CS2336 Discrete Mathematics

Homework 3
Tutorial: November 30, 2020
Exam 2: December 07, 2020 (2.5 hours)

Problems marked with * will be explained in the tutorial.

1. Prove each of the following for all integer $n \geq 1$ by mathematical induction.

   (a)
$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

   (b)
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

2. Use strong induction to prove that $\sqrt{2}$ is irrational.

   Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.

3. (*, Challenging, UKMT MOG 2016) Show that any odd positive integer can be expressed as a product of fractions with each fraction of the form
$$\frac{4q-1}{2q+1}, \qquad \text{where } q \text{ is a positive integer?}$$
   For instance,
$$1 = \frac{3}{3} = \frac{4 \times 1 - 1}{2 \times 1 + 1}$$
   and
$$3 = \frac{15}{9} \times \frac{27}{15} = \frac{4 \times 4 - 1}{2 \times 4 + 1} \times \frac{4 \times 7 - 1}{2 \times 7 + 1}.$$
   Hint: On Page 3; try your best without using the hint.

4. (Challenging, Adapted from AIME 1987) Show that the following expression is always a positive integer, for any $k \geq 1$, by expressing it in terms of k:
$$10 \left(\frac{10^4 + 324}{4^4 + 324} \right) \left(\frac{22^4 + 324}{16^4 + 324} \right) \cdots \left(\frac{(12k-2)^4 + 324}{(12k-8)^4 + 324} \right)$$
   Hint: On Page 3; try your best without using the hint.

5. (*) Chef Nicholas is a very talented cook. Give him a frying pan, and a stack of pancakes and waffles, he can flip freely any pieces of pancakes and waffles at the top of the stack.

   For instance, suppose that the frying pan has a stack of 6 pieces of pancakes and waffles as follows (`P` for pancake, `W` for waffle):

   `P W P P W P`   (reading from bottom to top)

If Nicholas flips the top 5 pieces, the stack would become:

`P P W P P W` (reading from bottom to top)

Furthermore, if Nicholas continues to flip the top 3 pieces, the stack would become:

`P P W W P P` (reading from bottom to top)

And, with one more flip of top 4 pieces, the stack would become:

`P P P P W W` (reading from bottom to top)

Show that Nicholas can use at most $n - 1$ flips to flip any stack of n pieces of pancakes and waffles, so that all the pancakes are arranged together, while all the waffles are also arranged together.

6. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$ and so on.

   [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

7. (*, Challenging) Let n be a positive integer, and consider an array with 2 rows and $2n$ columns. Each entry in the array is either `0` or `1`. It is known that for each row, exactly n entries are `0` and exactly n entries are `1`.

   For a particular column, if both entries are `0`, we call it a `0`-column; else, if both entries are `1`, we call it a `1`-column.

   Show that the number of `0`-columns is the same as the number of `1`-columns.

   For instance, suppose $n = 3$. Suppose the array looks like the following:

1	0	1	0	0	1
0	0	1	1	0	1

   Each row contains exactly n `0`s and exactly n `1`s. Also, we see that there are two `0`-columns (the 2nd one and the 5th one), and there are two `1`-columns (the 3rd one and the 6th one).

8. Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them.

   [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

9. There are 50 baskets of apples. Each basket contains at most 24 apples. Show that there are at least 3 baskets containing the same number of apples.

10. Suppose $n + 1$ integers are chosen from 1 to $2n$. Show that there exist two of the chosen numbers which have no common factor.

11. Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$, there are at least two selected integers whose sum is 26.

12. A lecture lasts 50 minutes and 6 students were sleeping for at least 10 minutes during the lecture. Show that two students were sleeping simultaneously at some point during the lecture.

13. (*) Show that in a group of 10 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

14. (**) Show that in a group of 9 people, either there are 3 mutual friends, or 4 mutual enemies, or both.

15. (*) Show that among a group of 100 people, if any two will shake hands at most once, then at least two people will shake hands for the same number of times.

16. (*, Challenging) Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $(b_1, b_2, b_3, b_4, b_5, b_6)$ be two arrangements of the integers $1, 2, 3, 4, 5, 6$. Consider the six pairs of differences $|a_i - b_i|$. Is it possible that all of these differences are not the same?

Hint for Question 3:

$$(2q+1) \times \frac{4q-1}{2q+1} = 4q - 1; \qquad (2q+1) \times \frac{12q+3}{6q+3} = 4q+1.$$

Hint for Question 4:

$$\text{Sophie Germain Identity: } a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$$