Homework 3 of Probabilistic Methods and Random Graph Theory

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1. Lower and Upper Bound of Balls and Bins

Assume that there are m balls, the probability that no two balls land in the same bin will be

$$\begin{aligned} P &= 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-m+1}{n} \\ &= (1-0) \cdot (1-\frac{1}{n}) \cdot (1-\frac{2}{n}) \cdots (1-\frac{m-1}{n}) \end{aligned}$$

With the fact that $e^{-x} \geq 1-x$ and $e^{-x-x^2} \leq 1-x$ for all $0 \leq x \leq \frac{1}{2}$,

$$\begin{aligned} P &= (1-0) \cdot (1-\frac{1}{n}) \cdot (1-\frac{2}{n}) \cdots (1-\frac{m-1}{n}) \\ &\leq e^{-0} \cdot e^{-\frac{1}{n}} \cdot e^{-\frac{2}{n}} \cdots e^{-\frac{m-1}{n}} \\ &= e^{-\frac{m(m-1)}{2n}} \end{aligned}$$

Let $c_1 = 2$,

$$\begin{aligned} P &\leq e^{-\frac{m(m-1)}{2n}} \\ &= e^{-\frac{2\sqrt{n}(2\sqrt{n}-1)}{2n}} \\ &\leq e^{-1} \end{aligned}$$

On the other hand, $\frac{m-1}{n} \leq \frac{1}{2}$ holds for sufficiently large n.

$$\begin{aligned} P &= (1-0) \cdot (1-\frac{1}{n}) \cdot (1-\frac{2}{n}) \cdots (1-\frac{m-1}{n}) \\ &\geq e^{-0} \cdot e^{-\frac{1}{n}-(\frac{1}{n})^2} \cdot e^{-\frac{2}{n}-(\frac{2}{n})^2} \cdots e^{-\frac{m-1}{n}-(\frac{m-1}{n})^2} \\ &= e^{-\frac{m(m-1)}{2n}-\frac{m(m-1)(2m-1)}{6n^2}} \\ &\geq e^{-\frac{m(m-1)}{n}} \\ &\geq e^{-\frac{m^2}{n}} \end{aligned}$$

Let $c_2 = \frac{1}{2}$,

$$\begin{aligned} P &\geq e^{-\frac{m^2}{n}} \\ &= e^{-\frac{1}{4}} \\ &\geq \frac{1}{2} \end{aligned}$$

2. Chernoff Bound for k-gaps

The second part of this solution refers to the notes by Dr. Christine Chung, Connecticut College.

Let the random variable X be the number of k-gaps in the n bins. Define random variable

$$X_i = \begin{cases} 1, & \text{if there exists a } k\text{-gap starting at bin } i \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$
\begin{aligned}
EX_i &= Pr(X_i = 1) \\
&= (\frac{n-k}{n})^m
\end{aligned}
$$

Thus,

$$
\begin{aligned}
EX &= \sum_{i=0}^{n-k} EX_i \\
&= \sum_{i=0}^{n-k} (\frac{n-k}{n})^m \\
&= (n-k+1)(\frac{n-k}{n})^m
\end{aligned}
$$

Let the random variable x_i be the number of balls in bin i and let its corresponding independent Poisson random variable be y_i. Similarly, define an independent Poisson random variable Y_i that corresponds to X_i. Note that X_i can equivalently be defined as a function of $x_i, x_{i+1}, \ldots, x_{i+k-1}$ as follows:

$$
X_i = \begin{cases} 1, & \text{if } \sum_{j=i}^{i+k-1} x_j = 0 \\ 0, & \text{otherwise} \end{cases}
$$

Similarly, Y_i can be defined as $f(y_i, y_{i+1}, \ldots, y_{i+k-1})$. Let

$$
Z_i = \sum_{j=0}^{\frac{n}{k}} X_{i+jk}
$$

and let

$$
Z_i^P = \sum_{j=0}^{\frac{n}{k}} Y_{i+jk}
$$

With this in mind, we can express the total number of k-gaps as

$$
X = \sum_{i=1}^{k-1} Z_i
$$

Since Z_i is a function of X_i, which is a function of x_i, and

$$
EZ_i = \frac{EX}{k}
$$

by linearity of expectation, we can let

$$
g(x_0, \ldots, x_{n-1}) = \begin{cases} 1, & \text{if } Z_0 \geq (1+\delta)\frac{EX}{k} \\ 0, & \text{otherwise} \end{cases}
$$

and similarly,

$$
g(y_0, \ldots, y_{n-1}) = \begin{cases} 1, & \text{if } Z_0^P \geq (1+\delta)\frac{EX}{k} \\ 0, & \text{otherwise} \end{cases}
$$

Then,

$$
Eg(x_0, \ldots, x_{n-1}) = Pr(Z_0 \geq (1+\delta)\frac{EX}{k})
$$

and

$$Eg(y_0, \ldots, y_{n-1}) = Pr(Z_0^P \geq (1+\delta)\frac{EX}{k})$$

By Theorem 5.7, Page 101, Book "Probability and Computing" by Michael Mitzenmacher and Eli Upfal,

$$Eg(x_0, \ldots, x_{n-1}) \leq e\sqrt{m}Eg(y_0, \ldots, y_{n-1})$$

Thus,

$$Pr(Z_0 \geq (1+\delta)\frac{EX}{k}) \leq e\sqrt{m}Pr(Z_0^P \geq (1+\delta)\frac{EX}{k})$$

By Theorem 4.4, Page 64, Book "Probability and Computing" by Michael Mitzenmacher and Eli Upfal,

$$Pr(Z_0^P \geq (1+\delta)\frac{EX}{k}) \leq e^{-\frac{EX\delta^2}{3k}}$$

Finally, by observing that if $X \geq (1+\delta)EX$ then there exists an i such that

$$Z_i \geq (1+\delta)\frac{EX}{k}$$

and then applying union bound and substitution, we know

$$\begin{aligned} Pr(X \geq (1+\delta)EX) &\leq Pr(\bigcup_{j=0}^{k-1} Z_i \leq (1+\delta)\frac{EX}{k}) \\ &\leq \sum_{j=0}^{k-1} Pr(Z_i \leq (1+\delta)\frac{EX}{k}) \\ &\leq ke\sqrt{m}Pr(Z_i^P \leq (1+\delta)\frac{EX}{k}) \\ &= k\sqrt{m}e^{1-\frac{EX\delta^2}{3k}} \end{aligned}$$

3. Multi-players of Balls and Bins

This solution refers to the notes by Prof. Jyrki Kivinen, Department of Computer Science, University of Helsinki.

We divide the bins into $N = \log n$ continuous blocks, such that block $i \in \{0, \ldots, N-1\}$ contains bins

$$\frac{in}{N}, \ldots, \frac{(i+1)n}{N} - 1$$

Now, if a player places a ball in a bin contained in block i, he must have placed his first ball in a bin contained either in the same block i or in the previous block $i-1 \bmod N$. Thus, the number of first balls placed either in the same block or in the previous block is an upper bound for the number of balls placed in the bin.

Now we model first balls and blocks using the balls-and-bins model, such that blocks correspond to bins and first balls correspond to balls. There are N first balls which are placed independently and uniformly at random into N blocks. Now Lemma 5.1, Page 93, Book "Probability and Computing" by Michael Mitzenmacher and Eli Upfal says, that, for a sufficiently large N, with probability at least

$$1 - \frac{1}{N}$$

no block contains more than

$$\frac{3\log N}{\log\log N}$$

first balls. As a consequence, using the observation in the previous paragraph, none of the bins contains more than

$$\frac{6\log N}{\log\log N} = O(\frac{\log\log n}{\log\log\log n})$$

balls.

4. Random String

The random string is: 11000001101111001001. The program is the same as the previous one.