Statistical Computing HW3

周聖諺

4/23/2021

Problem 1:

Two random variables are defined X_1, X_2 and combined as a set $X = \{X_1, X_2\}$

$$X_1 = \sigma_{X_1} Z_1 + \mu_{X_1}$$

$$X_2 = \sigma_{X_2} \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2 \right) + \mu_{X_2}$$

where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ and Z_1, Z_2 are independent

The Expectation

$$\mathbb{E}[X_1] = \mathbb{E}[\sigma_{X_1} Z_1 + \mu_{X_1}] = \sigma_{X_1} \mathbb{E}[Z_1] + \mu_{X_1} = \mu_{X_1}$$

$$\mathbb{E}[X_2] = \mathbb{E}[\sigma_{X_2} (\rho Z_1 + \sqrt{1-\rho^2} Z_2 + \mu_{X_2}]$$

$$= \sigma_{X_2} \mathbb{E}[\rho Z_1 + \sqrt{1-\rho^2} Z_2] + \mu_{X_2}$$

$$= \sigma_{X_2} (\rho \mathbb{E}[Z_1] + \sqrt{1-\rho^2} \mathbb{E}[Z_2]) + \mu_{X_2} = \mu_{X_2}$$

$$\mathbb{E}[X] = \{\mu_{X_1}, \mu_{X_2}\}$$

The Covariance

$$\begin{gathered}
\sigma_{X_1, X_2} = \sigma_{X_2, X_1} = \mathbb{E}[(X_1 - \mu_{X_1})(X_2 - \mu_{X_2})] \\
= \mathbb{E}[(\sigma_{X_1} Z_1 + \mu_{X_1} - \mu_{X_1})(\sigma_{X_2} (\rho Z_1 + \sqrt{1-\rho^2} Z_2 + \mu_{X_2} - \mu_{X_2})] \\
= \mathbb{E}[(\sigma_{X_1} Z_1)(\sigma_{X_2} (\rho Z_1 + \sqrt{1-\rho^2} Z_2)] \\
= \mathbb{E}[\sigma_{X_1} \sigma_{X_2} (\rho Z_1^2 + \sqrt{1-\rho^2} Z_1 Z_2)]
\end{gathered}$$

$$= \sigma_{X_1} \sigma_{X_2} \left(\rho \mathbb{E}[Z_1^2] + \sqrt{1-\rho^2} \mathbb{E}[Z_1 Z_2] \right)$$

With the definition of variance, we can derive $\mathbb{E}[Z_1^2] = Var[Z_1] + E[Z_1]^2 = 1$. Since Z_1, Z_2 are independent, $\mathbb{E}[Z_1 Z_2] = \mathbb{E}[Z_1] \mathbb{E}[Z_2] = 0$