Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

# Problem Set 12.12

No. 1

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$w(x,t) = \sin\left(t - \frac{x}{c}\right) \quad \text{if} \quad \frac{x}{c} < t < \frac{x}{c} + 2\pi$$

$$\frac{\partial^2 w}{\partial t^2} = -\sin\left(t - \frac{x}{c}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = -\frac{1}{c^2}\sin\left(t - \frac{x}{c}\right)$$

$$\therefore \frac{\partial^2 w}{\partial t} = c^2 \cdot \left(-\frac{1}{c^2}\right)\sin\left(t - \frac{x}{c}\right) = -\sin\left(t - \frac{x}{c}\right).$$

$\therefore$ $w(x,t) = \sin\left(t - \frac{x}{c}\right)$ is solution of $\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$

No. 2

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \quad c = 1.$$

$$w(0,t) = f(t), \quad \lim_{x \to \infty} w(x,t) = 0$$

$$f(t) = \begin{cases} t\ ; & 0 < t < \frac{1}{2} \\ 1 - t\ ; & \frac{1}{2} < t < 1. \end{cases}$$

$$w(x,t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right)$$
$$= f(t - x)\, u(t - x)$$

$$w(x,t)=\begin{cases}(t-x)\,u(t-x) & ;\ 0<t<\frac{1}{2}\\ (1-t+x)\,u(t-x) & ;\ \frac{1}{2}<t<1.\end{cases}$$

No. 3

問答或證明題，不解

No. 4

$w = w(x, t)$, $W = \mathcal{L}\{w(x, t)\} = W(x, s)$. The subsidiary equation is

$$\frac{\partial W}{\partial x} + x\mathcal{L}\{w_t(x, t)\} = \frac{\partial W}{\partial x} + x(sW - w(x, 0)) = x\mathcal{L}(1) = \frac{x}{s} \text{ and } w(x, 0) = 1.$$

By simplification,

$$\frac{\partial W}{\partial x} + xsW = x + \frac{x}{s}.$$

By integration of this first-order ODE with respect to $x$ we obtain

$$W = c(s)e^{-sx^2/2} + \frac{1}{s^2} + \frac{1}{s}.$$

For $x = 0$ we have $w(0, t) = 1$ and

$$W(0, s) = \mathcal{L}\{w(0, t)\} = \mathcal{L}\{1\} = \frac{1}{s} = c(s) + \frac{1}{s^2} + \frac{1}{s}.$$

Hence $c(s) = -1/s^2$, so that

$$W = -\frac{1}{s^2}e^{-sx^2/2} + \frac{1}{s^2} + \frac{1}{s}.$$

The inverse Laplace transform of this solution of the subsidiary equation is

$$w(x, t) = -(t - \tfrac{1}{2}x^2)\,u(t - \tfrac{1}{2}x^2) + t + 1 = \begin{cases} t + 1 & \text{if } t < \frac{1}{2}x^2 \\ \frac{1}{2}x^2 + 1 & \text{if } t > \frac{1}{2}x^2. \end{cases}$$

No. 5

$$x\frac{\partial w}{\partial x}+\frac{\partial w}{\partial t}=xt$$

$w(x,0)=0 \quad ; \quad x \geq 0 \qquad ; \qquad w(0,t)=0 \quad ; \quad t \geq 0.$

$$\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\}=s\mathcal{L}\{w\}-w(x,0)=sW \qquad (W=\mathcal{L}\{w\})$$

$$\mathcal{L}\left\{\frac{\partial w}{\partial x}\right\}=\frac{\partial}{\partial x}\mathcal{L}\{w\}=\frac{\partial W}{\partial x}$$

$$x\frac{\partial W}{\partial x}+sW=x\frac{s^2}{2}$$

$$\frac{\partial W}{\partial x}+\frac{s}{x}W=\frac{s^2}{2}$$

$$W=\frac{c(s)}{x^s}+\frac{x}{s^2(s+1)}$$

$$W(0,s)=\mathcal{L}\{w(0,t)\}=0 \quad \Rightarrow \quad c(s)=0$$

$$W=\frac{x}{s^2(s+1)}$$

$$w(x,t)=\mathcal{L}^{-1}\{W\}=x(t-1+e^{-t})$$

No. 6

$w(x, t) = 1/2 + 1/2t - 1/2\, u(t - 2x^2)(-1 + t - 2x^2)$
as obtained from

$$W(x, s) = \frac{s+1}{2s^2} + e^{-2x^2 s}\, c(s)$$

with $c(s) = (s - 1)/(2s^2)$ as obtained from $w(0, t) = 1$, $W(0, s) = 1/s$.

No. 7

$w = f(x)g(t)$, $xf' + \dot{g} = 2xt$, take $f(x) = x$ to get $g = C_1e^{-t} + 2t - 2$ and $C_1 = 2$ using the initial condition $w(x, 0) = 0$, *i.e*, $g(0) = 0$.

No. 8

$W = \mathcal{L}\{w\}$, $W_{xx} = (100s^2 + 100s + 25)W = (10s + 5)^2 W$. The solution of this ODE is

$$W = c_1(s)e^{-(10s+5)x} + c_2(s)e^{(10s+5)x}$$

with $c_2(s) = 0$, so that the solution is bounded. $c_1(s)$ follows from

$$W(0, s) = \mathcal{L}\{w(0, t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1} = c_1(s).$$

Hence

$$W = \frac{1}{s^2 + 1} e^{-(10s+5)x}.$$

The inverse Laplace transform (the solution of our problem) is

$$w = \mathcal{L}^{-1}\{W\} = e^{-5x} u(t - 10x) \sin(t - 10x),$$

a traveling wave decaying with $x$. Here $u$ is the unit step function (the Heaviside function).

No. 9

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} \quad ; \quad w(x, t)\Big|_{x\to\infty} \to 0 \ , \quad w(0, t) = f(t)$$

Let $\mathcal{L}\{w(x, t)\} = W(x, s)$

$$\mathcal{L}\left\{\frac{\partial w}{\partial t}\right\} = c^2 \mathcal{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\}$$

$$sW - 0 = c^2 \frac{\partial^2 W}{\partial x^2}$$

$$sW = c^2 \frac{\partial^2 W}{\partial x^2}$$

$$\frac{\partial^2 W}{\partial x^2} - \frac{s}{c^2} W = 0$$

$$W = A(s) e^{-\frac{\sqrt{s}}{c}x} + B(s) e^{+\frac{\sqrt{s}}{c}x}$$

At $x \to \infty$ ; $w(x, t) \to 0$ $\qquad W(x, s) \to 0$

$$\Rightarrow B(s) = 0$$

$$W(x, s) = A(s) e^{-\frac{\sqrt{s}}{c}x}$$

I.C :

$$W(0, s) = \mathcal{L}\{f(t)\} = F(s) = A(s)$$

$$\therefore W(x, s) = F(s) e^{-\frac{\sqrt{s}}{c}x}$$

No. 10

From $W = F(s)e^{-(x/c)\sqrt{s}}$ and the convolution theorem we have

$$w = f * \mathcal{L}^{-1}\{e^{-k\sqrt{s}}\}, \qquad k = \frac{x}{c}.$$

From this and formula 39 in Sec. 6.9 we get, as asserted,

$$w = \int_0^t f(t-\tau)\frac{k}{2\sqrt{\pi\tau^3}}e^{-k^2/(4\tau)}\,d\tau.$$

No. 11

Setting $\frac{x^2}{4c^2\tau} = z^2$ . $\tau = \frac{x^2}{4c^2z^2}$

$$d\tau = \frac{-x^2}{2c^2} z^{-3} dz$$

$$w_0(x,t) = \frac{x}{2c\sqrt{\pi}}\int_\infty^{\frac{x}{2c\sqrt{t}}} \frac{8c^3z^3}{x^3} \cdot \frac{-x^2}{2c^2} z^{-3} \cdot e^{-z^2} dz$$

$$= \frac{-2}{\sqrt{\pi}}\int_\infty^{\frac{x}{2c\sqrt{t}}} e^{-z^2} dz$$

$$= -\operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right) + \operatorname{erf}(\infty)$$

$$= 1 - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)$$

No. 12

$W_0(x, s) = s^{-1}e^{-\sqrt{s}x/c}$, $\mathcal{L}\{u(t)\} = 1/s$, and since $w(x, 0) = 0$,

$$\begin{aligned} W(x, s) &= F(s)sW_0(x, s) \\ &= F(s)[sW_0(x, s) - w(x, 0)] \\ &= F(s)\,\mathcal{L}\left\{\frac{\partial w_0}{\partial t}\right\}. \end{aligned}$$

Now apply the convolution theorem.