Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.4

No. 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, t) = \phi(x + ct) + \varphi(x - ct)$$

$$\frac{\partial u}{\partial t} = c\phi'(x + ct) - c\varphi'(x - ct)$$

No. 2

$u(0, t) = \frac{1}{2}[f(ct) + f(-ct)] = 0, f(-ct) = -f(ct)$, so that f is odd. Also

$$u(L, t) = \frac{1}{2}[f(ct + L) + f(-ct + L)] = 0$$

hence

$$f(ct + L) = -f(-ct + L) = f(ct - L).$$

This proves the periodicity.

No. 3

$$c^2 = \frac{T}{\rho} = \frac{300}{\left(\frac{0.9}{2 \cdot 9.8}\right)} = (80.83)^2 \ [m^2/sec^2]$$

No. 4

$$f = \frac{\lambda_n}{2\pi} = \frac{cn}{2L} = \frac{(80.83) \cdot n}{2 \cdot 2} = 20.2\,n.$$

No. 5

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$c=1$, $L=1$, $u_t(x,0)=0$, $u(x,0)=f(x)$

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)]$$

$$= \frac{1}{2}[k\sin\pi(x+t) + k\sin\pi(x-t)]$$

$$= \frac{k}{2}[2\sin\pi x\cos\pi t]$$

$$= k\cos\pi t\sin\pi x$$

No. 6

$f(x) = k(1-\cos\pi x)$

$$u(x,t) = \frac{1}{2}[f(x+t)+f(x-t)]$$

$$= \frac{1}{2}\{k[1-\cos\pi(x+t)] + k[1-\cos\pi(x-t)]\}$$

$$= \frac{1}{2}[2k - k(2\cos\pi x\cos\pi t)]$$

$$= k - k\cos\pi t\cos\pi x$$

No. 7

$f(x) = k \sin 2\pi x$

$u(x, t) = \frac{1}{2}[f(x+t) + f(x-t)]$

$= \frac{1}{2}[k \sin 2\pi (x+t) + k \sin 2\pi (x-t)]$

$= \frac{1}{2}[2k \sin 2\pi x \cos 2\pi t]$

$= k \cos 2\pi t \sin 2\pi x$

No. 8

$f(x) = kx(1-x)$

$u(x, t) = \frac{1}{2}[f(x+t) + f(x-t)]$

$= \frac{1}{2}[k(x+t)(1-x-t) + k(x-t)(1-x+t)]$

$= \frac{1}{2}[2k(x+t)(1-x)]$

$= k(x+t)(1-x)$

No. 9

$U_{xx} + 4U_{yy} = 0$

$4 \cdot 1 - 0^2 = 4 > 0 \implies$ Elliptic

Let $v = y + 2ix$; $w = y - 2ix$

$U_x = 2iU_v - 2iU_w$

$U_{xx} = -4U_{vv} - 4U_{ww}$

$U_y = U_v + U_w$

$U_{yy} = U_{vv} + 2U_{vw} + U_{ww}$

$(-4U_{vv} - 4U_{ww}) + 4(U_{vv} + 2U_{vw} + U_{ww}) = 0$

$U_{vw} = 0$

$U = f_1(v) + f_2(w)$

$u(y, x) = [f_1(y+2ix) + f_2(y-i2x)]$

No.10

Hyperbolic, wave equation. Characteristic equation

$$y'^2 - 16 = (y' + 4)(y' - 4) = 0.$$

New variables are

$$v = \phi = y + 4x, \qquad w = \Psi = y - 4x.$$

By the chain rule,

$$\begin{aligned} u_x &= 4u_v - 4u_w \\ u_{xx} &= 16u_{vv} - 16u_{vw} - 16u_{wv} + 16u_{ww} \end{aligned}$$

and

$$-16u_{yy} = -16u_{vv} - 16u_{vw} - 16u_{wv} - 16u_{ww}.$$

Assuming $u_{vw} = u_{wv}$, as usual, we have

$$u_{vw} = 0,$$

solvable by two integrations, as shown in the text.

No.11

$$U_{xx} + 2U_{xy} + U_{yy} = 0$$

$$\cancel{1 - \left(\tfrac{2}{2}\right)^2 = 0}$$

$1 \cdot 1 - 1^2 = 0 \implies$ Parabolic

Let $v = x$, $w = x - y$

$$U_x = U_v V_x + U_w W_x = U_v + U_w$$

$$U_{xx} = (U_v + U_w)_x = (U_v + U_w)_v V_x + (U_v + U_w)_w W_x = U_{vv} + 2U_{vw} + U_{ww}$$

$$U_{xy} = (U_v + U_w)_y = -U_{vw} - U_{ww}$$

$$U_{yy} = U_{ww}$$

$$(U_{vv} + 2U_{vw} + U_{ww}) + 2(-U_{vw} - U_{ww}) + U_{ww} = 0$$

$$U_{vv} = 0$$

$$U = v f_1(w) + f_2(w)$$

$$= x f_1(x-y) + f_2(x-y)$$

No.12

Parabolic. Characteristic equation

$$y'^2 + 2y' + 1 = (y' + 1)^2 = 0.$$

New variables $v = \Phi = x$, $w = \Psi = x + y$. By the chain rule,

$$\begin{aligned} u_x &= u_v + u_w \\ u_{xx} &= u_{vv} + 2u_{vw} + u_{ww} \\ u_{xy} &= u_{vw} + u_{ww} \\ u_{yy} &= u_{ww}. \end{aligned}$$

Substitution of this into the PDE gives the expected normal form

$$u_{vv} = 0.$$

No.13

$$U_{xx} + 5U_{xy} + 4U_{yy} = 0$$

$$1 \cdot 4 - \left(\frac{5}{2}\right)^2 = -\frac{9}{4} < 0 \quad ; \quad \text{Hyperbolic}$$

Let $v = y - 4x$, $w = y - x$.

$u_x = -4u_v - u_w$

$u_{xx} = 16u_{vv} + 5u_{vw} + u_{ww}$

$u_{xy} = -4u_{vv} - 2u_{vw} - u_{ww}$

$u_y = u_v + u_w$

$u_{yy} = u_{vv} + 2u_{vw} + u_{ww}$

$(16u_{vv} + 8u_{vw} + u_{ww}) + 5(-4u_{vv} - 2u_{vw} - u_{ww}) + 4(u_{vv} + 2u_{vw} + u_{ww}) = 1$

$$u_{vw} = 0$$

$$u = f_1(v) + f_2(w)$$
$$= f_1(y - 4x) + f_2(y - x)$$

No.14

Hyperbolic. New variables $x = v$ and $xy = w$. The latter is obtained from

$$-xy' - y = 0, \qquad \frac{y'}{y} = -\frac{1}{x}, \qquad \ln|y| = -\ln|x| + c.$$

By the chain rule we obtain, in these new variables from the given PDE by cancellation of $-yu_{yy}$ against a term in xu_{xy} and division of the remaining PDE by x, the PDE

$$u_w + xu_{vw} = 0.$$

(The normal form is $u_{vw} = -u_w/x = -u_w/v$.) We set $u_w = z$ and obtain

$$z_v = -\frac{1}{v}z, \qquad z = \frac{c(w)}{v}.$$

By integration with respect to w we obtain the solution

$$u = \frac{1}{v}f_1(w) + f_2(v) = \frac{1}{x}f_1(xy) + f_2(x).$$

Note that the solution of the next problem (Problem 15) is obtained by interchanging x and y in the present problem.

No.15

$$xu_{xx} - yu_{xy} = 0$$

$$0 \cdot x - \left(-\frac{y}{2}\right)^2 = -\frac{y^2}{4} < 0 \quad \therefore \text{Hyperbolic}$$

Let $v = y$, $xy = w$

$$u = \frac{1}{y} f_1(w) + f_2(\cancel{v})$$

$$= \frac{1}{y} f_1(xy) + f_2(v).$$

No.16

Elliptic. The characteristic equation is

$$y'^2 - 2y' + 10 = [y' - (1 - 3i)][y' - (1 + 3i)] = 0.$$

Complex solutions are

$$\Phi = y - (1 - 3i)x = \text{const}, \qquad \Psi = y - (1 + 3i)x = \text{const}.$$

This gives the solutions of the PDE:

$$u = f_1(y - (1 - 3i)x) + f_2(y - (1 + 3i)x).$$

Since the PDE is linear and homogeneous, real solutions are the real and the imaginary parts of u.

No.17

$$U_{xx} - 4U_{xy} + 5U_{yy} = 0$$

$$1\cdot 5 - (-2)^2 = 1 > 0 \qquad ; \text{ Elliptic}$$

Let : $v = y-(2-i)x$. $w = y-(2+i)x$

$$U_x = -(2-i)U_v - (2+i)U_w$$

$$U_{xx} = (2-i)^2 U_{vv} + 10U_{vw} + (2+i)^2 U_{ww}$$

$$U_{xy} = -(2-i)U_{vv} - 4U_{vw} - (2+i)U_{ww}$$

$$U_y = U_v + U_w$$

$$U_{yy} = U_{vv} + 2U_{vw} + U_{ww}$$

$$[(2-i)^2 U_{vv} + 10U_{vw} + (2+i)^2 U_{ww}) - 4[-(2-i)U_{vv} - 4U_{vw} - (2+i)U_{ww}] +$$

$$5[U_{vv} + 2U_{vw} + U_{ww}] = 0$$

$$U_{vw} = 0$$

$$U = f_1(v) + f_2(w)$$

$$= f_1(y-(2-i)x) + f_2(y-(2+i)x).$$

No.18

Parabolic. Characteristic equation

$$y'^2 + 6y' + 9 = (y' + 3)^2.$$

New variables $v = \Phi = x$, $w = \Psi = y = 3x$. By the chain rule,

$$\begin{aligned}
u_x &= u_v + 3u_w \\
u_{xx} &= u_{vv} + 6u_{vw} + 9u_{ww} \\
u_{xy} &= u_{vw} + 3u_{ww} \\
u_{yy} &= u_{ww}.
\end{aligned}$$

Substitution into the PDE gives the expected normal form

$$\begin{aligned}
u_{vv} + 6u_{vw} &+ \ \ 9u_{ww} \\
-6u_{vw} &- 18u_{ww} \\
&+ \ \ 9u_{ww} = u_{vv} = 0.
\end{aligned}$$

Solution

$$u = f_1(v) + f_2(w) = f_1(x) + f_2(y + 3x)$$

where f_1 and f_2 are any twice differentiable functions of the respective variables.

No.19

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t)=0$, $u_x(L,t)=0$

$u(x,0)=f(x)$, $u_t(x,0)=0$

Let $u(x,t)=F(x)G(t)$

$$\frac{F''}{F} = \frac{\ddot{G}}{c^2 G} = -p^2$$

$F'' + p^2 F = 0$

$F(x) = A\cos px + B\sin px$

$F(0)=0 \Rightarrow A=0$

$F'(L)=0 \Rightarrow Bp\cos pL = 0$, $B \neq 0$, $pL = \frac{(2n+1)\pi}{2}$ $(n=0,1,\cdots)$

$$p_n = \frac{(2n+1)}{2L}\pi$$

Let $B=1$

$F(x) = F_n(x) = \sin\left(\frac{(2n+1)\pi}{2L}\right)x = \sin p_n x$ $(n=0,1,\cdots)$

$$\ddot{G} + c^2 p_n^2 G = 0$$

$G_n(t) = A_n \cos cp_n t + A_n^* \sin cp_n t$

$G_n'(0)=0 \Rightarrow A_n^*=0$

$\therefore u_n(x,t) = F_n(x)G_n(t) = A_n \sin p_n x \cos cp_n t$ $(n=0,1,\cdots)$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n \sin p_n x \cos cp_n t$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \sin p_n x$$

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin p_n x\, dx$$

No.20

The Tricomi equation is elliptic in the upper half-plane and hyperbolic in the lower, because of the coefficient y.

$u = F(x)G(y)$ gives

$$yF''G = -FG'', \qquad \frac{F''}{F} = -\frac{G''}{yG} = -k$$

and $k = 1$ gives Airy's equation.