# CHAPTER 15 Power Series, Taylor Series

Power series and, in particular, Taylor series, play a much more fundamental role in complex analysis than they do in calculus. The student may do well to review what has been presented about power series in calculus but should become aware that many new ideas appear in complex, mainly owing to the use of complex integration.

## SECTION 15.1. Sequences, Series, Convergence Tests, page 671

**Purpose.** The beginnings on sequences and series in complex is similar to that in calculus (differences between real and complex appear only later). Hence this section can almost be regarded as a review from calculus plus a presentation of convergence tests for later use.

**Main Content, Important Concepts**

Sequences, series, convergence, divergence

Comparison test (Theorem 5)

Ratio test (Theorem 8)

Root test (Theorem 10)

## SOLUTIONS TO PROBLEM SET 15.1, page 679

**2.** Converges to 0, hence bounded, since $n!$ grows faster than $|1 + 2i|^n = (\sqrt{5})^n$. Convergence implies that there are no limit points other than the limit itself. Note that $z_n$ is the $n$th term of the Maclaurin series of $e^{1+2i}$.

**3.** $z_n = -\dfrac{1}{4}\pi i(1/1 + 2/4ni)$ by algebra; convergent to $-\pi i/4$

**4.** Unbounded, hence divergent, $|z| = 5^{n/2}$.

**5.** Bounded, divergent, $\pm 1 + 5i$.

**6.** Unbounded since $\cos 2n\pi i = \cosh 2n\pi > n$ hence divergent.

**7.** Unbounded, hence divergent.

**8.** Divergent; all terms have absolute value 1, and a graph suggests that every point on the unit circle is a limit point.

**9.** Convergent to 0, hence bounded.

**10.** Bounded, divergent, $\pm 1/\sqrt{2} \pm i, 0, 1, -2$

**12.** For any $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$|z_n - l| < \epsilon/3, \qquad |z_n^* - l^*| < \epsilon/3.$$

Hence by the triangle inequality, for all $n > N$ we have

$$\begin{aligned} |(z_n + z_n^*) - (l + l^*)| &= |(z_n - l) + (z_n^* - l^*)| \\ &\leqq |z_n - l| + |z_n^* - l^*| \\ &\leqq \epsilon/3 + \epsilon/3 \\ &< \epsilon. \end{aligned}$$

This proof is typical of many similar ones.

Much less important is termwise *multiplication* of sequences, but a similar theorem holds true for this case. Namely, under the assumptions just made on the convergence of the two sequences, it follows that the sequence $z_1z_1^*, z_2z_2^*, \cdots$ is convergent and has the limit $ll^*$. The proof is more complicated, as follows.

The two sequences are bounded, $|z_n| < K$, $|z_n^*| < K$. Since they converge, for an $\epsilon > 0$ there is an $N = N(\epsilon)$ (such that $|z_n - l| < \epsilon/(3K)$, $|z_n^* - l^*| < \epsilon/(3|l|)$ ($l \neq 0$; the case $l = 0$ is rather trivial), hence

$$\begin{aligned} |z_nz_n^* - ll^*| &= |(z_n - l)z_n^* + (z_n^* - l^*)l| \\ &\leqq |z_n - l||z_n^*| + |z_n^* - l^*||l| \\ &< \epsilon/3 + \epsilon/3 < \epsilon \quad (n > N). \end{aligned}$$

**16.** Convergent. In connection with MacLaurin series in Sec. 15.4 the sum will turn out to be $e^{20+30i}$, of absolute value $0.5 \cdot 10^9$.

**18.** Divergent

**20.** Divergent because

$$\left|\frac{n+i}{3n^2+2i}\right| = \frac{1}{3n}\left|\frac{1+i/n}{1+2i/(3n^2)}\right| > \frac{\frac{1}{3}}{n}$$

and the harmonic series diverges.

**22.** Divergent; $1/\sqrt{2n} = 1/\sqrt{2} \cdot 1/\sqrt{n}$; Series with $1/\sqrt{n}$ is divergent, since $1/\sqrt{n} > 1/n$ for $n = 2, 3, \cdots$ and the harmonic series diverges.

**24.** Divergent by the ratio test because

$$\begin{aligned} \left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{(3i)^{n+1}(n+1)!/(n+1)^{n+1}}{(3i)^n n!/n^n}\right| \\ &= 3(n+1)\frac{n^n}{(n+1)^{n+1}} \\ &= 3\left(\frac{n}{n+1}\right)^n \\ &= \frac{3}{\left(\frac{n+1}{n}\right)^n} \\ &= \frac{3}{\left(1+\frac{1}{n}\right)^n} \quad \rightarrow \quad \frac{3}{e} > 1. \end{aligned}$$

**25.** Divergent

**26.** It is essential that, from some $n$ on, the test ratio does not become greater than a fixed $q < 1$ instead of coming arbitrarily close to 1, as is the case, for instance, for the harmonic series, which *diverges*.

**30.** The form of the estimate of $R_n$ suggests we use the fact that the ratio test is a comparison test based on the geometric series. This gives

$$R_n = z_{n+1} + z_{n+2} + \cdots = z_{n+1}\left(1 + \frac{z_{n+2}}{z_{n+1}} + \frac{z_{n+3}}{z_{n+1}} + \cdots\right),$$

$$\left|\frac{z_{n+2}}{z_{n+1}}\right| \leqq q, \quad \left|\frac{z_{n+3}}{z_{n+1}}\right| = \left|\frac{z_{n+3}}{z_{n+2}}\,\frac{z_{n+2}}{z_{n+1}}\right| \leqq q^2, \qquad \text{etc.,}$$

$$|R_n| \leqq |z_{n+1}|(1 + q + q^2 + \cdots) = \frac{|z_{n+1}|}{1 - q}.$$

For the given series we obtain the test ratio

$$\frac{1}{2}\left|\frac{n+1+i}{n+1} \cdot \frac{n}{n+i}\right| = \frac{n}{2(n+1)}\sqrt{\frac{(n+1)^2+1}{n^2+1}}$$

$$= \frac{1}{2}\sqrt{\frac{n^4+2n^3+2n^2}{n^4+2n^3+2n^2+2n+1}} < \frac{1}{2};$$

from this with $q = \frac{1}{2}$ we have

$$|R_n| \leqq \frac{|z_{n+1}|}{1-q} = \frac{|n+1+i|}{2^n(n+1)} = \frac{\sqrt{(n+1)^2+1}}{2^n(n+1)} < 0.05.$$

Hence $n = 5$ (by computation), and

$$s = \frac{31}{32} + \frac{661}{960}\,i = 0.96875 + 0.688542i.$$

Exact to 6 digits is $1 + 0.693147i$.

## SECTION 15.2. Power Series, page 680

**Purpose.** To discuss the convergence behavior of power series, which will be basic to our further work (and which is simpler than that of series having arbitrary complex functions as terms).

**Comment.** Most complex power series appearing in practical work and applications have real coefficients because most of the complex functions of practical interest are obtained from calculus by replacing the real variable $x$ with the complex variable $z = x + iy$, retaining the real coefficients. Accordingly, in the problem set we consider primarily power series with real coefficients, also because complex coefficients would neither provide additional difficulties nor contribute new ideas.

**Proof of the Assertions in Example 6**

$R = 1/\tilde{L}$ follows from $R = 1/\tilde{l}$ by noting that in the case of convergence, $\tilde{L} = \tilde{l}$ (the only limit point). $\tilde{l}$ exists by the Bolzano–Weierstrass theorem, assuming boundedness of $\{\sqrt[n]{|a_n|}\}$. Otherwise, $\sqrt[n]{|a_n|} > K$ for infinitely many $n$ and any given $K$. Fix $z \neq z_0$ and take $K = 1/|z - z_0|$ to get

$$\sqrt[n]{|a_n(z-z_0)^n|} > K|z - z_0| = 1$$

and divergence for every $z \neq z_0$ by Theorem 9, Sec. 15.1.

Now, by the definition of a limit point, for a given $\epsilon > 0$ we have, for infinitely many $n$,

$$\tilde{l} - \epsilon < \sqrt[n]{|a_n|} < \tilde{l} + \epsilon;$$

hence for all $z \neq z_0$ and those $n$,

$$(*) \qquad (\tilde{l} - \epsilon)|z - z_0| < \sqrt[n]{|a_n(z - z_0)^n|} < (\tilde{l} + \epsilon)|z - z_0|.$$

The right inequality holds even for *all* $n > N$ ($N$ sufficiently large), by the definition of a *greatest* limit point.

Let $\tilde{l} = 0$. Since $\sqrt[n]{|a_n|} \geqq 0$, we then have convergence to 0. Fix any $z = z_1 \neq z_0$. Then for $\epsilon = 1/(2|z_1 - z_0|) > 0$ there is an $N$ such that $\sqrt[n]{|a_n|} < \epsilon$ for all $n > N$; hence

$$|a_n(z_1 - z_0)^n| > \epsilon^n |z_1 - z_0|^n = \frac{1}{2^n},$$

and convergence for all $z_1$ follows by the comparison test.

Let $\tilde{l} > 0$. We establish $1/\tilde{l}$ as the radius of convergence of (1) by proving

convergence of the series (1) if $|z - z_0| < 1/\tilde{l}$,

divergence of the series (1) if $|z - z_0| > 1/\tilde{l}$.

Let $|z - z_0| < 1/\tilde{l}$. Then, say, $|z - z_0|\tilde{l} = 1 - b < 1$. With this and $\epsilon = b/(2|z - z_0|) > 0$ in (*), for all $n > N$,

$$\sqrt[n]{|a_n(z - z_0)^n|} < \tilde{l}\,|z - z_0| + \epsilon|z - z_0| = 1 - b + \tfrac{1}{2}b < 1.$$

Convergence now follows from Theorem 9, Sec. 15.1.

Let $|z - z_0| > 1/\tilde{l}$. Then $|z - z_0|\tilde{l} = 1 + c > 1$. With this and $\epsilon = c/(2|z - z_0|) > 0$ in (*), for infinitely many $n$,

$$\sqrt[n]{|a_n(z - z_0)^n|} > \tilde{l}|z - z_0| - \epsilon|z - z_0| = 1 + c - \tfrac{1}{2}c > 1,$$

and divergence follows.

## SOLUTIONS TO PROBLEM SET 15.2, page 684

**4.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{6^n}(z - 4 + 3\pi i)^n$

**6.** Center $-1$, radius of convergence $\dfrac{1}{2}$ because in (6)

$$\frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

**7.** $\pi/4$, $\infty$ $\dfrac{2^n}{2^{n+1}} = \dfrac{1}{2}$

**8.** The center is $\pi i$. In (6) we obtain

$$\frac{n^n/n!}{(n+1)^{n+1}/(n+1)!} = \frac{n^n(n+1)}{(n+1)^{n+1}} = \frac{1}{\left(\dfrac{n+1}{n}\right)^n} = \frac{1}{\left(1 + \dfrac{1}{n}\right)^n} \quad \longrightarrow \quad \frac{1}{e}.$$

**9.** $-i, \sqrt{2}$

**10.** Center $2i$, radius of convergence $\infty$ because

$$\frac{1/n^n}{1/(n+1)^{n+1}} = (n+1)\left(\frac{n+1}{n}\right)^n = (n+1)\left(1+\frac{1}{n}\right)^n \quad \to \quad \infty.$$

**11.** $0, \sqrt{\frac{29}{10}}$

**12.** Center 0, radius of convergence 8

**14.** Center 0, radius of convergence $\infty$ because by (6),

$$\frac{\dfrac{(-1)^n}{4^{2n}(n!)^2}}{\dfrac{(-1)^{n+1}}{4^{2n+2}((n+1)!)^2}} \to -16(n+1)^2 \to \infty.$$

From Sec. 5.5 you see that this is the complex analog of the MacLaurin series of the Bessel function $J_0(x/2)$, so that is the MacLaurin series of $J_0(z/2)$ for complex $z$, as will follow in Sec. 15.4.

**16.** Center 0. From (6) we obtain the radius of convergence $2/27$ because

$$\frac{(3n)!/(2^n(n!)^3)}{(3n+3)!/(2^{n+1}((n+1)!)^3)} = 2\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \quad \to \quad \frac{2}{27}.$$

**17.** $1/\sqrt{3}$

**18.** Center 0, radius of convergence $\infty$. We mention that this is the series (36) of the error function in App. 3.1, extended to a complex variable $z$. Formula (6) gives

$$\frac{(2n+3)(n+1)!}{(2n+1)n!} = \frac{(2n+3)(n+1)}{2n+1} \quad \to \quad \infty.$$

**20. Team Project. (a)** The faster the coefficients go to zero, the larger $|a_n/a_{n+1}|$ becomes.

**(b)** (i) Nothing. (ii) $R$ is multiplied by $1/k$. (iii) The new series has radius of convergence $1/R$.

**(c)** In Example 6 we took the first term of one series, then the first term of the other, and so on alternately. We could have taken, for instance, the first three terms of one series, then the first five terms of the other, then again three terms and five terms, and so on; or we could have mixed three or more series term by term.

**(d)** No, because $|30 + 10i| > |31 - 6i|$.

## SECTION 15.3. Functions Given by Power Series, page 685

**Purpose.** To show what operations on power series are mathematically justified and to prove the basic fact that power series represent analytic functions.

**Main Content**

Termwise addition, subtraction, and multiplication of power series

Termwise differentiation and integration (Theorems 3, 4)

Analytic functions and derivatives (Theorem 5)

**Comment on Content**

That a power series is the Taylor series of its sum will be shown in the next section.

## SOLUTIONS TO PROBLEM SET 15.3, page 689

**5.** 4

**6.** $|z/(2\pi)|^2 < 1$ by integrating the geometric series. Thus $|z| < R = 2\pi$.

**7.** 5

**8.** $\frac{1}{3}$, where $1/n(n + 1)$ can be produced by two integrations of the geometric series.

**9.** $1/\sqrt{3}$

**10.** The binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

consists of the fixed $k!$, which has no effect on $R$, and factors

$$n(n-1)\cdots(n-k+1)$$

as obtained by differentiation. Since $\sum(z/2)^n$ has $R = 2$, the *answer* is 2.

**11.** $\sqrt{\dfrac{5}{2}}$

**12.** $\infty$, because $2n(2n - 1)$ results from differentiation, and for the coefficients without these factors we have in the Cauchy–Hadamard formula

$$\frac{1/n^n}{1/(n+1)^{n+1}} = \left(\frac{n+1}{n}\right)^n (n+1) \quad \to \quad \infty \qquad \text{as } n \to \infty.$$

**14.** 1, by applying Theorem 3 to $\sum z^{n+m}$.

**15.** $\dfrac{3}{5}$

**16.** This follows from

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \\ = f(-z) &= a_0 - a_1 z + a_2 z^2 - a_3 z^3 + - \cdots. \end{aligned}$$

**18.** This is a useful formula for binomial coefficients. It follows from

$$\begin{aligned} (1+z)^p(1+z)^q &= \sum_{n=0}^{p} \binom{p}{n} z^n \sum_{m=0}^{q} \binom{q}{m} z^m \\ &= (1+z)^{p+q} = \sum_{r=0}^{p+q} \binom{p+q}{r} z^r \end{aligned}$$

by equating the coefficients of $z^r$ on both sides. To get $z^n z^m = z^r$ on the left, we must have $n + m = r$; thus $m = r - n$, and this gives the formula in the problem.

**20. Team Project. (a)** Division of the recursion relation by $a_n$ gives

$$\frac{a_{n+1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}.$$

Take the limit on both sides, denoting it by $L$:

$$L = 1 + \frac{1}{L}.$$

Thus $L^2 - L - 1 = 0, L = (1 + \sqrt{5})/2 = 1.618$, an approximate value reached after just ten terms.

**(b)** The list is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233.$$

In the recursion, $a_n$ is the number of pairs of rabbits present and $a_{n-1}$ is the number of pairs of offsprings from the pairs of rabbits present at the end of the preceding month.

**(c)** Using the hint, we calculate

$$(1 - z - z^2)\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (a_n - a_{n-1} - a_{n-2})z^n = 1$$

where $a_{-1} = a_{-2} = 0$, and Theorem 2 gives $a_0 = 1$, $a_1 - a_0 = 0$, $a_n - a_{n-1} - a_{n-2} = 0$ for $n = 2, 3, \cdots$. The converse follows from the uniqueness of a power series representation (see Theorem 2).

## SECTION 15.4. Taylor and Maclaurin Series, page 690

**Purpose.** To derive and explain Taylor series, which include those for real functions known from calculus as special cases.

**Main Content**

Taylor series (1), integral formula (2) for the coefficients

Singularity, radius of convergence

Maclaurin series for $e^z$, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, $\operatorname{Ln}(1 + z)$

Theorem 2 connecting Taylor series to the last section

**Comment**

The series just mentioned, with $z = x$, are familiar from calculus.

## SOLUTIONS TO PROBLEM SET 15.4, page 697

**3.** $\frac{1}{2}z^2 - \frac{1}{48}z^6 + \cdots, R = \infty$

**4.** $(2 + z)\sum_{n=0}^{\infty} z^{2n} = 2 + z + 2z^2 + z^3 + 2z^4 + \cdots, \quad R = 1$

**5.** $\frac{1}{8} - \frac{1}{64}z^4 + \frac{1}{512}z^8 - \frac{1}{4096}z^{12} + \cdots, R = \sqrt[4]{8}$

**6.** $\frac{1}{1 + 2iz} = \frac{1}{1 - (-2iz)} = 1 - 2iz - 4z^2 + 8iz^3 + 16z^4 - 32iz^5 + \cdots, R = 1/2$

**7.** $1 - \cos z = \frac{1}{2}z^2 - \frac{1}{24}z^4 + \frac{1}{720}z^6 - \cdots, R = \infty$

**8.** $\frac{1}{2} - \frac{1}{2}\cos 2z = z^2 - \frac{1}{3}z^4 + \frac{2}{45}z^6 - \frac{1}{315}z^8 + - \cdots, \quad R = \infty$

**9.** $\displaystyle\int_0^z \left(1 - t^2 + \frac{1}{2}t^4 + \cdots\right) dt = z - 1/3\, z^3 + 1/10\, z^5, R = \infty$

**10.** The series is

$$f = z + \frac{2}{1 \cdot 3} z^3 + \frac{2^2}{1 \cdot 3 \cdot 5} z^5 + \frac{2^3}{1 \cdot 3 \cdot 5 \cdot 7} z^7 + \cdots, \quad R = \infty.$$

It can be obtained in several ways. **(a)** Integrate the Maclaurin series of the integrand termwise and form the Cauchy product with the series of $e^{z^2}$. **(b)** $f$ satisfies the differential equation $f' = 2zf + 1$. Use this, its derivatives $f'' = 2(f + zf')$, etc., $f(0) = 0, f'(0) = 1$, etc., and the coefficient formulas in (1). (c) Substitute $f = \sum_{n=0}^{\infty} a_n z^n$ and $f' = \sum_{n=0}^{\infty} n a_n z^{n-1}$ into the differential equation and compare coefficients; that is, apply the power series method (Sec. 5.1).

**12.** $z - \dfrac{z^5}{2!5} + \dfrac{z^9}{4!9} - \dfrac{z^{13}}{6!13} + - \cdots; \quad R = \infty$

**14.** $z - \dfrac{z^3}{3!3} + \dfrac{z^5}{5!5} - \dfrac{z^7}{7!7} + - \cdots; \quad R = \infty$

**16.** First of all, since $\sin(w + 2\pi) = \sin w$ and $\sin(\pi - w) = \sin w$, we obtain all values of $\sin w$ by letting $w$ vary in a suitable vertical strip of width $\pi$, for example, in the strip $-\pi/2 \leqq u \leqq \pi/2$. Now since

$$\sin\left(\frac{\pi}{2} - iy\right) = \sin\left(\frac{\pi}{2} + iy\right) = \cosh y$$

and

$$\sin\left(-\frac{\pi}{2} - iy\right) = \sin\left(-\frac{\pi}{2} + iy\right) = -\cosh y,$$

we have to exclude a part of the boundary of that strip, so we exclude the boundary in the lower half-plane. To solve our problem we have to show that the value of the series lies in that strip. This follows from $|z| < 1$ and

$$\left|\mathrm{Re}\left(z + \frac{1}{2}\frac{z^3}{3} + \cdots\right)\right| \leqq \left|z + \frac{1}{2}\frac{z^3}{3} + \cdots\right| \leqq |z| + \frac{1}{2}\frac{|z|^3}{3} + \cdots$$

$$= \arcsin |z| < \frac{\pi}{2}.$$

**18.** We obtain from the sum of the geometric series

$$\frac{1}{i(1 - i(z - i))} = -i \sum_{n=0}^{\infty} i^n (z - i)^n$$
$$= -i - i \cdot i(z - i) - i(-1)(z - i)^2 - \cdots, \quad R = 1.$$

**19.** $\dfrac{1}{2} + 1/2i - 1/2\, i(z + i) + \left(-\dfrac{1}{4} + 1/4\, i\right)(z + i)^2 + \dfrac{1}{4}(z + i)^3 + \cdots, R = \sqrt{2}$

**20.** We obtain

$$\begin{aligned}\cos^2 z &= \frac{1}{2} + \frac{1}{2}\cos 2z \\ &= \frac{1}{2} - \frac{1}{2}\cos(2z - \pi) \\ &= \frac{1}{2}\left[\frac{4}{2!}\left(z - \frac{1}{2}\pi\right)^2 - \frac{4^2}{4!}\left(z - \frac{1}{2}\pi\right)^4 + - \cdots\right], \quad R = \infty.\end{aligned}$$

**21.** $-1 + \frac{1}{2}(z - \pi)^2 - \frac{1}{24}(z - \pi)^4 + \cdots, \quad R = \infty$

**22.** $1 + \frac{1}{2!}(z - \pi i)^2 + \frac{1}{4!}(z - \pi i)^4 + \frac{1}{6!}(z - \pi i)^6 + \cdots, \quad R = \infty$

**23.** $-\frac{1}{4} + 1/4\, i(z + i) + \frac{3}{16}(z + i)^2 - 1/8\, i(z + i)^3 - \frac{5}{64}(z + i)^4 + \frac{3}{64} i(z + i)^5 + \cdots, R = 2$

**24.** We obtain

$$\begin{aligned}e^{z(z-2)} &= e^{(z-1+1)(z-1-1)} \\ &= e^{(z-1)^2-1} \\ &= \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z-1)^{2n}}{n!} \\ &= \frac{1}{e}\left(1 + (z-1)^2 + \frac{1}{2!}(z-1)^4 + \frac{1}{3!}(z-1)^6 + \cdots\right), \quad R = \infty.\end{aligned}$$

## SECTION 15.5. Uniform Convergence. *Optional,* page 698

**Purpose.** To explain the concept of uniform convergence. To show that power series have the advantage that they converge uniformly (exact formulation in Theorem 1). To discuss properties of general uniformly convergent series.

**Main Content**

Uniform convergence of power series (Theorem 1)

Continuous sum (Theorem 2)

Termwise integration (Theorem 3) and differentiation (Theorem 4)

Weierstrass test for uniform convergence (Theorem 5)

The test in Theorem 5 is very simple, conceptually and technically in its application.

## SOLUTIONS TO PROBLEM SET 15.5, page 704

**2.** $R = 7$, uniform convergence for $|z| \leqq 7 - \delta, \delta > 0$

**3.** $|z + i| \leq \sqrt{5} - \delta, \delta > 0$

**4.** $R = \infty$, uniform convergence on any bounded set

**6.** $|\tanh n^2| = 1$. Convergence for $|z^2| < \frac{1}{2}$. Uniform convergence for $|z| = 1/\sqrt{2} - \delta$, $\delta > 0$

**7.** Nowhere

**8.** $R = 1/\sqrt{3}$, uniform convergence for $|z - 1| \leqq 1/\sqrt{3} - \delta, \delta > 0$

**9.** $|z - 2i| \leq 3 - \delta, \delta > 0$

**10.** The MacLaurin series of $\sinh z$ converges for all $z$. Use Theorem 1.

**12.** $|z| \leq 1$, $1/(n^3 \sinh|z|) \leq 1/n^3 < 1/n^2$ and $\sum 1/n^2$ converges. Use the Weierstrass M-test.

**14.** $\left|\dfrac{z^n}{|z|^{2n} + 1}\right| \leqq \dfrac{1}{|z|^n} \leqq \dfrac{1}{2^n}$ and $\sum 2^{-n}$ converges. Use the Weierstrass $M$-test.

**16.** $|\tanh |z|| \leqq 1$ (see App. 3.1), $1/(n(n + 1)) < 1/n^2$ and $\sum 1/n^2$ converges. Use the Weierstrass $M$-test.

**17.** $R = 1/\pi > 0.25$; Use Theorem 1.

**18.** **Team Project.** **(a)** Convergence follows from the comparison test (Sec. 15.1). Let $R_n(z)$ and $R_n^*$ be the remainders of (1) and (5), respectively. Since (5) converges, for given $\epsilon > 0$ we can find an $N(\epsilon)$ such that $R_n^* < \epsilon$ for all $n > N(\epsilon)$. Since $|f_m(z)| \leqq M_m$ for all $z$ in the region $G$, we also have $|R_n(z)| \leqq R_n^*$ and therefore $|R_n(z)| < \epsilon$ for all $n > N(\epsilon)$ and all $z$ in the region $G$. This proves that the convergence of (1) in $G$ is uniform.

**(b)** Since $f_0' + f_1' + \cdots$ converges uniformly, we may integrate term by term, and the resulting series has the sum $F(z)$, the integral of the sum of that series. Therefore, the latter sum must be $F'(z)$.

**(c)** The converse is not true.

**(d)** Noting that this is a geometric series in powers of $q = (1 + z^2)^{-1}$, we have $q = |1 + z^2|^{-1} < 1$, $1 < |1 + z^2|^2 = (1 + x^2 - y^2)^2 + 4x^2y^2$, the exterior of a lemniscate. The series converges also at $z = 0$.

**(e)** We obtain (add and subtract 1)

$$x^2 \sum_{m=1}^{\infty} \frac{1}{(1 + x^2)^m} = x^2\left(-1 + \sum_{m=0}^{\infty} \frac{1}{(1 + x^2)^m}\right)$$
$$= -x^2 + \frac{x^2}{1 - \dfrac{1}{1 + x^2}} = -x^2 + 1 + x^2 = 1.$$

**20.** We obtain

$$|B_n| = \frac{2}{L}\left|\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx\right| < \frac{2}{L} ML$$

where $M$ is such that $|f(x)| < M$ on the interval of integration. Thus

$$|B_n| < K \,(= 2M).$$

Now when $t \geqq t_0 > 0$,

$$|u_n| = \left|B_n \sin \frac{n\pi x}{L}\, e^{-\lambda_n^2 t}\right| < Ke^{-\lambda_n^2 t_0}$$

because $\left|\sin \dfrac{n\pi x}{L}\right| \leqq 1$ and the exponential function decreases in a monotone fashion as $t$ increases. From this,

$$\left|\frac{\partial u_n}{\partial t}\right| = |-\lambda_n^2 u_n| = \lambda_n^2 |u_n| < \lambda_n^2 K e^{-\lambda_n^2 t_0} \quad \text{when} \quad t \geqq t_0.$$

Consider

$$\sum_{n=1}^{\infty} \lambda_n^2 K e^{-\lambda_n^2 t_0}.$$

Since $\lambda_n = \dfrac{cn\pi}{L}$, for the test ratio we have

$$\frac{\lambda_{n+1}^2 K \exp(-\lambda_{n+1}^2 t_0)}{\lambda_n^2 K \exp(-\lambda_n^2 t_0)} = \left(\frac{n+1}{n}\right)^2 \exp\left[-(2n+1)\left(\frac{c\pi}{L}\right)^2 t_0\right] \quad \to \quad 0$$

as $n \to \infty$, and the series converges. From this and the Weierstrass test it follows that $\sum \dfrac{\partial u_n}{\partial t}$ converges uniformly and, by Theorem 4, has the sum $\dfrac{\partial u}{\partial t}$, etc.

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**11.** 1.

**12.** $R = \frac{1}{4}$

**13.** 4

**14.** $R = \infty$

**15.** $\dfrac{1}{3}$.

**16.** $R = 1, -\ln(1 - z)$

**17.** $\infty, \exp(-2z)$.

**18.** $R = \infty, \quad \sin \pi z$

**19.** $\infty, \sinh\sqrt{z}$

**20.** $R = 5, \quad \left(1 - \dfrac{z}{3+4i}\right)^{-1}$

**21.** $\sum_{n=0}^{\infty} \dfrac{(z^2)^{2\,n}}{(2\,n)!}, R = \infty$

**22.** $\dfrac{1}{2} \sum\limits_{n=2}^{\infty} n(n-1)z^{n-2}, \quad R = 1$

**23.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n (z^2)^{2\,n}}{(2\,n)!}$

**24.** $\sum\limits_{n=0}^{\infty} (-\pi z)^n, \quad R = 1/\pi$

**25.** $\sum_{n=0}^{\infty} \dfrac{(z^{2n-2})}{n!} - 1$

**26.** $[(z - i) + i]^5 = (i + 5(z - i) - 10i(z - i)^2 - 10(z - i)^3 + 5i(z - i)^4 + (z - i)^5)$, a binomial expansion readily obtainable from Taylor's theorem.

**27.** $-z + \pi + 1/6\,(z - \pi)^3 - \dfrac{1}{120}(z - \pi)^5 + \cdots$

**28.** We obtain

$$\begin{aligned}\frac{1}{z - 2i + 2i} &= \frac{1}{2i\left(1 + \dfrac{z - 2i}{2i}\right)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2i)^{n+1}}(z - 2i)^n \\ &= -\tfrac{1}{2}i + \tfrac{1}{4}(z - 2i) + \tfrac{1}{8}i(z - 2i)^2 - \cdots, \quad R = 2.\end{aligned}$$

**30.** $e^{z-\pi i+\pi i} = -\displaystyle\sum_{n=0}^{\infty} \frac{(z - \pi i)^n}{n!}, \quad R = \infty$; here we used $e^{-\pi i} = -1$.