CHAPTER 17 Conformal Mapping

This chapter was new in the ninth edition.

SECTION 17.1. Geometry of Analytic Functions: Conformal Mapping, page 737

Purpose. To show conformality (preservation of angles in size and sense) of the mapping by an analytic function $w = f(z)$; exceptional are points with $f'(z) = 0$.

Main Content, Important Concepts

Concept of mapping. Surjective, injective, bijective

Conformal mapping (Theorem 1)

Magnification, Jacobian

Examples. Joukowski airfoil

Comment on the Proof of Theorem 1

The crucial point is to show that $w = f(z)$ rotates all straight lines (hence all tangents) passing through a point z_0 through the same angle $\alpha = \arg f'(z_0)$, but this follows from (3). This in a nutshell is the proof, once the stage has been set.

SOLUTIONS TO PROBLEM SET 17.1, page 740

2. $|w| > 2, \quad \operatorname{Im} w < 0$. This is the exterior of the circle $|w| = 2$ in the lower half-plane $\operatorname{Im} w < 0$.

4. Annulus $1/e \leqq |w| \leqq e^2$ cut along the negative real axis.

6. The rectangle $-\ln 2 \leqq u \leqq 0, \quad 0 \leqq v < \pi/2$

7. $2z - 2z^{-3} = 0, \quad z^4 = 1$, hence $\pm 1, \pm i$

8. The derivative

$$(5z^4 - 80) \exp (z^5 - 80z)$$

is zero at $z = \pm 2$ and $\pm 2i$ since the exponential function has no zeros.

10. $M = |w'| = 2/|z|^3 = 1$ on $|z| = \sqrt[3]{2} = 1.26$

12. $M = |w'| = 1/|z - 1|^2 = 1$ when $|z - 1|^2 = (x - 1)^2 + y^2 = 1$, the circle with center $z = 1$ and radius 1, which passes through the origin.

SECTION 17.2. Linear Fractional Transformations (M bius Transformations), page 741

Purpose. Systematic discussion of linear fractional transformations (Möbius transformations), which owe their importance to a number of interesting properties shown in this section and the next one.

Main Content

Definition (1)

Special cases (3), Example 1

Images of circles and straight lines (Theorem 1)

One-to-one mapping of the extended complex plane

Inverse mapping (4)

Fixed points

SOLUTIONS TO PROBLEM SET 17.2, page 744

3. $w = \bar{z} = x - iy = z = x + iy$, $y = 0$ (the x-axis), $w = 1/z = z$, $z^2 = 1$, $z = \pm 1$. The center of a rotation is a fixed point.

5. We obtain

$$w(-\tfrac{1}{2}iz - 1) = z - \tfrac{1}{2}i,$$

hence

$$z = \frac{-2w + i}{iw + 2}.$$

Mappings of this kind will occur in the next section in connection with mapping disks onto disks.

7. No fixed point when $a = 1$, that is, a translation has no fixed points in the finite plane; $z = -b/(a - 1)$ when $a \neq 1$, for instance, $z = 0$ when $b = 0$, which is a rotation about the origin and, for $|a| \neq 1$, combined with a uniform dilatation or contraction.

10. We have

$$k(z - 1)(z + 1) = k(z^2 - 1)$$

and, by comparing with (5),

$$c = k, \qquad a - d = 0, \qquad b = k$$

so that

$$w = \frac{az + b}{bz + a},$$

including the identity mapping $w = z$ when $b = 0$.

SECTION 17.3. Special Linear Fractional Transformations, page 745

Purpose. Continued discussion to show that linear fractional transformations map "standard domains" conformally onto each other.

Main Content

Determination by three points and their images

Mappings of standard domains (disks, half-planes)

Angular regions

SOLUTIONS TO PROBLEM SET 17.3, page 749

2. The inverse is

$$z = \frac{4w - 1}{-2w + 1}.$$

The fixed points $z = -\frac{3}{4} \pm \frac{1}{4}\sqrt{17}$ are obtained as solutions of

$$2z^2 - (1 - 4)z - 1 = 0.$$

5. Formula (2) gives

$$\frac{w + 1}{w - \infty} \cdot \frac{-\infty}{1} = \frac{z}{z - i} \cdot \frac{-2i}{-i} = \frac{2z}{z - i}.$$

Replacing infinity on the left as indicated in the text, we get $w + 1$ on the left, so that the answer is

$$w = \frac{2z}{z - i} - 1 = \frac{z + i}{z - i}.$$

Caution! In setting up further problems by starting from the result, which is quite easy, one should check how complicated the solution, starting from (2), will be; this may often involve substantial work until one reaches the final form.

8. The requirement is that

$$w = u = \frac{ax + b}{cx + d}$$

must come out real for all real x. Hence the four coefficients must be real, except possibly for a common complex factor.

SECTION 17.4. Conformal Mapping by Other Functions, page 749

Purpose. So far we have discussed mapping properties of z^n, e^z, and linear fractional transformations. We now add to this a discussion of trigonometric and hyperbolic functions.

SOLUTIONS TO PROBLEM SET 17.4, page 752

2. The portion of $1 < |w| < \infty$ in the first quadrant

5. The region in the right half-plane bounded by the v-axis and the hyperbola $4u^2 - \frac{4}{3}v^2 = 1$ because

$$\sin z = u + iv = \sin x \cosh y + i \cos x \sinh y$$

reduces to

$$\sin iy = i \sinh y$$

when $x = 0$; thus $u = 0$ (the v-axis) is the left boundary of that region. For $x = \pi/6$ we obtain

$$\sin(\pi/6 + iy) = \sin(\pi/6)\cosh y + i\cos(\pi/6)\sinh y;$$

thus

$$u = \tfrac{1}{2}\cosh y, \qquad v = \tfrac{1}{2}\sqrt{3}\sinh y$$

and we obtain the right boundary curve of that region from

$$1 = \cosh^2 y - \sinh^2 y = 4u^2 - \tfrac{4}{3}v^2,$$

as asserted.

6. $w = \cos z = \cos x \cosh y - i \sin x \sinh y$ (Sec. 13.6) gives

$$u = \cos x \cosh k, \qquad v = -\sin x \sinh k.$$

For $k \neq 0$ we thus obtain

$$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = \cos^2 x + \sin^2 x = 1.$$

For $k = 0$ we have $u = \cos x$, hence

$$-1 \leqq u \leqq 1, \qquad v = 0.$$

8. The upper boundary maps onto the ellipse

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1$$

and the lower boundary onto the ellipse

$$\frac{u^2}{\cosh^2 \frac{1}{2}} + \frac{v^2}{\sinh^2 \frac{1}{2}} = 1.$$

Since $0 < x < 2\pi$, we get the entire ellipses as boundaries of the image of the given domain, which therefore is an elliptical ring.

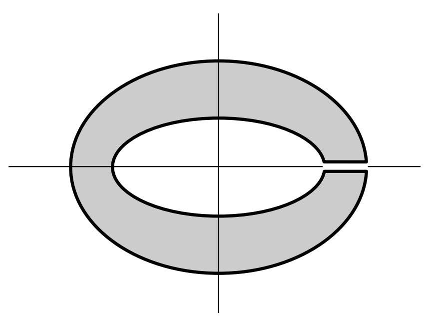
Now the vertical boundaries $x = 0$ and $x = 2\pi$ map onto the same segment

$$\sinh \tfrac{1}{2} \leqq u \leqq \sinh 1$$

of the u-axis because for $x = 0$ and $x = 2\pi$ we have

$$u = \cosh y, \qquad v = 0.$$

Answer: Elliptical annulus between those two ellipses and cut along that segment. See the figure.



Section 17.4. Problem 8

10. The lower boundary segment maps onto $\cos 1 \leqq u \leqq 1$ $(v = 0)$. The left and right boundary segments map onto portions of the hyperbola

$$\frac{u^2}{\cos^2 1} - \frac{v^2}{\sin^2 1} = 1.$$

The upper boundary segment maps onto a portion of the ellipse

$$\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} = 1.$$

SECTION 17.5. Riemann Surfaces. *Optional*, page 753

Purpose. To introduce the idea and some of the simplest examples of Riemann surfaces, on which multivalued relations become single-valued, that is, functions in the usual sense.

Short Courses. This section may be omitted.

SOLUTIONS TO PROBLEM SET 17.5, page 754

2. $-1 - 2i$, two sheets

4. $i/3$, infinitely many sheets

SOLUTIONS TO CHAPTER 17 REVIEW QUESTIONS AND PROBLEMS, page 755

2. $1/\pi < |w| < \pi, \quad v > 0$

4. $y = 0$ maps onto the nonnegative real axis $u = x^2$, $v = 0$. The other boundary $y = 2$ gives $u = x^2 - 4$, $v = 4x$. Elimination of x gives the parabola

$$u = v^2/16 - 4,$$

with apex at $u = -4$ and opening to the right.

Answer: The domain to the right of that parabola except for the nonnegative u-axis.

6. $u = \frac{1}{16}v^2 - 4$. See Example 1 and Fig. 379 in Sec. 17.1.

$y = -2$ and 2 are mapped onto the same parabola. The formulas in Example 1 of Sec. 17.1 show that this is true for any straight lines $x = \pm c$ (see the previous Prob. 15) and $y = \pm k$ since these formulas contain $c^2 = (-c)(-c)$ and k^2, respectively.

8. $z = re^{i\theta}, \quad w = \dfrac{1}{r}e^{-i\theta}$; hence the first quadrant of the interior of the unit disk in the z-plane maps onto the exterior of $|w| = 1$ in the fourth quadrant of the w-plane.

10. $-\pi/4 \leqq \arg w \leqq 0$

12. We have

$$w = \frac{1}{1 + iy} = \frac{1 - iy}{1 + y^2}$$

and thus obtain

$$u^2 + v^2 = \frac{1 + y^2}{(1 + y^2)^2} = \frac{1}{1 + y^2} = u$$

that is,

$$(u - \tfrac{1}{2})^2 + v^2 = \tfrac{1}{4},$$

the circle with center $z = \frac{1}{2}$ and radius $\frac{1}{2}$, thus passing through $w = 0$ and 1.

In the text it is shown that geometrical arguments may often replace calculations. In the present problem, that is as follows.

Since $z = 1 + iy$ lies outside $|z| = 1$, is a straight line, extends to infinity, and maps $z = 1$ onto $w = 1$, its image is a circle, passing through 0 and 1. Since this image lies inside $|w| = 1$, its center must lie on the u-axis, otherwise it would intersect $|w| = 1$ at some point different from 1; hence that center lies at $u = \frac{1}{2}$.

14. $w = 1/z$ by inspection. This illustrates that, in many cases, one hardly needs the general formula (2) in Sec. 17.3. Sometimes, one can also use points and their images, one after another, and determine the coefficents of the mapping function stepwise.

16. Translation $w = z + 2i$

18. $w = 1 + iz$, a translation combined with a rotation, first rotating, then translating

20. $z = \pm 2\sqrt{2}, \pm 2\sqrt{2}i$

22. $\pm i$

24. $z = \pm 1$

26. To map a quarter-disk onto a disk, we have to quadruple angles; thus $w = z^4$ maps the given quarter-disk onto the unit disk $|w| = 1$, and the exterior is obtained by taking the reciprocal.

Answer: $w = 1/z^4$

28. $w = 2/z$ maps $|z| = 1$ onto $|w| = 2$ and the interior onto the exterior. Hence $w + 2 = 2/z$ will give the answer

$$w = -2 + \frac{2}{z} = \frac{2 - z}{z}.$$

30. $z^2/4$ maps the semidisk onto the unit disk; $4/z^2$ maps its interior onto the exterior; $4\pi/z^2$ maps this onto the exterior of the circle with center 0 and radius π, so that the *answer* is

$$w = \pi + 4\pi/z^2.$$