CS5314 Randomized Algorithms

Assignment 1 (Suggested Solution)

1 Questions

1. **Ans:**

   (a) $\Pr(A) = (26 \times 25)/(52 \times 51)$.

   (b) $\Pr(B) = 26/52$. *(Reasoning: Each of the 52 cards has the same chance as being the third card, out of which 26 are black.)*

   (c) $\Pr(B \mid A) = 26/50$.

   (d) $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = \Pr(B \mid A)\Pr(A)/\Pr(B) = (26 \times 25)/(51 \times 50)$.

2. **Ans:**

   (a) 16/36. *(Reasoning: 16 cases with the desired outcome, each occurs with probability 1/36.)*

   (b) 6/(6+12+18). *(Reasoning: Denote #(a,b,c) as the number of cases with the outcomes of dice A, B, and C be a, b, and c, respectively. By counting, we have #(2,2,3) = 6, #(3,2,2) = 12, and #(2,3,2) = 18.)*

3. **Ans:** Let $\pi = 1 - (1-p)(1-q) = p + q - pq$, which is the probability that either X or Y (or both) succeeds in one trial.

   (a) $\Pr(X = Y) = \sum_{i=0}^{\infty}((1-p)(1-q))^i pq = pq/\pi$.

   (b) Firstly, observe that $\min(X, Y)$ is a geometric random variable with parameter π, since it is the number of trials needed to have either X or Y succeeds. Also, $\max(X, Y) = X - Y - \min(X, Y)$. Thus, by linearity of expectation, we have:

   $$\mathrm{E}[\max(X, Y)] = \mathrm{E}[X] + \mathrm{E}[Y] - \mathrm{E}[\min(X, Y)] = 1/p + 1/q - 1/\pi.$$

   (c) $\Pr(\min(X, Y) = k) = (1-\pi)^{k-1}\pi$.

   (d) Firstly, we have:

   $$\Pr(X = k \mid X \le Y) = \frac{\Pr(X = k \cap X \le Y)}{\Pr(X \le Y)} = \frac{(1-\pi)^{k-1}p}{p/\pi} = (1-\pi)^{k-1}\pi.$$

   In other words, under the condition $X \le Y$, X becomes a geometric random variable with parameter π, so that the desired answer is $1/\pi$.

   Alternatively, we can see that under the condition $X \le Y$, X is equal to $\min(X, Y)$, so that we can also derive the desired expected value as $1/\pi$ based on the result of part (c). *How about the value $E[Y \mid Y \le X]$?*

4. **Ans:**

   (a) Let X_i be an indicator such that $X_i = 1$ if and only if the ith card is not chosen. Thus, $\mathrm{E}[X_i] = (1 - 1/n)^{2n}$. Consequently, the expected number of cards not chosen is $n\mathrm{E}[X_i] = n(1 - 1/n)^{2n}$.

(b) Let Y_i be an indicator such that $Y_i = 1$ if and only if the ith card is chosen exactly once. Thus, $\mathrm{E}[Y_i] = \binom{2n}{1}(1/n)(1-1/n)^{2n-1}$. Consequently, the expected number of cards chosen exactly once is $n\mathrm{E}[Y_i] = 2n(1-1/n)^{2n-1}$.

5. **Ans:**

(a) Firstly, we see that $\mathrm{E}[Y_0] = 1$ and $\mathrm{E}[Y_1] = 2p$. Next, for $i \geq 1$, we have

$$\mathrm{E}[Y_i \mid Y_{i-1} = j] = 2pj,$$

so that

$$\mathrm{E}[Y_i] = \mathrm{E}[\mathrm{E}[Y_i \mid Y_{i-1}]] = \sum_j \Pr(Y_{i-1} = j) 2pj = 2p\,\mathrm{E}[Y_{i-1}].$$

Thus, we have $\mathrm{E}[Y_i] = (2p)^i$.

(b) The total number of copies is $\sum_i (2p)^i$, which is bounded if and only if $p < 1/2$.

6. **Ans:**

(a) For $i \leq m$, $\Pr(E_i) = 0$ since we can never choose the best candidate. For $i > m$, E_i occurs if and only if ith is the best candidate, while the best of the first $i-1$ candidates is among the first m persons. Thus,

$$\Pr(E_i) = \frac{1}{n} \times \frac{m}{i-1}.$$

Based on this, we have :

$$\Pr(E) = \sum_{i>m} \Pr(E_i) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

(b) The desired answer follows immediately from the fact that:

$$\sum_{j=m+1}^{n} \frac{1}{j-1} \geq \int_m^n \frac{1}{x} dx = \ln n - \ln m.$$

(c) Let $f(m) = (m/n)(\ln n - \ln m)$. Then we have $f'(m) = (1/n)(\ln n - \ln m) + (m/n)(-1/m)$. By setting $f'(m) = 0$, we have $m = n/e$, which can maximize $f(m)$ since $f''(m) < 0$. When $m = n/e$, $\Pr(E) \geq (m/n)(\ln n - \ln m) = 1/e$.