Part F. Optimization. Graphs

CHAPTER 22 Unconstrained Optimization. Linear Programming

SECTION 22.1. Basic Concepts. Unconstrained Optimization: Method of Steepest Descent, page 949

Purpose. To explain the concepts needed throughout this chapter. To discuss Cauchy's method of steepest descent or gradient method, a popular method of unconstrained optimization.

Main Content, Important Concepts

Objective function

Control variables

Constraints, unconstrained optimization

Cauchy's method

SOLUTIONS TO PROBLEM SET 22.1, page 951

1. $f(x) = (x_1 - 2.5)^2 + 0.5(x_2 - 3.0)^2 + 14.2$. The computation gives

Step	x_1	x_2	$f(\mathbf{x})$
1	2.3333	3.3333	14.2833
2	2.5556	3.1111	14.2093
3	2.4815	3.0370	14.2010
4	2.5062	3.0125	14.2001
5	2.4979	3.0041	14.2000

SECTION 22.2. Linear Programming, page 952

Purpose. To discuss the basic ideas of linear programming in terms of very simple examples involving two variables, so that the situation can be handled graphically and the solution can be found geometrically. To prepare conceptually for the case of three or more variables $x_1, \cdots, x_n$.

Main Content, Important Concepts

Linear programming problem

Its normal form. Slack variables

Feasible solution, basic feasible solution

Optimal solution

Comments on Content

Whereas the function to be maximized (or minimized) by Cauchy's method was arbitrary (differentiable), but we had no constraints, we now simply have a linear objective function, but constraints, so that calculus no longer helps.

No systematic method of solution is discussed in this section; these follow in the next sections.

SOLUTIONS TO PROBLEM SET 22.2, page 955

1. Ordinarily a vertex of a **region** is the intersection of only *two* straight lines given by inequalities taken with the equality sign. Here, (5, 4) is the intersection of *three* such lines. This may merit special attention in some cases, as we discuss in Sec. 22.4.

4. Location not unique; $f = 225$ on the segment from (3, 4) to (0, 10).

5. $f(\frac{5}{6}, \frac{7}{6}) = \frac{100}{3}$

6. $f(0, 5) = 10$

8. $x_1 =$ Number of days of operation of Kiln I, $x_2 =$ Number of days of operation of Kiln II. Objective function $f = 400x_1 + 600x_2$. Constraints:

$$\begin{aligned} 3000x_1 + 2000x_2 &\geqq 18{,}000 && \text{(Gray bricks)} \\ 2000x_1 + 5000x_2 &\geqq 34{,}000 && \text{(Red bricks)} \\ 300x_1 + 1500x_2 &\geqq 9000 && \text{(Glazed bricks).} \end{aligned}$$

$f_{\min} = f(2, 6) = 4400$, as can be seen from a sketch of the region in the x_1x_2-plane resulting from the constraints in the first quadrant. Operate Kiln I two days and Kiln II six days in filling that order. Note that the region determined by the constraints in the first quadrant of the x_1x_2-plane is unbounded, which causes no difficulty because we minimize (not maximize) the objective function.

SECTION 22.3. Simplex Method, page 956

Purpose. To discuss the standard method of linear programming for systematically finding an optimal solution by a finite sequence of transformations of matrices.

Main Content, Important Concepts

Normal form of the problem

Initial simplex table (initial augmented matrix)

Pivoting, further simplex tables (augmented matrices)

Comment on Concepts and Method

The given form of the problem involves inequalities. By introducing slack variables we convert the problem to the normal form. This is a linear system of equations. The initial simplex table is its augmented matrix. It is transformed by first selecting the column of a pivot and then the row of that pivot. The rules for this are entirely different from those for pivoting in connection with solving a linear system of equations. The selection of a pivot is followed by a process of elimination by row operations similar to that in the Gauss–Jordan method (Sec. 7.8). This is the first step, leading to another simplex table (another augmented matrix). The next step is done by the same rules, and so on. The process comes to an end when the first row of the simplex table obtained contains no more negative entries. From this final simplex table one can read the optimal solution of the problem because the first row corresponds to the objective function $f(x)$ to be maximized (or minimized).

SOLUTIONS TO PROBLEM SET 22.3, page 959

2. The matrices and pivot selections are

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 & 550 \\ 0 & 5 & 4 & 0 & 1 & 650 \end{bmatrix}$$

$550/3 > 650/5$, pivot 5

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 130 \\ 0 & 0 & \frac{8}{5} & 1 & -\frac{3}{5} & 160 \\ 0 & 5 & 4 & 0 & 1 & 650 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{5}\text{ Row 3} \\ \text{Row 2} - \frac{3}{5}\text{ Row 3} \\ \end{matrix}$$

$160/\frac{8}{5} < 650/4$, pivot $\frac{8}{5}$

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 150 \\ 0 & 0 & \frac{8}{5} & 1 & -\frac{3}{5} & 160 \\ 0 & 5 & 0 & -\frac{5}{2} & \frac{5}{2} & 250 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{8}\text{ Row 2} \\ \\ \text{Row 3} - \frac{5}{2}\text{ Row 2} \end{matrix}$$

$f_{\max} = 150$ at $x_1 = 250/5 = 50, x_2 = 160/(\frac{8}{5}) = 100$.

3. From the given data we have the augmented matrix (the initial simplex table)

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1200 \\ 0 & 4 & 2 & 0 & 1 & 1600 \end{bmatrix}.$$

The pivot is 4 since $1600/4 < 1200/2$. The indicated calculations give

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{4} & 400 \\ 0 & 0 & 2 & 1 & -\frac{1}{2} & 400 \\ 0 & 4 & 2 & 0 & 1 & 1600 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{4}\text{ Row 3} \\ \text{Row 2} - \frac{2}{4}\text{ Row 3} \\ \text{Row 3.} \end{matrix}$$

The pivot is 2 in Row 2. The indicated calculations give

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{8} & 500 \\ 0 & 0 & 2 & 1 & -\frac{1}{2} & 400 \\ 0 & 4 & 0 & -1 & \frac{3}{2} & 1200 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{4}\text{ Row 2} \\ \text{Row 2} \\ \text{Row 3} - \frac{2}{2}\text{ Row 2.} \end{matrix}$$

This shows that the solution is

$$f\left(\frac{1200}{4}, \frac{400}{2}\right) = 500.$$

5. Remember that we are looking for a *minimum* rather than for a maximum! The matrices and pivot selections are

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -4 & 10 & 20 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 & 1 & 0 & 0 & 60 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 20 \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix}$$

$\frac{60}{4} = 15 < \frac{20}{1} = 20$, pivot 4

$$\mathbf{T}_1 = \begin{bmatrix} 1 & -\frac{23}{2} & 0 & \frac{15}{2} & -\frac{5}{2} & 0 & 0 & -150 \\ 0 & 3 & 4 & 5 & 1 & 0 & 0 & 60 \\ 0 & \frac{5}{4} & 0 & -\frac{5}{4} & -\frac{1}{4} & 1 & 0 & 5 \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix} \begin{array}{l} \text{Row 1} - \frac{10}{4}\text{ Row 2} \\ \text{Row 2} \\ \text{Row 3} - \frac{1}{4}\text{ Row 2} \\ \text{Row 4} \end{array}$$

$\frac{60}{5} > \frac{30}{3}$, pivot 3

$$\mathbf{T}_2 = \begin{bmatrix} 1 & -\frac{33}{2} & 0 & 0 & -\frac{5}{2} & 0 & -\frac{5}{2} & -225 \\ 0 & -\frac{1}{3} & 4 & 0 & 1 & 0 & -\frac{5}{3} & 10 \\ 0 & \frac{25}{12} & 0 & 0 & -\frac{1}{4} & 1 & \frac{5}{12} & \frac{35}{2} \\ 0 & 2 & 0 & 3 & 0 & 0 & 1 & 30 \end{bmatrix} \begin{array}{l} \text{Row 1} - \frac{15}{6}\text{ Row 4} \\ \text{Row 2} - \frac{5}{3}\text{ Row 4} \\ \text{Row 3} + \frac{5}{12}\text{ Row 4} \\ \text{Row 4} \end{array}$$

$f_{\min} = -225$ at $x_1 = 0, x_2 = \frac{10}{4} = 2.5, x_3 = \frac{30}{3} = 10.$

6 From the given data we obtain the augmented matrix

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -2 & -3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 4.8 \\ 0 & 10 & 0 & 1 & 0 & 1 & 0 & 9.9 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0.2 \end{bmatrix}.$$

The pivot is 10. The indicated calculations give

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -3 & -\frac{4}{5} & 0 & \frac{1}{5} & 0 & 1.98 \\ 0 & 0 & 1 & \frac{9}{10} & 1 & -\frac{1}{10} & 0 & 3.81 \\ 0 & 10 & 0 & 1 & 0 & 1 & 0 & 9.9 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0.2 \end{bmatrix} \begin{array}{l} \text{Row 1} + \frac{2}{10}\text{ Row 3} \\ \text{Row 2} - \frac{1}{10}\text{ Row 3} \\ \text{Row 3} \\ \text{Row 4} \end{array}$$

The next pivot is 1 in Row 4 and Column 3. The indicated calculations give

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & -\frac{19}{5} & 0 & \frac{1}{5} & 3 & 2.58 \\ 0 & 0 & 0 & \frac{19}{10} & 1 & -\frac{1}{10} & -1 & 3.61 \\ 0 & 10 & 0 & 1 & 0 & 1 & 0 & 9.9 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0.2 \end{bmatrix} \quad \begin{matrix} \text{Row 1 + 3 Row 4} \\ \text{Row 2 − Row 4} \\ \text{Row 3} \\ \text{Row 4} \end{matrix}$$

The last pivot needed is $\frac{19}{10}$ in Row 2 and Column 4. We obtain

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 1 & 9.8 \\ 0 & 0 & 0 & \frac{19}{10} & 1 & -\frac{1}{10} & -1 & 3.61 \\ 0 & 10 & 0 & 0 & -\frac{10}{19} & \frac{20}{19} & \frac{10}{19} & 8 \\ 0 & 0 & 1 & 0 & \frac{10}{19} & -\frac{1}{19} & \frac{9}{19} & 2.1 \end{bmatrix} \quad \begin{matrix} \text{Row 1 + 2 Row 2} \\ \text{Row 2} \\ \text{Row 3} - \frac{10}{19}\text{ Row 2} \\ \text{Row 4} + \frac{10}{19}\text{ Row 2} \end{matrix}$$

Hence a solution of our problem is

$$f\left(\frac{8}{10}, \frac{2.1}{1}, \frac{3.61}{19/10}\right) = f(0.8, 2.1, 1.9) = 9.8.$$

Actually, all solutions are

$$(x_1, \quad 2.5 - 0.5x_1, \quad 2.3 - 0.5x_1)$$

where x_1 is arbitrary, satisfying $0 \leqq x_1 \leqq 0.8$, thus giving a straight segment with endpoints (0, 2.5, 2.3) and (0.8, 2.1, 1.9), where $x_1 = 0.8$ results from solving the system of three equations of the constraints taken with equality signs. The reason for the nonuniqueness is that the plane $f(x_1, x_2, x_3) = 9.8$ contains an edge of the region to which x_1, x_2, x_3 are restricted, whereas in general it will have just a single point (a vertex) in common with that region.

8 From the given data we obtain the augmented matrix

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -2 & -3 & 0 & 0 & 0 \\ 0 & 5 & 3 & 1 & 0 & 105 \\ 0 & 3 & 6 & 0 & 1 & 126 \end{bmatrix}.$$

The pivot is 5 in Row 2. The indicated calculations give

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{9}{5} & \frac{2}{5} & 0 & 42 \\ 0 & 5 & 3 & 1 & 0 & 105 \\ 0 & 0 & \frac{21}{5} & -\frac{3}{5} & 1 & 63 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{2}{5}\text{ Row 2} \\ \\ \text{Row 3} - \frac{3}{5}\text{ Row 2.} \end{matrix}$$

Since $63/(\frac{21}{5}) < \frac{105}{3}$, the pivot is $\frac{21}{5}$ in Row 3. The indicated calculations give

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{7} & \frac{3}{7} & 69 \\ 0 & 5 & 0 & \frac{10}{7} & -\frac{5}{7} & 60 \\ 0 & 0 & \frac{21}{5} & -\frac{3}{5} & 1 & 63 \end{bmatrix} \quad \begin{array}{l} \text{Row 1} + \frac{9}{21}\text{ Row 3} \\ \text{Row } w - \frac{5}{7}\text{ Row 3.} \\ \end{array}$$

Hence the result is

$$f_{\max} = f\left(\frac{60}{5}, \frac{63}{21/5}\right) = f(12, 15) = 69.$$

SECTION 22.4. Simplex Method: Difficulties, page 960

Purpose. To explain ways of overcoming difficulties that may arise in applying the simplex method.

Main Content, Important Concepts

Degenerate feasible solution

Artificial variable (for overcoming difficulties in starting)

Short Courses. Omit this section because these difficulties occur only quite infrequently in practice.

SOLUTIONS TO CHAPTER 22 REVIEW QUESTIONS AND PROBLEMS, page 965

6. The augmented matrix of the given data is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 2 & 1 & 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

The pivot is 2 in Row 3 and Column 2. The calculation gives

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 5 \\ 0 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & 5 \\ 0 & 2 & 1 & 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{array}{l} \text{Row 1} + \frac{1}{2}\text{Row 3} \\ \text{Row 2} - \frac{1}{2}\text{Row 3} \\ \text{Row 3} \\ \text{Row 4.} \end{array}$$

The next pivot is $\frac{3}{2}$ in Row 2. The calculation gives

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{20}{3} \\ 0 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & 5 \\ 0 & 2 & 0 & -\frac{2}{3} & \frac{4}{3} & 0 & \frac{20}{3} \\ 0 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{1}{3}\text{Row 2} \\ \text{Row 2} \\ \text{Row 3} - \frac{2}{3}\text{Row 2} \\ \text{Row 4} - \frac{2}{3}\text{Row 2.} \end{matrix}$$

We see from the last matrix that for the maximum we have

$$f\left(\frac{20/3}{2}, \frac{5}{3/2}\right) = f\left(\frac{10}{3}, \frac{10}{3}\right) = \frac{20}{3}.$$

8. The matrix of the given data is

$$\mathbf{T}_0 = \begin{bmatrix} 1 & -60 & -30 & 0 & 0 & 0 \\ 0 & 40 & 40 & 1 & 0 & 1800 \\ 0 & 200 & 20 & 0 & 1 & 6300 \end{bmatrix}$$

The pivot is 200. The calculation gives

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -24 & 0 & \frac{3}{10} & 1890 \\ 0 & 0 & 36 & 1 & -\frac{1}{5} & 540 \\ 0 & 200 & 20 & 0 & 1 & 6300 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{60}{200}\text{Row 3} \\ \text{Row 2} - \frac{1}{5}\text{Row 3} \\ \text{Row 3} \end{matrix}$$

The next pivot is 36. The calculation gives

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & 2250 \\ 0 & 0 & 36 & 1 & -\frac{1}{5} & 540 \\ 0 & 200 & 0 & -\frac{5}{9} & \frac{10}{9} & 6000 \end{bmatrix} \quad \begin{matrix} \text{Row 1} + \frac{24}{36}\text{Row 2} \\ \text{Row 2} \\ \text{Row 3} - \frac{20}{36}\text{Row 2} \end{matrix}$$

Hence the solution is

$$f\left(\frac{6000}{200}, \frac{540}{36}\right) = f(30, 15) = 2250.$$