\sinh(n\pi)}\int_0^{24}f(y)\cos\frac{n\pi y}{24}\,dy$$

No.24

$u = F(x)G(y)$, $F = A\cos px + B\sin px$, $u_x(0, y) = F'(0)G(y) = 0$, $B = 0$, $G = C\cosh py + D\sinh py$, $u_y(x, b) = F(x)G'(b) = 0$, $C = \cosh pb$, $D = -\sinh pb$, $G = \cosh(pb - py)$. For $u = \cos px \cosh p(b - y)$ we get

$$u_x(a, y) + hu(a, y) = (-p\sin pa + h\cosh pa)\cosh p(b - y) = 0.$$

Hence p must satisfy $\tan ap = h/p$, which has infinitely many positive real solutions $p = \gamma_1, \gamma_2, \cdots$, as you can illustrate by a simple sketch. *Answer:*

$$u_n = \cos\gamma_n x \cosh\gamma_n(b - y),$$

where $\gamma = \gamma_n$ satisfies $\gamma\tan\gamma a = h$.

To determine coefficients of series of u_n's from a boundary condition at the lower side is difficult because that would not be a Fourier series, the γ_n's being only approximately regularly spaced. See [C3], pp. 114–119, 167.

No.25

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x, 0) = f(x), \quad u(x, b) = u(0, y) = u(a, y) = 0.$$

$$u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\left(\frac{n\pi(b-y)}{a}\right)$$

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right) \sin\frac{n\pi x}{a}$$

$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$