CHAPTER 5 Series Solutions of ODEs. Special Functions

Changes of Text

Extensive changes have been made in this chapter. Section 5.1 has been rewritten and the material on the theory of the power series method (the previous Sec. 5.2) has been incorporated into it. The material on the Sturm–Liouville problem, orthogonal functions, and orthogonal eigenfunction expansions (the previous Secs. 5.7 and 5.8) has been moved to Chap. 11 where it is a natural extension of the discussions about Fourier series. The overview of some techniques required for **higher special functions** and the frequent need for a CAS in exploring them remains.

SECTION 5.1. Power Series Method, page 167

Purpose. A simple introduction to the technique of the power series method in terms of simple examples whose solution the student knows very well. Of course, one should emphasize repeatedly that for simple ODEs, such as that in Example 2, one does not need the present method, whereas Example 3 is a first case in which we do need it.

Main Content, Important Concepts

Power series (1) and (2)

Basic examples known from calculus (Example 1, geometric series)

ODE (4) [see also (12)] to be solved by inserting (2), (3), (5)

Special Legendre equation (Example 3)

Partial sum, remainder, convergence

Convergence, convergence interval

Radius of convergence (Example 4)

Theorem 1: Existence of power series solutions

Operations on power series

Problem Set 5.1

For reviewing power series in more detail the student should use a calculus book, preferably his or her own.

To mention Airy functions in Prob. 10 will do no harm and will give the student the impression that much research has been done on special functions and their power series, most of which are known to the usual CAS.

This impression if further deepened and expanded in CAS Probs. 18–19.

A figure such as 106 should be familiar to the student from calculus.

SOLUTIONS TO PROBLEM SET 5.1, page 174

2. 1

4. ∞

6. $y(x) = A(1 + 2x + x^2)$

7. $y(x) = A(1 - 2x^2 + 2x^4 + \cdots) = Ae^{-2x^2}$

8. $y(x) = -\frac{1}{4}k + Ax^4$

10. We obtain

$$a_0(1 - \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 - \tfrac{1}{120}x^5 + \cdots) + a_1(x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{24}x^4 - \tfrac{1}{30}x^5 - \cdots).$$

This solution can be expressed in terms of Airy functions,

$$y = e^{x/2}(a_0 \operatorname{Ai}(\tfrac{1}{4} - x) + a_1 \operatorname{Bi}(\tfrac{1}{4} - x));$$

but this representation is rather complicated, so that for numeric purposes it will be practical to use the power series (a partial sum with sufficiently many terms) directly.

11. $y(x) = A(1 - 1/12x^4 + \frac{1}{60}x^5 + \cdots)$
$\quad + B(x - 1/2x^2 + 1/6x^3 - 1/24x^4 - 1/24x^5 + \cdots)$

12. $y = a_1x + a_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - \cdots)$. [This is a particular case of Legendre's equation ($n = 1$), which we consider in Sec. 5.2.]

14. $y = (a_0 + a_1x)\left(1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \cdots\right) = (a_0 + a_1x)e^{x^2}$

16. $s = \frac{5}{4} - 4x + 8x^2 - \frac{32}{3}x^3 + \frac{32}{3}x^4 - \frac{128}{15}x^5$, $\quad s(0.2) = 0.69900$

17. $s(x) = 1 + x - x^2 - x^3 + 5/6x^4 + \frac{7}{10}x^5$, $s(1/4) = \frac{36121}{30720} = 1.1758$

18. $s = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$. This is the Legendre polynomial P_5, a solution of this Legendre equation with parameter $n = 5$, to be discussed in Sec. 5.2. The initial conditions were chosen accordingly, so that a second linearly independent solution (a Legendre function) does not appear in the answer. $s(0.5) = 0.089844$.

19. $s(x) = 4 - 4x^2 - 8/3x^3 + \frac{16}{15}x^5$, $s(2) = \frac{4}{5}$, but $x = 2$ is large to give good values. Exact solution is $4\exp(2x)(x - 1)^2$.

SECTION 5.2. Legendre s Equation. Legendre Polynomials $P_n(x)$, page 175

Purpose. This section on Legendre's equation, one of the most important ODEs, and its solution is more than just an exercise on the power series method. It should give the student a feel for the usefulness of power series in exploring properties of **special functions** and for the wealth of relations between functions of a one-parameter family (with parameter n).

Legendre's equation occurs again in Secs. 11.5 and 11.6.

Comment on Literature and History

For literature on Legendre's equation and its solutions, see Refs. [GenRef 1] and [GenRef 10].

Legendre's work on the subject appeared in 1785 and Rodrigues's contribution (see Prob. 12), in 1816.

Problem Set 5.2

Problems 1–9 on Legendre polynomials and functions are straightforward illustrations of the simpler examples of these functions, also showing how a CAS can be used to discover properties.

In particular, Prob. 9 is of general interest in connection with any coefficient recursion similar to (3).

The problem set also provides a good place to discuss the idea of a **generating function** and illustrate it in terms of Legendre polynomials.

Problems 11–15 discusses a small portion of particularly important basic formulas selected from a large set that can be found in reference books as well as in classical monographies (see [GenRef1] in App. 1).

SOLUTIONS TO PROBLEM SET 5.2, page 179

6. We know that at the endpoints of the interval $-1 \leqq x \leqq 1$ all the Legendre polynomials have the values ± 1. It is interesting that in between they are strictly less than 1 in absolute value (P_0 excluded). Furthermore, absolute values between $\frac{1}{2}$ and 1 are taken only near the endpoints, so that in an interval, say $-0.8 \leqq x \leqq 0.8$, they are less than $\frac{1}{2}$ in absolute value (P_0, P_1, P_2 excluded).

10. Team Project. (a) Following the hint, we obtain

$$\text{(A)} \qquad (1 - 2xu + u^2)^{-1/2} = 1 + \tfrac{1}{2}(2xu - u^2) + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}(2xu - u^2)^n + \cdots$$

and for the general term on the right,

$$\text{(B)} \qquad (2xu - u^2)^m = (2x)^m u^m - m(2x)^{m-1}u^{m+1} + \frac{m(m-1)}{2!}(2x)^{m-2}u^{m+2} + \cdots .$$

Now u^n occurs in the first term of the expansion (B) of $(2xu - u^2)^n$, in the second term of the expansion (B) of $(2xu - u^2)^{n-1}$, and so on. From (A) and (B) we see that the coefficients of u^n in those terms are

$$\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}(2x)^n = a_n x^n \qquad \text{[see (8)]},$$

$$-\frac{1 \cdot 3 \cdots (2n-3)}{2 \cdot 4 \cdots (2n-2)}(n-1)(2x)^{n-2} = -\frac{2n}{2n-1}\,\frac{n-1}{4}\,a_n x^{n-2} = a_{n-2}x^{n-2}$$

and so on. This proves the assertion.

(b) Set $u = r_1/r_2$ and $x = \cos\theta$.

(c) Use the formula for the sum of the geometric series and set $x = 1$ and $x = -1$. Then set $x = 0$ and use

$$(1 + u^2)^{-1/2} = \sum \binom{-\frac{1}{2}}{m} u^{2m}$$

12. We have

$$(x^2 - 1)^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (x^2)^{n-m}.$$

Differentiating n times, we can express the product of occurring factors $(2n - 2m)(2n - 2m - 1) \cdots$ as a quotient of factorials and get

$$\frac{d^n}{dx^n}[(x^2 - 1)^n] = \sum_{m=0}^{M} (-1)^m \frac{n!}{m!(n-m)!}\,\frac{(2n-2m)!}{(n-2m)!}x^{n-2m}$$

with M as in (11). Divide by $n!2^n$. Then the left side equals the right side in Rodrigues's formula, and the right side equals the right side of (11).

14. Abbreviate $1 - 2xu + u^2 = U$. Differentiation of (13) with respect to u gives

$$-\frac{1}{2}U^{-3/2}(-2x + 2u) = \sum_{n=0}^{\infty} nP_n(x)u^{n-1}.$$

Multiply this equation by U and represent $U^{-1/2}$ by (13):

$$(x - u)\sum_{n=0}^{\infty} P_n(x)u^n = (1 - 2xu + u^2)\sum_{n=0}^{\infty} nP_n(x)u^{n-1}.$$

In this equation, u^n has the coefficients

$$xP_n(x) - P_{n-1}(x) = (n + 1)P_{n+1}(x) - 2nxP_n(x) + (n - 1)P_{n-1}(x).$$

Simplifying gives the asserted Bonnet recursion.

SECTION 5.3. Extended Power Series Method: Frobenius Method, page 180

Purpose. To introduce the student to the Frobenius method (an extension of the power series method), which is important for ODEs with coefficients that have **singularities**, notably **Bessel's equation**, so that the power series method can no longer handle them. This extended method requires more patience and care.

Main Content, Important Concepts

Theorem 1 characterizes the ODEs for which the Frobenius method can be used and what form of solutions can be expected.

Indicial Equation (4): its role in determining the kind of series solutions to be expected.

Theorem 2: Extension of Theorem 1, three cases of roots of the indicial equations and corresponding forms of solution, illustrative Examples 1–3 of these forms.

Regular and singular points

Short Courses. Take a quick look at those bases in Frobenius's theorem, see how it fits with the Euler–Cauchy equation, and omit everything else.

Comment on "Regular Singular" and "Irregular Singular"

These terms are used in some books and papers, but there is hardly any need for confusing the student by using them, simply because we cannot do (and don't do) anything about "irregular singular points." A simple use of "regular" and "singular" (as in complex analysis, where holomorphic functions are also known as "regular analytic functions") may thus be the best terminology.

Comment on Footnote 5

Gauss was born in Braunschweig (Brunswick) in 1777. At the age of 16, in 1793 he discovered the method of least squares (Secs. 20.5, 25.9). From 1795 to 1798 he studied at Göttingen. In 1799 he obtained his doctor's degree at Helmstedt. In 1801 he published his first masterpiece, *Disquisitiones arithmeticae* (*Arithmetical Investigations,* begun in 1795), thereby initiating modern number theory. In 1801 he became generally known when his calculations enabled astronomers (Zach, Olbers) to rediscover the planet Ceres, which had been discovered in 1801 by Piazzi at Palermo but had been visible only very briefly. He became the director of the Göttingen observatory in 1807 and remained there until his death. In 1809 he published his famous *Theoria motus corporum coelestium in sectionibus conicis solem ambientium* (*Theory of the Heavenly Bodies Moving About the Sun in Conic Sections*; Dover Publications, 1963), resulting from his further work in astronomy. In 1814 he developed his method of numeric integration (Sec. 19.5). His *Disquisitiones generales circa superficies curvas* (*General Investigations Regarding Curved Surfaces*, 1828) represents the foundation of the differential geometry of surfaces and contributes to conformal mapping

(Sec. 17.1). His clear conception of the complex plane dates back to his thesis, whereas his first publication on this topic was not before 1831. This is typical: Gauss left many of his most outstanding results (non-Euclidean geometry, elliptic functions, etc.) unpublished. His paper on the hypergeometric series published in 1812 is the first systematic investigation into the convergence of a series. This series, generalizing the geometric series, allows a study of many special functions from a common point of view.

Problem Set 5.3

Problems 2–13: Only in simpler cases will it be possible to recognize the series as one of a known function; this task is included to make students aware that even an unfamiliar series may be an expansion of an elementary function.

Gauss's hypergeometric ODE (15), series (16), and function $F(a, b, c; x)$ play a central role in special functions, simply because they include an incredible number of familiar elementary and higher special functions. Perhaps it is worth mentioning that (15) may be regarded as a "natural" extension of the Euler–Cauchy equation.

SOLUTIONS TO PROBLEM SET 5.3, page 186

2. $y_1 = x + 1$, $y_2 = \frac{1}{x+1}$. Check: Set $x + 1 = z$, to get an Euler-Cauchy equation.

3. $y_1 = (1 + \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots) = \frac{\sinh x}{x}$

$y_2 = \frac{1}{x}(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots) = \frac{\cosh x}{x}$

4. $y_1 = x(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + \frac{1}{2880}x^4 + \frac{1}{86400}x^5 + \cdots)$

$y_2 = \ln(x) + 1 - 3/4x^2 - \frac{7}{36}x^3 - \frac{35}{1728}x^4 - \frac{101}{86400}x^5$

5. $b_0 = -1, c_0 = 1, (r-1)^2\, y_1 = x(1 - 3x + \frac{15}{4}x^2 - \frac{35}{12}x^3 + \frac{105}{64}x^4 - \frac{231}{320}x^5 + \cdots)$,

$y_2 = x\ln(x)y_1 + x(4x - \frac{29}{4}x^2 + \frac{27}{4}x^3 - \frac{1633}{384}x^4 + \frac{779}{384}x^5 + \cdots)$

6. $b_0 = 0, c_0 = -2, r(r-1) - 2 = (r-2)(r+1), r_1 = 2, r_2 = -1$,

$$y_1 = x^2(1 - \tfrac{1}{2}x^2 + \tfrac{9}{56}x^4 - \tfrac{13}{336}x^6 + \cdots)$$

$$y_2 = \frac{1}{x}(12 - 6x^2 + \frac{9}{2}x^4 - \frac{7}{4}x^6 + \cdots).$$

7. $y_1 = 1 + 1/4x^2 - 1/6x^3 + \frac{1}{96}x^4 - \frac{1}{60}x^5 + \cdots$.

$y_2 = x + 1/12x^3 - 1/12x^4 + \frac{1}{480}x^5 + \cdots$

8. $y_1 = 1 + \dfrac{x^2}{2^2} + \dfrac{x^4}{(2\cdot 4)^2} + \dfrac{x^6}{(2\cdot 4\cdot 6)^2} + \cdots$,

$y_2 = y_1 \ln x - \dfrac{x^2}{4} - \dfrac{3x^4}{8\cdot 16} - \dfrac{11x^6}{64\cdot 6\cdot 36} - \cdots$

10. $y_1 = \frac{\sin 4x}{x}$, $y_2 = \frac{\cos 4x}{x}$

12. $b_0 = 6, c_0 = 6, r_1 = -2, r_2 = -3$; the series are

$$y_1 = \frac{1}{x^2} - \frac{2}{3} + \frac{2}{15}x^2 - \frac{4}{315}x^4 + - \cdots = \frac{1}{2}\frac{\sin 2x}{x^3}$$

$$y_2 = \frac{1}{x^3} - \frac{2}{x} + \frac{2}{3}x - \frac{4}{45}x^3 + - \cdots = \frac{\cos 2x}{x^3}.$$

13. $y_1 = e^{-x}$, $y_2 = e^{-x} \ln x$.

14. Team Project. (b) In (11b) of Sec. 5.1,

$$\frac{a_{n+1}}{a_n} = \frac{(a+n)(b+n)}{(n+1)(c+n)} \to 1,$$

hence $R = 1$.

(c) In the third and fourth lines,

$$\arctan x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + - \cdots \qquad (|x| < 1)$$

$$\arcsin x = x + \frac{1}{2 \cdot 3}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \cdots \qquad (|x| < 1).$$

(d) The roots can be read off from (15), brought to the form (1′) by multiplying it by x and dividing by $1 - x$; then $b_0 = c$ in (4) and $c_0 = 0$.

16. $y = A(1 + 4x) + B/xF(-\tfrac{1}{2}, -\tfrac{3}{2}, \tfrac{3}{2}, x)$

18. $t^2 - 3t + 2 = (t - 1)(t - 2) = 0$. Hence the transformation is $x = t - 1$. It gives the ODE

$$4(x^2 - x)y'' - 2y' + y = 0.$$

To obtain the standard form of the hypergeometric equation, multiply this ODE by $-\tfrac{1}{4}$. It is clear that the factor 4 must be absorbed, but don't forget the factor -1; otherwise your values for a, b, c will not be correct. The result is

$$x(1 - x)y'' + \tfrac{1}{2}y' - \tfrac{1}{4}y = 0.$$

Hence $ab = \tfrac{1}{4}$, $b = \dfrac{1}{4a}$, $a + b + 1 = a + \dfrac{1}{4a} + 1 = 0$, $a = -\tfrac{1}{2}$, $b = -\tfrac{1}{2}$, $c = \tfrac{1}{2}$.

This gives

$$y_1 = F(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}; t - 1).$$

In y_2 we have $a - c + 1 = -\tfrac{1}{2} - \tfrac{1}{2} + 1 = 0$; hence y_2 terminates after the first term, and, since $1 - c = \tfrac{1}{2}$,

$$y_2 = x^{1/2} \cdot 1 = \sqrt{t - 1}.$$

20. $y = c_1F(-1, \tfrac{1}{3}, \tfrac{1}{3}; t + 1) + c_2(t + 1)^{2/3}F(-\tfrac{1}{3}, 1, \tfrac{5}{3}; t + 1)$

SECTION 5.4. Bessel s Equation. Bessel Functions $J_\nu(x)$, page 187

Purpose. To derive the Bessel functions of the first kind J_ν and $J_{-\nu}$ by the Frobenius method. (This is a major application of that method.) To show that these functions constitute a basis if ν is not an integer but are linearly dependent for integer $\nu = n$ (so that we must look later, in Sec. 5.5, for a second linearly independent solution). To show that various ODEs can be reduced to Bessel's equation (see Problem Set 5.4).

Main Content, Important Concepts

Derivation just mentioned

Linear independence of J_ν and $J_{-\nu}$ if ν is not an integer

Linear dependence of J_ν and $J_{-\nu}$ if $\nu = n = 1, 2, \cdots$

Gamma function as a tool

Short Courses. No derivation of any of the series. Discussion of J_0 and J_1 (which are similar to cosine and sine). Mention Theorem 2.

Comment on Special Functions

Since various institutions no longer find time to offer a course in special functions, Bessel functions may give another opportunity (together with Sec. 5.2) for getting at least some feel for the flavor of the theory of special functions, which will continue to be of significance to the engineer and physicist. For this reason we have added some material on basic relations for Bessel functions in this section.

Problem Set 5.4

Problems 2–10 concern a few of the large number of ODEs reducible to Bessel's equation. The latter contains a single parameter, and Prob. 5 shows how a second parameter can be introduced.

CAS Experiment 12 and Prob. 14 give an impression of the accuracy of the asymptotic formula (14).

Elimination of the first derivative from an ODE (Probs. 16–18) is a standard process used for various practical and theoretical purposes.

Problems 19–25 show that (21) is a backbone of the whole theory; in particular, it can be used to obtain Bessel's equation (1).

SOLUTIONS TO PROBLEM SET 5.4, page 195

2. $c_1 J_{1/3}(x) + c_2 J_{1/3}(x)$.

3. $c_1 J_0(2\sqrt{x}) + c_2 Y_0(2\sqrt{x})$

4. $c_1 J_{1/4}(e^{-x}) + c_2 J_{1/4}(x)$

6. $c_1 \sqrt{\pi x} J_{1/2}(2\sqrt{x}) + c_2 \sqrt{\pi x} J_{-1/2}(2\sqrt{x})$ or $c_1 \sin(2\sqrt{x}) \sqrt[4]{x} + c_2 \sqrt[4]{x} \cos(2\sqrt{x})$

7. $c_1 J_{1/4}(x) + c_2 Y_{1/4}(x)$

8. $c_1 J_1(x - 1) + c_2 Y_1(x - 1)$.

10. $x^\nu (c_1 J_\nu(x^\nu) + c_2 J_{-\nu}(x^\nu))$, $\nu \neq 0, \pm 1, \pm 2, \cdots$

12. **CAS Experiment. (b)** $x_0 = 1, x_1 = 2.5, x_2 = 20$, approximately. It increases with n **(c)** (14) is exact. **(d)** It oscillates. **(e)** Formula (24b) with $\nu = 0$.

14. This problem should give the student a feel for the applicability and accuracy of an asymptotic formula. A corresponding formula for y_ν is included in the next problem set.

Zeros of $J_0(x)$

Approximation (14)	Exact Value	Error
2.35619	2.40483	0.04864
5.49779	5.52008	0.02229
8.63938	8.65373	0.01435
11.78097	11.79153	0.01056

Zeros of $J_1(x)$

Approximation (14)	Exact Value	Error
3.92699	3.83171	−0.09528
7.06858	7.01559	−0.05299
10.21018	10.17347	−0.03671
13.35177	13.32369	−0.02808

18. For $\nu = \pm\frac{1}{2}$ the ODE (27) in Prob. 17 becomes

$$u'' + u = 0.$$

Hence for y we obtain the general solution

$$y = x^{-1/2}u = x^{-1/2}(A \cos x + B \sin x).$$

We can now obtain A and B by comparing with the first term in (20). Using $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ (see (23)) we obtain, for $\nu = \frac{1}{2}$, the first term

$$x^{1/2}/(2^{1/2} \cdot \tfrac{1}{2}\Gamma(\tfrac{1}{2})) = \sqrt{2x/\pi}.$$

This gives (22a) because the series of $\sin x$ starts with the power x.

For $\nu = -\frac{1}{2}$ the first term is $x^{-1/2}/(2^{1/2} \cdot \frac{1}{2}\Gamma(\frac{1}{2})) = \sqrt{2/\pi x}$. This gives (22b).

20. (21b) with $\nu - 1$ instead of ν is

$$(x^{-\nu+1}J_{\nu-1})' = -x^{-\nu+1}J_\nu.$$

Now use (21a) $(x^\nu J_\nu)' = x^\nu J_{\nu-1}$; solve it for $J_{\nu-1}$ to obtain

$$J_{\nu-1} = x^{-\nu}(x^\nu J_\nu)' = x^{-\nu}(\nu x^{\nu-1}J_\nu + x^\nu J_\nu') = \nu x^{-1}J_\nu + J_\nu'.$$

Substitute this into the previous equation on the left. Then perform the indicated differentiation:

$$\begin{aligned}(x^{-\nu+1}(\nu x^{-1}J_\nu + J_\nu'))' &= (\nu x^{-\nu}J_\nu + x^{-\nu+1}J_\nu')' \\ &= -\nu^2 x^{-\nu-1}J_\nu + \nu x^{-\nu}J_\nu' + (-\nu+1)x^{-\nu}J_\nu' + x^{-\nu+1}J_\nu''.\end{aligned}$$

Equating this to the right side of the first equation and dividing by $x^{-\nu+1}$ gives

$$J_\nu'' + \frac{1}{x}J_\nu' - \frac{\nu^2}{x^2}J_\nu = -J_\nu.$$

Taking the term on the right to the left (with a plus sign) and multiplying by x^2 gives (1).

22. Integrate (21b) and (21d).

24. $-\frac{1}{x}J_1(x)$

25. Use (21d) to get

$$\begin{aligned}\int J_3(x)dx &= -2J_2(x) + \int J_1(x)dx &(1)\\ &= -2J_2(x) - J_0(x) + C &(2)\end{aligned}$$

SECTION 5.5. Bessel Functions $Y_\nu(x)$. General Solution, page 196

Purpose. Derivation of a second independent solution, which is still missing in the case of $\nu = n = 0, 1, \cdots$.

Main Content

Detailed derivation of $Y_0(x)$

Cursory derivation of $Y_n(x)$ for any n

General solution (9) valid for all ν, integer or not

Short Courses. Omit this section.

Comment on Hankel Functions and Modified Bessel Functions

These are included for completeness, but will not be needed in our further work.

SOLUTIONS TO PROBLEM SET 5.5, page 200

1. $c_1 J_3(x) + c_2 Y_3(x)$

2. $x^{-1}(c_1 J_1(x) + c_2 Y_1(x))$. Here J_{-1} could not be used, because of linear dependence.

3. $c_1 J_{1/3}(\frac{x^2}{2}) + c_2 Y_{1/3}(\frac{x^2}{2})$

4. Substitute $y = ux^{1/2}$ and its derivatives into the given equation and multiply the resulting equation by $x^{3/2}$ to get

$$x^2 u'' + xu' + \left(x^3 - \tfrac{1}{4}\right)u = 0.$$

Now introduce z as given in the problem statement to get the answer

$$y = \sqrt{x}\left[c_1 J_{1/3}\left(\tfrac{2}{3}x^{3/2}\right) + c_2 J_{-1/3}\left(\tfrac{2}{3}x^{3/2}\right)\right].$$

6. $c_1 J_0(4\sqrt{x}) + c_2 Y_0(\sqrt{(x)})$

8. $\sqrt{x}(c_1 J_{1/6}(\frac{1}{3}kx^3) + c_2 Y_{1/6}(\frac{1}{3}kx^3))$. $J_{-1/6}$ could be used instead of $Y_{1/6}$.

9. $x^2(c_1 J_2(x) + c_2 Y_2(x))$.

10. CAS Experiment. (a) Y_0 and Y_1, similarly as for J_0 and J_1.

(b) Accuracy is best for Y_0. x_n increases with n; actual values will depend on the scales used for graphing.

(c), (d)

	Y_0		Y_1		Y_2	
m	By (11)	Exact	By (11)	Exact	By (11)	Exact
1	0.785	0.894	2.356	2.197	3.927	3.384
2	3.9270	3.958	5.498	5.430	7.0686	6.794
3	7.0686	7.086	8.639	8.596	10.210	10.023
4	10.210	10.222	11.781	11.749	13.352	13.210
5	13.352	13.361	14.923	14.897	16.493	16.379
6	16.493	16.501	18.064	18.043	19.635	19.539
7	19.635	19.641	21.206	21.188	22.777	22.694
8	22.777	22.782	24.347	24.332	25.918	25.846
9	25.918	25.923	27.489	27.475	29.060	28.995
10	29.060	29.064	30.631	30.618	32.201	32.143

These values show that the accuracy increases with x (for fixed n), as expected. For a fixed m (number of zero) it decreases with increasing n (order of Y_n).

12. Set $x = is$ in (1), Sec. 5.4, to get the present ODE (12) in terms of s.

14. For $x \neq 0$ all the terms of the series are real and positive.

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11. $\cos 2x$, $\sin 2x$

12. e^x, $e^x \ln x$

14. $(x-1)^{3/4}$, $(x-1)^{1/4}$; Euler-Cauchy equation with independent variable $x - 1$.

16. $b_0 = 0$, $c_0 = -2$, $r(r-1) - 2 = 0$, $r_1 = 2$, $r_2 = -1$,

$y_1 = x^2(1 - \frac{9}{10}x^2 + \frac{153}{280}x^4 - \frac{85}{336}x^6 + \cdots)$

$y_2 = x^{-1}(12 - 18x^2 + \frac{45}{2}x^4 - \frac{65}{4}x^6 + \cdots)$

17. e^x, $1 - x$

18. $x^{-2}\sin\frac{x^2}{2}$, $x^{-2}\cos\frac{x^2}{2}$

19. $\sqrt{x}J_1(2\sqrt{x})$, $\sqrt{x}Y_1(2\sqrt{x})$

20. $$y_1 = 1 + \frac{x^2}{2^2} + \frac{x^4}{(2\cdot 4)^2} + \frac{x^6}{(2\cdot 4\cdot 6)^2} + \cdots$$

$$y_2 = y_1 \ln x - \frac{x^2}{4} - \frac{3x^4}{8\cdot 16} - \frac{11x^6}{64\cdot 6\cdot 36} - \cdots$$