Part B   LINEAR ALGEBRA.
VECTOR CALCULUS

Part B consists of

Chap. 7   Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

Chap. 8   Linear Algebra: Matrix Eigenvalue Problems

Chap. 9   Vector Differential Calculus. Grad, Div, Curl

Chap. 10 Vector Integral Calculus. Integral Theorems

Hence we have retained the previous subdivision of Part B into four chapters.

Chapter 9 is self-contained and completely independent of Chaps. 7 and 8. Thus, Part B consists of two large **independent** units, namely, Linear Algebra (Chaps. 7, 8) and Vector Calculus (Chaps. 9, 10). Chapter 10 depends on Chap. 9, mainly because of the occurrence of div and curl (defined in Chap. 9) in the Gauss and Stokes theorems in Chap. 10.

CHAPTER 7   Linear Algebra: Matrices, Vectors, Determinants. Linear Systems

Changes

The order of the material in this chapter and its subdivision into sections has been retained, but various local changes have been made to increase the usefulness of this chapter for applications.

SECTION 7.1. Matrices, Vectors: Addition and Scalar Multiplication, page 257

Purpose. Explanation of the basic concepts. Explanation of the two basic matrix operations. The latter derive their importance from their use in defining vector spaces, a fact that should perhaps not be mentioned at this early stage. Its systematic discussion follows in Sec. 7.4, where it will fit nicely into the flow of thoughts and ideas.

No Prerequisites.   Although most of the students may have some working knowledge on the simplest parts of linear algebra, we make no prerequisites.

Main Content, Important Concepts

Matrix, square matrix, main diagonal

Double subscript notation

Row vector, column vector, transposition

Equality of matrices

Matrix addition

Scalar multiplication (multiplication of a matrix by a scalar)

Comments on Important Facts

One should emphasize that vectors are always included as special cases of matrices and that those two operations have properties [formulas (3), (4)] similar to those of operations for numbers, which is a great practical advantage.

Content and Significance of the Examples

Example 1 of the text gives a first impression of the main application of matrices, that is, to linear systems, whose significance and systematic discussion will be explained later, beginning in Sec. 7.3.

Example 2 gives a simple application showing the usefulness of matrix addition.

Example 3 elaborates on equality of matrices.

Examples 4 and 5 concern the two basic algebraic operations of addition and scalar multiplication.

Purpose and Structure of Problem Set

The questions in Probs. 1–7 should help the student to reflect on the basic concepts in this section.

Problems 8–16 should help the student in gaining technical skill.

Problems 17–20 show applications in connection with forces in mechanics and with electrical networks.

SOLUTIONS TO PROBLEM SET 7.1, page 261

2. 100,   810,   960,   0

4. 4,   0,   1; a_{11}, a_{22};   4, -1

6. $\mathbf{B} = (1/1.609344)\mathbf{A}$; see inside of the front cover.

8.

$$\begin{bmatrix} 13 & -2 & 15 \\ 2 & 18 & 16 \\ -17 & 7 & -8 \end{bmatrix}, \quad \begin{bmatrix} 13 & -2 & 15 \\ 2 & 18 & 16 \\ -17 & 7 & -8 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ -5 & 3 & -4 \\ -4 & 2 & -4 \end{bmatrix} \begin{bmatrix} -1.40 & 5.20 & -12.0 \\ -10.60 & -9.0 & -20.0 \\ 6.40 & -2.0 & -0.80 \end{bmatrix}$$

9. $$\begin{bmatrix} 5 & -10 & 25 \\ 20 & 20 & 40 \\ -15 & 5 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1.25 & 0.50 & 0.0 \\ -1.25 & 0.75 & -1.0 \\ -1.0 & 0.50 & -1.0 \end{bmatrix}, \quad \begin{bmatrix} 6.25 & -9.50 & 25.0 \\ 18.75 & 20.75 & 39.0 \\ -16.0 & 5.50 & -1.0 \end{bmatrix}, \text{ undefined}$$

10. $$\begin{bmatrix} 8 & -16 & 40 \\ 32 & 32 & 64 \\ -24 & 8 & 0 \end{bmatrix}, \quad \begin{bmatrix} 8 & -16 & 40 \\ 32 & 32 & 64 \\ -24 & 8 & 0 \end{bmatrix} \begin{bmatrix} 30 & 12 & 0 \\ -30 & 18 & -24 \\ -24 & 12 & -24 \end{bmatrix}, \quad \begin{bmatrix} 30 & 12 & 0 \\ -30 & 18 & -24 \\ -24 & 12 & -24 \end{bmatrix}$$

11. $$\begin{bmatrix} 12 & -4 \\ 28 & -24 \\ -8 & 10 \end{bmatrix} \begin{bmatrix} 12 & -4 \\ 28 & -24 \\ -8 & 10 \end{bmatrix}, \quad \begin{bmatrix} 12 & -4 \\ 28 & -24 \\ -8 & 10 \end{bmatrix} \begin{bmatrix} 3.60 & -1.20 \\ 0.0 & -1.60 \\ 0.40 & -1.20 \end{bmatrix} \begin{bmatrix} 3.60 & -1.20 \\ 0.0 & -1.60 \\ 0.40 & -1.20 \end{bmatrix}$$

12. $\begin{bmatrix} 5 & -1 \\ 0 & -1 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 5 & -1 \\ 0 & -1 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} -12 & 4 \\ 8 & 0 \\ -4 & 8 \end{bmatrix},$ undefined

13. $\begin{bmatrix} 90 & -30 \\ 30 & -60 \\ 0 & -15 \end{bmatrix}, \begin{bmatrix} 90 & -30 \\ 30 & -60 \\ 0 & -15 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -2 & 0 \\ 1 & -2 \end{bmatrix},$ undefined

14. $\begin{bmatrix} 18.0 \\ 0.0 \\ -1.50 \end{bmatrix}, \begin{bmatrix} -24.0 \\ -15.0 \\ 13.50 \end{bmatrix},$ undefined, $\begin{bmatrix} 21.60 \\ -18.0 \\ 12.0 \end{bmatrix}$

15. $\begin{bmatrix} 7.20 \\ 9.0 \\ -7.50 \end{bmatrix}, \begin{bmatrix} 7.20 \\ 9.0 \\ -7.50 \end{bmatrix},$ undefined, undefined

16. $\begin{bmatrix} 42.0 \\ 15.0 \\ 21.0 \end{bmatrix}, \begin{bmatrix} 42 \\ 15 \\ 21 \end{bmatrix},$ undefined, $\begin{bmatrix} -19.04 \\ -45.20 \\ 39.60 \end{bmatrix}$

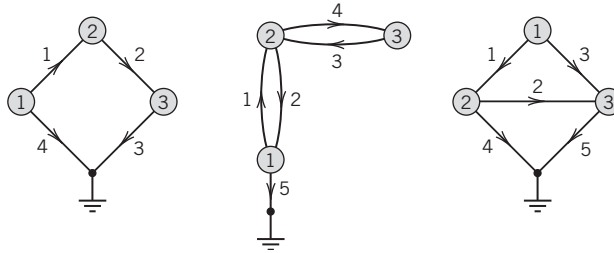**17.** $\begin{bmatrix} -0.80 \\ -11.0 \\ 8.50 \end{bmatrix}$

18. From Prob. 17 we have

$$\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{p} = \mathbf{0}, \quad \mathbf{p} = -(\mathbf{u} + \mathbf{v} + \mathbf{w}) = \begin{bmatrix} 4.5 \\ 27.0 \\ -9.0 \end{bmatrix}.$$

20. TEAM PROJECT. (b) The nodal incidence matrices are

$$\begin{bmatrix} -1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

(c) The networks with these incidence matrices are

SECTION 7.2. Matrix Multiplication, page 263

Purpose. Matrix multiplication, the third and last algebraic operation, is defined and discussed, with emphasis on its "unusual" properties; this also includes its representation by inner products of row and column vectors.

   The motivation of this multiplication is given by formulas (6)–(8) in connection with linear transformations.

Main Content, Important Facts

   Definition of matrix multiplication ("rows times columns")

   Properties of matrix multiplication

   Matrix products in terms of inner products of vectors

   Linear transformations motivating the definition of matrix multiplication

   $\mathbf{AB} \neq \mathbf{BA}$ in general, so the order of factors is important

   $\mathbf{AB} = \mathbf{0}$ does not imply $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$

   $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$

Short Courses.   Products in terms of row and column vectors and the discussion of linear transformations could be omitted.

Comment on Notation

For transposition, T seems preferable over a prime, which is often used in the literature, but will be needed to indicate differentiation in Chap. 9.

Comments on Content and Significance of the Examples in the Text

Matrix multiplication is shown for the three possible cases, namely, for products of two matrices (in Example 1), for a matrix times a vector (the most important case in the next sections, in Example 2), and for products of row and column vectors (in Example 3).

   Most important, matrix multiplication is not commutative (Example 4).

   Example 5 shows how a matrix product can be expressed in terms of row and column vectors.

   Example 6 illustrates the computation of matrix products on parallel processors.

   The operation of transposition (Example 7) transforms, as a special case, row vectors into column vectors and conversely.

   It is also used in the definition of the very important symmetric and skew-symmetric matrices (Example 8).

Further special square matrices are triangular matrices (Example 9), diagonal and scalar matrices, including unit matrices of various sizes n (Example 10).

Examples 11–13 show some typical applications.

Formula (10d) for the transposition of a product should be memorized.

In motivating matrix multiplication by linear transformations, one may also illustrate the geometric significance of noncommutativity by combining a rotation with a stretch in the x-direction in both orders and show that a circle transforms into an ellipse with main axes in the direction of the coordinate axes or rotated, respectively.

SOLUTIONS TO PROBLEM SET 7.2, page 270

2. The 3×3 zero matrix, as follows directly from the definitions.

4. $n(n - 1) + l$, where l is the number of different main diagonal entries (which are all 0), hence 13 when $n = 4$.

6. Triangular are $\mathbf{U}_1 + \mathbf{U}_2$, $\mathbf{U}_1\mathbf{U}_2$, hence $\mathbf{U}_1^2$, which, by transposition, implies the same for $\mathbf{L}_1 + \mathbf{L}_2$, $\mathbf{L}_1\mathbf{L}_2$, and $\mathbf{L}_1^2$.

8. $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$, $\begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$, etc. Problems 7 and 8 should serve as eye openers, to see what can happen under multiplication.

10. The entry c_{kj} of $(\mathbf{AB})^{\mathsf{T}}$ is c_{jk} of $\mathbf{AB}$, which is Row j of $\mathbf{A}$ times Column k of $\mathbf{B}$. On the right, c_{kj} is Row k of $\mathbf{B}^{\mathsf{T}}$, hence Column k of $\mathbf{B}$, times Column j of $\mathbf{A}^{\mathsf{T}}$, hence Row j of $\mathbf{A}$.

11. $\begin{bmatrix} 1 & 5 & 6 \\ -1 & -5 & 8 \\ -7 & 5 & -4 \end{bmatrix}$, $\begin{bmatrix} -5 & -7 & 6 \\ 5 & 7 & 8 \\ 5 & -1 & -4 \end{bmatrix}$, $\begin{bmatrix} -8 & 4 & 9 \\ -8 & 4 & -5 \\ 2 & 4 & -4 \end{bmatrix}$, $\begin{bmatrix} 4 & -2 & -15 \\ 4 & -2 & 13 \\ 2 & 4 & -4 \end{bmatrix}$

12. $\begin{bmatrix} 14 & 7 & -6 \\ 7 & 21 & -8 \\ -6 & -8 & 9 \end{bmatrix}$, $\begin{bmatrix} 9 & 3 & -4 \\ -2 & 11 & -10 \\ -4 & -3 & 15 \end{bmatrix}$, $\begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -4 & 2 \\ -4 & 8 & -4 \\ 2 & -4 & 4 \end{bmatrix}$, $\begin{bmatrix} -7 & 7 \\ -5 & 5 \\ 4 & 0 \end{bmatrix}$, undefined, $\begin{bmatrix} 5 & 1 & 4 \\ -5 & -1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} 8 & -9 & 9 \\ 0 & 1 & 12 \\ 3 & 6 & -10 \end{bmatrix}$, $\begin{bmatrix} 8 & 0 & 3 \\ -9 & 1 & 6 \\ 9 & 12 & -10 \end{bmatrix}$, $\begin{bmatrix} 8 & 0 & 3 \\ -9 & 1 & 6 \\ 9 & 12 & -10 \end{bmatrix}$, $\begin{bmatrix} -8 \\ 7 \\ -33 \end{bmatrix}$

15. undefined, $\begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$, $[10 \quad -3 \quad -1]$, $[10 \quad -3 \quad -1]$

16. $\begin{bmatrix} -7 & 7 \\ -5 & 5 \\ 4 & 0 \end{bmatrix}$, undefined, $\begin{bmatrix} -6 \\ -10 \\ 2 \end{bmatrix}$ $[0 \quad 8 \quad 2]$

17. $\begin{bmatrix} 3 & 9 \\ 25 & -9 \\ -25 & 17 \end{bmatrix}$, undefined, $\begin{bmatrix} 4 \\ 10 \\ -30 \end{bmatrix}$, undefined

18. $[-1]$, $\begin{bmatrix} -3 & -6 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 0 \end{bmatrix}$, $[2 \quad -1 \quad -11]$, $\begin{bmatrix} -6 \\ -10 \\ 2 \end{bmatrix}$

19. undefined, $\begin{bmatrix} 7.50 \\ -6.0 \\ 3.0 \end{bmatrix}$, $\begin{bmatrix} 16 \\ 7 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 16 \\ 7 \\ -3 \end{bmatrix}$

20. $[32]$, $[3]$, $[6 \quad -12 \quad 6]$, $\begin{bmatrix} -7 & -14 & 0 \\ 5 & 10 & 0 \end{bmatrix}$

24. $\mathbf{M} = \mathbf{AB} - \mathbf{BA}$ must be the 2×2 zero matrix. It has the form

$$\mathbf{M} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 2a_{11} + 3a_{12} - 2a_{11} - 3a_{21} & 3a_{11} + 4a_{12} - 2a_{12} - 3a_{22} \\ 2a_{21} + 3a_{22} - 3a_{11} - 4a_{21} & 3a_{21} + 4a_{22} - 3a_{12} - 4a_{22} \end{bmatrix}.$$

$a_{21} = a_{12}$ from $m_{11} = 0$ (also from $m_{22} = 0$). $a_{22} = a_{11} + \frac{2}{3}a_{12}$ from $m_{12} = 0$ (also from $m_{21} = 0$). *Answer:*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} + \frac{2}{3}a_{12} \end{bmatrix}.$$

26. The transition probabilities can be given in a matrix

$$\text{From } N \quad \text{From } T$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \quad \begin{matrix} \text{To } N \\ \text{To } T \end{matrix}$$

and multiplication of $[1 \quad 0]^\mathsf{T}$ by $\mathbf{A}$, $\mathbf{A}^2$, $\mathbf{A}^3$ gives $[0.8 \quad 0.2]^\mathsf{T}$, $[0.74 \quad 0.26]^\mathsf{T}$, and $[0.722 \quad 0.278]^\mathsf{T}$.

28. The matrix of the transition probabilities is

$$\mathbf{A} = \begin{bmatrix} 0.9 & 0.002 \\ 0.1 & 0.998 \end{bmatrix}.$$

The initial state is $[1200 \quad 98{,}800]^\mathsf{T}$. Hence multiplication by $\mathbf{A}$ gives the further states (rounded) $[1278 \quad 98{,}722]^\mathsf{T}$, $[1347 \quad 98{,}653]^\mathsf{T}$, $[1410 \quad 98{,}590]^\mathsf{T}$, indicating that a substantial increase is likely.

30. Team Project. (b) Use induction on n. True if $n = 1$. Take the formula in the problem as the induction hypothesis, multiply by $\mathbf{A}$, and simplify the entries in the product by the addition formulas for the cosine and sine to get $\mathbf{A}^{n+1}$.

(c) These formulas follow directly from the definition of matrix multiplication.

(d) A scalar matrix would correspond to a stretch or contraction by the same factor in all directions.

(e) Rotations about the x_1-, x_2-, x_3-axes through θ, φ, ψ, respectively.

SECTION 7.3. Linear Systems of Equations. Gauss Elimination, page 272

Purpose. This section centers around the Gauss elimination for solving linear systems of m equations in n unknowns $x_1, \cdots, x_n$, its practical use as well as its mathematical justification (leaving the—more demanding—general existence theory to the next sections).

Main Content, Important Concepts

Nonhomogeneous, homogeneous, coefficient matrix, augmented matrix

Gauss elimination in the case of the existence of

    I. a unique solution (Example 2)

   II. infinitely many solutions (Example 3)

  III. no solutions (Example 4)

Pivoting

Elementary row operations, echelon form

Background Material.   All one needs here is the multiplication of a matrix and a vector.

Comments on Content

The student should become aware of the following facts:

1. Linear systems of equations provide a major application of matrix algebra and justification of the definitions of its concepts.

2. The Gauss elimination (with pivoting) gives meaningful results in each of the Cases I−III.

3. This method is a *systematic* elimination that does not look for unsystematic "shortcuts" (depending on the size of the numbers involved and still advocated in some older precomputer-age books).

Algorithms for programs of Gauss's and related methods are discussed in Sec. 20.1, which is independent of the rest of Chap. 20 and can thus be taken up along with the present section in case of time and interest.

Comments on Examples in the Text

Example 1 and Fig. 158 show geometric interpretations of linear systems in 2 and 3 unknowns.

Example 2 on an electrical network of Ohm's resistors shows Gauss elimination with pivoting and back substitution in the case of a unique solution.

Theorem 1 is central; it proves that the three kinds of elementary row operations leave solution sets unchanged, thus justifying Gauss elimination.

Examples 2, 3, and 4 illustrate the Gauss elimination for the three possible cases that a linear system has a unique solution, or infinitely many solutions, or no solution, respectively.

The section closes with a few comments on the row-echelon form, that is, on the form into which Gauss elimination transforms linear systems.

Comments on Problems

Problems 1–14 on Gauss elimination give further illustrations of those three cases.

Problem 15 concerns equivalence (its general definition), which is of general mathematical interest, involving reflexivity, symmetry, and transitivity.

Electrical networks of Ohm's resistors (no inductances or capacitances) are discussed as linear systems in Probs.17–20 and some further models in Probs. 21–23.

Project 24 presents the idea of representing matrix operations (as discussed before) in terms of standard matrices, called *elementary matrices.* Such representations are helpful, for instance, in designing algorithms, whereas computations themselves generally proceed directly.

SOLUTIONS TO PROBLEM SET 7.3, page 280

1. $x = -1, y = \frac{1}{4}$

2. $x = 0.4, \quad y = 1.2$

3. $x = -1, y = 1, z = -2$

4. No solution

5. $x = -6, y = 7.$

6. $x = 2y + 1 = 2t + 1, \quad y = t$ arbitrary, $\quad z = 4$

7. $x = 3t, y = t$ (arbitrary), $z = -2t$

8. No solution

9. $x = 8 + 3t, y = -4 - t, z = t$ (t arbitrary)

10. No solution

11. $w = t_1, x = 1, y = -t_2 + 2t_1, z = t_2, t_1$, and t_2 arbitrary.

12. $w = x - 2y = t_1 - 2t_2, \quad x = t_1$ arbitrary, $\quad y = t_2$ arbitrary, $\quad z = 3$

13. $w = 2, x = 0, y = 4, z = 1$

14. $w = 0, \quad x = 3z = 3t, \quad y = 2z + 1 = 2t + 1, \quad z = t$ arbitrary

18. Currents at the lower node:

$$-I_1 + I_2 + I_3 = 0$$

(minus because I_1 flows out). Voltage in the left circuit:

$$4I_1 + 12I_2 = 12 + 24$$

and in the right circuit

$$12I_2 - 8I_3 = 24$$

(minus because I_3 flows against the arrow of E_2). Hence the augmented matrix of the system is

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 4 & 12 & 0 & 36 \\ 0 & 12 & -8 & 24 \end{bmatrix}.$$

The solution is

$$I_1 = \tfrac{27}{11} \text{ A}, \qquad I_2 = \tfrac{24}{11} \text{ A}, \qquad I_3 = \tfrac{3}{11} \text{ A}.$$

22. $P_1 = 5, P_2 = 5, D_1 = 13, D_2 = 21, S_1 = 13, S_2 = 21$

24. PROJECT. (a) B and **C** are different. For instance, it makes a difference whether we first multiply a row and then interchange, and then do these operations in reverse order.

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} - 5a_{11} & a_{22} - 5a_{12} \\ 8a_{41} & 8a_{42} \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} - 5a_{11} & a_{32} - 5a_{12} \\ a_{21} & a_{22} \\ 8a_{41} & 8a_{42} \end{bmatrix}$$

(b) Premultiplying **A** by **E** makes **E** operate on *rows* of **A**. The assertions then follow almost immediately from the definition of matrix multiplication.

SECTION 7.4. Linear Independence. Rank of a Matrix. Vector Space, page 282

Purpose. This section introduces some theory centered around linear independence and rank, in preparation for the discussion of the existence and uniqueness problem for linear systems of equations (Sec. 7.7).

Main Content, Important Concepts

Linear independence

Real vector space R^n, dimension, basis

Rank defined in terms of row vectors

Rank in terms of column vectors

Invariance of rank under elementary row operations

Short Courses.   For the further discussion in the next sections, it suffices to define linear independence and rank.

Comments on Rank and Vector Spaces

Of the three possible equivalent definitions of **rank**,

(i) By row vectors (our definition),

(ii) By column vectors (our Theorem 3),

(iii) By submatrices with nonzero determinant (Sec. 7.7),

the first seems to be most practical in our context.

Introducing **vector spaces** here, rather than in Sec. 7.1, we have the advantage that the student immediately sees an application (row and column spaces). Vector spaces in full generality follow in Sec. 7.9.

Comments on Text and Problem Set

Examples 1–5 concern the same three vectors, in Examples 2–5 as row vectors of a matrix.
 Theorem 1 states that rank is invariant under row reduction.
 Example 3 determines rank by Gauss reduction to echelon form.
 Since we defined rank in terms of row vectors, rank in terms of column vectors becomes a theorem (Theorem 3).
 Theorem 4 results from Theorems 2 and 3, as indicated.
 The text then continues with the definition of vector space in general, of vector space R^n, and of row space and column space of a matrix **A**, both of which have the same dimension, equal to rank **A**.
 This is immediately illustrated in Probs. 1–10 of the problem set.
 Problems 12–16 give a further discussion of rank.
 Problems 17–25 concern linear independence and dependence.
 Sets of vectors that form, or do not form, vector spaces follow in Probs. 27–35.
 The discussion of vector space will be continued and extended in the last section (Sec. 7.9) of this chapter, which we leave optional since we shall not make further use of it.

SOLUTIONS TO PROBLEM SET 7.4, page 287

1. 1, $[1, -2, 3]$, $[-2, 1]^T$

2. Rank is 2. Row reduction gives

$$
\begin{bmatrix}
2a & -b \\
0 & 1/2\dfrac{2a^2 - b^2}{a}
\end{bmatrix}
$$

Hence for general $a \neq 0$ and $2a^2 \neq b^2$ the rank is 2. A basis is $[2a - b]$, $[0, \frac{2a^2 - b^2}{2a}]$.

3. 3, $[3, 5, 0]$, $[0, -25/3, 0]$, $[0, 0, 5]$

4. 3, Row reduction gives

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & -9 & 1 \\ 0 & 0 & \frac{37}{9} \end{bmatrix}$$

A basis of the row space is $\{[2, -3, 0], [0, -9, 1], [0, 0, 37]\}$. The matrix is symmetric and has the eigenvalues 4 and $2 \pm \sqrt{41}$.

5. 3, $\{[2, -2, 1], [0, 1, 2], [0, 0, 1]\}$, $\{[1, 0, 1], [0, 2, 1], [0, 0, 1]\}$

6. The matrix is skew-symmetric. Row reduction gives

$$\begin{bmatrix} -1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as can be seen without calculation. Hence the rank of the matrix is 2. Bases are $\{[1 \quad 0 \quad 4], [0 \quad 1 \quad 0]\}$ and the same vectors transposed (as column vectors).

7. 2, $\{[6, 0, -3, 0], [0, -1, 0, 5]\}$, $\{[6, 0, 2] [0, -1, 0]\}$.

8. The matrix has rank 4. Row reduction gives as a basis of the row space:

$$[1 \quad 2 \quad 4 \quad 8], \quad [0 \quad 4 \quad 10 \quad 21], \quad [0 \quad 0 \quad 2 \quad 5], \quad [0 \quad 0 \quad 0 \quad 1].$$

Row reduction of the tranpose gives a basis of the column space of the given matrix.

9. 3, $\{[5, 0, 1, 0], [5, 4, 5], [0, 0, 1, 0]\}$, $\{[5, 0, 1], [0, -5, 4, 5], [0, 0, 1, 0]\}$

10. The matrix is symmetric. Row reduction gives

$$\begin{bmatrix} 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the rank is 3, and a basis is

$$[5 \quad -2 \quad 1 \quad 0], \quad [0 \quad 1 \quad 2 \quad 0], \quad [0 \quad 0 \quad 2 \quad -1].$$

12. $\mathbf{AB}$ and its transpose $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$ have the same rank.

14. This follows directly from Theorem 3.

16. A proof is given in Ref. [B3], vol 1, p. 12. (See App. 1 of the text.)

17. No.

18. Yes. We mention that these are the row vectors of the 4×4 Hilbert matrix; see the Index of the textbook.

19. Yes.

20. No. It is remarkable that $\mathbf{A} = [a_{jk}]$ with $a_{jk} = j + k - 1$ has rank 2 for any size n of the matrix.

22. No. Quite generally, if one of the vectors $\mathbf{v}_{(1)}, \cdots, \mathbf{v}_{(m)}$ is $\mathbf{0}$, then (1) holds with any $c_1 \neq 0$ and $c_2, \cdots, c_m$ all zero.

24. No, by Theorem 4.

26. Three steps; the first two vectors remain; these form a linearly independent subset of the given set.

27. 2, [4, 0, 1], [0, −4, 1]

28. No, if $k \neq 0$; yes, if $k = 0$, dimension 2, basis [1   0   0],   [0   1   −3].

29. No, it is a sub-space.

30. Yes, dimension 2, basis [0 $\cdots$ 0   1   0],   [0 $\cdots$ 0   1].

31. No.

32. Yes, dimension 1. The two given equations form a linear system with coefficient matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 4 & 5 & 0 \end{bmatrix}.$$

The solution is $x_1 = 5t_1, x_2 = -4t_1, x_3 = -23t_1$ with arbitrary t_1.
Hence a basis is [5   −4   −23].

34. No

SECTION 7.5. Solutions of Linear Systems: Existence, Uniqueness, page 288

Purpose. The student should see that the totality of solutions (including the existence and uniqueness) can be characterized in terms of the ranks of the coefficient matrix and the augmented matrix.

Main Content, Important Concepts

Augmented matrix

Necessary and sufficient conditions for the existence of solutions

Implications for homogeneous systems

rank $\mathbf{A}$ + nullity $\mathbf{A} = n$

Background Material.   Rank (Sec. 7.4)

Short Courses.   Brief discussion of the first two theorems, illustrated by some simple examples.

Comments on Content

This section should make the student aware of the great importance of rank. It may be good to have students memorize the condition

$$\text{rank } \mathbf{A} = \text{rank } \widetilde{\mathbf{A}}$$

for the existence of solutions.

Students familiar with ODEs may be reminded of the analog of Theorem 4 (see Sec. 2.7).

This section may also provide a good opportunity to point to the roles of existence and uniqueness problems throughout mathematics (and to the distinction between the two).

SECTION 7.7. Determinants. Cramer s Rule, page 293

For second- and third-order determinants see the reference Sec. 7.6.

Main Content of This Section

nth-order determinants

General properties of determinants

Rank in terms of determinants (Theorem 3)

Cramer's rule for solving linear systems by determinants (Theorem 4)

General Comments on Determinants

Our definition of a determinant seems more practical than that in terms of permutations (because it immediately gives those general properties), at the expense of the proof that our definition is unambiguous (see the proof in App. 4).

General properties are given for order n, from which they can be easily seen for $n = 3$ when needed.

The importance of determinants has decreased with time, but determinants will remain in eigenvalue problems (characteristic determinants), ODEs (Wronskians!), integration and transformations (Jacobians!), and other areas of practical interest.

Comments on Examples

Examples 1–3 show expansions of determinants in the simplest cases.

Example 4 illustrates the role of triangular matrices in the present context.

The theorems show properties of determinants, in particular the relation to rank (Theorem 3) and Cramer's rule for n equations in n unknowns. Note that the cases $n = 2$ and $n = 3$ were considered in Sec. 7.6.

Comments on Problems

Problems 1–15 illustrate general properties of determinants and their evaluation.

Problems 17–19 compare the (impractical) determination of rank by determinants and by row reduction.

Team Project 20 concerns some applications of linear systems to analytic geometry all using the vanishing of determinants.

SOLUTIONS TO PROBLEM SET 7.7, page 300

8. -1.10

10. 1

11. 48

12. $a^3 + b^3 + c^3 - 3abc$

13. 17

14. 216. Note that

$$\begin{vmatrix} 4 & 7 \\ 2 & 8 \end{vmatrix} \cdot \begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} = 18 \cdot 12 = 216.$$

15. 1

16. $\det \mathbf{A}_n = (-1)^{n-1}(n-1)$. True for $n = 2$, a 2-simplex on R^1, that is, a segment (an interval), because

$$\det \mathbf{A}_2 = (-1)^{2-1}(2-1) = -1.$$

Assume true for n as just given. Consider $\mathbf{A}_{n+1}$. To get the first row with all entries 0, except for the first entry, subtract from Row 1 the expression

$$\frac{1}{n-1}(\text{Row } 2 + \cdots + \text{Row } (n+1)).$$

The first component of the new row is $-n/(n-1)$, whereas the other components are all 0. Develop $\det \mathbf{A}_{n+1}$ by this new first row and notice that you can then apply the above induction hypothesis,

$$\det \mathbf{A}_{n+1} = -\frac{n}{n-1}(-1)^{n-1}(n-1) = (-1)^n n,$$

as had to be shown.

18. 3 because interchange of Rows 1 and 2 and row reduction gives

$$\begin{bmatrix} 4 & 0 & 10 \\ 0 & 4 & -6 \\ 0 & 0 & 30 \end{bmatrix}.$$

19. 2

20. Team Project. (a) Use row operation (subtraction of rows) on D to transform the last column of D into the form $[0 \quad 0 \quad 1]^\mathsf{T}$ and then develop $D = 0$ by this column.
(b) For a plane the equation is $ax + by + cz + d \cdot 1 = 0$, so that we get the determinantal equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

The plane is $3x + 4y - 2z = 5$.
(c) For a circle the equation is

$$a(x^2 + y^2) + bx + cy + d \cdot 1 = 0,$$

so that we get

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The circle is $x^2 + y^2 - 4x - 2y = 20$.

(**d**) For a sphere the equation is

$$a(x^2 + y^2 + z^2) + bx + cy + dz + e \cdot 1 = 0,$$

so that we obtain

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

The sphere through the given points is $x^2 + y^2 + (z - 1)^2 = 16$.
(**e**) For a general conic section the equation is

$$ax^2 + bxy + cy^2 + dx + ey + f \cdot 1 = 0,$$

so that we get

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0.$$

21. $x = 2.5, y = -0.15$

22. In Cramer's rule we have

$$D = \begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix} = 24$$

$$D_1 = \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -48$$

$$D_2 = \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 120.$$

Hence $x = D_1/D = -48/24 = -2$, and $y = D_2/D = 120/24 = 5$.

23. $x = -1, y = 1, z = 0$

24. $D = -60, D_1 = -60, D_2 = 180,$ and $D_3 = -240.$ Hence $x = 1, y = -3,$ and $z = 4.$

25. $w = -1, x = 1, y = 2, z = -2$

SECTION 7.8. Inverse of a Matrix. Gauss—Jordan Elimination, page 301

Purpose. To familiarize the student with the concept of the inverse $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A}$, its conditions for existence, and its computation.

Main Content, Important Concepts

$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Nonsingular and singular matrices

Existence of $\mathbf{A}^{-1}$ and rank

Gauss–Jordan elimination

$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$

Cancellation laws (Theorem 3)

$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}$

Short Courses. Theorem 1 without proof, Gauss–Jordan elimination, formulas (4*) and (7).

Comments on Content

Although in this chapter we are not concerned with operations count (Chap. 20), it would make no sense to first mislead the student by using Gauss–Jordan for solving $\mathbf{Ax} = \mathbf{b}$ and then later, in numerics, correct the false impression by explaining why Gauss elimination is better because back substitution needs fewer operations than the diagonalization of a triangular matrix. Thus Gauss–Jordan should be applied only when $\mathbf{A}^{-1}$ is wanted.

The "unusual" properties of matrix multiplication, briefly mentioned in Sec. 7.2 can now be explored systematically by the use of rank and inverse.

Formula (4*) is worth memorizing.

SOLUTIONS TO PROBLEM SET 7.8, page 308

1. $\begin{bmatrix} -3 & 1/2 \\ -10 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

Note that the given matrix corresponds to a rotation through an angle 2θ. If 2θ is replaced by -2θ (rotation in the opposite sense), this gives the inverse, which corresponds to a rotation through -2θ.

3. $\begin{bmatrix} 12.5 & 2.5 & -1.80 \\ -5.0 & 0.0 & 0.47 \\ 0.0 & 0.0 & 0.12 \end{bmatrix}$

4. $\begin{bmatrix} 0.0 & 0.0 & 1.25 \\ 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.20 & 0.0 & 0.0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$

6. Note that, due to the special form of the given matrix, the 2×2 minor in the right lower corner of the inverse has the form of the inverse of a 2×2 matrix; the inverse is

$$\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix}.$$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

8. The matrix is singular. It is interesting that this is not the case for the 2×2 matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

9. $\begin{bmatrix} 0 & 0 & 8 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$

10. The inverse equals the transpose. This is the defining property of orthogonal matrices to be discussed in Sec. 8.3.

12. $\mathbf{I} = (\mathbf{A}^2)^{-1}\mathbf{A}^2$. Multiply this by $\mathbf{A}^{-1}$ from the right on both sides of the equation. This gives $\mathbf{A}^{-1} = (\mathbf{A}^2)^{-1}\mathbf{A}$. Do the same operation once more to get the formula to be proved.

14. $\mathbf{I} = \mathbf{I}^\mathsf{T} = (\mathbf{A}^{-1}\mathbf{A})^\mathsf{T} = \mathbf{A}^\mathsf{T}(\mathbf{A}^{-1})^\mathsf{T}$. Now multiply the first and last expression by $(\mathbf{A}^\mathsf{T})^{-1}$ from the left, obtaining $(\mathbf{A}^\mathsf{T})^{-1} = (\mathbf{A}^{-1})^\mathsf{T}$.

16. Rotation through 2θ. The inverse represents the rotation through -2θ. Replacement of 2θ by -2θ in the matrix gives the inverse.

18. Multiplication by $\mathbf{A}$ from the right interchanges Rows 1 and 2 of $\mathbf{A}$. The inverse of this is the interchange that gives back the original matrix. Hence the inverse of the given matrix should equal the matrix itself, as is the case.

20. Straightforward calculation, particularly simple because of the zeros. And instructive because we now see distinctly why the inverse has zeros at the same positions as the given matrix does.

SECTION 7.9. Vector Spaces, Inner Product Spaces, Linear Transformations, *Optional*, page 309

Purpose. In this optional section we extend our earlier discussion of vector spaces R^n and C^n, define inner product spaces, and explain the role of matrices in linear transformations of R^n into R^m.

Main Content, Important Concepts

Real vector space, complex vector space

Linear independence, dimension, basis

Inner product space

Linear transformation of R^n into R^m

Background Material. Vector spaces R^n (Sec. 7.4)

Comments on Content

The student is supposed to see and comprehend how concrete models (R^n and C^n, the inner product for vectors) lead to abstract concepts, defined by axioms resulting from basic properties of those models. Because of the level and general objective of this chapter, we have to restrict our discussion to the illustration and explanation of the abstract concepts in terms of some simple typical examples.

Most essential from the viewpoint of matrices is our discussion of *linear transformations,* which, in a more theoretically oriented course of a higher level, would occupy a more prominent position.

Comment on Footnote 4

Hilbert's work was fundamental to various areas in mathematics; roughly speaking, he worked on number theory 1893–1898, foundations of geometry 1898–1902, integral equations 1902–1912, physics 1910–1922, and logic and foundations of mathematics 1922–1930. Closest to our interests here is the development in integral equations, as follows. In 1870 Carl Neumann (Sec. 5.6) had the idea of solving the Dirichlet problem for the Laplace equation by converting it to an integral equation. This created general interest in integral equations. In 1896 Vito Volterra (1860–1940) developed a general theory of these equations, followed by Erik Ivar Fredholm (1866–1927) in 1900–1903 (whose papers caused great excitement), and Hilbert since 1902. This gave the impetus to the development of inner product and Hilbert spaces and operators defined on them. These spaces and operators and their spectral theory have found basic applications in quantum mechanics since 1927. Hilbert's great interest in mathematical physics is documented by Ref. [GenRef3], a classic full of ideas that are of importance to the mathematical work of the engineer. For more details, see G. Birkhoff and E. Kreyszig. The establishment of functional analysis. *Historia Mathematica* **11** (1984), pp. 258–321.

Further Comments on Content and Comments on Problems

It is important to understand that matrices form vector spaces (Example 1, etc.) and so do polynomials up to a fixed degree *n* (Example 2).

An inner product space is obtained from a vector space *V* by defining an inner product on $V \times V$. In addition to the usual dot product notation the student should perhaps also become aware of the standard notation (**a**, **b**), used, e.g., in functional analysis and applications.

The last part of the section concerns the role of matrices in connection with linear transformations.

The fundamental importance of (3)–(5) (Cauchy–Schwarz, triangle inequalities, parallelogram equality) will not appear on our level, but should perhaps be mentioned in passing, along with the verifications required in Probs. 23–25.

The problems center around vector spaces, supplementing Problem Set 7.4, linear spaces and inverse matrices, as well as norm and inner product (to be substantially extended in numerics in Chap. 20).

SOLUTIONS TO PROBLEM SET 7.9, page 318

2. Take the difference of the given representation and another representation $\mathbf{v} = k_1\mathbf{a}_{(1)} + \cdots + k_n\mathbf{a}_{(n)}$, obtaining

$$\mathbf{v} - \mathbf{v} = \sum (c_j - k_j)\,\mathbf{a}_{(j)} = \mathbf{0}.$$

Hence $c_j - k_j = 0$ because of the linear independence of the basis vectors. This implies $k_j = c_j$ for $j = 1, \cdots, n$, the uniqueness.

3. $1, [3, 2, 1]^T$

4. Yes, dimension 3 because of the skew symmetry, at most three entries of the nine entries of such a matrix can be different (and not zero). A basis is

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

6. Yes. Dimension 2. Basis $\cos 2x$, $\sin 2x$. Note that these functions are solutions of the ODE $y'' + 4y = 0$. To mention this connection with vector spaces would not have added much to our discussion in Chap. 2. Similarly for the next problem (Prob. 7).

8. No, because $\det (\mathbf{A}_1 + \mathbf{A}_2) \neq \det \mathbf{A}_1 + \det \mathbf{A}_2$ in general.

10. Yes, dimension 4, basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

11. $x_1 = -2y_1 + y_2, x_2 = -3y_1 + 0.50y_2$

12. The inverse transformation is obtained by calculating the inverse matrix. This gives

$$x_1 = -\tfrac{1}{5}y_1 + \tfrac{2}{5}y_2$$

$$x_2 = \tfrac{4}{5}y_1 - \tfrac{3}{5}y_2.$$

14. $x_1 = \quad 4y_1 - 2y_2 + 2y_3$

$x_2 = -2y_1 - 4y_2 + 4y_3$

$x_3 = -4y_1 + 2y_2 + 8y_3$

15. $\sqrt{14}$

16. $\sqrt{\tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{4} + \tfrac{1}{9}} = \sqrt{\tfrac{1}{2} + \tfrac{2}{9}} = \sqrt{\tfrac{13}{18}}$

17. $2\sqrt{2}$

18. 9

19. $\frac{\sqrt{14}}{5}$ (0.75).

20. 1

21. 7

22. Yes. Vectors $[v_1 \quad v_2 \quad v_3]^T$ orthogonal to the given vector must satisfy $2v_1 + v_3 = 0$. A basis of this two-dimensional vector space is $[0 \quad 1 \quad 0]^T, [1 \quad 0 \quad -2]^T$.

23. $a = [2, -1, -3]^T$, $b = [-4, 8, -1]^T$, $\|a + b\| = 8.307$, $\|a\| + \|b\| = 12.742$

24. $a = [\frac{1}{2}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{3}]^T$,

$b = [\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, 0]^T$,

$|(\mathbf{a}, \mathbf{b})| = |\mathbf{a}^T \mathbf{b}| = |-\frac{1}{15}| = 0.067 \leq \|\mathbf{a}\| \|\mathbf{b}\| = \frac{1}{6}\sqrt{26} \cdot \frac{1}{5}\sqrt{14} = 0.636$

SOLUTIONS TO CHAPTER 7 REVIEW QUESTIONS AND PROBLEMS, page 318

11. $\begin{bmatrix} -1 & 2 & 5 \\ 0 & 3 & -1 \\ -1 & -5 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -2 & 3 \\ 5 & -2 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

13. $\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $[4 \quad 0 \quad -1]$

14. -1, $\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

15. $5, -2$

16. $\begin{bmatrix} 1/5 & 1/5 & -1/5 \\ 1/5 & 6/5 & 4/5 \\ -1/5 & 4/5 & 1/5 \end{bmatrix}$, $\begin{bmatrix} -1/5 & -1 & 2/5 \\ 2/5 & 0 & 1/5 \\ 1/5 & 0 & -2/5 \end{bmatrix}$

17. $-5, 25, 25, -5$

18. $\begin{bmatrix} \frac{3}{25} & \frac{3}{25} & \frac{2}{25} \\ \frac{3}{25} & \frac{53}{25} & \frac{27}{25} \\ \frac{2}{25} & \frac{27}{25} & \frac{18}{25} \end{bmatrix}$, $\begin{bmatrix} \frac{3}{25} & \frac{3}{25} & \frac{2}{25} \\ \frac{3}{25} & \frac{53}{25} & \frac{27}{25} \\ \frac{2}{25} & \frac{27}{25} & \frac{18}{25} \end{bmatrix}$

19. $\begin{bmatrix} -1 & 1 & 4 \\ 0 & 5 & -4 \\ -6 & -3 & -4 \end{bmatrix}$

20. $\begin{bmatrix} -2 & 4 & 0 \\ -2 & 10 & 2 \\ -4 & -16 & -8 \end{bmatrix}$

22. $x = 1, \quad y = t$ arbitrary, $\quad z = 3t + 2$

24. $x = \frac{13}{2}t_1 - \frac{3}{2}t_2 + 2, \quad y = t_1$ arbitrary, $\quad z = t_2$ arbitrary

26. No solution

27. $x = -1, y = -\frac{1}{2}.$

28. $x = \frac{1}{4}, \quad y = -\frac{1}{2}, \quad z = \frac{3}{2}$

30. Ranks 1, 1. The first row of the matrices equals -3 times the second row.

31. Ranks 2, 2, one solution.

32. Ranks 1, 2, so that there exists no solution.

34. $I_1 = 12$ A, $\quad I_2 = 4$ A, $\quad I_3 = 16$ A