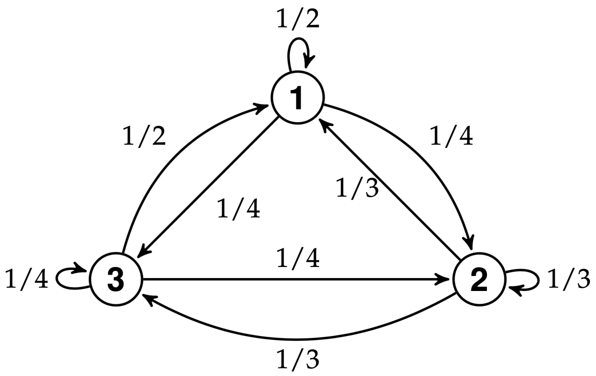
CS5314 Randomized Algorithms

Homework 5
Due: June 23, 2020 (before 11:59 pm)

Exam 3 will be held on June 23, 2020 (10:00–12:00)

1. Consider a Markov chain $\{X_0, X_1, X_2, \ldots\}$ whose transition diagram is shown in the following figure.



   (a) Suppose that $\Pr(X_0 = 1) = 1/2$. What is $\Pr((X_0 = 1) \cap (X_1 = 2))$?

   (b) Suppose that $\Pr(X_0 = 1) = 1/2$. What is $\Pr((X_0 = 1) \cap (X_1 = 2) \cap (X_2 = 3))$?

2. Suppose that we have a Markov chain with three states, 0, 1, and 2. For state 0, we have probability 0.1 to stay and 0.9 to go to state 1. When we are at state 1, the probability to go to state 2 is 0.3, and the probability to go back to state 0 is 0.7. We would stay for probability 0.4 while we are at state 2 and go back to state 1 with probability 0.6.

   (a) Argue that the Markov chain is aperiodic and irreducible.

   (b) Find the stationary probability.

3. A bug starts at a vertex of an equilateral triangle. On each move, it selects one of the two vertices where it is not currently located, with equal probability, and crawls along a side of the triangle to that vertex. Let p_k denote the probability that the bug moves to its starting vertex on its kth move. What are the values of p_1, p_2, p_3, p_4, and p_5? Express the answers in the simplest fractional form.

4. (Further studies: No marks) What is the general form of p_k in Question 3?

5. (Further studies: No marks) We have considered the gambler's ruin problem in the case where the game is fair. Consider the case where the game is not fair; instead, the probability of losing a dollar each game is 2/3 and the probability of winning a dollar each game is 1/3. Suppose that you start with i dollars and finish either when you reach n or lose it all. Let W_t be the amount you have gained after t rounds of play.

   (a) Show that $E[2^{W_t}] = E[2^{W_{t+1}}]$.

   (b) Find the probability that you are winning.