106033233 周聖諺 Randomized Algo Mid 2

(一)

1、

$X \sim Bin(n, p)$, $X = \sum_{i=1}^{n} X_i$; $X_i \sim Ber(p)$

$M_X(t) = E[e^{tX}] = \prod_{i=1}^{n} E[e^{tX_i}] = \prod_{i=1}^{n} (e^{t \cdot 1} p + e^{t \cdot 0} \cdot (1-p))$

$= \prod_{i=1}^{n} (1 + (e^t - 1)p) = (1 + (e^t - 1)p)^n$ #

2、

$n(1+(e^t-1)p)^{n-1} \times e^t p$

$= n(1)^{n-1} \times p = np$

$Y \sim Po(\lambda)$

$\lambda e^t - \lambda$

$= (e^t - 1)\lambda$

$M_Y(t) = E[e^{tY}] = \sum_{k=0}^{\infty} e^{tk} \cdot \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!}$

$= e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$E[Y] = M_Y'(0) = \lambda e^t \cdot e^{\lambda(e^t - 1)} \Rightarrow \lambda \cdot 1 = \lambda$

$\Rightarrow$ (a) : $M_Y(t) = e^{\lambda(e^t - 1)}$ # (b) λ #

(二)

1. $I \sim Ber(p)$ $Pr(I=1) = p$

(a) $M_I(t) = E[e^{tI}] = e^t \cdot p + e^{t \cdot 0} \cdot (1-p) = e^t \cdot p + (1-p)$

$= 1 + p(e^t - 1)$ #

$E[(X - E[X])^2]$

$= X^2 - E[X]^2$

(b) $M_I'(t) = \frac{d}{dt} 1 + pe^t - p = pe^t$ $M_I'(0) = f'(0) = p$ #

$M_I''(t) = \frac{d}{dt} pe^t = pe^t$ $M_I''(0) = f''(0) = p$ #

(c) $f''(0) - (f'(0))^2 = E[I^2] - (E[I])^2 = Var[X] = p - p^2$ #

It means the variance of I

II-2,

$$E[e^{tX}] = (0.2 + 0.3e^t + 0.5e^{4t})(0.4e^{2t} + 0.6e^{3t})$$

$$= 0.08e^{2t} + 0.12e^{3t} + 0.12e^{3t} + 0.18e^{4t} + 0.2e^{6t} + 0.3e^{7t}$$

$$= 0.08e^{2t} + 0.24e^{3t} + 0.18e^{4t} + 0.2e^{6t} + 0.3e^{7t}$$

(a)

Y is a RV that

$Pr(Y=2) = 0.08$, $Pr(Y=3) = 0.24$,

$Pr(Y=4) = 0.18$, $Pr(Y=6) = 0.2$,

$Pr(Y=7) = 0.3$, $Pr(Y=\text{others}) = 0$ #

$$E[e^{tY}] = \sum_k e^{tk} \cdot Pr(Y=k) = E[e^{tX}]$$

(b)

$$E[Y] = E[X] = \sum_{k=1}^{7} k \cdot Pr(Y=k)$$

$$= 2 \times 0.08 + 3 \times 0.24 + 4 \times 0.18 + 6 \times 0.2 + 7 \times 0.3$$

$$= 4.9 \ \#$$

$$Var[X] = E[X^2] - E[X]^2 = 27.26 - (4.9)^2$$

$$= \frac{13}{4} = 3.25 \ \#$$

(c) $Pr(X \geq 3) = 1 - Pr(X < 3) = 1 - 0.08 = 0.92$ #

$$E[X^2] = E[Y^2] = 4 \times 0.08 + 9 \times 0.24 + 16 \times 0.18 + 36 \times 0.2 + 49 \times 0.3 = 27.26$$

II-3.

For sum of independent Poisson RV: Y, Z

Let $X = Y + Z$, for $Y \sim Po(\mu_y)$, $Z \sim Po(\mu_z)$

we get $X \sim Po(\mu_y + \mu_z) = Po(3)$

$$Pr(Z=0) = \frac{e^{-3} 3^0 \to 1}{0! \to 1} = e^{-3} \#$$

Proof: $Pr(X = k) = \sum_{r=0}^{k} Pr(Y = r \cap Z = k-r)$

$$= \sum_{r=0}^{k} \frac{e^{-\mu_y} \mu_y^r}{r!} \times \frac{e^{-\mu_z} \mu_z^{(k-r)}}{(k-r)!} = \frac{e^{-(\mu_y+\mu_z)}}{k!} \cdot \sum_{r=0}^{k} \frac{k!}{r!(k-r)!} \mu_y^r \mu_z^{(k-r)}$$

$$= \frac{e^{-(\mu_y+\mu_z)}}{k!} \times (\mu_y + \mu_z)^k = \frac{e^{-3} \cdot 3^k}{k!}$$

II-4.

Chernoff bound: $Pr(Y \le X) = Pr(e^{tY} \ge e^{tX})$

$$\le \frac{E[e^{tY}]}{e^{tX}}, \text{ for } t<0$$

$$\frac{E[e^{tY}]}{e^{tX}} = \frac{e^{\mu(e^t-1)}}{e^{tX}} \le e^{-\lambda}\left(\frac{e\lambda}{X}\right)^X \quad \#$$

For chernoff bound of Poisson:

For $X<\mu$, $Pr(Y \le X) = e^{-\mu}\left(\frac{e\mu}{X}\right)^X$

$$Pr(e^{tY} \ge e^{tX}) \le \frac{E[e^{tY}]}{e^{tX}} = \frac{e^{\mu(e^t-1)}}{e^{tX}}$$

$$= \frac{e^{\mu(\frac{X}{\mu}-1)}}{(\frac{X}{\mu})^X} = \frac{e^{X-\mu}}{(\frac{X}{\mu})^X} = e^{-\mu}\left(\frac{e\mu}{X}\right)^X \quad \#$$

$$\text{Min}(\mu(e^t-1)-tX) = \mu\left(\frac{X}{\mu}-1\right) - X\ln\frac{X}{\mu}$$

$$= X - \mu - X(\ln X - \ln\mu)$$

$$\frac{d}{dt}\mu(e^t-1)-tX = \mu e^t - X = 0$$

$$t = \ln\left(\frac{X}{\mu}\right)$$

II - 5. Y_i is a RV, $Y_i \sim Po(m)$

$$E[f(Y_1^{(m)} \cdots Y_n^{(m)})] = \sum_{k=1}^{\infty} E[f(Y_1^{(m)} \cdots Y_n^{(m)}) \mid \sum_{i=1}^{n} Y_i^{(m)} = k]$$

$$\times Pr(\sum_{i=1}^{n} Y_i^{(m)} = k)$$

$$= \sum_{k=1}^{\infty} E[f(X_1^{(k)} \cdots X_n^{(k)})] \cdot Pr(\sum_{i=1}^{n} Y_i^{(m)} = k)$$

$$\geq \sum_{k \leq m} E[f(X_1^{(k)} \cdots X_n^{(k)})] \times Pr(\sum_{i=1}^{n} Y_i^{(m)} = k)$$

$$\geq E[f(X_1^{(m)} \cdots X_n^{(m)})] \sum_{k \leq m} Pr(\sum_{i=1}^{n} Y_i^{(m)} = k)$$

$$= E[f(X_1^{(m)} \cdots X_n^{(m)})] \, Pr(\sum_{i=1}^{n} Y_i^{(m)} \leq m)$$

$$= E[f(X_1^{(m)} \cdots X_n^{(m)})] \, Pr(Y \leq m), \quad \text{Let } Y = \sum_{i=1}^{n} Y_i^{(m)}$$

$$E[f(Y_1^{(m)} \cdots Y_n^{(m)})] \geq E[f(X_1^{(m)} \cdots X_n^{(m)})] \, Pr(Y \leq m)$$

$$\geq E[f(X_1^{(m)} \cdots X_n^{(m)})] \cdot \frac{1}{2}$$

$$\Rightarrow E[f(X_1^{(m)} \cdots X_2^{(m)})] \leq 2 E[f(Y_1^{(m)} \cdots Y_n^{(m)})]$$

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