CS5314 Randomized Algorithms

Homework 4
Due: June 02, 2020 (before 23:59)

1. Let Z be a Poisson random variable with mean μ, where $\mu \geq 1$ is an integer.
   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.
   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

2. Let X be a Poisson random variable with mean μ, representing the number of criminals in a city. There are two types of criminals: For the first type, they are not too bad and are reformable. For the second type, they are flagrant. Suppose each criminal is independently reformable with probability p (so that flagrant with probability $1-p$). Let Y and Z be random variables denoting the number of reformable criminals and flagrant criminals (respectively) in the city. Show that Y and Z are independent Poisson random variables.

3. We consider another way to obtain Chernoff-like bound in the balls-and-bins setting. Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the ith bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$ be the number of empty bins.

   Let Y_i be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.

   (a) Show that $\mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.
   (b) Show that $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.
   (c) Show that $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$ for all $t \geq 0$.
   Hint: Use the expansion for e^x and compare $\mathrm{E}[e^{tX}]$ to $\mathrm{E}[e^{tY}]$.
   (d) Derive a Chernoff bound for $\Pr(X \geq (1+\delta)\mathrm{E}[X])$.

4. In the lecture, we showed that, for any nonnegative function f,

$$\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = m\right).$$

   (a) Now suppose we further know that $\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})]$ is monotonically increasing in m. Show that

$$\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} \geq m\right).$$

   (b) Combining part (a) with the result in Question 1, show that:

$$\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \leq 2\,\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})].$$