HW04

Partial Differential Equations and Complex Variables (EE 2021 Summer)

<u>Due date: 10/14 (Thur.)</u> Bring it to the class Delta 209.

I. Problems in Problem Set 2.8 (page 92) :**22** and **24** and **25 (plot with matlab)**

II. Problems in the Problem Set 11.3 (page 494): **15** and **16** (**plot with matlab**)

[Hint: Use matlab to program and graph the response y(t), together with its input r(t). Feel free to set your initial position y(0), velocity y'(0), as well as the damping factor c if the problem does not provide them].

6. $(D^2 + 4D + 3I)y = \cos t + \frac{1}{3}\cos 3t$

7. $(4D^2 + 12D + 9I)y = 225 - 75\sin 3t$

8–15 TRANSIENT SOLUTIONS

Find the transient motion of the mass–spring system modeled by the ODE. Show the details of your work.

8. $2y'' + 4y' + 6.5y = 4\sin 1.5t$

9. $y'' + 3y' + 3.25y = 3\cos t - 1.5\sin t$

10. $y'' + 16y = 56\cos 4t$

11. $(D^2 + 2I)y = \cos\sqrt{2}t + \sin\sqrt{2}t$

12. $(D^2 + 2D + 5I)y = 4\cos t + 8\sin t$

13. $(D^2 + I)y = \cos\omega t, \omega^2 \neq 1$

14. $(D^2 + I)y = 5e^{-t}\cos t$

15. $(D^2 + 4D + 8I)y = 2\cos 2t + \sin 2t$

16–20 INITIAL VALUE PROBLEMS

Find the motion of the mass–spring system modeled by the ODE and the initial conditions. Sketch or graph the solution curve. In addition, sketch or graph the curve of $y - y_p$ to see when the system practically reaches the steady state.

16. $y'' + 25y = 24\sin t, \quad y(0) = 1, \quad y'(0) = 1$

17. $(D^2 + 4I)y = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t,$
$y(0) = 0, \quad y'(0) = \frac{3}{35}$

18. $(D^2 + 8D + 17I)y = 474.5\sin 0.5t, \quad y(0) = -5.4,$
$y'(0) = 9.4$

19. $(D^2 + 2D + 2I)y = e^{-t/2}\sin\frac{1}{2}t, \quad y(0) = 0,$
$y'(0) = 1$

20. $(D^2 + 5I)y = \cos\pi t - \sin\pi t, \quad y(0) = 0, \quad y'(0) = 0$

21. Beats. Derive the formula after (12) from (12). Can we have beats in a damped system?

22. Beats. Solve $y'' + 25y = 99\cos 4.9t$, $y(0) = 2$, $y'(0) = 0$. How does the graph of the solution change if you change **(a)** $y(0)$, **(b)** the frequency of the driving force?

23. TEAM EXPERIMENT. Practical Resonance.
(a) Derive, in detail, the crucial formula (16).

(b) By considering dC^*/dc show that $C^*(\omega_{max})$ increases as c ($\leqq \sqrt{2mk}$) decreases.

(c) Illustrate practical resonance with an ODE of your own in which you vary c, and sketch or graph corresponding curves as in Fig. 57.

(d) Take your ODE with c fixed and an input of two terms, one with frequency close to the practical resonance frequency and the other not. Discuss and sketch or graph the output.

(e) Give other applications (not in the book) in which resonance is important.

24. Gun barrel. Solve $y'' + y = 1 - t^2/\pi^2$ if $0 \leqq t \leqq \pi$ and 0 if $t \to \infty$; here, $y(0) = 0$, $y'(0) = 0$. This models an undamped system on which a force F acts during some interval of time (see Fig. 59), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). *Hint:* At π both y and y' must be continuous.

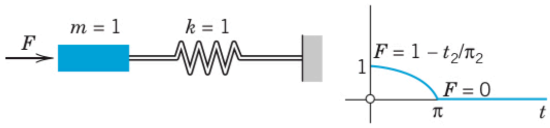

Fig. 59. Problem 24

25. CAS EXPERIMENT. Undamped Vibrations. **(a)** Solve the initial value problem $y'' + y = \cos\omega t$, $\omega^2 \neq 1$, $y(0) = 0$, $y'(0) = 0$. Show that the solution can be written

$$y(t) = \frac{2}{1 - \omega^2}\sin\left[\tfrac{1}{2}(1 + \omega)t\right]\sin\left[\tfrac{1}{2}(1 - \omega)t\right].$$

(b) Experiment with the solution by changing ω to see the change of the curves from those for small ω (>0) to beats, to resonance, and to large values of ω (see Fig. 60).

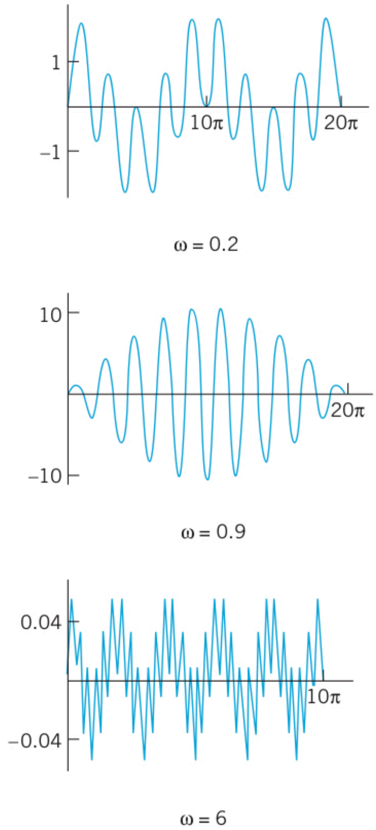

Fig. 60. Typical solution curves in CAS Experiment 25

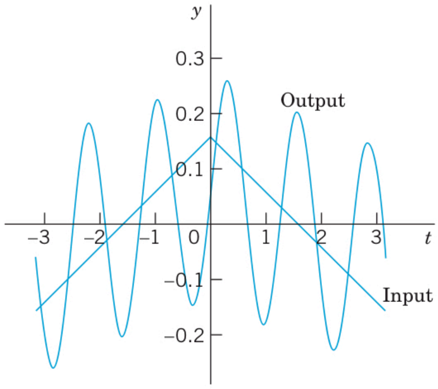

Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

1. **Coefficients C_n.** Derive the formula for C_n from A_n and B_n.
2. **Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?
3. **Phase shift.** Explain the role of the B_n's. What happens if we let $c \to 0$?
4. **Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
5. **Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

6. $r(t) = \sin \alpha t + \sin \beta t$, $\omega^2 \neq \alpha^2, \beta^2$
7. $r(t) = \sin t$, $\omega = 0.5, 0.9, 1.1, 1.5, 10$
8. **Rectifier.** $r(t) = \pi/4\,|\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$, $|\omega| \neq 0, 2, 4, \cdots$
9. What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \cdots$?
10. **Rectifier.** $r(t) = \pi/4\,|\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t)$, $|\omega| \neq 0, 2, 4, \cdots$
11. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi, \end{cases} \qquad |\omega| \neq 1, 3, 5, \cdots$

12. **CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$. $(c = 0.1)$

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$
14. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$

* 15. $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$
16. $r(t) = \begin{cases} t & \text{if} \quad -\pi/2 < t < \pi/2 \\ \pi - t & \text{if} \quad \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 275, where $R = 10\ \Omega$, $L = 1$ H, $C = 10^{-1}$ F and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

17. $E(t) = \begin{cases} -50t^2 & \text{if} \quad -\pi < t < 0 \\ 50t^2 & \text{if} \quad 0 < t < \pi \end{cases}$