CS2336 Discrete Mathematics

Exam 3
December 17, 2018 (10:10–12:30)

Answer all questions. Total marks = 120. Maximum Score = 110. Large portion of marks may be deducted for incomplete proofs. Q7 and Q8 require some thinking.

1. (15%) Let A, B, and C be three sets. It is known that the following are true.

   - $|A| = 14$;
   - $|A - C| = 10$;
   - $|B - C| = 8$;
   - $|(A \cap B) - C| = 4$;
   - $|(B \cap C) - A| = 9$.

   Find $|A \cup B|$. (Hint: Draw a Venn diagram to help.)

2. (15%) Let r, s, k be integers with $r \geq s \geq k \geq 0$. Use combinatorial argument to show

$$\binom{r}{s}\binom{s}{k} = \binom{r}{k}\binom{r-k}{s-k}.$$

   Remark 1: No marks if the proof is not a combinatorial argument.
   Remark 2: We assume that $\binom{0}{0} = 1$.

3. (20%)

   (a) (10%) Show that the set
$$\{ (x, k) \mid x, k \in \mathbb{Z}^+ \}$$
   is countable.

   (b) (10%) Using the above result (or otherwise), show that the set
$$\{ x^{1/k} \mid x, k \in \mathbb{Z}^+ \}$$
   is countable.

4. (15%) Consider the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ with
$$f(x) = x^2.$$
   Determine if f is one-one, or onto, or none, or both. Give explanation to your answer.

5. (15%) Let G be a simple undirected graph with 3 vertices a, b, c. There are 2 edges in G, where one connects a and b, and the other connects b and c. Draw all the subgraphs of G.

6. (20%)

   (a) (10%) Draw two non-isomorphic simple undirected graphs H_1 and H_2, each with 5 vertices, and the degrees of these vertices are 1, 2, 2, 2, 3, respectively.

   (b) (10%) Show that H_1 and H_2 are non-isomorphic.

7. (10%)

   (a) (5%) Let X be a graph with 5 vertices. Show that either X or the complement of X must be non-bipartite.

   (b) (5%) Show that for any graph Y with at least 5 vertices, either Y or the complement of Y must be non-bipartite.

8. (10%) Let A and B be two non-empty sets of real numbers. Define $A + B$ to be the set

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

For instance, if $A = \{1, 3, 4\}$ and $B = \{1, 3\}$, then $A + B = \{2, 4, 5, 6, 7\}$.

Show that in any case, we have

$$|A + B| \geq |A| + |B| - 1.$$