Assignment 3

Q3/Q5/Q7

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Q3

(*, Challenging, UKMT MOG 2016) Show that any odd positive integer can be expressed as a product of fractions with each fraction of the form

$$\frac{4q-1}{2q+1}, \qquad \text{where } q \text{ is a positive integer?}$$

For instance,

$$1 = \frac{3}{3} = \frac{4 \times 1 - 1}{2 \times 1 + 1}$$

and

$$3 = \frac{15}{9} \times \frac{27}{15} = \frac{4 \times 4 - 1}{2 \times 4 + 1} \times \frac{4 \times 7 - 1}{2 \times 7 + 1}.$$

Q3

- In Q3, we need to show any odd positive integer can be expressed as a product of fractions with each fraction of the form $\frac{4q-1}{2q+1}$.
- We set $f(q) = \frac{4q-1}{2q+1}$

Hint for Question 3:

$$(2q+1) \times \frac{4q-1}{2q+1} = 4q-1; \qquad (2q+1) \times \frac{12q+3}{6q+3} = 4q+1.$$

- $(2q+1) \times \frac{4q-1}{2q+1} = 4q-1 \rightarrow (2q+1) \times f(q) = 4q-1$ *…equation 1*
- $(2q+1) \times \frac{12q+3}{6q+3} = 4q+1 \rightarrow (2q+1) \times \frac{4(3q+1)-1}{2(3q+1)+1} = 4q+1$

  $\rightarrow (2q+1) \times f(3q+1) = 4q+1$ *…equation 2*

Q3

$$f(q) = \frac{4q-1}{2q+1}$$

- $4q - 1 = (2q+1) \times f(q)$*...equation 1*
- $4q + 1 = (2q+1) \times f(3q+1)$*...equation 2*
- According to the information provided by the problem, we know that:
- $1 = f(1)$ $\quad 1 = \frac{3}{3} = \frac{4 \times 1 - 1}{2 \times 1 + 1}$
- $3 = f(4) \times f(7)$ $\quad 3 = \frac{15}{9} \times \frac{27}{15} = \frac{4 \times 4 - 1}{2 \times 4 + 1} \times \frac{4 \times 7 - 1}{2 \times 7 + 1}.$
- When $q = 1$, we can use *equation 2* to get that $5 = 3 \times f(4) = f(4) \times f(7) \times f(4)$
- We see that 1,3,5 can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$).

Q3

$$f(q) = \frac{4q-1}{2q+1}$$

- $4q - 1 = (2q+1) \times f(q)$ *…equation 1*
- $4q + 1 = (2q+1) \times f(3q+1)$ *…equation 2*
- For any odd number which is larger than 7, we can always express this odd number in the form of $4q-1$ or $4q+1$ (Where $q \geq 2$)
- We assume we check these odd numbers from small to large.
- When $q \geq 2$, $4q-1 > 2q+1$, that means we can express any odd number (≥ 7) as the product of a smaller odd number (which has been proved that it can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$)) and ($f(q)$ or $f(3q+1)$), that means this odd number can also be expressed as a product of many fractions(where each fraction is in the form of $f(q)$).

We use the concept of strong induction.

Q3

$$f(q)=\frac{4q-1}{2q+1}$$

- P(x)= statement “$x \in odd\ positive\ number$, x can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$)”
- We already know that P(1)=T,P(3)=T,P(5)=T
- We check odd number 7 now:
- $4q-1=(2q+1)\times f(q)$…*equation 1*
- When $q=2$:
- $7 \quad = \quad 5 \quad \times \quad f(2)$ ⟶ P(7) = T

Smaller odd number which has been checked that P(x)=T

X where we have checked that P(x)=T
1,3,5

Q3

$$\boldsymbol{f(q) = \frac{4q-1}{2q+1}}$$

- P(x)= statement "$x \in odd\ positive\ number$, x can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$)"
- We check odd number 9 now:
- $4q + 1 = (2q + 1) \times f(3q + 1)$…*equation 2*
- When $q = 2$:
- $9 = 5 \times f(7)$ → P(9) = T

Smaller odd number which has been checked that P(x)=T

X where we have checked that P(x)=T
1,3,5,7

Q3

$$\boldsymbol{f(q) = \frac{4q-1}{2q+1}}$$

- P(x)= statement "$x \in odd\ positive\ number$, x can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$)"
- We check odd number 11 now:
- $4q - 1 = (2q+1) \times f(q)$...*equation 1*
- When $q = 3$:
- $11 = 7 \times f(3)$

P(11) = T

Smaller odd number which has been checked that P(x)=T

X where we have checked that P(x)=T

1,3,5,7,9

Q3

$$f(q) = \frac{4q-1}{2q+1}$$

- P(x)= statement "$x \in odd\ positive\ number$, x can be expressed as a product of many fractions(where each fraction is in the form of $f(q)$)"
- We check odd number 13 now:
- $4q+1 = (2q+1) \times f(3q+1)$…*equation 2*
- When $q = 3$:
- $13 = 7 \times f(10)$ → P(13) = T

Smaller odd number which has been checked that P(x)=T

X where we have checked that P(x)=T

1,3,5,7,9,11

And so on….

Q3

$$\boldsymbol{f(q) = \frac{4q-1}{2q+1}}$$

- When $q \geq 2$,
- $4q - 1 = (2q + 1) \times f(q)$*...equation 1*
- $4q + 1 = (2q + 1) \times f(3q + 1)$*...equation 2*
- By *equation 1* : $4 \times 2 - 1 = (2 \times 2 + 1) \times f(2) \rightarrow 7 = 5 \times f(2)$
- By *equation* 2 : $4 \times 2 + 1 = (2 \times 2 + 1) \times f(3 \times 2 + 1) \rightarrow 9 = 5 \times f(7)$

$q = 2$

- By *equation 1* : $4 \times 3 - 1 = (2 \times 3 + 1) \times f(3) \rightarrow 11 = 7 \times f(3)$
- By *equation* 2 : $4 \times 3 + 1 = (2 \times 3 + 1) \times f(3 \times 3 + 1) \rightarrow 13 = 7 \times f(10)$

$q = 3$

- By *equation 1* : $4 \times 4 - 1 = (2 \times 4 + 1) \times f(4) \rightarrow 15 = 9 \times f(4)$
- By *equation* 2 : $4 \times 4 + 1 = (2 \times 4 + 1) \times f(3 \times 4 + 1) \rightarrow 17 = 9 \times f(13)$

$q = 4$

$\vdots$

And so on....

We can show that all odd positive integers can be expressed as a product of many fractions where each of the fraction is in the form of $\frac{4q-1}{2q+1}$ $(f(q))$.

Q5

(*) Chef Nicholas is a very talented cook. Give him a frying pan, and a stack of pancakes and waffles, he can flip freely any pieces of pancakes and waffles at the top of the stack.

For instance, suppose that the frying pan has a stack of 6 pieces of pancakes and waffles as follows (`P` for pancake, `W` for waffle):

`P W P P W P` (reading from bottom to top)

If Nicholas flips the top 5 pieces, the stack would become:

`P P W P P W` (reading from bottom to top)

Furthermore, if Nicholas continues to flip the top 3 pieces, the stack would become:

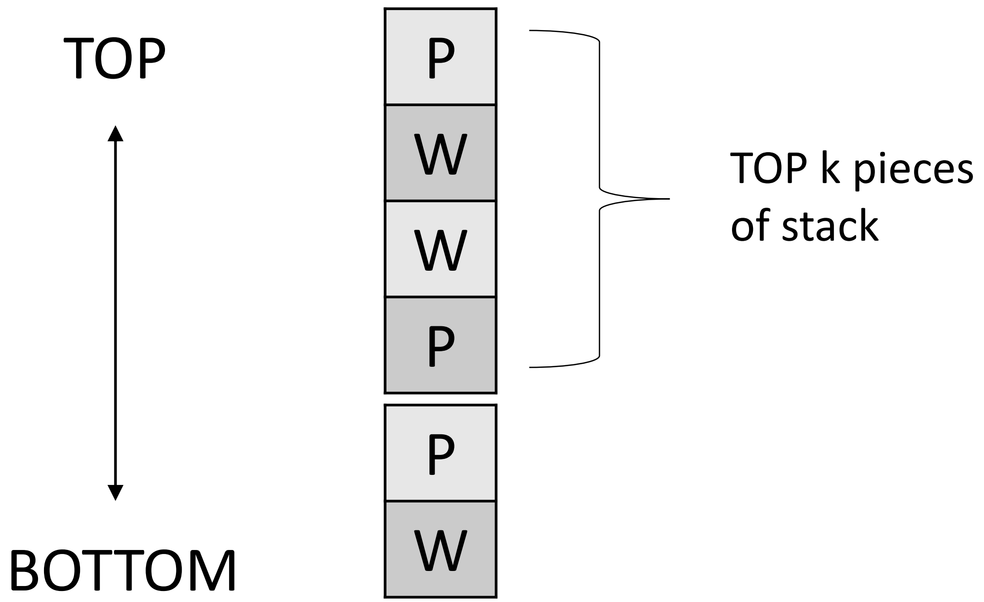
`P P W W P P` (reading from bottom to top)

And, with one more flip of top 4 pieces, the stack would become:

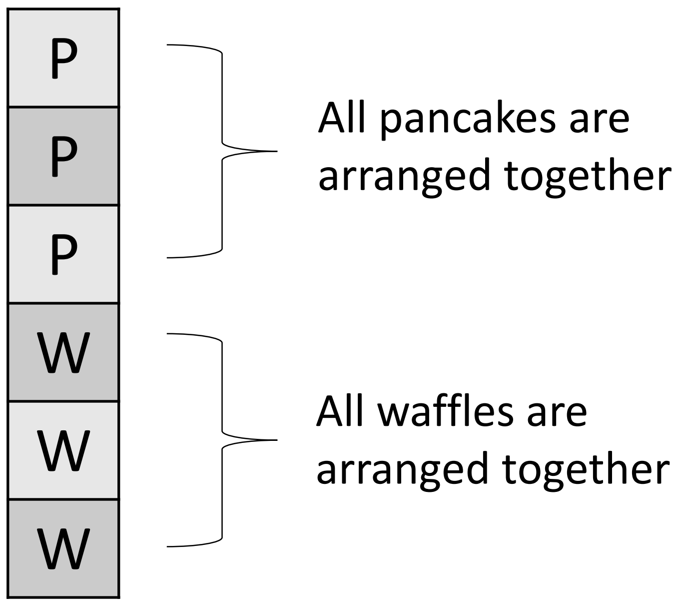
`P P P P W W` (reading from bottom to top)

Show that Nicholas can use at most $n-1$ flips to flip any stack of n pieces of pancakes and waffles, so that all the pancakes are arranged together, while all the waffles are also arranged together.

Q5.



In every flip, we can flip any k pieces on the top of the stack



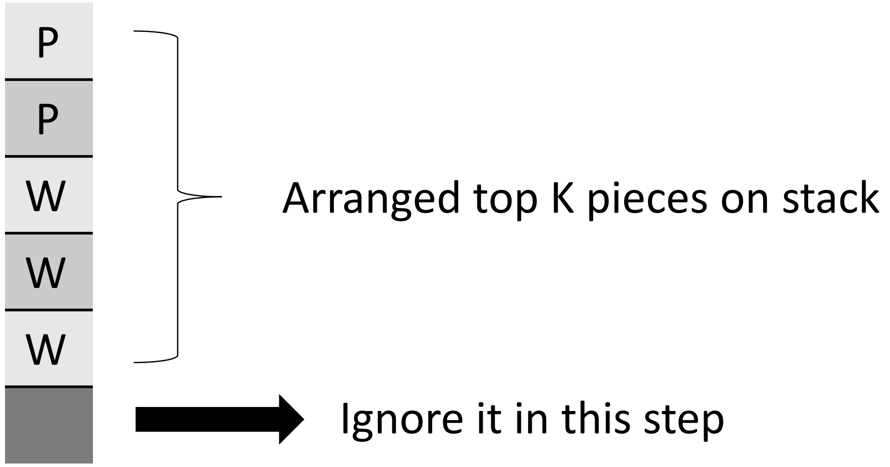
Final result we want : All pancakes are arranged together, and all waffles are arranged together

Q5

- We need to show that we can use at most n-1 flips to flip any stack of n pieces of pancakes and waffles, so that all the pancakes are arranged together, while all the waffles are also arranged together.
- We can just use Mathematical Induction to solve this problem:
- Basis step: If n=1, then there's only 1 piece of pancake or waffle, we don't need to flip -> we can use at most n-1 (0) flip to make the pancakes arranged together and the waffles arranged together.

Q5

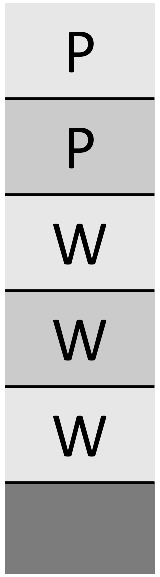
- Induction step: If when n=k (There are k pieces of pancakes and waffles) , we can use at most k-1 flips to make the pancakes arranged together and the waffles arranged together.
- Then when n=k+1.
- First , we use k-1 flips to make the top k pieces of pancakes and waffles on the stack arranged together respectively (ignore the bottom piece)



Q5

- Second,we just check if the k-th piece from the top is different from the (k+1)-th piece from the top (the bottom).

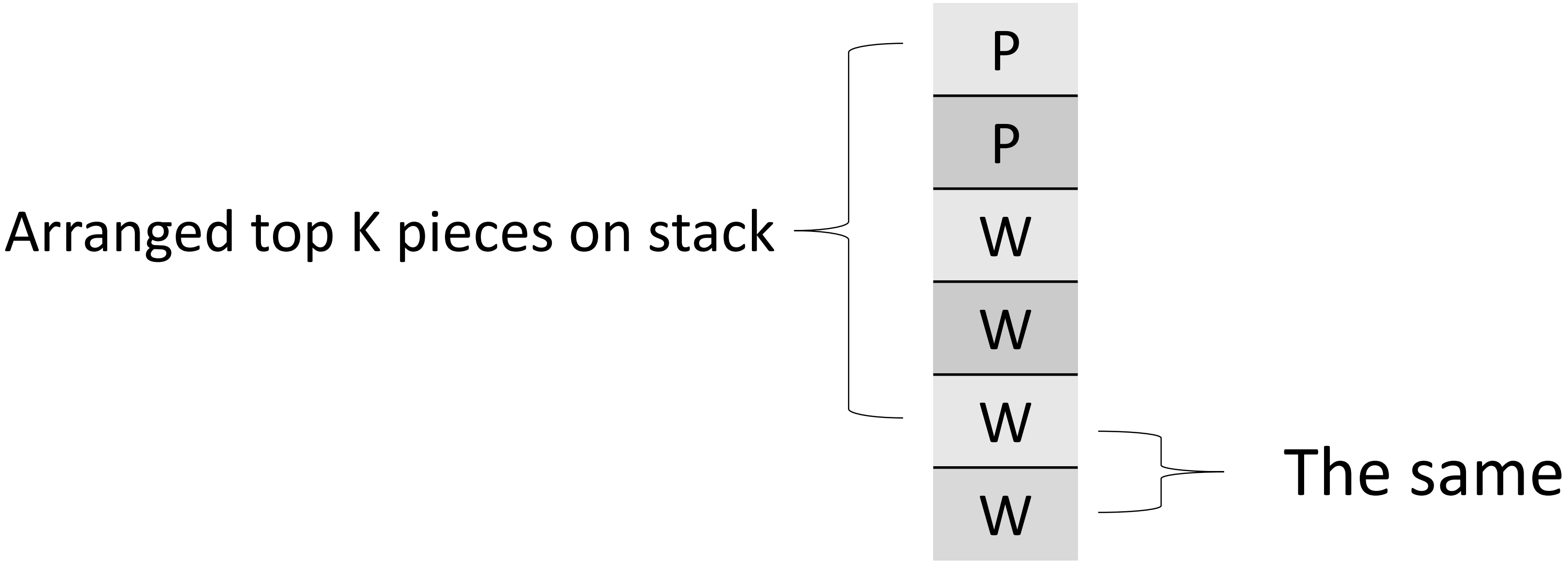
Arranged top K pieces on stack



Check if they are different

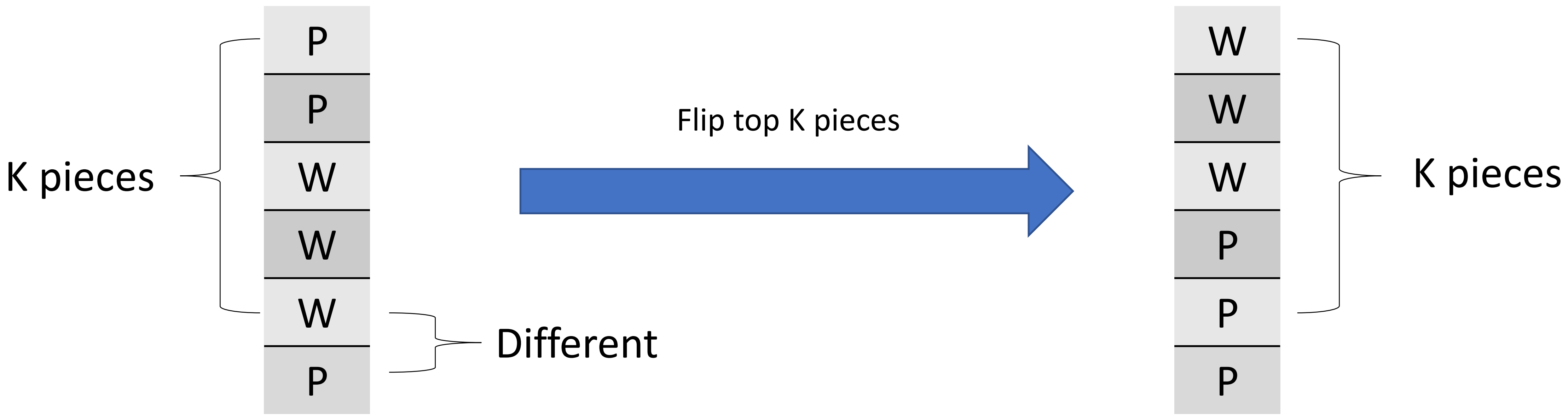
Q5

- If they are the same, then we don't have to do any more flip to make the pancakes/waffles arranged together.

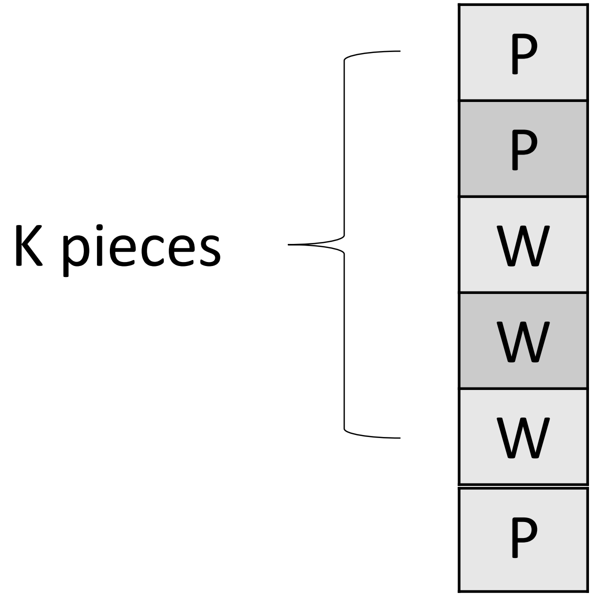


Q5

- If they are different ,we just flip the top k pieces (do 1 more flip) to make the pancakes/waffles arranged together.



Q5



- flip the top k pieces (do 1 more flip) to make the pancakes/waffles arranged together.

Q5

- In conclusion, when n=k+1, we we can use at most (k-1)+1 flips, equals to (k+1)-1 flips to make the pancakes arranged together and the waffles arranged together.
- Proof completed, we can show that we can use at most n-1 flips to flip any stack of n pieces of pancakes and waffles, so that all the pancakes are arranged together, while all the waffles are also arranged together.

We can also prove that when $n \geq 2$,we can use at most $n-2$ flips to do the same task, just set the basis case that when $\mathrm{n}=2$, we don't need to do any flip to complete the task, so that we can use at most $n-2$ flips to do the task, and for the induction step , we use the same method.

Q7

(*, Challenging) Let n be a positive integer, and consider an array with 2 rows and $2n$ columns. Each entry in the array is either 0 or 1. It is known that for each row, exactly n entries are 0 and exactly n entries are 1.

For a particular column, if both entries are 0, we call it a 0-column; else, if both entries are 1, we call it a 1-column.

Show that the number of 0-columns is the same as the number of 1-columns.

For instance, suppose $n = 3$. Suppose the array looks like the following:

1	0	1	0	0	1
0	0	1	1	0	1

Each row contains exactly n 0s and exactly n 1s. Also, we see that there are two 0-columns (the 2nd one and the 5th one), and there are two 1-columns (the 3rd one and the 6th one).

Q7-Induction Method

$n \in N$

- We can use Mathematical Induction to solve this problem:
- Basis step: If n=1, then there will be an array of 2 rows and 2 columns, and for each row , exactly one entry is 1, and one entry is 0.
- So that the two columns can only be consisted of $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ or $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$
- Either option , the number of 0-column remains the same as the number of 1-column.

Q7-Induction Method

$n, k \in N$

- Induction step:If when n = k, there already exists an array of 2 rows and 2k columns, and for each row , exactly k entries are 1, and k entries are 0, and the number of 0-column is the same as the number of 1-column.
- Then when n = k+1 , we need to add an array of 2 rows and 2 columns (to form an array of 2 rows and 2(k+1) columns)
- There should be (k+1)×'1', (k+1)× '0' in each row, in the added 2 columns , there should be exactly 1× '1', 1× '0' in each row .
- So that the added two columns can only be consisted of $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ or $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

Q7-Induction Method

- Either option , the number of added 0-column is the same as the number of added 1-column.
- So we can apparently tell that after adding, the number of 0-column remains the same as the number of 1-column.
- Proof is completed.

Q7-Counting Method

$a, b, c, d, n, k \in N$

- The sum of all entries is 2n (2 rows , $n \times$ '1', $n \times$ '0' in each row)
- Suppose there are k× $\begin{bmatrix}1\\0\end{bmatrix}$ columns,then there must have k× $\begin{bmatrix}0\\1\end{bmatrix}$ columns.
- (Proof: if we have a× $\begin{bmatrix}1\\0\end{bmatrix}$, b× $\begin{bmatrix}0\\1\end{bmatrix}$, c× $\begin{bmatrix}1\\1\end{bmatrix}$, d× $\begin{bmatrix}0\\0\end{bmatrix}$,then we have a

sum of a+c in the first row, a sum of b+c in the second row , a+c = b+c, so a =b)

Q7-Counting Method

$\boxed{a, b, n, k \in N}$

- Remove these k× $\begin{bmatrix}1\\0\end{bmatrix}$ columns and the k× $\begin{bmatrix}0\\1\end{bmatrix}$ columns, we have 2n-2k columns left.
- We assume these 2n-2k columns are consisted of (a × $\begin{bmatrix}1\\1\end{bmatrix}$, b× $\begin{bmatrix}0\\0\end{bmatrix}$) , the sum of the the entries left is 2n-2k.
- 2n-2k = 2× a + 0× b(the sum of entries left)
- 2n-2k = a + b (the sum of columns left)
- a = n - k , b = n – k.
- a = b
- The number of 1-column and 0-column is the same.

13 14

王瑞恩

13. (*) Show that in a group of 10 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

14. (**) Show that in a group of 9 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

If we solve Q14, then we solve Q13. Since Q13 is just add an additional node into Q14, so we focus on Q14 first.

Based on generalized pigeonhole principle, there exist at least 4 friends or 4 enemies for a node, and there are three cases.

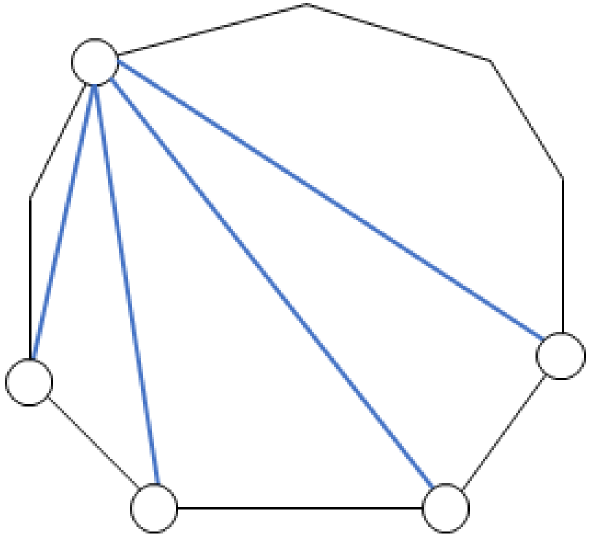
case 1. a node with at least 4 friends

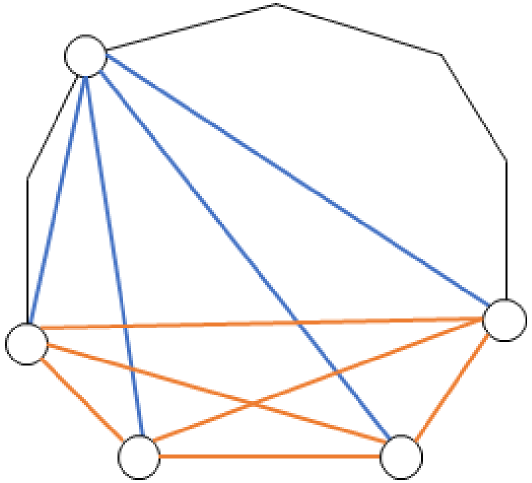
case 2. a node with at least 6 enemies

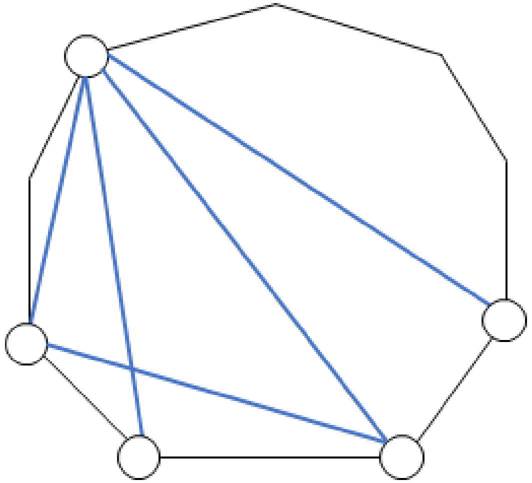
case 3. a node with exactly 3 friends and 5 enemies

case 1. a node with at least 4 friends

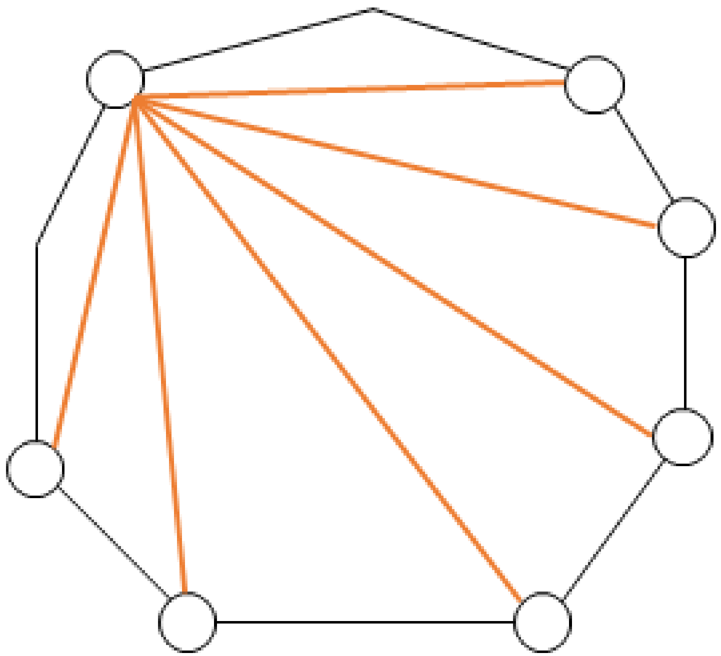
blue: friend
orange: enemy







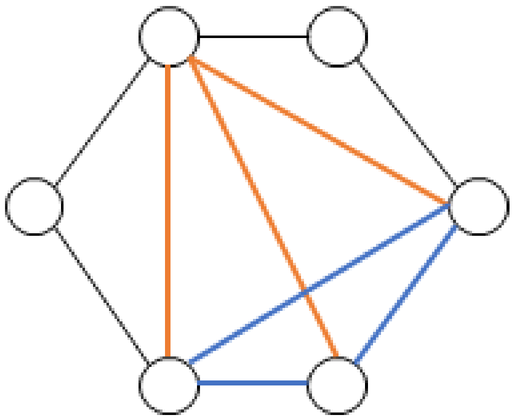
case 2. a node with at least 6 enemies

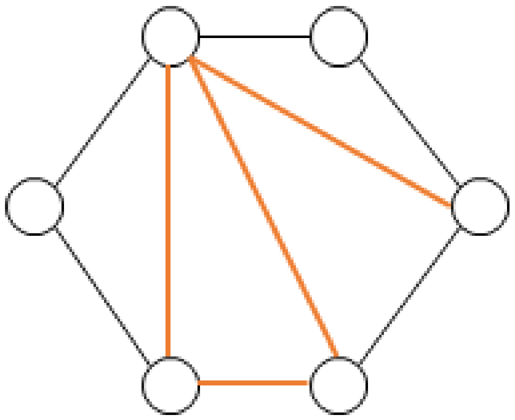


We first break the problem into 6 nodes have either 3 mutual friends or 3 mutual enemies

case 2. a node with at least 6 enemies

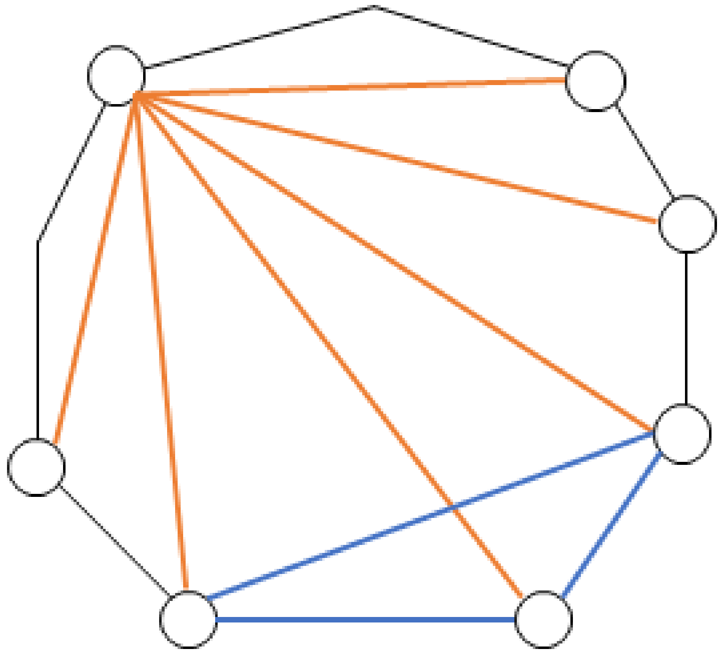
We first break the problem into 6 nodes have either 3 mutual friends or 3 mutual enemies



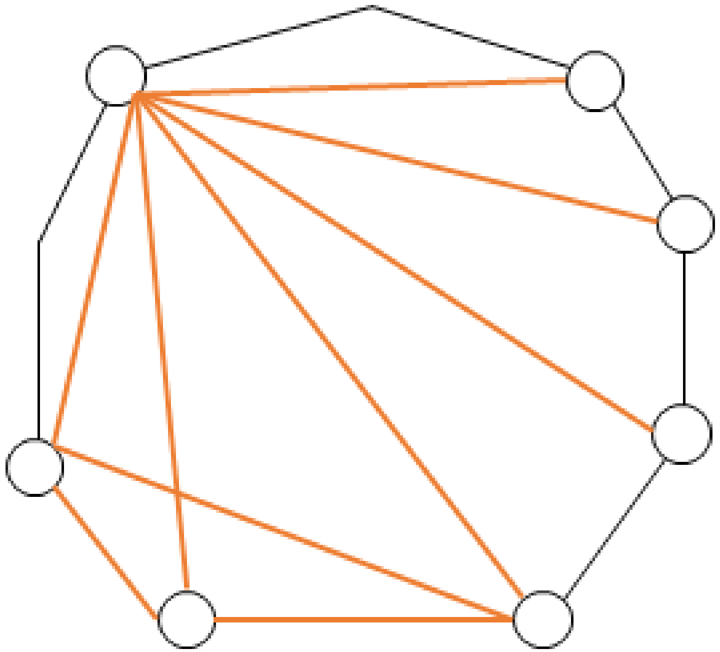


case 2. a node with at least 6 enemies

Based on we know that 6 nodes have either 3 mutual friends
or 3 mutual enemies

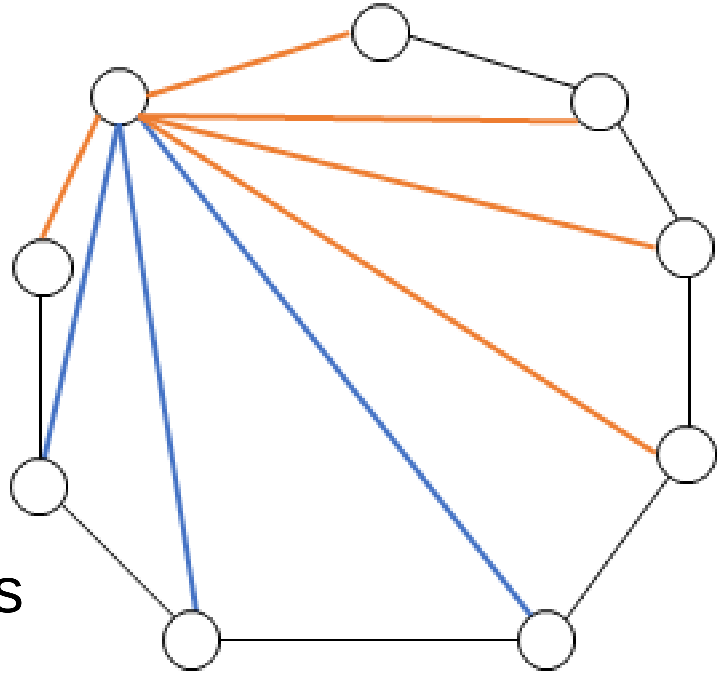


if that 6 nodes contain 3 mutual friends, then done



if that 6 nodes contain 3 mutual enemies, then
since there is a node is the enemies of all the other
6 nodes, so this case also done

case 3. a node with exactly 3 friends and 5 enemies



This is a special case, and if all nodes are with exactly 3 friends and 5 enemies, then we can't prove it.

However, this situation wouldn't happen. Prove it by observe degree of all node. If all nodes with exactly 3 friends and 5 enemies, then degree of all nodes for friends is 9*3 = 27. Since it is not a directed graph, so degree of all vertices is equal to 2*|E| = 27. |E| = 13.5 for friends, that is impossible. Based on this, we can prove that it is impossible that all nodes with exactly 3 friends and 5 enemies.

Since, there exist nodes not have exactly 3 friends and 5 enemies, so we can apply case1 or case2 for these nodes, and done!

15

王瑞恩

15. (*) Show that among a group of 100 people, if any two will shake hands at most once, then at least two people will shake hands for the same number of times.

There are two possible situation.

case 1. everyone shake hands with others.

case 2. there exist someone doesn’t shake hads with others.

case 1. everyone shake hands with others.

The possible times of handshakes in this situation are varies from 1 to 99, and there are 100 people, so based on Pigeonhole principle there have at least two people will have same number of times.

case 2. there exist someone doesn’t shake hads with others.

The possible times of handshakes in this situation are varies from 0 to 98, and there are 100 people, so based on Pigeonhole principle there have at least two people will have same number of times.

16

王瑞恩

16. (*, Challenging) Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $(b_1, b_2, b_3, b_4, b_5, b_6)$ be two arrangements of the integers $1, 2, 3, 4, 5, 6$. Consider the six pairs of differences $|a_i - b_i|$. Is it possible that all of these differences are not the same?

Two method.

method 1. brute method.

method 2. tricky method.

Brute method

Just list all possible combinations, and you will find that it is impossible.

Tricky method

If all differences between a and b are not the same, then differences are (0, 1 ,2 ,3 ,4 ,5). Sum of the differences is 1+2+3+4+5 = 15.
And here we observe that

$\sum_{i=1}^{6}|a_i - b_i| - \sum_{i=1}^{6}(a_i - b_i)$ is an even number, since $|a_i - b_i| - (a_i - b_i) = 0\ or\ 2|a_i - b_i|$.

And in this case $\sum_{i=1}^{6}(a_i - b_i)$ is 0, since $\sum_{i=1}^{6} a_i = \sum_{i=1}^{6} b_i$.

However, we got 15, and 15 is not an even number, so 15 not possible be the sum of differences and it is impossible that all differences are not the same.

Homework4
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Q2. Three integers are selected from the integers {1, 2, . . . , 1000}. In how many ways can these integers be selected such their sum is divisible by 4?

Q2.

{1, 2, . . . , 1000}. can be divided into 4 groups.

1000 numbers divided by 4. There are 250 numbers that is divisible by 4 and 250 numbers left 1,2,3 after they mod 4. So the reminder of the selected numbers can be (0,0,0),(0,1,3),(0,2,2),(1,1,2),(2,3,3).

So there are $250!/3!247!+250^3+250!/2!248!*250*3=41541750$

Q3.How many arrangements of the letters in MISSISSIPPI have no consecutive S's ?

Q3.

The arrangements of the above letters=7!/(4!2!)=105

's' has combinations=8!/(4!4!)=70

The total arrangements of the letters in Mississippi having no consecutive s's=70X105=7350.

Q11. LEFT

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}=\binom{n}{1}\binom{n}{0}+\binom{n}{2}\binom{n}{1}+\cdots+\binom{n}{n}\binom{n}{n-1}$$

$$=\binom{n}{n-1}\binom{n}{0}+\binom{n}{n-2}\binom{n}{1}+\cdots+\binom{n}{0}\binom{n}{n-1}$$

$$=\binom{n+n}{n-1}=\frac{(2n)!}{(n+1)!\,(n-1)!}$$

RIGHT

$$\binom{2n+2}{n+1}/2-\binom{2n}{n}=\frac{(2n+2)!}{(n+1)!(n+1)!\times 2}-\frac{(2n)!}{n!\,n!}$$

$$=\frac{(2n+2)(2n+1)(2n!)}{2(n+1)(n)(n-1)!(n+1)!}-\frac{(2n)!(n+1)}{(n)(n-1)!(n+1)!}$$

$$=\left[\frac{(2n+2)(2n+1)}{2(n+1)(n)}-\frac{n+1}{n}\right]\cdot\frac{(2n!)}{(n+1)!(n-1)!}$$

$$=\frac{(2n)!}{(n+1)!(n-1)!}$$

Discrete Mathematics

Homework 4
Question 6, 7, 10

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Question 6

How many ways are there to distribute five distinguishable objects into three indistinguishable boxes so that each of the boxes contains at least one object?

Question 6

How many ways are there to distribute five distinguishable objects into three indistinguishable boxes so that each of the boxes contains at least one object?

Question 6

Solution:

Let $(\cdot , \cdot , \cdot)$ be the number of objects in 3 boxes.

e.g.
(1, 2, 2) means there are 2 boxes with 2 objects and 1 box with 1 object.

$(1, 2, 2) = (2, 1, 2) = (2, 2, 1)$

Question 6

Solution:

First, suppose that all objects are identical.

Question 6

Solution:

First, suppose that all objects are identical.

There are 2 possible groupings:

(3, 1, 1)

(2, 2, 1)

Question 6

Solution:

Then, consider that all objects are different.

There are 2 possible groupings:

$(3, 1, 1)$: $C(5,3)C(2,1)C(1,1)/2!$

$(2, 2, 1)$: $C(5,2)C(3,2)C(1,1)/2!$

Question 6

Solution:

Then, consider that all objects are different.

There are 2 possible groupings:

$(3, 1, 1) : C(5,3)C(2,1)C(1,1)/2! = 10$

$(2, 2, 1) : C(5,2)C(3,2)C(1,1)/2! = 15$

total: $10 + 15 =$ 25 ways

Question 7

How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

Question 7

How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

Question 7

Solution:

Let $(\cdot , \cdot , \cdot)$ be the number of objects in 3 boxes.

e.g.
(0, 2, 3) means there is 1 boxes with 0 objects, 1 box with 2 objects and 1 box with 3 objects.

(0, 2, 3) = (0, 3, 2) = (2, 0, 3) = ……

Question 7

Solution:

First, suppose that all objects are identical.

There are 5 possible groupings:

Question 7

Solution:

(5, 0, 0)

(4, 1, 0)

(3, 2, 0)

(3, 1, 1)

(2, 2, 1)

Question 7

Solution:

$(5, 0, 0) : 1$

$(4, 1, 0) : C(5,4)C(1,1) = 5$

$(3, 2, 0) : C(5,3)C(2,2) = 10$

$(3, 1, 1) : C(5,3)C(2,1)C(1,1)/2! = 10$

$(2, 2, 1) : C(5,2)C(3,2)C(1,1)/2! = 15$

total: $1 + 5 + 10 + 10 + 15 =$ 41 ways

Question 10

There are 6 boys and 4 girls. How many ways can they be divided into groups of 2 persons, such that there is no group with two girls?

Question 10

Solution:

They can be divided into 5 groups.

Let the groups be Group1, Group2, Group3, Group4, Group5.

Before grouping, all groups are considered the same group.

Question 10

Solution:

Let the girls be Girl1, Girl2, Girl3, Girl4.

We can assume Girl1 is in Group1, Girl2 is in Group2, Girl3 is in Group3, Girl4 is in Group4.

After allocate all girls, all groups will be considered different groups.

Question 10

Solution:

Now, we split 6 boys into the 5 different groups.

$C(6,2) \times 4! =$ 360 ways