CHAPTER 18 Complex Analysis and Potential Theory

This is perhaps the most important justification for teaching complex analysis to engineers, and it also provides for nice applications of conformal mapping.

SECTION 18.1. Electrostatic Fields, page 756

Purpose. To show how complex analysis can be used to discuss and solve two-dimensional electrostatic problems and to demonstrate the usefulness of **complex potential**, a major concept in this chapter.

Main Content, Important Concepts

Equipotential lines

Complex potential (2)

Combination of potentials by superposition

The main reason for using complex methods in two-dimensional potential problems is the possibility of using the complex potential, whose real and imaginary parts both have a physical meaning, as explained in the text. This fact should be emphasized in teaching from this chapter.

SOLUTIONS TO PROBLEM SET 18.1, page 760

1. We have $\Phi = a \ln r + b$ (see Example 2 of the text) and obtain from this and the boundary conditions

$$\Phi(1) = a \ln 1 + b = b = 400$$
$$\Phi(2) = a \ln 2 + b = a \ln 2 + 400 = 0.$$

Hence $a = -400/\ln 2$. Together this gives the *answer*

$$\Phi(r) = 400\left(1 - \frac{1}{\ln 2} \ln r\right).$$

The complex potential is

$$F(z) = 400\left(1 - \frac{1}{\ln 2} \ln z\right).$$

4. $\Phi = 500 - 100xy$

SECTION 18.2. Use of Conformal Mapping. Modeling, page 761

Purpose. To show how conformal mapping helps in solving potential problems by mapping given domains onto simpler ones or onto domains for which the solution of the problem (subject to the transformed boundary conditions) is known.

The theoretical basis of this application of conformal mapping is given by Theorem 1, characterizing the behavior of harmonic functions under conformal mapping.

Problem 7 gives a hint on possibilities of generalizing potential problems for which the solution is known or can be easily obtained. The idea extends to more sophisticated situations.

SOLUTIONS TO PROBLEM SET 18.2, page 764

1. Figure 386 in Sec. 17.1 shows D, which is a semi-infinite horizontal strip, and D^*, which is the upper half of the unit circular disk. Since $u = e^x \cos y$ and $v = e^x \sin y$, the potential in D is

$$\begin{aligned}\Phi(x, y) &= \Phi^*(u(x, y), v(x, y)) \\ &= 4u(x, y)v(x, y) \\ &= 4e^{2x} \cos y \sin y \\ &= 2e^{2x} \sin 2y.\end{aligned}$$

This is a harmonic function. Its boundary values are 0 on the horizontal lines $y = 0$ and $y = \pi$ and $2 \sin 2y$ on the vertical segment $x = 0$ of D.

4. $\pm i$ are fixed points, and straight lines are mapped onto circles or straight lines. From this the assertion follows.

Alternatively, it also follows very simply by setting $x = 0$ and calculating $|w|$.

6. The potential on the y-axis is

$$\begin{aligned}\Phi &= \frac{6000}{\pi} \operatorname{Arg} \frac{1 + iy}{1 - iy} \\ &= \frac{6000}{\pi} \operatorname{Arg} \frac{1 + 2iy - y^2}{1 + y^2} \\ &= \frac{6000}{\pi} \arctan \frac{2y}{1 - y^2}\end{aligned}$$

and its derivative $1200/(\pi(1 + y^2))$ has a maximum at $y = 0$.

7. Apply $w = z^4$.

9. By Theorem 1 in Sec. 17.2.

10. $\Phi = \dfrac{V_0}{\pi}(-\operatorname{Arg}(z - a) + \operatorname{Arg}(z + a)), \quad F = \dfrac{V_0}{\pi} i \operatorname{Ln} \dfrac{z - a}{z + a}$

SECTION 18.3. Heat Problems, page 765

Purpose. To show that previous examples and new ones can be interpreted as potential problems in time-independent heat flow.

Comment on Interpretation Change

Boundary conditions of importance in one interpretation may be of no interest in another; this is about the only handicap in a change of interpretation. In other words, one should emphasize that, whereas the unifying underlying theory remains the same, problems of interest will change from field to field of application. This can be seen most distinctly by comparing the problem sets in this chapter.

SOLUTIONS TO PROBLEM SET 18.3, page 767

1. This is an approximate model of a long slender metal plate with insulated faces and edges kept at the indicated temperatures.

By inspection we find that the temperature distribution is

$$T(x, y) = 10 + 7.5(y - x).$$

This is the real part of the complex potential

$$F(z) = 10 - 7.5(1 + i)z.$$

A systematic derivation is as follows. The boundary and boundary values suggest that $T(x, y)$ is linear in x and y,

$$T(x, y) = ax + by + c.$$

From the boundary conditions,

$$T(x, x - 4) = ax + b(x - 4) + c = -20. \tag{1}$$

$$T(x, x + 4) = ax + b(x + 4) + c = 40. \tag{2}$$

By addition,

$$2ax + 2bx + 2c = 20.$$

Since this is an identity in x, we must have $a = -b$ and $c = 10$. From this and (1),

$$-bx + bx - 4b + 10 = -20.$$

Hence $b = -7.5$. This agrees with our result obtained by inspection.

2. $\dfrac{150}{\pi} \arctan \dfrac{y}{x} = \operatorname{Re}\left(-\dfrac{150i}{\pi} \operatorname{Ln} z\right)$

4. $\operatorname{Arg} z = \arctan \dfrac{y}{x} = \operatorname{Re}\,(-i \operatorname{Ln} z)$ is a ***basic building block*** when the ***boundary values have jumps***. The reason is that it is a harmonic function, so that we may use it for the present purpose, and that it is discontinuous at 0, that is, it is 0 for positive x and π for negative x. Accordingly, we obtain a jump j at $x = 0$ by using j/π. And we obtain such a jump at $x = a$, as in the figure, by using $\operatorname{Ln}(z - a)$ instead of $\operatorname{Ln} z$. To obtain an arbitrary jump with arbitrary values on both sides, we just have to include a few constants: in the figure we add T_1 and obtain the right size of the jump by taking $(T_2 - T_1)/\pi$. Together we thus have in the figure

$$T(x, y) = T_1 + \frac{T_2 - T_1}{\pi} \operatorname{Ln}(z - a),$$

where the minus sign results from $i \cdot i$.

With this in mind, it is not difficult to produce arbitrary discontinuous boundary potentials by going along the real axis from right to left.

6. This generalizes Online Prob. 5. We obtain, using the idea explained in the solution to Online Prob. 4,

$$\begin{aligned} T(x, y) &= T_1 + \frac{T_2 - T_1}{\pi} \operatorname{Arg}(z - b) + \frac{T_3 - T_2}{\pi} \operatorname{Arg}(z - a) \\ &= \operatorname{Re}\left(T_1 - \frac{i}{\pi}(T_2 - T_1) \operatorname{Ln}(z - b) - \frac{i}{\pi}(T_3 - T_2) \operatorname{Ln}(z - a)\right). \end{aligned}$$

8. The temperature distribution is

$$T(x, y) = \frac{T_0}{\pi} \operatorname{Arg} \frac{\cosh z - 1}{\cosh z + 1}.$$

The boundary in the figure is mapped onto the u-axis; $\cosh x$, $0 \leqq x < \infty$, gives $u \geqq 1$; $\cosh(x + \pi i) = -\cosh x$, $0 \leqq x < \infty$, gives $u < -1$; the vertical segment $(x = 0)$ is mapped onto $-1 < u < 1$. Now use the boundary values in Online Prob. 7.

11. The lines of heat flow are perpendicular to the isotherms, and heat flows from higher to lower temperatures. Accordingly, heat flows from the portion of higher temperature of the unit circle $|Z| = 1$ to that kept at a lower temperature, along the circular arcs that intersect the isotherms at right angles.

Of course, as temperatures on the boundary, we must choose values that are physically possible, for example, 10°C and 300°C.

SECTION 18.4. Fluid Flow, page 768

Purpose. To give an introduction to complex analysis in potential problems of fluid flow. These two-dimensional flows are given by their velocity vector field, and our presentation in the text begins with an explanation of handling this field by complex methods.

It is interesting that we use complex potentials as before, but whereas in electrostatics the real part (the real potential) is of central interest, here it is the imaginary part of the complex potential which gives the *streamlines* of the flow.

Important Concepts

Stream function Ψ, streamlines $\Psi = \text{const}$

Velocity potential Φ, equipotential lines $\Phi = \text{const}$

Complex potential $F = \Phi + i\Psi$

Velocity $V = \overline{F'(z)}$

Circulation (6), vorticity, rotation (9)

Irrotational, incompressible

Flow around a cylinder (Example 2, Team Project 16)

SOLUTIONS TO PROBLEM SET 18.4, page 775

1. Flow against a horizontal wall (the x-axis)

2. $F = (1 - i)Kz/\sqrt{2}$ because $V = \overline{F'} = (1 + i)K/\sqrt{2}$ and $|V| = K$.

4. $F(z) = iz^2 = i(x^2 - y^2) - 2xy$ gives the streamlines

$$x^2 - y^2 = \text{const.}$$

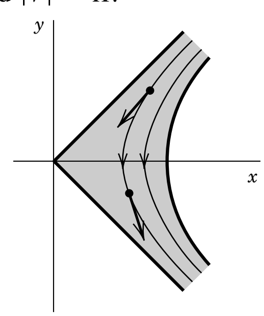
The equipotential lines are

$$xy = \text{const.}$$

The velocity vector is

$$V = \overline{F'} = -2i\bar{z} = -2y - 2ix.$$

See the figure.



Section 18.4. Problem 4

6. $F(z) = iz^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3) = -3x^2y + y^3 + i(x^3 - 3xy^2)$ gives the streamlines

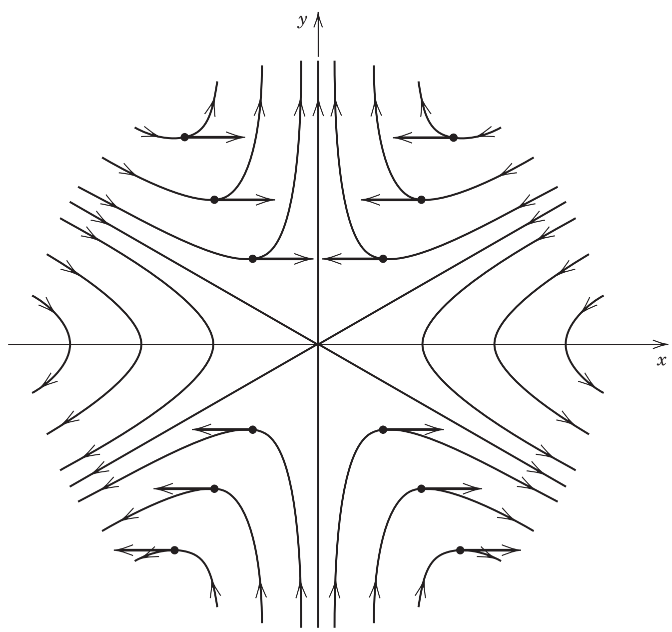
$$x(x^2 - 3y^2) = \text{const.}$$

This includes the three straight-line asymptotes $x = 0$ and $y = \pm x/\sqrt{3}$ (which make 60° angles with one another, dividing the plane into six angular regions of angle 60° each), and we could interpret the flow as a flow in such a region. This is similar to the case $F(z) = z^2$, where we had four angular regions of 90° opening each (the four quadrants of the plane) and the streamlines were hyperbolas. In the present case the streamlines look similar but they are "squeezed" a little so that each stays within its region, whose two boundary lines it has for asymptotes.

The velocity vector is

$$V = -6xy + 3i(y^2 - x^2)$$

so that $V_2 = 0$ on $y = x$ and $y = -x$. See the figure.



Section 18.4. Problem 6

9. $w = \operatorname{arccosh} z$ implies

$$z = x + iy = \cosh w = \cos iw = \sin\left(iw + \tfrac{1}{2}\pi\right).$$

Along with an interchange of the roles of the z- and w-planes, this reduces the present problem to the consideration of the sine function in Sec. 17.4 (compare with Fig. 391). We now have the hyperbolas

$$\frac{x^2}{\sin^2 c} - \frac{y^2}{\cos^2 c} = 1$$

as streamlines, where c is different from the zeros of sine and cosine, and as limiting cases the y-axis and the two portions of the aperture.

SECTION 18.5. Poisson's Integral Formula for Potentials, page 774

Purpose. To represent the potential in a standard region (a disk $|z| \leqq R$) as an integral (5) over the boundary values; to derive from (5) a series (7) that gives the potential and for $|z| = R$ is the Fourier series of the boundary values. So here we see another important application of Fourier series, much less obvious than that of vibrational problems, where one can "see" the cosine and sine terms of the series.

Comment on Footnote 2

Poisson's discovery (1812) that Laplace's equation holds only outside the masses (or charges) resulted in the Poisson equation (Sec. 12.1). The publication on the Poisson distribution (Sec. 24.7) appeared in 1837.

SOLUTIONS TO PROBLEM SET 18.5, page 778

2. $\Phi(r, \theta) = 5 - r^2 \cos 2\theta$

4. $\Phi(r, \theta) = 3r \sin \theta - r^3 \sin 3\theta$, as follows from

$$\sin^3 \theta = \tfrac{3}{4} \sin \theta - \tfrac{1}{4} \sin 3\theta.$$

7. This is a sine series plus the constant term $k/2$ because $\Phi(1, \theta) - k/2$ is odd. *Answer:*

$$\Phi(r, \theta) = \frac{k}{2} + \frac{2k}{\pi}\left(r \sin \theta + \frac{1}{3} r^3 \sin 3\theta + \frac{1}{5} r^5 \sin 5\theta + \cdots\right).$$

9. The given boundary potential is an even function. *Answer:*

$$\Phi(r, \theta) = \frac{1}{2} - \frac{4}{\pi^2}\left(r \cos \theta + \frac{1}{9} r^3 \cos 3\theta + \frac{1}{25} r^5 \cos 5\theta + \cdots\right).$$

SECTION 18.6. General Properties of Harmonic Functions. Uniqueness Theorem for the Dirchlet Problem, page 778

Purpose. We derive general properties of analytic functions and from them corresponding properties of harmonic functions.

Main Content, Important Properties

Mean value of analytic functions over circles (Theorem 1)

Mean value of harmonic functions over circles, over disks (Theorem 2)

Maximum modulus theorem for analytic functions (Theorem 3)

Maximum principle for harmonic functions (Theorem 4)

Uniqueness theorem for the Dirichlet problem (Theorem 5)

Comment on Notation

Recall that we introduced F to reserve f for conformal mappings (beginning in Sec. 18.2), and we continue to use F also in this last section of Chap. 18.

SOLUTIONS TO PROBLEM SET 18.6, page 781

1. Use (2), obtaining

$$
\begin{aligned}
F(2) = 32 &= \frac{1}{2\pi}\int_0^{2\pi} 2(-2 + e^{i\alpha})^4\, d\alpha \\
&= \frac{2}{2\pi}\int_0^{2\pi} ((-2)^4 + \cdots)\, d\alpha \\
&= \frac{2}{2\pi}((-2)^4 \cdot 2\pi + 0 + \cdots) \\
&= 32.
\end{aligned}
$$

4. $|F(z)| = \exp(x^2 - y^2), \qquad z = 1, \quad \text{Max} = e$

SOLUTIONS TO CHAPTER 18 REVIEW QUESTIONS AND PROBLEMS, page 782

2. In this case, the general form of the potential is (derivation in Sec. 18.1)

$$\Phi = a \ln r + b.$$

From the boundary conditions,

$$
\begin{aligned}
U_1 &= 0 + b = 200, \qquad b = 200 \\
U_2 &= a \ln 10 + 200 = 2000.
\end{aligned}
$$

Hence $a = 1800/\ln 10 = 781.7$. The *answer* is

$$\Phi = a \ln r + b = \operatorname{Re}(a \operatorname{Ln} z + b) = \frac{1800}{\ln 10} \ln r + 200.$$

4. Since the logarithmic curve is convex, the value in Prob. 2 is larger than that in the present problem, which involves a linear function.