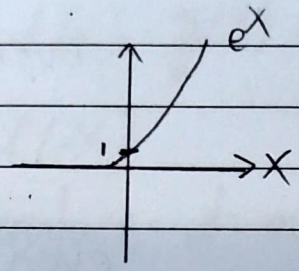
PDE HW2

106033233 周聖豪 No P1 Date

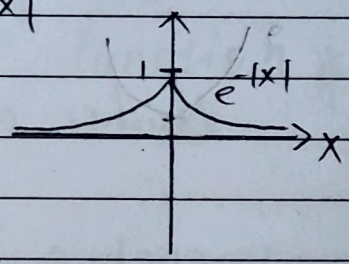
P49.

1. (a) e^x



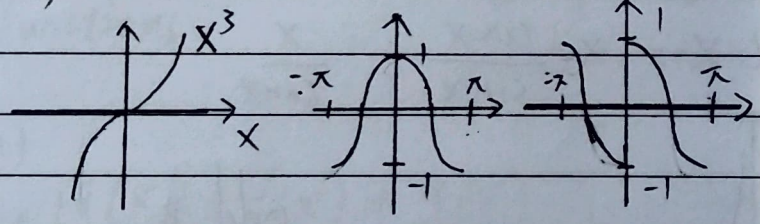
Neither odd nor even #

(b) $e^{-|x|}$



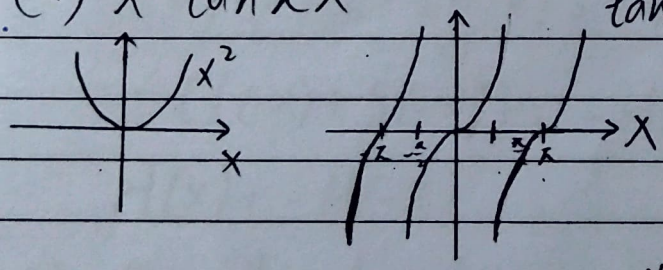
Even #

(c) $x^3 \cos \pi x$



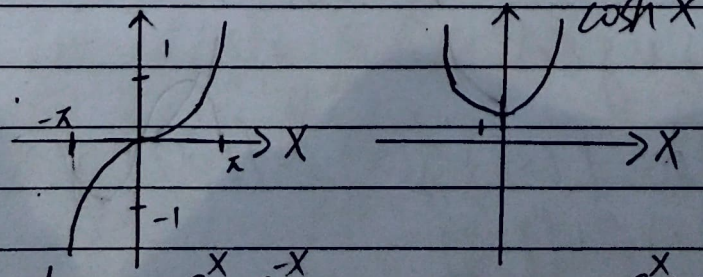
Odd #

(d) $x^2 \tan \pi x$



odd #

(e) $\sinh x - \cosh x = \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} = -e^{-x}$



Neither odd nor even #

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

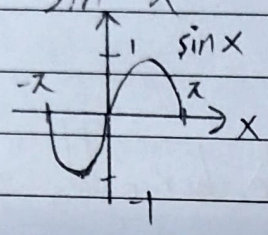
KENKO® 30 Lines, 6 mm

odd: $f(x) = -f(-x)$ $H(x) = f(x) \cdot g(x) = -f(-x) \cdot g(-x)$

even: $g(x) = g(x)$ $= -H(-x)$

2.

(a) $\sin^2 x$



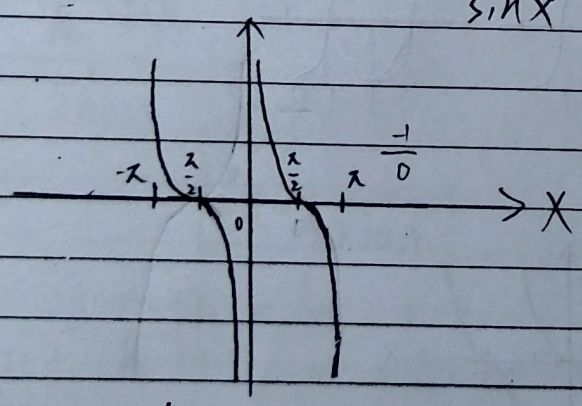
even #

(b) $\sin(x^2)$ even #

(c) $\ln x$ Neither odd nor even #

(d) $\dfrac{x^2}{x^2+1}$ even #

(e) $x \cot x = x \dfrac{\cos x}{\sin x} = \dfrac{x}{\tan x}$



odd #

3.

(a) let $f(x) = f(-x)$, $g(x) = g(-x)$ are even functions

let $H(x) = f(x) + g(x)$

$\because H(x) = f(x) + g(x) = f(-x) + g(-x) = H(-x)$

$\therefore H(x) = H(-x)$

$\Rightarrow$ Sum of even function is an even function #

(b) let $l(x) = f(x) \cdot g(x)$

$\because l(x) = f(x) \cdot g(x) = f(-x) \cdot g(-x) = l(-x)$

$\therefore l(x) = l(-x)$

$\Rightarrow$ The product of even functions is an even function #

4. $f_i(x) = -f_i(-x)$

let $f(x) = -f(-x)$, $g(x) = -g(-x)$ are odd functions

(a)

let $H(x) = f(x) + g(x)$

$\because H(x) = f(x) + g(x) = -f(-x) - g(-x)$

$= -(f(-x) + g(-x)) = -H(-x)$

$\therefore H(x) = -H(-x)$

$\Rightarrow$ Sum of odd functions is an odd function #

(b) let $l(x) = \prod_{i=1}^{N} f_i(x)$

$\because l(x) = \prod_{i=1}^{N} f_i(x) = (-1)^N \cdot \prod_{i=1}^{N} f_i(-x)$

$\therefore \begin{cases} l(x) = -l(x) & \text{if } N \text{ is odd} \\ l(x) = l(x) & \text{if } N \text{ is even} \end{cases}$

KENKO® 30 Lines, 6 mm

$\Rightarrow$ If time of product is odd, it's an odd function.
If time of product is even, it's an even function.

No
Date

5,

let $f(x) = -f(-x)$ be an odd function

let $H(x) = |f(x)|$

$\because H(x) = |f(x)| = |-f(-x)| = |f(-x)|$
$= H(-x)$

$\therefore H(x) = H(-x)$

$\Rightarrow$ The absolute value of odd functions is an Even function #

6.

let $f(x) = -f(-x)$ be an odd function
let $g(x) = g(-x)$ be an even function

let $H(x) = f(x) \cdot g(x)$

$\because H(x) = f(x) \cdot g(x) = -f(-x) \cdot g(-x)$
$= -H(-x)$

$\therefore H(x) = -H(-x)$

$\Rightarrow$ Product of an odd times an even function is an odd function #

9.

Let $f(x)$ be an both odd and even function. $f(x): \mathbb{R} \to \mathbb{R}$

$\because f(x) = f(-x)$ and $f(x) = -f(-x)$

$\therefore f(-x) = -f(-x)$

$\Rightarrow f(x) = f(-x) = 0$

$f(x) = 0$ is all functions that are both even and odd.

10.

$$f(x) = \begin{cases} 4-x, & 0 \le x < 4 \\ -4-x, & -4 < x < 0 \end{cases}$$

odd

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x) \cdot dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \cos\frac{n\pi}{L}x \cdot dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \sin\frac{n\pi}{L} \cdot x \cdot dx$$

$$a_0 = \frac{1}{8}\left(\int_0^4 (4-x) \cdot dx + \int_{-4}^{0} (-4-x) \cdot dx\right)$$

$$= \frac{1}{8}\left[\left(4x - \frac{x^2}{2}\right)\Big|_0^4 + \left(-4x - \frac{x^2}{2}\right)\Big|_{-4}^{0}\right]$$

$$= \frac{1}{8}\left[(16-8) - \cancel{(0)} + \cancel{(0)} - (16-8)\right] = \frac{1}{8}[8-8] = 0$$

KENKO® 30 Lines, 6 mm

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\sin x = \cos x$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(x)\cdot\cos\frac{n\pi}{4}\cdot x\cdot dx$$

$\Rightarrow$ Since $f(x)$ is an odd function,

$a_n = 0$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(x)\cdot\sin\frac{n\pi}{4}x\cdot dx$$

$$= \frac{1}{4}\left(\int_{0}^{4}(4-x)\cdot\sin\frac{n\pi}{4}x\cdot dx + \int_{-4}^{0}(-4-x)\cdot\sin\frac{n\pi}{4}x\cdot dx\right)$$

$$= \frac{1}{4}\left[(4-x)\cdot\frac{-\cos\frac{n\pi}{4}x}{\frac{n\pi}{4}} - \int_{0}^{4} -\frac{-\cos\frac{n\pi}{4}x}{\frac{n\pi}{4}}\cdot dx\right]$$

$$+\left[(-4-x)\cdot\frac{-\cos\frac{n\pi}{4}x}{\frac{n\pi}{4}} - \int_{-4}^{0} -\frac{-\cos\frac{n\pi}{4}x}{\frac{n\pi}{4}}\cdot dx\right]$$

$$= \frac{1}{4}\left\{\left[(x-4)\frac{4\cos\frac{n\pi}{4}x}{n\pi} + \frac{-\sin\frac{n\pi}{4}x}{\left(\frac{n\pi}{4}\right)^2}\right]\Bigg|_{0}^{4} + \left[(x+4)\cdot\frac{4\cos\frac{n\pi}{4}x}{n\pi} + \frac{-\sin\frac{n\pi}{4}x}{\left(\frac{n\pi}{4}\right)^2}\right]\Bigg|_{-4}^{0}\right\}$$

$$= \frac{1}{4}\left\{\left[\cancel{0\cdot\frac{4\cos 0}{n\pi}} + \cancel{\frac{-\sin n\pi}{\left(\frac{n\pi}{4}\right)^2}}\right] - \left[-4\cdot\frac{4\cos 0}{n\pi} + \cancel{\frac{-\sin 0}{\left(\frac{n\pi}{4}\right)^2}}\right] + \left[4\cdot\frac{4\cos 0}{n\pi} + \cancel{\frac{-\sin 0}{\left(\frac{n\pi}{4}\right)^2}}\right] - \left[\cancel{0\cdot\frac{4\cos n\pi}{n\pi}} + \cancel{\frac{-\sin n\pi}{\left(\frac{n\pi}{4}\right)^2}}\right]\right\}$$

KENKO® 30 Lines, 6 mm

$$= \frac{1}{4}\left\{ 4 \cdot \frac{4}{n\pi} + 4 \cdot \frac{4}{n\pi} \right\} = \frac{8}{n\pi}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$= 0 + \sum_{n=1}^{\infty} \frac{8}{n\pi} \sin \frac{n\pi}{4} x$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{4} x \quad \text{; Odd Function}$$

$$= \frac{8}{\pi}\left(\sin \frac{\pi x}{4} + \frac{1}{2} \sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{4} + \frac{1}{4} \sin \pi x + \frac{1}{5} \sin \frac{5\pi x}{4} + \cdots \right)$$

11.

$$f(x) = x^2 \quad -1 < x < 1, \; p = 2 \Rightarrow \text{even}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \cdot dx = \frac{1}{2} \int_{-1}^{1} x^2 \cdot dx = \frac{1}{2} \cdot \frac{x^3}{3} \Big|_{-1}^{1}$$

$$= \frac{1}{6} x^3 \Big|_{-1}^{1}$$

$$= \frac{1}{6} - \left(-\frac{1}{6}\right) = \frac{1}{3}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos \frac{n\pi}{L} x \cdot dx$$

$$= \frac{1}{1} \int_{-1}^{1} x^2 \cdot \cos n\pi \cdot x \cdot dx$$

$$= x^2 \cdot \frac{\sin n\pi x}{n\pi} - \int_{-1}^{1} 2x \cdot \frac{\sin n\pi x}{n\pi} \cdot dx$$

$$= x^2 \cdot \frac{\sin n\pi x}{n\pi} - \left(2x \cdot \frac{-\cos n\pi x}{(n\pi)^2} - \int_{-1}^{1} 2 \cdot \frac{-\cos n\pi x}{(n\pi)^2} \cdot dx \right)$$

$$= x^2 \cdot \frac{\sin n\pi x}{n\pi} - \left(2x \cdot \frac{-\cos n\pi x}{(n\pi)^2} - 2 \cdot \frac{-\sin n\pi x}{(n\pi)^3} \right) \Big|_{-1}^{1}$$

KENKO® 30 Lines, 6 mm

$$= x^2 \cdot \frac{\sin n\pi x}{n\pi} + 2x \cdot \frac{\cos n\pi x}{(n\pi)^2} - 2 \cdot \frac{\sin n\pi x}{n\pi^3} \Big|_{-1}^{1}$$

$$= \left(\cancel{\frac{\sin n\pi}{n\pi}} + 2 \cdot \frac{\cos n\pi}{(n\pi)^2} - 2 \cdot \cancel{\frac{\sin n\pi}{(n\pi)^3}} \right) -$$

$$\left(\cancel{\frac{\sin -n\pi}{n\pi}} - 2 \cdot \frac{\cos -n\pi}{(n\pi)^2} - 2 \cdot \cancel{\frac{\sin -n\pi}{(n\pi)^3}} \right)$$

$$= \frac{2\cos n\pi}{(n\pi)^2} + \frac{2\cos -n\pi}{(n\pi)^2}$$

$$= \frac{4\cos n\pi}{(n\pi)^2}$$

$\frac{4}{\pi^2} \cdot \frac{1}{n^2}$

$\frac{4}{\pi^2} \cdot \frac{-1}{n^2}$

$$a_n = \begin{cases} \frac{4}{(n\pi)^2}, & \text{if } n \text{ is even} \\ -\frac{4}{(n\pi)^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi}{c} x$$

$$= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} \cdot \cos n\pi \cdot x$$

$$= \frac{1}{3} + \frac{4}{\pi^2} \left(-\cos \pi x + \frac{1}{4} \cos 2\pi x - \frac{1}{9} \cos 3\pi \cdot x + \frac{1}{16} \cos 4\pi x - \frac{1}{25} \cos 5\pi x \ldots \right)$$

24.

$$f(x)\begin{cases} 1, & 2<x<4 \\ 0, & 0<x<2 \end{cases} \qquad P=2L=8$$

(a) Odd Ext

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi}{L}x \cdot dx$$

$$a_0 = \frac{1}{4}\int_0^4 f(x)\cdot dx$$

$-\cos\pi + \cos\frac{\pi}{2} = 1$

$-\cos 2\pi + \cos\pi = -2$

$-\cos 3\pi + \cos\frac{3\pi}{2} = 1$

$-\cos 4\pi + \cos\frac{4\pi}{2} = 0$

$-\cos 5\pi + \cos\frac{5\pi}{2} = 1$

$-\cos 6\pi + \cos\frac{6\pi}{2} = -2$

$$= \frac{1}{4}\left[\int_2^4 1\cdot dx + \int_0^2 0\cdot dx\right] = \frac{1}{4}\left(x\Big|_2^4\right) = \frac{1}{4}(4-2) = \frac{1}{2}$$

$$b_n = \frac{1}{2}\int_0^4 f(x)\cdot \sin\frac{n\pi}{4}x \cdot dx$$

$$= \frac{1}{2}\left[\int_2^4 1\cdot \sin\frac{n\pi}{4}x\cdot dx + \cancel{\int_0^2 0\cdot \sin\frac{n\pi}{4}x\cdot dx}\right]$$

$$= \frac{1}{2}\left(\frac{-\cos\frac{n\pi}{4}x}{\frac{n\pi}{4}}\Bigg|_2^4\right)$$

$$= \frac{-2\cos\frac{n\pi}{4}x}{n\pi}\Bigg|_2^4 = \frac{-2\cos n\pi}{n\pi} - \frac{-2\cos\frac{n\pi}{2}}{n\pi}$$

$$= -\frac{-2\cos n\pi}{n\pi} + \frac{-2\cos\frac{n\pi}{2}}{n\pi} = \frac{2}{n\pi}\left(-\cos n\pi + \cos\frac{n\pi}{2}\right)$$

KENKO® 30 Lines, 6 mm

$$f(x) = \frac{2}{\pi}\left(\sin\frac{\pi x}{4} - \sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{4} + \frac{1}{5}\sin\frac{5\pi x}{4} - \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right)$$

Odd Extension

(b) Even Ext

$$a_0 = \frac{1}{L}\int_0^L f(x)\cdot dx = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi}{L}x\cdot dx$$

$$= \frac{1}{2}\int_0^4 f(x)\cdot\cos\frac{n\pi}{4}x\cdot dx$$

$$= \frac{1}{2}\left(\int_2^4 1\cdot\cos\frac{n\pi}{4}\cdot x\cdot dx + \cancel{\int_0^2 0\cdot\cos\frac{n\pi}{4}\cdot x\cdot dx}\right)$$

$$= \frac{1}{2}\left(\frac{\sin\frac{n\pi}{4}x}{\frac{n\pi}{4}}\Big|_2^4\right) = \frac{2\sin\frac{n\pi}{4}x}{n\pi}\Big|_2^4$$

$$= \frac{2}{n\pi}\left(\cancel{\sin n\pi} - \sin\frac{n\pi}{2}\right) = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$a_1 = -\frac{2}{\pi}\cdot\sin\frac{\pi}{2} = \frac{2}{\pi}\cdot(-1)$ $\quad a_2 = -\frac{2}{\pi}\cdot\frac{1}{2}\sin\pi = \frac{2}{\pi}\cdot 0$

$a_3 = -\frac{2}{\pi}\cdot\frac{1}{3}\sin\frac{3\pi}{2} = \frac{2}{\pi}\cdot\frac{1}{3}$ $\quad a_4 = -\frac{2}{\pi}\cdot\frac{1}{4}\cdot\sin 2\pi = \frac{2}{\pi}\cdot 0$

$a_5 = -\frac{2}{\pi}\cdot\frac{1}{5}\sin\frac{5\pi}{2} = \frac{2}{\pi}\cdot\left(-\frac{1}{5}\right)$ $\quad a_6 = -\frac{2}{\pi}\cdot\frac{1}{6}\sin 3\pi = \frac{2}{\pi}\cdot 0$

$a_7 = -\frac{2}{\pi}\cdot\frac{1}{7}\sin\frac{7\pi}{2} = \frac{2}{\pi}\cdot\frac{1}{7}$ $\quad a_9 = -\frac{2}{\pi}\cdot\frac{1}{9}\sin\frac{9\pi}{2} = \frac{2}{\pi}\cdot\left(-\frac{1}{9}\right)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\cdot\sin\frac{n\pi}{2}\right)\cdot\cos\frac{n\pi}{L}x$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(-\cos\frac{\pi}{4}x + \frac{1}{3}\cos\frac{3\pi}{4}x - \frac{1}{5}\cos\frac{5\pi}{4}x + \frac{1}{7}\cos\frac{7x}{4} - \frac{1}{9}\cos\frac{9x}{4} + \cdots\right)$$

Even Extension

KENKO® 30 Lines, 6 mm

28.

$$f(x) = x;\ 0 \le x \le L,\ P = 2L$$

(a) Odd Extension

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi}{L} x \cdot dx$$

$$b_n = \frac{2}{L} \int_0^L x \cdot \sin \frac{n\pi}{L} \cdot x \cdot dx$$

$$= \frac{2}{L} \left(x \cdot \frac{-\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}} - \int \frac{-\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}} \cdot dx \right) \Bigg|_0^L$$

$$= \frac{2}{L} \left(-x \cdot \frac{\cos \frac{n\pi}{L} x}{\frac{n\pi}{L}} + \frac{\sin \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)^2} \Bigg|_0^L \right)$$

$$= \frac{2}{L} \left[\left(-L^2 \cdot \frac{\cos n\pi}{n\pi} + \cancel{\frac{L^2 \cdot \sin n\pi}{(n\pi)^2}} \right) - \left(\cancel{-0 \cdot \frac{\cos 0}{\frac{n\pi}{L}}} + \cancel{\frac{L^2 \cdot \sin 0}{(n\pi)^2}} \right) \right]$$

$$= \frac{2}{L} \cdot (-L^2) \cdot \frac{\cos n\pi}{n\pi} = -\frac{2L}{n\pi} \cos n\pi$$

$$b_n = \begin{cases} \frac{2L}{n\pi}, & \text{if } n \text{ is odd} \\ -\frac{2L}{n\pi}, & \text{if } n \text{ is even} \end{cases}$$

No
Date

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$= \frac{2L}{\pi}\left(\sin \frac{\pi}{L}x - \frac{1}{2}\sin \frac{2\pi}{L}x + \frac{1}{3}\sin \frac{3\pi}{L}x - \frac{1}{4}\sin \frac{4\pi}{L}x + \frac{1}{5}\sin \frac{5\pi}{L}x - \cdots \right)$$

Odd Extention

(b) Even Extension

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{L}\int_0^L f(x) \cdot dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cdot \cos \frac{n\pi}{L} x \cdot dx$$

$$a_0 = \frac{1}{L}\int_0^L x \cdot dx = \frac{1}{L} \cdot \frac{x^2}{2}\Big|_0^L = \frac{1}{2L}(L^2 - 0)$$

$$= \frac{L}{2}$$

$$a_n = \frac{2}{L}\int_0^L x \cdot \cos \frac{n\pi}{L} x \cdot dx$$

$$= \frac{2}{L}\left(x \cdot \frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}} - \int \frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}} \cdot dx \right)\Big|_0^L$$

$$= \frac{2}{L}\left(x \cdot \frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}} + \frac{\cos \frac{n\pi}{L} x}{\left(\frac{n\pi}{L}\right)^2} \right)\Big|_0^L$$

$$= \frac{2}{L}\left[\left(\cancel{L^2 \cdot \frac{\sin n\pi}{n\pi}} + L^2 \cdot \frac{\cos n\pi}{(n\pi)^2} \right) - \left(\cancel{0 \cdot \frac{\sin 0}{\frac{n\pi}{L}}} + \frac{\cos 0}{\left(\frac{n\pi}{L}\right)^2} \right) \right]$$

KENKO® 30 Lines, 6 mm

$$= \frac{2}{l}\left[l^2 \cdot \frac{\cos n\pi}{(n\pi)^2} - l^2 \cdot \frac{1}{(n\pi)^2} \right]$$

$$= \frac{2l}{(n\pi)^2}(\cos n\pi - 1)$$

$$a_n = \begin{cases} \frac{2l}{(n\pi)^2} \cdot (-2) = -\frac{4l}{(n\pi)^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{l} \cdot x$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos \frac{\pi}{l}x + \frac{1}{9}\cos \frac{3\pi}{l}x + \frac{1}{25}\cos \frac{5\pi}{l}x + \frac{1}{49}\cos \frac{7\pi}{l}x + \frac{1}{81}\cos \frac{9\pi}{l}x + \cdots \right)$$

Problem III

$f(t) = \cos(2t) + 3\sin(2t)$

$f(t) = X \cdot \cos(2t + Y)$

$\because \cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$\therefore f(t) = X \cdot \cos(2t + Y)$

$= X \cdot [\cos(Y) \cdot \cos(2t) - \sin(Y) \cdot \sin(2t)]$

$= X \cdot \cos(Y) \cdot \cos(2t) - X \cdot \sin(Y) \cdot \sin(2t)$

Thus $\begin{cases} X \cdot \cos(Y) = 1 \\ -X \cdot \sin(Y) = 3 \end{cases}$

To solve Y

$\frac{-X \cdot \sin(Y)}{X \cdot \cos(Y)} = 3$, $\tan(Y) = -3$, $Y = \arctan(-3)$

To solve X

$X \cdot \cos(\arctan(-3)) = 1$

$X = \frac{1}{\cos(\arctan(-3))}$

$\Rightarrow X = \frac{1}{\cos(\arctan(-3))}$

$Y = \arctan(-3)$

KENKO® 30 Lines, 6 mm