CS5314 Randomized Algorithms

Homework 1 Suggested Solution

(Homework due date was April 7, 2020)

1 Part I: Multiple Choice

1. We roll two standard six-sided dice. Compute the following probabilities.

   (a) Pr(the second die is 5 | sum is odd) = ?   (A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$

   (b) Pr(the second die is 4 | sum is 6) = ?   (A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$

   Ans.

   (a) D

$$\begin{aligned}&\Pr(\text{the second die is 5} \mid \text{sum is odd})\\&= \frac{\Pr(\text{the second die is 5 } \cap \text{ sum is odd})}{\Pr(\text{sum is odd})}\\&= \frac{\Pr(\text{the second die is 5}) \cdot \Pr(\text{the first die is even})}{\Pr(\text{sum is odd})}\\&= \frac{(1/6)\cdot(1/2)}{1/2} = \frac{1}{6}\end{aligned}$$

   (b) C

$$\begin{aligned}&\Pr(\text{the second die is 4} \mid \text{sum is 6})\\&= \frac{\Pr(\text{the second die is 4 } \cap \text{ sum is 6})}{\Pr(\text{sum is 6})}\\&= \frac{\Pr(\text{the second die is 4}) \cdot \Pr(\text{the first die is 2})}{\Pr(\text{sum is 6})}\\&= \frac{(1/6)\cdot(1/6)}{5/36} = \frac{1}{5}\end{aligned}$$

2. After lunch one day, Alice suggests to Bob the following method to determine who pays. Alice pulls four six-sided dice from her pocket. These dice are not the standard dice, but have the following numbers on their faces:

   - die A: 1, 1, 1, 5, 5, 5;
   - die B: 3, 3, 3, 3, 3, 3;
   - die C: 0, 0, 4, 4, 4, 4;
   - die D: 2, 2, 2, 2, 6, 6;

The dice are fair, so each side comes up with equal probability. Alice explains that Alice and Bob will each pick up one of the dice. They will each roll their die, and the one who rolls the lowest number loses and will buy lunch. So as to take no advantage, Alice offers Bob the first choice of the dice.
In this question, we will see an amazing fact: no matter which die Bob chooses, Alice can always chooses another die such that the probability that Alice wins is greater than 1/2. (In the following questions, we assume that Alice will choose a die such that the probability that Alice wins as high as possible. Please answer A, B, C, or D.)

(a) Suppose that Bob chooses die A, which die should Alice choose?

(b) Suppose that Bob chooses die B, which die should Alice choose?

(c) Suppose that Bob chooses die C, which die should Alice choose?

(d) Suppose that Bob chooses die D, which die should Alice choose?

Ans. Note that

$$\begin{aligned}
\Pr(\text{die A beats die B}) &= \frac{1}{2} \\
\Pr(\text{die A beats die C}) &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} = \frac{2}{3} \\
\Pr(\text{die A beats die D}) &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \\
\Pr(\text{die B beats die C}) &= \frac{1}{3} \\
\Pr(\text{die B beats die D}) &= \frac{2}{3} \\
\Pr(\text{die C beats die D}) &= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}
\end{aligned}$$

Thus, the answers to the four questions would be: (a) D, (b) C, (c) A, (d) B.

2 Part II: Calculation (Express your answer in simplest form)

1. There are four red balls and four blue balls, to be randomly (with equal probability) distributed into two boxes, so that each box contains exactly four balls.[1] What is the probability that balls in the first box are all red?

   Ans. 1/70. There are $\binom{8}{4} = 70$ ways to select 4 balls into the first bin, each way equally likely, and exactly one of these ways corresponds to having four red balls in the first bin.

2. Three rooks are independently put on a 8×8 chess board, where each rook chooses a square with equal probability. That is, it could happen that two, or even three, rooks are placed at the same square. What is the probability that there are two (or more) rooks attacking each other? *Hint:* What is the physical meaning about one rook attacking another?

[1] That is, we repeatedly select a ball to put in either box, with equal probability, until one box has four balls; then, we put the remaining balls in the other box.

Ans. 583/1024. The case when some rook is attacking another is the complement of the case when no rooks are attacking one another. By brute-force counting, we see that the desired probability can be calculated as follows:

$$\begin{aligned}
&\Pr(\text{at least two rooks attacking each other}) \\
&= 1 - \Pr(\text{no rooks attacking each other}) \\
&= 1 - \frac{64 \cdot 49 \cdot 36}{64 \cdot 64 \cdot 64} = 1 - \frac{441}{1024} = \frac{583}{1024}
\end{aligned}$$

Alternatively, the placement of each rook corresponds to selecting an x-coordinate and a y-coordinate, randomly. where the latter case is exactly all rooks are picking distinct x- and y-coordinates. So,

$$\Pr(\text{no rooks attacking each other}) = \frac{\binom{8}{3}}{8^3} \times \frac{\binom{8}{3}}{8^3} = \frac{441}{1024}$$

and we can get the same result.

3. Two persons are playing a match in which they stop as soon as anyone of them wins 10 games. Suppose that the probability for each person to win any game is 1/2, independent of other games. What is the probability that they have played exactly 11 games when the match is over?

   Ans. 5/512. Assume that the two persons are A and B.

$$\begin{aligned}
&\Pr(\text{match is over in exactly 11 games}) \\
&= \Pr(A \text{ wins 9 games in the first 10 games and } A \text{ wins the 11th game}) \\
&\quad + \Pr(B \text{ wins 9 games in the first 10 games and } B \text{ wins the 11th game}) \\
&= 2 \cdot \Pr(A \text{ wins 9 games in the first 10 games and } A \text{ wins the 11th game}) \\
&= 2 \cdot \Pr(A \text{ wins 9 games in the first 10 games}) \cdot \Pr(A \text{ wins the 11th games}) \\
&= 2 \cdot \binom{10}{9} \cdot \left(\frac{1}{2}\right)^{10} \cdot \frac{1}{2} = \frac{5}{512}
\end{aligned}$$