1、Let Z be poisson RV with mean μ, $\mu \geq 1$, $\mu \in \mathbb{N}$

(a) $Pr(Z=\mu+h) = \dfrac{e^{-\mu} \cdot \mu^{(\mu+h)}}{(\mu+h)!} \geq Pr(Z=\mu-h-1) = \dfrac{e^{-\mu} \cdot \mu^{(\mu-h-1)}}{(\mu-h-1)!}$

$Pr(X=r) = \dfrac{e^{-\mu}\mu^{r}}{r!}$ for $0 \leq h \leq \mu-1$

$$Pr(Z=\mu+h) = \frac{\cancel{e^{-\mu}} \cdot \mu^{(\mu+h)}}{(\mu+h)!} \geq Pr(Z=\mu-h-1) = \frac{\cancel{e^{-\mu}} \cdot \mu^{(\mu-h-1)}}{(\mu-h-1)!}$$

$$\mu^{2h+1} \geq \frac{(\mu+h)!}{(\mu-h-1)!} \qquad \mu^{(2h+1)} \geq \prod_{k=\mu-h}^{\mu+h} k$$

$$\mu^{2h} \geq (\mu-h)(\mu-h+1)\cdots(\mu-1)(\mu+1)\cdots(\mu+h)$$

$$\mu^{2h} \geq (\mu^2-h^2)(\mu^2-(h-1)^2)\cdots(\mu^2-1) \quad \#$$

(b)

$$Pr(Z \geq \mu) = \sum_{h \geq 0} Pr(Z=\mu+h) \geq \sum_{h \geq 0} Pr(Z=\mu-h-1)$$

$$= 1 - \sum_{h \geq 0} Pr(Z=\mu+h)$$

$$2\sum_{h \geq 0} Pr(Z=\mu+h) \geq 1$$

$$\sum_{h \geq 0} Pr(Z=\mu+h) = Pr(Z \geq \mu) \geq \frac{1}{2} \quad \#$$

2.

$X \sim Po(\mu)$

$X = Y + Z$ $\quad$ Assume $n \to \infty$

$\Rightarrow Y \sim Bin(n, p)$, $np = \mu_Y$

$Z \sim Bin(n, 1-p)$ $n(1-p) = \mu_Z$

$X \sim Po(\mu)$

$Pr(Y=k) = Pr(Y=k \mid X=n) Pr(X=n)$

$Pr(Y=k \cap Z=n-k) = Pr(Y=k \mid Z=n-k) Pr(Z=n-k)$

$= Pr(Y=k) Pr(Z=n-k)$

$$Pr(Y=k) = \sum_{n=k}^{\infty} Pr(Y=k \mid X=n) Pr(X=n)$$

$$= \sum_{n=k}^{\infty} C_k^n p^k (1-p)^{n-k} \cdot \frac{e^{-\mu}\mu^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{(n-k)!\,k!} p^k (1-p)^{n-k} \cdot \frac{e^{-\mu}\mu^n}{\cancel{n!}}$$

$$= \frac{\mu^k e^{-\mu} p^k}{k!} \sum_{n=k}^{\infty} \frac{[(1-p)\mu]^{n-k}}{(n-k)!} = \frac{(\mu p)^k e^{-\mu}}{k!} \sum_{i=0}^{\infty} \frac{[(1-p)\mu]^i}{i!}$$ $\quad i = n-k$

let $k+h=n$ $\quad$ $$= \frac{(\mu p)^k e^{-\mu} \cdot e^{(1-p)\mu}}{k!} = \frac{(\mu p)^k e^{-p\mu}}{k!}$$

$$Pr(Z=n-k) = Pr(Z=h) = \sum_{n=h}^{\infty} C_h^n (1-p)^h p^{n-h} \frac{e^{-\mu}\mu^n}{n!}$$

$$= \sum_{n=h}^{\infty} \frac{\cancel{n!}}{(n-h)!\,h!} (1-p)^h p^{n-h} \frac{e^{-\mu}\mu^n}{\cancel{n!}} = \frac{\mu^h (1-p)^h e^{-\mu}}{h!} \sum_{n=h}^{\infty} \frac{(p\mu)^{n-h}}{(n-h)!} = \frac{(\mu(1-p))^h e^{-\mu}}{h!} \cdot e^{p\mu}$$

$$= \frac{[\mu(1-p)]^h e^{\mu(p-1)}}{h!}$$

$p\mu - \mu = \mu(p-1)$

$$Pr(Z=h) Pr(Y=k) = \frac{[\mu(1-p)]^h e^{\mu(p-1)} \cdot (\mu p)^k \cdot e^{-p\mu}}{h! \cdot k!} = \frac{e^{-\mu} \cdot \mu^n (1-p)^h p^k}{h!\,k!}$$

$$= Pr(Z=h \mid X=h+k) Pr(X=h+k) = \frac{\cancel{n!}}{(n-h)!\,h!} (1-p)^h p^{n-h} \cdot \frac{e^{-\mu}\mu^n}{\cancel{n!}} = \frac{e^{-\mu}\mu^n (1-p)^h p^k}{k!\,h!}$$ #

let $h+k=n$

3.

(a) $E[X_1 X_2 \cdots X_k] = \sum_{a_1=0,1} \sum_{a_2=0,1} \cdots \sum_{a_k=0,1} X_1 X_2 \cdots X_k \cdot Pr(X_1=a_1) \cap Pr(X_2=a_2) \cap \cdots \cap Pr(X_k=a_k)$

$= 1 \cdot Pr(X_1=1) \cap Pr(X_2=1) \cap \cdots \cap Pr(X_k=1) = (1-\frac{k}{n})^n$

$E[Y_1 Y_2 \cdots Y_k] = \sum_{a_1=0,1} \sum_{a_2=0,1} \cdots \sum_{a_k=0,1} Y_1 Y_2 \cdots Y_k \cdot Pr(Y_1=a_1) \cap Pr(Y_2=a_2) \cdots \cap Pr(Y_k=a_k)$

$= 1 \cdot Pr(Y_1=1) \cap Pr(Y_2=1) \cap \cdots \cap Pr(Y_k=1)$

$= ((1-\frac{1}{n})^n)^k = (1-\frac{1}{n})^{nk}$

$\Rightarrow (1-\frac{k}{n}) \le (1-\frac{1}{n})^k$ $\quad \frac{d}{dk} 1-\frac{k}{n} = -\frac{1}{n}$

$\frac{d}{dk}(1-\frac{1}{n})^k = \frac{d}{dk} e^{k(\ln(1-\frac{1}{n}))}$

$= e^{k(\ln(1-\frac{1}{n}))} \cdot \ln(1-\frac{1}{n})$

(b) $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k$

since $X_i^{j_i} = 1$ or 0, $1^k = 1$, $0^k = 0$

(c)

$E[e^{tX}] = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(X^i)$ $\quad E[e^{tY}] = \sum_{i=0}^{\infty} \frac{t^i}{i!} E[Y^i]$

Taylor : $\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$

$(x-a)$

Goal : $E[X^i] \le E[Y^i]$

$E[(\sum_{j=1}^{n} X_j)^i] = E[\sum(\prod_{j=1}^{n} X_j^{c_j})] = E[\sum(\prod_{j=1}^{n} X_j)]$

$= \sum(E[\prod_{j=1}^{n} X_j]) \le \sum(E[\prod_{j=1}^{n} Y_j])$

(D)

$Pr(X \ge (1+\delta)E[X]) = Pr(e^{tX} \ge e^{t(1+\delta)E[X]}) \le \frac{E[e^{tX}]}{e^{t(1+\delta)E[X]}}$

$= \frac{\prod_{i=1}^{n} E[e^{tX_i}]}{e^{t(1+\delta)E[X]}} = \frac{\prod_{i=1}^{n} 1+(e^t-1)p}{e^{t(1+\delta)E[X]}} = \frac{e^{(e^t-1)np}}{e^{t(1+\delta)}} = \frac{e^{(\frac{1+\delta}{np}-1)np}}{(\frac{1+\delta}{np})^{(1+\delta)}} = \frac{e^{1+\delta-np}}{(\frac{1+\delta}{np})^{(1+\delta)}}$

$\min(\frac{e^{(e^t-1)np}}{e^{t(1+\delta)}}) = \min((e^t-1)np - t(1+\delta))$ $\quad \frac{d}{dt}(e^t-1)np - t(1+\delta)$ #

$t = \ln\frac{1+\delta}{np}$ $\quad t+\ln np = \ln(1+\delta)$ $\ln e^t np = \ln(1+\delta)$ $= e^t np - (1+\delta)$

$\leftarrow e^t np - (1+\delta) = 0$

4.

(a) $E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})] Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m)$

$$E[f(Y_1^{(m)}, \cdots Y_n^{(m)})]$$
$$= \sum_{k=0}^{\infty} E[f(Y_1^{(m)}, \cdots Y_n^{(m)}) \mid \sum_{i=1}^{n} Y_i^{(m)} = k] \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} = k)$$
$$\geq E[f(Y_1^{(m)}, \cdots Y_n^{(m)}) \mid \sum_{i=1}^{n} Y_i^{(m)} \geq m] \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m)$$
$$= \sum f(Y_1^{(m)}, \cdots Y_n^{(m)}) \cdot Pr((Y_1^{(m)} \cdots Y_n^{(m)}) = (k_1 \cdots k_n) \mid \sum_{i=1}^{n} Y_i^{(m)} \geq m) \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m)$$
$$= \sum f(X_1^{(m)} \cdots X_n^{(m)}) \cdot Pr((X_1^{(m)} \cdots X_n^{(m)}) = (k_1 \cdots k_n)) \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m)$$
$$= E[f(X_1^{(m)} \cdots X_n^{(m)})] \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m)$$

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$$\Rightarrow Pr((Y_1^{(m)} \cdots Y_n^{(m)}) = (k_1 \cdots k_n) \mid \sum_{i=1}^{n} k_i = k)$$
$$= \frac{Pr(Y_1^{(m)} = k_1) \cap Pr(Y_2^{(m)} = k_2) \cdots \cap Pr(Y_n^{(m)} = k_n)}{Pr(\sum_{i=1}^{n} k_i = k)}$$
$$= \frac{\prod_{i=1}^{n} \frac{e^{-\frac{m}{n}} \cdot (\frac{m}{n})^{k_i}}{k_i!}}{\frac{e^{-m} \cdot m^k}{k!}} = \frac{\frac{\cancel{e^{-m}} \cancel{m^k}}{k_1! k_2! \cdots k_n! \cdot n^k}}{\frac{\cancel{e^{-m} \cdot m^k}}{k!}} = \frac{k!}{k_1! k_2! \cdots k_n! \cdot n^k}$$

$$\Rightarrow Pr((X_1^{(m)} \cdots X_n^{(m)}) = (k_1 \cdots k_n) \mid \sum_{i=1}^{n} k_i = k)$$
$$= \frac{Pr(X_1^{(m)} = k_1) \cap \cdots \cap Pr(X_n^{(m)} = k_n)}{Pr(\sum_{i=1}^{n} k_i = k)} = \frac{k!}{k_1! k_2! \cdots k_n! \, n^k}$$

$$\therefore Pr((Y_1^{(m)} \cdots Y_n^{(m)}) = (k_1 \cdots k_n) \mid \sum_{i=1}^{n} k_i = k) = Pr((X_1^{(m)} \cdots X_n^{(m)}) = (k_1 \cdots k_n)$$

(b)

According to 1.

$$\Pr(Z \geq \mu) \geq \frac{1}{2}, \quad Z \sim Po(\mu), \mu \geq 1$$

Let $\sum_{i=1}^{n} Y_i^{(m)} = Y^{(m)}$, since $Y_i^{(m)} \sim Po(\frac{m}{n})$

$$E[Y^{(m)}] = n \cdot \frac{m}{n} = m \Rightarrow Y^{(m)} \sim Po(m)$$

$$\therefore \Pr(Y^{(m)} \geq m) \geq \frac{1}{2}$$

$$\begin{aligned} E[f(Y_1^{(m)}, \cdots Y_n^{(m)})] &\geq E[f(X_1^{(m)}, \cdots X_n^{(m)})] \cdot \Pr(\sum_{i=1}^{n} Y_i^{(m)} \geq m) \\ &= E[f(X_1^{(m)} \cdots X_n^{(m)})] \cdot \Pr(Y^{(m)} \geq m) \\ &\geq \frac{1}{2} E[f(X_1^{(m)} \cdots X_n^{(m)})] \end{aligned}$$

$$\therefore E[f(X_1^{(m)} \cdots X_n^{(m)})] \leq 2E[f(Y_1^{(m)}, \cdots Y_n^{(m)})]$$

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