CHAPTER 9 Vector Differential Calculus. Grad, Div, Curl

This chapter is independent of the previous two chapters (7 and 8).

Formulas for grad, div, and curl in **curvilinear coordinates** are placed for reference in App. A3.4.

SECTION 9.1. Vectors in 2-Space and 3-Space, page 354

Purpose. We introduce vectors in 3-space given geometrically by (families of parallel) directed segments or algebraically by ordered triples of real numbers, and we define addition of vectors and scalar multiplication (multiplication of vectors by numbers).

Main Content, Important Concepts

Vector, norm (length), unit vector, components

Addition of vectors, scalar multiplication

Vector space R^3, linear independence, basis

Comments on Content

Our discussions in the whole chapter will be independent of Chaps. 7 and 8, and there will be no more need for writing vectors as columns and for distinguishing between row and column vectors. Our notation $\mathbf{a} = [a_1, a_2, a_3]$ is compatible with that in Chap. 7. Engineers seem to like both notations

$$\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

preferring the first for "short" components and the second in the case of longer expressions.

The student is supposed to understand that the whole vector algebra (and vector calculus) has resulted from applications, with concepts that are practical, that is, they are "made to measure" for standard needs and situations; thus, in this section, the two algebraic operations resulted from forces (forming resultants and changing magnitudes of forces); similarly in the next sections. The restriction to three dimensions (as opposed to n dimensions in the previous two chapters) allows us to "visualize" concepts, relations, and results and to give geometrical explanations and interpretations.

On a higher level, the equivalence of the geometric and the algebraic approach (Theorem 1) would require a consideration of how the various triples of numbers for the various choices of coordinate systems must be related (in terms of coordinate transformations) for a vector to have a norm and direction independent of the choice of coordinate systems.

Teaching experience makes it advisable to cover the material in this first section rather slowly and to assign relatively many problems, so that the student gets a feel for vectors in R^3 (and R^2) and the interrelation between algebraic and geometric aspects.

Comments on Problems

Problems 1–10 illustrate components and length.

Operations on vectors (addition, scalar multiplication) in Probs. 11–20 are followed by applications to forces and velocities in Probs. 31–37. This includes questions on equilibrium and relative velocity.

SOLUTIONS TO PROBLEM SET 9.1, page 360

1. 3, 2, 0; $\sqrt{13}$, $[3/\sqrt{13}, 2/\sqrt{13}, 0]$

2. $\mathbf{v} = [1, 1, -1]$; $|\mathbf{v}| = \sqrt{3}$; $|\mathbf{u}| = [1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}]$

3. $11, -4, 3, \sqrt{146}$, $[0.911, -0.332, 0.248]$

4. $\mathbf{v} = [-2, -8, -4]$. A line segment in space, of length $\sqrt{4 + 64 + 16} = \sqrt{84}$, with the origin as midpoint. The unit vector is

$$\mathbf{u} = [-1/\sqrt{21}, -4/\sqrt{21}, -2/\sqrt{21}].$$

5. $3, \sqrt{7}, -3$, $\mathbf{u} = [3/5, \sqrt{7}/5, -3/5]$, poition vector of Q.

6. Q: (4, 2, 13), $|\mathbf{v}| = \sqrt{189}$

7. Q : [3/2, 0, 5/4]; $|\mathbf{v}| = \sqrt{61}/4$

8. Position vector of Q: (13.1, 0.8, −2.0), $|\mathbf{v}| = \sqrt{176.25}$

9. Q : [0, 0, −4]; $|\mathbf{v}| = 4$

10. Q: (0, 0, 0), $|\mathbf{v}| = \sqrt{18}$

11. [8, 12, 0], [1/2, 3/4, 0], [−2, −3, 0]

12. [5, 2, 3], the same.

13. [3, −1, 3], the same.

14. [−6, 30, 12], the same.

15. [−25, 55, 15], the same.

16. [8, −1, −6], the same.

17. [6, 9, 0], the same.

18. [24, −24, 0], [−24, 24, 0]

22. $\mathbf{0}$, equilibrium

24. [1, 1, 0]

26. $\mathbf{v} = -(\mathbf{p} + \mathbf{q} + \mathbf{u}) = [-4, -9, 3]$

28. $[1/\sqrt{2}, 1/\sqrt{2}, 0]$. Unit vectors will play a role in fixing (determining) directions.

30. $\mathbf{v} = [-1, -1, v_3]$ with arbitrary v_3

32. $2 \leqq |\mathbf{p} + \mathbf{q}| \leqq 10$; nothing about direction. Application: suitable lengths of the portions of an arm.

34. $\mathbf{v}_B - \mathbf{v}_A = [-450/\sqrt{2}, 450/\sqrt{2}] - [-550/\sqrt{2}, -550/\sqrt{2}] = [100/\sqrt{2}, 1000/\sqrt{2}]$

36. Choose a coordinate system whose axes contain the mirrors. Let $\mathbf{u} = [u_1, u_2]$ be incident. Then the first reflection gives, say, $\mathbf{v} = [u_1, -u_2]$, and the second $\mathbf{w} = [-u_1, -u_2] = -\mathbf{u}$. The reflected ray is parallel to the incoming ray, with the direction reversed.

38. Team Project. (a) The idea is to write the position vector of the point of intersection P in two ways and then to compare them, using that $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors. Thus

$$\lambda(\mathbf{a} + \mathbf{b}) = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a}).$$

$\lambda = 1 - \mu$ are the coefficients of $\mathbf{a}$ and $\lambda = \mu$ those of $\mathbf{b}$. Together, $\lambda = \mu = \frac{1}{2}$, expressing bisection.

(b) The idea is similar to that in part (a). It gives

$$\lambda(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}\mathbf{a} + \mu\tfrac{1}{2}(\mathbf{b} - \mathbf{a}).$$

$\lambda = \frac{1}{2} - \frac{1}{2}\mu$ from $\mathbf{a}$ and $\lambda = \frac{1}{2}\mu$ from $\mathbf{b}$, resulting in $\lambda = \frac{1}{4}$, thus giving a ratio $\left(\frac{3}{4}\right):\left(\frac{1}{4}\right) = 3:1$.

(c) Partition the parallelogram into four congruent parallelograms. Part (a) gives 1:1 for a small parallelogram, hence 1: (1 + 2) for the large parallelogram.

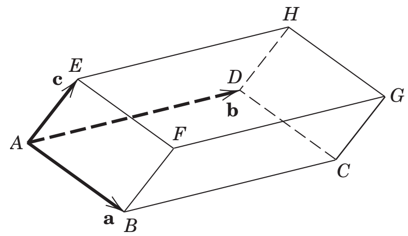
(d) $\mathbf{v}(P) = \frac{1}{2}\mathbf{a} + \lambda(\mathbf{b} - \frac{1}{2}\mathbf{a}) = \frac{1}{2}\mathbf{b} + \mu(\mathbf{a} - \frac{1}{2}\mathbf{b})$ has the solution $\lambda = \mu = \frac{1}{3}$, which gives by substitution $\mathbf{v}(P) = \frac{1}{3}(\mathbf{a} + \mathbf{b})$ and shows that the third median OQ passes through P and OP equals $\frac{2}{3}$ of $|\mathbf{v}(Q)| = \frac{1}{2}|\mathbf{a} + \mathbf{b}|$, dividing OQ in the ratio 2:1, too.

(e) In the figure in the problem set, $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$; hence $\mathbf{c} + \mathbf{d} = -(\mathbf{a} + \mathbf{b})$. Also, $AB = \frac{1}{2}(\mathbf{a} + \mathbf{b})$, $CD = \frac{1}{2}(\mathbf{c} + \mathbf{d}) = -\frac{1}{2}(\mathbf{a} + \mathbf{b})$, and for DC we get $+\frac{1}{2}(\mathbf{a} + \mathbf{b})$, which shows that one pair of sides is parallel and of the same length. Similarly for the other pair.

(f) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be edge vectors with a common initial point (see the figure below). Then the four (space) diagonals have the midpoints

$$\begin{aligned} AG&: \quad \tfrac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ BH&: \quad \mathbf{a} + \tfrac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{a}) \\ EC&: \quad \mathbf{c} + \tfrac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c}) \\ DF&: \quad \mathbf{b} + \tfrac{1}{2}(\mathbf{a} + \mathbf{c} - \mathbf{b}), \end{aligned}$$

and these four position vectors are equal.



Section 9.1. Parallelepiped in Team Project 38(f)

(g) Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be the vectors. Their angle is $\alpha = 2\pi/n$. The interior angle at each vertex is $\beta = \pi - (2\pi/n)$. Put $\mathbf{v}_2$ at the terminal point of $\mathbf{v}_1$, then $\mathbf{v}_3$ at the terminal point of $\mathbf{v}_2$, etc. Then the figure thus obtained is an n-sided regular polygon, because the angle between two sides equals $\pi - \alpha = \beta$. Hence

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \mathbf{0}.$$

(Of course, for *even n* the truth of the statement is immediately obvious.)

SECTION 9.2. Inner Product (Dot Product), page 361

Purpose. We define, explain, and apply a first kind of product of vectors, the dot product $\mathbf{a} \bullet \mathbf{b}$, whose value is a scalar.

Main Content, Important Concepts

Definition (1)

Dot product in terms of components

Orthogonality

Length and angle between vectors in terms of dot products

Cauchy–Schwarz and triangle inequalities

Comment on Dot Product

This product is motivated by work done by a force (Example 2), by the calculation of components of forces (Example 3), and by geometric applications such as those given in Examples 5 and 6.

"Inner product" is more modern than "dot product" and is also used in more general settings (see Sec. 7.9).

Comments on Text

Figure 178 shows geometrically why the inner product can be positive or negative or—this is the most important case—zero, in which the vectors are called orthogonal. This includes the case of two zero vectors in the definition, in which case the angle is no longer defined.

Equations (6)–(8) concern relationships that also extend to more abstract setting, where they turn out to be of basic importance; see [GenRef7].

Examples 2–6 in the text show some simple applications of inner products in mechanics and geometry that motivate these products. Further applications will appear as we proceed.

Comments on Problems

Problems 1–10 illustrate the various laws for inner products.

Problems 11–16 include a modest amount of theory.

Problems 17–40 add further applications of inner products in mechanics and geometry, including a generalization of the concept of component in Probs. 36–40 that is quite useful.

SOLUTIONS TO PROBLEM SET 9.2, page 367

1. 6, 6, 16

2. 24, -120

3. $\sqrt{14}, 4\sqrt{5}, \sqrt{22}$

4. $\sqrt{46}, \sqrt{14} + 2\sqrt{5}$

5. $\sqrt{74}, 2\sqrt{5} + \sqrt{22}$

6. 0 by (8). The left side of (8) is the sum of the squares of the diagonals; the right side equals the sum of the squares of the four sides of the parallelogram.

7. $5, 2\sqrt{7}\sqrt{11}$

8. 168, 168.

9. 12, 12

10. 11, -21

12. $\mathbf{u} \bullet (\mathbf{v} - \mathbf{w}) = 0$ implies nothing if $\mathbf{u} = \mathbf{0}$ and implies orthogonality of $\mathbf{u}$ and $\mathbf{v} - \mathbf{w}$ if $\mathbf{u} \neq \mathbf{0}$.

16. $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) = |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)$

17. $[5, 3, 0] \cdot [1, 3, 3] = 14$

18. 0; the vectors $\mathbf{p}$ and $\mathbf{v} = [6, 7, 5]$ are orthogonal.

20. $[6, -3, -3] \cdot [2, -1, -1] = 18 = |\mathbf{p}||\mathbf{v}|$ because $\mathbf{p}$ has the direction of $\overline{AB} = \mathbf{v} = [2, -1, -1]$.

22. $\arccos(5/\sqrt{14 \cdot 5}) = \arccos 0.9449 = 0.3335 = 19.1°$

24. $\mathbf{a} + \mathbf{c} = [2, 1, 2], \quad \mathbf{b} + \mathbf{c} = [4, 2, 3]$, hence

$$\gamma = \arccos(16/\sqrt{9 \cdot 29}) = 0.1389 = 7.96°.$$

26. $|\mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \gamma$

28. $\gamma_A = \arccos(\overline{AB}, \overline{AC}) = \arccos(3/(3 \cdot \sqrt{3})) = 54.74°$,
$\gamma_B = \arccos(\overline{BA}, \overline{BC}) = 35.26°, \quad \gamma_3 = \arccos(\overline{CA}, \overline{CB}) = 90°$

30. The distance of P: $3x + y + z = 9$ from the origin is

$$9/|\mathbf{a}| = 9/\sqrt{11} = 2.714$$

(Hesse's normal form). The plane parallel to P through A is

$$3x + y + z = 3 \cdot 1 + 1 \cdot 0 + 1 \cdot 2 = 5.$$

Its distance from the origin is

$$5/\sqrt{11} = 1.508.$$

This gives the *answer* $2.714 - 1.508 = 1.206$.

32. Necessary and sufficient is the orthogonality of the normal vectors $[3, 0, 1]$ and $[8, -1, c]$. Hence

$$[3, 0, 1] \cdot [8, -1, c] = 0, \qquad c = -24.$$

33. $\pm[\frac{4}{5}, -\frac{3}{5}]$

34. Let the mirrors correspond to the coordinate planes. If the ray $[v_1, v_2, v_3]$ first hits the *yz*-plane, then the *xz*-plane, and then the *xy*-plane, it will be reflected to $[-v_1, v_2, v_3], [-v_1, -v_2, v_3], [-v_1, -v_2, -v_3]$; hence the angle is 180°, the reflected ray will be parallel to the incident ray but will have the opposite direction.

Corner reflectors have been used in connection with missiles; their aperture changes if the axis of the missile deviates from the tangent direction of the path. See E. Kreyszig, On the theory of corner reflectors with unequal faces. Ohio State University: *Antenna Lab Report* 601/19.

36. $6/\sqrt{14}$

38. $-34/\sqrt{17} = -2\sqrt{17}$. Note that the vectors have exactly opposite directions; this is a case in which the component will have a minus sign. Also $|\mathbf{a}|/|\mathbf{b}| = \sqrt{68/17}$ gives the factor 2.

40. Nothing because $|\mathbf{b}|$ appears in the numerator $\mathbf{a} \cdot \mathbf{b}$ as well as in the denominator.

SECTION 9.3. Vector Product (Cross Product), page 368

Purpose. We define and explain a second kind of product of vectors, the cross product $\mathbf{a} \times \mathbf{b}$, which is a vector perpendicular to both given vectors (or the zero vector in some cases).

Main Content, Important Concepts

Definition of cross product, its components (2), (2**)

Right- and left-handed coordinate systems

Properties (anticommutative, not associative)

Scalar triple product

Prerequisites. Elementary use of second- and third-order determinants (see Sec. 7.6)

Comment on Motivations

Cross products were suggested by the observation that, in certain applications, one associates with two given vectors a third vector perpendicular to the given vectors (illustrations in Examples 4–6). Scalar triple products can be motivated by volumes and linear independence (Theorem 2 and Example 6).

Comments on Problems

Problems 1–10 should help in obtaining an intuitive understanding of the cross product and give further motivation of this concept by applications.

Problems 11–23 compare various products, with emphasis on those of three factors.

Team Project 24 concerns standard formulas needed in working with dot and cross products and their combination.

Problems 25–35 show some further applications in mechanics and geometry, to emphasize further that the definitions of these products are motivated by applications.

SOLUTIONS TO PROBLEM SET 9.3, page 374

2. $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, thus $\mathbf{b} = \mathbf{c}$ or $\mathbf{b} - \mathbf{c}$ has the direction of $\mathbf{a}$ or $-\mathbf{a}$.

4. The cross product is

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix} = (8 - 0)\mathbf{i} + (2 - 6)\mathbf{j} + (0 - 4)\mathbf{k}$$

$$= [8, -4, -4].$$

Its length is $|\mathbf{v}| = \sqrt{96}$, which equals the right side of (12),

$$\sqrt{29 \cdot 5 - 7^2} = \sqrt{145 - 49}.$$

Using the definition of the length of a vector product and the given hint, we obtain (12) by taking the square roots of

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \gamma = |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2 \gamma) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2.$$

6. Instead of ωd you now have $2\omega d = 2|\mathbf{w} \times \mathbf{r}|$, hence $|\mathbf{v}|$ doubles.

8. We obtain

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10/\sqrt{2} & 10/\sqrt{2} & 0 \\ 4 & 2 & -2 \end{vmatrix} = [-10\sqrt{2}, 10\sqrt{2}, -10\sqrt{2}]$$

so that the speed is $\sqrt{600}$.

11. $[0, 0, -1], [0, 0, 1], -8$

12. $[-126, -78, 60], \quad [126, 78, -60], 30, 30$

13. $[1, 1, -2], [-1, -1, 2]$

14. 0 because of anticommutativity

15. 0

16. The first expression gives

$$[-3, -2, 2] \cdot [3, -1, 5] = 3.$$

The second expression looks totally different but, of course, gives the same value:

$$[-2, 3, 0] \cdot [-21, -13, 10] = 3.$$

17. $[-8, 21, 9]$, $[30, 20, 89]$

18. $[-2, -1, 0]$. The student should note and understand why both product vectors lie in the plane of **a** and **b**, why neither of the zero, and why they are the same. This should become clear by drawing little sketches of the factors and products.

19. $-1, 1$

20. Formula (14) shows that the two expressions are equal, namely, equal to $[-13, 21, 0]$. The intermediate calculation of the second expression is

$$(-5)[2, -4, -1] - (1)[3, -1, 5] = [-10, 20, 5] - [3, -1, 5] = [-13, 21, 0].$$

21. $[-24, -16, 16]$, $8\sqrt{17} = 32.985$, 32.985

22. 8, 5

23. 0, 0, 13

24. **Team Project.** To prove (13), we choose a right-handed Cartesian coordinate system such that the x-axis has the direction of **d** and the xy-plane contains **c**. Then the vectors in (13) are of the form

$$\mathbf{b} = [b_1, b_2, b_3], \qquad \mathbf{c} = [c_1, c_2, 0], \qquad \mathbf{d} = [d_1, 0, 0].$$

Hence by (2**),

$$\mathbf{c} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & 0 \\ d_1 & 0 & 0 \end{vmatrix} = -c_2 d_1 \mathbf{k}, \qquad \mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ 0 & 0 & -c_2 d_1 \end{vmatrix}.$$

The "determinant" on the right equals $[-b_2c_2d_1, \quad b_1c_2d_1, \quad 0]$. Also,

$$\begin{aligned}(\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d} &= b_1 d_1 [c_1, c_2, 0] - (b_1 c_1 + b_2 c_2)[d_1, 0, 0] \\ &= [-b_2c_2d_1, \quad b_1d_1c_2, \quad 0].\end{aligned}$$

This proves (13) for our special coordinate system. Now the length and direction of a vector and a vector product, and the value of an inner product, are independent of the choice of the coordinates. Furthermore, the representation of $\mathbf{b} \times (\mathbf{c} \times \mathbf{d})$ in terms of **i**, **j**, **k** will be the same for right-handed and left-handed systems, because of the double cross multiplication. Hence, (13) holds in any Cartesian coordinate system, and the proof is complete.

Equation (14) follows from (13) with **b** replaced by $\mathbf{a} \times \mathbf{b}$.

To prove (15), we note that $\mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})]$ equals

$$(\mathbf{a} \quad \mathbf{b} \quad [\mathbf{c} \times \mathbf{d}]) = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$$

by the definition of the triple product, as well as $(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$ by (13) (take the dot product by **a**).

The last formula, (16), follows from familiar rules of interchanging the rows of a determinant.

26. $\mathbf{m} = [2, 3, 2] \times [1, 0, 3] = [9, -4, -3]; \quad m = \sqrt{106} = 10.3$

28. The midpoints are

$$\begin{array}{lll} M_1: & (3.5, 0, 0) & \text{Midpoint of } \overline{AB}, \\ M_2: & (6.5, 0.5, 0) & \text{Midpoint of } \overline{BC}, \\ M_3: & (6, 2.5, 0) & \text{Midpoint of } \overline{CD}, \\ M_4: & (3, 2, 0) & \text{Midpoint of } \overline{DA}. \end{array}$$

The cross product of adjacent sides of Q is

$$\begin{aligned} \overline{M_1M_4} \times \overline{M_1M_2} &= [0.5, 2, 0] \times [3, 0.5, 0] \\ &= [0, 0, -6.25]. \end{aligned}$$

Its length 6.25 is the area of Q.

30. A normal vector is

$$\begin{aligned} \mathbf{N} = \overline{AB} \times \overline{AC} &= [3, 0, -2.25] \times [-1, 6, 3.75] \\ &= [13.5, -9, 18]. \end{aligned}$$

Hence the plane is represented by

$$\mathbf{N} \cdot \mathbf{r} = 13.5x - 9y + 18z = c$$

with c obtained by substituting the coordinates of C (or of A or B)

$$c = -9 \cdot 8 + 18 \cdot 4 = 0.$$

32. 10

34. Edge vectors are

$$\begin{aligned} &[3 - 1, 7 - 3, 12 - 6], \\ &[8 - 1, 8 - 3, 9 - 6], \\ &[2 - 1, 2 - 3, 8 - 6]. \end{aligned}$$

The mixed triple product of these vectors is -90 (or +90). This gives the answer 15.

SECTION 9.4. Vector and Scalar Functions and Their Fields. Vector Calculus: Derivatives, page 375

Purpose. To get started on vector differential calculus, we discuss vector functions and their continuity and differentiability.

Differentiation of scalar and vector functions will be needed throughout the rest of the chapter for developing the differential geometry of curves with application to mechanics (Sec. 9.5) and the three operators, gradient, with application to directional derivatives (Sec. 9.7), divergence (Sec. 9.8), and curl (Sec. 9.3).

The form of these operators in curvilinear coordinates is given in App. A3.4.

Main Content, Important Concepts

Vector and scalar functions and fields

Continuity, derivative of vector functions (9), (10) Differentiation of dot, cross, and triple products, (11)–(13)

Partial derivatives

Comment on Content

This parallels calculus of functions of one variable and can be surveyed quickly.

Further Comments on Text

A vector field (or scalar field) may be given along a straight line, along a curve (Fig. 195) or a surface (Fig. 196) or in a three-dimensional region of space. In practice, these are the most important cases for the engineer.

Important applications are scalar fields in space (Example 1), velocity fields of rotations (Example 2), and the gravitational field of masses (Example 3).

Convergence, continuity, and differentiability of vector functions are defined in connection with (4), (8), and (9), and these concepts relating to vector functions can be expressed in terms of components. In particular, formula (10) states that a vector function can be differentiated componentwise.

Formulas (11)–(13) are immediate consequences of familiar differentiation rules.

An extension of this to partial differentiation is illustrated in Example 5.

Comments on Problems

Although there is practically not much difference in working in the plane and in space, we begin in Probs. 1–8 with the former case, where visualization and graphing (sketching) is much simpler.

Extension to space follows in Probs. 9–14.

Those first problems concern scalar fields, which are simpler than vector fields, which may technically be regarded as triples of (coordinate-dependent!) scalar fields (which conceptually they are *not*!).

The set ends with a few problems (22–25) in differential calculus. Here the student should consult and review material from his or her calculus text.

SOLUTIONS TO PROBLEM SET 9.4, page 380

2. Hyperbolas $T = xy = \text{const}$ with the coordinate axes as asymptotes.

4. Straight lines through the origin (planes through the z-axis) $y/x = \text{const}$.

6. $x/(2x^2 + 2y^2) = c$, hence $x = 2c(x^2 + y^2)$. Division by c ($\neq 0$) gives

$$\frac{x}{2c} = x^2 + y^2, \qquad \text{thus} \qquad \left(x - \frac{1}{4c}\right)^2 + y^2 = \frac{1}{16c^2}.$$

These are circles with center at $(1/(4c), 0)$ and radius $1/(4|c|)$, so that they all pass through the origin.

7. Ellipses.

8. CAS Project. A CAS can graphically handle these more complicated functions, whereas the paper-and-pencil method is relatively limited. This is the point of this project.

Note that all these functions occur in connection with Laplace's equation, so that they are real or imaginary parts of complex analytic functions.

9. Parallel planes

10. Ellipsoids of revolution. The ellipsoid

$$4x^2 + 4y^2 + z^2 = c^2$$

intersects the axes at $c/2$, $c/2$, and c, respectively.

11. Elliptic cyclinders

12. Congruent circular cones $z = \sqrt{x^2 + y^2} + c$ with apex at $z = c$ on the z-axis.

14. Congruent parabolic cylinders with vertical generators and the xz-plane as plane of symmetry.

16. This could be the velocity field of a counterclockwise rotation about the origin. Indeed, at a point (x, y) the vector $\mathbf{v}$ is perpendicular to the segment from the origin to (x, y). Also, $|\mathbf{v}| = \sqrt{x^2 + y^2}$, that is, the speed is proportional to the distance of the point from the origin (the axis of rotation in space), as it should be for such a rotation.

18. $\mathbf{v}$ has radial direction away from the origin.

20. Clockwise rotation; compare with Prob. 16.

22. $\mathbf{r}' = [-6 \sin 2t, 6 \cos 2t, 4]$. The second derivative is

$$\mathbf{r}'' = [-12 \cos 2t, -12 \sin 2t, 0].$$

This problem has to do with a helix, as we shall see in the next section.

24. $\mathbf{y}_{1x} = [e^x \cos y, e^x \sin y]$, $\mathbf{v}_{1y} = [-e^x \sin y, e^x \cos y]$. Similarly, for the second given function,

$$\mathbf{v}_2x = [-\sin x \cosh y, -\cos x \sinh y]$$

and

$$\mathbf{v}_{2y} = [\cos x \sinh y, -\sin x \cosh y].$$

SECTION 9.5. Curves. Arc Length. Curvature. Torsion, page 381

Purpose. Discussion of space curves as an application of vector functions of one variable, the use of curves as paths in mechanics (and as paths of integration of line integrals in Chapter 10). Role of parametric representations, interpretation of derivatives in mechanics, completion of the discussion of the foundations of differential–geometric curve theory.

Main Content, Important Concepts

Parametric representation (1)

Orientation of a curve

Circle, ellipse, straight line, helix

Tangent vector (7), unit tangent vector (8), tangent (9)

Length (10), arc length (11)

Arc length as parameter [cf. (14)]

Velocity, acceleration (16)–(19)

Centripetal acceleration, Coriolis acceleration

Curvature, torsion, Frenet formulas (Prob. 50)

Short Courses. This section can be omitted.

Comments on Text

This long section gives an overview of the differential geometry of curves in space, as needed in mechanics, where velocity and tangential and normal acceleration are basic; see (17), (18), and (18*).

The discussion begins with parametric representations (Examples 1–4), tangents (Example 5), and arc length (for the helix in Example 6).

Then the section turns to mechanics, discussing centripetal and centrifugal forces (Example 7) and Coriolis acceleration appearing in the superposition of rotations, as for the motion of missiles (Example 8 and Fig. 211).

We finally discuss curvature κ and torision τ and related concepts shown in Fig. 212; since this is of minor interest to the engineer, we leave this last part of the section optional. The culmination of this are the Frenet formulas (Probs. 54 and 55), which imply that $\kappa(s)$ and $\tau(s)$, if sufficiently differentiable, determine a curve uniquely, except for its position in space.

Comments on Problems

These follow the train of thoughts in the text, discussing first parametric representations in detail (Probs. 1–23). Here, Prob. 23 shows a list of classical curves the engineer may need from time to time.

Problems 24–28 concern the representations of tangents.

Problems 29–32 involve only integrals that are simple, which is generally not the case in connection with lengths of curves.

Problems 35–46 concern mechanics.

The remaining Probs. 47–55 correspond to the optional parts of the text regarding curvature and torsion.

SOLUTIONS TO PROBLEM SET 9.5, page 390

1. Circle, center (0, 2), radius 4.

2. Straight line through (a, b, c) in the direction of the vector $[1, 3, -5]$.

3. Cubic parabola $x = 0, z = 2t^3$

4. Circle of radius 5 and center $(2, -1)$ in the plane $x = -2$, which is parallel to the yz-plane.

5. Ellipse

6. This is an ellipse with center (a, b) and semi-axes 3 and 2, oriented clockwise because of the minus sign. Because of the factor π the whole curve is obtained if we let t vary from 0 to $2\pi/\pi = 2$.

7. Helix

8. $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$ gives a hyperbola in the plane $z = 2$.

9. A "Lissajous curve"

10. Hyperbola $xz = 1$ in the plane $y = 2$.

11. $\mathbf{r} = [1 + \sqrt{2} \cos t, -1 + \sqrt{2} \sin t, 2]$

12. The yz-plane is $x = 0$. The center $(4, 0)$ has the distance 5 from $(0, 3)$. Hence a representation is

$$\mathbf{r} = [0, 4 + 5 \cos t, 0 + 5 \sin t].$$

13. $\mathbf{r} = [3 + t, 1, 2 + 4t]$

14. A vector from $(1, 1, 1)$ to $(4, 0, 2)$ is $\mathbf{b} = [3, -1, 1]$. Hence a representation is

$$\mathbf{r} = [1, 1, 1] + \mathbf{b}t = [1 + 3t, 1 - t, 1 + t].$$

15. $\mathbf{r} = [t, 2t - 1, 3t]$

16. Ellipse $\mathbf{r} = [\cos t, \sin t, \sin t]$. Since the plane makes an angle of 45° with the xy-plane, the semi-axes of the ellipse are $\sqrt{2}$ (in the y-direction) and 1 (in the x-direction); indeed, the apex at $(0, 1, 1)$ has distance $\sqrt{2}$ from the origin.

17. $\mathbf{r} = [\sqrt{3}\cos t, \sin t, \sin t]$

18. $\mathbf{r} = [5 \cos t, 5 \sin t, 2t]$.

19. $\mathbf{r} = [\cosh t, \frac{1}{\sqrt{2}} \sinh (t), -2]$.

20. This linear system of two equations in three unknowns has the solution

$$\mathbf{r} = [\tfrac{7}{5} - t, \tfrac{4}{5} + t, t]$$

where $z = t$ remains arbitrary. Hence this may be regarded as a parametric representation of the straight line of intersection of the two planes given by the two equations.

Obviously, this line is determined by a point through which it passes and a direction, given by a vector $\mathbf{v}$. As a point we can choose the intersection of the line with the plane $z = 0$ (the xy-plane), for which the given equations, with $z = 0$, yield

$$2x - y = 2, \qquad x + 2y = 3$$

and have the solution $x = \frac{7}{5}$, $y = \frac{4}{5}$. Hence $\mathbf{a} = [\frac{7}{5}, \frac{4}{5}, 0]$ is a point on the line of intersection, call it L. The direction of the latter is given by the vector product for the two normal vectors of the given planes, that is

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = [-5, 5, 5].$$

Hence a parametric representation of the straight line of intersection of the two planes is

$$\begin{aligned} \mathbf{r} = \mathbf{a} + t\mathbf{v} &= \mathbf{a} + [-5t, 5t, 5t] \\ &= [\tfrac{7}{5} - 5t, \tfrac{4}{5} + 5t, 5t]. \end{aligned}$$

24. P corresponds to $t = 2$; indeed, $\mathbf{r}(2) = [2, 1, 2]$. Differentation gives

$$\mathbf{r}'(t) = [1, t/2, 0] \qquad \text{and at } P, \qquad \mathbf{r}'(2) = [1, 1, 0].$$

The unit tangent vector in the direction of $\mathbf{r}'(t)$ is

$$\mathbf{u}'(t) = [2/\sqrt{4 + t^2}, t/\sqrt{4 + t^2}, 0] \qquad \text{and at } P,\ \mathbf{u}'(2) = [\sqrt{8}/4, \sqrt{8}/4, 0].$$

This gives the representation of the tangent of C at P in the form

$$\mathbf{q}(w) = \mathbf{r}(2) + w\mathbf{r}'(2) = [2 + w, 1 + w, 2].$$

26. Differentiation gives a tangent vector

$$\mathbf{r}'(t) = [-\sin t, \cos t, 9].$$

P corresponds to the parametric value $t = 2\pi$. The value of $\mathbf{r}'$ at P is $[0, 1, 9]$. A representation of the tangent at P is

$$\mathbf{q}(w) = [1 + 0, 0 + w, 18\pi + 9w].$$

27. $\mathbf{q}(w) = [4 + w, 1 - w/4, 0]$.

28. A tangent vector is

$$\mathbf{r}'(t) = [1, 2t, 3t^2].$$

The corresponding unit tangent vector is

$$\mathbf{u} = (1 + 4t^2 + 9t^4)^{-1/2}[1, 2t, 3t^2].$$

P corresponds to $t = 1$. Hence at P we have

$$\mathbf{r}' = [1, 2, 3]$$

and

$$\mathbf{u} = 14^{-1/2}[1, 2, 3].$$

A representation of the tangent at P is

$$\mathbf{q}(w) = [1 + w, 1 + 2w, 1 + 3w].$$

29. $\sqrt{\mathbf{r}' \cdot \mathbf{r}'} = \cosh t$, $l = \sinh(2) = 3.627$.

30. The initial and terminal point of the arc correspond to $t = 0$ and $t = 2\pi$. Differentiation gives a tangent vector

$$\mathbf{r}' = [-4 \sin t, 4 \cos t, 5].$$

The integrand needed is $\sqrt{\mathbf{r}' \bullet \mathbf{r}'} = \sqrt{41}$. Hence the length is $l = 2\pi\sqrt{41}$.

31. $\sqrt{\mathbf{r}' \cdot \mathbf{r}'} = a$, $l = a\pi$

32. $\mathbf{r}' = [-3a \cos^2 t \sin t, 3a \sin^2 t \cos t]$. Taking the dot product and applying trigonometric simplification gives

$$\begin{aligned}\mathbf{r}' \bullet \mathbf{r}' &= 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t \\ &= 9a^2 \cos^2 t \sin^2 t \\ &= \frac{9a^2}{4} \sin^2 2t.\end{aligned}$$

From this we obtain as the length in the first quadrant

$$l = \frac{3}{2} a \int_0^{\pi/2} \sin 2t \, dt = -\frac{3a}{4}(\cos \pi - \cos 0) = \frac{3a}{2}. \qquad \textit{Answer: } 6a$$

34. We obtain

$$\begin{aligned}ds^2 &= dx^2 + dy^2 \\ &= (d\rho \cos\theta - \rho \sin\theta \, d\theta)^2 + (d\rho \sin\theta + \rho \cos\theta \, d\theta)^2 \\ &= d\rho^2 + \rho^2 \, d\theta^2 \\ &= (\rho'^2 + \rho^2) \, d\theta^2.\end{aligned}$$

For the cardioid.

$$\begin{aligned}\rho^2 + \rho'^2 &= a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta \\ &= 2a^2(1 - \cos\theta) \\ &= 4a^2 \sin^2 \tfrac{1}{2}\theta\end{aligned}$$

so that

$$l = 2a \int_0^{2\pi} \sin \frac{1}{2}\theta \, d\theta = 8a.$$

35. $\mathbf{v} = \mathbf{r}' = [1, 8t, 0]$, $|\mathbf{v} = \sqrt{1 + 64t^2}$, $\mathbf{a} = [0, 8, 0]$

36. $\mathbf{v} = \mathbf{r}' = [2, 4, 0]$, $|\mathbf{v} = 2\sqrt{5}$, $\mathbf{a} = [0, 0, 0]$, a nonaccelerated motion (uniform motion, motion of constant speed).

38. $\mathbf{v} = \mathbf{r}' = [-\sin t, \quad 2\cos t, \quad 0]$, $|\mathbf{v}| = (\sin^2 t + 4\cos^2 t)^{1/2}$,
$\mathbf{a} = [-\cos t, \quad -2\sin t, \quad 0]$. Hence the tangential acceleration is

$$\mathbf{a}_{\tan} = \frac{-3\sin t\cos t}{\sin^2 t + 4\cos^2 t}[-\sin t, \quad 2\cos t, \quad 0]$$

and has the magnitude $|\mathbf{a}_{\tan}|$, where

$$|\mathbf{a}_{\tan}|^2 = \frac{9\sin^2 t\cos^2 t}{\sin^2 t + 4\cos^2 t}.$$

39. $\mathbf{v} = [2\cos 2t, -\sin t]$, $|\mathbf{v}|^2 = 8\cos 4t - 2\cos 2t + 10$
$\mathbf{a} = [-4\sin 2t, -\cos t]$, $\mathbf{a}_{tan} = \frac{8\sin 4t - \sin 2t}{-4\cos 4t + \cos 2t - 5}\mathbf{v}$

40. The velocity is

$$\mathbf{v} = [-2\sin t - 2\sin 2t, 2\cos t - 2\cos 2t].$$

From this we obtain the square of the speed

$$|\mathbf{v}|^2 = \mathbf{v} \bullet \mathbf{v} = (-2\sin t - 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2.$$

Performing the squares and simplifying gives

$$\begin{aligned} |\mathbf{v}|^2 &= 8(1 + \sin t\sin 2t - \cos t\cos 2t) \\ &= 8(1 - \cos 3t) \\ &= 16\sin^2\frac{3t}{2}. \end{aligned}$$

Hence

$$|\mathbf{v}| = 4\sin\frac{3t}{2}.$$

$$\mathbf{a} = [-2\cos t - 4\cos 2t, \quad -2\sin t + 4\sin 2t].$$

We use (18*). By straightforward simplification (four terms cancel),

$$\begin{aligned} \mathbf{a} \bullet \mathbf{v} &= 12(\cos t\sin 2t + \sin t\cos 2t) \\ &= 12\sin 3t. \end{aligned}$$

Hence (18*) gives

$$\mathbf{a}_{\tan} = \frac{12\sin 3t}{16\sin^2(3t/2)}\mathbf{v}$$

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\tan}.$$

41. $\mathbf{v} = [\cos t, -\sin t, -2\sin 2t]$, $|\mathbf{v}|^2 = 3 - 2\cos 4t$
$\mathbf{a} = [-\sin t, -\cos t, -4\cos 2t]$, $\mathbf{a}_{tan} = \frac{4\sin 4t}{-3 + 2\cos 4t}\mathbf{v}$

42. The velocity vector is

$$\mathbf{v} = [c\cos t - ct\sin t, c\sin t + ct\cos t, c].$$

Hence the square of the speed is

$$|\mathbf{v}|^2 = c^2(t^2 + 2).$$

Another differentiation gives the acceleration

$$\mathbf{a} = [-2c \sin t - ct \cos t,\, 2c \cos t - ct \sin t,\, 0].$$

The tangential acceleration is

$$\mathbf{a}_{\tan} = \frac{ct}{t^2 + 2} [\cos t - t \sin t,\, \sin t + t \cos t,\, 1]$$

and the normal acceleration is

$$\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\tan}.$$

This is a spiral on a cone.

44. $\mathbf{R} = 3.85 \cdot 10^8$ m, $|\mathbf{v}| = 2\pi R/(2.36 \cdot 10^6) = 1025$ [m/sec], $|\mathbf{v}| = \omega R, |\mathbf{a}| = \omega^2/R = |\mathbf{v}|^2/R = 0.0027$ [m/sec^2], which is only $2.8 \cdot 10^{-4}\, g$, where g is the acceleration due to gravity at the Earth's surface.

46. $\mathbf{R} = 3960 + 450 = 4410$ [mi], $2\pi R = 100|\mathbf{v}|$, $|\mathbf{v}| = 277.1$ mi/min,

$$g = |\mathbf{a}| = \omega^2 R = |\mathbf{v}|^2/R = 17.41 \text{ [mi/min}^2] = 25.53 \text{ [ft/sec}^2] = 7.78 \text{ [m/sec}^2].$$

Here we used $|\mathbf{v}| = \omega R$.

48. We denote derivatives with respect to t by primes. In (22),

$$\mathbf{u} = \frac{d\mathbf{r}}{ds} = \mathbf{r}' \frac{dt}{ds}, \qquad \frac{dt}{ds} = \frac{1}{s'} = (\mathbf{r}' \bullet \mathbf{r}')^{-1/2}. \qquad \text{[See (12).]}$$

Thus in (22),

$$\frac{d\mathbf{u}}{ds} = \mathbf{r}''\left(\frac{dt}{ds}\right)^2 + \mathbf{r}' \frac{d^2t}{ds^2} = \mathbf{r}''(\mathbf{r}' \bullet \mathbf{r}')^{-1} + \mathbf{r}' \frac{d^2t}{ds^2}$$

where

$$\begin{aligned} \frac{d^2t}{ds^2} = \frac{d}{dt}\left(\frac{dt}{ds}\right)\frac{dt}{ds} &= -\frac{1}{2}(\mathbf{r}' \bullet \mathbf{r}')^{-3/2}\, 2(\mathbf{r}'' \bullet \mathbf{r}')(\mathbf{r}' \bullet \mathbf{r}')^{-1/2} \\ &= -(\mathbf{r}'' \bullet \mathbf{r}')(\mathbf{r}' \bullet \mathbf{r}')^{-2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d\mathbf{u}}{ds} &= \mathbf{r}''(\mathbf{r}' \bullet \mathbf{r}')^{-1} - \mathbf{r}'(\mathbf{r}'' \bullet \mathbf{r}')(\mathbf{r}' \bullet \mathbf{r}')^{-2} \\ \frac{d\mathbf{u}}{ds} \bullet \frac{d\mathbf{u}}{ds} &= (\mathbf{r}'' \bullet \mathbf{r}'')(\mathbf{r}' \bullet \mathbf{r}')^{-2} - 2(\mathbf{r}'' \bullet \mathbf{r}')^2(\mathbf{r}' \bullet \mathbf{r}')^{-3} + (\mathbf{r}' \bullet \mathbf{r}')^{-3}(\mathbf{r}'' \bullet \mathbf{r}')^2 \\ &= (\mathbf{r}'' \bullet \mathbf{r}'')(\mathbf{r}' \bullet \mathbf{r}')^{-2} - (\mathbf{r}'' \bullet \mathbf{r}')^2(\mathbf{r}' \bullet \mathbf{r}')^{-3}. \end{aligned}$$

Taking square roots, we get (22*).

50. $\tau = -\mathbf{p} \bullet (\mathbf{u} \times \mathbf{p})' = -\mathbf{p} \bullet (\mathbf{u}' \times \mathbf{p} + \mathbf{u} \times \mathbf{p}') = 0 - (\mathbf{p}\ \ \mathbf{u}\ \ \mathbf{p}') = +(\mathbf{u}\ \ \mathbf{p}\ \ \mathbf{p}')$. Now $\mathbf{u} = \mathbf{r}'$, $\mathbf{p} = (1/\kappa)\mathbf{r}''$; hence $\mathbf{p}' = (1/\kappa)\mathbf{r}''' + (1/\kappa)'\mathbf{r}''$. Inserting this into the triple product (the determinant), we can simplify the determinant by familiar rules and let the last term in $\mathbf{p}'$ disappear. Pulling out $1/\kappa$ from both $\mathbf{p}$ and $\mathbf{p}'$, we obtain the second formula in (23**).

52. From $\mathbf{r}(t) = [a \cos t,\ \ a \sin t,\ \ ct]$ we obtain

$$\mathbf{r}' = [-a \sin t,\ \ a \cos t,\ \ c], \qquad \mathbf{r}' \bullet \mathbf{r}' = a^2 + c^2 = K^2.$$

Hence, by integration, $s = Kt$. Consequently, $t = s/K$. This gives the indicated representation of the helix with arc length s as parameter. Denoting derivatives with respect to s also by primes, we obtain

$$\mathbf{r}(s) = \left[a \cos \frac{s}{K},\quad a \sin \frac{s}{K},\quad \frac{cs}{K}\right], \qquad K^2 = a^2 + c^2$$

$$\mathbf{u}(s) = \mathbf{r}'(s) = \left[-\frac{a}{K} \sin \frac{s}{K},\quad \frac{a}{K} \cos \frac{s}{K},\quad \frac{c}{K}\right]$$

$$\mathbf{r}''(s) = \left[-\frac{a}{K^2} \cos \frac{s}{K},\quad -\frac{a}{K^2} \sin \frac{s}{K},\quad 0\right]$$

$$\kappa(s) = |\mathbf{r}''| = \sqrt{\mathbf{r}'' \bullet \mathbf{r}''} = \frac{a}{K^2} = \frac{a}{a^2 + c^2}$$

$$\mathbf{p}(s) = \frac{1}{\kappa(s)} \mathbf{r}''(s) = \left[-\cos \frac{s}{K},\quad -\sin \frac{s}{K},\quad 0\right]$$

$$\mathbf{b}(s) = \mathbf{u}(s) \times \mathbf{p}(s) = \left[\frac{c}{K} \sin \frac{s}{K},\quad -\frac{c}{K} \cos \frac{s}{K},\quad \frac{a}{K}\right]$$

$$\mathbf{b}'(s) = \left[\frac{c}{K^2} \cos \frac{s}{K},\quad \frac{c}{K^2} \sin \frac{s}{K},\quad 0\right]$$

$$\tau(s) = -\mathbf{p}(s) \bullet \mathbf{b}'(s) = \frac{c}{K^2} = \frac{c}{a^2 + c^2}.$$

Positive c gives a right-handed helix and positive torsion; negative c gives a left-handed helix and negative torsion.

54. $\mathbf{p}' = (1/\kappa)\mathbf{u}'$ implies the first formula, $\mathbf{u}' = \kappa\mathbf{p}$. The third Frenet formula was given in the text before (23). To obtain the second Frenet formula, use

$$\mathbf{p}' = (\mathbf{b} \times \mathbf{u})' = \mathbf{b}' \times \mathbf{u} + \mathbf{b} \times \mathbf{u}' = -\tau\mathbf{p} \times \mathbf{u} + \mathbf{b} \times \kappa\mathbf{p} = +\tau\mathbf{b} - \kappa\mathbf{u}.$$

In differential geometry (see [GenRef8] in App. 1) it is shown that the whole differential–geometric theory of curves can be obtained from the Frenet formulas, whose solution shows that the **natural equations** $\kappa = \kappa(s), \tau = \tau(s)$ determine a curve uniquely, except for its position in space.

SECTION 9.6. Calculus Review: Functions of Several Variables. *Optional,* page 392

Purpose. To give students a handy reference and some help on material known from calculus that they will need in their further work.

SECTION 9.7. Gradient of a Scalar Field. Directional Derivative, page 395

Purpose. To discuss gradients and their role in connection with directional derivatives, surface normals, and the generation of vector fields from scalar fields (potentials).

Main Content, Important Concepts

Gradient, nabla operator

Directional derivative, maximum increase, surface normal

Vector fields as gradients of potentials

Laplace's equation

Comments on Content

This is probably the first section in which one should no longer rely on knowledge from calculus, although relatively elementary calculus books usually include a passage on gradients.

Potentials are important; they will occur at a number of places in our further work.

Further Comments on Text

Figure 215 illustrates the directional derivatives geometrically. Note that s can be positive, zero, or negative.

Theorem 1 is needed because the gradient in (1) involves coordinates.

Figure 216 illustrates a major geometric application of the gradient.

The notion of potential is basic, and Theorem 3 states one of the most important examples.

Coulomb's law (12) is of the same form as Newton's law of graviation in (8); thus the two are governed by the same theory.

Comments on Problems

Problems 1–17 require specific calculations and show some general foumulas for the gradient and the Laplacian.

Problems 18–23 and 43–45 concern vector fields and their potentials.

Problems 30–35 show applications to curve and surface theory.

Directional derivatives are considered in Probs. 36–42.

Hence the problem set reflects the many-sided aspects of the gradient and its applications.

SOLUTIONS TO PROBLEM SET 9.7, page 402

1. $[4y - 2, 4x - 4]$

2. $bfv = \text{grad} f = [4x, 10y]$

3. $[1/y, -x/y^2]$

4. $\mathbf{v} = \text{grad} f = [2x - 4, 8y + 16]$

5. $[5x^4, 5y^4]$

6. $(x^2 - y^2)^{-2}[-4xy^2, 4yx^2]$

8. Applying the product rule to each component of $\nabla(fg)$ and collecting terms, the formula follows,

$$[(fg)_x, (fg)_y, (fg)_z] = [f_x g, f_y g, f_z g] + [fg_x, fg_y, fg_z].$$

10. Apply the product rule twice to each of the three terms of ∇^2, obtaining

$$(fg)_{xx} = f_{xx}g + 2f_x g_x + fg_{xx}$$

and so on, and reorder and collect terms into three sums that make up the right side of the formula.

11. $[y, x]$, $[-4, 3]$

12. $\mathbf{v} = \nabla f = \left[-\dfrac{x^2 - y^2}{(x^2 + y^2)^2}, -\dfrac{2xy}{(x^2 + y^2)^2}\right]$, $\mathbf{v}(1, 1) = [0, -\frac{1}{2}]$, so that the gradient at (1, 1) is pointing in the negative y-direction.

13. $(x^2 + y^2)^{-1}[2x, 2y]$, $[2/5, 1/5]$

14. $\mathbf{v} = \text{grad} f = -(x^2 + y^2 + z^2)^{-3/2}[x, y, z]$. Its value at P is $[-0.0015, 0, -0.0020]$.

15. $[4x, 8y, 18z]$, $[-4, 16, -72]$

16. $\mathbf{v} = \nabla f = [50x, 18y, 32z] = -k[x, y, z]$ has solutions precisely for the points on the three principal axes of the ellipsoids, that is, for P on the coordinate axes.

17. For P on the $x-$ and $y-$ axes.

18. $\mathbf{v} = \nabla f = [2x - 6, -2y]$. The value at P is $[-8, -10]$. The curves $f = \text{const}$ are hyperbolas with asymptotes $y = \pm(x - 3)$.

20. $\mathbf{v} = \nabla f = \left[1 + \dfrac{1}{x^2 + y^2} - \dfrac{2x^2}{(x^2 + y^2)^2}, -\dfrac{2xy}{(x^2 + y^2)^2}\right]$. At $(1, 1)$ this equals $[1, -0.5]$.

We mention that this is the real part of the complex analytic function $\zeta + 1/\zeta$, where $\zeta = x + iy$ (we write ζ since z is used as coordinate in space), giving the flow around the circle $x^2 + y^2 = 1$, that is, a cylinder in space with axis intersecting the xy-plane at the origin. This flow and extensions of it will be discussed in the chapter on complex analysis and potential theory (in Sec. 18.4).

22. The x-component of $\mathbf{v} = \nabla f = [e^x \cos y, -e^x \sin y]$ must be zero; thus $y = \pm(2n + 1)\pi/2$. Then $\sin y = \pm 1$. We must have $\sin y = -1$ to obtain $-e^x \sin y > 0$ (upward flow), hence $y = -\frac{1}{2}\pi + 2n\pi$.

24. $-\nabla T = [-6x, 4y]$. At P this gives $[-15.0, 7.2]$.

26. $-\nabla T = [-2x, -2y, -8z]$ at P is $[-4, 2, -16]$.

28. $-\nabla z = [-2x, -18y]$, $\nabla z(P) = [-8, -18]$. Hence a vector in the direction of steepest ascent is $[-1, -2.25]$.

30. $\nabla f = [8x, 18y]$, $\nabla f(p) = [16, 12\sqrt{14}]$

32. $[a, b, c]$. Planes have constant normal direction.

34. $\nabla f = [4x^3, 4y^3, 4z^3]$, $\nabla f(p) = [32, 4, 256]$. The intersection of this surface with planes parallel to the coordinate planes are curves each of which is between a circle and a square of portions of four tangents to that circle whose center is the origin of the plane of the circle.

36. $\nabla f = [4x, 4y]$. From (5*) we thus obtain the answer

$$[-1, -3] \bullet [12, 12]/\sqrt{10} = -48/\sqrt{10}.$$

38. $D_{\mathbf{a}} f = [1, 1, 3] \bullet [4, 4, -2]/\sqrt{11} = 2/\sqrt{11}$

40. $(\nabla f) \bullet \mathbf{a}/|\mathbf{a}| = (x^2 + y^2)^{-1}[2x, 2y] \bullet [1, -1] = (x^2 + y^2]^{-1}(2x - 2y)/\sqrt{2}$ at $(3, 0)$ equals $\sqrt{2}/3$.

42. 0 without calculation because on the axes an ellipsoid $f = \text{const}$ has a tangent plane perpendicular to the axis, whereas $\mathbf{a}$ lies in that plane at the x-axis, so that ∇f and $\mathbf{a}$ are perpendicular to each other.

44. $f = ye^x + \frac{1}{3}z^3$

SECTION 9.8. Divergence of a Vector Field, page 402

Purpose. To explain the divergence (the second of the three concepts grad, div, curl) and its physical meaning in fluid flows.

Main Content, Important Concepts

Divergence of a vector field

Continuity equations (5), (6)

Incompressibility condition (7)

Comment on Content

The interpretation of the divergence in Example 2 depends essentially on our assumption that there are no sources or sinks in the box. From our calculations it becomes plausible that, in the case of sources or sinks, the divergence may be related to the net flow across the boundary surfaces of the box. To confirm this and to make it precise we need integrals; we shall do this in Sec. 10.8 (in connection with Gauss's divergence theorem).

Moving div and curl to Chap. 10?

Experimentation has shown that this would perhaps not be a good idea, simply because it would combine two substantial difficulties, that of understanding div and curl themselves and that of understanding the nature and role of the two basic integral theorems by Gauss and Stokes, in which div and curl play the key role.

Comments on Problems

Project 9 concerns some standard formulas useful in working with the divergence.

CAS Experiment should help the student in gaining an intuitive understanding of the divergence.

Formula (3) is basic, as the problems should further emphasize.

SOLUTIONS TO PROBLEM SET 9.8, page 405

1. $4x - 6y + 16z$, 15

2. $\cos(x^2yz)\, x^2z - \sin(xy^2z)\, xy^2$, $\sqrt{2}/8$

3. 0, after simplification, solenoidal.

4. 0. Hence this field is solenoidal, regardless of the special form of v_1, v_2, v_3.

5. $6xyz$, 36

6. 0, hence the vector field is solenoidal.

7. $-2e^y \cos xz$

8. $\operatorname{div} \mathbf{v} = 2 + \dfrac{\partial v_3}{\partial z}$. Of course, there are many ways of satisfying the conditions. For instance, (a) $v_3 = 0$, (b) $v_3 = -z - \frac{1}{3}z^3$. The point of the problem is that the student gets used to the definition of the divergence and recognizes that div **v** can have different values and also the sign can differ in different regions of space.

10. **(a)** Parallel flow.

(b) Outflow on the left and right, no flow across the other sides; hence $\operatorname{div} \mathbf{v} > 0$.

(c) Outflow left and right, inflow from above and below, balance perhaps zero; by calculation, $\operatorname{div} \mathbf{v} = 0$. Etc.

12. $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k} = x\mathbf{i}$. Hence $\operatorname{div} \mathbf{v} = 1$, and

$$\frac{dx}{dt} = x, \qquad \frac{dy}{dt} = 0, \qquad \frac{dz}{dt} = 0.$$

By integration, $x = c_1e^t$, $y = c_2$, $z = c_3$, and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence

$$\mathbf{r}(0) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \qquad \text{and} \qquad \mathbf{r}(1) = c_1e\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

This shows that the cube in Prob. 9 is now transformed into the rectangular parallelepiped bounded by $x = 0$, $x = e$, $y = 0$, $y = 1$, $z = 0$, $z = 1$, whose volume is e.

14. No. $\mathbf{u} = \mathbf{v}$ concerns components, whereas div **u** and div **v** are sums of contributions from all three components.

16. The gradient is

$$\nabla f = e^{xyz}\,[yz, zx, xy].$$

Hence (3) gives

$$\operatorname{div}(\nabla f) = e^{xyz}(y^2z^2 + z^2x^2 + x^2y^2).$$

The work in the direct calculation is practically the same. From Probs. 15–20 the student should understand that relation (3) is quite natural.

18. The gradient is

$$\nabla f = [-x(x^2 + y^2)^{-1/2}, -y(x^2 + y^2)^{-1/2}, 1].$$

Application of the definition of the divergence now gives

$$\nabla^2 f = -(x^2 + y^2)^{-1/2} + x^2(x^2 + y^2)^{-3/2} - (x^2 + y^2)^{-1/2} + y^2(x^2 + y^2)^{-3/2}$$

which simplifies to $-(x^2 + y^2)^{-1/2}$.

20. The gradient is

$$\nabla f = [2e^{2x}\cosh 2y, 2e^{2x}\sinh 2y],$$

so that for the Laplacian we obtain

$$\operatorname{div}(\nabla f) = 4e^{2x}\cosh 2y + 4e^{2x}\cosh 2y = 8e^{2x}\cosh 2y,$$

whereas for $\tilde{f} = e^{2x}\cos 2y$ we have $\nabla^2\tilde{f} = 0$.

SECTION 9.9. Curl of a Vector Field, page 406

Purpose. We introduce the curl of a vector field (the last of the three concepts grad, div, curl) and interpret it in connection with rotations (Example 2 and Theorem 1). A main application of the curl follows in Sec. 10.9 in Stokes's integral theorem.

Experience has shown that it is generally didactically preferable to defer Stokes's theorem to a later section and first to give the student a feel for the curl independent of an integral theorem.

Main Content

Definition of the curl (1)

Curl and rotations (Theorem 1)

Gradient fields are irrotational (Theorem 2)

Irrotational fields, conservative fields

Comments on Text

The curl is suggested by rotations; see Theorem 1.

We have now reached the point at which we can state basic relations among the three operators grad, div, curl (Theorem 2).

Since Definition 1 involves coordinates, we have to prove that curl **v** is a vector; see Theorem 3.

Comments on Problems

Calculations (Probs. 4–8) are followed by typical applications in fluid mechanics (Probs. 9–13).

As in the previous two sections, we finally present general formulas, this time for div and curl, and request some corresponding calculations (Probs. 14–20).

SOLUTIONS TO PROBLEM SET 9.9, page 408

2. **(a)** Nothing, in general. **(b)** curl $\mathbf{v}$ is parallel to the x-axis or $\mathbf{0}$.

4. curl $\mathbf{v} = [0, 0, (6x - 8y)]$

5. $[x(z^3 - y^3), y(x^3 - z^3), z(y^3 - x^3)]$

6. curl $\mathbf{v} = \mathbf{0}$. Recall from Theorem 3 in Sec. 9.7 with $\mathbf{r}_0 = \mathbf{0}$ and $x^2 + y^2 + z^2 = r^2$ that the present vector field is a gradient field, so that we must have curl $\mathbf{v} = \mathbf{0}$.

8. curl $\mathbf{v} = [-2ye^{-y^2}, -2ze^{-z^2}, -2xe^{-x^2}]$

10. curl $\mathbf{v} = [0, 0, -\cos x \csc^2 x]$, div $\mathbf{v} = \sec x \tan x$, compressible. Streamlines are obtained as follows. By the definition of the velocity vector and its present given form,

$$\mathbf{v} = [x', y', z'] = [\sec x, \csc x, 0].$$

Equating the first components gives

$$x' = \frac{dx}{dt} = \sec x, \qquad \cos x\, dx = dt.$$

By integration,

$$\sin x = t + c_1.$$

Hence

$$x = \arcsin (t + c_1).$$

From this and the second components,

$$y' = \operatorname{cosec} x = \frac{1}{1 + c_1}.$$

By integration,

$$y = \ln (t + c_1) + c_2.$$

Equating the third components and integrating, we finally have $z = c_3$.

12. curl $\mathbf{v} = [0, 0, 2]$. Also div $\mathbf{v} = 0$, incompressible. Streamlines are helices obtained as follows. As in Prob. 10 we first have

$$\mathbf{v} = [x', y', z'] = [-y, x, \pi].$$

In components,

$$x' = -y, \qquad y' = x, \qquad z' = \pi.$$

From the first two components, by differentiation and substitution,

$$x'' = -y' = -x.$$

A general solution is

$$x = a \cos t + b \sin t.$$

From this and the first components,

$$y = -x' = a \sin t - b \cos t.$$

From the third component,

$$z = \pi t + c_3.$$

The helix obtained lies on the cylinder of radius $\sqrt{a^2 + b^2}$ and axis the z-axis. Indeed,

$$x^2 + y^2 = a^2 \cos^2 t + 2ab \cos t \sin t + b^2 \sin^2 t$$
$$+ a^2 \sin^2 t - 2ab \cos t \sin t + b^2 \sin^2 t = a^2 + b^2.$$

14. Project. Parts (b) and (d) are basic. They follow from the definitions by direct calculation. Part (a) follows by decomposing each component accordingly.

(c) In the first component in (1) we now have fv_3 instead of v_3, etc. Product differentiation gives $(fv_3)_y = f_y v_3 + f \cdot (v_3)_y$. Similarly for the other five terms in the components. $f_y v_3$ and the corresponding five terms give $(\text{grad}\, f) \times \mathbf{v}$ and the other six terms $f \cdot (v_3)_y$, etc. give f curl $\mathbf{v}$.

(d) For twice continuously differentiable f the mixed second derivatives are equal, so that the result follows from $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ and (1), which gives

$$\text{curl}\,(\nabla f) = [(f_z)_y - (f_y)_z]\mathbf{i} + [(f_x)_z - (f_z)_x]\mathbf{j} + [(f_y)_x - (f_x)_y]\mathbf{k}.$$

(e) Write out and compare the 12 terms on either side.

15. [1, 1, 1], same (Why?)

16. $[x^2(z - y), y^2(x - z), z^2(y - x)]$. Confirmation by (c) Project 14:

$$(\nabla g) \times v = [yz, zx, xy] \times [y + z, z + x, x + y] = [x^2(z - y), y^2(x - z), z^2(y - x)].$$

Note that g curl $\mathbf{v} = 0$ because $\mathbf{v}$ is a gradient field, namely, $\mathbf{v} = \text{grad}\, f$. Hence the result is confirmed.

17. $2y + 2z + 2x$, 0 (why?), $x + y + z$

18. $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \text{div}\,[(x^2 - yz), (y^2 - zx), (z^2 - xy)]$
$= 2x + 2y + 2z$

Confirmation by (e) in Project 14:

$$\begin{aligned}\text{div}\,(\mathbf{u} \times \mathbf{v}) &= \mathbf{v} \cdot \text{curl}\, u - u \cdot \text{curl}\, \mathbf{v}\\ &= [(y + z), (z + x), (x + y)] \cdot [1, 1, 1] - \mathbf{u} \cdot \text{curl}\,(\nabla f)\\ &= y + z + z + x + x + y - 0\\ &= 2x + 2y + 2z\end{aligned}$$

19. $[2xy + z - x, 2z + xy - y^2, 2xy + z - zx]$, same (why?)

20. $\text{div}\,([(xyz + (x + y - z)yz) + (xyz + (x + y - z)\,xz) + (-xyz$
$+ (x + y - z)\,xy)]) = 2(yz + zx - xy)$

Confirmation. By Problem Set 9.7,

$$\nabla(fg) = f\nabla g + g\nabla f.$$

From this and Problem Set 9.8,

$$\begin{aligned}\text{div}\,(\nabla(fg)) &= \text{div}\,(f\nabla g) + \text{div}\,(g\nabla f)\\ &= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g\\ &= 0 + 0 + 2[1, 1, -1] \cdot [yz, xz, xy]\\ &= 2(yz + zx - xy).\end{aligned}$$

SOLUTIONS TO CHAPTER 9 REVIEW QUESTIONS AND PROBLEMS, page 409

12. [0, 0, 50], [2, −19, −5], [−2, 19, 5], **0**

14. −1250, −1250, −1250, undefined

16. $[4/\sqrt{65}, 7/\sqrt{65}, 0]$, $[3/\sqrt{35}, -1/\sqrt{35}, 5/\sqrt{35}]$; $5/\sqrt{35}$ is the projection of **a** in the direction of **b**. Similarly, $5/\sqrt{65}$ is the projection of **b** in the direction of **a**.

18. $\sqrt{110} = 10.49 < \sqrt{65} + \sqrt{35} = 13.98$ illustrates the triangle inequality (7) in Sec. 9.2.

20. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Always.

22. $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{v}$ should have no x-component; thus $4 + 3 - 6 + v_1 = 0, v_1 = -1$, v_2 and v_3 arbitrary.

24. We are looking for the normal vectors [−1, 1, 4] and [1, −1, 2]. We obtain

$$\gamma = \arccos \frac{6}{\sqrt{108}} = 0.9553 = 54.73°.$$

25. $[2, 3, 0] \cdot [6, 7, 0] = 33$

26. The condition is $\mathbf{v} \cdot \mathbf{w}/|\mathbf{w}| = \mathbf{w} \cdot \mathbf{v}/|\mathbf{v}|$. The answer is $|\mathbf{v}| = |\mathbf{w}|$ or **v** and **w** are orthogonal, so that the numerators are zero and the size of the denominators does not matter.

28. The moment $m = |\mathbf{m}| = |\mathbf{r} \times \mathbf{p}|$ of a force **p** about a point Q is zero if $\mathbf{p} = \mathbf{0}$ or **p** is acting in a straight line through Q, which makes **p** and **r** parallel (or exactly opposite or $\mathbf{r} = \mathbf{0}$).

30. This is a helix. P corresponds to $t = \pi/3$. By differentiation,

$$\mathbf{r}'(t) = [-4 \sin t, 4 \cos t, 3].$$

At P the velocity is $\mathbf{v} = \mathbf{r}' = [-2\sqrt{3}, 2, 3]$. The speed is $|\mathbf{v}| = 5$. The acceleration vector is $\mathbf{r}''(t) = [-4 \cos t, -4 \sin t, 0]$. It is parallel to the xy-plane. Its absolute value, the acceleration, is constant, just as the speed, namely, $|\mathbf{r}''(t)| = 4$.

31. 5/3

32. $\text{grad} f = [z, -z, (x - y)]$ at $P: [0, 3, 1]$ is $[1, -1, -3]$. The value of f at P is -3. Hence the value of f grad f at P is $[-3, 3, 9]$.

33. $1, 2y + 4z$

34. $[0, 0, 3]$, $[0, 0, -2x - 2y]$

35. 0, same (why?), $2z^2 + 2x(x - y)$

36. −16

37. [0, 2, 4]

38. $D_v f = \dfrac{\text{grad} f \cdot \mathbf{v}}{|\mathbf{v}|} = \dfrac{[4z, 2y, (x - z)] \cdot [z, -z, (x - y)]}{\sqrt{17z^2 + 4y^2 + x^2 - 2zx}}$ has at P the value

$$\frac{[4, 2, 0] \cdot [1, -1, 0]}{\sqrt{17 + 4 + 1 - 2}} = 2/\sqrt{20} = 1/\sqrt{5}$$

39. $3/\sqrt{5}$

40. 0 since **v** appears in two rows (component wise) of this scalar triple product.