1. Gamma Function

We will prove that the improper integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

exists for every $x > 0$. The function $\Gamma(x)$ is called the Gamma function. Let us recall the comparison test for improper integrals.

Theorem 1.1. (Comparison Test for Improper Integral of Type I) Let $f(x), g(x)$ be two continuous functions on $[a, \infty)$ such that $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

(1) If $\int_a^\infty g(x)dx$ is convergent, so is $\int_a^\infty f(x)dx$.
(2) If $\int_a^\infty f(x)dx$ is divergent to infinity, so is $\int_a^\infty g(x)dx$.

Theorem 1.2. (Limit Comparison Test) Let $f(x), g(x)$ be two nonnegative continuous functions on $[a, \infty)$. Suppose that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \text{ with } L > 0.$$

Then both $\int_a^\infty g(x)dx$ and $\int_a^\infty f(x)dx$ are convergent or divergent.

Lemma 1.1. For every $s > 0$, the improper integral $\int_0^\infty e^{-st}dt$ converges.

Proof. Let us compute

$$\int_0^N e^{-st}dt = \frac{e^{-sM} - 1}{-s}.$$

By definition,

$$\int_0^\infty e^{-st}dt = \lim_{N\to\infty} \int_0^N e^{-st}dt = \lim_{N\to\infty} \frac{e^{-sM} - 1}{-s} = \frac{1}{s}.$$

The limit exists and hence the improper integral converges. □

Now let us study the case when $x \geq 1$.

Lemma 1.2. Let n be a natural number. Then

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = 0.$$

Proof. By L' Hospital rule,

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{(n-1)t^{n-2}}{\frac{1}{2}e^{\frac{1}{2}t}}.$$

Since t^{n-1} is a polynomial of degree $n-1$, we know

$$\frac{d^n}{dt^n} t^{n-1} = 0.$$

Inducitvely, we find

$$\lim_{t\to\infty} \frac{t^{n-1}}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{0}{\left(\frac{1}{2}\right)^n e^{\frac{1}{2}t}} = 0.$$

□

By the definition of limit, we choose $\epsilon = 1$, there exists $M > 0$ such that for all $t \geq M$

$$\left| \frac{t^{n-1}}{e^{\frac{1}{2}t}} \right| < \epsilon = 1.$$

Hence for $t \geq M$, $0 \leq t^{n-1} < e^{\frac{1}{2}t}$. This implies that for $t \geq M$,

$$0 \leq e^{-t}t^{n-1} \leq e^{-t} \cdot e^{\frac{1}{2}t} = e^{-\frac{1}{2}t}. \tag{1.1}$$

Lemma 1.1 implies that $\int_0^\infty e^{-\frac{1}{2}t}dt$ is convergent. (1.1) and Comparison test implies that $\int_0^\infty e^{-t}t^{n-1}dt$ is convergent for every $n \in \mathbb{N}$.

Let $x \geq 1$ be any real number. Let $[x]$ be the largest integer so that $[x] \leq x < [x] + 1$. Then for $t \geq 0$,

$$0 \leq e^{-t}t^{x-1} \leq e^{-t}t^{[x]}. \tag{1.2}$$

Since $\int_0^\infty e^{-t}t^{[x]}dx$ is convergent, by comparison test and (1.2), we find $\int_0^\infty e^{-t}t^{x-1}dt$ is convergent.

Now let us study the case when $0 < x < 1$. Then we know

$$\frac{1}{e^{\frac{1}{2}t}} \leq \frac{t^{x-1}}{e^{\frac{1}{2}t}} \leq \frac{t}{e^{\frac{1}{2}t}}.$$

We know that

$$\lim_{t\to\infty} \frac{t}{e^{\frac{1}{2}t}} = \lim_{t\to\infty} \frac{1}{e^{\frac{1}{2}t}} = 0.$$

By the Sandwich principle,

$$\lim_{t\to\infty} \frac{t^{x-1}}{e^{\frac{1}{2}t}} = 0$$

for $0 < x < 1$. (Of course, this statement is true for all $x > 0$.) This implies that

$$0 \leq e^{-t}t^{x-1} \leq e^{-\frac{1}{2}t}, \quad t \geq 1.$$

By comparison test, $\int_1^\infty e^{-t}t^{x-1}dt$ is convergent for $0 < x < 1$. Now, we need to verify that

$$\int_0^1 e^{-t}t^{x-1}dt = \int_0^1 \frac{e^t}{t^{1-x}}dt$$

is convergent. Notice that $\lim_{t\to 0} \frac{e^{-t}}{t^{1-x}} = \infty$. Hence the integral $\int_0^1 e^{-t}t^{x-1}dt$ is a type II improper integral.

Theorem 1.3. Let $f, g \in C(a, b]$ and $0 \leq f(x) \leq g(x)$ for all $a < x \leq b$.

(1) If $\int_a^b g(x)dx$ is convergent, so is $\int_a^b f(x)dx$.

(2) If $\int_a^b f(x)dx$ is divergent to infinity, so is $\int_a^b g(x)dx$.

Note that

$$0 < e^{-t}t^{x-1} \leq et^{x-1}.$$

We know that

$$\begin{aligned}\int_0^1 t^{x-1}dt &= \lim_{b\to 0}\int_b^1 t^{x-1}dt \\ &= \lim_{b\to 0}\left.\frac{t^x}{x}\right|_b^1 \\ &= \lim_{b\to 0}\left(\frac{1}{x}-\frac{b^x}{x}\right) \\ &= \frac{1}{x}.\end{aligned}$$

By the comparison test, $\int_0^1 e^{-t}t^{x-1}dt$ is convergent.

Theorem 1.4. For every $x > 0$, the improper integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

is convergent.

Proposition 1.1. For $x > 0$,

$$\Gamma(x+1) = x\Gamma(x).$$

Proof. For every $N > 0$, using integration by parts,

$$\begin{aligned}\int_0^N e^{-t}t^x dt &= -t^x e^{-t}\Big|_0^N + x\int_0^N e^{-t}t^{x-1}dt \\ &= -N^x e^{-N} + x\int_0^N e^{-t}t^{x-1}dt.\end{aligned}$$

We have seen that $\lim_{N\to\infty} N^x e^{-N} = \lim_{N\to\infty}\frac{N^x}{e^N} = 0$ for all $x > 0$. Then

$$\begin{aligned}\Gamma(x+1) &= \lim_{N\to\infty}\int_0^N e^{-t}t^x dt \\ &= \lim_{N\to\infty}(-N^x e^{-N} + x\int_0^N e^{-t}t^{x-1}dt) \\ &= x\lim_{N\to\infty}\int_0^N e^{-t}t^{x-1}dt \\ &= x\Gamma(x).\end{aligned}$$

□

Corollary 1.1. For every $n \geq 0$, $\Gamma(n+1) = n!$.

Proof. We know $\int_0^\infty e^{-t}dt = 1$. Hence $\Gamma(1) = 1$. We can prove the formula by induction. Assume that the statement is true for some nonnegative integer k, i.e. $\Gamma(k+1) = k!$. By the previous proposition,

$$\Gamma(k+2) = \Gamma((k+1)+1) = (k+1)\Gamma(k+1) = (k+1)\cdot k! = (k+1)!.$$

□

Proposition 1.2. For any $s > 0$, $x > 0$, we have

$$\int_0^\infty e^{-st} t^{x-1} dt = \frac{\Gamma(x)}{s^x}.$$

Proof. This formula can be proved by using substitution $u = st$. □

Example 1.1.

$$\int_0^\infty e^{-st}(2 - 3t + 5t^2) dt = \frac{2}{s} - \frac{3}{s^2} + \frac{10}{s^3}.$$

This can be proved by the previous proposition.

Here comes a question, if $f(t)$ is a continuous function on $[0, \infty)$ such that

$$\int_0^\infty e^{-st} f(t) dt = \frac{2}{s} - \frac{3}{s^2} + \frac{10}{s^3},$$

what can you say about $f(t)$? Let us introduction the notion of Laplace transformation.

2. Laplace Transformation, its inverse transformation, and linear homogeneous o.d.e of constant coefficients

Definition 2.1. Let $f(t)$ be a continuous function on $[0,\infty)$. Suppose $\displaystyle\int_0^\infty e^{-st}f(t)dt$ converges for $s\geq a$ for some $a\in\mathbb{R}$. Denote

$$\mathfrak{L}(f)(s)=\int_0^\infty e^{-st}f(t)dt$$

called the Laplace transform of f.

Now, let us assume that the Laplace transform of f_1,f_2 exist on some domain $D\subset\mathbb{R}$ where f_1,f_2 are continuous functions on $[0,\infty)$. For any real numbers a_1,a_2, we know that for any $s\in D$,

$$\begin{aligned}\mathfrak{L}(a_1f_1+a_2f_2)(s)&=\int_0^\infty (a_1f_1(t)+a_2f_2(t))e^{-st}dt\\&=\lim_{M\to\infty}\int_0^M (a_1f_1(t)+a_2f_2(t))e^{-st}dt\\&=\lim_{M\to\infty}\left(a_1\int_0^M f_1(t)e^{-st}dt+a_2\int_0^M f_2(t)e^{-st}dt\right)\\&=a_1\lim_{M\to\infty}\int_0^M f_1(t)e^{-st}dt+a_2\lim_{M\to\infty}\int_0^M f_2(t)e^{-st}dt\\&=a_1\mathfrak{L}(f_1)(s)+a_2\mathfrak{L}(f_2)(s).\end{aligned}$$

Proposition 2.1. Suppose that f_1,f_2 are two continuous functions on $[0,\infty)$ such that their Laplace transform exist on some domain $D\subset\mathbb{R}$ for s. Then

$$\mathfrak{L}(a_1f_1+a_2f_2)(s)=a_1\mathfrak{L}(f_1)(s)+a_2\mathfrak{L}(f_2)(s),\quad s\in D,$$

for any $a_1,a_2\in\mathbb{R}$. (We say that the Laplace transform is a linear transformation.)

Theorem 2.1. Suppose f_1,f_2 are two continuous functions on $[0,\infty)$ such that their Laplace transform exist. If $\mathfrak{L}(f_1)=\mathfrak{L}(f_2)$, then $f_1=f_2$.

Hence if $g(s)=\mathfrak{L}(f)(s)$, we denote

$$f(t)=\mathfrak{L}^{-1}(g)(t)$$

called then Laplace inverse transform.

Proposition 2.2. Suppose $f(t)$ is a smooth function on $[0,\infty)$ ($f^{(k)}(t)$ exists for all $k\geq 1$. Set $f^{(0)}(t)=f(t)$.) Assume that $\lim_{t\to\infty}f^{(k)}(t)e^{-st}=0$ for all $k\geq 0$. Then

$$\mathfrak{L}(f')(s)=-f(0)+s\mathfrak{L}(f)(s).$$

Proof. Using integration by parts,

$$\begin{aligned}\int_0^N e^{-st}f'(t)dt&=f(t)e^{-st}\big|_0^N+s\int_0^N e^{-st}f(t)dt\\&=f(N)e^{-sN}-f(0)+s\int_0^N e^{-st}f(t)dt.\end{aligned}$$

By assumption, $\lim_{N\to\infty} f(N)e^{-sN} = 0$ (consider $k = 0$.) Hence

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)dt &= \lim_{N\to\infty} \int_0^N e^{-st} f'(t)dt \\ &= \lim_{N\to\infty} \left(f(N)e^{-sN} - f(0) + s\int_0^N e^{-st} f(t)dt \right) \\ &= -f(0) + s\lim_{N\to\infty} \int_0^N e^{-st} f(t)dt \\ &= -f(0) + s\int_0^\infty e^{-st} f(t)dt.\end{aligned}$$

□

Corollary 2.1. Under the same assumption as above, we have

$$\mathfrak{L}(f^{(n)})(s) = s^n F(s) - \sum_{k=0} s^{n-k-1} f^{(k)}.$$

When $n = 2$, we have $\mathfrak{L}(f'')(s) = s^2F(s) - sf'(0) - f(0)$.

Now let us use Laplace transform to solve ordinary differential equation with constant coefficients.

Example 2.1. Solve for $y' + 2y = 0$ with initial condition $y(0) = 1$.

Take Laplace transform, we obtain

$$\mathfrak{L}(y' + 2y)(s) = \mathfrak{L}(y')(s) + 2\mathfrak{L}(y)(s) = 0.$$

Let $g(s) = \mathcal{L}(y)(s)$. Using the previous proposition,

$$\mathfrak{L}(y')(s) = -y(0) + s\mathfrak{L}(y)(s) = -1 + sg(s).$$

Hence $(-1 + sg(s)) + 2g(s) = 0$. This implies that $g(s) = \dfrac{1}{s+2}$. Then

$$y = \mathfrak{L}^{-1}(g)(t) = e^{-2t}.$$

Example 2.2. Solve for $y'' + y = 0$ with initial condition $y(0) = a$ and $y'(0) = b$.

Let $g(s)$ be the Laplace transform of y. Then

$$\mathfrak{L}(y'')(s) + g(s) = 0.$$

On the other hand, $\mathfrak{L}(y'')(s) = s^2g(s) - sy(0) - y'(0) = s^2g(s) - as - b$. This implies that

$$(s^2 + 1)g(s) = as + b \Longrightarrow g(s) = a\frac{s}{s^2+1} + b\frac{1}{s^2}.$$

We know $\dfrac{s}{s^2+1} = \mathfrak{L}(\cos t)(s)$, and $\dfrac{1}{s^2+1} = \mathfrak{L}(\sin t)(s)$. Hence $y(t) = a\cos t + b\sin t$.

In general, we can solve for the linear (homogeneous) differential equation of constant coefficients through Laplace transformation. A linear homogeneous differential equation of constant coefficients is a differential equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0, \tag{2.1}$$

where $a_0, \cdots, a_n \in \mathbb{R}$. If $a_n \neq 0$, we say that the equation is of order n.

Let us study the case when $n = 2$:

$$ay'' + by' + cy = 0. \tag{2.2}$$

Here $a \neq 0$. Taking the Laplace transformation of the equation, we have

$$a(s^2F(s) - sy'(0) - y(0)) + b(sF(s) - y(0)) + cF(s),$$

where $F(s) = \mathfrak{L}(y)(s)$. Therefore $F(s)$ is a rational function in s and given by

$$F(s) = \frac{As + B}{as^2 + bs + c}.$$

where $A = ay'(0)$ and $B = ay'(0) + by(0)$. We can use partial fraction expansion to decompose $F(s)$.

Definition 2.2. The polynomial $\chi(s) = as^2 + bs + c$ is called the characteristic polynomial of (2.2).

We assume that $a = 1$ and let $D = b^2 - 4c$ be the discriminant of the characteristic polynomial.

case 1: If $D > 0$, $\chi(s)$ has two distinct real roots. We denote the root of $\chi(s)$ by λ_1 and λ_2. Thus we may write

$$F(s) = C_1\frac{1}{s - \lambda_1} + C_2\frac{1}{s - \lambda_2}$$

for some $C_1, C_2 \in \mathbb{R}$. We know that

$$\mathfrak{L}(e^{at})(s) = \frac{1}{s - a}.$$

Using the inverse transform and the linearity of $\mathfrak{L}^{-1}$, we see that

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}(F)(t) \\ &= C_1\mathfrak{L}^{-1}\left(\frac{1}{s - \lambda_1}\right) + C_2\mathfrak{L}^{-1}\left(\frac{1}{s - \lambda_2}\right) \\ &= C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}. \end{aligned}$$

Hence y is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

case 2: If $D = 0$, $\chi(s)$ has one repeated root, called λ. Then we write

$$F(s) = C_1\frac{1}{s - \lambda} + C_2\frac{1}{(s - \lambda)^2}.$$

Recall that

$$\mathfrak{L}(t^ne^{\lambda t})(s) = \int_0^\infty e^{-st}t^ne^{t\lambda}dt = \frac{\Gamma(n+1)}{(s - \lambda)^{n+1}} = \frac{n!}{(s - \lambda)^{n+1}}.$$

Thus we obtain

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}(F)(t) \\ &= C_1\mathfrak{L}^{-1}\left(\frac{1}{s - \lambda}\right) + C_2\mathfrak{L}^{-1}\left(\frac{1}{(s - \lambda)^2}\right) \\ &= C_1e^{\lambda t} + C_2te^{\lambda t} \\ &= (C_1 + C_2t)e^{\lambda t}. \end{aligned}$$

case 3: If $D < 0$, by completing the square, we write

$$\chi(s) = (s - \alpha)^2 + \beta^2.$$

We write

$$F(s) = \frac{As}{(s-\alpha)^2 + \beta^2} + \frac{B}{(s-\alpha)^2 + \beta^2} = C_1\frac{s - \alpha}{(s-\alpha)^2 + \beta^2} + C_2\frac{\beta}{(s-\alpha)^2 + \beta^2}.$$

Notice that

$$\mathfrak{L}(e^{\alpha t}\cos\beta t)(s) = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad \mathfrak{L}(e^{\alpha t}\sin\beta t) = \frac{\beta}{(s-\alpha)^2+\beta^2}.$$

Thus

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}(F)(t) \\ &= C_1\mathfrak{L}^{-1}\left(\frac{s-\alpha}{(s-\alpha)^2+\beta^2}\right) + C_2\mathfrak{L}^{-1}\left(\frac{\beta}{(s-\alpha)^2+\beta^2}\right) \\ &= C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t \\ &= e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t). \end{aligned}$$

Remark. Let V be the subset of $C^2[0,\infty)$ consisting of y such that (2.2) holds. The above observation implies that V is in fact a two dimensional real vector (sub)space (of $C^2[0,\infty)$).

This method can be applied to (2.1) for any $n \in \mathbb{N}$.

Definition 2.3. The characteristic polynomial of (2.1) is defined to be

$$\chi(s) = a_n s^n + \cdots + a_1 s + a_0.$$

Taking the Laplace transformation of (2.1) and using Corollary 2.1, we find that

$$F(s) = \frac{P(s)}{\chi(s)}$$

for some polynomial $P(s) \in \mathbb{R}[s]$. Thus $F(s)$ is a rational function and we can solve for y using the partial fraction for $F(s)$. Thus we reduce our problems to the cases when $\chi(s) = (s-a)^n$ or $\chi(s) = ((s-\alpha)^2+\beta^2)^m$. When $\chi(s) = (s-a)^n$, we write

$$\begin{aligned} F(s) &= \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n} \\ &= C_0\frac{1}{s-a} + C_1\frac{1!}{(s-a)^2} + \cdots + C_{n-1}\frac{(n-1)!}{(s-a)^n}, \end{aligned}$$

where $C_{i-1} = A_i/(i-1)!$ for $i = 1, \cdots, n$. By taking the inverse transform, we obtain

$$y(t) = (C_0 + C_1 t + \cdots + C_{n-1}t^{n-1})e^{at}.$$

When $\chi(s) = ((x-\alpha)^2+\beta^2)^m$, we write

$$\begin{aligned} F(s) &= \frac{A_1 s + B_1}{(x-\alpha)^2+\beta^2} + \cdots + \frac{A_m s + B_m}{((s-\alpha)^2+\beta^2)^m} \\ &= \frac{C_1(s-a)+D_1\beta}{(x-\alpha)^2+\beta^2} + \cdots + \frac{C_m(s-a)+D_m\beta}{((s-\alpha)^2+\beta^2)^m}. \end{aligned}$$

Here C_i, D_j are constants. By finding the inverse transform of $(C_i(s-a)+D_i\beta)/(((s-\alpha)^2+\beta^2)^i)$, we obtain y.

3. Beta Function

In this section, x, y are positive real numbers. Let us define

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The function $B(x,y)$ is called the Gamma function. It follows immediately from the definition that

$$B(x,y) = B(y,x), \quad \forall x,y > 0.$$

Using the property of Gamma function: $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, we obtain:

Proposition 3.1. Suppose $p, q > 0$. Then

(1) $B(p,q) = B(p+1,q) + B(p,q+1)$.
(2) $B(p,q+1) = \frac{q}{p}B(p+1,q) = \frac{q}{p+q}B(p,q)$.

Theorem 3.1. For $x, y > 0$,

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt.$$

Let us postpone the proof of this equation.

Let us consider the substitution $t = \tan^2\theta$ with $0 \leq \theta \leq \pi/2$. Then $dt = 2\tan\theta\sec^2\theta d\theta$. Using $\sec^2\theta = \tan^2\theta + 1$, Beta function $B(x,y)$ can be rewritten as

$$\begin{aligned}
B(x,y) &= \int_0^{\frac{\pi}{2}} \frac{(\tan^2\theta)^{x-1}}{(1+\tan^2\theta)^{x+y}} \cdot 2\tan\theta\sec^2\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{\tan^{2x-2}\theta}{\sec^{2x+2y}\theta} \cdot \tan\theta\sec^2\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \tan^{2x-1}\theta \cdot \frac{1}{\sec^{2x+2y-2}\theta} d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\cos\theta}\right)^{2x-1} \cdot \cos^{2x+2y-2}\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta.
\end{aligned}$$

Using $B(y,x) = B(x,y)$, we also obtain

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\sin^{2y-1}\theta d\theta.$$

This is the second form of Beta function. Consider substitution $t = \sin^2\theta$ for $0 \leq \theta \leq \pi/2$. Then $dt = 2\sin\theta\cos\theta d\theta$, we obtain the third form of the Beta function:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

We conclude that

Theorem 3.2. The Beta function has the following forms:

(1) $B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$,

(2) $B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta d\theta$,

(3) $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

Example 3.1. Compute $\int_0^\infty e^{-x^2}dx$.

Let us make a change of variable $t = x^2$. Then $dt = 2xdx$ and thus $dx = \frac{t^{-\frac{1}{2}}dt}{2}$.The integral can be rewritten as

$$\int_0^\infty e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2}.$$

Now, we only need to compute $\Gamma(1/2)$. Using Beta function,

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \Gamma\left(\frac{1}{2}\right)^2$$

On the other hand,

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \cos^{2\cdot\frac{1}{2}-1}\theta \sin^{2\cdot\frac{1}{2}-1}\theta d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = \pi.$$

Hence $\Gamma(1/2) = \sqrt{\pi}$. We obtain that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Example 3.2. Compute $\int_0^{2\pi} \sin^4\theta d\theta$.

We know

$$\int_0^{2\pi} \sin^4\theta d\theta = 4\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = 2\cdot 2\int_0^{\frac{\pi}{2}} \sin^{2\cdot\frac{5}{2}-1}\theta \cos^{2\cdot\frac{1}{2}}\theta d\theta = 2B\left(\frac{5}{2},\frac{1}{2}\right).$$

We compute

$$B\left(\frac{5}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}.$$

Using $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

Hence $B\left(\frac{5}{2},\frac{1}{2}\right) = \frac{\frac{3}{4}\sqrt{\pi}\cdot\sqrt{\pi}}{2!} = \frac{3}{8}\pi$. Thus $\int_0^{2\pi}\sin^4\theta d\theta = \frac{3}{4}\pi$.

Now, let us go back to the proof of Theorem 3.1.

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \left(\int_0^\infty e^{-t}t^{x-1}dt\right)\left(\int_0^\infty e^{-s}s^{x-1}ds\right)\\ &= \int_0^\infty\int_0^\infty e^{-(t+s)}t^{x-1}s^{y-1}dtds.\end{aligned}$$

Let us compute $\int_0^\infty e^{-(t+s)} t^{x-1} dt$. Consider $t = su$, we rewrite

$$\int_0^\infty e^{-(t+s)} t^{x-1} dt = s^x \int_0^\infty e^{-(u+1)s} u^{x-1} du.$$

Hence the integral becomes

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty \left(\int_0^\infty e^{-(u+1)s} s^{x+y-1} ds \right) u^{x-1} du \\ &= \int_0^\infty \frac{\Gamma(x+y)}{(u+1)^{x+y}} \cdot u^{x-1} du \\ &= \Gamma(x+y) \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du.\end{aligned}$$

Here we use Proposition 1.2. This shows that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du.$$

Example 3.3. Compute $\int_1^3 (x-1)^{10}(x-3)^3 dx$.

Let $t = (x-1)/2$. Then the integral becomes

$$\int_1^3 (x-1)^{10}(x-3)^3 dx = \int_0^1 (2t)^{10}(2t-2)^3 2dt = -2^{14} \int_0^1 t^{10}(1-t)^3 dt.$$

We obtain

$$\int_1^3 (x-1)^{10}(x-3)^3 dx = -2^{14} B(11,4) = -2^{14} \cdot \frac{10!3!}{14!}.$$