CHAPTER 6 Laplace Transforms

In the ninth edition, this chapter underwent major changes, which have been retained and further extended. This concerns a more natural order of the material and changing emphasis placed on the various topics. This has made the chapter more teachable and simpler, a goal reached by placing Dirac's delta function in a separate section, discussing partial fractions earlier, and in terms of practical problems rather than in terms of impractical and lengthy general formulas.

Also, to have a better flow of ideas, convolution and nonhomogeneous linear ODEs now appear earlier, whereas differentiation and integration of transforms (not of functions!) are now presented near the end of the chapter and with less emphasis.

SECTION 6.1. Laplace Transform. Linearity. First Shifting Theorem (*s*-Shifting), page 204

Purpose. To explain the basic concepts, to present a short list of basic transforms, and to show how these are derived from the definition.

Main Content, Important Concepts

Transform, inverse transform, linearity

First shifting theorem

Table 6.1

Existence and its practical significance

Comment on Table 6.1

After working for a while in this chapter, the student should be able to memorize these transforms. Further transforms in Sec. 6.9 are derived as we go along, many of them from Table 6.1.

Problem Set 6.1

The problem set addresses the two main tasks in connection with Laplace transform, namely, to find transforms of given functions (Probs. 1–16 and 33–36), and to find functions for given transforms (that is, to find inverse transforms, Probs. 25–32 and 37–45). This includes the task of finding formulas for graphically given functions, and, as techniques, integration, reduction by partial fractions, the use of Table 6.1, and s-shifting (in Probs. 33–45).

Problems 17–24 include a modest amount of theory compatible with the level of this book.

SOLUTIONS TO PROBLEM SET 6.1, page 210

1. $2/s^2 + 8/s$

2. $a^2/s - 2ab/s^2 + 2b^2/s^3$

3. $s/(s^2 + 4\pi^2)$

4. $\mathcal{L}(\cos^2 \omega t) = \mathcal{L}\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega t\right) = \frac{1}{2s} + \frac{1}{2} \cdot \frac{s}{s^2 + 4\omega^2}$

5. $\frac{1}{(s-4)(s-2)}$

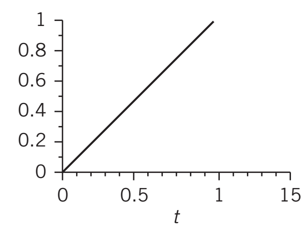
6. $e^{-t}\sinh 4t = \frac{1}{2}(e^{3t} - e^{-5t})$; transform $\frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+5}\right) = \frac{4}{(s+1)^2 - 16}$

This can be checked by the first shifting theorem.

7. $\frac{\cos(\theta)s - \sin(\theta)\omega}{s^2 + \omega^2}$

8. $1.5\sin(3t - \pi/2) = 1.5(\sin 3t \cos \pi/2 - \cos 3t \sin \pi/2) = -1.5\cos 3t$; transform $-1.5s/(s^2 + 9)$

9. $\frac{1 - (s+1)e^{-s}}{s^2}$



Problem 9

10. $\frac{k}{s}(1 - e^{-cs})$

12. $\frac{1 - e^{-s}}{s^2} - \frac{1}{se^{2s}}$

14. $k\int_a^b e^{-st}\,dt = \frac{k}{s}(e^{-as} - e^{-bs})$

16. $\int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt$. Integration by parts gives

$$-\frac{e^{-st}t}{s}\bigg|_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{e^{-st}(2-t)}{s}\bigg|_1^2 - \frac{1}{s}\int_1^2 e^{-st}\,dt$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1) + \frac{e^{-s}}{s} + \frac{1}{s^2}(e^{-2s} - e^{-s})$$

$$= \frac{1}{s^2}(-e^{-s} + 1 + e^{-2s} - e^{-s}) = \frac{(1 - e^{-s})^2}{s^2}.$$

18. Use Prob. 10 with $k = 1$ and $c = 2$ to obtain

$$\mathcal{L}(f_1) = \mathcal{L}(1) - \mathcal{L}(f) = \frac{1}{s} - \frac{1}{s}(1 - e^{-2s}) = \frac{e^{-2s}}{s}.$$

20. No matter how large we choose M and k, we have $e^{t^2} > Me^{kt}$ for all t greater than some t_0 because $t^2 > \ln M + kt$ for all sufficiently large t (and fixed positive M and k).

22. Let $st = \tau$, $t = \tau/s$, $dt = d\tau/s$. Then

$$\begin{aligned}\mathcal{L}(1/\sqrt{t}) &= \int_0^\infty e^{-st} t^{-1/2}\,dt \\ &= \int_0^\infty e^{-\tau}\left(\frac{\tau}{s}\right)^{-1/2} \frac{1}{s}\,d\tau \\ &= s^{-1/2}\int_0^\infty e^{-\tau}\tau^{-1/2}\,d\tau \\ &= s^{-1/2}\Gamma(\tfrac{1}{2}) = \sqrt{\pi/s}.\end{aligned}$$

24. Let $f = \mathcal{L}^{-1}(F)$, $g = \mathcal{L}^{-1}(G)$. Since the *transform* is linear, we obtain

$$aF + bG = a\mathcal{L}(f) + b\mathcal{L}(g) = \mathcal{L}(af + bg).$$

Now apply $\mathcal{L}^{-1}$ on both sides to get the desired result,

$$\mathcal{L}^{-1}(aF + bG) = \mathcal{L}^{-1}\mathcal{L}(af + bg) = af + bg = a\mathcal{L}^{-1}(F) + b\mathcal{L}^{-1}(G).$$

Note that we have proved much more than just the claim, namely, the following.

Theorem. *If a linear transformation has an inverse, the inverse is linear.*

25. $0.2 \cos(1.4t) + \sin(1.4t)$

26. The inverse transform is

$$5 \cosh 5t + \tfrac{1}{5}\sinh 5t = \tfrac{13}{5}e^{5t} + \tfrac{12}{5}e^{-5t}.$$

27. $\dfrac{\cos(\frac{1}{2}\frac{\pi t}{L})}{L^2}$

28. In terms of partial fractions, the given function is

$$\frac{1}{\sqrt{3} + \sqrt{2}}\left(\frac{1}{s - \sqrt{3}} - \frac{1}{s + \sqrt{2}}\right).$$

Hence the inverse transform is

$$\frac{1}{\sqrt{3} + \sqrt{2}}\left(e^{t\sqrt{3}} - e^{-t\sqrt{2}}\right).$$

29. $-1/15t^3(-5 + 6t^2)$

30. The inverse transform is

$$4 \cosh 4t + 8 \sinh 4t = 6e^{4t} - 2e^{-4t}.$$

31. $2e^{3t} - 3e^{-t}$

32. If $a \neq b$, then $\dfrac{1}{b - a}(e^{-at} - e^{-bt})$. If $a = b$, then te^{-bt} by the shifting theorem (or from the first result by l'Hôpital's rule, taking derivatives with respect to a).

33. $\frac{6}{(s + 2)^4}$

34. $\dfrac{k(s + a)}{(s + a)^2 + \omega^2}$

35. $32\frac{\pi}{(2s + 1)^2 + 64\pi^2}$

36. $$\mathcal{L}\left(\frac{1}{2}(e^t - e^{-t})\cos t\right) = \frac{1}{2}\left(\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right)$$
$$= \frac{1}{2}\left(\frac{2s^2-4}{s^4+4}\right)$$
$$= \frac{s^2-2}{s^4+4}$$

37. $\pi t^2 \mathrm{e}^{-\pi t}$

38. $3t^2 e^{-t}$

39. $\frac{3}{4} t^5 \mathrm{e}^{-\sqrt{3}t}$

40. $2e^t \sinh 2t = e^{3t} - e^{-t}$

41. $\mathrm{e}^{-2\pi t} \sinh(\pi t)$

42. $(a_0 + a_1 t + \frac{1}{2} a_2 t^2) e^{-t}$

43. $\frac{1}{4}\mathrm{e}^{-t}(12\cos(2t) + \sin(2t))$

44. $e^{-kt}(a\cos \pi t + b \sin \pi t)$

45. $k_0 + k_1 t\mathrm{e}^{at}$

SECTION 6.2. Transforms of Derivatives and Integrals. ODEs, page 211

Purpose. To get a first impression of how the Laplace transform solves ODEs and initial value problems, the task for which it is designed.

Main Content, Important Concepts

(1) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

Extension of (1) to higher derivatives [(2), (3)]

Solution of an ODE, subsidiary equation

Transform of the integral of a function (Theorem 3)

Transfer function (6)

Shifted data problems (Example 6)

Comment on ODEs

The last of the three steps of solution is the hardest, but we shall derive many general properties of the Laplace transform (collected in Sec. 6.8) that will help, along with formulas in Table 6.1 and those in Sec. 6.9, so that we can proceed to ODEs for which the present method is superior to the classical one.

Problem Set 6.2

Problems 1–15 concern IVPs, including those with shifted data (12–15).

Problems 16–22 show how to obtain new transforms from known ones by differentiation, the basic formulas being (1) and (2).

Conversely, Probs. 23–29 show how to obtain new inverse transforms by integration (Theorem 3).

Project 30 extends Theorem 1 to the case when a discontinuity (finite jump) occurs.

SOLUTIONS TO PROBLEM SET 6.2, page 216

1. $y = -\frac{18}{5}\sin(x)\cos(x) + 3/5 - 6/5\,(\cos(x))^2 + 3/5\mathrm{e}^{-2/3x}$

2. The subsidiary equation is $sY - 1.5 + 2Y = 0$. Hence

$$(s + 2)Y = 1.5, \qquad Y = 1.5/(s + 2), \qquad y = 1.5e^{-2t}.$$

3. $s^2Y + (Y - 1)s - 6Y - 2 = 0$, $Y = \frac{s+2}{s^2+s-6}$, $y = 4/5\,\mathrm{e}^{2t} + 1/5\,\mathrm{e}^{-3t}$.

4. We obtain

$$(s^2 + 9)Y = \frac{10}{s + 1}.$$

The solution of this subsidiary equation is

$$Y = \frac{10}{(s + 1)(s^2 + 9)} = \frac{1}{s + 1} - \frac{s - 1}{s^2 + 9}.$$

This gives the solution $y = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t$.

6. The subsidiary equation is

$$(s^2 - 6s + 5)Y = 3.2s + 6.2 - 6 \cdot 3.2 + 29s/(s^2 + 4).$$

The solution Y is

$$Y = \frac{(3.2s - 13)(s^2 + 4) + 29s}{(s - 1)(s - 5)(s^2 + 4)}.$$

In terms of partial fractions, this becomes

$$Y = \frac{1}{s - 1} + \frac{2}{s - 5} + \frac{0.2s - 2.4 \cdot 2}{s^2 + 4}.$$

The inverse transform of this gives the solution

$$y = e^t + 2e^{5t} + 0.2\cos 2t - 2.4\sin 2t.$$

8. The subsidiary equation is

$$(s - 2)^2Y = 8.1s + 3.9 - 4 \cdot 8.1.$$

Its solution is

$$Y = \frac{8.1s - 28.5}{(s - 2)^2} = \frac{8.1(s - 2) + 8.1 \cdot 2 - 28.5}{(s - 2)^2}.$$

This can be written

$$Y = \frac{8.1}{s - 2} - \frac{12.3}{(s - 2)^2}.$$

The inverse transform (the solution of the IVP) is

$$y = 8.1e^{2t} - 12.3te^{2t}.$$

It has the form expected in the case of a double root of the characteristic equation.

9. $y = 2e^{2t} + e^t - 1 + 2t$

10. $(s^2 + 0.04)Y = -25s + 0.04/s^3$. The solution is

$$Y = -\frac{25}{s} + \frac{1}{s^3}.$$

The inverse transform of Y (the solution of the IVP) is

$$y = -25 + 0.5t^2.$$

Note that the initial values are such that the general solution of the homogeneous ODE $y_h = c_1 \cos 0.2t + c_2 \sin 0.2t$ does not contribute to the solution of the IVP.

12. $t = \tilde{t} + 2$, so that the "shifted problem" is

$$\tilde{y}'' + 2\tilde{y}' - 3\tilde{y} = 0, \tilde{y}(0) = -3, \tilde{y}'(0) = -5$$

The subsidiary equation is

$$(s^2 + 2s - 3)\,\tilde{Y} + 11 + 3s = 0$$

Its solution is

$$\tilde{Y} = \frac{-11 + 3s}{s^2 + 2s - 3} = \frac{1}{2(s + 3)} - \frac{7}{2(s - 1)}$$

Inversion gives,

$$\tilde{y} = -7/2e^{\tilde{t}} + 1/2e^{-3\tilde{t}}$$

Hence the solution to the given problem is

$$y = -7/2e^{(t-2)} + 1/2e^{-3(t-2)}$$

14. $t = \tilde{t} + 2$, so that the "shifted equation" is

$$\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = 50\tilde{t}.$$

The corresponding subsidiary equation is

$$[(s + 1)^2 + 4]\tilde{Y} = -4s + 14 - 2 \cdot 4 + 50/s^2.$$

Its solution is

$$\tilde{Y} = \frac{10}{s^2} - \frac{4}{s} + \frac{4}{(s + 1)^2 + 4}.$$

The inverse transform is

$$\tilde{y} = 10\tilde{t} - 4 + 2e^{-\tilde{t}} \sin 2\tilde{t}.$$

Hence the given problem has the solution

$$y = 10(t - 2) - 4 + 2e^{-(t-2)} \sin 2(t - 2).$$

16. $f = t \cos 4t, f' = \cos 4t - 4t \sin 4t, f'' = -8 \sin 4t - 16t \cos 4t$. Hence

$$\mathcal{L}(f'') = \frac{-32}{s^2 + 16} - 16\mathcal{L}(f) = s^2\mathcal{L}(f) - s \cdot 0 - 1$$

and thus

$$(s^2 + 16)\mathcal{L}(f) = \frac{-32}{s^2 + 16} + 1 = \frac{s^2 - 16}{s^2 + 16}.$$

Hence the *answer* is

$$\mathcal{L}(f) = \frac{s^2 - 16}{(s^2 + 16)^2}.$$

17. $\frac{1}{(s-a)^2}$

18. $f = \cos^2 2t, f' = -4\cos 2t \sin 2t = -2\sin 4t$. Hence

$$\mathcal{L}(f') = \frac{-8}{s^2 + 16} = s\mathcal{L}(f) - 1.$$

Solving for $s\mathcal{L}(f)$ gives

$$s\mathcal{L}(f) = \frac{s^2 + 8}{s^2 + 16}.$$

Division by s gives the *answer*

$$\mathcal{L}(f) = \frac{s^2 + 8}{s(s^2 + 16)}.$$

19. $\frac{2\omega^2 + s^2}{s(s^2 + 4\omega^2)}$

20. $f = \sin^4 t, f' = 4\sin^3 t\cos t, f'' = 12\sin^2 t\cos^2 t - 4\sin^4 t$. From this and (2), it follows that

$$\mathcal{L}(f'') = 12\mathcal{L}(\sin^2 t\cos^2 t) - 4\mathcal{L}(f) = s^2\mathcal{L}(f).$$

Collecting terms and using Prob. 19 with $\omega = 2$ and $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we obtain

$$(s^2 + 4)\mathcal{L}(f) = 3\mathcal{L}(\sin^2 2t) = \frac{3 \cdot 8}{s(s^2 + 16)}.$$

Answer

$$\mathcal{L}(f) = \frac{24}{s(s^2 + 4)(s^2 + 16)}.$$

21. $\mathcal{L}(f') = \mathcal{L}(\sinh 2t) = s\mathcal{L}(f) - 0$. Hence, $\mathcal{L}f = \frac{2}{s(s^2 - 4)}$.

22. **Project.** We derive **(a)**. We have $f(0) = 0$ and

$$f'(t) = \cos\omega t - \omega t\sin\omega t, \qquad f'(0) = 1$$
$$f''(t) = -2\omega\sin\omega t - \omega^2 f(t).$$

By (2),

$$\mathcal{L}(f'') = -2\omega\frac{\omega}{s^2 + \omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f) - 1.$$

Collecting $\mathcal{L}(f)$-terms, we obtain

$$\mathcal{L}(f)(s^2 + \omega^2) = \frac{-2\omega^2}{s^2 + \omega^2} + 1 = \frac{s^2 - \omega^2}{s^2 + \omega^2}.$$

Division by $s^2 + \omega^2$ on both sides gives (a).

In (b) on the right we get from (a)

$$\mathcal{L}(\sin\omega t - \omega t\cos\omega t) = \frac{\omega}{s^2 + \omega^2} - \omega\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$$

Taking the common denominator and simplifying the numerator,

$$\omega(s^2 + \omega^2) - \omega(s^2 - \omega^2) = 2\omega^3$$

we get **(b)**.

(c) is shown in Example 1.

(d) is derived the same way as (b), with $+$ instead of $-$, so that the numerator is

$$\omega(s^2 + \omega^2) + \omega(s^2 - \omega^2) = 2\omega s^2,$$

which gives (d).

(e) is similar to (a). We have $f(0) = 0$ and obtain

$$f'(t) = \cosh at + at \sinh at, \qquad f'(0) = 1$$
$$f''(t) = 2a \sinh at + a^2 f(t).$$

By (2) we obtain

$$\mathcal{L}(f'') = \frac{2a^2}{s^2 - a^2} + a^2\mathcal{L}(f) = s^2\mathcal{L}(f) - 1.$$

Hence

$$\mathcal{L}(f)(s^2 - a^2) = \frac{2a^2}{s^2 - a^2} + 1 = \frac{s^2 + a^2}{s^2 - a^2}.$$

Division by $s^2 - a^2$ gives (e).

(f) follows similarly. We have $f(0) = 0$ and, furthermore,

$$f'(t) = \sinh at + at \cosh at, \qquad f'(0) = 0$$
$$f''(t) = 2a \cosh at + a^2 f(t)$$
$$\mathcal{L}(f''(t)) = 2a \frac{s}{s^2 - a^2} + a^2\mathcal{L}(f) = s^2\mathcal{L}(f)$$
$$\mathcal{L}(f)(s^2 - a^2) = \frac{2as}{s^2 - a^2}.$$

Division by $s^2 - a^2$ gives formula (f).

23. $6 - 6e^{-1/3t}$.

24. We start from

$$\mathcal{L}^{-1}\left(\frac{1}{s - 2\pi}\right) = e^{2\pi t}$$

and integrate twice. The first integration gives

$$\frac{1}{2\pi}(e^{2\pi t} - 1).$$

Multiplication by 20 and another integration from 0 to t gives the *answer*

$$\frac{20}{4\pi^2}(e^{2\pi t} - 1) - \frac{20t}{2\pi} = \frac{1}{5\pi^2}(e^{2\pi t} - 1) - \frac{10t}{\pi}.$$

25. $4\frac{1 - \cos(1/2\omega t)}{\omega^2}$.

26. The inverse of $1/(s^2 - 1)$ is $\sinh t$. A first integration from 0 to t gives $\cosh t - 1$. Another integration from 0 to t yields $\sinh t - t$. This can be confirmed as follows.

$$\mathcal{L}\left(\frac{1}{s^4 - s^2}\right) = \mathcal{L}\left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right) = \sinh t - t.$$

27. $-\frac{1}{4}\cos 2t - \sin 2t + \frac{1}{4} + 2t.$

28. The transform of

$$\frac{3s + 4}{s^2 + k^2}$$

is $3\cos kt + \dfrac{4}{k}\sin kt$. A first integration from 0 to t gives

$$\frac{3}{k}\sin kt - \frac{4}{k^2}(\cos kt - 1).$$

Another integration from 0 to t gives the *answer*

$$-\frac{3}{k^2}(\cos kt - 1) - \frac{4}{k^2}\left(\frac{1}{k}\sin kt - t\right).$$

30. Project. **(a)** Theorems 1 and 2 are crucial in solving ODEs, whereas Theorem 3 serves as a tool for obtaining new transforms, that is, as one of various tools for this purpose.

(b) In the integration by parts shown in the proof of Theorem 1 we now have to integrate from 0 to a and then from a to ∞, thus obtaining $f(a - 0)e^{-as}$ from the upper limit of integration of the first integral and $-f(a + 0)e^{-as}$ from the lower limit of integration of the second integral.

(c) Direct integration of the defining integral formulas gives

$$\mathcal{L}(f) = (1 - e^{-(s+1)})/(s + 1)$$

and

$$\mathcal{L}(f') = (e^{-(s+1)} - 1)/(s + 1).$$

Formula (1*) confirms this by straightforward calculation and simplification since $f(0) = 1$, $a = 1$, and

$$f(a + 0) - f(a - 0) = 0 - e^{-1}.$$

SECTION 6.3. Unit Step Function (Heaviside Function). Second Shifting Theorem (*t*-Shifting), page 217

Purpose

1. To introduce the unit step function $u(t - a)$, which, together with Dirac's delta (Sec. 6.4), greatly increases the usefulness of the Laplace transform.
2. To find the transform of

$$\tilde{f}(t) = 0 \text{ if } f < a \quad \text{and} \quad \tilde{f}(t) = f(t - a) \text{ if } t > a$$

where the transform of $f(t)$ $(0 < t < \infty)$ is known (Theorem 1, called Second Shifting Theorem).

Main Content, Important Concepts

Unit step function (1), its transform (2)

Second shifting theorem (Theorem 1)

Comment on the Unit Step Function

Problem Set 6.3 shows that $u(t - a)$ is the basic function for representing discontinuous functions.

Problem Set 6.3

Problems 2–11 concern the role of the unit step function in t-shifting in the second shifting theorem.

Problems 12–17 show the application of that theorem in finding inverse transforms.

IVPs, some with discontinuous inputs (right sides of the ODEs) are solved in Probs. 18–27; this is a typical task using the Laplace transform.

Models of electric circuits follow in Probs. 28–40; these have a discontinuous EMF (electromotive force); the most general of these circuits are *RLC*-circuits (Probs. 38–40).

SOLUTIONS TO PROBLEM SET 6.3, page 223

2. $t(1 - u(t-2)) = t - [(t-2) + 2]u(t-2)$. Hence the transform is

$$\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}.$$

3. $\frac{e^{-3s}}{s^2}$

4. $\cos 2t(1 - u(t - \pi)) = \frac{s(1 - e^{-s\pi})}{s^2 + 4}$

5. $e^{-t}(1 - u(\pi/2)) = (1 - e^{-\frac{1}{2}\pi(1+s)})/(1 + s)$

6. $\sin (t-2)\pi = \sin (t-4)\pi = \sin \pi t$ because of periodicity. The representation in terms of unit step functions is

$$(\sin \pi t)(u(t-2) - u(t-4))$$

and thus gives the transform

$$\frac{\pi}{s^2 + \pi^2}(e^{-2s} - e^{-4s}).$$

7. $2\frac{\mathrm{e}^{-s-1/2\pi} - \mathrm{e}^{-3/2\pi-3s}}{2s + \pi}$

8. The given function is represented by

$$\begin{aligned} t^2(u(t-1) - u(t-2)) &= [(t-1)^2 + 2(t-1) + 1]u(t-1) \\ &\quad - [(t-2)^2 + 4(t-2) + 4]u(t-2) \end{aligned}$$

and thus has the transform

$$\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)e^{-s} - \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)e^{-2s}$$

9. $1/2\frac{\mathrm{e}^{-5/2s}(25s^2 + 20s + 8)}{s^3}$

10. $\frac{1}{2}(e^t - e^{-t})(1 - u(t-2)) = \frac{1}{2}(e^t - e^{-t}) - \frac{1}{2}(e^{(t-2)+2} - e^{(-t+2)-2})u(t-2)$. Hence the transform is

$$\frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{2}\left(\frac{e^2}{s-1} - \frac{e^{-2}}{s+1}\right)e^{-2s}.$$

Alternatively, by using the addition formula (22) in App. 3.1 we obtain the transform in the form

$$\frac{1}{s^2-1} - \left(\frac{\cosh 2}{s^2-1} + \frac{s \sinh 2}{s^2-1}\right)e^{-2s}.$$

12. s^{-3} has the inverse $t^2/2$, hence $(s-1)^{-3}$ has the inverse $e^t t^2/2$ (first shifting), and $e^{-2s}/(s-1)^3$ has the inverse $\frac{1}{2}u(t-2)(t-2)^2 e^{t-2}$ (second shifting).

13. $\sin 2tu(\pi - t) \sin 2t$

14. $4u(t-2) - 8u(t-5)$

15. $\frac{1}{120}(t-2)^5 u(t-2)$

16. $(\sinh(2t-2))u(t-1) - (\sinh(2t-6))u(t-3)$

18. $(4s^2 - 12s + 9)\,Y = \frac{8}{3}s - 4$, $Y = \frac{4}{3(-3+2s)}$. After applying the inverse Laplace transform, we get $\frac{2}{3}e^{3t}2$.

19. $1/15\mathrm{e}^{-x} + \frac{83}{80}\mathrm{e}^{4x} - 1/48\mathrm{e}^{-4x} - 1/12\mathrm{e}^{2x}$

20. $(s^2 + 10s + 24)Y = (s+4)(s+6)Y = \dfrac{288}{s^3} + \dfrac{19}{12}s - 5 + \dfrac{190}{12}$. Division by $s^2 + 10s + 24$ and expansion in terms of partial fractions gives

$$Y = \frac{12}{s^3} - \frac{5}{s^2} + \frac{\frac{19}{12}}{s}.$$

Hence the *answer* is $y = 6t^2 - 5t + \frac{19}{12}$. Note that it does not contain a contribution from the general solution of the homogeneous ODE.

21. $\frac{1}{2}\sin 2t + \frac{4}{3}\cos t + \frac{4}{3}\cos 2t$, $(0 < t < \pi)$; $\frac{1}{2}\sin 2t$, $(t > \pi)$.

22. In terms of unit step functions, the function on the right is

$$r(t) = 4t[1 - u(t-1)] + 8u(t-1) = 4t - [4(t-1) - 4]u(t-1).$$

Answer:

$$y = \begin{cases} 4e^{-t} - e^{-2t} + 2t - 3 & \text{if} \quad 0 < t < 1 \\ (4 - 8e)e^{-t} + (3e^2 - 1)e^{-2t} + 4 & \text{if} \quad t > 1. \end{cases}$$

23. $-\sin t$, $(0 < t < 2\pi)$, $-\frac{1}{2}\sin 2t$, $(t > 2\pi)$.

24. The subsidiary equation is

$$(s^2 + 3s + 2)Y = \frac{1}{s} - \frac{e^{-s}}{s}.$$

It has the solution

$$Y = \frac{1 - e^{-s}}{s(s^2 + 3s + 2)} = \left(\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1}\right)(1 - e^{-s}).$$

This gives the *answer*

$$y = \tfrac{1}{2} + \tfrac{1}{2}e^{-2t} - e^{-t} - (\tfrac{1}{2} + \tfrac{1}{2}e^{-2(t-1)} - e^{-(t-1)})u(t-1);$$

that is,

$$y = \begin{cases} \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} & \text{if} \quad 0 < t < 1 \\ \frac{1}{2}e^{-2t}(1 - e^2) - e^{-t}(1 - e) & \text{if} \quad t > 1. \end{cases}$$

25. $2t - 4 \sin t$, $(t > 1)$; $2 - 4 \sin t + 2 \sin (t - 1)$, $(t > = 1)$.

26. $t = \pi + \tilde{t}, \tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = \tilde{r}, \tilde{r} = 10[-1 + u(\tilde{t} - \pi)] \sin \tilde{t}, \tilde{y}(0) = 1, \tilde{y}'(0) = -2 + 2e^{-\pi}$. The solution in terms of $\tilde{t}$ is

$$\begin{aligned} \tilde{y}(\tilde{t}) &= e^{-t-\pi} \sin 2\tilde{t} + \cos \tilde{t} - 2 \sin \tilde{t} \\ &\quad + u(\tilde{t} - \pi)[e^{-\tilde{t}+\pi}(-\cos 2\tilde{t} + \tfrac{1}{2}\sin 2\tilde{t}) - \cos \tilde{t} + 2 \sin \tilde{t}]. \end{aligned}$$

In terms of t,

$$\begin{aligned} y(t) &= e^{-t} \sin 2t - \cos t + 2 \sin t \\ &\quad + u(t - 2\pi)[e^{-t+2\pi}(-\cos 2t + \tfrac{1}{2} \sin 2t) + \cos t - 2 \sin t]. \end{aligned}$$

28. $i' + 1000i = 40 \sin t u(t - \pi)$. The subsidiary equation is

$$sI + 1000I = -40 \frac{e^{-\pi s}}{s^2 + 1}.$$

Its solution is

$$I = \frac{-40e^{-\pi s}}{(s^2 + 1)(s + 1000)} = \frac{-40e^{-\pi s}}{1{,}000{,}001}\left(\frac{1}{s + 1000} - \frac{s - 1000}{s + 1}\right).$$

The inverse transform is

$$i = u(t - \pi)\left[-\frac{40}{1{,}000{,}001}(\cos t + e^{-1000(t-\pi)}) + \frac{40{,}000}{1{,}000{,}001} \sin t\right];$$

hence $i = 0$ if $t < \pi$, and $i \approx 0.04 \sin t$ if $t > \pi$.

30. $(0.5i' + 10i = 200t(1 - u(t - 2))$. The subsidiary equation is

$$(0.5s + 10)I(s) = 200(1 - e^{-2s}(1 + 2s))/s^2.$$

Its solution is

$$I = 400(-1 + e^{-2s} + 2se^{-2s})/(s^2(s + 20)).$$

The solution of the problem (the current in the circuit) is

$$i = e^{-20t} + 20t - 1 + u(t - 2)[-20t + 1 + 39e^{-20(t-2)}].$$

32. We obtain

$$I = 14 \cdot 10^5 \frac{se^{-4s-12}}{(s + 10)(s + 3)} = \left(\frac{2 \cdot 10^6}{s + 10} - \frac{6 \cdot 10^5}{s + 3}\right)e^{-4s-12}$$

hence

$$i = u(t - 4)\,[2 \cdot 10^6 e^{-10(t-4)-12} - 6 \cdot 10^5 e^{-3(t-4)-12}].$$

34. $10i + 100\int_0^t i(\tau)\,d\tau = 100(u(t-0.5) - u(t-0.6))$. Divide by 10 and take the transform, using Theorem 3 in Sec. 6.2,

$$I + \frac{10}{s} I = \frac{10}{s}\,(e^{-0.5s} - e^{-0.6s}).$$

Solving for $I = \mathscr{L}(i)$ gives

$$I = \frac{10}{s+10}\,(e^{-0.5s} - e^{-0.6s}).$$

The inverse transform is

$$i(t) = 10(e^{-10(t-0.5)}\,u(t-0.5) - e^{-10(t-0.6)}\,u(t-0.6)).$$

Hence

$$\begin{aligned}
i(t) &= 0 && \text{if} \quad t < 0.5\\
i(t) &= 10e^{-10(t-0.5)} && \text{if} \quad 0.5 < t < 0.6\\
i(t) &= 10(e^{-10(t-0.5)} - e^{-10(t-0.6)}) &&\\
&= 10e^{-10t}(e^5 - e^6) &&\\
&= -2550e^{-10t} && \text{if} \quad t > 0.6.
\end{aligned}$$

Jumps occur at $t = 0.5$ (upward) and at $t = 0.6$ (downward) because the right side has those jumps and the term involving the integral (representing the charge on the capacitor) cannot change abruptly; hence the first term, $Ri(t)$, must jump by the amounts of the jumps on the right, which have size 100, and since $R = 10$, the current has jumps of size 10.

36. $i'' + 4i = 200(1 - t^2)(1 - u(t-1))$. Observing that

$$t^2 = (t-1)^2 + 2(t-1) + 1,$$

we obtain the subsidiary equation

$$(s^2 + 4)I = 200\left(\frac{1}{s} - \frac{2}{s^3}\right) + 200e^{-s}\left(\frac{2}{s^2} + \frac{2}{s^3}\right).$$

Its solution is

$$\begin{aligned}
I = 200\,\frac{s^2 - 2 + e^{-s}(2s+2)}{s^3(s^2+4)} &= 25\left(\frac{3}{s} - \frac{4}{s^3} - \frac{3s}{s^2+4}\right)\\
&\quad + 25e^{-s}\left(-\frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3} + \frac{s-4}{s^2+4}\right).
\end{aligned}$$

Its inverse transform is

$$\begin{aligned}
i &= 75 - 50t^2 - 75\cos 2t\\
&\quad + u(t-1)[-75 + 50t^2 + 25\cos(2t-2) - 50\sin(2t-2)].
\end{aligned}$$

38. $i' + 4i + 20\int_0^t i(\tau)\,d\tau = 34e^{-t}(1 - u(t-4))$. The subsidiary equation is

$$\left(s + 4 + \frac{20}{s}\right)I = \frac{34}{s+1}(1 - e^{-4s-4}).$$

Its solution is

$$I = \frac{34s}{(s+1)(s^2+4s+20)}(1 - e^{-4s-4}).$$

The inverse transform is

$$\begin{aligned} i = &-2e^{-t} + e^{-2t}(2\cos 4t + 9\sin 4t) \\ &+ u(t-4)\big[2e^{-t} - e^{2(2-t)}(9\sin 4(t-4) + 2\cos 4(t-4))\big]. \end{aligned}$$

40. The model

$$i' + 2i + 10\int_0^t i(\tau)\,d\tau = 255(1 - u(t-2\pi))\sin t.$$

This gives the subsidiary equation

$$\left[(s+2) + \frac{10}{s}\right]I = \frac{255}{s^2+1}(1 - e^{-2\pi s}).$$

Its solution is

$$\begin{aligned} I &= \frac{255s(1 - e^{-2\pi s})}{(s^2+2s+10)(s^2+1)} \\ &= \left(\frac{27s+6}{s^2+1} - \frac{27(s+1)+33}{(s+1)^2+9}\right)(1 - e^{-2\pi s}). \end{aligned}$$

The inverse transform (the solution of the problem) is

$$\begin{aligned} i = {}&27\cos t + 6\sin t - e^{-t}(27\cos 3t + 11\sin 3t) \\ &+ u(t-2\pi)\big[e^{-(t-2\pi)}(27\cos 3t + 11\sin 3t) \\ &\qquad - (27\cos t + 6\sin t)\big]. \end{aligned}$$

SECTION 6.4. Short Impulses. Dirac s Delta Function. Partial Fractions, page 225

Purpose. Modeling of short impulses by **Dirac's delta** function (also known as **unit impulse** function). The text includes a remark that this is not a function in the usual sense of calculus but a "generalized function" or "distribution." Details cannot be discussed on the level of this book; they can be found in books on functional analysis or on PDEs. See, e.g., L. Schwartz, *Mathematics for the Physical Sciences*, Paris: Hermann, 1966. The French mathematician LAURENT SCHWARTZ (1915–2002) created and popularized the theory of distributions. See also footnote 2.

Main Content

Definition of Dirac's delta (3)

Sifting property (4)

Transform of delta (5)

Application to mass–spring systems and electric networks

More on partial fractions (Example 4)

For the beginning of the discussion of partial fractions in the present context, see Sec. 6.2.

Problem Set 6.4

CAS Project 1 involves exploring the nature of solutions with the damping or spring constant being changed in the presence of one or two Dirac delta functions on the right.

CAS Experiment 2 models a wave of constant area acting for shorter and shorter times.

Problems 3–12 model vibrating systems with driving forces consisting of unit step functions, Dirac's delta functions, and other functions.

Problems 14–15 concern rectifiers, sawtooth waves, and staircase functions.

SOLUTIONS TO PROBLEM SET 6.4, page 230

2. CAS Experiment. Students should become aware that careful observation of graphs may lead to discoveries or to more information about conjectures that they may want to prove or disprove. The curves branch from the solution of the homogeneous ODE at the instant at which the impulse is applied, which by choosing, say $a = 1, 2, 3, \cdots$, gives an interesting joint graph.

3. $y = 2\cos(3t) + 1/3u(t - 1/2\pi)\cos(3t)$

4. The subsidiary equation is

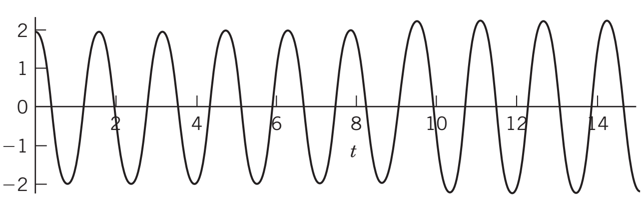
$$(s^2 + 16)Y = 2s + 4e^{-3\pi s}$$

where $2s$ comes from $y(0)$. The solution is

$$Y = \frac{2s + 4e^{-3\pi s}}{s^2 + 16}.$$

The inverse transform (the solution of the IVP) is

$$y = 2\cos 4t + u(t - 3\pi)\sin 4t.$$



Section 6.4. Problem 4

5. $$y = \begin{cases} 1/2\sin(2t) & 0 < t < \pi \\ \sin(2t) & \pi < t \le 2\pi \\ 1/2\sin(2t) & t > 2\pi \end{cases}$$

6. The subsidiary equation is

$$(s^2 + 4s + 5)Y = 3 + e^{-s}$$

where 3 comes from $y'(0)$. The solution is

$$Y = \frac{3 + e^{-s}}{(s + 2)^2 + 1}.$$

The inverse transform (the solution of the initial value problem) is

$$y = 3e^{-2t} \sin t + u(t - 1)e^{-2(t-1)} \sin (t - 1).$$

7. $y = 3/5\, \mathrm{e}^{-t} + 1/2\, u\,(t - 1/4)\, \mathrm{e}^{-2t+1/2} \sin (1/2t - 1/8)$

8. The subsidiary equation is

$$(s^2 + 3s + 2)Y = s - 1 + 3 + \frac{10}{s^2 + 1} + 10e^{-s}.$$

In terms of partial fractions, its solution is

$$Y = \frac{-2}{s + 2} + \frac{6}{s + 1} - \frac{3s - 1}{s^2 + 1} + 10\left(\frac{1}{s + 1} - \frac{1}{s + 2}\right)e^{-s}.$$

Its inverse transform is

$$y = -2e^{-2t} + 6e^{-t} - 3 \cos t + \sin t + 10u(t - 1)[e^{-t+1} - e^{-2(t-1)}].$$

Without the δ-term, the solution is $-3 \cos t + \sin t - 2e^{-2t} + 6e^{-t}$ and approaches a harmonic oscillation fairly soon. With the δ-term the first half-wave has a maximum amplitude of about 5, but from about $t = 8$ or 10 on its graph practically coincides with the graph of that harmonic oscillation (whose maximum amplitude is $\sqrt{10}$). This is physically understandable, since the system has damping that eventually consumes the additional energy due to the δ-term.

9. $y(t) = 1/5\, \mathrm{e}^{t} u(2 - t) + 1/5\, \mathrm{e}^{4-t} u\,(t - 2)\,(\cos\,(t - 2) - 3 \sin\,(t - 2))$
$\quad + 1/5(-\cos\,(t) + 3 \sin\,(t))\, \mathrm{e}^{-t}$

10. The subsidiary equation is

$$(s^2 + 5s + 6)Y = e^{-\pi s/2} - \frac{s}{s^2 + 1} e^{-\pi s}.$$

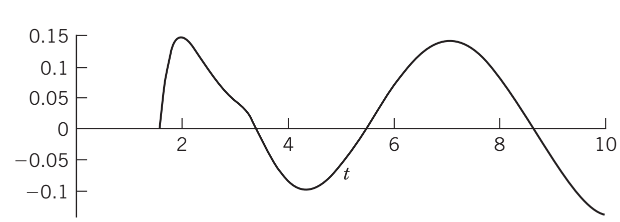
Its solution is

$$Y = \left(\frac{1}{s + 2} - \frac{1}{s + 3}\right)e^{-\pi s/2} - \left(-\frac{0.4}{s + 2} + \frac{0.3}{s + 3} + \frac{0.1(s + 1)}{s^2 + 1}\right)e^{-\pi s}.$$

The inverse transform of Y is

$$\begin{aligned} y = {} & u(t - \tfrac{1}{2}\pi)\left[e^{-2t+\pi} - e^{-3t+3\pi/2}\right] \\ & - 0.1u(t - \pi)\left[-4e^{-2t+2\pi} + 3e^{-3t+3\pi} - \cos t - \sin t\right]. \end{aligned}$$

This solution is zero from 0 to $\frac{1}{2}\pi$ and then increases rapidly. Its first negative half-wave has a smaller maximum amplitude (about 0.1) than the continuation as a harmonic oscillation with maximum amplitude of about 0.15.



Section 6.4. Problem 10

11. $y(t) = \mathrm{e}^{-t} - \mathrm{e}^{-2t} + u(t-2)\,(\mathrm{e}^{2-t} - \mathrm{e}^{-2t+4})$
$+ 1/2(1 - 2\mathrm{e}^{1-t} + \mathrm{e}^{-2t+2})u(t-1)$.

12. The subsidiary equation is

$$(s^2 + 2s + 5)Y = 1 - 2s + \frac{25}{s^2} - 100e^{-\pi s}.$$

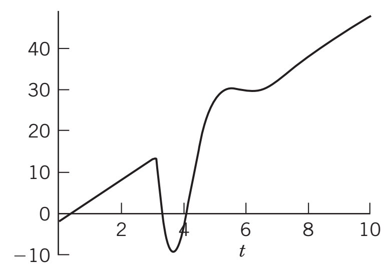
Its solution is

$$Y = \frac{-s^2 + 2s^3 - 25 + 100s^2 e^{-\pi s}}{s^2((s+1)^2 + 4)}.$$

Its inverse transform (the solution of the IVP) is

$$y = 5t - 2 - 50u(t - \pi)e^{-t+\pi} \sin 2t.$$

This is essentially a straight line, sharply deformed between π and about 8.



Section 6.4. Problem 12

14. TEAM PROJECT. (a) If $f(t)$ is piecewise continuous on an interval of length p, then its Laplace transform exists, and we can write the integral from zero to infinity as the series of integrals over successive periods:

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^p e^{-st} f\,dt + \int_p^{2p} e^{-st} f\,dt + \int_{2p}^{3p} e^{-st} f\,dt + \cdots.$$

If we substitute $t = \tau + p$ in the second integral, $t = \tau + 2p$ in the third integral, $\cdots$, $t = \tau + (n-1)p$ in the nth integral, $\cdots$, then the new limits in every integral are 0 and p. Since

$$f(\tau + p) = f(\tau), \qquad f(\tau + 2p) = f(\tau),$$

etc., we thus obtain

$$\mathcal{L}(f) = \int_0^p e^{-s\tau} f(\tau)\,d\tau + \int_0^p e^{-s(\tau+p)} f(\tau)\,d\tau + \int_0^p e^{-s(\tau+2p)} f(\tau)\,d\tau + \cdots.$$

The factors that do not depend on τ can be taken out from under the integral signs; this gives

$$\mathcal{L}(f) = [1 + e^{-sp} + e^{-2sp} + \cdots] \int_0^p e^{-s\tau} f(\tau)\,d\tau.$$

The series in brackets $[\cdots]$ is a geometric series whose sum is $1/(1 - e^{-ps})$. The theorem now follows.
(b) From (11) we obtain

$$\mathscr{L}(f) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt.$$

Using $1 - e^{-2\pi s/\omega} = (1 + e^{-\pi s/\omega})(1 - e^{-\pi s/\omega})$ and integrating by parts or noting that the integral is the imaginary part of the integral

$$\int_0^{\pi/\omega} e^{(-s+i\omega)t} \, dt = \frac{1}{-s + i\omega} e^{(-s+i\omega)t} \Big|_0^{\pi/\omega} = \frac{-s - i\omega}{s^2 + \omega^2} (-e^{-s\pi/\omega} - 1)$$

we obtain the result.
(c) From (11) we obtain the following equation by using $\sin \omega t$ from 0 to π/ω and $-\sin \omega t$ from π/ω to $2\pi/\omega$:

$$\begin{aligned} \frac{\omega}{s^2 + \omega^2} \frac{1 + e^{\pi s/\omega}}{e^{\pi s/\omega} - 1} &= \frac{\omega}{s^2 + \omega^2} \frac{e^{-\pi s/2\omega} + e^{\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/2\omega}} \\ &= \frac{\omega}{s^2 + \omega^2} \frac{\cosh (\pi s/2\omega)}{\sinh (\pi s/2\omega)}. \end{aligned}$$

This gives the result.
(d) The sawtooth wave has the representation

$$f(t) = \frac{k}{p} t \qquad \text{if} \quad 0 < t < p, \qquad f(t + p) = f(t).$$

Integration by parts gives

$$\begin{aligned} \int_0^p e^{-st} t \, dt &= -\frac{t}{s} e^{-st} \Big|_0^p + \frac{1}{s} \int_0^p e^{-st} \, dt \\ &= -\frac{p}{s} e^{-sp} - \frac{1}{s^2} (e^{-sp} - 1) \end{aligned}$$

and thus from (11) we obtain the result

$$\mathscr{L}(f) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1 - e^{-ps})} \qquad (s > 0).$$

SECTION 6.5. Convolution. Integral Equations, page 232

Purpose. To find the inverse $h(t)$ of a product $H(s) = F(s)G(s)$ of transforms whose inverses are known.

Main Content, Important Concepts

Convolution $f * g$, its properties

Convolution theorem

Application to ODEs and integral equations

Comment on Occurrence

In an ODE the transform $R(s)$ of the right side $r(t)$ is known from Step 1. Solving the subsidiary equation algebraically for $Y(s)$ causes the transform $R(s)$ to be multiplied by the reciprocal of the factor of $Y(s)$ on the left (the transfer function $Q(s)$; see Sec. 6.2). This calls for the convolution theorem, unless one sees some other way or shortcut.

Very Short Courses. This section can be omitted.

Problem Set 6.5

This set concerns integrations needed to obtain convolutions (Probs. 1–7), the use of the latter in solving a certain class of integrations (Probs. 8–14), the effect of varying a parameter in an integral equation (CAS Experiment 15), a problem (Prob. 16) on general properties of convolution.

Problems 17–26 show how to obtain inverse transforms by evaluating integrals that define convolutions.

SOLUTIONS TO PROBLEM SET 6.5, page 237

1. $-t$

2. $1 * \sin \omega t = \displaystyle\int_0^t \sin \omega\tau \, d\tau = -\frac{\cos \omega\tau}{\omega}\Bigg|_0^t = \frac{1 - \cos \omega t}{\omega}$

3. $\frac{1}{2}(e^t + e^{-t}) = \sinh t$

4. We obtain

$$\int_0^t \cos \omega\tau \cos (\omega t - \omega\tau)\, d\tau = \frac{1}{2}\int_0^t [\cos \omega t + \cos (2\omega\tau - \omega t)]\, d\omega$$

$$= \frac{1}{2}\left[t \cos \omega t + \frac{\sin \omega t - \sin(-\omega t)}{2\omega}\right] = \frac{1}{2} t \cos \omega t + \frac{1}{2\omega} \sin \omega t.$$

5. $\frac{\sin \omega t}{\omega}$

6. $e^{at} * e^{bt} = \displaystyle\int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt}\int_0^t e^{(a-b)\tau}\, d\tau = \frac{e^{at} - e^{bt}}{a - b}$

7. $te^{-t} - te^{-2t}$.

8. The integral equation can be written

$$y(t) + 4y(t) * t = 2t.$$

By the convolution theorem it has the transform

$$Y + 4Y/s^2 = 2/s^2.$$

The solution is $Y = 2/(s^2 + 4)$. Its inverse transform is $y = \sin 2t$.

9. $y + 1 * y = 2;\ Y = \frac{2}{s+1};\ y = 2e^{-t}$

10. In terms of convolution the given integral equation is

$$y(t) - y(t) * \sin 2t = \sin 2t.$$

By the convolution theorem its transform is

$$Y - 2Y/(s^2 + 4) = 2/(s^2 + 4).$$

Multiplication by $s^2 + 4$ gives

$$Y(s^2 + 2) = 2; \qquad \text{thus} \qquad Y = \frac{2}{s^2 + 2}.$$

The inverse transform (the solution of the integral equation) is

$$y = \sqrt{2} \sin (t\sqrt{2}).$$

11. $Y = \frac{s}{s^2 - 1}$; $y = \cosh t$.

12. The integral equation can be written

$$y(t) + y(t) * \cosh t = t + e^t.$$

This implies, by the convolution theorem, that its transform is

$$Y + \frac{s}{s^2 - 1} Y = \frac{1}{s^2} + \frac{1}{s - 1}.$$

The solution is

$$Y = \frac{s^2 - 1}{s^2 + s - 1}\left(\frac{1}{s^2} + \frac{1}{s - 1}\right) = \frac{1}{s^2} + \frac{1}{s}.$$

Hence its inverse transform gives the *answer* $y(t) = t + 1$. This result can easily be checked by substitution into the given equation and integration.

14. $Y\left(1 - \frac{1}{s^2}\right) = \frac{2}{s} - \frac{1}{s^3}$, hence

$$Y = \frac{2s^2 - 1}{s(s^2 - 1)} = \frac{1}{s} + \frac{s}{s^2 - 1}.$$

The *answer* is $y = 1 + \cosh t$.

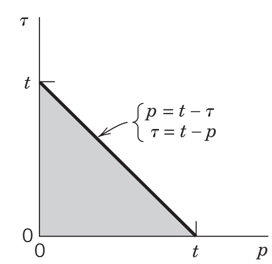
16. Team Project. (a) Setting $t - \tau = p$, we have $\tau = t - p, d\tau = -dp$, and p runs from t to 0; thus

$$f * g = \int_0^t f(\tau)g(t - \tau)\, d\tau = \int_t^0 g(p)f(t - p)(-dp)$$

$$= \int_0^t g(p)f(t - p)\, dp = g * f.$$

(b) Interchanging the order of integration and noting that we integrate over the shaded triangle in the figure, we obtain

$$\begin{aligned}(f * g) * v &= v * (f * g)\\ &= \int_0^t v(p)\int_0^{t-p} f(\tau)g(t-p-\tau)\,d\tau\,dp\\ &= \int_0^t f(\tau)\int_0^{t-\tau} g(t-\tau-p)v(p)\,dp\,d\tau\\ &= f * (g * v).\end{aligned}$$



Section 6.5. Team Project 16(b)

(c) This is a simple consequence of the additivity of the integral.

(d) Let $t > k$. Then $(f_k * f)(t) = \displaystyle\int_0^k \frac{1}{k} f(t-\tau)\,d\tau = f(t-\tilde{t})$ for some $\tilde{t}$ between 0 and k. Now let $k \to 0$. Then $\tilde{t} \to 0$ and $f_k(t-\tilde{t}) \to \delta(t)$, so that the formula follows.

(e) $s^2Y - sy(0) - y'(0) + \omega^2 Y = \mathcal{L}(r)$ has the solution

$$Y = \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right)\mathcal{L}(r) + y(0)\frac{s}{s^2+\omega^2} + \frac{y'(0)}{\omega}\frac{\omega}{s^2+\omega^2}$$

etc.

18. $e^{at} * e^{at} = \displaystyle\int_0^t a^{a\tau}e^{a(t-\tau)}\,d\tau = e^{at}\int_0^t d\tau = te^{at}$

20. $9 * e^{-3t} = 9\displaystyle\int_0^t e^{-3\tau}\,d\tau = 3 - 3e^{-3t}$

21. $-\frac{t}{w} + \frac{\sinh(wt)}{w^2}$.

22. $u(t-a) * e^{2t} = \displaystyle\int_a^t e^{2(t-\tau)}\,d\tau = e^{2t}\int_a^t e^{-2\tau}\,d\tau = \frac{1}{2}(e^{2(t-a)} - 1)$ if $t > a$ and 0 if $t < a$

24. $48\,(\sin t) * (\sin 5t) = 48 \int_0^t \sin\tau \sin 5(t-\tau)\,d\tau$. Using formula (11) in App. 3.1, convert the product in the integrand to a sum and integrate, obtaining

$$
\begin{aligned}
24 \int_0^t & [-\cos(5t - 4\tau) + \cos(\tau - 5t + 5\tau)]\,d\tau \\
&= 24[-\tfrac{1}{4}\sin(-5t + 4\tau) + \tfrac{1}{6}\sin(6\tau - 5t)]\Big|_0^t \\
&= 3(\sin t - \sin 5t) + 2(\sin t + \sin 5t) \\
&= 10 \sin t - 2 \sin 5t.
\end{aligned}
$$

SECTION 6.6. Differentiation and Integration of Transforms. ODEs with Variable Coefficients, page 238

Purpose. To show that, ***roughly***, differentiation and integration of transforms (not of functions, as before!) correspond to multiplication and division, respectively, of functions by t, with application to the derivation of further transforms and to the solution of Laguerre's differential equation.

Comment on Application to Variable-Coefficient Equations

This possibility is rather limited; our Example 3 is perhaps the best elementary example of practical interest.

Very Short Courses. This section can be omitted.

Problem Set 6.6

Problems 2–11 concern single or twofold applications of differentiation with respect to s, as the only method wanted for solving these problems, whereas in Probs. 14–20 the student has first to select a suitable one among several possible methods suggested.

Problem 13 is an invitation to study Laguerre polynomials in somewhat more detail, in particular, to compare the locations of the extrema depending on the parameter n.

SOLUTIONS TO PROBLEM SET 6.6, page 241

2. $F(s) = -3\left(\dfrac{4}{s^2 - 16}\right)' = \dfrac{24s}{(s^2 - 16)^2}$

3. $\frac{1}{4}(s + 2)^{-2}$

4. We have

$$\mathcal{L}(e^{-t}\cos t) = \frac{s+1}{(s+1)^2 + 1} = \frac{s+1}{s^2 + 2s + 2}.$$

Differentiation and simplification gives the answer

$$
\begin{aligned}
F'(s) = -\left(\frac{s+1}{s^2+2s+2}\right)' &= -\frac{s^2 + 2s + 2 - (s+1)(2s+2)}{(s^2 + 2s + 2)^2} \\
&= \frac{s^2 + 2s}{(s^2 + 2s + 2)^2}.
\end{aligned}
$$

5. $2\frac{s\omega}{(s^2 + \omega^2)^2}$

6. We need two differentiations. We can drop the two minus signs. Starting from the transform of sin $3t$, we obtain

$$\begin{aligned}\left(\frac{3}{s^2+9}\right)'' &= \left(\frac{-6s}{(s^2+9)^2}\right)' \\ &= \frac{-6(s^2+9)^2 + 6s\cdot 2(s^2+9)\cdot 2s}{(s^2+9)^4} \\ &= \frac{-6s^2 - 54 + 24s^2}{(s^2+9)^3} \\ &= \frac{18(s^2-3)}{(s^2+9)^3}.\end{aligned}$$

7. $(s - 2)^{-3} - (s + 2)^{-3}$.

8. $-\left(\dfrac{1}{(s+k)^2+1}\right)' = \dfrac{2(s+k)}{((s+k)^2+1)^2}$

9. $16\frac{s(4s^2 - 3\pi^2)}{(4s^2 + \pi^2)^3}$

10. $\mathcal{L}(t^n e^{kt}) = \dfrac{n!}{(s-k)^{n+1}}$ can be obtained from $\mathcal{L}(e^{kt}) = \dfrac{1}{s-k}$ by n subsequent differentiations,

$$\left(\frac{1}{s-k}\right)^{(n)} = \left(\frac{-1}{(s-k)^2}\right)^{(n-1)} = \cdots = \frac{(-1)^n n!}{(s-k)^{n+1}}$$

and multiplication by $(-1)^n$ (to take care of the minus sign in (1) in each of the n steps), or much more simply, by the first shifting theorem, starting from $\mathcal{L}(t^n) = n!/s^{n+1}$.

11. $\frac{\pi}{s^2 + 4\pi^2}$.

12. CAS Project. Students should become aware that usually there are several possibilities for calculations, and they should not rush into numeric work before carefully selecting formulas.

(b) Use the usual rule for differentiating a product n times. Some of the Laguerre polynomials are

$$\begin{aligned}l_2 &= 1 - 2t + \tfrac{1}{2}t^2 \\ l_3 &= 1 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3 \\ l_4 &= 1 - 4t + 3t^2 - \tfrac{2}{3}t^3 - \tfrac{1}{24}t^4 \\ l_5 &= 1 - 5t + 5t^2 - \tfrac{5}{3}t^3 - \tfrac{5}{24}t^4 - \tfrac{1}{120}t^5.\end{aligned}$$

14. By differentiation we have

$$\left(\frac{4}{s^2+16}\right)' = \frac{-8s}{(s^2+16)^2}.$$

Hence the *answer* is $\frac{1}{8}t \sin 4t$. By integration we see that

$$\int_s^\infty \frac{\tilde{s}}{(\tilde{s}^2+16)^2}\, d\tilde{s} = \frac{\frac{1}{2}}{s^2+16}$$

has the inverse transform $\frac{1}{8}\sin 4t$ and gives the same answer. By convolution,

$$(\cos 4t) * (\tfrac{1}{4}\sin 4t) = \int_0^t \cos 4\tau \sin (4t - 4\tau)\, d\tau$$

and gives the same answer.

15. $F(s) = -\frac{1}{2}\left(\frac{1}{s^2 - 4}\right)'$; $f(t) = \frac{1}{4}\sinh 2t$.

16. By (6),

$$\int_s^\infty \frac{2\tilde{s} + 6}{(\tilde{s}^2 + 6\tilde{s} + 10)^2}\, d\tilde{s} = \frac{1}{s^2 + 6s + 10} = \mathcal{L}\left(\frac{f}{t}\right).$$

The inverse transform of the integral is $e^{-3t}\sin t$. *Answer*:

$$te^{-3t}\sin t.$$

18. $\left(\operatorname{arccot}\dfrac{s}{\pi}\right)' = -\dfrac{1/\pi}{1 + \left(\dfrac{s}{\pi}\right)^2} = \dfrac{-\pi}{s^2 + \pi^2}$ shows that the *answer* is $(\sin \pi t)/t$.

20. $\ln \dfrac{s + a}{s + b} = \ln (s + a) - \ln (s + b) = -\displaystyle\int_s^\infty \frac{d\sigma}{\sigma + a} + \int_s^\infty \frac{d\sigma}{\sigma + b}$. This shows that the *answer* is $(-e^{-at} + e^{-bt})/t$.

SECTION 6.7. Systems of ODEs, page 242

Purpose. This section explains the application of the Laplace transform to systems of ODEs in terms of three typical examples: a mixing problem, an electrical network, and a system of several (two) masses on elastic springs.

Note that Examples 1 and 2 in the text concern first-order systems, whereas the system in Example 3 is of second order.

Problem Set 6.7

Team Project 1 concerns Examples 1 and 2 of Sec. 4.1, namely, a comparison of the usual method and the present transform method.

Problems 2–10 involve IVPs for first-order systems, Probs. 11–13 second-order systems, and Probs. 14–15 systems of three ODEs.

SOLUTIONS TO PROBLEM SET 6.7, page 246

2. $y_1(t) = t\sin(t) + \cos(t)$, $y_2(t) = t\cos(t)$

3. $y_1(t) = \cosh(t) + 2\sinh(t)$, $y_2(t) = \sinh(t)$

4. The subsidiary equations are

$$sY_1 = 4Y_2 - \frac{8s}{s^2 + 16}$$

$$sY_2 = 3 - 3Y_1 - \frac{36}{s^2 + 16}.$$

The solution is

$$Y_1 = \frac{4}{s^2 + 16}, \qquad Y_2 = \frac{3s}{s^2 + 16}.$$

The inverse transform is $y_1 = \sin 4t$, $y_2 = 3\cos 4t$.

5. $y_1(t) = 1 - u(t-1)(2(\sin(1/2t - 1/2)^2 + \sin(t-1)) + 2\sin(t)$,
$y_2(t) = -2 + 2\cos(t) + (-\sin(t-1) + 2(\sin(1/2t - 1/2))^2)u(t-1)$

6. The subsidiary equations are

$$\begin{aligned} sY_1 - 1 &= 5Y_1 + Y_2 \\ sY_2 + 3 &= Y_1 + 5Y_2. \end{aligned}$$

The solution is

$$\begin{aligned} Y_1 &= \frac{s-8}{(s-5)^2 - 1} = \frac{(s-5) - 3}{(s-5)^2 - 1} \\ Y_2 &= \frac{-3s + 16}{(s-5)^2 - 1} = \frac{-3(s-5) + 1}{(s-5)^2 - 1}. \end{aligned}$$

The inverse transform is

$$\begin{aligned} y_1 &= e^{5t}\cosh t - 3e^{5t}\sinh t = 2e^{4t} - e^{6t} \\ y_2 &= -3e^{5t}\cosh t + e^{5t}\sinh t = -2e^{4t} - e^{6t}. \end{aligned}$$

8. The subsidiary equations are

$$\begin{aligned} sY_1 &= 4 - 2Y_1 + 3Y_2 \\ sY_2 &= 3 + 4Y_1 - Y_2. \end{aligned}$$

The solution is

$$\begin{aligned} Y_1 &= \frac{4s + 13}{s^2 + 3s - 10} = \frac{3}{s-2} + \frac{1}{s+5} \\ Y_2 &= \frac{3s + 22}{s^2 + 3s - 10} = \frac{4}{s-2} - \frac{1}{s+5}. \end{aligned}$$

The inverse transform is

$$y_1 = 3e^{2t} + e^{-5t}, \qquad y_2 = 4e^{2t} - e^{-5t}.$$

9. $y_1(t) = -(t-1)e^{2t}$, $y_2(t) = -te^{2t}$

10. The subsidiary equations are

$$\begin{aligned} sY_1 &= 1 - Y_2 \\ sY_2 &= -Y_1 + \frac{2s}{s^2 + 1} - \frac{2e^{-2\pi s}s}{s^2 + 1}. \end{aligned}$$

The solution is

$$Y_1 = \frac{s(s^2 - 1 + 2e^{-2\pi s})}{s^4 - 1}$$
$$= \frac{s}{s^2 + 1} + \frac{2se^{-2\pi s}}{s^4 - 1}$$
$$= \frac{s}{s^2 + 1} + e^{-2\pi s}\left(\frac{\frac{1}{2}}{s - 1} + \frac{\frac{1}{2}}{s + 1} - \frac{s}{s^2 + 1}\right)$$
$$Y_2 = \frac{s^2 - 1 - 2s^2e^{-2\pi s}}{s^4 - 1}$$
$$= \frac{1}{s^2 + 1} + e^{-2\pi s}\left(\frac{1}{2}\left(\frac{1}{s + 1} - \frac{1}{s - 1}\right) - \frac{1}{s^2 + 1}\right).$$

The inverse transform is

$$y_1 = \cos t + u(t - 2\pi)\left[-\cos t + \tfrac{1}{2}(e^{-t+2\pi} + e^{t-2\pi})\right]$$
$$y_2 = \sin t + u(t - 2\pi)\left[-\sin t + \tfrac{1}{2}(e^{-t+2\pi} - e^{t-2\pi})\right].$$

Thus $y_1 = \cos t$ and $y_2 = \sin t$ if $0 < t < 2\pi$, $y_1 = \cosh(t - 2\pi)$, and $y_2 = -\sinh(t - 2\pi)$ if $t > 2\pi$.

12. The subsidiary equations are

$$s^2Y_1 = s - 4Y_1 + 5Y_2 \qquad (1)$$
$$s^2Y_2 = 2s - Y_1 + 2Y_2 \qquad (2)$$

The solution is

$$Y_1(t) = \frac{s(8 + s^2)}{-3 + s^4 + 2s^2} = -\frac{5}{4}\frac{s}{s^2 + 3} + \frac{9}{8(s + 1)} + \frac{9}{8(s - 1)}$$
$$Y_2(t) = \frac{s(2s^2 + 7)}{-3 + s^4 + 2s^2} = -\frac{1}{4}\frac{s}{s^2 + 3} + \frac{9}{8(s + 1)} + \frac{9}{8(s - 1)}$$

Hence the inverse transform is

$$y_1 = -5/4\cos(\sqrt{3}t) + 9/4\cosh(t),$$
$$y_2 = -1/4\cos(\sqrt{3}t) + 9/4\cosh(t)$$

14. The subsidiary equations are

$$4sY_1 - 8 + sY_2 - 2sY_3 = 0$$
$$-2sY_1 + 4 \qquad + \quad sY_3 = \frac{1}{s}$$
$$2sY_2 - 4sY_3 = -\frac{16}{s^2}.$$

The solution is

$$Y_1 = 2\left(\frac{1}{s} + \frac{1}{s^3}\right), \qquad Y_2 = \frac{2}{s^2}, \qquad Y_3 = \frac{1}{s^2} + \frac{4}{s^3}.$$

The inverse transform is

$$y_1 = 2 + t^2, \qquad y_2 = 2t, \qquad y_3 = t + 2t^2.$$

15. $y_1 = -3/2 \cosh(t) + 1/2 \sinh(t) + 3/2$, $y_2 = -3/2 \cosh(t) + 5/2 \sinh(t) + 3/2$, $y_3 = -1/2 \cosh(t) + 3/2 \sinh(t) + 3/2$.

16. The subsidiary equations are

$$s^2Y_1 - s - 1 = -8Y_1 + 4Y_2 + \frac{11}{s^2 + 1}$$
$$s^2Y_2 - s + 1 = -8Y_2 + 4Y_1 - \frac{11}{s^2 + 1}.$$

The solution, in terms of partial fractions, is

$$Y_1 = \frac{s}{s^2 + 4} + \frac{1}{s^2 + 1}$$
$$Y_2 = \frac{s}{s^2 + 4} - \frac{1}{s^2 + 1}.$$

The inverse transform is

$$y_1 = \cos 2t + \sin t$$
$$y_2 = \cos 2t - \sin t.$$

18. The new salt contents are

$$y_1 = 100 - 62.5e^{-0.24t} - 37.5e^{-0.08t}$$
$$y_2 = 100 + 125e^{-0.24t} - 75e^{-0.08t}.$$

Setting $2t = \tau$ gives the old solution, except for notation.

20. For $0 \leqq t \leqq 2\pi$ the solution is as in Prob. 19; for i_1 we have

$$i_1 = -26e^{-2t} - 16e^{-8t} + 42 \cos t + 15 \sin t.$$

For $t > 2\pi$ we have to add to this further terms whose form is determined by this solution and the second shifting theorem,

$$u(t - 2\pi)[26e^{-2t+4\pi} + 16e^{-8t+16\pi} - 42 \cos t - 15 \sin t].$$

The cosine and sine terms cancel, so that

$$i_1 = -26(1 - e^{4\pi})e^{-2t} - 16(1 - e^{16\pi})e^{-8t} \qquad \text{if} \quad t > 2\pi.$$

Similarly, for i_2 we obtain

$$i_2 = \begin{cases} -26e^{-2t} + 8e^{-8t} + 18 \cos t + 12 \sin t \\ -26(1 - e^{4\pi})e^{-2t} + 8(1 - e^{16\pi})e^{-8t}. \end{cases}$$

SOLUTIONS TO CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS, page 251

11. $\frac{3s}{s^2 - 1} - \frac{10}{s^2 - 4}$.

12. $\frac{s - 6}{(s + 2)^2 + 4}$

13. $\frac{1}{2} \frac{\pi^2 + 2s^2}{s(s^2 + \pi^2)}$

14. We have

$$16t^2 = 16(t - \tfrac{1}{4})^2 + 8(t - \tfrac{1}{4}) + 1.$$

The transform is

$$\frac{32}{s^3} + \frac{8}{s^2} + \frac{1}{s}.$$

This times $e^{-s/4}$ gives the answer (the transform of the given function).

15. $2\frac{e^{-1-2s}}{1 + 2s}$

16. $\frac{e^{-2s\pi}s}{s^2 + 4}$

17. $\frac{s(s^2 - 1)}{(s^2 + 1)^2}$

18. The transform of $\sin \omega t$ is $\omega/(s^2 + \omega^2)$, and that of $\cos \omega t$ is $s/(s^2 + \omega^2)$. Hence, by convolution, the given function (use (11) in Appendix 3.1 in integration)

$$(\sin \omega t) * (\cos \omega t) = \tfrac{1}{2} t \sin \omega t$$

has the transform

$$\frac{\omega s}{(s^2 + \omega^2)^2}.$$

19. $\frac{8}{(s + 2)s^3}$

20. $1.25(e^{4t} - e^{-2t}) = 2.5e^t \sinh 3t$

21. $(2 - t)\, u\, (t - 1)$

22. $\dfrac{\frac{1}{16}}{s^2 + s + \frac{1}{2}} = \dfrac{\frac{1}{2} \cdot \frac{1}{8}}{(s + \frac{1}{2})^2 + \frac{1}{4}}$. Hence $\frac{1}{8} e^{-t/2} \sin \frac{1}{2} t$.

23. $\sin(\theta - \omega t)$

24. The given transform suggests the differentiation

$$\left(\frac{s}{s^2 + 6.25}\right)' = \frac{s^2 + 6.25 - s \cdot 2s}{(s^2 + 6.25)^2} = -\frac{s^2 - 6.25}{(s^2 + 6.25)^2}$$

and shows that the *answer* is $t \cos 2.5t$.

25. $t(t - 2)$

26. $\dfrac{2s - 10}{s^3} = \dfrac{2}{s^2} - \dfrac{10}{s^3}$. This shows that the inverse transform is

$$u(t - 5)\left[2(t - 5) - 5(t - 5)^2\right] = u(t - 5)\left[-5t^2 + 52t - 135\right].$$

27. $1/2e^{-t}(4 \cos 2t - \sin 2t)$

28. $\dfrac{3s}{s^2 - 2s + 2} = \dfrac{3(s-1) + 3}{(s-1)^2 + 1}$. Hence the inverse transform is

$$3e^t(\cos t + \sin t).$$

29. $y = 5t - 2 - 5e^{-t} \sin 2t$

30. $y = -\cos 4t + u(t - \pi) \sin 4t$. To see the impact of $4\delta(t - \pi)$, graph both the solution and the term $-\cos 4t$, perhaps in a short t-interval with midpoint π.

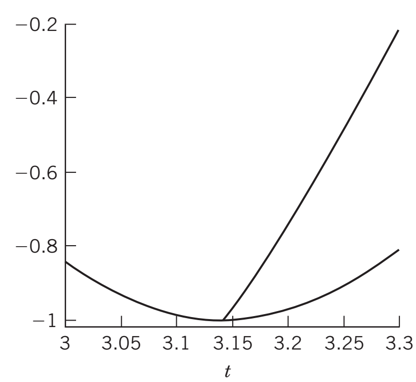
31. $y = \frac{1}{2}e^{-2t} + 5\,(u\,(\pi - t) - 1)e^{t-\pi} + (s \sin t - 9 \cos t - 4e^{-2t+2\pi})\,u\,(t - \pi)$

32. $y = \cos 2t + \frac{1}{2}[u(t - \pi) - u(t - 2\pi)] \sin 2t$. The curve has cusps at $t = \pi$ and 2π (abrupt changes of the tangent direction).

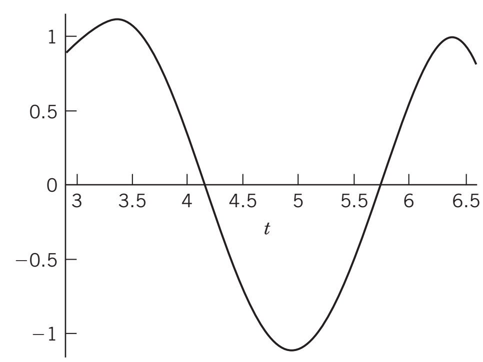
33. $y = \begin{cases} 0 & t < 1 \\ 1 - 3e^{2t-2} + 2e^{3t-3} & 1 \le t \end{cases}$

34. $y_1 = \frac{1}{2}u(t - \pi) \sin 2t$, $y_2 = u(t - \pi) \cos 2t$. Hence y_1 is continuous at π, whereas y_2 has an upward jump of 1 at that point.

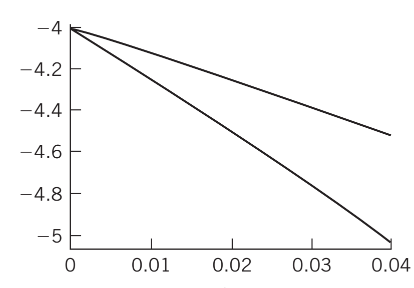
36. $y_1 = -6e^{4t} + 2$, $y_2 = -3e^{4t} - 1$. Note that, from the given system, we see that $y_2' = \frac{1}{2}y_1'$, so that $y_2 = \frac{1}{2}y_1 + \text{const.}$



Problem 30



Problem 32



Problem 36

37. $y_1 = 2\cos(t) - u\,(t - \pi) \sin(t) - 1,$
$y_2 = 2\sin(t) + 2u\,(t - \pi)(\cos(1/2t))^2$

38. $-k_1y_1$ is the force in the first spring when extended by an amount y_1; it tends to pull m_1 to the left, hence the minus. Similarly, a positive $y_2 - y_1$ causes the spring force $k_2(y_2 - y_1)$ in the second spring, pulling m_1 to the right. The spring force $-k_2(y_2 - y_1)$ in the second equation pulls m_2 to the left. Finally, a $y_2 > 0$ causes a compression of the third spring and a spring force $-k_3y_2$ in the negative direction, pushing m_2 to the left.

40. We now have masses m_1, m_2, m_3 in a row, connected to each other by two springs and connected to the two walls at the ends by the other two springs. The two springs pulling or pushing at each of the three masses give two terms in each of the three ODEs

$$\begin{aligned} m_1y_1'' &= -k_1y_1 + k_2(y_2 - y_1) \\ m_2y_2'' &= -k_2(y_2 - y_1) + k_3(y_3 - y_2) \\ m_3y_3'' &= -k_3(y_3 - y_2) - k_4y_3. \end{aligned}$$

42. $$q = \begin{cases} 1 - \frac{1}{2}(e^{-t} + \cos t + \sin t) & \text{if} \quad 0 < t < \pi \\ \frac{1}{2}[(e^{-\pi} - 3)\cos t - (e^{-\pi} + 1)\sin t] & \text{if} \quad t > \pi. \end{cases}$$

44. $i_1 = 2(1 - e^{-t}), \quad i_2 = 2e^{-t}.$

Author Query

AQ1 Please provide the updated image.