CS2336 Discrete Mathematics

Exam 2
December 2, 2019 (10:10–12:30)

Answer all questions. Total marks = 110. Maximum Score = 110/100. For all the proofs, if it is incomplete, a large portion of marks may be deducted.

1. Peter is studying a recurrence as shown below:

$$a_n = a_{n-1} + 2a_{n-2}.$$

He tries to plug in $a_0 = 8$ and $a_1 = 1$, and finds that using the recurrence, he will obtain $a_2 = 17$, $a_3 = 19$, $a_4 = 53$, and so on.

Peter conjectures that for any integer $n \geq 0$, a_n would obey the following formula:

$$a_n = 3 \times (2)^n + 5 \times (-1)^n.$$

Indeed, the recurrence for a_n is called a *linear recurrence*, and there is a standard method to find a formula for a_n (without referring to other terms like a_{n-1} or a_{n-2}); however, Peter does not know this. Fortunately, Peter has learnt induction technique a few weeks ago.

(15%) Use induction to show that Peter's formula is correct.

2. Let n be a positive integer, and consider an array with 2 rows and $2n$ columns. Each entry in the array is either `0` or `1`. It is known that for each row, exactly n entries are `0` and exactly n entries are `1`.

For a particular column, if both entries are `0`, we call it a `0`-column; else, if both entries are `1`, we call it a `1`-column.

(15%) Show that the number of `0`-columns is the same as the number of `1`-columns.

For instance, suppose $n = 3$. Suppose the array looks like the following:

1	0	1	0	0	1
0	0	1	1	0	1

Each row contains exactly n `0`s and exactly n `1`s. Also, we see that there are two `0`-columns (the 2nd one and the 5th one), and there are two `1`-columns (the 3rd one and the 6th one).

3. (15%) How many binary strings of length 6 we can find, such that each string does not contain three or more contiguous `1`s?

For instance, `011011` is counted, but `011110` is not.

4. (a) (15%) How many positive integral solutions are there for $x + y + z = 99$?

   (b) (10%) How many positive integral solutions are there for $x + y + z = 99$, if x, y, z are restricted to be all odd integers?

5. (15%) Use a combinatorial argument to show that for any positive integers r, k with $r > k$:

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}.$$

Note: No marks will be given if your proof is not a combinatorial proof.

6. Consider the triangular grid as shown in Figure 1, which contains 21 nodes. If two nodes u and v in the grid are connected directly by an edge, we say u and v are *adjacent*.

   (a) (15%) Show that if we color any 9 of these nodes as black, we can always find two black nodes that are adjacent.

   (b) (5%, Challenging) Show that if we color any 8 of these nodes as black, we can always find two black nodes that are adjacent.

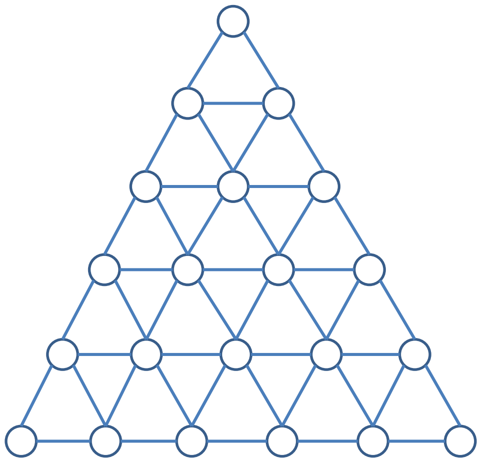

Figure 1: Figure for Question 6

7. (Extremely Challenging; Estimated time to solve is more than 1 hour)
   Let $f : \mathbb{N} \to \mathbb{N}$ be a function, such that for any positive integer n, $f(n)$ is equal to some positive integer. It is known that for any positive integer n, $f(n+1) > f(f(n))$.

   (5%) Show that $f(n) = n$. Justify your answer.