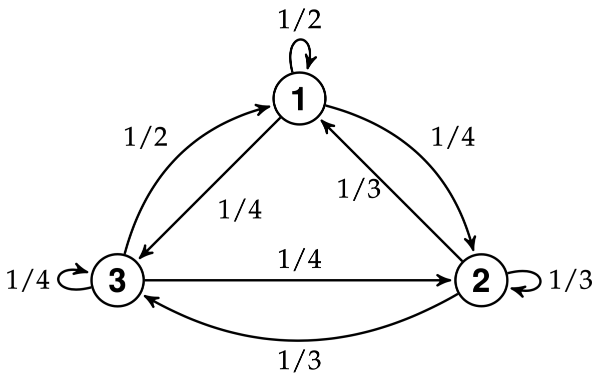
CS5314 Randomized Algorithms

Homework 5 Suggested Solution

(Original due date was June 23, 2020)

1. Consider a Markov chain $\{X_0, X_1, X_2, \ldots\}$ whose transition diagram is shown in the following figure.



(a) Suppose that $\Pr(X_0 = 1) = 1/2$. What is $\Pr((X_0 = 1) \cap (X_1 = 2))$?

Ans. $\frac{1}{8}$

Let p_{ij} denote the transition probability from state i to state j.

$$\begin{aligned}\Pr((X_0 = 1) \cap (X_1 = 2)) &= \Pr(X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1) \\ &= \frac{1}{2} \cdot\ p_{12} \\ &= \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}\end{aligned}$$

(b) Suppose that $\Pr(X_0 = 1) = 1/2$. What is $\Pr((X_0 = 1) \cap (X_1 = 2) \cap (X_2 = 3))$?

Ans. $\frac{1}{24}$

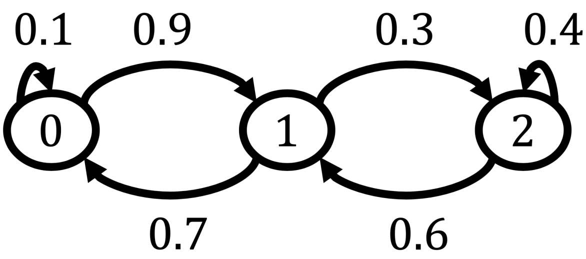
$$\begin{aligned}&\Pr((X_0 = 1) \cap (X_1 = 2) \cap (X_2 = 3)) \\ &= \Pr(X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1)\Pr(X_2 = 3 \mid (X_1 = 2) \cap (X_0 = 1)) \\ &= \Pr(X_0 = 1)\Pr(X_1 = 2 \mid X_0 = 1)\Pr(X_2 = 3 \mid X_1 = 2) \quad \text{(by Markov chain property)} \\ &= \frac{1}{2} \cdot\ p_{12} \cdot p_{23} \\ &= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{24}\end{aligned}$$

2. Suppose that we have a Markov chain with three states, 0, 1, and 2. For state 0, we have probability 0.1 to stay and 0.9 to go to state 1. When we are at state 1, the probability to go to state 2 is 0.3, and the probability to go back to state 0 is 0.7. We would stay for probability 0.4 while we are at state 2 and go back to state 1 with probability 0.6.

   (a) Argue that the Markov chain is aperiodic and irreducible.

   Ans.

   The transition diagram of this Markov chain is shown as below.



   In the diagram, there is a directed cycle that connects all the states (from state 0 to state 1 to state 2 to state 1 to state 0). This shows that every state is accessible from every other state (including itself), or in other words, the graph is strongly connected. Hence, this Markov chain is irreducible.

   Let $p_{i,j}^m$ be the transition probability from state i to state j in exactly m steps, and let $d(k)$ be the period of state k. By definition,

$$d(k) = gcd\{m \geqslant 1 : p_{i,i}^m > 0\}$$

$$d(0) = gcd\{m \geqslant 1 : p_{0,0}^m > 0\} = gcd\{1, 2, 3, ...\} = 1$$
$$d(1) = gcd\{m \geqslant 1 : p_{1,1}^m > 0\} = gcd\{2, 3, 4, ...\} = 1$$
$$d(2) = gcd\{m \geqslant 1 : p_{2,2}^m > 0\} = gcd\{1, 2, 3, ...\} = 1$$

   Because $d(0)$ is equal to 1, state 0 is aperiodic. In a similar way, state 1 and state 2 are also aperiodic. Since all three states are aperiodic, this Markov chain is aperiodic.

   (b) Find the stationary probability.

Ans. $\begin{bmatrix} \frac{14}{41} & \frac{18}{41} & \frac{9}{41} \end{bmatrix}$

Let $\pi = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$ be the stationary distribution of this Markov chain.

In the above question we know that this Markov chain is irreducible. Every irreducible finite state space Markov chain has a unique stationary distribution. Recall that the stationary distribution π is the vector such that

$$\pi = \pi P$$

where $P = \begin{bmatrix} 0.1 & 0.9 & 0 \\ 0.7 & 0 & 0.3 \\ 0 & 0.6 & 0.4 \end{bmatrix}$ is the probability transition matrix.

Therefore, we can find our stationary distribution by solving the following linear system:

$$\begin{aligned} 0.1\pi_0 + 0.7\pi_1 &= \pi_0 \\ 0.9\pi_0 + 0.6\pi_2 &= \pi_1 \\ 0.3\pi_1 + 0.4\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

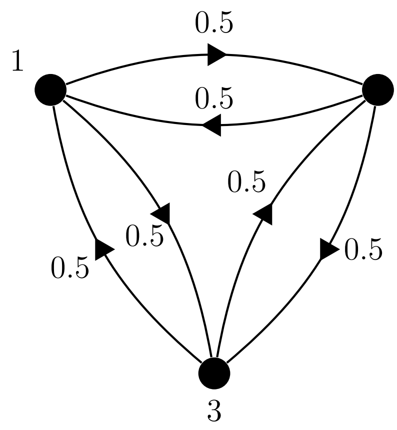
We find that:

$$\pi_0 = \frac{14}{41}, \quad \pi_1 = \frac{18}{41}, \quad \pi_2 = \frac{9}{41}$$

3. A bug starts at a vertex of an equilateral triangle. On each move, it selects one of the two vertices where it is not currently located, with equal probability, and crawls along a side of the triangle to that vertex. Let p_k denote the probability that the bug moves to its starting vertex on its kth move. What are the values of p_1, p_2, p_3, p_4, and p_5? Express the answers in the simplest fractional form.

   Ans. $p_1 = 0$, $p_2 = \frac{1}{2}$, $p_3 = \frac{1}{4}$, $p_4 = \frac{3}{8}$, $p_5 = \frac{5}{16}$

   We can change the question into the following Markov chain.

   The transition matrix P is $\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$.

   Let X_m be the probability distribution of states after m moves, $m \in \{0, 1, 2, 3, ...\}$. Without lose of generality, assume the chain starts from state 1, i.e., the initial probability vector X_0 is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

In order to find the distribution of states after the first moves, we multiply the initial probability vector X_0 and the transition matrix P:

$$X_1 = X_0 \cdot P = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}$$

and p_1 is equal to the first entry of X_1. We obtain

$$p_1 = 0$$

Then, we multiply X_1 and P to get X_2:

$$X_2 = X_0 \cdot P^2 = X_1 \cdot P = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \end{bmatrix}$$

We obtain

$$p_2 = \frac{1}{2}$$

In a similar way, we multiply X_2 and P to get X_3:

$$X_3 = X_0 \cdot P^3 = X_2 \cdot P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$$

We obtain

$$p_3 = \frac{1}{4}$$

Again, we multiply X_3 and P to get X_4:

$$X_4 = X_0 \cdot P^4 = X_3 \cdot P = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \end{bmatrix}$$

We obtain

$$p_4 = \frac{3}{8}$$

Finally, we multiply X_4 and P to get X_5:

$$X_5 = X_0 \cdot P^5 = X_4 \cdot P = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{11}{32} & \frac{11}{32} \end{bmatrix}$$

We obtain

$$p_5 = \frac{5}{16}$$

4. (Further studies: No marks) What is the general form of p_k in Question 3?

   Ans. $p_k = \frac{2}{3} \cdot (-\frac{1}{2})^k + \frac{1}{3}$

   In Question 3, if the bug is on its starting vertex after k moves, then it must be not on its starting vertex after $k-1$ moves. At this point it has $\frac{1}{2}$ chance of reaching the starting vertex in the next move. Thus we can get the following recurrence relation:

$$\begin{cases} p_k = \frac{1}{2} \cdot (1 - p_{k-1}), & \text{if } k > 0 \\ p_0 = 1 \end{cases}$$

   This is a non-homogeneous linear recurrence relation, where the associated homogeneous equation is

$$p_k = -\frac{1}{2} \cdot p_{k-1}$$

   The characteristic equation of its associated homogeneous relation is

$$\alpha + \frac{1}{2} = 0, \text{ or } \alpha = -\frac{1}{2}$$

   Then, p_k is of the form $p_k = (-\frac{1}{2})^k \cdot a + b$ for some constants a and b.
   In previous question, we have

$$1 = p_0 = a + b$$

$$0 = p_1 = (-\frac{1}{2}) \cdot a + b$$

   Solving these two equations, we get $a = \frac{2}{3}$ and $b = \frac{1}{3}$.
   Hence,

$$p_k = \frac{2}{3} \cdot (-\frac{1}{2})^k + \frac{1}{3}$$

5. (Further studies: No marks) We have considered the gambler's ruin problem in the case where the game is fair. Consider the case where the game is not fair; instead, the probability of losing a dollar each game is 2/3 and the probability of winning a dollar each game is 1/3. Suppose that you start with i dollars and finish either when you reach n or lose it all. Let W_t be the amount you have gained after t rounds of play.

   (a) Show that $E[2^{W_t}] = E[2^{W_{t+1}}]$.
   Ans.

   We need to consider two cases:

   i. When $W_t = 0$ or $W_t = n$. In this case, $W_t = W_{t+1}$, and hence, $E[2^{W_t}] = E[2^{W_{t+1}}]$.

ii. When $0 < w_t < n$, we have

$$\begin{aligned} E[2^{W_{t+1}} \mid W_t] &= \frac{2}{3} \cdot 2^{W_t - 1} + \frac{1}{3} \cdot 2^{W_t + 1} \\ &= \frac{1}{3} \cdot 2^{W_t} + \frac{2}{3} \cdot 2^{W_t} \\ &= 2^{W_t} \end{aligned}$$

Thus, in either case we have

$$E[2^{W_{t+1}}] = E[E[2^{W_{t+1}} \mid W_t]] = E[2^{W_t}]$$

(b) Find the probability that you are winning.

Ans.

Let p be the probability of finishing with n dollars, and let t be the stopping time (i.e. the first time we reach n dollars). Form part (a), we have

$$p \cdot 2^n + (1-p) \cdot 2^0 = E[2^{W_t}] = E[2^{W_{t-1}}] = E[2^{W_{t-2}}] = ... = E[2^{W_1}] = E[2^{W_0}] = 2^i$$

We obtain

$$p = \frac{2^i - 1}{2^n - 1}$$