CS2336 Discrete Mathematics

Homework 1
Tutorial: October 28, 2020
Exam 1: November 2, 2020

Problems marked with * will be explained in the tutorial.

1. Determine which of the following statements are propositions and which are non-propositions. If a statement is a proposition, then decide the truth value of it.
   (a) What time is it?
   (b) $4 + x = 5$.
   (c) The moon is made of green cheese.
   (d) $2 + 3 = 5$
   (e) $5 + 7 = 10$.
   (f) This sentence is false.
2. What is the negation of each of theses propositions?
   (a) Mei has an MP3 player.
   (b) There is no pollution in Taipei.
   (c) $2 + 1 = 3$.
   (d) The summer in Kenting is hot and sunny.
3. Use a truth table to verify the distributive law
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r).$$
4. (*) Consider the expression $(p \wedge q) \vee \neg(p \to q)$. In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.
5. A compound proposition is *satisfiable* if there is a way to assign truth values to each of the propositions, such that the truth value of the compound proposition is true. In other words, a compound proposition is satisfiable if and only if it is not a contradiction. For example, the compound proposition $p \wedge \neg q$ is satisfiable, as setting $p = \mathtt{T}$ and $q = \mathtt{F}$ will make the $p \wedge \neg q$ true. In contrast, $p \wedge (q \wedge \neg q)$ is not satisfiable.

   Determine whether each of these compound propositions is satisfiable.
   (a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
   (b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$
6. (*) The following exercises involve the logical operator $\uparrow$ (read as NAND). The proposition $p \uparrow q$ is true when either p, or q, or both, are false.
   (a) Show that $p \uparrow q \equiv \neg(p \wedge q)$.

(b) Show that $p \uparrow p \equiv \neg p$.

(c) Express $p \wedge q$ by using only $\uparrow$ operators.

(d) Express $p \vee q$ by using only $\uparrow$ operators.

7. What is wrong with this argument? Let $H(x)$ be "x is happy." Given that $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore Lola is happy.

8. (*) What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$ it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

9. Which rule of inference is used in each of these arguments?

   (a) Alice is a mathematics major. Therefore, Alice is either a mathematics major and a computer science.

   (b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

   (c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

   (d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

   (e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

10. Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

(1) $\forall x(P(x) \vee Q(x))$	Premise.
(2) $P(c) \vee Q(c)$	Universal instantiation from (1).
(3) $P(c)$	Simplification from (2).
(4) $\forall x P(x)$	Universal generalization from (3).
(5) $Q(c)$	Simplification from (2).
(6) $\forall x Q(x)$	Universal generalization from (5).
(7) $\forall x P(x) \vee \forall Q(x)$	Conjunction from (4) and (6).

11. (*) Determine whether these are valid arguments.

   (a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

   (b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

12. To describe the various restaurants in the city, we let p denote the statement "The food is good," q denote the statement "The service is good," and r denote the statement "The rating is three-star." Write the following statements in symbolic form.

   (a) Either the food is good, or the service is good, or both.

   (b) Either the food is good, or the service is good, but not both.

(c) If both the food and services are good, then the rating will be three-star.

(d) It is not true that a three-star rating always means good food and good service.

13. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

    (a) The product of two negative real numbers is positive.

    (b) The difference of a real number and itself is zero.

    (c) A negative real number does not have a square root that is a real number.

14. (*) Determine the truth value of each of these statements if the domain for all variables consists of all integers.

    (a) $\forall n \exists m\ (n^2 < m)$

    (b) $\exists n \forall m\ (n < m^2)$

    (c) $\forall n \exists m\ (n + m = 0)$

    (d) $\exists n \forall m\ (nm = m)$

    (e) $\exists n \exists m\ (n^2 + m^2 = 5)$

    (f) $\exists n \exists m\ (n^2 + m^2 = 6)$

    (g) $\exists n \exists m\ (n + m = 4 \wedge n - m = 1)$

    (h) $\exists n \exists m\ (n + m = 4 \wedge n - m = 2)$

15. Express the following definition of *limit*,

$$\lim_{x \to x_0} f(x) = L,$$

using predicates, quantifiers, logical connectives, and mathematical operators:

For every $\varepsilon > 0$, there exists $\delta > 0$, such that for every x with $0 < |x - x_0| < \delta$, $|f(x) - L| < \varepsilon$.

16. In the following figure, each English letter represents a distinct digit. Find out the digit for each letter so that the addition is correct.

$$\begin{array}{r} \texttt{SEND} \\ +\ \texttt{MORE} \\ \hline \texttt{MONEY} \end{array}$$

Figure 1: The figure for Question 16.

17. Figure 2 shows a long division process that a number is properly divided by some number y. In the figure, each * symbol represents a single digit and y is a positive integer (may have multiple digits). What are the values of y and each * symbol?

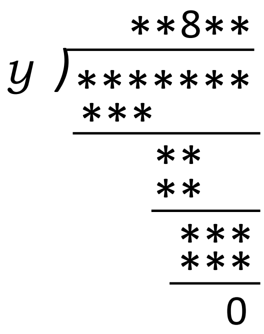

Figure 2: A division process for Question 17.

18. (*) Four cards are displayed on the table as shown in Figure 3. It is known that for each card, both faces are drawn with geometric shapes, such that one is solid while the other is empty. For instance, Card 1 shows a solid circle, which implies its other face will be some empty shape. Similarly, Card 2 shows an empty square, which implies its other face will be some solid shape.

    Peter took a look at the other face of each card, and said, “if one face is drawn with a solid circle, then the other face must be drawn with an empty triangle”.

    (a) You want to double check about Peter’s claim. One way is to look at the other face of every card. However, you want to save time. Is it possible to check only some (but not all) of these cards, so that you can be 100% sure that Peter’s claim is correct?

    (b) What is the minimum number of cards you need to check?

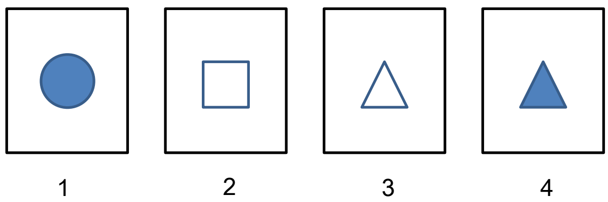

Figure 3: The four cards for Question 18.

19. Determine the answers of the following questions so that all can be answered correctly.

    Q1. Which is the first question where (c) is the correct answer?

    (a) Q3
    (b) Q4
    (c) Q1
    (d) Q2

    Q2. Which is the first question where (a) is the correct answer?

    (a) Q4
    (b) Q2
    (c) Q3
    (d) Q1

    Q3. Which is the first question where (d) is the correct answer?

(a) Q1
(b) Q2
(c) Q4
(d) Q3

Q4. Which is the first question where (b) is the correct answer?

(a) Q2
(b) Q4
(c) Q3
(d) Q1

20. (*) (Adapted from a logical puzzle in an online competition)

Raymond is visiting the famous country, *Pureland*, where citizens there are either honest (always tell the truth) or dishonest (always lie). Raymond sees three people, let us identify them as A, B, and C, and chats with them. Suddenly, one of them said "A and B are liars", and then another one of them said "A and C are liars."

How many liars are there among these three people?

21. (Extremely Challenging)

Two super smart boys, Sam and Peter, are present in a room. Teacher Hans goes to them, one by one, secretly telling each of them something. Now, Teacher Hans says: I have chosen two distinct integers, x and y, such that $1 < x < y$ and $x + y \leq 65$. I have just told Sam the sum $x + y$, and told Peter the product xy.

Next, the boys begin the following interesting conversation:

Peter: I don't know the numbers x and y.
Sam: I already knew that you didn't know.
Peter: Oh, now I know x and y.
Sam: Oh, I also know x and y now.

What are the numbers x and y?