Randomized Algo HW3.

1.

(a) $M_X(t) = E[e^{Xt}]$ for $X \sim Bin(n,p) \Rightarrow Pr(X=k) = C_k^n p^k (1-p)^{n-k}$

$\Rightarrow M_X(t) = E[e^{Xt}] \qquad X = X_1 + X_2 + \cdots X_n$

$= E[e^{\sum_{i=1}^n X_i \cdot t}] = E[\prod_{i=1}^n e^{X_i \cdot t}] \qquad X_i \sim Ber(p) \Rightarrow Pr(X_i=1) = p$

$\qquad Pr(X_i = 0) = 1-p$

$= \prod_{i=1}^n E[e^{X_i \cdot t}] = \prod_{i=1}^n (p \cdot e^t + (1-p)\cancel{e^{0t}})$

$= \prod_{i=1}^n (1 + (e^t - 1)p) = (1+(e^t-1)p)^n$ ※

(b)

$Z = X + Y$

$X \sim Bin(n,p)$

$Y \sim Bin(m,p)$

$M_Z(t) = E[e^{Zt}] = E[e^{Xt} \cdot e^{Yt}] = E[e^{Xt}] \cdot E[e^{Yt}]$

$= (1+(e^t-1)p)^{n+m}$ ※

$\frac{d}{dt}(1+(e^t-1)p)^n$

$= n(1+(\cancel{e^t}-\overset{0}{1})\cancel{p})^{n-1} \cdot p\cancel{e^t}1$

$= n \cdot p$

(c)

$E[Z] = M_Z'(0) \Rightarrow \frac{d}{dt}(1+(e^t-1)p)^{n+m} = (n+m)(1+(e^t-1)p)^{n+m-1} \cdot pe$

$\Rightarrow M_Z'(0) = (n+m)p$ ※

$E[Z^2] = M_Z''(0) \Rightarrow \frac{d}{dt} M_Z'(t) = \frac{d}{dt}(n+m)(1+(e^t-1)p)^{n+m-1} \cdot pe^t$

$= (n+m)(n+m-1)\cancel{(1+(e^t-1)p)^{n+m-2}} \cdot pe^t \cdot pe^t$

$+ (n+m)\cancel{(1+(e^t-1)p)^{n+m-1}} \cdot pe^t$

$M_Z''(0) = (n+m)(n+m-1)p^2 + (n+m)p$

$Var[Z] = E[(Z-E[Z])^2] = E[Z^2] - E[Z]^2$

$= (n+m)p^2(n+m-1) + (n+m)p - (n+m)^2p^2$

$= (n+m)p^2[(\cancel{n}+\cancel{m}-1) - (\cancel{n}+\cancel{m})] + (n+m)p$

$= (n+m)p - (n+m)p^2 = (n+m)p(1-p)$ ※

2.

(a) $Pr(W \geq (1+\delta)v) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^v$

$a_1, \cdots a_n$ in $[0,1]$ $\quad W = \sum_{i=1}^{n} a_i X_i \quad v = E[W]$ for $\delta > 0$

$Pr(W \geq (1+\delta)v) = Pr(e^{tW} \geq e^{t(1+\delta)v}) \leq \frac{E[e^{tW}]}{e^{t(1+\delta)v}}$

$\frac{E[e^{tW}]}{e^{t(1+\delta)v}} = \frac{E[e^{t \cdot \sum_{i=1}^{n} a_i X_i}]}{e^{t(1+\delta)v}} = \frac{\prod_{i=1}^{n} E[e^{t a_i X_i}]}{e^{t(1+\delta)v}}$

$= \frac{\prod_{i=1}^{n} e^{(e^t-1)P_A P_X}}{e^{t(1+\delta)v}} \leq \frac{e^{(e^t-1)v}}{e^{t(1+\delta)v}} = \frac{e^{\delta v}}{(1+\delta)^{(1+\delta)v}} = \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\delta$ #

$E[e^{ta_i x_i}] = \sum_{B=0}^{1} \sum_{A=0}^{1} e^{t(a_i \cdot x_i)} Pr(a_i = A) \cdot Pr(X_i = B)$

$= e^t \cdot P_A \cdot P_X + \cancel{e^{t \cdot 0 \cdot 1}}^{1} \cdot (1-P_A) P_X + e^{t \cdot 1 \cdot 0 \to 1} \cdot P_A \cdot (1-P_X) + \cancel{e^{t \cdot 0 \cdot 0}}^{1} \cdot (1-P_A)(1-P_X)$

$= e^t P_A P_X + (1 - P_A P_X) = 1 + P_A P_X (e^t - 1) \leq e^{(e^t-1)P_A P_X}$

$\min\left(\frac{e^{(e^t-1)v}}{e^{t(1+\delta)v}}\right) = \min\left((e^t-1) - t(1+\delta)\right) \Rightarrow \frac{e^{(1+\delta-1)v}}{(1+\delta)^{(1+\delta)v}}$

$\frac{d}{dt}\left((e^t-1) - t(1+\delta)\right) = e^t - (1+\delta)$

to minimize $(e^t-1) - t(1+\delta)$, let $e^t - (1+\delta) = 0$

$e^t = 1+\delta \quad t = \ln(1+\delta)$ Also $(e^t-1) - t(1+\delta)$ is an increasing function

(b)

$Pr(W \leq (1-\delta)v) = Pr(e^{tW} \leq e^{t(1-\delta)v}) = Pr(e^{tW} \geq e^{t(1-\delta)v})$

for $t < 0$

$\Rightarrow Pr(e^{tW} \geq e^{t(1-\delta)v}) \leq \frac{E[e^{tW}]}{e^{t(1-\delta)v}} = \left(\frac{e^\delta}{(1-\delta)^{(1-\delta)}}\right)^v$ ※

$\frac{E[e^{tW}]}{e^{t(1-\delta)v}} = \frac{\prod_{i=1}^{n} E[e^{ta_i x_i}]}{e^{t(1-\delta)v}} \leq \frac{\prod_{i=1}^{n} e^{(e^t-1)P_A P_X}}{e^{t(1-\delta)v}} = \frac{e^{(e^t-1)v}}{e^{t(1-\delta)v}}$

$\min\left(\frac{e^{(e^t-1)v}}{e^{t(1-\delta)v}}\right) = \min\left((e^t-1) - t(1-\delta)\right) \Rightarrow \frac{e^{\delta v}}{(1-\delta)^{(1-\delta)v}} = \left(\frac{e^\delta}{(1-\delta)^{(1-\delta)}}\right)^v$

$\frac{d}{dt}(e^t-1) - t(1-\delta) = e^t - (1-\delta)$, let $e^t - (1-\delta) = 0 \quad t = \log_e(1-\delta)$

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For $X_1, X_2, \cdots X_n$ be independent RV, $Pr(X_i = 1-P_i) = P_i$ and $Pr(X_i = -P_i) = 1-P_i$, Let $X = \sum_{i=1}^{n} X_i$, prove that $Pr(|X| \geq a) \leq 2e^{\frac{-2a^2}{n}}$

Supplement: Hoeffding's Lemma: $P_i e^{\lambda(1-P_i)} + (1-P_i)e^{-\lambda P_i} \leq e^{\frac{\lambda^2}{8}}$

$$Pr(|X| \geq a) = 2Pr(X \geq a) = 2Pr(e^{tX} \geq e^{ta}) \leq 2 \cdot \frac{E[e^{tX}]}{e^{ta}}$$

$$\frac{E[e^{tX}]}{e^{ta}} = \frac{\prod_{i=1}^{n} E[e^{tX_i}]}{e^{ta}} \leq \frac{\prod_{i=1}^{n} e^{\frac{t^2}{8}}}{e^{ta}} = \frac{e^{\frac{nt^2}{8}}}{e^{ta}} = e^{-\frac{2a^2}{n}}$$

$$E[e^{tX_i}] = e^{t(1-P_i)} \cdot Pr(X_i = 1-P_i) + e^{t(-P_i)} \cdot Pr(X_i = -P_i)$$

$$= e^{t(1-P_i)} \cdot P_i + e^{t(-P_i)} \cdot (1-P_i) \leq e^{\frac{t^2}{8}}$$

with Hoeffding's Lemma $\uparrow$

$$P_i e^{t(1-P_i)} + (1-P_i)e^{-tP_i} \leq e^{\frac{t^2}{8}}$$

$$\min\left(\frac{e^{\frac{nt^2}{8}}}{e^{ta}}\right) = \min\left(\frac{nt^2}{8} - ta\right) \Rightarrow \frac{e^{\frac{2a^2}{n}}}{e^{\frac{4a^2}{n}}} = \left(\frac{e^2}{e^4}\right)^{\frac{a^2}{n}} = e^{\frac{-2a^2}{n}}$$

$\frac{nt^2}{8} - ta$ is a convex funtion, has a minimum point

$$\Rightarrow \frac{d}{dt} \frac{nt^2}{8} - ta = \frac{2nt}{8} - a = 0 \quad 2nt = 8a \quad t = \frac{8a}{2n} = \frac{4a}{n}$$

$$\Rightarrow \text{Result: } Pr(|X| \geq a) = 2 \cdot \frac{E[e^{tX}]}{e^{ta}} = 2e^{-\frac{2a^2}{n}} \quad \divideontimes$$