Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.1

No. 1

Let $L\{\ \}$ is second-order PDE operator.

u_1, u_2 are solutions of PDE

$L\{u_1\} = 0, \quad L\{u_2\} = 0.$

$$L\{c_1u_1 + c_2u_2\} = L\{c_1u_1\} + L\{c_2u_2\} = c_1L\{u_1\} + c_2L\{u_2\}$$

$$= 0 + 0$$

$$= 0$$

$\therefore$ $c_1u_1 + c_2u_2$ is also solution of PDE.

No. 2

$c = 1, u_{tt} = 4 = u_{xx}$.

Problems 2–13 should give the student a first impression of what kind of solutions to expect, and of the great variety of solutions compared with those of ODEs. It should be emphasized that although the wave and the heat equations look so similar, their solutions are basically different. It could be mentioned that the boundary and initial conditions are basically different, too. Of course, this will be seen in great detail in later sections, so one should perhaps be cautious not to overload students with such details before they have seen a problem being solved.

No. 3

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u = \cos 4t \sin 2x$ is solution.

$$\frac{\partial^2 u}{\partial t^2} = -16 \cos 4t \sin 2x$$

$$\frac{\partial^2 u}{\partial x^2} = -4 \cos 4t \sin 2x$$

$$-16 \cos 4t \sin 2x = -4c^2 \cos 4t \sin 2x$$

$$c^2 = 4$$

$$c = \pm 2.$$ (負不合)

No. 4

$u = \sin kct \cos kx$

$\frac{\partial^2 u}{\partial t^2} = -(kc)^2 \sin kct \cos kx$

$\frac{\partial^2 u}{\partial x^2} = -k^2 \sin kct \cos kx$

$-(kc)^2 \sin kct \cos kx = -k^2 c^2 \sin kct \cos kx$

c = arbitary.

No. 5

$\frac{\partial^2 u}{\partial t^2} = -a^2 \sin at \sin bx$

$\frac{\partial^2 u}{\partial x^2} = -b^2 \sin at \sin bt$

$-a^2 \sin at \sin bx = -c^2 b^2 \sin at \sin bt$

$a^2 = c^2 b^2$

$c^2 = \left(\frac{a}{b}\right)^2$

$c = \pm \frac{a}{b}$ (負不合)

No. 6

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u = e^{-t} \sin x$ is solution

$$\frac{\partial u}{\partial t} = -e^{-t} \sin x$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-t} \sin x$$

$$-e^{-t} \sin x = -c^2 e^{-t} \sin x$$

$$c^2 = 1$$

$c = \pm 1$ ([illegible])

No. 7

$$u = e^{-w^2 c^2 t} \cos wx$$

$$\frac{\partial u}{\partial t} = -w^2 c^2 e^{-w^2 c^2 t} \cos wx$$

$$\frac{\partial^2 u}{\partial x^2} = -w^2 e^{-w^2 c^2 t} \cos wx$$

$$-w^2 c^2 e^{-w^2 c^2 t} \cos wx = -c^2 w^2 e^{-w^2 c^2 t} \cos wx$$

C = arbitrary.

No. 8

$$u = e^{-9t}\sin wx$$

$$\frac{\partial u}{\partial t} = -9e^{-9t}\sin wt$$

$$\frac{\partial^2 u}{\partial x^2} = -w^2 e^{-9t}\sin wt$$

$$-9e^{-9t}\sin wt = -c^2w^2e^{-9t}\sin wt$$

$$c^2w^2 = 9$$

$$c^2 = \frac{9}{w^2}$$

$$c = \pm\frac{3}{w} \quad (\text{負不合})$$

No. 9

$$u = e^{-\pi^2 t}\cos 25x$$

$$\frac{\partial u}{\partial t} = -\pi^2 e^{-\pi^2 t}\cos 25x$$

$$\frac{\partial^2 u}{\partial x^2} = -(25)^2 e^{-\pi^2 t}\cos 25x$$

$$-\pi^2 e^{-\pi^2 t}\cos 25x = -c^2(25)^2 e^{-\pi^2 t}\cos 25x$$

$$c^2(25)^2 = \pi^2$$

$$c^2 = \left(\frac{\pi}{25}\right)^2$$

$$c = \pm\frac{\pi}{25} \quad (\text{負不合})$$

No.10

Laplace equation: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

(1) $u=e^x\cos y$

$\frac{\partial u}{\partial x}=e^x\cos y$, $\frac{\partial^2 u}{\partial x^2}=e^x\cos y$

$\frac{\partial u}{\partial y}=-e^x\sin y$, $\frac{\partial^2 u}{\partial y^2}=-e^x\cos y$

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x\cos y-e^x\cos y=0$

$u=e^x\cos y$ is solution.

(2) $u=e^x\sin y$

$\frac{\partial u}{\partial x}=e^x\sin y$, $\frac{\partial^2 u}{\partial x^2}=e^x\sin y$

$\frac{\partial u}{\partial y}=e^x\cos y$, $\frac{\partial^2 u}{\partial y^2}=-e^x\sin y$

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x\sin y-e^x\sin y=0$

$u=e^x\sin y$ is solution.

No.11

$u=\arctan\left(\frac{y}{x}\right)$

$\frac{\partial u}{\partial x}=\frac{-\frac{y}{x^2}}{\left[1+\left(\frac{y}{x}\right)^2\right]}=\frac{-y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial x^2}=\frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y}=\frac{\frac{1}{x}}{\left[1+\left(\frac{y}{x}\right)^2\right]}=\frac{x}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{+2xy}{(x^2+y^2)^2}$

故 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}=0$

$u = \arctan\left(\frac{y}{x}\right)$ is solution.

No.12

(1) $u = \cos y \sinh x$

$\frac{\partial u}{\partial x} = \cos y \cosh x$, $\frac{\partial^2 u}{\partial x^2} = \cos y \sinh x$

$\frac{\partial u}{\partial y} = -\sin y \sinh x$, $\frac{\partial^2 u}{\partial y^2} = -\cos y \sinh x$

故 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cos y \sinh x - \cos y \sinh x = 0$

$u = \cos y \sinh x$ is solution.

(2) $u = \sin y \cosh x$

$\frac{\partial u}{\partial x} = \sin y \sinh x$, $\frac{\partial^2 u}{\partial x^2} = \sin y \cosh x$

$\frac{\partial u}{\partial y} = \cos y \cosh x$, $\frac{\partial^2 u}{\partial y^2} = -\sin y \cosh x$

故 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin y \cosh x - \sin y \cosh x = 0$

$u = \sin y \cosh x$ is solution.

No.13

(1) $u = \dfrac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}$$

$$\text{故} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2x^3-6xy^2}{(x^2+y^2)^3} + \frac{-2x^3+6xy^2}{(x^2+y^2)^3} = 0$$

$u = \dfrac{x}{(x^2+y^2)}$ is solution.

(2) $u = \dfrac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}$$

$$\text{故} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{6x^2y-2y^3}{(x^2+y^2)^3} + \frac{2y^3-6x^2y}{(x^2+y^2)^3} = 0$$

$u = \dfrac{y}{(x^2+y^2)}$ is solution.

No.14

Team Project. (a) Denoting derivatives with respect to the entire argument $x + ct$ and $x - ct$, respectively, by a prime, we obtain by differentiating twice

$$u_{xx} = v'' + w'', \qquad u_{tt} = v''c^2 + w''c^2$$

and from this the desired result.

(c) The student should realize that $u = 1/\sqrt{x^2 + y^2}$ is not a solution of Laplace's equation in two variables, but satisfies the remarkable Poisson equation shown under (b).

No.15

(1)

Laplace equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u(x,y) = a \ln(x^2+y^2) + b.$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{+a[2(x^2+y^2) - 2x \cdot 2x]}{(x^2+y^2)^2} + \frac{a[2(x^2+y^2) - 2y \cdot 2y]}{(x^2+y^2)^2}$$

$$= 0.$$

$\therefore u(x,y) = a \ln(x^2+y^2) + b$ is solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(2) B.C (1) $x^2+y^2 = 1$, $u = 110$

(2) $x^2+y^2 = 100$, $u = 0$

$$\begin{cases} 110 = a \ln 1 + b & \Rightarrow b = 110 \\ 0 = a \ln 100 + b & \Rightarrow a = \frac{-110}{\ln 100} \end{cases}$$

No.16

Integrate twice with respect to y,

$$u_y = c_1(x), \qquad u = c_1(x)y + c_2(x)$$

With the "constants" of integration $c_1(x)$ and $c_2(x)$ arbitrary.

Problems 16–25 will help the student get used to the notations in this chapter; in particular, y will now occur as an *independent* variable. Second-order PDEs in this set will also help review the solution methods in Chap. 2, which will play a role in separating variables.

No.17

$$U_{xx} + 16\pi^2 U = 0$$

no y- derivatives occur.

$$U = A(y)\cos 4\pi x + B(y)\sin 4\pi x$$

No.18

$$25\,U_{yy} - 4U = 0$$

$$U_{yy} - \frac{4}{25}U = 0$$

no x-derivatives occur.

$$u(x,y) = A(x)e^{\frac{2}{5}y} + B(x)e^{-\frac{2}{5}y}.$$

No.19

$$U_y + y^2 u = 0$$

no x-derivatives occur

$$u' + y^2 u = 0$$

$$\frac{u'}{u} = -y^2$$

$$\ln|u| = -\frac{y^3}{3} + c(x)$$

$$u(x,y) = c^*(x)e^{-\frac{y^3}{3}}$$

No.20

The characteristic equation is

$$2\lambda^2 + 9\lambda + 4 = 2(\lambda + 4)(\lambda + \tfrac{1}{2}) = 0.$$

Hence a general solution of the homogeneous PDE is

$$u_h(x, y) = c_1(y)e^{-4x} + c_2(y)e^{-0.5x}.$$

A particular solution u_p of the nonhomogeneous PDE is obtained by the method of undetermined coefficients,

$$u_p(x, y) = 3\cos x - \sin x.$$

No.21

$u_{yy} + 6u_y + 13u = 4e^{3y}$.

no x- derivatives occur.

$u_h(x, y) = e^{-3y}[A(x)\cos 2y + B(x)\sin 2y]$

$u_p(x, y) = ke^{3y}$.

$9ke^{3y} + 18ke^{3y} + 13ke^{3y} = 4e^{3y}$.

$40k = 4$

$k = 0.1$

$\therefore u_p(x, y) = 0.1e^{3y}$

$u(x, y) = u_h(x, y) + u_p(x, y) = e^{-3y}[A(x)\cos 2y + B(x)\sin 2y] + 0.1e^{3y}$

No.22

Set $u_x = v$ to get $v_y = v$, $v_y/v = 1$, $v = c(x)e^y$, and

$$u = \int v\,dx = c_1(x)e^y + c_2(y).$$

No.23

$$x^2 u_{xx} + 2x u_x - 2u = 0$$

no y-derivatives occur

Euler equation :

$$x^2 u'' + 2x u' - 2u = 0$$

Let $u = x^m$

$$m(m-1) + 2m - 2 = 0$$

$$m = 1, -2$$

$$u(x,y) = C_1(y) x^1 + C_2(y) \frac{1}{x^2}$$

No.24

By the given PDE and the chain rule,

(A) $$yz_x - xz_y = y(z_r r_x + z_\theta \theta_x) - x(z_r r_y + z_\theta \theta_y) = 0.$$

Differentiate $r^2 = x^2 + y^2$ by parts and divide by $2r$,

(B) $$r_x = x/r, \qquad r_y = y/r.$$

Now z_r has in (A) the coefficients (use (B))

$$yr_x - xr_y = yx/r - xy/r = 0$$

so that (A) reduces to $z_\theta = 0$. That is, $z(r, \theta)$ depends only on r, not on the angle θ, as for a sphere, a circular cylinder, and so on.

No.25

$$\begin{cases} u_{xx} = 0 & \text{—— ①} \\ u_{yy} = 0 & \text{—— ②} \end{cases}$$

由 ①. $u(x,y) = f(y)x + g(y)$.

代入 ② $f''(y)x + g''(y) = 0$.

$$\begin{cases} f''(y) = 0 \Rightarrow f(y) = ay + b. \\ g''(y) = 0 \Rightarrow g(y) = cy + k \end{cases}$$

$\therefore u(x,y) = (ay+b)x + cy + k$

$= axy + bx + cy + k$

a, b, c, k : arbitrary