Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.7

No. 1

問答或證明題，不解

No. 2

From (8) and (6) we obtain

$$A = \frac{1}{\pi}\int_{-a}^{a} \pi \cos pv\, dv = 2\frac{\sin pa}{p}, B = 0$$

and thus

$$u = \int_0^\infty 2\frac{\sin(pa)\cos(px)e^{-e^2p^2t}}{p}\,dp$$

No. 3

$$u(x,t) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, e^{-c^2p^2t}\, dp$$

$$u(x,0) = f(x) = \frac{1}{1+x^2}$$

$f(x)$ is even function

$$u(x,0) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\, dp = \frac{1}{1+x^2}$$

$B(p) = 0.$

$$A(p) = \frac{2}{\pi}\int_0^\infty \frac{\cos pv}{1+v^2}\,dv = \frac{2}{\pi}\cdot\frac{\pi}{2}e^{-p} = e^{-p}$$

$$\therefore u(x,t) = \int_0^\infty e^{-p}\cos px\cdot e^{-c^2p^2t}\,dp$$

$$= \int_0^\infty e^{-p-c^2p^2t}\cos px\, dp.$$

No. 4

$$u(x,0) = f(x) = e^{-|x|}$$

$f(x)$ is even function

$B(p) = 0$

$$A(p) = \frac{2}{\pi}\int_0^{\infty} e^{-v} \cos pv \, dv =$$

No. 5

$$u(x,0) = f(x) = \begin{cases} |x| & ; \quad |x| < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$f(x)$ is even function.

$B(p) = 0$

$$A(p) = \frac{2}{\pi}\int_0^{\infty} v \cos pv \, dv = \frac{2}{\pi}\int_0^{1} v \cos pv \, dv$$

$$= \frac{2}{\pi p^2}(\cos p + p \sin p - 1)$$

$$u(x,t) = \int_0^{\infty} \frac{2}{\pi p^2}(\cos p + p \sin p - 1) \cos px \, e^{-c^2 p^2 t} \, dp$$

No. 6

$$u(x,0) = f(x) = \begin{cases} x & ; \quad |x| < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$f(x)$ is odd function.

$A(p) = 0$

$$B(p) = \frac{2}{\pi}\int_0^\infty v \sin pv \, dv = \frac{2}{\pi}\int_0^1 v \sin pv \, dv$$

$$= \frac{2}{\pi p^2}(\sin p - p\cos v)$$

$$u(x,t) = \int_0^\infty \frac{2}{\pi p^2}(\sin p - p\cos v)\sin px \, e^{-c^2p^2t} \, dp$$

No. 7

$$u(x,0) = f(x) = \frac{\sin x}{x}$$

$f(x)$ is even function

$B(p) = 0$

$$A(p) = \frac{2}{\pi}\int_0^\infty \frac{\sin v}{v}\cos pv \, dv$$

$$= \begin{cases} \frac{2}{\pi}\cdot\frac{\pi}{2} = 1 & ; \quad 0 < p < 1 \\ 0 & ; \quad p > 1 \end{cases}$$

$$\therefore \ u(x,t) = \int_0^\infty A(p)\cos px \, e^{-c^2p^2t} \, dp$$

$$= \int_0^1 \cos px \, e^{-c^2p^2t} \, dp$$

No. 8

$$u(x,t)=\int_0^1 \cos px\, e^{-c^2p^2t}\, dp$$

$$u(x.0)=\int_0^1 \cos px\, dp$$

$$=\frac{\sin px}{x}\Big|_0^1$$

$$=\frac{\sin x}{x}$$

No. 9

(1) $$\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-w^2}dw$$

$$\operatorname{erf}(-x)=\frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-w^2}dw=\frac{-2}{\sqrt{2}}\int_0^x e^{-w^2}dw=-\operatorname{erf}(x).$$

so erf x is odd.

(2). $$\int_a^b e^{-w^2}dw=\int_0^b e^{-w^2}dw-\int_0^a e^{-w^2}dw$$

$$=\frac{\sqrt{\pi}}{2}(\operatorname{erf} b-\operatorname{erf} a).$$

(3) $$\int_{-b}^b e^{-w^2}dw=\frac{\sqrt{\pi}}{2}(\operatorname{erf} b-\operatorname{erf}(-b))$$

$$=\frac{\sqrt{\pi}}{2}(\operatorname{erf} b+\operatorname{erf} b)$$

$$=\sqrt{\pi}\operatorname{erf} b.$$

No.10

See (36) in App. A3.1.

No.11

問答或證明題，不解

No.12

問答或證明題，不解

No.13

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\,e^{-z^2}\,dz$$

$$z=\frac{v-x}{2c\sqrt{t}}$$

$$f(x)=\begin{cases}1 & ;\ x>0\\ 0 & ;\ x<0\end{cases} \Rightarrow f(x+2cz\sqrt{t})=\begin{cases}1 & ;\ x>-2cz\sqrt{t}\\ 0 & ;\ x<-2cz\sqrt{t}\end{cases}$$

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2c\sqrt{t}}}^{\infty} e^{-x^2}\,dz$$

$$=\frac{1}{2}-\frac{1}{2}\,erf\left(-\frac{x}{2c\sqrt{t}}\right)$$

No.14

$$u(x,t)=\frac{U_0}{\sqrt{\pi}}\int_{\frac{-(1+x)}{2c\sqrt{t}}}^{\frac{(1-x)}{2c\sqrt{t}}} e^{-z^2}\,dz$$

$$=\frac{U_0}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}\left(erf\left(\frac{1-x}{2c\sqrt{t}}\right)-erf\left(\frac{-(1+x)}{2c\sqrt{t}}\right)\right)$$

$$=\frac{U_0}{2}\left(erf\left(\frac{1-x}{2c\sqrt{t}}\right)+erf\left(\frac{1+x}{2c\sqrt{t}}\right)\right)$$

No.15

$$\Phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

Let $w = \frac{s}{\sqrt{2}}$ $\quad ds = \sqrt{2}\, dw$

$$\Phi = \frac{\sqrt{2}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-w^2} dw$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}}\left(\frac{\sqrt{\pi}}{2}\right)\left(erf\left(\frac{x}{\sqrt{2}}\right) - erf(-\infty)\right)$$

$$= \frac{1}{2}\left(erf\left(\frac{x}{\sqrt{2}}\right) + erf(\infty)\right)$$

$$= \frac{1}{2} + \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right).$$