CHAPTER 4 Systems of ODEs. Phase Plane. Qualitative Methods

Major Changes

This chapter was completely rewritten in the eighth edition, on the basis of suggestions by instructors who have taught from it and my own recent experience. The main reason for rewriting was the increasing emphasis on **linear algebra** in our standard curricula, so that we can expect that students taking material from Chap. 4 have at least some working knowledge of 2×2 matrices.

Accordingly, Chap. 4 makes modest use of 2×2 matrices. $n \times n$ matrices are mentioned only in passing and are immediately followed by illustrative examples of systems of two ODEs in two unknown functions, involving 2×2 matrices only. Section 4.2 and the beginning of Sec. 4.3 are intended to give the student the impression that, for first-order systems, one can develop a theory that is conceptually and structurally similar to that in Chap. 2 for a single ODE. Hence if the instructor feels that the class may be disturbed by $n \times n$ matrices, omission of the latter and explanation of the material in terms of two ODEs in two unknown functions will entail no disadvantage and will leave no gaps of understanding or skill.

To be completely on the safe side, Sec. 4.0 is included for reference, so that the student will have no need to search through Chap. 7 or 8 for a concept or fact needed in Chap. 4.

Basic throughout Chap. 4 is the **eigenvalue problem** (for 2×2 matrices), consisting first of the determination of the eigenvalues λ_1, λ_2 (not necessarily numerically distinct) as solutions of the characteristic equation, that is, the quadratic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$
$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0,$$

and then an eigenvector corresponding to λ_1 with components x_1, x_2 from

$$(a_{11} - \lambda_1)x_1 + a_{12}x_2 = 0$$

and an eigenvector corresponding to λ_2 from

$$(a_{11} - \lambda_2)x_1 + a_{12}x_2 = 0.$$

It may be useful to emphasize early that eigenvectors are determined only up to a nonzero factor and that, in the present context, normalization (to obtain unit vectors) is hardly of any advantage.

If there are students in the class who have not seen eigenvalues before (although the elementary theory of these problems does occur in every up-to-date introductory text on beginning linear algebra), they should not have difficulties in readily grasping the meaning of these problems and their role in this chapter, simply because of the numerous examples and applications in Sec. 4.3 and in later sections.

Section 4.5 includes three famous applications, namely, the **pendulum** and **van der Pol equations** and the **Lotka–Volterra predator–prey population model**.

SECTION 4.0. For Reference: Basics of Matrices and Vectors, page 124

Purpose. This section is for reference and review only, the material being restricted to what is actually needed in this chapter, to make it self-contained.

Main Content

Matrices, vectors

Algebraic matrix operations

Differentiation of vectors

Eigenvalue problems for 2×2 matrices

Important Concepts and Facts

Matrix, column vector and row vector, multiplication

Linear independence

Eigenvalue, eigenvector, characteristic equation

Some Details in Content

Most of the material is explained in terms of 2×2 matrices, which play the major role in Chap. 4; indeed, $n \times n$ matrices for general n occur only briefly in Sec. 4.2 and at the beginning in Sec. 4.3. Hence the demand of linear algebra on the student in Chap. 4 will be very modest, and Sec. 4.0 is written accordingly.

In particular, eigenvalue problems lead to quadratic equations only, so that nothing needs to be said about difficulties encountered with 3×3 or larger matrices.

Example 1. Although the later sections include many eigenvalue problems, the complete solution of such a problem (the determination of the eigenvalues and corresponding eigenvectors) is given in Sec. 4.0.

Emphasize to your students that the eigen*values* of a given square matrix are uniquely determined (and some of them can very well be 0), whereas eigen*vectors* must not be zero vectors and are determined only up to a nonzero multiplicative constant.

SECTION 4.1. Systems of ODEs as Models in Engineering Applications, page 130

Purpose. In this section the student will gain a first impression of the importance of systems of ODEs in physics and engineering and will learn why they occur and why they lead to eigenvalue problems.

Main Content

Mixing problem

Electrical network

Conversion of single equations to system (*Theorem* 1)

The possibility of switching back and forth between systems and single ODEs is practically quite important because, depending on the situation, the system or the single ODE will be the better source for obtaining the information sought in a specific case.

Background Material. Secs. 2.4, 2.8.

Short Courses. Take a quick look at Sec. 4.1, skip Sec. 4.2 and the beginning of Sec. 4.3, and proceed directly to solution methods in terms of the examples in Sec. 4.3.

Some Details on Content

Example 1 extends the physical system in Sec. 1.3, consisting of a single tank, to a system of two tanks. The principle of modeling remains the same. The problem leads to a typical eigenvalue problem, and the solutions show typical exponential increases and decreases to a constant value.

Problem Set 4.1

The mixing problems (Probs. 1–6) should lead to an understanding of the physical parameters involved (tank size, flow rate, amount of fertilizer), similarly in the networks in Probs. 7–9.

Problems 10–13 show conversions from ODEs to first-order systems.

Problem 14 shows the principle of extending the physical system in Sec. 2.5 to a system of more than one mass and spring, with (11) probably best understood by looking at Fig. 81.

SOLUTIONS TO PROBLEM SET 4.1, page 136

2. The system is

$$\begin{aligned} y_1' &= 0.02y_2 - 0.01y_1 \\ y_2' &= 0.01y_1 - 0.02y_2 \end{aligned}$$

where 0.01 appears because we divide by the content of the tank T_1, which is twice the old value. In proper order, the system becomes

$$\begin{aligned} y_1' &= -0.01y_1 + 0.02y_2 \\ y_2' &= 0.01y_1 - 0.02y_2. \end{aligned}$$

As a single vector equation,

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -0.01 & 0.02 \\ 0.01 & -0.02 \end{bmatrix}.$$

$\mathbf{A}$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -0.03$ and corresponding eigenvectors

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

respectively. The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}e^{-0.03t}.$$

From the initial values,

$$\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 150 \end{bmatrix}.$$

In components this is $c_1 + c_2 = 0$, $0.5c_1 - c_2 = 150$. Hence $c_1 = 100$, $c_2 = -100$. This gives the solution

$$\mathbf{y} = 100\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} - 100\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-0.03t}.$$

In components,

$$\begin{aligned} y_1 &= 100(1 - e^{-0.03t}) \\ y_2 &= 100(\tfrac{1}{2} + e^{-0.03t}). \end{aligned}$$

4. With

$$a = \frac{\text{Flow rate}}{\text{Tank size}}$$

we can write the system that models the process in the following form:

$$\begin{aligned} y_1' &= ay_2 - ay_1 \\ y_2' &= ay_1 - ay_2, \end{aligned}$$

ordered as needed for the proper vector form

$$\begin{aligned} y_1' &= -ay_1 + ay_2 \\ y_2' &= ay_1 - ay_2. \end{aligned}$$

In vector form,

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} -a & a \\ a & -a \end{bmatrix}.$$

The characteristic equation is

$$(\lambda + a)^2 - a^2 = \lambda^2 + 2a\lambda = 0.$$

Hence the eigenvalues are 0 and $-2a$. Corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively. The corresponding "general solution" is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2at}.$$

This result is interesting. It shows that the solution depends only on the ratio a, not on the tank size or the flow rate alone. Furthermore, the larger a is, the more rapidly y_1 and y_2 approach their limit.

The term "general solution" is in quotation marks because this term has not yet been defined *formally*, although it is clear what is meant.

6. The matrix of the system is

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 & 0 \\ 0.02 & -0.04 & 0.02 \\ 0 & 0.02 & -0.02 \end{bmatrix}.$$

The characteristic polynomial is

$$\lambda^3 + 0.08\lambda^2 + 0.0012\lambda = \lambda(\lambda + 0.02)(\lambda + 0.06).$$

This gives the eigenvalues and corresponding eigenvectors

$$\lambda_1 = 0, \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -0.02, \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \lambda_3 = -0.06, \quad \mathbf{x}^{(3)} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Hence a "general solution" is

$$\mathbf{y} = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-0.02t} + c_3\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{-0.06t}.$$

We use quotation marks since the concept of a general solution has not yet been defined *formally*, although it is clear what is meant.

8. The first ODE remains as before. The second ODE is obviously changed to

$$I_2' = 0.4I_1' - 0.54I_2.$$

Substitution of the first ODE into the new second one, as in the text, gives

$$I_2' = -1.6I_1 + 1.06I_2 + 4.8.$$

Hence the matrix of the new system is

$$\mathbf{A} = \begin{bmatrix}-4 & 4\\-1.6 & 1.06\end{bmatrix}.$$

Its eigenvalues are -1.5 and -1.44. The corresponding eigenvectors are $\mathbf{x}^{(1)} = [1 \quad 0.625]^{\mathsf{T}}$ and $\mathbf{x}^{(2)} = [1 \quad 0.64]^{\mathsf{T}}$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{-1.5t} + c_2\mathbf{x}^{(2)}e^{-1.44t}.$$

10. The system is

$$\begin{aligned} y_1' &= \qquad\quad y_2 \\ y_2' &= -3y_1 - 4y_2 \end{aligned}$$

The matrix has eigenvalues -1 and -3 and corresponding eigenvectors $[1, -1]^T$ and $[1, -3]^T$, respectively. From this $y = c_1e^{-t} + c_2e^{-3t}$.

11. $y_1' = y_2, y_2' = y_1 + \frac{3}{2}y_2, y = c_1[1/2,1]^Te^{2t} + [-2,1]^Te^{-t/2}$

12. The system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3 &= -2y_1 + y_2 + 2y_3 \end{aligned}$$

The eigenvalues of its matrix are $-1, 1, 2$. Eigenvalues are $[1, -1, 1]^T$, $[1, 1, 1]^T$, $[1, 2, 4]^T$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1[1, -1, 1]^Te^{-t} + c_2[1, 1, 1]^Te^{t} + c_3[1, 2, 4]^Te^{2t}.$$

13. $y_1' = y_2, y_2' = 12y_1 - y_2, y_1 = c_1e^{-4t} + c_2e^{3t}, y_2 = y_1'$

14. TEAM PROJECT. (a) From Sec. 2.5 we know that the undamped motions of a mass on an elastic spring are governed by $my'' + ky = 0$ or

$$my'' = -ky$$

where $y = y(t)$ is the displacement of the mass. By the same arguments, for the two masses on the two springs in Fig. 81 we obtain the linear homogeneous system

$$
\begin{aligned}
m_1 y_1'' &= -k_1 y_1 + k_2(y_2 - y_1) \\
m_2 y_2'' &= -k_2(y_2 - y_1)
\end{aligned} \tag{11}
$$

for the unknown displacements $y_1 = y_1(t)$ of the first mass m_1 and $y_2 = y_2(t)$ of the second mass m_2. The forces acting on the first mass give the first equation, and the forces acting on the second mass give the second ODE. Now $m_1 = m_2 = 1$, $k_1 = 3$, and $k_2 = 2$ in Fig. 81 so that by ordering (11) we obtain

$$
\begin{aligned}
y_1'' &= -5y_1 + 2y_2 \\
y_2'' &= 2y_1 - 2y_2
\end{aligned}
$$

or, written as a single vector equation,

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

(b) As for a single equation, we try an exponential function of t,

$$\mathbf{y} = \mathbf{x}e^{\omega t}. \qquad \text{Then} \qquad \mathbf{y}'' = \omega^2 \mathbf{x} e^{\omega t} = \mathbf{A}\mathbf{x}e^{\omega t}.$$

Then, writing $\omega^2 = \lambda$ and dividing by $e^{\omega t}$, we get

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

Eigenvalues and eigenvectors are

$$\lambda_1 = -1, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = -6, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Since $\omega = \pm\sqrt{\lambda}$ and $\sqrt{-1} = i$ and $\sqrt{-6} = i\sqrt{6}$, we get

$$\mathbf{y} = \mathbf{x}^{(1)}(c_1 e^{it} + c_2 e^{-it}) + \mathbf{x}^{(2)}(c_3 e^{i\sqrt{6}t} + c_4 e^{-i\sqrt{6}t})$$

or, by (7) in Sec. 2.3,

$$\mathbf{y} = a_1 \mathbf{x}^{(1)} \cos t + b_1 \mathbf{x}^{(1)} \sin t + a_2 \mathbf{x}^{(2)} \cos \sqrt{6}t + b_2 \mathbf{x}^{(2)} \sin \sqrt{6}t$$

where $a_1 = c_1 + c_2$, $b_1 = i(c_1 - c_2)$, $a_2 = c_3 + c_4$, $b_2 = i(c_3 - c_4)$. These four arbitrary constants can be specified by four initial conditions. In components, this solution is

$$
\begin{aligned}
y_1 &= a_1 \cos t + b_1 \sin t + 2a_2 \cos \sqrt{6}t + 2b_2 \sin \sqrt{6}t \\
y_2 &= 2a_1 \cos t + 2b_1 \sin t - a_2 \cos \sqrt{6}t - b_2 \sin \sqrt{6}t.
\end{aligned}
$$

(c) The first two terms in y_1 and y_2 give a slow harmonic motion, and the last two a fast harmonic motion. The slow motion occurs if, at some instant, both masses are moving downward or both upward. For instance, if $a_1 = 1$ and the three other constants are zero, we get $y_1 = \cos t$, $y_2 = 2 \cos t$; this is an example of such a motion. The fast motion occurs if, at each instant, the two masses are moving in opposite directions, so that one of the two springs is extended, whereas the other is simultaneously compressed. For instance, if $a_2 = 1$ and the other constants are zero, we have $y_1 = 2 \cos \sqrt{6}t$, $y_2 = -\cos \sqrt{6}t$; this is a fast motion of the indicated type. Depending on the initial conditions, one or the other motion will occur or a superposition of both.

SECTION 4.2 Basic Theory of Systems of ODEs. Wronskian, page 137

Purpose. This survey of some basic concepts and facts on nonlinear and linear systems is intended to give the student an impression of the conceptual and structural similarity of the theory of systems to that of single ODEs.

Content, Important Concepts

Standard form (1) of first-order systems, nonlinear or linear. The point is that each equation contains only one derivative, which appears on the left. For instance, Kirchhoff's Voltage Law (KVL) for electric systems gives a system of ODEs that can be transformed into the form (1) by algebra and differentiation.

Form of corresponding **initial value problems (1), (2)**. If an ODE is converted to a system, using Theorem 1 of Sec. 4.1, a corresponding IVP written as in Chaps. 2 and 3 converts to the form (1), (2) and conversely.

Existence and Uniqueness Theorem 1 for solutions of IVPs (1), (2). Note that the (sufficient) conditions correspond to those for single ODEs.

Standard form and notations (3) for **linear systems** of ODEs. These notations agree with the usual notations for matrices involving double subscripts.

Homogeneous and nonhomogeneous linear systems of ODEs.

Basis, **general solution (5)**, Wronskian (7), which is the determinant of the fundamental matrix $\mathbf{Y}$ (see (6)) such that a general solution can be written (8) $\mathbf{y} = \mathbf{Y}\mathbf{c}$, $\mathbf{c}$ the vector with the arbitrary constants as components.

Background Material. Sec. 2.6 contains the analogous theory for single equations. See also Sec. 1.7.

Short Courses. This section may be skipped, as mentioned before.

SECTION 4.3. Constant-Coefficient Systems. Phase Plane Method, page 140

Purpose. Typical examples show the student the rich variety of pattern of solution curves (trajectories) near critical points in the phase plane, along with the process of actually solving homogeneous linear systems. This will also prepare the student for a good understanding of the systematic discussion of critical points in the phase plane in Sec. 4.4.

Main Content

Solution method for homogeneous linear systems

Examples illustrating types of critical points

Solution when no basis of eigenvectors is available (Example 6)

Important Concepts and Facts

Trajectories as solution curves in the phase plane

Phase plane as a means for the simultaneous (qualitative) discussion of a large number of solutions

Basis of solutions obtained from basis of eigenvectors

Background Material. Short review of eigenvalue problems from Sec. 4.0, if needed.

Short Courses. Omit Example 6.

Some Details on Content

In addition to developing skill in solving homogeneous linear systems, the student is supposed to become aware that it is the kind of eigenvalues that determines the type of critical point. The examples show important cases. (A systematic discussion of *all* cases follows in the next section.)

Example 1. Two negative eigenvalues give a **node**.

Example 2. A real double eigenvalue gives a **node**.

Example 3. Real eigenvalues of opposite sign give a **saddle point**.

Example 4. Pure imaginary eigenvalues give a **center**, and working in complex is avoided by a standard trick, which can also be useful in other contexts.

Example 5. Genuinely complex eigenvalues give a **spiral point**. Some work in complex can be avoided, if desired, by differentiation and elimination. The first ODE is

$$y_2 = y_1' + y_1. \tag{a}$$

By differentiation and from the second ODE as well as from (a),

$$y_1'' = -y_1' + y_2' = -y_1' - y_1 - (y_1' + y_1) = -2y_1' - 2y_1.$$

Complex solutions $e^{(-1\pm i)t}$ give the real solution

$$y_1 = e^{-t}(A \cos t + B \sin t).$$

From this and (a) there follows the expression for y_2 given in the text.

Example 6 shows that the present method can be extended to include cases when **A** does not provide a basis of eigenvectors, but then becomes substantially more involved. In this way the student will recognize the importance of bases of eigenvectors, which also play a role in many other contexts.

A further illustration of the situation is given in Probs. 8 and 16.

SOLUTIONS TO PROBLEM SET 4.3, page 147

1. $y_1 = c_1e^{-t} + c_2e^t$, $y_2 = 3c_1e^{-t} + c_2e^t$

2. The eigenvalues are 3 and 9. Eigenvectors are $[3 \quad -1]^\mathsf{T}$ and $[3 \quad 1]^\mathsf{T}$, respectively. The corresponding general solution is

$$\begin{aligned} y_1 &= 3c_1e^{3t} + 3c_2e^{9t} \\ y_2 &= -c_1e^{3t} + c_2e^{9t}. \end{aligned}$$

3. $y_1 = c_1 + c_2e^t$, $y_2 = 2c_2e^t + \frac{4}{3}c_1$

4. The matrix has the double eigenvalue -6. An eigenvector is $[1 \quad -1]^\mathsf{T}$. Hence the vector **u** needed is obtained from

$$(\mathbf{A} + 6\mathbf{I})\mathbf{u} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We can take $\mathbf{u} = [0, \quad -\frac{1}{2}]$. With this we obtain as a general solution

$$\begin{aligned} y_1 &= c_1e^{-6t} + c_2te^{-6t} \\ y_2 &= -c_1e^{-6t} - c_2(t + \tfrac{1}{2})e^{-6t}. \end{aligned}$$

6. The eigenvalues are complex, $2 + 2i$ and $2 - 2i$. Corresponding complex eigenvectors are $[1 \quad -i]^{\mathsf{T}}$ and $[1 \quad i]^{\mathsf{T}}$, respectively. Hence a complex general solution is

$$\begin{aligned} y_1 &= c_1 e^{(2+2i)t} + c_2 e^{(2-2i)t} \\ y_2 &= -ic_1 e^{(2+2i)t} + ic_2 e^{(2-2i)t}. \end{aligned}$$

From this and the Euler formula we obtain a real general solution

$$\begin{aligned} y_1 &= e^{2t}[(c_1 + c_2)\cos 2t + i(c_1 - c_2)\sin 2t] \\ &= e^{2t}(A\cos 2t + B\sin 2t), \\ y_2 &= e^{2t}[(-ic_1 + ic_2)\cos 2t + i(-ic_1 - ic_2)\sin 2t] \\ &= e^{2t}(-B\cos 2t + A\sin 2t) \end{aligned}$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$.

7. $y_1 = -(1/2)c_2\sqrt{2}\cos(\sqrt{2}ax) + (1/2)c_3\sin(\sqrt{2}ax)\sqrt{2} + c_1,$
$y_2 = c_2\sin(\sqrt{2}ax) + c_3\cos(\sqrt{2}ax),$
$y_3 = (1/2)c_2\sqrt{2}\cos(\sqrt{2}ax) - (1/2)c_3\sin(\sqrt{2}ax)\sqrt{2} + c_1.$

8. The eigenvalue of algebraic multiplicity 2 is 9. An eigenvector is $[1 \quad -1]^{\mathsf{T}}$. There is no basis of eigenvectors. A first solution is

$$\mathbf{y}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{9t}.$$

A second linearly independent solution is (see Example 6 in the text)

$$\mathbf{y}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{9t} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{9t}$$

with $[u_1 \quad u_2]^{\mathsf{T}}$ determined from

$$(\mathbf{A} - 9\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus $u_1 + u_2 = -1$. We can take $u_1 = 0$, $u_2 = -1$. This gives the general solution

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = (c_1 + c_2 t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{9t} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{9t}.$$

10. The eigenvalues are 1 and -3. The corresponding general solution is

$$y_1 = c_1 e^t + 5c_2 e^{-3t} \text{ and } y_2 = c_1 e^t + c_2 e^{-3t}$$

Using the initial conditions we obtain

$$y_1(0) = c_1 + 5c_2 = 0 \text{ and } y_2(0) = c_1 + c_2 = 4$$

Hence the solution of the IVP is

$$\begin{aligned} y_1 &= 5e^t - 5e^{-3t} \\ y_2 &= 5e^t - e^{-3t} \end{aligned}$$

11. $y_1 = \frac{8}{5}e^t - \frac{18}{5}e^{-t/4}$ and $y_2 = \frac{8}{5}e^t - \frac{8}{5}e^{-t/4}$

12. The eigenvalues are 0 and 2. The corresponding general solution is

$$\begin{aligned} y_1 &= \quad c_1 + \; c_2e^{2t} \\ y_2 &= -\tfrac{1}{3}c_1 + \tfrac{1}{3}c_2e^{2t}. \end{aligned}$$

From this and the initial values we obtain

$$\begin{aligned} y_1(0) &= \quad c_1 + \; c_2 = 12 \\ y_2(0) &= -\tfrac{1}{3}c_1 + \tfrac{1}{3}c_2 = \;\; 2. \end{aligned}$$

Hence the solution of the IVP is

$$\begin{aligned} y_1 &= \quad 3 + 9e^{2t} \\ y_2 &= -1 + 3e^{2t}. \end{aligned}$$

13. $y_1 = \sinh 2x$, $y_2 = \cosh 2x$

14. The eigenvalues are $-1 + i$ and $-1 - i$. The corresponding real general solution is

$$\begin{aligned} y_1 &= e^{-t}(c_1 \cos t + c_2 \sin t) \\ y_2 &= e^{-t}(-c_2 \cos t + c_1 \sin t). \end{aligned}$$

From this and the initial conditions we obtain the solutions of the IVP

$$\begin{aligned} y_1 &= e^{-t} \cos t \\ y_2 &= e^{-t} \sin t. \end{aligned}$$

15. $y_1 = 0.25e^{-t}$, $y_2 = -0.25e^{-t}$

16. The system is

(a) $$y_1' = 8y_1 - y_2$$

(b) $$y_2' = y_1 + 10y_2.$$

10(a) + (b) gives $10y_1' + y_2' = 81y_1$; hence

(c) $$y_2' = -10y_1' + 81y_1.$$

Differentiating (a) and using (c) gives

$$\begin{aligned} 0 &= y_1'' - 8y_1' + y_2' \\ &= y_1'' - 8y_1' - 10y_1' + 81y_1 \\ &= y_1'' - 18y_1' + 81y_1. \end{aligned}$$

A general solution is

$$y_1 = (c_1 + c_2t)e^{9t}.$$

From this and (a) we obtain

$$\begin{aligned} y_2 = -y_1' + 8y_1 &= [-c_2 - 9(c_1 + c_2t) + 8(c_1 + c_2t)]e^{9t} \\ &= (-c_2 - c_1 - c_2t)e^{9t}, \end{aligned}$$

in agreement with the answer to Prob. 8.

18. The restriction of the inflow from outside to pure water is necessary to obtain a homogeneous system. The principle involved in setting up the model is

$$\text{Time rate of change} = \text{Inflow} - \text{Outflow}.$$

For Tank T_1 this is (see Fig. 88)

$$y_1' = \left(12 \cdot 0 + \frac{4}{200} y_2\right) - \frac{16}{200} y_1.$$

For Tank T_2 it is

$$y_2' = \frac{16}{200} y_1 - \frac{4 + 12}{200} y_2.$$

Performing the divisions and ordering terms, we have

$$\begin{aligned} y_1' &= -0.08y_1 + 0.02y_2 \\ y_2' &= 0.08y_1 - 0.08y_2. \end{aligned}$$

The eigenvalues of the matrix of this system are -0.04 and -0.12. Eigenvectors are $[1 \quad 2]^\mathsf{T}$ and $[1 \quad -2]^\mathsf{T}$, respectively. The corresponding general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.04t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-0.12t}.$$

The initial conditions are $y_1(0) = 100$, $y_2(0) = 200$. This gives $c_1 = 100$, $c_2 = 0$. In components the answer is

$$\begin{aligned} y_1 &= 100e^{-0.04t} \\ y_2 &= 200e^{-0.04t}. \end{aligned}$$

Both functions approach zero as $t \to \infty$, a reasonable result because pure water flows in and mixture flows out.

SECTION 4.4. Criteria for Critical Points. Stability, page 148

Purpose. Systematic discussion of critical points in the phase plane from the standpoints of both geometrical shapes of trajectories and **stability**.

Main Content

Table 4.1 for the types of critical points

Table 4.1 for the stability behavior

Stability chart (Fig. 92), giving Tables 4.1 and 4.2 graphically

Important Concepts

Node, saddle point, center, spiral point

Stable and attractive, stable, unstable

Background Material. Sec. 2.4 (needed in Example 2).

Short Courses. Since all these types of critical points already occurred in the previous section, one may perhaps present just a short discussion of stability.

Some Details on Content

The types of critical points in Sec. 4.3 now recur, and the discussion shows that they exhaust all possibilities. With the examples of Sec. 4.3 fresh in mind, the student will acquire a deeper understanding by discussing the **stability chart** and by reconsidering those examples from the viewpoint of stability. This gives the instructor an opportunity to emphasize that the general importance of stability in engineering can hardly be overestimated.

Example 2, relating to the familiar free vibrations in Sec. 2.4, gives a good illustration of stability behavior, namely, depending on c, attractive stability, stability (and instability if one includes "negative damping," with $c < 0$, as it will recur in the next section in connection with the famous van der Pol equation).

Problem Set 4.4

Problems 1–10 give straightforward applications of the criteria in this section.

Problems 11–12 are related to previous material, for instance, to oscillations of a mass on a spring.

For Prob. 13 let the student reconsider the discussion of (10)–(13) in Sec. 4.3 as discussed there and then, in addition, from the new viewpoint of stability.

Problems 15–17 concern the basic concept of perturbation, which, in practice, may be due to inaccuracies of measurement.

The purpose of Probs. 18–20 is fairly obvious.

SOLUTIONS TO PROBLEM SET 4.4, page 151

1. Unstable improper node, $y_1 = c_1e^t$, $y_2 = e^{\frac{1}{2}t}$.

2. $p = -7$, $q = 12 > 0$, $\Delta = 49 - 48 > 0$, stable and attractive improper node, $y_1 = c_1e^{-4t}$, $y_2 = c_2e^{-3t}$

3. Center, always stable, $y_1 = c_1 \sin 2t + c_2 \cos 2t$, $y_2 = 2c_1 \cos 2t - 2c_2 \sin 2t$

4. $p = 0$, $q = -9$, saddle point, always unstable. A general solution is

$$\begin{aligned} y_1 &= c_1e^{-3t} + c_2e^{3t} \\ y_2 &= -5c_1e^{-3t} + c_2e^{3t}. \end{aligned}$$

5. Stable spiral, $y_1 = e^{-t}(c_1 \sin t + c_2 \cos t)$, $y_2 = -e^{-t}(-c_1 \cos t + c_2 \sin t)$

6. $p = -12$, $q = 27$, $\Delta = 144 - 108 > 0$, stable and arrtactive node. A general solution is

$$\begin{aligned} y_1 &= c_1e^{-3t} + c_2e^{-9t} \\ y_2 &= -3c_1e^{-3t} + 3c_2e^{-9t}. \end{aligned}$$

7. Saddle point, always unstable, $y_1 = c_1e^{-2t} + c_2e^{3t}$, $y_2 = c_1e^{-2t} - 4c_2e^{3t}$.

8. $p = -3$, $q = -10$, saddle point, always unstable. A general solution is

$$\begin{aligned} y_1 &= c_1e^{-5t} + 4c_2e^{2t} \\ y_2 &= -c_1e^{-5t} + 3c_2e^{2t}. \end{aligned}$$

9. Unstable node, $y_1 = c_1e^{2t} + c_2e^{3t}$, $y_2 = -\frac{4}{3}c_1e^{2t} - c_2e^{3t}$.

10. $p = -2, \quad q = 5, \quad \Delta = 4 - 20 < 0$, stable and attractive spiral. The components of a general solution are

$$\begin{aligned}
y_1 &= c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t} \\
&= e^{-t}((c_1 + c_2)\cos 2t + i(c_1 - c_2)\sin 2t) \\
&= e^{-t}(A\cos 2t + B\sin 2t), \\
y_2 &= (-1 + 2i)c_1 e^{(-1+2i)t} + (-1 - 2i)c_2 e^{(-1-2i)t} \\
&= e^{-t}((-c_1 - c_2 + 2ic_1 - 2ic_2)\cos 2t + i(2ic_1 + 2ic_2 - c_1 + c_2)\sin 2t) \\
&= e^{-t}((-A + 2B)\cos 2t - (2A + B)\sin 2t)
\end{aligned}$$

where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$.

11. $y = e^{-t}(c_1 \sin 2t + c_2 \cos 2t)$. Stable and attractive spirals.

12. $y = A\cos \frac{1}{3}t + B\sin \frac{1}{3}t$. The trajectories are the ellipses

$$\tfrac{1}{9}y_1^2 + y_2^2 = \text{const.}$$

This is obtained as in Example 4 in Sec. 4.3.

14. $y_1' = -dy_1/d\tau, \quad y_2' = -dy_2/d\tau$, reversal of the direction of motion; to get the usual form, we have to multiply the transformed system by -1, which amounts to multiplying the matrix by -1, changing p into $-p$, but leaving q and Δ unchanged. In the example, we get an unstable node.

16. At a center, $p = a_{11} + a_{22} = 0$, $q = \det \mathbf{A} > 0$, hence $\Delta < 0$. Under the change, p changes into $a_{11} + k + a_{22} + k = 2k \neq 0$; q remains positive because

$$(a_{11} + k)(a_{22} + k) - a_{12}a_{21} = q + k^2 > 0.$$

Finally, Δ remains unchanged because

$$(p + 2k)^2 - 4(q + k^2) = (2k)^2 - 4(q + k^2) = -4q < 0.$$

Hence we obtain a spiral point, which is unstable if $k > 0$ and stable and attractive if $k < 0$.

We can reason more simply as follows. For a center, the eigenvalues are pure imaginary (to have closed trajectories). An eigenvalue λ of $\mathbf{A}$ gives an eigenvalue $\lambda + k$ of $\mathbf{A}$, causing a damped oscillation (when $k < 0$) or an increasing one (when $k > 0$), thus a spiral.

SECTION 4.5 Qualitative Methods for Nonlinear Systems, page 152

Purpose. As a most important step, in this section we extend phase plane methods from linear to nonlinear systems and nonlinear ODEs.

The particular importance of phase plane methods for nonlinear ODEs and systems results from the difficulty (or impossibility) of solving them, explicitly as explained in the text.

Main Content

Critical points of nonlinear systems

Their discussion by linearization

Transformation of single autonomous ODEs

Applications of linearization and transformation techniques

Important Concepts and Facts

Linearized system (3), condition for applicability

Linearization of pendulum equations

Self-sustained oscillations, van der Pol equation

Short Courses. Linearization at different critical points seems the main issue that the student is supposed to understand and handle in practice. Examples 1 and 2 may help students to gain skill in that technique. The other material can be skipped without loss of continuity.

Some Details on Content

This section is very important, because from it the student should learn not only techniques (linearization, etc.) but also the fact that phase plane methods are particularly powerful and important in application to systems or single ODEs that cannot be solved explicitly. The student should also recognize that it is quite surprising how much information these methods can give. This is demonstrated by the **pendulum equation** (Examples 1 and 2) for a relatively simple system, and by the famous **van der Pol equation** for a single ODE, which has become a prototype for self-sustained oscillations of electrical systems of various kinds.

We also discuss the famous **Lotka–Volterra predator–prey model**.

For the **Rayleigh** and **Duffing equations**, see the problem set.

Problem Set 4.5

Problems 1–3 concern the undamped pendulum and limit cycles and their deformation in the famous van der Pol equation.

Problems 4–8 are formal and concern linearization of systems in the case of several limit points, usually of a different type.

Problems 9–13 concern linearization of single nonlinear second-order ODEs, which are first converted to systems.

Team Project 14 discusses further self-sustained oscillations.

SOLUTIONS TO PROBLEM SET 4.5, page 159

2. A limit cycle is approached by trajectories (from inside and outside). No such approach takes place for a closed trajectory.

4. Writing the system in the form

$$\begin{aligned} y_1' &= y_1(4 - y_1) \\ y_2' &= y_2 \end{aligned}$$

we see that the critical points are (0, 0) and (4, 0). For (0, 0) the linearized system is

$$\begin{aligned} y_1' &= 4y_1 \\ y_2' &= y_2. \end{aligned}$$

The matrix is

$$\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $p = 5$, $q = 4$, $\Delta = 25 - 16 = 9$. This shows that the critical point at (0, 0) is an unstable node.

For (4, 0) the translation to the origin is

$$y_1 = 4 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

This gives the transformed system

$$\begin{aligned} \tilde{y}_1' &= (4 + \tilde{y}_1)(-\tilde{y}_1) \\ \tilde{y}_2' &= \tilde{y}_2 \end{aligned}$$

and the corresponding linearized system

$$\begin{aligned} \tilde{y}_1' &= -4\tilde{y}_1 \\ \tilde{y}_2' &= \tilde{y}_2. \end{aligned}$$

Its matrix is

$$\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\tilde{p} = -3$, $\tilde{q} = -4$. Thus the critical point at (4, 0) is a saddle point, which is always unstable.

5. Center at (0, 0). Saddle point at (4, 0). At (4, 0) set $y_1 = 4 + \tilde{y}_1$. Then $\tilde{y}'_2 = \tilde{y}_1$.

6. At (0, 0), $y_1' = y_2$, $y_2' = -y_1$, $p = 0$, $q = 1$, $\Delta = -4$, center. The other critical point is at $(-1, 0)$. We set $y_1 = -1 + \tilde{y}_1$, $y_2 = \tilde{y}_2$. Then $-y_1 - y_1^2 \approx \tilde{y}_1$. Hence $\tilde{y}_1' = \tilde{y}_2$, $\tilde{y}_2' = \tilde{y}_1$. This gives a saddle point.

7. The point (0, 0), $y_1' = -y_1 + y_2$, $y_2' = -y_1 - \frac{1}{2}y_2$, is stable and an attractive spiral point. The point $(-1, 2)$ is a saddle point, $y_1 = -1 + \tilde{y}_1$, $y_2 = 2 + \tilde{y}_2$, $\tilde{y}_1' = -2\tilde{y}_1 - 3\tilde{y}_2$, $\tilde{y}_2' = -\tilde{y}_1 - \frac{1}{2}\tilde{y}_2$.

8. The system may be written

$$\begin{aligned} y_1' &= y_2(1 - y_2) \\ y_2' &= y_1(1 - y_1). \end{aligned}$$

From this we see immediatly that there are four critical points, at (0, 0), (0, 1), (1, 0), (1, 1).

At (0, 0) the linearized system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1. \end{aligned}$$

The matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence $p = 0$, $q = -1$, so that we have a saddle point.

At (0, 1) the transformation is $y_1 = \tilde{y}_1$, $y_2 = 1 + \tilde{y}_2$. This gives the transformed system

$$\begin{aligned} \tilde{y}_1' &= (1 + \tilde{y}_2)(-\tilde{y}_2) \\ \tilde{y}_2' &= \tilde{y}_1(1 - \tilde{y}_1). \end{aligned}$$

Linearization gives

$$\begin{aligned} \tilde{y}_1' &= -\tilde{y}_2 \\ \tilde{y}_2' &= \tilde{y}_1 \end{aligned} \qquad \text{with matrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

for which $\tilde{p} = 0, \tilde{q} = 1, \tilde{\Delta} = -4$, and we have a center.

At (1, 0) the transformation is $y_1 = 1 + \tilde{y}_1, y_2 = \tilde{y}_2$. The transformed system is

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2(1 - \tilde{y}_2) \\ \tilde{y}_2' &= (1 + \tilde{y}_1)(-\tilde{y}_1). \end{aligned}$$

Its linearization is

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2 \\ \tilde{y}_2' &= -\tilde{y}_1 \end{aligned} \qquad \text{with matrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for which $\tilde{p} = 0, \tilde{q} = 1, \tilde{\Delta} = -4$, so that we get another center.

At (1, 1) the transformation is

$$y_1 = 1 + \tilde{y}_1, \qquad y_2 = 1 + \tilde{y}_2.$$

The transformed system is

$$\begin{aligned} \tilde{y}_1' &= (1 + \tilde{y}_2)(-\tilde{y}_2) \\ \tilde{y}_2' &= (1 + \tilde{y}_1)(-\tilde{y}_1). \end{aligned}$$

Linearization gives

$$\begin{aligned} \tilde{y}_1' &= -\tilde{y}_2 \\ \tilde{y}_2' &= -\tilde{y}_1 \end{aligned} \qquad \text{with matrix} \qquad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

for which $\tilde{p} = 0, \tilde{q} = -1$, and we have another saddle point.

9. (0, 0) saddle point, $(-2, 0)$ and (2, 0) centres.

10. $y'' + y - y^3 = 0$ written as a system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + y_1^3. \end{aligned}$$

Now $-y_1 + y_1^3 = y_1(-1 + y_1^2) = 0$ shows that there are three critical points, at $(y_1, y_2) = (0, 0), (-1, 0)$, and $(1, 0)$.

The linearized system at (0, 0) is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1. \end{aligned} \qquad \text{Matrix:} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

From the matrix we see that $p = a_{11} + a_{22} = 0, q = 1$. Hence (0, 0) is a center (see Sec. 3.4).

For the next critical point we have to linearize at $(-1, 0)$ by setting

$$y_1 = -1 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

Then

$$y_1(-1 + y_1^2 = (-1 + \tilde{y}_1)[-1 + (-1 + \tilde{y}_1)^2]$$
$$= (-1 + \tilde{y}_1)[-2\tilde{y}_1 + \tilde{y}_1^2] \approx 2\tilde{y}_1.$$

Hence the linearized system is

$$\tilde{y}_1' = \tilde{y}_2 \qquad \text{Matrix: } \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$
$$\tilde{y}_2' = 2\tilde{y}_1.$$

Hence $q = \det \mathbf{A} = -2 < 0$, that is, the critical point at $(-1, 0)$ is a saddle point.
Similarly, to linearize at $(1, 0)$, set

$$y_1 = 1 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

Then

$$-y_1 + y_1^3 \approx 2\tilde{y}_1$$

and we obtain another saddle point, as just before.

11. $(\frac{\pi}{4} \pm n\pi, 0)$ saddle points; $(-\frac{\pi}{4} \pm n\pi, 0)$ centres. For the critical point $(\frac{\pi}{4}, 0)$, after linearizing and using the substitutions $y_1' = \frac{\pi}{4} + \tilde{y}_1$, and $y_2' = \tilde{y}_2$, we get

$$\begin{aligned} y'_1 &= \tilde{y}'_1 \\ y'_2 &= -\cos 2(\tfrac{\pi}{4} + \tilde{y}'_1) \\ &= \sin 2\tilde{y}_1 \\ &= 2\tilde{y}_1 \end{aligned}$$

where we have used the fact $2y = 2y - \frac{4}{3}y^3 + \cdots$.

12. $y_1' = y_2, y_2' = y_1(-9 - y_1)$. $(0, 0)$ is a critical point. The linearized system at $(0, 0)$ is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -9y_1 \end{aligned} \qquad \text{with matrix} \qquad \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$$

for which $p = 0$ and $q = 9 > 0$, so that we have a center.
A second critical point is at $(-9, 0)$. The transformation is

$$y_1 = -9 + \tilde{y}_1, \qquad y_2 = \tilde{y}_2.$$

This gives the transformed system

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2 \\ \tilde{y}_2' &= (-9 + \tilde{y}_1)(-\tilde{y}_1). \end{aligned}$$

Its linearization is

$$\begin{aligned} \tilde{y}_1' &= \tilde{y}_2 \\ \tilde{y}_2' &= 9\tilde{y}_1 \end{aligned} \qquad \text{with matrix} \qquad \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$$

for which $\tilde{q} = -9 < 0$, so that we have a saddle point.

13. $(\pm 2n\pi, 0)$ centres; $y_1 = (2n + 1)\pi + \tilde{y}_1$, $(\pi \pm 2n\pi, 0)$ saddle points.

14. TEAM PROJECT. (a) Unstable node if $\mu \geqq 2$, unstable spiral point if $2 > \mu > 0$, center if $\mu = 0$, stable and attractive spiral point if $0 > \mu > -2$, stable and attractive node if $\mu \leqq -2$.

(c) As a system we obtain

$$\text{(A)} \qquad y_1' = y_2$$
$$\text{(B)} \qquad y_2' = -(\omega_0^2 y_1 + \beta y_1^3).$$

The product of the left side of (A) and the right side of (B) equals the product of the right side of (A) and the left side of (B):

$$y_2' y_2 = -(\omega_0^2 y_1 + \beta y_1^3) y_1'.$$

Integration on both sides and multiplication by 2 gives

$$y_2^2 + \omega_0^2 y_1^2 + \tfrac{1}{2}\beta y_1^4 = \text{const.}$$

For positive β these curves are closed because then βy^3 is a proper restoring term, adding to the restoring due to the y-term. If β is negative, the term βy^3 has the opposite effect, and this explains why then some of the trajectories are no longer closed but extend to infinity in the phase plane.

For generalized van der Pol equations, see e.g., K. Klotter and E. Kreyzig, On a class of self-sustained oscillations. *J. Appl. Math.* **27**(1960), 568–574.

SECTION 4.6. Nonhomogeneous Linear Systems of ODEs, page 160

Purpose. We now turn from homogeneous linear systems considered so far to solution methods for nonhomogeneous systems.

Main Content

Method of undetermined coefficients

Modification for special right sides

Method of variation of parameters

Short Courses. Select just one of the preceding methods.

Some Details on Content

In addition to understanding the solution methods as such, the student should observe the conceptual and technical similarities to the handling of nonhomogeneous linear ODEs in Secs. 2.7–2.10 and understand the reason for this, namely, that systems can be converted to single equations and conversely. For instance, in connection with Example 1 in this section, one may point to the Modification Rule in Sec. 2.7, or, if time permits, establish an even more definite relation by differentiation and elimination of y_2,

$$\begin{aligned} y_1'' &= -3y_1' + y_2' + 12e^{-2t} \\ &= -3y_1' + (y_1 - 3y_2 + 2e^{-2t}) + 12e^{-2t} \\ &= -3y_1' + y_1 - 3(y_1' + 3y_1 + 6e^{-2t}) + 14e^{-2t} \\ &= -6y_1' - 8y_1 - 4e^{-2t}, \end{aligned}$$

solving this for y_1 and then getting y_2 from the solution.

Problem Set 4.6

Problems 2–7 concern general solutions of nonhomogeneous systems, where the particular solution needed is obtained by the method of undetermined coefficients.

Problems 8 and 9 are general questions on the method of undetermined coefficients.

Problems 10–15 concern initial value problems; here it is important to emphasize that the initial conditions are to be used only after a general solution of the *nonhomogeneous* system has been obtained.

Typical applications to nonhomogeneous systems are shown in Probs. 17–20.

SOLUTIONS TO PROBLEM SET 4.6, page 163

2. The matrix of the system has the eigenvalues 2 and -2. Eigenvectors are $[1 \quad 1]^{\mathsf{T}}$ and $[1 \quad -3]^{\mathsf{T}}$, respectively. Hence a general solution of the homogeneous system is

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}.$$

We determine $\mathbf{y}^{(p)}$ by the method of undetermined coefficients, starting from

$$\mathbf{y}^{(p)} = \begin{bmatrix} a_1 \cos t + b_1 \sin t \\ a_2 \cos t + b_2 \sin t \end{bmatrix}.$$

Substituting this and its derivative into the given nonhomogeneous system, we obtain, in terms of components,

$$\begin{aligned} -a_1 \sin t + b_1 \cos t &= (a_1 + a_2) \cos t + (b_1 + b_2) \sin t + 10 \cos t \\ -a_2 \sin t + b_2 \cos t &= (3a_1 - a_2) \cos t + (3b_1 - b_2) \sin t - 10 \sin t. \end{aligned}$$

By equating the coefficients of the cosine and of the sine in the first of these two equations we obtain

$$b_1 = a_1 + a_2 + 10, \qquad -a_1 = b_1 + b_2.$$

Similarly from the second equation,

$$b_2 = 3a_1 - a_2, \qquad -a_2 = 3b_1 - b_2 - 10.$$

The solution is $a_1 = -2$, $b_1 = 4$, $a_2 = -4$, $b_2 = -2$. This gives the *answer*

$$\begin{aligned} y_1 &= c_1 e^{2t} + c_2 e^{-2t} - 2 \cos t + 4 \sin t \\ y_2 &= c_1 e^{2t} - 3c_2 e^{-2t} - 4 \cos t - 2 \sin t. \end{aligned}$$

3. $y_1 = c_1 e^t + c_2 e^{-t}$, $y_2 = c_1 e^t - c_2 e^{-t} - \frac{5}{3} e^{2t}$

4. From the characteristic equation,

$$\lambda_1 = 2, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad\qquad \lambda_2 = -4, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\mathbf{y}^{(p)}$ can be obtained by the method of undetermined coefficients, starting from

$$\mathbf{y}^{(p)} = \mathbf{a} \cosh t + \mathbf{b} \sinh t.$$

Substitution gives

$$\begin{aligned}\mathbf{y}^{(p)\prime} &= \mathbf{a}\sinh t + \mathbf{b}\cosh t\\ &= \begin{bmatrix}4 & -8\\ 2 & -6\end{bmatrix}(\mathbf{a}\cosh t + \mathbf{b}\sinh t) + \begin{bmatrix}2\\ 1\end{bmatrix}\cosh t + \begin{bmatrix}0\\ 2\end{bmatrix}\sinh t.\end{aligned}$$

Comparing sinh terms and cosh terms (componentwise), we get from this

$$\left.\begin{aligned}a_1 &= 4b_1 - 8b_2\\ a_2 &= 2b_1 - 6b_2 + 2\end{aligned}\right\}\text{sinh terms}$$
$$\left.\begin{aligned}b_1 &= 4a_1 - 8a_2 + 2\\ b_2 &= 2a_1 - 6a_2 + 1\end{aligned}\right\}\text{cosh terms.}$$

To solve this, one can substitute the first two equations into the last two, solve for $b_1 = 2$, $b_2 = 1$, and then get from the first two equations $a_1 = a_2 = 0$. This gives the general solution

$$y_1 = 4c_1e^{2t} + c_2e^{-4t} + 2\sinh t, \quad y_2 = c_1e^{2t} + c_2e^{-4t} + \sinh t.$$

5. $y_1 = -\frac{13}{4} - \frac{5}{2}t + c_1e^{2t} + c_2e^{t}$, $y_2 = \frac{7}{2} - \frac{2}{3}c_1e^{2t} - c_2e^{t} + 3t$

6. From the characteristic equation we obtain

$$\lambda_1 = 4, \qquad \mathbf{x}^{(1)} = \begin{bmatrix}1\\ 1\end{bmatrix}, \qquad \lambda_2 = -4, \qquad \mathbf{x}^{(2)} = \begin{bmatrix}1\\ -1\end{bmatrix}.$$

By the method of undetermined coefficients, we set

$$\mathbf{y}^{(p)} = \mathbf{u} + \mathbf{v}t + \mathbf{w}t^2.$$

By substitution,

$$\mathbf{y}^{(p)\prime} = \mathbf{v} + 2\mathbf{w}t = \begin{bmatrix}0 & 4\\ 4 & 0\end{bmatrix}(\mathbf{u} + \mathbf{v}t + \mathbf{w}t^2) + \begin{bmatrix}0\\ 2\end{bmatrix} + \begin{bmatrix}0\\ -16\end{bmatrix}t^2.$$

We now compare componentwise the constant terms, linear terms, and quadratic terms:

$$\left.\begin{aligned}v_1 &= 4u_2\\ v_2 &= 4u_1 + 2\end{aligned}\right\}\text{constant terms}$$
$$\left.\begin{aligned}2w_1 &= 4v_2\\ 2w_2 &= 4v_1\end{aligned}\right\}\text{linear terms}$$
$$\left.\begin{aligned}0 &= 4w_2\\ 0 &= 4w_1 - 16\end{aligned}\right\}\text{quadratic terms.}$$

We then obtain, in this order,

$$w_1 = 4, \quad w_2 = 0, \quad v_1 = 0, \quad v_2 = 2, \quad u_1 = \tfrac{1}{4}(v_2 - 2) = 0, \quad u_2 = 0.$$

The corresponding general solution is

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{4t} + c_2\mathbf{x}^{(2)}e^{-4t} + \mathbf{v}t + \mathbf{w}t^2;$$

in components,

$$y_1 = c_1e^{4t} + c_2e^{-4t} + 4t^2, \qquad y_2 = c_1e^{4t} - c_2e^{-4t} + 2t.$$

7. $y_1 = c_1e^{-2t} + c_2e^{3t} + \frac{3}{4}e^{-t} + \frac{1}{6}t - \frac{1}{36}$
$y_2 = c_1e^{-2t} - 4c_2e^{3t} + \frac{47}{6}t + \frac{175}{36}$

10. $\mathbf{y} = c_1\mathbf{x}^{(1)}e^t + c_2\mathbf{x}^{(2)}e^{2t}$, $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$. Now e^t on the right is a solution of the homogeneous system. Hence, to find $\mathbf{y}^{(p)}$, we have to proceed as in Example 1, setting

$$\mathbf{y}^{(p)} = \mathbf{u}te^t + \mathbf{v}e^t.$$

Substitution gives

$$\mathbf{y}^{(p)\prime} = \mathbf{u}(t+1)e^t + \mathbf{v}e^t = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}(\mathbf{u}te^t + \mathbf{v}e^t) + \begin{bmatrix} 5 \\ -6 \end{bmatrix}e^t.$$

Equating the terms in e^t (componentwise) gives

$$\begin{aligned} u_1 + v_1 &= -3v_1 - 4v_2 + 5 \\ u_2 + v_2 &= 5v_1 + 6v_2 - 6 \end{aligned}$$

and the terms in te^t give

$$\begin{aligned} u_1 &= -3u_1 - 4u_2 \\ u_2 &= 5u_1 + 6u_2. \end{aligned}$$

Hence $u_1 = 1, u_2 = -1, v_1 = 1, v_2 = 0$. This gives the general solution

$$y_1 = c_1e^t + 4c_2e^{2t} + te^t + e^t, \qquad y_2 = -c_1e^t - 5c_2e^{2t} - te^t.$$

From the initial conditions we obtain $c_1 = -2, c_2 = 5$. *Answer:*

$$y_1 = -2e^t + 20e^{2t} + te^t + e^t, \qquad y_2 = 2e^t - 25e^{2t} - te^t.$$

11. $y_1 - \cosh t + 2\sinh t + e^t, y_2 = 2\cosh t - \sinh t - e^{-t}$

12. A general solution of the homogeneous system is

$$\mathbf{y}^{(h)} = c_1\mathbf{x}^{(1)}e^{3t} + c_2\mathbf{x}^{(2)}e^{-t}, \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Answer:

$$y_1 = 2e^{-t} + t^2, \qquad y_2 = -e^{-t} - t.$$

13. $y_1: = -\sin 2t - \cos 2t + 2\sin t, y_2: = 2\cos 2t - 2\sin 2t$

14. The matrix of the homogeneous system

$$\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$$

has the eigenvalues $2i$ and $-2i$ and eigenvectors $[2 \quad i]^\mathsf{T}$ and $[2 \quad -i]^\mathsf{T}$, respectively. Hence a complex general solution is

$$\mathbf{y}^{(h)} = c_1\begin{bmatrix} 2 \\ i \end{bmatrix}e^{2it} + c_2\begin{bmatrix} 2 \\ -i \end{bmatrix}e^{-2it}.$$

By Euler's formula this becomes, in components,

$$\begin{aligned} y_1^{(h)} &= (2c_1 + 2c_2) \cos 2t + i(2c_1 - 2c_2) \sin 2t \\ y_2^{(h)} &= \ \ (ic_1 - ic_2) \cos 2t + \ \ (-c_1 - c_2) \sin 2t. \end{aligned}$$

Setting $A = c_1 + c_2$ and $B = i(c_1 - c_2)$, we can write

$$\begin{aligned} y_1^{(h)} &= 2A \cos 2t + 2B \sin 2t \\ y_2^{(h)} &= \ \ B \cos 2t - \ \ A \sin 2t. \end{aligned}$$

Before we can consider the initial conditions, we must determine a particular solution $\mathbf{y}^{(p)}$ of the given system. We do this by the method of undetermined coefficients, setting

$$\begin{aligned} y_1^{(p)} &= a_1 e^t + b_1 e^{-t} \\ y_2^{(p)} &= a_2 e^t + b_2 e^{-t}. \end{aligned}$$

Differentiation and substitution gives

$$\begin{aligned} a_1 e^t - b_1 e^{-t} &= 4a_2 e^t + 4b_2 e^{-t} + 5e^t \\ a_2 e^t - b_2 e^{-t} &= -a_1 e^t - b_1 e^{-t} - 20e^{-t}. \end{aligned}$$

Equating the coefficients of e^t on both sides, we get

$$a_1 = 4a_2 + 5, \qquad a_2 = -a_1, \qquad \text{hence} \qquad a_1 = 1, \qquad a_2 = -1.$$

Equating the coefficients of e^{-t}, we similarly obtain

$$-b_1 = 4b_2, \qquad -b_2 = -b_1 - 20, \qquad \text{hence} \qquad b_1 = -16, \qquad b_2 = 4.$$

Hence a general solution of the given nonhomogeneous system is

$$\begin{aligned} y_1 &= 2A \cos 2t + 2B \sin 2t + e^t - 16e^{-t} \\ y_2 &= \ \ B \cos 2t - \ \ A \sin 2t - e^t + 4e^{-t}. \end{aligned}$$

From this and the initial conditions we obtain

$$y_1(0) = 2A + 1 - 16 = 1, \qquad y_2(0) = B - 1 + 4 = 0.$$

The solution is $A = 8$, $B = -3$. This gives the *answer* (the solution of the initial value problem)

$$\begin{aligned} y_1 &= 16 \cos 2t - 6 \sin 2t + e^t - 16e^{-t} \\ y_2 &= -3 \cos 2t - 8 \sin 2t - e^t + 4e^{-t}. \end{aligned}$$

15. $y_1 = e^{2t} + 4t + 4 - 6e^t$, $y_2 = \frac{2}{3}e^{2t} + 9t + 1 - 6e^t + \frac{7}{3}e^{-t}$.

18. The equations are

(a) $$I_1' = -2I_1 + 2I_2 + 440 \sin t$$

and

$$8I_2 + 2\int I_2\,dt + 2(I_2 - I_1) = 0.$$

Thus

$$I_2 = -0.25\int I_2\,dt + 0.25(I_1 - I_2),$$

which, upon differentiation and insertion of I_1' from (a) and simplification, gives

(b) $$I_2' = -0.4I_1 + 0.2I_2 + 88 \sin t.$$

The general solution of the homogeneous system is as in Prob. 17, and the method of undetermined coefficients gives, as a particular solution,

$$-\frac{1}{3}\begin{bmatrix} 352 \\ 44 \end{bmatrix} \cos t + \frac{1}{3}\begin{bmatrix} 616 \\ 132 \end{bmatrix} \sin t.$$

20. $\mathbf{A} = \begin{bmatrix} -3 & 1.25 \\ 1 & -1 \end{bmatrix}$,

$$\mathbf{J} = -\frac{125}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5t} - \frac{125}{21}\begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{-3.5t} + \frac{500}{7}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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11. $y_1 = c_1e^{-3t} + c_2e^{3t}$, $y_2 = -3c_1e^{-3t} + 3c_2e^{3t}$. Saddle point.

12. The matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

has the eigenvalues 2 and 1 with eigenvalues $[1, 0]^T$ and $[0, 1]^T$, respectively. Hence, a general solution is

$$\mathbf{y} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t}.$$

Since $p = 3$, $q = 2$, $\Delta = p^2 - 4q = 9 - 8 = 1 > 0$, the critical point is an unstable node.

13. $y_1 = e^{-2t}(c_1 \sin t + c_2 \cos t)$, $y_2 = \frac{1}{5}e^{-2t}(3c_1 \sin t + c_1 \cos t + 3c_2 \cos t - c_2 \sin t)$. Asymptotically stable spiral point.

14. Eigenvalues -1, 6. Eigenvectors $[1 \quad -1]^{\mathsf{T}}$, $[4 \quad 3]^{\mathsf{T}}$. The corresponding general solution is

$$y_1 = c_1e^{-t} + 4c_2e^{6t}, \quad y_2 = -c_1e^{-t} + 3c_2e^{6t}.$$

The critical point at (0, 0) is a saddle point, which is always unstable.

16. Eigenvalues $-2i$ and $2i$. $y_1 = c_1 \sin 2t + c_2 \cos 2t$, $y_2 = c_1 \cos 2t - c_2 \sin 2t$. Center, which is always stable.

18. Saddle point; see Table 4.1 **(b)** in Sec. 4.4.

20. $y_1 = 3 \cosh t + \sinh t + c_1t + c_2$, $y_2 = -3 \cosh t + \sinh t + c_1 - c_1t - c_2$

21. $y_1 = c_1e^{-2t} + c_2e^{2t} - 4 - 8t^2$, $y_2 = -c_1e^{-2t} + c_2e^{2t} - 8t$.

22. The matrix

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

has the eigenvalues 3 and -1 with eigenvectors $[1 \quad 2]^\mathsf{T}$ and $[1 \quad -2]^\mathsf{T}$, respectively. A particular solution of the nonhomogeneous system is obtained by the method of undetermined coefficients. We set

$$y_1^{(p)} = a_1 \cos t + b_1 \sin t, \qquad y_2^{(p)} = a_2 \cos t + b_2 \sin t.$$

Differentiation and substitution gives the following two equations, where $c = \cos t$ and $s = \sin t$.

$$(1) \qquad -a_1 s + b_1 c = a_1 c + b_1 s + a_2 c + b_2 s + s$$

$$(2) \qquad -a_2 s + b_2 c = 4a_1 c + 4b_1 s + a_2 c + b_2 s.$$

From the cosine terms and the sine terms in (1) we obtain

$$b_1 = a_1 + a_2, \qquad -a_1 = b_1 + b_2 + 1.$$

Similarly from (2),

$$b_2 = 4a_1 + a_2, \qquad -a_2 = 4b_1 + b_2.$$

Hence $a_1 = -0.3$, $a_2 = 0.4$, $b_1 = 0.1$, $b_2 = -0.8$. This gives the *answer*

$$y_1 = c_1 e^{3t} + c_2 e^{-t} - 0.3 \cos t + 0.1 \sin t$$
$$y_2 = 2c_1 e^{3t} - 2c_2 e^{-t} + 0.4 \cos t - 0.8 \sin t.$$

24. The balance equations are

$$y_1' = -\frac{16}{400} y_1 + \frac{6}{200} y_2$$
$$y_2' = \frac{16}{400} y_1 - \frac{16}{200} y_2$$

Note that the denominators differ. Note further that the outflow to the right must be included in the balance equation for T_2. The matrix is

It has the eigenvalues -0.1 and -0.02 with the eigenvectors $[-1, 2]^T$ and $[3, 2]^T$ respectively. The initial condition is $y_1(0) = 120$, $y_2(0) = 0$. This gives the solution

$$y_1 = 90e^{-0.02t} + 30e^{-0.10t}$$
$$y_2 = 60e^{-0.02t} - 60e^{-0.10t}$$

25. $I_1' + 2.5(I_1 - I_2) = 169 \sin 2t$, $2.5(I_2' - I_1') + 25I_2 = 0$,
$I_1(t) = (31.8 + 58.3t)\mathrm{e}^{-5.0t} - 31.8 \cos(2.0t) + 50.2 \sin(2.0t)$,
$I_2(t) = (-8.44 - 58.3t)\mathrm{e}^{-5.0t} + 8.04 \sin(2.0t) + 8.44 \cos(2.0t)$

26. The ODEs of the system are

$$I_1' - I_2' + 5I_1 = 0$$
$$I_2 - I_1 + 1.25I_2' = 0,$$

written in the usual form

$$\mathbf{I}' = \begin{bmatrix} A - B & -A \\ A & -A \end{bmatrix} \mathbf{I}$$

where $A = R/L = 0.8$ and $B = 1/(RC) = 5$. A general solution is

$$I = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 4 \\ -1 \end{bmatrix} e^{-4t}.$$

The initial conditions give $c_1 = -\frac{1}{3}$ and $c_2 = \frac{1}{3}$. Hence the particular solution satisfying the initial conditions is

$$\begin{aligned} I_1 &= -\tfrac{1}{3}e^{-t} + \tfrac{4}{3}e^{-4t} \\ I_2 &= \tfrac{4}{3}e^{-t} - \tfrac{1}{3}e^{-4t}. \end{aligned}$$

28. $\cos y_2 = 0$ when $y_2 = (2n + 1)\pi/2$, where n is any integer. This gives the location of the critical points, which lie on the y_2-axis $y_1 = 0$ in the phase plane.

For $(0, \frac{1}{2}\pi)$ the transformation is $y_2 = \frac{1}{2}\pi + \tilde{y}_2$. Now

$$\cos y_2 = \cos(\tfrac{1}{2}\pi + \tilde{y}_2) = -\sin \tilde{y}_2 \approx -\tilde{y}_2.$$

Hence the linearized system is

$$\begin{aligned} \tilde{y}_1' &= -\tilde{y}_2 \\ \tilde{y}_2' &= 3\tilde{y}_1. \end{aligned}$$

For its matrix

$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

we obtain $\tilde{p} = 0, \tilde{q} = 3 > 0$, so that this is a center. Similarly, by periodicity, the critical points at $(4n + 1)\pi/2$ are centers.

For $(0, -\frac{1}{2}\pi)$ the transformation is $y_2 = -\frac{1}{2}\pi + \tilde{\tilde{y}}_2$. From

$$\cos y_2 = \cos(-\tfrac{1}{2}\pi + \tilde{\tilde{y}}_2) = \sin \tilde{\tilde{y}}_2 \approx \tilde{\tilde{y}}_2$$

we obtain the linearized system

$$\begin{aligned} \tilde{\tilde{y}}_1' &= \tilde{\tilde{y}}_2 \\ \tilde{\tilde{y}}_2' &= 3\tilde{\tilde{y}}_1 \end{aligned}$$

with matrix

$$\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$$

for which $\tilde{\tilde{q}} = -3 < 0$, so that this point and the points with $y_2 = (4n - 1)\pi/2$ on the y_2-axis are saddle points.

29. $(\frac{n\pi}{2}, 0)$, Center when n is odd and saddle point when n is even.

30. Critical points at $(0, 0)$ and $(0, -1)$. The linearized systems are:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -2y_1 \end{aligned}$$

and

$$\begin{aligned} y_1' &= -y_2 \\ y_2' &= -2y_1 \end{aligned}$$