CS5314 Randomized Algorithms

Homework 2 (Suggested Solution)

1. **Ans:**

   (a) Firstly, by linearity of expectation, we have

   $$\mathrm{E}[X^2] = \mathrm{E}[X \sum_i X_i] = \sum_i \mathrm{E}[XX_i].$$

   Then,

   $$\begin{aligned}\mathrm{E}[XX_i] &= \mathrm{E}[\mathrm{E}[XX_i \mid X_i]] \\ &= \Pr(X_i = 1)\mathrm{E}[XX_i \mid X_i = 1] + \Pr(X_i = 0)\mathrm{E}[XX_i \mid X_i = 1] \\ &= \Pr(X_i = 1)\mathrm{E}[X \mid X_i = 1] + \Pr(X_i = 0) \cdot 0 = \Pr(X_i)\mathrm{E}[X \mid X_i = 1].\end{aligned}$$

   Combining the above gives the desired result.

   (b) $\mathrm{E}[X \mid X_i = 1] = (n-1)p + 1$.

   (c) $\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = np((n-1)p+1) - (np)^2 = np(1-p)$.

2. **Ans:**

   (a) $\Pr(Y_i = 0) = \Pr(HH) + \Pr(TT) = 1/2$, and $\Pr(Y_i = 1) = 1 - \Pr(Y_i = 0) = 1/2$.

   (b) If ith pair and jth pair do not share any bit,

   $$\mathrm{E}[Y_iY_j] = \Pr(Y_i = 1 \cap Y_j = 1) = \Pr(Y_i = 1)\Pr(Y_j = 1) = \mathrm{E}[Y_i]\mathrm{E}[Y_j].$$

   Otherwise, they share exactly one bit, so that

   $$\begin{aligned}\mathrm{E}[Y_iY_j] &= \Pr(Y_i = 1 \cap Y_j = 1) \\ &= \Pr(\text{unshared bits have same value, and opposite to shared bit}) \\ &= 1/4 = \mathrm{E}[Y_i]\mathrm{E}[Y_j].\end{aligned}$$

   (c) $\mathrm{Cov}(X, Y) = \mathrm{E}[(Y_i - \mathrm{E}[Y_i])(Y_j - \mathrm{E}[Y_j])] = \mathrm{E}[Y_iY_j] - \mathrm{E}[Y_i]\mathrm{E}[Y_j] = 0$.

   (d)

   $$\begin{aligned}\Pr(|Y - \mathrm{E}[Y]| \geq n) &\leq \frac{\mathrm{Var}(Y)}{n^2} \\ &= \frac{\sum_i \mathrm{Var}(Y_i) + \sum_{i,j} \mathrm{Cov}(Y_i, Y_j)}{n^2} \\ &= \frac{\binom{n}{2} \cdot 1/4 + 0}{n^2} \leq 1/8.\end{aligned}$$

3. **Ans.** For each i, we have

   $$E[X_i] = \Pr(X_i = 1) = 1/n \quad \text{and} \quad E[X_i^2] = \Pr(X_i^2 = 1) = 1/n.$$

For any i, j with $i \neq j$, we have

$$E[X_iX_j] = \Pr(X_iX_j = 1) = \Pr(X_i = 1 \cap X_j = 1)$$
$$= \Pr(X_i = 1)\Pr(X_j = 1 \mid X_i = 1) = \frac{1}{n(n-1)}.$$

Let X be the number of fixed points. So, $X = \sum_{i=1}^{n} X_i$, and $E[X] = \sum_{i=1}^{n} E[X_i] = 1$. Then, we have

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - 1$$

$$= \sum_{i=1}^{n} E[X_i^2] + \sum_{i \neq j} E[X_iX_j] - 1 = n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n(n-1)} - 1 = 1.$$

4. (a) **Ans.** Let $X = n\tilde{p}$ denote the number of heads that came up. So, $X = \mathrm{Bin}(n, p)$ and $\mathrm{E}[X] = np$. Then we have:

$$\begin{aligned}
\Pr(|p - \tilde{p}| > \varepsilon p) &= \Pr(|np - n\tilde{p}| > n\varepsilon p) \\
&= \Pr(np - n\tilde{p} > n\varepsilon p) + \Pr(n\tilde{p} - np > n\varepsilon p) \\
&= \Pr(X < (1-\varepsilon)\mathrm{E}[X]) + \Pr(X > (1+\varepsilon)\mathrm{E}[X]) \\
&\leq \exp\left(\frac{-na\varepsilon^2}{2}\right) + \exp\left(\frac{-na\varepsilon^2}{3}\right).
\end{aligned}$$

(b) **Ans.** When

$$n > \frac{3\ln(2/\delta)}{a\varepsilon^2} \quad ,$$

we have:

$$\frac{na\varepsilon^2}{3} > \ln(2/\delta) \quad \text{so that} \quad \delta > 2\,\exp\left(\frac{-na\varepsilon^2}{3}\right).$$

Combining this with the result of part (a), we have:

$$\Pr(|p - \tilde{p}| > \varepsilon p) \leq \exp\left(\frac{-na\varepsilon^2}{2}\right) + \exp\left(\frac{-na\varepsilon^2}{3}\right) < 2\exp\left(\frac{-na\varepsilon^2}{3}\right) < \delta.$$

5. (a) **Ans.** We shall make use of the following claim:

Claim 1. *For any $r \in [0, 1]$, $e^{tr} - 1 \leq r(e^t - 1)$.*

Proof. Let $f(r) = r(e^t - 1) - e^{tr} + 1$. Then we have $f'(r) = (e^t - 1) - te^{tr}$, and $f''(x) = -t^2e^{tr} \leq 0$. This implies that f is a concave function.
In other words, for $r \in [0, 1]$, f achieves minimum value either at the boundaries $f(0)$ or $f(1)$. Thus, $f(r) \geq \min\{f(0), f(1)\} = 0$ for all $r \in [0, 1]$, and the claim follows. $\square$

Back to the answer. Since $W = \sum_{i=1}^{n} a_i X_i$, we have

$$\nu = E[W] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i p_i.$$

For any i,

$$E\left[e^{ta_i X_i}\right] = p_i e^{ta_i} + (1 - p_i) = 1 + p_i(e^{ta_i} - 1) \leq 1 + p_i a_i (e^t - 1),$$

where the last inequality is from Claim 1.
Hence,

$$E\left[e^{ta_i X_i}\right] \leq e^{p_i a_i (e^t - 1)},$$

and by the independence of X_i's and property of MGF,

$$E\left[e^{tW}\right] = \prod_{i=1}^{n} E\left[e^{ta_i X_i}\right] \leq \prod_{i=1}^{n} e^{a_i p_i (e^t - 1)} = e^{\nu(e^t - 1)}.$$

For any $t > 0$, we have

$$\Pr(W \geq (1+\delta)\nu) = \Pr(e^{tW} \geq e^{t(1+\delta)\nu}) \leq \frac{E[e^{tW}]}{e^{t(1+\delta)\nu}} \leq \frac{e^{\nu(e^t - 1)}}{e^{t(1+\delta)\nu}}.$$

Then, for any $\delta > 0$, we can set $t = \ln(1+\delta) > 0$ and obtain:

$$\Pr(W \geq (1+\delta)\nu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\nu}.$$

(b) **Ans.** For any $t < 0$, we have

$$\Pr(W \leq (1-\delta)\nu) = \Pr(e^{tW} \geq e^{t(1-\delta)\nu}) \leq \frac{E[e^{tW}]}{e^{t(1-\delta)\nu}} \leq \frac{e^{\nu(e^t - 1)}}{e^{t(1-\delta)\nu}}.$$

Then, for any $0 < \delta < 1$, we can set $t = \ln(1-\delta) < 0$ and obtain:

$$\Pr(W \leq (1-\delta)\nu) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\nu}.$$