CHAPTER 21 Numerics for ODEs and PDEs

SECTION 21.1. Methods for First-Order ODEs, page 898

Purpose. To explain three numerical methods for solving initial value problems $y' = f(x, y)$, $y(x_0) = y_0$ by stepwise computing approximations to the solution at $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, etc.

Main Content, Important Concepts

Euler's method (3)

Automatic variable step size selection

Improved Euler method (8), Table 21.1

Classical Runge–Kutta method (Table 21.3)

Error and step size control

Runge–Kutta–Fehlberg method

Backward Euler's method

Stiff ODEs

Comments on Content

Euler's method is good for explaining the principle but is too crude to be of practical value.

The improved Euler method is a simple case of a predictor–corrector method.

The classical Runge–Kutta method is of order h^4 and is of great practical importance.

Principles for a good choice of h are important in any method.

f in the equation must be such that the problem has a unique solution (see Sec. 1.7).

SOLUTIONS TO PROBLEM SET 21.1, page 907

2. $y = \sin \frac{1}{2}\pi x$. Since the values obtained give $y_9 = 1.01170 > 1$, y_{10} comes out complex and is meaningless.

x_n	y_n	Error
0.1	0.15708	−0.00065
0.2	0.31221	−0.00319
0.3	0.46144	−0.00745
0.4	0.60079	−0.01301
0.5	0.72636	−0.01926
0.6	0.83433	−0.02531
0.7	0.92092	−0.02991
0.8	0.98214	−0.03109
0.9	1.01170	−0.02401
1.0	—	—

4. This is a special Riccati equation. Set $y + x = u$, then $u' = u^2 + 1$ and

$$u'/(u^2 + 1) = 1.$$

By integration, $\arctan u = x + c$ and

$$u = \tan(x + c) = y + x$$

so that

$$y = \tan(x + c) - x$$

and $c = 0$ from the initial condition. Hence $y = \tan x - x$. The calculation is

x_n	y_n	$y(x_n)$	Error
0.1	0.000000	0.000335	0.000335
0.2	0.001000	0.002710	0.001710
0.3	0.005040	0.009336	0.004296
0.4	0.014345	0.022793	0.008448
0.5	0.031513	0.046302	0.014789
0.6	0.059764	0.084137	0.024373
0.7	0.103293	0.142288	0.038996
0.8	0.167821	0.289639	0.061818
0.9	0.261488	0.360158	0.098670
1.0	0.396394	0.557408	0.161014

Although the ODE is similar to that in Prob. 3, the error is greater by about a factor of 10. This is understandable because $\tan x$ becomes infinite as $x \to \frac{1}{2}\pi$.

6. $y = \tan 2x$. Note that the error is first negative and then positive and rapidly increasing, due to the behavior of the tangent.

x_n	y_n	Error $\times 10^5$
0.05	0.10050	-17
0.10	0.20304	-33
0.15	0.30981	-48
0.20	0.42341	-62
0.25	0.54702	-72
0.30	0.68490	-76
0.35	0.84295	-66
0.40	1.02989	-25
0.45	1.25930	$+86$
0.50	1.55379	362

8. The solution is $y = 1/(1 + 4e^{-x})$. The 10S-computation, rounded to 6D, gives

x_n	y_n	$y(x_n)$	Error $y(x_n) - y_n$
0.1	0.216467	0.216481	0.000014
0.2	0.233895	0.233922	0.000028
0.3	0.252274	0.252317	0.000043
0.4	0.271587	0.271645	0.000058
0.5	0.291802	0.291875	0.000073
0.6	0.312876	0.312965	0.000089
0.7	0.334754	0.334858	0.000104
0.8	0.357366	0.357486	0.000119
0.9	0.380633	0.380767	0.000134
1.0	0.404462	0.404610	0.000148

10. We obtain the following with $h = 0.1$ in Prob. 7 and now with $h = 0.05$ and see that the error now decreases to about $\frac{1}{4}$ of its former value, as expected by halfing h in a second-order method.

x	$h = 0.1$	$h = 0.05$
0.1	0.00003	0.00001
0.2	0.00006	0.00001
0.3	0.00009	0.00001
0.4	0.00016	0.00003
0.5	0.00029	0.00005
0.6	0.00054	0.00011
0.7	0.00106	0.00023
0.8	0.00220	0.00051
0.9	0.00488	0.00119
1.0	0.01187	0.00305

12. The solution is $y = 1/(1 + 4e^{-x})$. The computation gives

x_n	y_n	Error $y(x_n) - y_n$
0.1	0.2164806848	$0.043 \cdot 10^{-8}$
0.2	0.2339223328	$0.085 \cdot 10^{-8}$
0.3	0.2523167036	$0.128 \cdot 10^{-7}$
0.4	0.2716446148	$0.170 \cdot 10^{-7}$
0.5	0.2918751118	$0.209 \cdot 10^{-7}$
0.6	0.3129648704	$0.249 \cdot 10^{-7}$
0.7	0.3348578890	$0.284 \cdot 10^{-7}$
0.8	0.3574855192	$0.318 \cdot 10^{-7}$
0.9	0.3807668746	$0.346 \cdot 10^{-7}$
1.0	0.4046096381	$0.370 \cdot 10^{-7}$

To apply (10), we need the calculation of five steps with $h = 0.2$. We obtain (the third column would not be needed)

x_n	y_n	Error $y(x_n) - y_n$
0.2	0.2339222103	$1.310 \cdot 10^{-7}$
0.4	0.2716443697	$2.621 \cdot 10^{-7}$
0.6	0.3129645099	$3.854 \cdot 10^{-7}$
0.8	0.3574850569	$4.941 \cdot 10^{-7}$
1.0	0.4046090919	$5.832 \cdot 10^{-7}$

Hence the required error estimate is

$$(0.4046096381 - 0.4046090919)/15 = 0.364 \cdot 10^{-7}.$$

The actual error is $0.370 \cdot 10^{-7}$; the estimate is much closer to the actual error than one can expect in general.

14. We obtain the solution $y = e^{x-1}/x$, and the computation gives

x_n	y_n	Error $\times 10^9$
1.1	1.0047008	68
1.2	1.0178355	113
1.3	1.0383528	144
1.4	1.0655889	168
1.5	1.0991473	188
1.6	1.1388240	206
1.7	1.1845602	224
1.8	1.2364116	241
1.9	1.2945277	260
2.0	1.3591406	281

It is interesting that, from $x = 1.2$ on, the error is almost a linear function.

16. The solution is $y = 3\cos x - 2\cos^2 x$. The computation gives

x_n	y_n	Error: $y(x_n) - y_n$	Error in Prob. 15
0.2	1.019137566	$0.1173 \cdot 10^{-5}$	$0.74 \cdot 10^{-7}$
0.4	1.066471079	$0.5194 \cdot 10^{-5}$	$0.328 \cdot 10^{-6}$
0.6	1.113634713	$0.14378 \cdot 10^{-4}$	$0.904 \cdot 10^{-6}$
0.8	1.119282901	$0.36749 \cdot 10^{-4}$	$0.228 \cdot 10^{-5}$
1.0	1.036951866	$0.101888 \cdot 10^{-3}$	$0.614 \cdot 10^{-5}$

Note that the ratio of the errors is about the same for all x_n, about 2^4, as can be expected in doubling the step size in the case of a fourth-order method.

18. From $y' = x + y$ and the given formula we get, with $h = 0.2$,

$$\begin{aligned}
k_1 &= 0.2(x_n + y_n) \\
k_2 &= 0.2[x_n + 0.1 + y_n + 0.1(x_n + y_n)] \\
&= 0.2[1.1(x_n + y_n) + 0.1] \\
k_3^* &= 0.2[x_n + 0.2 + y_n - 0.2(x_n + y_n) + 0.4[1.1(x_n + y_n) + 0.1]] \\
&= 0.2[1.24(x_n + y_n) + 0.24]
\end{aligned}$$

and from this

$$y_{n+1} = y_n + \tfrac{1}{6}[1.328(x_n + y_n) + 0.128].$$

The computed values are

x_n	y_n	Error
0.0	0.000000	0.000000
0.2	0.021333	0.000067
0.4	0.091655	0.000165
0.6	0.221808	0.000312
0.8	0.425035	0.000505
1.0	0.717509	0.000771

20. CAS Experiment. (b) The computation gives

x_n	y_n	Error Estimate (10) $\cdot 10^9$	Error $\cdot 10^9$
0.1	1.2003346725	3.0	−0.4
0.2	1.4027100374	3.7	−1.9
0.3	1.6093362546	5.9	−5.0
0.4	1.8227932298	9.4	−11.0
0.5	2.0463025124	13.1	−22.5
0.6	2.2841368531	14.4	−44.7
0.7	2.5422884689	+5.0	−88.4
0.8	2.8296387346	−38.8	−177.6
0.9	3.1601585865	−191.1	−369.0
1.0	3.5574085377	−699.9	−813.0

SECTION 21.2. Multistep Methods, page 908

Purpose. To explain the idea of a multistep method in terms of the practically important Adams–Moulton method, a predictor–corrector method that in each computation uses four preceding values.

Main Content, Important Concepts

Adams–Bashforth method (5)

Adams–Moulton method (7)

Self-starting and not self-starting

Numerical stability, fair comparison

Short Courses. This section may be omitted.

SOLUTIONS TO PROBLEM SET 21.2, page 912

2. The solution is $y = \exp(x^2)$. The computation gives

x_n	y_n	Error $\cdot 10^6$
0.4	1.173518	−7
0.5	1.284044	−19
0.6	1.433365	−35
0.7	1.632375	−59
0.8	1.896573	−92
0.9	2.248047	−139
1.0	2.718486	−205

4. The comparison shows that, in the present case, RK is somewhat better. The comparison is fair since we have four evaluations per step for RK but only two for AM. The errors are

x	0.4	0.6	0.8	1.0
AM	$-0.7 \cdot 10^{-5}$	$-3.5 \cdot 10^{-5}$	$-9.2 \cdot 10^{-5}$	$-20.5 \cdot 10^{-5}$
RK	$0.1 \cdot 10^{-5}$	$0.8 \cdot 10^{-5}$	$4.0 \cdot 10^{-5}$	$17.5 \cdot 10^{-5}$

6. The exact solution is $y = \tan x + x + 1$. The Adams–Moulton calculation gives

x_n	Starting y_n	Predicted y_n^*	Corrected y_n	Exact Values	Error $\cdot 10^6$ of y_n
0.0	1.000000			1.000000	0
0.1	1.200335			1.200335	0.08
0.2	1.402710			1.402710	0.16
0.3	1.609336			1.609336	0.2
0.4		1.822715	1.822798	1.822793	−4.9
0.5		2.046197	2.046315	2.046302	−12.4
0.6		2.283978	2.284161	2.284137	−24.2
0.7		2.542027	2.542332	2.542288	−43.5
0.8		2.829171	2.829714	2.829639	−75.6
0.9		3.159247	3.160288	3.160158	−130
1.0		3.555451	3.557626	3.557408	−218

8. The solution is $y = \frac{1}{2}\tanh 2x$. Computation gives

x_n	y_n	Exact	Error $\cdot 10^6$
0.1	0.098686	0.098688	1
0.2	0.189971	0.189974	3
0.3	0.268519	0.268525	6
0.4	0.332007	0.332018	11
0.5	0.380726	0.380797	71

x_n	y_n	Exact	Error $\cdot 10^6$
0.6	0.416701	0.416827	127
0.7	0.442532	0.442676	144
0.8	0.460706	0.460834	128
0.9	0.473306	0.473403	97
1.0	0.481949	0.482014	65

10. We obtain

x_n	y_n	Error $\cdot 10^9$
1.2	3.07246	−22
1.4	3.15595	−42
1.6	3.24962	−59
1.8	3.35261	−781
2.0	3.46410	−1273

x_n	y_n	Error $\cdot 10^9$
2.2	3.58330	−1564
2.4	3.70945	−1691
2.6	3.84188	−1696
2.8	3.97995	−1615
3.0	4.12311	−1480

14. $y = \exp(x^2)$. Some of the values and errors are

x_n	y_n $(h = 0.05)$	Error $\cdot 10^6$	y_n $(h = 0.1)$	Error $\cdot 10^6$
0.1	1.010050		1.01005	
0.2	1.040817	−6	1.040811	
0.3	1.094188	−14	1.094224	−50
0.4	1.173535	−24	1.173623	−112
0.5	1.284064	−38	1.284219	−194
0.6	1.433388	−58	1.433636	−307
0.7	1.632404	−87	1.632782	−466
0.8	1.896612	−131	1.897175	−694
0.9	2.248105	−197	2.248931	−1023
1.0	2.718579	−297	2.719785	−1503

The errors differ by a factor of 4 to 5, approximately.

SECTION 21.3. Methods for Systems and Higher Order ODEs, page 912

Purpose. Extension of the methods in Sec. 21.1 to first-order systems and to higher order ODEs.

Content

Euler's method for systems (5)

Classical Runge–Kutta method extended to systems (6)

Runge–Kutta–Nyström method (7)

Short Courses. Discuss merely Runge–Kutta (6), which shows that this "vectorial extension" of the method is conceptually quite simple.

SOLUTIONS TO PROBLEM SET 21.3, page 919

2. The exact solution is (see Fig. 86 in Sec. 4.3)

$$y_1 = 4e^{-x} \sin x$$
$$y_2 = 4e^{-x} \cos x.$$

Computation gives the following values and errors:

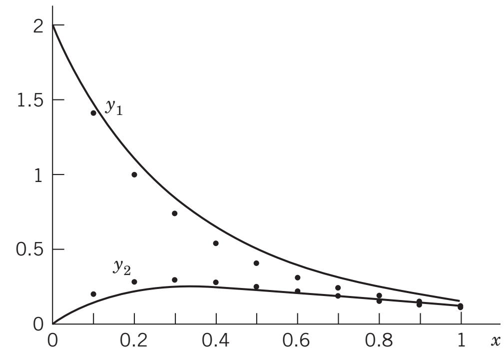
x	y_1	Error	y_2	Error
0	0	0	4	0
0.2	0.8	−0.149	3.2	0.01
0.4	1.28	−0.236	2.40	0.07
0.6	1.504	−0.264	1.664	0.148
0.8	1.5360	−0.2467	1.0304	0.2218
1.0	1.3488	−0.19664	0.51712	0.27794

We see that the values are much too inaccurate to be of any practical value, and we recall that we briefly mentioned this method only for illustrating that the extension from a single ODE to a system is quite straightforward.

4. The solution $y_1 = e^{-2x} + e^{-4x}$, $y_2 = e^{-2x} - e^{-4x}$. The computation is

x_n	$y_{1,n}$	Error	$y_{2,n}$	Error
0.1	1.4	0.0890	0.2	−0.0516
0.2	1.00	0.1196	0.28	−0.0590
0.3	0.728	0.1220	0.296	−0.0484
0.4	0.5392	0.1120	0.2800	−0.0326
0.5	0.4054	0.0978	0.2499	−0.0174

The figure shows (for $x = 0, \cdots, 1$) that these values give a qualitatively correct impression, although they are rather inaccurate. Note that the error of y_1 is not monotone.

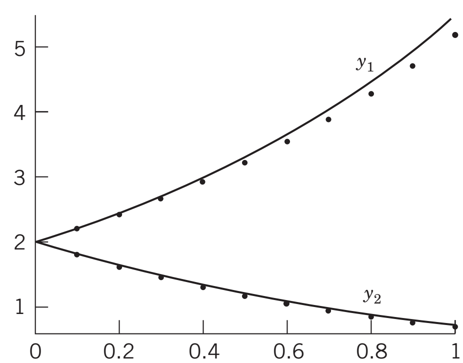


Section 21.3. Solution curves and computed values in Problem 4

6. Solution $y_1 = 2e^x$, $y_2 = 2e^{-x}$ (see also Example 3 in Sec. 4.3). The computation is

x_n	$y_{1,n}$	Error	$y_{2,n}$	Error
0.1	2.2	0.0104	1.8	0.0097
0.2	2.42	0.0228	1.62	0.0175
0.3	2.662	0.0378	1.458	0.0236
0.4	2.9282	0.0554	1.3122	0.0284
0.5	3.2210	0.0764	1.1810	0.0321
0.6	3.5431	0.1011	1.0629	0.0347
0.7	3.8974	0.1302	0.95659	0.03659
0.8	4.2872	0.1638	0.86093	0.03773
0.9	4.7159	0.2033	0.77484	0.03830
1.0	5.1875	0.2491	0.69736	0.03840

The figure illustrates that the error of y_1 is monotone increasing and is positive (the points lie below that curve), and similarly for y_2.



Section 21.3. Solution curves and computed values (the dots) in Problem 6

8. The errors are smaller by a factors 10^3 to 10^5. Computation gives

x	y_1	Error $\cdot\ 10^5$	y_2	Error $\cdot\ 10^5$
0	0	0	4	0
0.2	0.650667	−4.0	3.2096	4.3
0.4	1.044190	−5.0	2.469541	8.1
0.6	1.239570	−4.1	1.811705	11.0
0.8	1.289335	−2.0	1.252075	12.7
1.0	1.238233	+0.6	0.794934	13.1

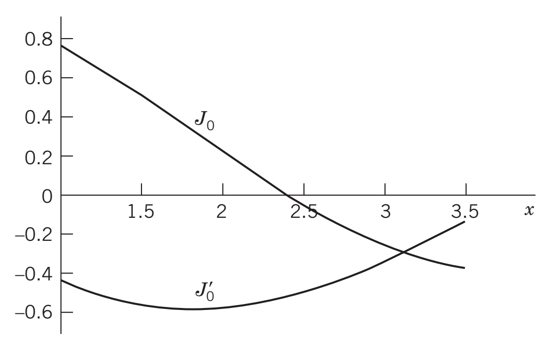
10. The errors are smaller by a factor 10^3, approximately. The computation gives

x_n	$y_{1,n}$	Error $\cdot\ 10^5$	$y_{2,n}$	Error $\cdot\ 10^5$
0	2	0	0	0
0.1	1.489133	−8	0.148333	8
0.2	1.119760	−11	0.220888	10
0.3	0.850119	−11	0.247515	10
0.4	0.651328	−10	0.247342	9
0.5	0.503301	−9	0.232469	7

12. Division by x gives $y'' + y'/x + y = 0$. The system is $y_1' = y_2, y_2' = y'' = -y_2/x - y_1$. Because of the factor $1/x$ we have to choose $x_0 \neq 0$. The computation gives, with the initial values taken from Ref. [GenRef1] in App. 1:

x_n	$J_0(x_n)$	$J_0'(x_n)$	$10^6 \cdot$ Error of $J_0(x_n)$
1	0.765198	−0.440051	0
1.5	0.511903	−0.558002	−76
2	0.224008	−0.576897	−117
2.5	−0.048289	−0.497386	−95
3	−0.260055	−0.339446	+3
3.5	−0.380298	−0.137795	170

Note that, although the step is very large, the values are relatively accurate. Also, recall that $J_0' = -J_1$.



Section 21.3. Solution curves in Problem 12. Computed values lie practically on the curves.

14. We obtain

$$
\begin{aligned}
k_1 &= 0.1x_n y_n \\
k_2 = k_3 &= 0.1(x_n + 0.1)(y_n + 0.1y_n' + 0.05k_1) \\
k_4 &= 0.1(x_n + 0.2)(y_n + 0.2y_n' + 0.2k_2)
\end{aligned}
$$

and the following numeric values, whose accuracy is about the same as that in Example 2, but the work was much less.

x_n	y_n	y_n'	$y(x)$ Exact (8D)	$10^8 \cdot$ Error of y_n
0.0	0.35502806	−0.25881940	0.35502806	0
0.2	0.30370304	−0.25240464	0.30370315	11
0.4	0.25474212	−0.23583070	0.25474235	23
0.6	0.20979975	−0.21279172	0.20980006	31
0.8	0.16984600	−0.18641134	0.16984632	32
1.0	0.13529219	−0.15914608	0.13529242	23

SECTION 21.4. Methods for Elliptic PDEs, page 919

Purpose. To explain numerical methods for the Dirichlet problem involving the Laplace equation, the typical representative of elliptic PDEs.

Main Content, Important Concepts

Elliptic, parabolic, hyperbolic equations

Dirichlet, Neumann, mixed problems

Difference analogs (7), (8) of Poisson's and Laplace's equations

Coefficient scheme (9)

Liebmann's method of solution (identical with Gauss–Seidel, Sec. 20.3)

Peaceman–Rachford's ADI method (15)

Short Courses. Omit the ADI method.

Comments on Content

Neumann's problem and the mixed problem follow in the next section, including the modification in the case of irregular boundaries.

The distinction between the three kinds of PDEs (elliptic, parabolic, hyperbolic) is not merely a formal matter because the solutions of the three types behave differently in principle, and the boundary and initial conditions are different; this necessitates different numerical methods, as we shall see.

SOLUTIONS TO PROBLEM SET 21.4, page 927

4. $u_{11} = 92.86$, $u_{21} = 90.18$, $u_{12} = 81.25$, $u_{22} = 75.00$, $u_{13} = 57.14$, $u_{23} = 47.32$, $u_{31} = u_{11}$, etc., by symmetry

6. Gauss gives the values of the exact solution $u(x, y) = x^3 - 3xy^2$. Gauss–Seidel needs about 10 steps for producing 5S-values:

n	u_{11}	u_{21}	u_{12}	u_{22}
1	50.25	44.062	31.062	5.031
2	19.031	12.516	−0.484	−10.742
3	3.2578	4.6289	−8.371	−14.686
4	−0.6855	2.6572	−10.343	−15.671
5	−1.6714	2.1643	−10.836	−15.918
6	−1.9178	2.0411	−10.959	−15.979
7	−1.9795	2.0103	−10.990	−15.995
8	−1.9949	2.0026	−10.997	−15.999
9	−1.9987	2.0006	−10.999	−16.000
10	−1.9997	2.0002	−11.000	−16.000

It is interesting that it takes only 4 or 5 steps to turn the values away from the starting values to values that are already relatively close to the respective limits.

8. 165, 165, 165, 165 by Gauss. The Gauss–Seidel computation gives

n	u_{11}	u_{21}	u_{12}	u_{22}
1	132.50	140.63	140.63	152.81
2	152.81	158.91	158.91	161.95
3	161.95	163.48	163.48	164.24
4	164.24	164.62	164.62	164.81
5	164.81	164.90	164.90	164.95
6	164.95	164.98	164.98	164.99
7	164.99	165.00	165.00	165.00
8	165.00	165.00	165.00	165.00

10. The values of the exact solution of the Laplace equation are

$$u(1, 1) = -4, \qquad u(2, 1) = u(1, 2) = -7, \qquad u(2, 2) = -64.$$

Gauss gives $-2, -5, -5, -62$. Corresponding errors are $-2, -2, -2, -2$. Gauss–Seidel needs about 10 steps for producing 5S-values:

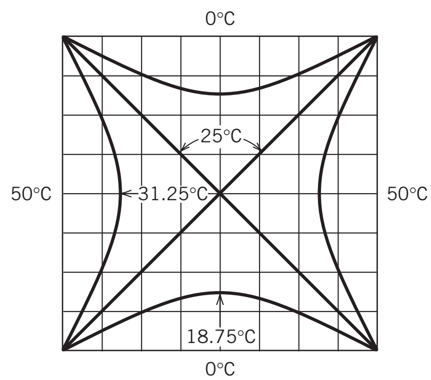
n	u_{11}	u_{21}	u_{12}	u_{22}
1	50.50	48.625	48.625	−35.188
2	24.8125	8.406	8.406	−55.297
3	4.7031	−1.648	−1.648	−60.324
4	−0.3242	−4.162	−4.162	−61.581
5	−1.5811	−4.791	−4.791	−61.895
6	−1.8953	−4.948	−4.948	−61.974
7	−1.9738	−4.987	−4.987	−61.993
8	−1.9935	−4.997	−4.997	−61.998
9	−1.9984	−4.999	−4.999	−62.000
10	−1.9996	−5.000	−5.000	−62.000

12. This shows the importance of good starting values; it then does not take long until the approximations come close to the solution. A rule of thumb is to take a rough estimate of the average of the boundary values at the points that enter the linear system. By starting from **0** we obtain

$$[0.09472 \quad 0.10149 \quad 0.31799 \quad 0.32138] \qquad \text{(Step 3)}$$
$$[0.10741 \quad 0.10783 \quad 0.32433 \quad 0.32455] \qquad \text{(Step 5).}$$

14. All the isotherms must begin and end at a corner. The diagonals are isotherms $u = 25$, because of the data obtained and for reasons of symmetry. Hence we obtain a qualitative picture as the figure shows.

In Prob. 7 the situation is similar.



Section 21.4. Problem 14

16. ***Step 1.*** First come ***rows*** $j = 1, j = 2$; for these, (14a) is

$$\begin{aligned}
j = 1, \quad i = 1. \quad & u_{01} - 4u_{11} + u_{21} = -u_{10} - u_{12} \\
i = 2. \quad & u_{11} - 4u_{21} + u_{31} = -u_{20} - u_{22} \\
j = 2, \quad i = 1. \quad & u_{02} - 4u_{12} + u_{22} = -u_{11} - u_{13} \\
i = 2. \quad & u_{12} - 4u_{22} + u_{32} = -u_{21} - u_{22}.
\end{aligned}$$

Six of the boundary values are zero, and the two on the upper edge are

$$u_{13} = u_{23} = \sqrt{3}/2 = 0.866025.$$

Also, on the right we substitute the starting values 0. With this, our four equations become

$$\begin{aligned}
-4u_{11} + \quad u_{21} &= 0 \\
u_{11} - 4u_{21} &= 0 \\
-4u_{12} + \quad u_{22} &= -0.866025 \\
u_{12} - 4u_{22} &= -0.866025.
\end{aligned}$$

From the first two equations,

$$u_{11} = 0, \qquad u_{21} = 0$$

and from the other two equations,

$$u_{12} = 0.288675, \qquad u_{22} = 0.288675.$$

Step 1. Now come ***columns***; for these, (14b) is

$$
\begin{aligned}
i = 1, \quad j = 1. \quad & u_{10} - 4u_{11} + u_{12} = -u_{01} - u_{21} \\
j = 2. \quad & u_{11} - 4u_{12} + u_{13} = -u_{02} - u_{22} \\
i = 2, \quad j = 1. \quad & u_{20} - 4u_{21} + u_{22} = -u_{11} - u_{31} \\
j = 2. \quad & u_{21} - 4u_{22} + u_{23} = -u_{12} - u_{32}.
\end{aligned}
$$

With the boundary values and the previous solution ***on the right***, this becomes

$$
\begin{aligned}
-4u_{11} + \quad u_{12} &= 0 \\
u_{11} - 4u_{12} &= -0.866025 - 0.288675 \\
-4u_{21} + \quad u_{22} &= 0 \\
u_{21} - 4u_{22} &= -0.866025 - 0.288675.
\end{aligned}
$$

The solution is

$$
\begin{aligned}
u_{11} &= 0.07698 \\
u_{21} &= 0.07698 \\
u_{12} &= 0.30792 \\
u_{22} &= 0.30792.
\end{aligned}
$$

Step 2. Rows. We can use the previous equations, changing only the right sides:

$$
\begin{aligned}
-4u_{11} + \quad u_{21} &= -0.30792 \\
u_{11} - 4u_{21} &= -0.30792 \\
-4u_{12} + \quad u_{22} &= -0.866025 - 0.07698 \\
u_{12} - 4u_{22} &= -0.866025 - 0.07698
\end{aligned}
$$

Solution:

$$
u_{11} = u_{21} = 0.102640, \qquad u_{12} = u_{22} = 0.314335.
$$

Step 2. Columns. The equations with the new right sides are

$$
\begin{aligned}
-4u_{11} + \quad u_{12} &= -0.102640 \\
u_{11} - 4u_{12} &= -0.866025 - 0.314335 \\
-4u_{21} + \quad u_{22} &= -0.102640 \\
u_{21} - 4u_{22} &= -0.866025 - 0.314335.
\end{aligned}
$$

Final result (solution of these equations):

$$
\begin{aligned}
u_{11} &= 0.106061 \\
u_{21} &= 0.106061 \\
u_{12} &= 0.321605 \\
u_{22} &= 0.321605.
\end{aligned}
$$

Exact 3D values:

$$
u_{11} = u_{21} = 0.108, \qquad u_{12} = u_{22} = 0.325.
$$

18. CAS Project. (b) The solution of the linear system (rounded to integers), with the values arranged as the points in the xy-plane, is

160	170	157	110
138	145	125	75
138	145	125	75
160	170	157	110

Twenty steps gave accuracies of 3S–5S, with slight variations, between the components of the output vector.

SECTION 21.5. Neumann and Mixed Problems. Irregular Boundary, page 928

Purpose. Continuing our discussion of elliptic PDEs, we explain the ideas needed for handling Neumann and mixed problems and the modifications required when the domain is no longer a rectangle.

Main Content, Important Concepts

Mixed problem for a Poisson equation (Example 1)

Modified stencil (6) (notation in Fig. 460)

Comments on Content

Neumann's problem can be handled as explained in Example 1 on the mixed problem.

In all the cases of an elliptic PDE we need only *one* boundary condition at each point (given u or given u_n).

SOLUTIONS TO PROBLEM SET 21.5, page 932

2. The exact solution of the Poisson equation is $u = x^2y^2$. The approximate solution results from $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 10 \\ 8 \\ 1 \\ -20 \\ -103 \end{bmatrix}$$

where the six equations correspond to $P_{11}, P_{21}, P_{31}, P_{12}, P_{22}, P_{32}$, in our usual order. The components of $\mathbf{b}$ are of the form $a - c$ with a resulting from $2(x^2 + y^2)$ and c from the boundary values; thus, $4 - 0 = 4$, $10 - 0 = 10$, $20 - 12 = 8$, $10 - 9 = 1$, $16 - 36 = -20$, $26 - 81 - 48 = -103$. The solution of this system agrees with the values obtained at the P_{jk} from the exact solution, $u_{11} = 1$, $u_{21} = u_{12} = 4$, $u_{22} = 16$, and $u_{31} = 9$, $u_{32} = 36$ on the boundary. $u_{41} = u_{21} + 12$ and $u_{42} = u_{22} + 48$ produced entries 2 in $\mathbf{A}$ and -12 and -48 in $\mathbf{b}$.

4. $0 = u_{01,x} = \dfrac{1}{2h}(u_{11} - u_{-1,1})$ gives $u_{-1,1} = u_{11}$. Similarly, $u_{41} = u_{21} + 3$ from the condition on the right edge, so that the equations are

$$\begin{aligned}
-4u_{01} + 2u_{11} \qquad\qquad &= 1\\
u_{01} - 4u_{11} + u_{21} \qquad &= -0.25 + 0.75 = 0.5\\
u_{11} - 4u_{21} + u_{31} &= -1\\
2u_{21} - 4u_{31} &= -2.25 - 1.25 - 3 = -6.5.
\end{aligned}$$

$u_{01} = -0.25$, $u_{11} = 0$, $u_{21} = 0.75$, $u_{31} = 2$; this agrees with the values of the exact solution $u(x, y) = x^2 - y^2$ of the problem.

6. Exact solution $u = 9y \sin \frac{1}{3}\pi x$. Linear system $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0\\ 1 & -4 & 0 & 1 & 0 & 0\\ 1 & 0 & -4 & 1 & 1 & 0\\ 0 & 1 & 1 & -4 & 0 & 1\\ 0 & 0 & 2 & 0 & -4 & 1\\ 0 & 0 & 0 & 2 & 1 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} a\\ a\\ 2a\\ 2a\\ 3a + c\\ 3a + c \end{bmatrix}$$

$a = -8.54733$, $c = -\sqrt{243} = -15.5885$. The solution of this system is (exact values of u in parentheses)

$$\begin{aligned}
u_{11} = u_{21} &= 8.46365 \quad (\text{exact } \tfrac{9}{2}\sqrt{3} = 7.79423)\\
u_{12} = u_{22} &= 16.8436 \quad (\text{exact } 9\sqrt{3} = 15.5885)\\
u_{13} = u_{23} &= 24.9726 \quad (\text{exact } \tfrac{27}{2}\sqrt{3} = 23.3827).
\end{aligned}$$

14. Let v denote the unknown boundary potential. Then v occurs in $\mathbf{Au} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & 1 & 0\\ 1 & -4 & 0 & 1\\ 1 & 0 & -4 & 1\\ 0 & \frac{2}{3} & \frac{2}{3} & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0\\ -v\\ -v\\ -\frac{8}{3}v \end{bmatrix}.$$

The solution of this linear system is $\mathbf{u} = \dfrac{v}{19}[5 \quad 10 \quad 10 \quad 16]^{\mathsf{T}}$. From this and $5v/19 = 220$ (the potential at P_{11}) we have $v = 836$ V as the constant boundary potential on the indicated portion of the boundary.

16. Two equations are as usual:

$$\begin{aligned}
-4u_{11} + u_{21} + u_{12} - 2 &= 2\\
u_{11} - 4u_{21} \qquad - 0.5 &= 2
\end{aligned}$$

where the right side is due to the fact that we are dealing with the Poisson equation. The third equation results from (6) with $a = p = q = 1$ and $b = \frac{1}{2}$. We get

$$2\left[\frac{u_{22}}{2} + \frac{u_{1.5/2}}{\frac{3}{4}} + \frac{u_{02}}{2} + \frac{u_{11}}{\frac{3}{2}} - \frac{\frac{3}{2}}{\frac{1}{2}}u_{12}\right] = 2.$$

The first two terms are zero and $u_{02} = -2$; these are given boundary values. There remains

$$\tfrac{2}{3}u_{11} - 3u_{12} = 1 + 1 = 2.$$

Our three equations for the three unknowns have the solution

$$u_{11} = -1.5, \qquad u_{21} = -1, \qquad u_{12} = -1.$$

SECTION 21.6. Methods for Parabolic PDEs, page 933

Purpose. To show the numerical solution of the heat equation, the prototype of a parabolic equation, on the region given by $0 \leqq x \leqq 1, t \geqq 0$, subject to one initial condition (initial temperature) and one boundary condition on each of the two vertical boundaries.

Content

Direct method based on (5), convergence condition (6)

Crank–Nicolson method based on (8)

Special case (9) of (8)

Comment on Content

Condition (6) restricts the size of time steps too much, a disadvantage that the Crank–Nicolson method avoids.

SOLUTIONS TO PROBLEM SET 21.6, page 938

4. CAS Experiment. $u(0, t) = u(1, t) = 0$, $u(0.2, t) = u(0.8, t)$, $u(0.4, t) = u(0.6, t)$, where

	$x = 0.2$	$x = 0.4$	
$t = 0$	0.587785	0.951057	
$t = 0.04$	0.393432	0.636586	Explicit
	0.399274	0.646039	CN
	0.396065	0.640846	Exact (6D)
$t = 0.08$	0.263342	0.426096	
	0.271221	0.438844	
	0.266879	0.431818	
$t = 0.12$	0.176267	0.285206	
	0.184236	0.298100	
	0.179830	0.290970	
$t = 0.16$	0.117984	0.190901	
	0.125149	0.202495	
	0.121174	0.196063	
$t = 0.2$	0.078972	0.127779	
	0.085012	0.137552	
	0.081650	0.132112	

6. Note that $h = 0.2$ and $k = 0.01$ gives $r = 0.25$. The computation gives

t	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
0.00	0.2	0.4	0.4	0.2
0.01	0.2	0.35	0.35	0.2
0.02	0.1875	0.3125	0.3125	0.1875
0.03	0.171875	0.281250	0.281250	0.171875
0.04	0.156250	0.253906	0.253906	0.156250
0.05	0.141602	0.229492	0.229492	0.141602
0.06	0.128174	0.207520	0.207520	0.128174
0.07	0.115967	0.187683	0.187683	0.115967
0.08	0.104904	0.169754	0.169754	0.104904

8. $u(x, 0) = u(1 - x, 0)$ and the boundary conditions imply $u(x, t) = u(1 - x, t)$ for all t. The calculation gives

$$(0, 0.2, 0.35, 0.35, 0.2, 0)$$
$$(0, 0.1875, 0.3125, 0.3125, 0.1875, 0)$$
$$(0, 0.171875, 0.28125, 0.28125, 0.171875, 0)$$
$$(0, 0.156250, 0.253906, 0.253906, 0.156250, 0)$$
$$(0, 0.141602, 0.229492, 0.229492, 0.141602, 0)$$

10. The boundary condition on the left is that the normal derivative is zero. We have, by (5),

$$u_{0,j+1} = (1 - 2r)u_{0j} + r(u_{1j} + u_{-1j}).$$

Now, by the central difference formula (see hint) for the normal derivative (partial derivative with respect to x) we get

$$0 = \frac{\partial u_{0j}}{\partial x} = \frac{1}{2h}(u_{1j} - u_{-1j}).$$

Hence, the previous formula becomes

$$u_{0,j+1} = (1 - 2r)u_{0j} + 2ru_{1j}.$$

We have $r = 0.25$ so

$$u_{0,j+1} = \tfrac{1}{2}(u_{0j} + u_{1j}).$$

The underlying idea is quite similar to that in Sec. 21.5. The computation gives

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1$
0	0	0	0	0	0	0
0.01	0	0	0	0	0	0.5
0.02	0	0	0	0	0.125	0.866
0.03	0	0	0	0.031	0.279	1
0.04	0	0	0.008	0.085	0.397	0.866
0.05	0	0.002	0.025	0.144	0.437	0.5
0.06	0.001	0.007	0.049	0.187	0.379	0
0.07	0.004	0.016	0.073	0.201	0.236	−0.5
0.08	0.010	0.027	0.091	0.178	0.043	−0.866
0.09	0.019	0.039	0.097	0.122	−0.150	−1
0.10	0.029	0.048	0.089	0.048	−0.295	−0.866
0.11	0.039	0.054	0.068	−0.028	−0.352	−0.5
0.12	0.046	0.054	0.041	−0.085	−0.308	0

12. We obtain (10S-computation, rounded to 4D)

t	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
0.04	0.1636	0.2545	0.2545	0.1636
0.08	0.1074	0.1752	0.1752	0.1074
0.12	0.0735	0.1187	0.1187	0.0735
0.16	0.0498	0.0807	0.0807	0.0498
0.20	0.0339	0.0548	0.0548	0.0339
Exact: 0.20	0.0331	0.0535	0.0535	0.0331

14. We need the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}.$$

3S-values computed by Crank–Nicolson for $x = 0.1$ (and 0.9), 0.2 (and 0.8), 0.3 (and 0.7), 0.4 (and 0.6), 0.5 and $t = 0.01, 0.02, \cdots, 0.05$ are

0.0754	0.141	0.190	0.220	0.230
0.0669	0.126	0.172	0.201	0.210
0.0600	0.114	0.156	0.182	0.192
0.0541	0.103	0.141	0.166	0.174
0.0489	0.093	0.128	0.150	0.158

5S-values for $t = 0.04$ are

$$0.054112 \qquad 0.10275 \qquad 0.14115 \qquad 0.16565 \qquad 0.17407.$$

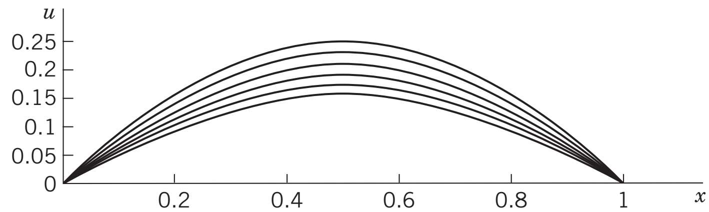
The corresponding values in Prob. 15 are

$$0.10182 \qquad\qquad 0.16727.$$

Exact 5S-values computed by (9) and (10) in Sec. 12.5 (two nonzero terms suffice) are

$$0.053946 \qquad 0.10245 \qquad 0.14074 \qquad 0.16519 \qquad 0.17358.$$

We see that the present values are better than those in Prob. 15.



Section 21.6. $u(x, t)$ for constant $t = 0.01, \cdots, 0.05$ as polygons with the Crank–Nicolson values as vertices in Problem 14

SECTION 21.7. Method for Hyperbolic PDEs, page 939

Purpose. Explanation of the numerical solution of the wave equation, the prototype of a hyperbolic PDE, on a region of the same type as in the last section, subject to initial and boundary conditions that guarantee the uniqueness of the solution.

Comments on Content

We now have two initial conditions (given initial displacement and given initial velocity), in contrast to the heat equation in the last section, where we had only one initial condition.

The computation by (6) is simple. Formula (8) gives the values of the first time-step in terms of the initial data.

SOLUTIONS TO PROBLEM SET 21.7, page 941

2. Computation, rounded to 3D, gives

t	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$
0	0.032	0.096	0.144	0.128
0.2	0.048	0.088	0.112	0.072
0.4	0.056	0.064	0.016	−0.016
0.6	0.016	−0.016	−0.064	−0.056
0.8	−0.072	−0.112	−0.088	−0.048
1.0	−0.128	−0.144	−0.096	−0.032
1.2	−0.072	−0.112	−0.088	−0.048
1.4	0.016	−0.016	−0.064	−0.056
1.6	0.056	0.064	0.016	−0.016
1.8	0.048	0.088	0.112	0.072
2.0	0.032	0.096	0.144	0.128

4. Since $u(x, 0) = f(x)$, the derivation is immediate. Formula (9) results if the integral equals $2kg_i$.

6. Exact solution $u(x, t) = (x + t)^2$. The values obtained in the computation are those of the exact solution. $u_{11}, u_{21}, u_{31}, u_{41}$ are obtained from (8) and the initial conditions

$$u_{i0} = (0.2i)^2, \quad g_i = 2(0.2i), \qquad i = 0, \cdots, 5.$$

In connection with the left boundary condition we can use the central difference formula

$$\frac{1}{2h}(u_{1,j} - u_{-1,j}) \approx u_x(0, jk) = 2jk$$

to obtain $u_{-1,j}$ and then (8) to compute u_{01} and (6) to compute $u_{0,j+1}$.

8. From (12), Sec. 12.4, with $c = 1$ we get the exact solution

$$u(x, t) = \frac{1}{2} \int_{x-ct}^{x+ct} \sin \pi s \, ds = \frac{1}{2\pi} [\cos \pi(x - ct) - \cos \pi(x + ct)].$$

From (8) we have $kg_i = 0.1g_i = 0.1 \sin 0.1\pi i$. Because of the symmetry with respect to $x = 0.5$ we need to list only the following values (with the exact values in parentheses):

t	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
0.0	0	0	0	0	0
0.1	0.030902	0.058779	0.080902	0.095106	0.100000
	(0.030396)	(0.057816)	(0.079577)	(0.093549)	(0.098363)
0.2	0.058779	0.111803	0.153884	0.180902	0.190211
	(0.057816)	(0.109973)	(0.151365)	(0.177941)	(0.187098)
0.3	0.080902	0.153884	0.211803	0.248990	0.261803
	(0.079577)	(0.151365)	(0.208337)	(0.244914)	(0.257518)
0.4	0.095106	0.180902	0.248990	0.292705	0.307768
	(0.093549)	(0.177941)	(0.244914)	(0.287914)	(0.302731)

10. By (13), Sec. 12.4, with $c = 1$ the left side of (6) is

(A) $$u_{i,j+1} = u(ih, (j + 1)h) = \tfrac{1}{2}[f(ih + (j + 1)h) + f(ih - (j + 1)h)]$$

and the right side is the sum of the six terms

$$\begin{aligned} u_{i-1,j} &= \tfrac{1}{2}[f((i - 1)h + jh) + f((i - 1)h - jh)], \\ u_{i+1,j} &= \tfrac{1}{2}[f(i + 1)h + jh) + f((i + 1)h - jh)], \\ -u_{i,j-1} &= -\tfrac{1}{2}[f(ih + (j - 1)h) + f(ih - (j - 1)h)]. \end{aligned}$$

Four of these six terms cancel in pairs, and the remaining expression equals the right side of (A).

SOLUTIONS TO CHAPTER 21 REVIEW QUESTIONS AND PROBLEMS, page 942

18. $y = e^x$. Computed values are

x_n	y_n	$y(x_n)$	Error $\cdot 10^6$	Error in Prob. 17
0.01	1.010000	1.010050	50	
0.02	1.020100	1.020201	101	
0.03	1.030301	1.030455	154	
0.04	1.040604	1.040811	207	
0.05	1.051010	1.051271	261	
0.06	1.061520	1.061837	312	
0.07	1.072135	1.072508	373	
0.08	1.082857	1.083287	430	
0.09	1.093685	1.094174	489	
0.10	1.104622	1.105171	549	0.005171

We see that the error of the last value has decreased by a factor of 10, approximately, due to the smaller step.

20. $y = 2e^{-x} + x^2 + 1$.

x_n	y_n	Error $\cdot 10^4$
0.1	2.8205	−8
0.2	2.6790	−15
0.3	2.5738	−22
0.4	2.5033	−27
0.5	2.4662	−32
0.6	2.4612	−36
0.7	2.4871	−39
0.8	2.5428	−42
0.9	2.6276	−44
1.0	2.7404	−46

22. **(a)** 0.1, 0.2034, 0.3109, 0.4217, 0.5348, 0.6494, 0.7649, 0.8806, error 0.0044, 0.0122, $\cdots$, 0.0527. **(b)** 0.2055, 0.4276, 0.6587, 0.8924, error 0.0101, 0.0221, 0.0322, 0.0409. **(c)** 0.4352, 0.9074, error 0.0145, 0.0258

24. Solution $y = \tan x - x + 4$.

x_n	y_n	Error $\cdot 10^6$
0.8	4.22969	−52
1.0	4.55686	+548

The starting values were obtained by classical Runge–Kutta.

26. Exact solution $4y_1^2 + y_2^2 = 16$ (ellipse). The computation gives

x_n	$y_{1,n}$	$y_{2,n}$
0	2	0
0.2	2	−1.6
0.4	1.68	−3.2
0.6	1.04	−4.544
0.8	0.1312	−5.376
1.0	−0.9440	−5.4810
1.2	−2.0402	−4.7258
1.4	−2.9853	−3.0936
1.6	−3.6041	−0.7053
1.8	−3.7451	2.1779
2.0	−3.3095	5.1740

28. The exact solution is $y_1 = -6e^{9t} + 3e^{3t}$, $y_2 = -2e^{9t} - e^{3t}$.
The computation gives

x_n	$y_{1,n}$	$(y_{1,n})$	$y_{2,n}$	$(y_{2,n})$
0	−3.00000	0	−3.00000	0
0.05	−5.92338	−0.00099	−4.29813	−0.00033
0.10	−10.70492	−0.00312	−6.26802	−0.00104
0.15	−18.43227	−0.00734	−9.28071	−0.00245

30. From the 3D-values given below we see that at each point $x > 0$ the temperature oscillates with a phase lag and a maximum amplitude that decreases with decreasing x.

t	$x = 0$	$x = 0.2$	$x = 0.4$	$x = 0.6$	$x = 0.8$	$x = 1.0$
0	0	0	0	0	0	0
0.02	0	0	0	0	0	0.5
0.04	0	0	0	0	0.250	0.866
0.06	0	0	0	0.125	0.433	1
0.08	0	0	0.063	0.217	0.563	0.866
0.10	0	0.031	0.108	0.313	0.541	0.5
0.12	0	0.054	0.172	0.325	0.406	0
0.14	0	0.086	0.189	0.289	0.162	−0.5
0.16	0	0.095	0.188	0.176	−0.105	−0.866
0.18	0	0.094	0.135	0.041	−0.345	−1
0.20	0	0.068	0.067	−0.105	−0.479	−0.866
0.22	0	0.034	−0.019	−0.206	−0.485	−0.5
0.24	0	−0.009	−0.086	−0.252	−0.353	0

32. $u(P_{21}) = 500$, $u(P_{22}) = 200$, $u = 100$at all other gridpoints

34. $u(P_{11}) = u(P_{22}) = u(P_{33}) = 35$, $u(P_{21}) = u(P_{32}) = 20$,
$u(P_{31}) = 10$, $u(P_{12}) = u(P_{23}) = 50$, $u(P_{13}) = 60$