CHAPTER 10 Vector Integral Calculus. Integral Theorems

SECTION 10.1. Line Integrals, page 413

Purpose. To explain line integrals in space and in the plane conceptually and technically with regard to their evaluation by using the representation of the path of integration.

Main Content, Important Concepts

Line integral (3), (3′), its evaluation

Its motivation by work done by a force ("work integral")

General properties (5)

Dependence on path (Theorem 2)

Background Material. Parametric representation of curves (Sec. 9.5); a couple of review problems may be useful.

Comments on Content

The integral (3) is more practical than (8) (more direct in view of subsequent material), and work done by a force motivates it sufficiently well.

Independence of path will be settled in the next section.

Further Comments on Text and Problem Set 10.1

Examples 1 and 2 show that the evaluation of line integrals in the plane and in space is conceptually the same, making the difference just a technical one.

Example 3 illustrates a main motivation of (3), which we take as a definition of line integral.

Theorem 1 and Project 12 in the problem set concern direction preserving and reversing transformations of line integrals, as they are needed in various applications.

Kinetic energy appears in connection with the work integral; see Example 4.

The basic fact of path dependence is emphasized in Theorem 2 and again in Project 12.

Problems 2–11 and 15–20 concern the evaluation of integrals (3) and (8), respectively.

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2. $\mathbf{r} = [t, t^2]$, so that the integrand is

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}' = [t^4, -t^2] \cdot [1, 2t] = t^4 - 2t^3.$$

Integration gives $\frac{1}{5}t^5 - \frac{1}{2}t^4$. The limits of integration are 0 and 1, so that the *answer* is $-3/10$.

3. 4

4. For instance, we may take parametric form of the line to be

$$\mathbf{r} = [2 - 2t, 2t], (0 \le t \le 1).$$

Then we obtain the integrand

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}' &= [2, (2 - 2t)^2 - 4t)^2] \cdot [-2, 2] \\ &= [2, 4 - 8t] \cdot [-2, 2]. \\ &= 4 - 16t\end{aligned}$$

Integration between the limits 0 and 1, gives -4

5. $\mathbf{r} = [2 \cos t, 2 \sin t], 0 \le t \le \pi/2; \frac{2}{3} - \pi$.

6. Helix on the cylinder $x^2 + z^2 = 4$, $\mathbf{F}(C) = [2 \cos t - t,\ t - 2 \sin t,\ 2 \sin t - 2 \cos t]$. *Answer*: $-8\pi + 2\pi^2$

8. C looks similar to the curve in Fig. 210, Sec. 9.5. The integrand is

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \quad \mathbf{r}'(t) &= [e^t, \cosh t^2, \sinh t^3] \quad [1, 2t, 3t^2] \\ &= e^t + 2t \cosh t^2 + 3t^2 \sinh t^3.\end{aligned}$$

Integration from $t = 0$ to $t = \frac{1}{2}$ gives the *answer*

$$e^{1/2} + \sinh \tfrac{1}{4} + \cosh \tfrac{1}{8} - 2.$$

9. 8.5, 0

10. Here we integrate around a triangle in space. For the three sides and corresponding integrals we obtain

$$\mathbf{r}_1 = [t, t, 0], \quad \mathbf{r}_1' = [1, 1, 0], \quad \mathbf{F}(\mathbf{r}_1(t)) = [t, 0, 2t], \quad \int_0^1 t\, dt = \frac{1}{2}$$

$$\mathbf{r}_2 = [1, 1, t], \quad \mathbf{r}_2' = [0, 0, 1], \quad \mathbf{F}(\mathbf{r}_2(t)) = [1, -t, 2], \quad \int_0^1 2\, dt = 2$$

$$\mathbf{r}_3 = [1 - t, \quad 1 - t, \quad 1 - t], \quad \mathbf{r}_3' = [-1, -1, -1],$$

$$\mathbf{F}(\mathbf{r}_3(t)) = [1 - t, \quad -1 + t, \quad 2 - 2t], \quad \int_0^1 (-2 + 2t)\, dt = -2 + 1.$$

Hence the *answer* is $\frac{3}{2}$.

11. "Exponential helix", $-1/3 + 1/3\, e^{6\pi}$.
$-3\, e^{-t} - e^{-t^2}, 4 - 3\, e^{-2} - e^{-4}$

12. Project. (a) $\mathbf{r} = [\cos t, \quad \sin t]$, $\mathbf{r}' = [-\sin t, \quad \cos t]$. From $\mathbf{F} = [xy, \quad -y^2]$ we obtain $\mathbf{F}(\mathbf{r}(t)) = [\cos t \sin t, \quad -\sin^2 t]$. Hence the integral is

$$-2\int_0^{\pi/2} \cos t \sin^2 t\, dt = -\frac{2}{3}.$$

Setting $t = p^2$, we have $\mathbf{r} = [\cos p^2, \quad \sin p^2]$ and

$$\mathbf{F}(\mathbf{r}(p)) = [\cos p^2 \sin p^2, \quad -\sin^2 p^2].$$

Now $\mathbf{r}' = [-2p \sin p^2, \quad 2p \cos p^2]$, so that the integral is

$$\int_0^{\sqrt{\pi/2}} (-2p \cos p^2 \sin^2 p^2 - 2p \cos p^2 \sin^2 p^2)\, dp = -\frac{2}{3}.$$

(b) $\mathbf{r} = [t, \quad t^n]$, $\mathbf{F}(\mathbf{r}(t)) = [t^{n+1}, \quad -t^{2n}]$, $\mathbf{r}' = [1, \quad nt^{n-1}]$. The integral is

$$\int_0^1 (t^{n+1} - nt^{3n-1})\, dt = \frac{1}{n + 2} - \frac{1}{3}.$$

(c) The limit is $-\frac{1}{3}$. The first portion of the path gives 0, since $y = 0$. The second portion is $\mathbf{r}_2 = [1, \quad t]$, so that $F(\mathbf{r}_2(t)) = [t, \quad -t^2]$, $\mathbf{r}' = [0, \quad 1]$. Hence the integrand is $-t^2$, which upon integration gives $-\frac{1}{3}$.

14. $\mathbf{r} = [t, \frac{4}{3}t]$, $0 \leqq t \leqq 3$, $\quad L = 5$, $\quad |\mathbf{F}| = \sqrt{t^4 + \frac{16}{9}t^2}$. The derivative is

$$\tfrac{1}{2}[t^4 + \tfrac{16}{9}t^2]^{-1/2}(4t^3 + \tfrac{32}{9}t).$$

The expression in parentheses $(\cdots)$ has the root $t = 0$, but no further real roots. Hence the maximum of $|\mathbf{F}|$ is taken at (3, 4), so that we obtain the bound

$$L|\mathbf{F}| \leqq 5\sqrt{81 + 16} = 5\sqrt{97} < 50.$$

Calculation gives $\mathbf{r}' = [1, \frac{4}{3}]$, $\quad \mathbf{F}(\mathbf{r}(t)) = [t^2, \frac{4}{3}t]$. The integral is

$$\int_0^3 \left(t^2 + \frac{16}{9}t\right) dt = 9 + 8 = 17.$$

The result is typical, that is, the point of the *ML*-inequality is generally not to obtain a very sharp upper bound for integrals, but to show that an integral remains bounded in some limit process.

Also, instead of L, you may sometimes have to be satisfied with using an upper bound for L if L itself is complicated.

16. $3t + \cosh t + 5 \sinh t$ integrated from 0 to 1 gives

$$\tfrac{3}{2} + \sinh 1 + 5 \cosh 1 - 5 = \sinh 1 + 5 \cosh 1 - \tfrac{7}{2}.$$

The projection of this space curve into the xy-plane is a cosh curve. Its projection into the xz-plane is a sinh curve. From these projections one can conclude the general shape of C.

18. $\mathbf{F}(\mathbf{r}(t)) = [\sin t, \cos t, 0]$ integrated from 0 to $\pi/4$ gives

$$\left[-\frac{1}{\sqrt{2}} + 1, \frac{1}{\sqrt{2}}, 0\right].$$

20. C is an exponentially increasing curve in the plane $x = y$ from (0, 0, 0) to $(5, 5, e^5)$. The representation of C gives

$$\mathbf{F}(\mathbf{r}(t)) = [te^t, te^t, t^4].$$

Integration from 0 to 5 gives

$$[4e^5 + 1, 4e^5 + 1, 625].$$

SECTION 10.2. Path Independence of Line Integrals, page 419

Purpose. Independence of path is a basic issue on line integrals, and we discuss it here in full.

Main Content, Important Concepts

Definition of independence of path

Relation to gradient (Theorem 1), potential theory

Integration around closed curves

Work, conservative systems

Relation to exactness of differential forms

Comment on Content

We see that our text pursues three ideas by relating path independence to (i) gradients (potentials Theorem 1), (ii) closed paths (Theorem 2), and (iii) exactness of the form under the integral sign (Theorem 3*). The complete proof of the latter needs Stokes's theorem, so here we leave a small gap to be easily filled in Sec. 10.9.

It would not be a good idea to delay introducing the important concept of path independence until Stokes's theorem is available.

Simple connectedness of domains is further emphasized in Example 4.

The determination of a potential is shown in Example 2.

Problem 1, writing a report on the concepts and relations in this section, should help to understand the various ideas.

The remaining problems shed light on path dependence and independence from several angles, with experimentation and computer application.

SOLUTIONS TO PROBLEM SET 10.2, page 425

2. No. The origin is a boundary point of this "degenerate annulus," which therefore is not simply connected (but doubly connected).

3. $\sin 3x \sin \frac{y}{2}$, 1.

4. The exactness test for path independence gives

$$(F_2)_x = \frac{e^{2y}}{\sqrt{x}} = (F_1)_y.$$

We find $\mathbf{F} = \operatorname{grad} f$, where

$$f(x, y) = \sqrt{x}e^{2y}$$

so that for the integral we have $f(9, 1) - f(4, 0) = \sqrt{9}e^2 - \sqrt{4}e^0 = 20.17$.

6. For the dx-term and the dy-term, the exactness test of path independence gives $2xye^{x^2} + y^2 + z^2 = 2xye^{x^2} + y^2 + z^2$, etc. We find

$$f = \tfrac{1}{2} \exp(x^2 + y^2 + z^2).$$

Evaluation at the limits gives

$$\tfrac{1}{2}(e^2 - 1).$$

8. The test for independence gives $f = x \sin yz$. Hence evaluating at the limits, we get

$$f(2, \pi, 1/2) - f(1, 3/4, \pi) = 2 - \frac{1}{\sqrt{2}} = 1.29.$$

10. Project. **(a)** $2y^2 \neq x^2$ from (6*).

(b) $r = [t, bt]$, $0 \leqq t \leqq 1$, represents the first part of the path. By integration we obtain $b/4 + b^2/2$. On the second part, $r = [1, t]$, $b \leqq t \leqq 1$. Integration gives $2(1 - b^3)/3$. Equating the derivative of the sum of the two expressions to zero gives $b = 1/\sqrt{2} = 0.70711$. The corresponding maximum value of I is $1/(6\sqrt{2}) + \frac{2}{3} = 0.78452$.

(c) The first part is $y = x/c$ or $\mathbf{r} = [t, \quad t/c]$, $0 \leqq t \leqq c$. The integral over this portion is $c^3/4 + c/2$. For the second portion $\mathbf{r} = [t, \quad 1]$, $c \leqq t \leqq 1$, the integral is $(1 - c^3)/3$. For $c = 1$ we get $I = 0.75$, the same as in (b) for $b = 1$. This is the maximum value of I for the present paths through $(c, 1)$ because the derivative of I with respect to c is positive for $0 \leqq c \leqq 1$.

12. CAS Experiment. The circle passes through (0, 0) and (1, 1) if its center P lies on the line $y = 1 - x$, so that P is $(a, 1 - a)$, a arbitrary. Then the radius is

$$\rho = \sqrt{a^2 + (1 - a)^2}.$$

One would perhaps except a value near $a = \frac{1}{2}$, for which the circle is smallest. The experiment gives for the circle

$$\mathbf{r} = [a + \rho \cos t, \quad 1 - a + \rho \sin t]$$

a minimal value $I_{\min} = 50.85137100$, approximately, of the integral when $a = 0.4556$, approximately.

13. $e^{-a^2} \sin 2b$

14. Dependent; indeed, we obtain

$$\sinh xz + zx \cosh xz \neq -\sinh xz - zx \cosh xz.$$

16. Independent, $f = ye^x - ze^y$, $be^a - ce^b$

17. Dependent, $2 \neq 0$, etc.

18. Dependent because for

$$\mathbf{F} = [yz \cos xy, xz \cos xy, -2 \sin xy]$$

we obtain

$$\text{curl } \mathbf{F} = [-3x \cos xy, 3y \cos xy, 0] \neq \mathbf{0}.$$

20. The point of the problem is to make the student think of the nature of (6) and (6′). The constructions are trivial. Start from an $\mathbf{F}$ satisfying (6), for instance, $\mathbf{F} = [1, 1, 1]$ and, to violate the third equation (6′), add to $F_1 = 1$ a function of y, e.g., y.

SECTION 10.3. Calculus Review: Double Integrals. *Optional,* page 426

Purpose. We need double integrals (and line integrals) in the next section and review them here for completeness, suggesting that the student go on to the next section.

Content

Definition, evaluation, and properties of double integrals

Some standard applications

Change of variables, Jacobians (6), (7)

Polar coordinates (8)

Historical Comment

The two ways of evaluating double integrals explained in the text give the same result. For continuous functions this was known at least to Cauchy. Some calculus books call this **Fubini's theorem**, after the Italian mathematician GUIDO FUBINI (1879–1943; 1939–1943 professor at New York University), who in 1907 proved the result for arbitrary Lebesgue-integrable functions (published in *Atti Accad. Naz. Lincei*, *Rend.*, $\mathbf{16_1}$, 608–614).

Comments on Text and Problems

We can present here only an absolute minimum of what we shall need, and students should be encouraged to supplement this by material from their calculus books, if needed.

Centers of gravity of other domains (cross sections appearing in engineering design) can be found in engineering handbooks.

SOLUTIONS TO PROBLEM SET 10.3, page 432

2. Integration over y gives $8/3x^3$. Integration over x now gives the *answer* $2/3$.

3. $-\frac{4}{3}y^3, -\frac{16}{3}$

4. This order of integration is less practical since it requires splitting the integral into two parts

$$\int_{-2}^{0}\int_{-x}^{2}(x^2 - y^2)\,dy\,dx + \int_{0}^{2}\int_{x}^{2}(x^2 - y^2)\,dy\,dx.$$

Integration over y, we get from the first part

$$\int_{-2}^{0}[x^2(2 + x) - 8/3 - 1/3\,x^3]\,dx = -8/3.$$

The other part gives $-8/3$, too. *Answer*: $-16/3$.

5. $\displaystyle\int_0^1 3\,x - 3\,x^2 + 4\,x^3 - 4\,x^5\,dx = 5/6$

6. $-\sinh y + \sinh 2y, \frac{1}{2} - \cosh 1 + \frac{1}{2}\cosh 2$

7. $-\sinh x + \sinh 2x, \frac{1}{2} - \cosh 1 + \frac{1}{2}\cosh 2$

8. After the integration over x, we have

$$\int_0^{\pi/4}\frac{1}{3}\cos^3 y\sin y\,dy = \frac{1}{16}.$$

10. $\displaystyle\int_0^1\int_0^{1-x^2}\int_0^{1-x^2}dy\,dz\,dx = \int_0^1\int_0^{1-x^2}(1 - x^2)\,dz\,dx = \frac{8}{15}$

12. $\bar{x} = b/2$ for reasons of symmetry. Since the given R and its left half (the triangle with vertices $(0, 0)$, $(b/2, 0)$, $(b/2, h)$) have the same $\bar{y}$, we can consider that half, for which $M = \frac{1}{4}bh$. We obtain

$$\bar{y} = \frac{4}{bh}\int_0^{b/2}\int_0^{2hx/b}y\,dy\,dx = \frac{4}{bh}\int_0^{b/2}\frac{1}{2}\left(\frac{2hx}{b}\right)^2 dx$$

$$= \frac{4}{bh}\cdot\frac{1}{2}\left(\frac{2h}{b}\right)^2\cdot\frac{1}{3}\left(\frac{b}{2}\right)^3 = \frac{h}{3}.$$

Note that $\bar{y}$ is the same value as in the next problem (Prob. 13), for obvious reasons.

14. The total mass is

$$M = \int_0^{\pi} \int_{r_1}^{r_2} r\, dr\, d\theta = \frac{\pi}{2}(r_2^2 - r_1^2).$$

We thus obtain $\bar{x} = 0$ by symmetry and

$$\bar{y} = \frac{2}{\pi(r_2^2 - r_1^2)} \int_{r_1}^{y_2} (r \sin\theta)\, r\, dr\, d\theta$$

$$= \frac{2}{(r_2^2 - r_1^2)} \cdot 2 \cdot \frac{1}{3}(r_2^3 - r_1^3)$$

$$= \frac{4(r_2^3 - r_1^3)}{3\pi(r_2^2 - r_1^2)}.$$

With $r_1 = 0$ and $r_2 = r$ this gives the *answer* to Prob. 15.

16. $\bar{x} = \bar{y} = 4r/3\pi$ from Prob. 15 without calculation.

18. We obtain

$$I_x = \int_0^{b/2} \int_0^{2hx/b} y^2\, dy\, dx + \int_{b/2}^{b} \int_0^{2h-2hx/b} y^2\, dy\, dx$$

$$= \int_0^{b/2} \frac{1}{3}\left(\frac{2hx}{3}\right)^3 dx + \int_{b/2}^{b} \frac{1}{3}\left[2h\left(1 - \frac{x}{b}\right)\right]^3 dx$$

$$= \frac{1}{24} bh^3 + \frac{1}{24} bh^3 = \frac{1}{12} bh^3.$$

Each of these two halves of R contributes half to the moment I_x about the x-axis. Hence we could have simplified our calculation and saved half the work. Of course, this ***does not hold*** for I_y. We obtain

$$I_y = \int_0^{b/2} \int_0^{2hx/b} x^2\, dy\, dx + \int_{b/2}^{b} \int_0^{2h-2hx/b} x^2\, dy\, dx$$

$$= \int_0^{b/2} x^2\left(\frac{2hx}{b}\right) dx + \int_{b/2}^{b} x^2\left(2h - \frac{2hx}{b}\right) dx$$

$$= \frac{1}{32} b^3 h + \frac{11}{96} b^3 h = \frac{7}{48} b^3 h.$$

20. We denote the right half of R by $R_1 \cup R_2$, where R_1 is the rectangular part and R_2 the triangular. The moment of interia I_{x1} of R_1 with respect to the x-axis is

$$I_{x1} = \int_0^{b/2} \int_0^{h} y^2\, dy\, dx = \int_0^{b/2} \frac{h^3}{3}\, dx = \frac{1}{6} bh^3.$$

Similarly for the triangle R_2 we obtain

$$I_{x2} = \int_{b/2}^{a/2} \int_0^{h(2x-a)/(b-a)} y^2\, dy\, dx$$

$$= \int_{b/2}^{a/2} \frac{1}{3} \frac{h^3(2x-a)^3}{(b-a)^3}\, dx$$

$$= \frac{1}{24} h^3(a-b).$$

Together,

$$\frac{1}{2} I_x = \frac{h^3}{24}(3b+a) \qquad \text{and} \qquad I_x = \frac{1}{12} h^3(3b+a).$$

I_y is the same as in Prob. 19; that is,

$$I_y = \frac{h(a^4-b^4)}{48(a-b)}.$$

This can be derived as follows, where we integrate first over x and then over y, which is simpler than integrating in the opposite order, where we would have to add the two contributions, one over the square and the other over the triangle, which would be somewhat cumbersome. Solving the equation for the right boundary

$$y = \frac{h}{a-b}(a-2x)$$

for x, we have

$$x = \frac{1}{2h}(ah-(a-b)y)$$

and thus

$$\frac{1}{2} I_y = \int_0^h \int_0^{(ah-(a-b)y)/2h} x^2\, dx\, dy$$

$$= \int_0^h \frac{1}{24h^3}(ah-(a-b)y)^3\, dy$$

$$= \frac{h}{96}(a^3+a^2b+ab^2+b^3) = \frac{h(a^4-b^4)}{96(a-b)}.$$

Now we multiply by 2, because we considered only the right half of the profile.

SECTION 10.4. Green s Theorem in the Plane, page 433

Purpose. To state, prove, and apply Green's theorem in the plane, relating line and double integrals.

Comment on the Role of Green's Theorem in the Plane

This theorem is a special case of each of the two "big" integral theorems in this chapter, Gauss's and Stokes's theorems (Secs. 10.7, 10.9), but we need it as the essential tool in the proof of Stokes's theorem.

The present theorem must not be confused with *Green's first and second theorems* in Sec. 10.8.

Comments on Text and Problems

Equation (1) is the basic formula in this section and later on in applications of Theorem 1. Eq. $(1')$ shows its vectorial form, and other forms of Green's theorem are included in Project 12 of the problem set.

The proof of Theorem 1 proceeds componentwise, (2) and (3) relating to F_1 when $F_2 = 0$. For F_2 when $F_1 = 0$ the calculation is similar, with a change of the direction of an integration as noted in the proof.

Cancelation of integrals over subdivisions, as it occurs here (see Fig. 238), is a standard idea that will occur quite often in the sequel.

The important Examples 2–4 need no further comments.

The resulting integral formula (9) has applications in theory and practice.

A basic formula for harmonic functions (solutions of Laplace's equation whose second partial derivatives are continuous) is considered in Probs. 18–20.

All these formulas and calculational problems emphasize the great importance of the present theorem.

SOLUTIONS TO PROBLEM SET 10.4, page 438

1. $\displaystyle\int_0^{2\pi} 1/4\,(\sin(t))^2 + 1/4\,(\cos(t))^2 = \pi/2$

2. Integrate $(F_2)_x - (F_1)_y = 20x^3 - 6y^2$ over y from -2 to 2 and the result over x from -2 to 2, obtaining -128.

3. $(F_2)_x - (F_1)_y = -y^2 e^{-x} + 3/8\, x^2 e^{-y}$, $-8 + 9e^{-2} - e^{-3}$

4. We obtain

$$\int_0^1\int_{x^2}^{x} 2x\cosh 2y\,dy\,dx =$$

$$= \int_0^1 (-x\sinh(2x^2) + x\sinh(2x))\,dx$$

$$= \frac{1}{4} + \frac{1}{4}\cosh 2 - \frac{1}{4}\sinh 2$$

5. $2x + 2y$, $2x(1 - x^2) + (2 - x^2)^2 - 1$, $x = -1..1$, $\frac{55}{15}$

6. $\displaystyle\int_1^3\int_x^{3x} (-\sinh x - \cosh y)\,dy\,dx$. Evaluation gives

$$\int_1^3 (-2\sinh(x)\,x + \sinh(x) - \sinh(3x))\,dx$$

$$= \cosh(1) - 2\sinh(1) - 14/3\cosh(3) + 2\sinh(3) - 1/3\cosh(9).$$

7. [Note that Prob. 7 is not affected, since the region R is not changed.]

8. $\mathbf{F} = \text{grad}\,(e^{-x}\cos y)$, so that the integral around a closed curve is zero. Also the integrand in (1) on the left is identically zero.

10. This is a portion of a circular ring (annulus) bounded by the circles of radii 1 and 2 centered at the origin, in the first quadrant bounded by $y = x$ and the y-axis. The integrand is $-1/y^2 - 2x^2y$. We use polar coordinates, obtaining

$$\begin{aligned}
&\int_{\pi/4}^{\pi/2}\int_1^2\left(-\frac{1}{r^2\sin^2\theta} - 2r^3\cos^2\theta\sin\theta\right) r\,dr\,d\theta \\
&= \int_{\pi/4}^{\pi/2}\left[-\frac{\ln 2}{\sin^2\theta} - \frac{2}{5}(32 - 1)\cos^2\theta\sin\theta\right] d\theta \\
&= (\ln 2)\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + \frac{62}{15}\left(\cos^3\frac{\pi}{2} - \cos^2\frac{\pi}{4}\right) \\
&= -\ln 2 - \frac{31}{15\sqrt{2}} \\
&= -2.155.
\end{aligned}$$

12. Project. We obtain div $\mathbf{F}$ in (11) from (1) if we take $\mathbf{F} = [F_2, \ \ -F_1]$. Taking $\mathbf{n} = [y', \ \ -x']$ as in Example 4, we get from (1) the right side in (11),

$$(\mathbf{F}\cdot\mathbf{n})\,ds = \left(F_2\frac{dy}{ds} + F_1\frac{dx}{ds}\right) ds = F_2\,dy + F_1\,dx.$$

Formula (12) follows from the explanation of (1′).

Furthermore, div $\mathbf{F} = 7 - 3 = 4$ times the area of the disk of radius 2 gives 16π. For the line integral in (11) we need

$$\mathbf{r} = \left[2\cos\frac{s}{2}, \quad 2\sin\frac{s}{2}\right], \qquad \mathbf{r}' = \left[-\sin\frac{s}{2}, \quad \cos\frac{s}{2}\right], \quad \mathbf{n} = [y', \quad -x']$$

where s varies from 0 to 4π. This gives

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C (7xy' + 3yx')\,ds = \int_0^{4\pi}\left(14\cos^2\frac{s}{2} - 6\sin^2\frac{s}{2}\right) ds = 16\pi.$$

In (12) we have curl $\mathbf{F} = \mathbf{0}$ and

$$\mathbf{F}\cdot\mathbf{r}' = -14\cos\frac{s}{2}\sin\frac{s}{2} - 6\cos\frac{s}{2}\sin\frac{s}{2} = -10\sin s$$

which gives zero upon integration from 0 to 4π.

14. $\nabla^2 w = 2y + 2x$, so that we obtain

$$\begin{aligned}
&\int_0^1\int_0^{\sqrt{1-x^2}} (2y + 2x)\,dy\,dx \\
&= \int_0^1 (1 - x^2 + 2x\sqrt{1 - x^2})\,dx \\
&= \left[x - \frac{1}{3}x^3 - \frac{2}{3}(1 - x^2)^{3/2}\right]\Bigg|_0^1 \\
&= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.
\end{aligned}$$

Confirmation in polar coordinates. We have $\nabla^2 w = 2r\cos\theta + 2r\sin\theta$, so that

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^1 2r(\cos\theta + \sin\theta)\, r\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{2}{3}(\cos\theta + \sin\theta)\, d\theta \\
&= \frac{2}{3}(\sin\theta - \cos\theta)\Big|_0^1 \\
&= \frac{2}{3}(1 + 1) = \frac{4}{3}.
\end{aligned}$$

16. $\nabla^2 w = 2 + 2 = 4$. *Answer*: $4 \cdot 4\pi$. *Confirmation:* For C, use

$$\mathbf{r} = \left[2\cos\frac{s}{2}, 2\sin\frac{s}{2}\right].$$

The corresponding unit tangent vector is

$$\dot{\mathbf{r}} = \left[-\sin\frac{s}{2}, \cos\frac{s}{2}\right].$$

The outer unit normal vector is

$$\mathbf{n} = \left[\cos\frac{s}{2}, \sin\frac{s}{2}\right].$$

Furthermore,

$$\operatorname{grad} w = [2x, 2y] = \left[2 \cdot 2\cos\frac{s}{2}, 2 \cdot 2\sin\frac{s}{2}\right],$$

so that

$$(\operatorname{grad} w) \bullet \mathbf{n} = 4\cos^2\frac{s}{2} + 4\sin^2\frac{s}{2} = 4$$

and

$$\int_{s=0}^{4\pi} 4\, ds = 16\pi.$$

18. Set $F_1 = -ww_y$ and $F_2 = ww_x$ in Green's theorem, where subscripts x and y denote partial derivatives. Then $(F_2)_x - (F_1)_y = w_x^2 + w_y^2$ because $\nabla^2 w = 0$, and

$$\begin{aligned}
F_1\, dx + F_2\, dy &= (-ww_y x' + ww_z y')\, ds \\
&= w(\operatorname{grad} w) \bullet (y'\mathbf{i} - x'\mathbf{j})\, ds \\
&= w(\operatorname{grad} w) \bullet \mathbf{n}\, ds = w\frac{\partial w}{\partial n}\, ds
\end{aligned}$$

where primes denote derivatives with respect to s.

20. The integrand on the left is $|\operatorname{grad} w|^2 = 4(x^2 + y^2)$. Integration over y gives

$$4x^2 y + \tfrac{4}{3}y^3.$$

We have to integrate over y from 0 to $y = 1 - x$. These limits give in the previous formula

$$4x^2(1 - x) + \tfrac{4}{3}(1 - x)^3.$$

Integration over x now gives

$$\frac{4}{3}x^3 - x^4 - \frac{4}{3} \cdot \frac{(1 - x)^4}{4}.$$

Inserting the limits 0 and 1 we finally have the *answer* $\frac{2}{3}$.

SECTION 10.5. Surfaces for Surface Integrals, page 439

Purpose. The section heading indicates that we are dealing with a tool in surface integrals, and we concentrate our discussion accordingly.

Main Content, Important Concepts

Parametric surface representation (2) (see also Fig. 241)

Tangent plane

Surface normal vector $\mathbf{N}$, unit surface normal vector $\mathbf{n}$

Short Courses. Discuss (2) and (4) and a simple example.

Comments on Text and Problems

The student should realize and understand that the present parametric representations are the two-dimensional analog of parametric curve representations.

Examples 1–3 and Probs. 1–8 and 14–19 concern some standard surfaces of interest in applications. We shall need only a few of these surfaces, but these problems should help students grasp the idea of a parametric representation and see the relation to representations (1). Moreover, it may be good to collect surfaces of practical interest in one place for possible reference.

SOLUTIONS TO PROBLEM SET 10.5, page 442

1. Straight lines $-\mathbf{k}$.

2. Circles, straight lines through the origin. A normal vector is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = [0, \quad 0, \quad u] = u\mathbf{k}.$$

At the origin this normal vector is the zero vector, so that (4) is violated at (0, 0). This can also be seen from the fact that all the lines $v =$ const pass through the origin, and the curves $u =$ const (the circles) shrink to a point at the origin. ***This is a consequence of the choice of the representation, not of the geometric shape of the present surface*** (in contrast with the cone, where the apex has a similar property, but for ***geometric*** reasons).

3. $z = \frac{c}{2}\sqrt{x^2 + y^2}$. circles, straight lines, $[-2cu \cos v, -2cu \sin v, 4u]$

4. The parameter curves $u =$ const are ellipses, namely, the intersections of the cylinder with planes $z =$ const; a and b are their semi-axes. The curves $v =$ const are the generating straight lines of the cylinder, which are perpendicular to the xy-plane. A normal vector is $[-b \cos v, \quad -a \sin v, \quad 0]$. Note that these normal vectors are parallel to the xy-plane, which is geometrically obvious.

For $b = a$ we obtain the representation in Example 1 of the text.

Note further that the normal vectors are independent of u; they are parallel along each generator $v = \text{const}$, which is also geometrically obvious.

6. $z = \arctan(y/x)$, helices (hence the name!), horizontal straight lines. This surface is similar to a spiral staircase, without steps (as in the Guggenheim Museum in New York). A normal vector is

$$[\sin v, \quad -\cos v, \quad u].$$

8. $z = x^2/a^2 - y^2/b^2$, hyperbolas, parabolas; a normal vector is

$$[-2bu^2 \cosh v, \quad 2au^2 \sinh v, \quad abu].$$

10. $\mathbf{r}_u$ is tangent to the curves $v = \text{const}$, and r_v is tangent to the curves $u = \text{const}$, and $\mathbf{r}_u \bullet \mathbf{r}_v = 0$ if and only if we have orthogonality, as follows directly from the definition of the inner product in Sec. 9.2.

12. $\mathbf{N}(0, 0) = \mathbf{0}$ in Prob. 2 (polar coordinates); see the answer to Prob. 2. $\mathbf{N} = \mathbf{0}$ in Prob. 3 at the apex of the cone, where no tangent plane and hence no normal exists. In Probs. 5 (paraboloid) and 7 (ellipsoid) the situation is similar to that in the case of polar coordinates. In Prob. 8 the origin is a saddle point. In each of these cases, one can find a representation for which $\mathbf{N}(0, 0) \neq \mathbf{0}$; for Prob. 5 this is shown in Prob. 11. See also Example 4 in the text for the sphere.

14. $z = 5 - 2x + 3y$, hence

$$\mathbf{r}(u, v) = [u, v, 5 - 2u - 3v].$$

A normal vector is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} = [2, \quad -3, \quad 1].$$

More simply, $[6, -9, 3]$ by applying grad to the given representation. Or $[6, -9, 3]$ by remembering that a plane can be represented by $\mathbf{N} \cdot [x, y, z] = c$.

15. $[1 + 4 \cos u, -2 + 4 \sin u, v]$, $[4 \cos u, 4 \sin u, 0]$.

16. Generalizing a representation of a sphere in the text suggests

$$\mathbf{r}(u, v) = [\cos v \cos u, \quad \cos v \sin u, \quad 3 \sin v]$$

because then

$$x^2 + y^2 = \cos^2 v\,(\cos^2 u + \sin^2 u) = \cos^2 v$$

and $\frac{1}{9}z^2 = \sin^2 v$, so that the sum of the two expressions is $\cos^2 v + \sin^2 v = 1$, as it should be.

17. For a sphere of radius a, $[a \cos v \cos u, 4.5 + a \cos v \sin u, -2.5 + a \sin v]$, $a = 1.5$, $[a^2 \cos^2 v \cos u, a^2 \cos^2 v \sin u, a^2 \cos v \sin v]$

18. $x^2 + 16y^2$ suggests the parametrization $x = 4u \cos v$, and $y = u \sin v$ because the $x^2 + 16y^2 = 16u^2\,(\cos^2 v + \sin^2 v) = 16u^2$, hence $z = 4u$. Together,

$$\mathbf{r}(u, v) = [4u \cos v, u \sin v, 4u]$$

and

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [-4u \cos v, -16u \sin v, 4u].$$

20. Project. (a) $\mathbf{r}_u(P)$ and $\mathbf{r}_v(P)$ span $T(P)$. $\mathbf{r}^*$ varies over $T(P)$. The vanishing of the scalar triple product implies that $\mathbf{r}^* - \mathbf{r}(P)$ lies in the tangent plane $T(P)$.

(b) Geometrically, the vanishing of the dot product means that $\mathbf{r}^* - \mathbf{r}(P)$ must be perpendicular to ∇g, which is a normal vector of S at P.

(c) Geometrically, $f_x(P)$ and $f_y(P)$ span $T(P)$, so that for any choice of x^*, y^* the point (x^*, y^*, z^*) lies in $T(P)$. Also, $x^* = x$, $y^* = y$ gives $z^* = z$, so that $T(P)$ passes through P, as it should.

SECTION 10.6. Surface Integrals, page 443

Purpose. We define and discuss surface integrals with and without taking into account surface orientations.

Main Content

Surface integrals (3) ≡ (4) ≡ (5)

Change of orientation (Theorem 1)

Integrals (6) without regard to orientation; also (11)

Comments on Content

The right side of (3) shows that we need only $\mathbf{N}$ but not the corresponding unit vector $\mathbf{n}$.

An orientation results automatically from the choice of a surface representation, which determines $\mathbf{r}_u$ and $\mathbf{r}_v$ and thus $\mathbf{N}$.

The existence of nonorientable surfaces is interesting but is not needed in our further work.

Further Comments and Suggestions

Emphasize to your students that the integral of (3) is a *scalar*, and the orientation results from $\mathbf{N}$, that is, from the choice of parameters u, v.

Note further that the three terms in (5) give three double integrals over regions in the coordinate planes.

(3) and (5) are compared in Example 1.

Example 2 is of a similar character.

Example 3 illustrates the effect of interchanging parameters, resulting in a factor minus, in the result.

Orientability is extended beyond our need, but the student should see this excursion into topology in Fig. 248 and perhaps also in Probs. 17 and 18.

Formula (8) and Examples 4 and 5 (sphere and doughnut) are typical applications of (6), in which orientation no longer appears.

Moments of inertia of surfaces appear in engineering construction work from time to time. A simple special case of constant distance (as well as of constant mass) is shown in Example 6.

The text concludes with a look at representations $z = f(x, y)$.

Problems 1–11 concern integrals (3) and Probs. 12–16, more briefly, integrals (6).

Problems 17 and 18 are worth noting, also for historical reasons.

Further applications are shown in Probs. 19–25.

Problem 26 is important; it defines the *first fundamental form* of a surface and shows that this form determines the metric on that surface, as was first shown in full generality by Gauss.

SOLUTIONS TO PROBLEM SET 10.6, page 450

1. $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = [u^2, -v^2, 0] \cdot [-2, 3, 1] = -2u^2 - 3y^2 = -\frac{33}{16}$

2. $z = 1 - x - y$, $\mathbf{r} = [u, v, 1 - u - v]$, hence

$$\begin{aligned}\mathbf{F}(\mathbf{r}) \bullet \mathbf{N} &= [e^v, e^u, 1] \bullet ([1, 0, -1] \times [0, 1, -1]) \\ &= [e^v, e^u, 1] \bullet [1, 1, 1] \\ &= 1 + e^u + e^v.\end{aligned}$$

Integration gives the *answer* $2e - 1$.

3. $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = \sin v\,(\cos v \sin^2 u \sin v + \cos v \cos^2 u \sin v)$ from (3). Section 10.5, $\pi/6$

4. Quarter of a circular cylinder of radius 5 and height 2 in the first octant with the z-axis as axis. A parametric representation is

$$\mathbf{r} = [5 \cos u, \quad 5 \sin u, \quad v], \qquad 0 \leqq u \leqq \tfrac{1}{2}\pi, \quad 0 \leqq v \leqq 2.$$

From this we obtain

$$\begin{aligned}\mathbf{N} &= [5 \cos u, \quad 5 \sin u, \quad 0] \\ \mathbf{F} \bullet \mathbf{N} &= 5e^{5 \sin u} \cos u \; - 5e^v \sin u.\end{aligned}$$

Integration over u from 0 to $\frac{1}{2}\pi$ gives $e^5 - 1 - 5e^v$. Integration of this over v from 0 to 2 gives the *answer*

$$2(e^5 - 1) - 5(e^2 - 1) = 2e^5 - 5e^2 + 3 = 262.88.$$

5. $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = -u^3(2 \sin 2v - 1)$, 8π

6. $\mathbf{r} = [u, v, u + v^2]$. This is a parabolic cylinder, the parameter curves being parabolas and straight line generators of the cylinder. For the normal vector we obtain

$$\mathbf{N} = [-1, \; -2v, 1].$$

$\mathbf{F}$ on the surface is

$$\mathbf{F}(\mathbf{r}(u, v)) = [\cosh v, 0, \sinh u].$$

This gives the integrand

$$\mathbf{F} \bullet \mathbf{N} = -\cosh v + \sinh u.$$

Integration over v from 0 to u gives $-\sinh u + u \sinh u$. Integration of this over u from 0 to 1 gives the *answer*

$$[-\cosh u + u \cosh u - \sinh u]\big|_0^1 = 1 - \sinh 1 = -0.1752.$$

7. $\mathbf{F}(\mathbf{r}) \cdot \mathbf{N} = [0, \cos u, \sin v] \cdot [1, -2, 0]$, $4 - \frac{1}{2}\pi^2 = -0.9348$

8. We may choose

$$\mathbf{r} = [u, \cos v, \sin v], \qquad 2 \leqq u \leqq 5, \quad 0 \leqq v \leqq \pi/2.$$

The integrand is

$$\mathbf{F} \bullet \mathbf{N} = [\tan (u \cos v), u, \cos v] \bullet [0, -\cos v, -\sin v].$$

Integration gives

$$\int_0^{\pi/2}\int_2^5 (-u\cos v - \cos v \sin v)\, dv\, du = -12.$$

9. $\mathbf{r} = [3\cos u, 3\sin v, v]$, $0 \le u \le \pi/3$, $0 \le v \le 3$. Integrate $3\cosh v \sin u$ to get $\frac{3}{2}\sinh 3 = 15.027$.

10. Portion of a circular cone with the z-axis as axis. A parametric representation is

$$\mathbf{r} = [u\cos v, \quad u\sin v, \quad 4u] \qquad (0 \leqq u \leqq 2, 0 \leqq v \leqq \pi).$$

From this,

$$\mathbf{N} = [-4u\cos v, \quad -4u\sin v, \quad u], \qquad \mathbf{F} = [u^2\sin^2 v, \quad u^2\cos^2 v, \quad 256u^4].$$

The integrand is

$$\mathbf{F} \bullet \mathbf{n} = -4u^3\sin^2 v\cos v - 4u^3\cos^2 v \sin v + 256u^5.$$

Integration over u from 0 to 2 gives

$$-16\sin^2 v\cos v - 16\cos^2 v\sin v + \frac{8192}{3}.$$

Integration of this over v from 0 to π gives

$$\left[-\frac{16}{3}\sin^3 v + \frac{16}{3}\cos^3 v + \frac{8192}{3}v\right]_0^{\pi} = -\frac{32}{3} + \frac{8192}{3}\pi = 8567.98.$$

12. $\mathbf{r} = [u, \quad v, \quad 1 - u - v]$, $G(r) = \cos u + \sin v$, $|\mathbf{N}| = \sqrt{3}$. Integration gives

$$\int_0^1\int_0^{1-u} (\cos u + \sin v)\sqrt{3}\, dv\, du = (2 - \cos 1 - \sin 1)\sqrt{3} = 1.0708.$$

14. $\mathbf{r} = [\cos v\cos u, \quad \cos v\sin u, \quad \sin v]$, $0 \leqq u \leqq \pi$, $0 \leqq v \leqq \pi/2$. A normal vector is

$$\mathbf{r} = [\cos^2 v\cos u, \quad \cos^2 v\sin u, \quad \cos v\sin v] \qquad \text{and} \qquad |\mathbf{N}| = \cos v.$$

On S,

$$G = a\cos v\cos u + b\cos v\sin u + c\sin v.$$

The integrand is this expression times $\cos v$. Integration over u from 0 to π gives

$$0 + 2b\cos^2 v + \pi c\sin v\cos v.$$

Integration over v from 0 to $\pi/2$ gives the answer

$$\tfrac{1}{2}\pi(b + c)$$

16. $\mathbf{r} = [u\cos v, \quad u\sin v, \quad u^2]$, hence

$$G(\mathbf{r}) = \arctan\frac{u\sin v}{u\cos v} = v.$$

Furthermore,

$$|\mathbf{N}| = \sqrt{4u^4 + u^2},$$

so that the integrand is

$$G(\mathbf{r})|\mathbf{N}(\mathbf{r})| = vu\sqrt{4u^2 + 1}.$$

Integration gives

$$\int_1^3 \int_0^{\pi/2} vu\sqrt{4u^2+1}\,dv\,du = \frac{\pi^2}{8}\int_1^3 u\sqrt{4u^2+1}\,du$$

$$= \frac{\pi^2}{8}\cdot\frac{1}{8}\cdot\frac{2}{3}(4u^2+1)^{3/2}\Big|_1^3$$

$$= \frac{1}{96}\pi^2(37^{3/2}-5^{3/2}) = 22.00.$$

18. Möbius reported that Gauss had shown him the "double ring," most likely in connection with counting the intertwinements of two curves, physically related to the electromagnetic field near two wires. Both Möbius and Listing independently published the invention of the Möbius strip in 1858, three years after Gauss had died. It is possible that they got the idea from Gauss, who usually published full theories, rather than isolated results.

Listing was a student of Gauss and the author of the earliest book on topology, which he (and Riemann and others) called *Analysis situs.*

22. B is represented by $y = 0$, $z = h/2$. The square of the distance of a point (x, y, z) on S from B is

$$(x-x)^2 + (y-0)^2 + (z-\tfrac{1}{2}h)^2 = y^2 + (z-\tfrac{1}{2}h)^2.$$

S can be represented by

$$\mathbf{r} = [\cos u, \sin u, v],$$

where $0 \leqq u \leqq 2\pi$, $0 \leqq v \leqq h$. Hence the moment of inertia is (with $dA = r\,d\theta\,dz = 1 \cdot du\,dv$)

$$I = \int_0^h\int_0^{2\pi}\left[\sin^2 u + \left(v-\frac{1}{2}h\right)^2\right]du\,dv$$

$$= \int_0^h\left(\pi + \left(v-\frac{1}{2}h\right)^2\cdot 2\pi\right)dv$$

$$= h\pi + \frac{h^3}{12}\cdot 2\pi$$

$$= h\pi\left(1+\frac{h^2}{6}\right).$$

24. Proof for a lamina S of density σ. Choose coordinates so that B is the z-axis and κ is the line $x = k$ in the xz-plane. Then

$$I_\kappa = \iint_S [(x-k)^2 + y^2]\sigma\,dA = \iint_S (x^2 - 2kx + k^2 + y^2)\sigma\,dA$$

$$= \iint_S (x^2+y^2)\sigma\,dA - 2k\iint_S x\sigma\,dA + k^2\iint_S \sigma\,dA$$

$$= I_B - 2k\cdot 0 + k^2 M,$$

the second integral being zero because it is the first moment of the mass about a line through the center of gravity.

For a mass distributed in a region in space the idea of proof is the same.

26. Team Project. **(a)** Use $d\mathbf{r} = \mathbf{r}_u\,du + \mathbf{r}_v\,dv$. This gives (13) and (14) because

$$d\mathbf{r} \bullet d\mathbf{r} = \mathbf{r}_u \bullet \mathbf{r}_u\,du^2 + 2\mathbf{r}_u \bullet \mathbf{r}_v\,du\,dv + \mathbf{r}_u \bullet \mathbf{r}_v\,dv^2.$$

(b) E, F, G appear if you express everything in terms of dot products. In the numerator,

$$\mathbf{a} \bullet \mathbf{b} = (\mathbf{r}_u g' + \mathbf{r}_v h') \bullet (\mathbf{r}_u p' + \mathbf{r}_v q') = Eg'p' + F(g'q' + h'p') + Gh'q'$$

and similarly in the denominator.

(c) This follows by Lagrange's identity (Problem Set 9.3),

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \times \mathbf{r}_v) \bullet (\mathbf{r}_u \times \mathbf{r}_v) = (\mathbf{r}_u \bullet \mathbf{r}_u)(\mathbf{r}_v \bullet \mathbf{r}_v) - (\mathbf{r}_u \bullet \mathbf{r}_v)^2$$
$$= EG - F^2.$$

(d) $\mathbf{r} = [u\cos v,\ \ u\sin v]$, $\mathbf{r}_u = [\cos v,\ \ \sin v]$, $\mathbf{r}_u \bullet \mathbf{r}_u = \cos^2 v + \sin^2 v = 1$, etc.

(e) By straightforward calculation, $E = (a + b\cos v)^2$, and $F = 0$ (the coordinate curves on the torus are orthogonal!), and $G = b^2$. Hence, as expected,

$$\sqrt{EG - F^2} = b(a + b\cos v).$$

SECTION 10.7. Triple Integrals. Divergence Theorem of Gauss, page 452

Purpose, Content

Proof and application of the first "big" integral theorem in this chapter, Gauss's theorem, preceded by a short discussion of triple integrals (probably known to most students from calculus).

Comment on Proof

The proof is simple:

1. Cut (2) into three components. Take the third, (5).
2. On the left, integrate $\iiint \frac{\partial F_3}{\partial z}\,dz\,dx\,dy$ over z to get

$$\iint [F_3(\text{upper surface}) - F_3(\text{lower surface})]\,dx\,dy \tag{8}$$

   integrated over the projection R of the region in the xy-plane (Fig. 252).

3. Show that the right side of (5) equals (8). Since the third component of $\mathbf{n}$ is $\cos\gamma$, the right side is

$$\iint F_3 \cos\gamma\,dA = \iint F_3\,dx\,dy$$
$$= \iint F_3(\text{upper})\,dx\,dy - \iint F_3(\text{lower})\,dx\,dy,$$

where minus comes from $\cos\gamma < 0$ in Fig. 252, lower surface. This is the proof. Everything else is (necessary) accessory.

Comments on Problems

Problems 1–8 concern triple integrals and Probs. 9–18 the divergence theorem itself. Masses in space lead to triple integrals over the region of mass distribution; an application to some standard regions is given in Probs. 19–25.

SOLUTIONS TO PROBLEM SET 10.7, page 457

1. 40

2. $\displaystyle\int_0^c\int_0^b\int_0^a xyz^2\,dx\,dy\,dz = \frac{1}{2}\int_0^c a^2yz^2\,dy\,dz = \frac{1}{4}\int_0^c a^2b^2z^2\,dz = \frac{1}{12}a^2b^2c^3$

4. Integration over x from 0 to $3 - y - z$ gives

$$e^{-y-z} - e^{-3}.$$

Integration over y from 0 to 3 – z then gives

$$e^{-z} - 4e^{-3} + ze^{-3}.$$

Integration over z from 0 to 3 finally gives the *answer*

$$1 - 8.5e^{-3}.$$

Note that the region T in space is bounded by portions of the coordinate planes and of the plane

$$x + y + z = 3,$$

from which one obtains the limits of integration.

6. We can represent the region T: $x^2 + z^2 \leqq 16$, $|y| \leqq 4$, a portion of a solid cylinder, by

$$[r\cos u, v, r\sin u], \qquad 0 \leqq r \leqq 4, \quad 0 \leqq u \leqq 2, \quad -4 \leqq v \leqq 4.$$

Then the volume element is

$$dV = r\,dr\,du\,dv$$

and the density is

$$\sigma = r^4(\cos^2 u \sin^2 u)v^2.$$

Integration over r from 0 to 4 gives

$$\frac{2048}{3}(\cos^2 u \sin^2 u)v^2.$$

Integration of this over u from 0 to 2π gives

$$\frac{512}{3}v^2\pi.$$

Integration of this over v from -4 to 4 finally gives the *answer*

$$\frac{65{,}536}{9}\pi = 22{,}876.4.$$

8. From (3) in Sec. 10.5 with variable r instead of constant a we have

$$x = r\cos v\cos u, \qquad y = r\cos v\sin u, \qquad z = r\sin v.$$

Hence $x^2 + y^2 = r^2\cos^2 v$. The volume element is

$$dV = r^2\cos v\,dr\,du\,dv.$$

The intervals of integration are $0 \leqq r \leqq a$, $0 \leqq u \leqq 2\pi$, $0 \leqq v \leqq \frac{1}{2}\pi$. The integrand is $r^4\cos^3 v$. Integration over r, u, and v gives $a^5/5$, 2π, and $\frac{2}{3}$. The product of these is the *answer* $4\pi a^5/15$.

10. The outer normal vectors of the faces $x = \pm 1$ are $\mathbf{n} = [\pm 1, 0, 0]$. This gives $\mathbf{F} \cdot \mathbf{n} = \pm 1$, integrated over the two faces $\pm 1 \cdot 6 \cdot 2 = \pm 12$, and $-12 + 12 = 0$. No contribution from the faces $y = \pm 3$ since $\mathbf{n} = [0, \pm 1, 0]$, so that

$$\mathbf{F} \cdot \mathbf{n} = [x^2, 0, z^2] \cdot [0, \pm 1, 0] = 0.$$

The face $z = 0$ gives $\mathbf{F} \cdot \mathbf{n} = [x^2, 0, 0] \cdot [0, 0, -1] = 0$. The face $z = 2$ gives

$$\mathbf{F} \cdot \mathbf{n} = [x^2, 0, 4] \cdot [0, 0, 1] = 4$$

times the area $2 \cdot 6 = 12$, hence 48, as expected. This calculation is so simple only because of the simplicity of the surface and the region.

Let the student make a sketch so that he/she understands the form of the outer normals.

12. div $\mathbf{F} = 2x^2 + 3y^2 + 3z^2 = 2r^2$, $dV = r^2 \cos v \, dr \, du \, dv$. Intervals of integration $0 \le r \le 3$, $0 \le 2\pi$, $0 \le v \le \pi 2$. The integrand is $3r^4 \cos v$. Integration over r, v, and u gives successively $\frac{729}{5} \cos v$, $\frac{729}{5}$, $\frac{1458}{5} \pi$.

14. div $\mathbf{F} = -\sin z$. Integration over z gives $\cos 2 - 1$. Multiplication by the cross-sectional area gives the *answer*

$$9\pi(\cos 2 - 1) = -40.04.$$

16. div $\mathbf{F} = -\sinh x$. Integration over z from 0 to $1 - x - y$ gives $(1 - x - y) \sinh x$. Integration of this over y from 0 to $1 - x$ gives $\frac{1}{2}(\sinh x)(1 - x)^2$. Integration of this over x from 0 to 1 gives the *answer* $\cosh 1 - \frac{3}{2} = 0.0431$.

18. div $\mathbf{F} = y + z + x$. A parametrization of the cone is

$$[r \cos v, r \sin v, u].$$

The volume element is $r \, dr \, du \, dv$. Integration over r from 0 to $2u$ gives

$$\tfrac{8}{3} (\sin v + \cos v) + 2u^3.$$

Integration over v from 0 to 2π gives $0 + 4\pi u^3$. Integration of this over u from 0 to 2 gives the *answer* 16π.

20. From (3) in Sec. 10.5 with variable r instead of constant a we have

$$x = r \cos v \cos u, \qquad y = r \cos v \sin v, \qquad z = r \sin v.$$

Hence $x^2 + v^2 = r^2 \cos^2 v$. The volume element is $dV = r^2 \cos v \, dr \, du \, dv$. The intervals of integration are $0 \leqq r \leqq a$, $0 \leqq u \leqq 2\pi$, $-\frac{1}{2}\pi \leqq v \leqq \frac{1}{2}\pi$. The integrand is $r^4 \cos^3 v$. Writing the integrals as a product of three integrals, integration over r, u, and v gives $a^5/5$, 2π, and $\frac{4}{3}$, respectively. The product of these is the *answer* $8\pi a^5/15$.

22. $r^2 = y^2 + z^2$. Integration over r from 0 to $\sqrt{x}$ gives $x^2/4$. Integration of this over x from 0 to h gives $h^3/12$. *Answer*: $h^3\pi/6$.

24. $\pi h^5 10 = \pi h^3/6$ gives $h = \sqrt{5/3}$. For $h > \sqrt{5/3}$ the moment I_x is larger for the cone because the mass of the cone is spread out farther than that of the paraboloid when $x > 1$.

SECTION 10.8. Further Applications of the Divergence Theorem, page 458

Purpose. To represent the divergence free of coordinates and to show that it measures the source intensity (Example 1); to use Gauss's theorem for deriving the **heat equation**

governing heat flow in a region (Example 2); to obtain basic properties of harmonic functions.

Main Content, Important Concepts

Total flow (1) out of a region

Divergence as the limit of a surface integral; see (2)

Heat equation (5) (to be discussed further in Chap. 12)

Properties of harmonic functions (Theorems 1–3)

Green's formulas (8), (9)

Short Courses. This section can be omitted.

Comments on (2)

Equation (2) is sometimes used as a *definition* of the divergence, giving independence of the choice of coordinates immediately. Also, Gauss's theorem follows more readily from this definition, but, since its proof is simple (see Sec. 10.7. in this Manual), that saving is marginal. Also, it seems that our Example 2 in Sec. 9.8 motivates the divergence at least as well (and without integrals) as (2) in the present section does for a beginner.

General Comments on Text and Problems

Team Project 12 shows how the ideas of the text in Theorems 1–3 can be further extended and supplemented.

These ideas are basic in the theory of partial differential equations and their applications in physics, where the latter have helped in the discovery of mathematical theorems on our present level as well as in more general and more abstract theories, using functional analysis and measure theory.

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2. A representation of the cylinder is

$$\mathbf{r} = [2\cos\theta, 2\sin\theta],$$

which also equals the normal vector $\mathbf{N}$, as is seen by geometry or by calculating $\mathbf{r}_\theta \times \mathbf{r}_v$. Here $v = z$ varies from 0 to h and θ from 0 to 2π. It follows that

$$\mathbf{n} = [\cos\theta, \sin\theta].$$

Hence the directional derivative is

$$\begin{aligned}\frac{\partial f}{\partial n} = \nabla f \bullet \mathbf{n} &= [-8\cos\theta\sin\theta - 8\sin\theta\cos\theta, 0] \bullet [\cos\theta, \sin\theta] \\ &= -16\cos^2\theta\sin\theta.\end{aligned}$$

The integral over θ from 0 to 2π or from $-\pi$ to π is zero since the integrand is odd. Integration of this over $v = z$ from 0 to h gives zero, as had to be shown.

3. The volume integral of $16y^2 + [0, 4y] \cdot [8x, 0] = 16y^2$ is $\frac{16}{3}$.
The surface integral of $f\,\partial g/\partial n = f \cdot 8x = 8xf = 16y^2x = 16y^2$ over $x = 1$ is $16y^3/3 = 16/3$. Others zero.

4. $\nabla^2 g = 4$, grad $f \cdot$ grad $g = [1, \quad 0, \quad 0] \cdot [0, \quad 2y, \quad 2z] = 0$. Integration of $4x$ over the box gives 12. Also, $f \partial g/\partial n$ for the six surfaces gives

$(x = 0)$	$0[-1, \quad 0, \quad 0] \cdot [0, \quad 2y, \quad 2z]$,	integral 0
$(x = 1)$	$1[1, \quad 0, \quad 0] \cdot [0, \quad 2y, \quad 2z]$,	integral 0
$(y = 0)$	$x[0, \quad -1, \quad 0] \cdot [0, \quad 0, \quad 2z]$,	integral 0
$(y = 2)$	$x[0, \quad 1, \quad 0] \cdot [0, \quad 4, \quad 2z]$,	integral of $4x$ gives $2 \cdot 3 = 6$
$(z = 0)$	$x[0, \quad 0, \quad -1] \cdot [0, \quad 2y, \quad 0]$,	integral 0
$(z = 3)$	$x[0, \quad 0, \quad 1] \cdot [0, \quad 2y, \quad 6]$,	integral of $6x$ gives $3 \cdot 2 = 6$.

6. The volume integral of $12x^2y^2 - 2y^4$ is $\frac{4}{3} - \frac{2}{5}$. The surface integral of

$$x^2\mathbf{n} \cdot [0, \quad 4y^3, \quad 0] - y^4\mathbf{n} \cdot [2x, \quad 0, \quad 0] = \mathbf{n} \cdot [2xy^4, \quad 4y^3, \quad 0]$$

is $-\frac{2}{5}(x = 1)$ and $\frac{4}{3}(y = 1)$ and 0 for the other faces.

8. $z = hr/a$ (make a sketch). For the disk S_1: $z = h$, $0 \leqq r \leqq a$,

$$\iint_{S_1} z\, dx\, dy = \int_0^{2\pi} \int_0^a hr\, dr\, d\theta = 2\pi h \frac{a^2}{2} = \pi h a^2.$$

For the conical portion, whose normal vector has a *negative* z-component, hence the minus sign in the formula,

$$\iint_{S_2} z\, dx\, dy = -\int_0^{2\pi} \int_0^a \frac{hr}{a} r\, dr\, d\theta = -2\pi \frac{h}{a} \frac{a^3}{3} = -\pi \frac{2}{3} h a^2.$$

Together $\pi h a^2/3$.

10. $\mathbf{F} = [x, y, z]$, div $\mathbf{F} = 3$. In (2), Sec. 10.7, we obtain

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= |\mathbf{F}||\mathbf{n}| \cos \phi \\ &= \sqrt{x^2 + y^2 + z^2} \cos \phi \\ &= r \cos \phi. \end{aligned}$$

12. Team Project. (a) Put $f = g$ in (8).

(b) Use (a).

(c) Use (9).

(d) $h = f - g$ is harmonic and $\partial h/\partial n = 0$ on S. Thus $h = $ const in T by (b).

(e) Use div (grad f) $= \nabla^2 f$ and (2).

SECTION 10.9. Stokes s Theorem, page 463

Purpose. To prove, explain, and apply Stokes's theorem, relating line integrals over closed curves and surface integrals.

Main Content

Formula (2) $\equiv$ (2*)

Further interpretation of the curl (see also Sec. 9.9)

Path independence of line integrals (leftover from Sec. 10.2)

Comment on Orientation

Since the choice of right-handed or left-handed coordinates is essential to the curl (Sec. 9.9), surface orientation becomes essential here (Fig. 253).

Comment on Proof

The proof is simple:

1. Cut (2*) into components. Take the first, (3).
2. Using N_1 and N_3, cast the left side of (3) into the form (7).
3. Transform the right side of (3) by Green's theorem in the plane into a double integral and show equality with the integral obtained on the left.

Further Comments on Text and Problems

Examples 1 and 3 show typical calculations in connection with Stoke's theorem.

In connection with Example 2, emphasize that Green's theorem in the plane (a special case of Stokes's theorem) was needed in the proof of the general case of Stokes's theorem, as is mentioned in Example 2 (but was perhaps missed by the student).

Basic applications of Stokes's theorem to fluid flow and to work done in displacements around closed curves are illustrated in Examples 4 and 5.

Problems 1–10 concern direct integration of surface integrals and Probs. 13–20 concern the evaluation of such integrals with Stokes's theorem as a tool for evaluating line integrals around closed curves.

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2. The rectangle is represented by $0 \le x \le 2$, $0 \le y \le \pi/4$, $z = 4$. The curl is

$$\operatorname{curl} \mathbf{F} = [(1 - \sinh z), 0, -3 \sin y].$$

A normal vector of S is $\mathbf{N} = [0, 0, 1]$. Hence

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{N} = -3 \sin y.$$

Integration over x from 0 to 2 and over y from 0 to $\pi/4$ gives the answer $\pm(-6 + 3\sqrt{2})$.

4. The curl is

$$\operatorname{curl} \mathbf{F} = [0, 2z, -2x] = [0, 2xy, -2x].$$

A normal vector of S is

$$\mathbf{N} = [-y, -x, 1].$$

Hence the integrand is

$$(\operatorname{curl} \mathbf{F}) \bullet \mathbf{N} = 0 - 2x^2y - 2x.$$

Integration over x from 0 to 1 and over y from 0 to 4 gives the *answer* $\mp 28/3$.

6. The curl is

$$\operatorname{curl} \mathbf{F} = [0, 0, -3x^2 - 3y^2].$$

A normal vector of S is

$$\mathbf{N} = [0, 0, 1].$$

Hence the integrand is

$$(\text{curl } \mathbf{F}) \cdot \mathbf{N} = -3x^2 - 3y^2 = -3r^2$$

where $x = r\cos\theta$, $y = r\sin\theta$ are polar coordinates. The integral is

$$-3\iint_S (x^2 + y^2)\, dx\, dy = -3\int_0^{2\pi}\int_0^1 r^2 \cdot r\, dr\, d\theta = -\frac{3}{4}r^4 \cdot 2\pi\Big|_0^1 = -\frac{3}{2}\pi.$$

Hence the *answer* is $\mp\frac{3}{2}\pi$.

8. This is a portion of a cone. A representation is

$$\mathbf{r} = [u\cos v, u\sin v, u] \qquad (0 \leqq u \leqq h, 0 \leqq v \leqq \pi).$$

The integrand is

$$\begin{aligned}(\text{curl } \mathbf{F}) \cdot \mathbf{N} &= [2y, 2z, 2x] \cdot (-x, -y, \sqrt{x^2 + y^2}] \\ &= [2u\sin v, 2u, 2u\cos v] \cdot [-u\cos v, -u\sin v, u] \\ &= -2u^2(\cos v\sin v + \sin v - \cos v).\end{aligned}$$

Integration over u from 0 to h and v from 0 to π (since $y \geqq 0$) gives

$$-\frac{2h^3}{3}(0 + 2 - 0) = -\frac{4h^3}{3}.$$

Hence the answer is $\mp 4h^3/3$.

10. $\mathbf{r} = [\cos\theta, \sin\theta]$, $\mathbf{F} \cdot \mathbf{r}' = [\sin^3\theta, -\cos^3\theta] \cdot [-\sin\theta, \cos\theta] = -\sin^4\theta - \cos^4\theta$. Integration over θ from 0 to 2π gives $-3\pi/4 - 3\pi/4$.

14. The curl is

$$\text{curl } \mathbf{F} = [3y^2, 3z^2, 3x^2].$$

The circle is the boundary curve of a circular disk of radius 3 in the yz-plane with normal [1, 0, 0]. Hence

$$(\text{curl } \mathbf{F}) \cdot \mathbf{N} = 3y^2.$$

To integrate, we introduce polar coordinates defined by

$$\mathbf{r} = [2, u\cos v, u\sin v]$$

and obtain, using $dy\, dz = u\, du\, dv$,

$$\int_0^3\int_0^{2\pi} 3u^2(\cos^2 v)\, u\, du\, dv = \frac{243\pi}{4}.$$

16. The curl is

$$\text{curl } \mathbf{F} = [0, -e^x, -e^y].$$

A normal vector is [0, 0, 1], as in Prob. 15. Hence the surface integral is (sketch the triangle!)

$$\int_0^1\int_0^x -e^y\, dy\, dx = -e + 2.$$

18. S bounded by C can be represented by

$$\mathbf{r} = [v, 2\cos u, 2\sin u] \qquad (0 \leqq v \leqq h, 0 \leqq u \leqq \pi).$$

A normal vector with a nonnegative z-component is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [0, 2\cos u, 2\sin u].$$

The curl is

$$\text{curl}\,\mathbf{F} = [-2, 0, 1].$$

The inner product is

$$(\text{curl}\,\mathbf{F}) \bullet \mathbf{N} = 2\sin u.$$

Integration over u from 0 to π (since $z \geqq 0$) and over v gives

$$\int_0^h \int_0^\pi 2\sin u\, du\, dv = 4h.$$

20. A representation of the portion of the cylinder bounded by C is

$$r = [v, 2\cos u, 2\sin u], \qquad (0 \leqq u \leqq \pi/2, 0 \leqq v \leqq \pi).$$

A normal vector of the cylinder is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [0, 2\cos u, 2\sin u].$$

The curl is

$$\text{curl}\,\mathbf{F} = [0, 0, -\sin v]$$

and

$$(\text{curl}\,\mathbf{F}) \bullet \mathbf{N} = -2\sin u \sin v.$$

Integration over u from 0 to $\pi/2$ and over v from 0 to π gives -4.

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11. $\mathbf{r} = [-4 + 6t, 3 + 5t]$, $\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = [-4(-4 + 6t)^2, 6(3 + 5t)^2] \cdot [6, 5]\, dt$; 874, Or using exactness.

12. Exact. By integration we find that

$$\mathbf{F} = \text{grad}\, f, \qquad \text{where} \qquad f = \sin xy + e^z.$$

Substituting the limits of integration gives

$$f(\tfrac{1}{2}, \pi, 1) - f(\pi, 1, 0) = \sin \tfrac{1}{2}\pi - \sin \pi + e^1 - e^0 = e.$$

13. Not exact, curl $\mathbf{F} = [0, -2x, -4 \sin x]$, ± 24.

14. Not exact. Use Green's theorem. We obtain

$$\text{curl}\,\mathbf{F} = [0, 0, 3x^2 + 3y^2], \qquad \mathbf{N} = [0, 0, 1].$$

Hence

$$(\text{curl}\,\mathbf{F}) \bullet \mathbf{N} = 3x^2 + 3y^2.$$

Using polar coordinates, we integrate

$$\int_0^{2\pi}\int_0^5 3r^2 \cdot r\, dr\, d\theta = \frac{1875}{2}\pi.$$

The *answer* is $\pm 1875\pi/2$.

15. 0 since curl $\mathbf{F} = \mathbf{0}$

16. Not exact. We obtain

$$\mathbf{r}' = [-2\sin t, \quad 2\cos t, \quad 3]$$

and

$$\mathbf{F}(\mathbf{r}) = [4\cos^2 t, \quad 4\sin^2 t, \quad 8\sin^2 t\cos t].$$

The inner product is

$$\mathbf{F}\bullet\mathbf{r}' = -8\cos^2 t\sin t + 32\sin^2 t\cos t.$$

Integration gives

$$\tfrac{8}{3}\cos^3 t + \tfrac{32}{3}\sin^3 t\Big|_0^{\pi} = -\tfrac{16}{3}.$$

18. Use Stokes's theorem. $\mathbf{r} = [u, v, u]$, $\mathbf{N} = [-1, 0, 1]$. Furthermore,

$$\begin{aligned}\text{curl}\,\mathbf{F} &= [0, \quad -\pi\cos\pi x, \quad -\pi(\sin\pi x + \cos\pi y)]\\ &= [0, \quad -\pi\cos\pi u, \quad -\pi(\sin\pi u + \cos\pi v)].\end{aligned}$$

The inner product is

$$(\text{curl}\,\mathbf{F}(\mathbf{r}))\bullet\mathbf{N} = -\pi(\sin\pi u + \cos\pi v).$$

Integration over u from 0 to 1, gives $-\pi\cos\pi v - 2$. Integration of this over v from 0 to 2 gives -4. *Answer*: ∓ 4.

20. Exact, $\mathbf{F} = \text{grad}(e^{xz} + \cosh 2y)$. Integration from $(-1, -1, 1)$ to $(1, 1, 1)$ gives

$$e + \cosh 2 - (e^{-1} + \cosh 2) = 2\sinh 1.$$

21. $M = 27/2, \bar{x} = \frac{9}{4}, \bar{y} = \frac{5}{4}$

22. $M = \pi a^4/4, \bar{x} = 0$ (why?), $\bar{y} = \dfrac{1}{M}\displaystyle\int_0^{\pi}\int_0^a (r\sin t)r^2 \cdot r\, dr\, dt = \frac{8a}{5\pi}$

24. $\bar{x} = 0$ by symmetry. Furthermore,

$$M = \int_{-1}^{1}(1 - x^4)\, dx = \frac{8}{5}$$

and

$$\bar{y} = \frac{1}{M} = \int_{-1}^{1}\int_0^{1-x^4} y\, dy\, dx = \int_{-1}^{1}\frac{5}{16}(1 - x^4)^2\, dx = \frac{4}{9}.$$

25. $M = \frac{1}{4}k, \bar{x} = \frac{8}{15}, \bar{y} = \frac{1}{3}$

26. k drops out in both integrands xf/M and yf/M.

28. div $\mathbf{F} = 3$, $\quad V = \frac{4}{3}\pi abc$. *Answer*: $4\pi abc$

30. Direct integration. We have

$$\mathbf{r} = [2\cos u \cos v, \quad 2\cos u \sin v, \quad \sin u] \qquad (0 \leqq u \leqq \tfrac{1}{2}\pi, \quad 0 \leqq v \leqq 2\pi).$$

From this,

$$\begin{aligned}
\mathbf{r}_u &= [-2\sin u \cos v, \quad -2\sin u \sin v, \quad \cos u] \\
\mathbf{r}_v &= [-2\cos u \sin v, \quad 2\cos u \cos v, \quad 0] \\
\mathbf{N} &= [-2\cos^2 u \cos v, \quad -2\cos u^2 \sin v, \quad -4\cos u \sin u].
\end{aligned}$$

The inner product is

$$\mathbf{F} \bullet \mathbf{N} = (-2\cos^2 u)(\cos v + \sin v) - 4 \quad \cos u \sin u.$$

Integration of $\cos v + \sin v$ over v from 0 to 2π gives 0. Integration of $-4\cos u \sin u$ over u from 0 to $\pi/2$ gives -2. Integration of this constant over v from 0 to 2π gives -4π (or $+4\pi$ if we change the orientation by interchanging u and v).

32. By direct integration. We can represent the paraboloid in the form

$$\mathbf{r} = [u\cos v, u\sin v, u^2].$$

A normal vector is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [-2u^2\cos v, -u^2\sin v, u].$$

On the paraboloid,

$$\mathbf{F} = [u^2\sin^2 v, u^2\cos^2 v, u^4].$$

The inner product is

$$\mathbf{F} \bullet \mathbf{N} = -2u^4\cos v \sin^2 v - 2u^4\cos^2 v \sin v + u^5.$$

Integration over v from 0 to 2π gives $0 + 0 + 2\pi u^5$. Integration of this over u from 0 to 3 (note that $z = u^2$ varies from 0 to 9) gives $2\pi 3^6/6 = 243\pi$.

34. By Gauss's theorem, T can be represented by

$$\mathbf{r} = [r\cos u, r\sin u, v], \qquad \text{where} \qquad 0 \le r \le 1, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 4.$$

The divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = y + z + x = r\cos u + r\sin u + v.$$

The integrand is $(r\cos u + r\sin u + v)r$. Integration over r from 0 to 1 gives $\frac{1}{3}\cos u + \frac{1}{3}\sin u + \frac{1}{2}v$. Integration of this over u from 0 to 2π gives $v\pi$. Integration of this over v from 0 to 4 gives the *answer* 8π.