Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.11

No. 1

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

spherical coordinates.

$x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z}$$

代入

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}.$$

No. 2

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)\right]$$

代入 $$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

No. 3

$P_n(\cos\theta)$, $0 \le \theta \le 2\pi$, $n = 0, 1, 2$

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

No. 4

By (11′) in Sec. 5.3 we have (cf. Fig. 312)

$$u_1 = A_1 r \cos\phi = 0 \qquad \text{if} \qquad \phi = \tfrac{1}{2}\pi.$$

This is the *xy*-plane. Similarly,

$$u_2 = A_2 \frac{r^2}{2}(3\cos^2\phi - 1) = 0 \qquad \text{if} \qquad \cos\phi = \frac{1}{\sqrt{3}}$$

and

$$u_3 = A_3 \frac{r^3}{2}(5\cos^3\phi - 3\cos\phi) = 0 \qquad \text{if} \qquad \cos\phi = 0 \quad \text{and} \quad \sqrt{\frac{3}{5}}.$$

No. 5

$$A_n = \frac{55(2n+1)}{2^n} \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{(2n-2m)!}{m!\,(n-m)!\,(n-2m+1)!}$$

$A_4 = A_6 = A_8 = A_{10} = 0$

$A_5 = \frac{605}{16}$, $A_7 = \frac{-4125}{128}$, $A_9 = \frac{7315}{256}$

No. 6

問答或證明題，不解

No. 7

$$u_n = A_n r^n P_n(\cos\phi)$$

$$u_n^* = \frac{B_n}{r^{n+1}} P_n(\cos\phi)$$

$$\nabla^2 u_n = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial(A_n r^n P_n(\cos\phi))}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial(A_n r^n P_n(\cos\phi))}{\partial\phi}\right)\right]$$

$$= 0$$

$$\nabla u_n^* = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial\left(\frac{B_n}{r^{n+1}} P_n(\cos\phi)\right)}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi \frac{\partial\left(\frac{B_n}{r^{n+1}} P_n(\cos\phi)\right)}{\partial\phi}\right)\right]$$

$$= 0$$

$\therefore$ u_n and u_n^* are solutions.

No. 8

$$u = \frac{c}{r}$$

$$\nabla^2 u = \frac{\partial^2}{\partial r^2}\left(\frac{c}{r}\right) + \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{c}{r}\right) + \frac{1}{r^2}\frac{\partial^2\left(\frac{c}{r}\right)}{\partial\phi^2} + \frac{\cot\phi}{r^2}\frac{\partial\left(\frac{c}{r}\right)}{\partial\phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2\left(\frac{c}{r}\right)}{\partial\theta^2}$$

$$= 2cr^{-3} - 2cr^{-3} + 0 + 0 + 0$$

$$= 0$$

$\therefore$ $u = \frac{c}{r}$ satisfies Laplace's equation in spherical coordinates.

No. 9

$$\nabla^2 u = u'' + \frac{2}{r} u' = 0$$

$$\frac{u''}{u'} = -\frac{2}{r}$$

$$\ln|u'| = -2\ln|r| + C_1$$

$$u' = \frac{\tilde{C}}{r^2}$$

$$u = -\frac{C}{r} + k.$$

No.10

By (5),

$$\nabla^2 u = u'' + u'/r = 0.$$

Separation and integration gives

$$u''/u' = -1/r, \qquad \ln|u'| = -\ln|r| + c_1.$$

Taking exponents and integrating again gives

$$u' = c/r \qquad \text{and} \qquad u = c \ln r + k.$$

No.11

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$u_{xx} + u_{yy} + u_{zz} = 0$$

$$u''(r) + \frac{2u'(r)}{r} = 0$$

No.12

$$u = c\ln r + k$$

$u(2) = 220$, $u(4) = 140$

$u(2) = c\ln 2 + k = 220$

$u(4) = c\ln 4 + k = 140$

$\Rightarrow c\ln 2 = -80 \Rightarrow c = \dfrac{-80}{\ln 2}$

$k = 300$

$\therefore u(r) = \dfrac{-80}{\ln 2}\ln r + 300 \quad ; \; 2 \leq r \leq 4$

No.13

$$u(r) = \frac{c}{r} + k$$

$u(2) = 220$, $u(4) = 140$

$u(2) = \dfrac{c}{2} + k = 220$

$u(4) = \dfrac{c}{4} + k = 140$

$\dfrac{c}{4} = 80 \Rightarrow c = 320$

$k = 60$

$u(r) = \dfrac{320}{r} + 60 \quad ; \; 2 \leq r \leq 4$

No.14

$v = F(r)G(t)$, $F'' + k^2F = 0$, $\dot{G} + c^2k^2G = 0$, $F_n = \sin(n\pi r/R)$,

$G_n = B_n \exp(-c^2n^2\pi^2t/R^2)$, $B_n = \dfrac{2}{R}\displaystyle\int_{-0}^{R} rf(r)\sin\frac{n\pi r}{R}dr$

No.15

問答或證明題，不解

No.16

$\tilde{f}(w) = w$, $A_n = \dfrac{2n+1}{2}\displaystyle\int_{-1}^{1} wP_n(w)\,dw$. Since $w = P_1(w)$ and the $P_n(w)$ are orthogonal on the interval $-1 \leqq w \leqq 1$, we obtain $A_1 = 1$, $A_n = 0$ $(n = 0, 2, 3, \cdots)$. *Answer:* $u = r\cos\phi$. Of course, this is at once seen by integration.

No.17

$f(\phi) = 1$

$w = \cos\phi$

$f(\phi) = 1 = P_0(w) = 1.$

$\therefore u = 1.$

No.18

By definition,

$$P_2(\cos\phi) = \tfrac{3}{2}\cos^2\phi - \tfrac{1}{2}.$$

Hence

$$1 - \cos^2\phi = -\tfrac{2}{3}P_2(\cos\phi) + \tfrac{2}{3}$$

and

$$\begin{aligned} u &= -\tfrac{2}{3}r^2P_2(\cos\phi) + \tfrac{2}{3} \\ &= r^2(-\cos^2\phi) + \tfrac{1}{3}) + \tfrac{2}{3}. \end{aligned}$$

No.19

$$f(\phi)=\cos 2\phi$$

$$w=\cos\phi$$

$$f(\phi)=\cos 2\phi=2\cos^2\phi-1=2w^2-1=\frac{4}{3}P_2(w)-\frac{1}{3}$$

$$=\frac{4}{3}\left(\frac{3}{2}w^2-\frac{1}{2}\right)-\frac{1}{2}$$

$$\therefore\ u=\frac{4}{3}r^2P_2(w)-\frac{1}{3}$$

$$=\frac{4}{3}r^2P_2(\cos\phi)-\frac{1}{3}$$

No.20

$$f(\phi)=10\cos^3\phi-3\cos^2\phi-5\cos\phi-1$$

$$w=\cos\phi$$

$$f(\phi)=10w^3-3w^2-5w-1$$

$$=4P_3(w)-2P_2(w)+P_1(w)-2$$

$$\therefore\ u=4r^3P_3(w)-2r^2P_2(w)+rP_1(w)-2$$

$$=4r^3P_3(\cos\phi)-2r^2P_2(\cos\phi)+rP_1(\cos\phi)-2$$

No.21

問答或證明題，不解

No.22

In Prob. 16, $f(\phi) = \cos\phi$; hence

$$u_{\text{int}} = r\cos\phi, \qquad u_{\text{ext}} = r^{-2}\cos\phi.$$

In Prob. 19, $f(\phi) = \cos 2\phi$; hence $f(\phi) = 2\cos^2\phi - 1$, so that

$$2\cos^2\phi - 1 = \tfrac{4}{3}P_2(\cos\phi) - \tfrac{1}{3}$$

and thus

$$u_{\text{int}} = \tfrac{4}{3}r^2 P_2(\cos\phi) - \tfrac{1}{3}$$

and

$$u_{\text{ext}} = \frac{4}{3r^3}P_2(\cos\phi) - \frac{1}{3r}.$$

No.23

問答或證明題，不解

No.24

Team Project. **(a)** The two drops over a portion of the cable of length Δx are $-Ri\Delta x$ and $-L(\partial i/\partial t)\Delta x$, respectively. Their sum equals the difference $u_{x+\Delta x} - u_x$. Divide by Δx and let $\Delta x \to 0$.

(c) To get the first PDE, differentiate the first transmission line equation with respect to x and use the second equation to replace i_x and i_{xt}:

$$\begin{aligned} -u_{xx} &= Ri_x + Li_{xt} \\ &= R(-Gu - Cu_t) + L(-Gu_t - Cu_{tt}). \end{aligned}$$

Now collect terms. Similarly for the second PDE.

(d) Set $\dfrac{1}{RC} = c^2$. Then $u_t = c^2 u_{xx}$, the heat equation. By (9), (10), Sec. 12.6,

$$u = \frac{4U_0}{\pi}\left(\sin\frac{\pi x}{l}\,e^{-\lambda_1^2 t} + \frac{1}{3}\sin\frac{3\pi x}{l}\,e^{-\lambda_3^2 t} + \cdots\right), \quad \lambda_n^2 = \frac{n^2\pi^2}{l^n RC}.$$

(e) $u = U_0 \cos(\pi t/(l\sqrt{LC}\sin(\pi x/l)$

No.25

Let $\frac{1}{r} = \rho$

$$u(\rho, \theta, \phi) = r\, v(r, \theta, \phi)$$

$$u_\rho = (v + r v_r) \cdot \left(\frac{-1}{\rho^2}\right)$$

$$u_{\rho\rho} = (2v_r + r v_{rr})\left(\frac{1}{\rho^4}\right) + (v + r v_r)\left(\frac{2}{\rho^3}\right)$$

$$u_{\rho\rho} + \frac{2}{\rho} u_\rho = r^5 \left(v_{rr} + \left(\frac{2}{r}\right) v_r\right)$$

$$u_{\phi\phi} = r\, v_{\phi\phi}$$