Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.9

No. 1

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t+B_{mn}^{*}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

$$\lambda_{mn}=c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\quad .\quad c^2=\frac{T}{\rho}$$

$$f=\frac{\lambda_{mn}}{2\pi}$$

(a) $T=2T_0$, $\quad f=\sqrt{2}f_0$

(b) $\rho=\frac{1}{2}\rho_0$, $\quad f=\sqrt{2}f_0$

(c) $a=2a_0$, $b=2b_0$, $\quad f=\frac{1}{2}f_0$

No. 2

Modeling is the art of recognizing and neglecting minor factors and circumstances, and formulating major factors so that they become mathematically accessible, leading to a model that can be solved. No assumption in any model can be satisfied exactly; in particular, in Assumption 2 the tension *will* change during the motion.

No. 3

Square membrance : $a=b$.

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t+B_{mn}^{*}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}.$$

$$\lambda_{mn}=c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{a^2}}=\frac{c\pi}{a}\sqrt{m^2+n^2}$$

No. 4

$B_{mn}=8/(mn\pi^2)$ if m, n odd, 0 otherwise

No. 5

$$B_{mn} = -\frac{8}{mn\pi^2}, \ m \text{ odd}, \ n \text{ even}$$

$$B_{mn} = -\frac{24}{mn\pi^2}, \text{ when both } m \text{ and } n \text{ are odd. } B_{mn} = 0 \text{ otherwise.}$$

No. 6

$$B_{mn} = -\frac{8}{mn\pi^2}, \text{ when } m \text{ is odd and } n \text{ even}$$

$$B_{mn} = \frac{8}{mn\pi^2}, \text{ when } m \text{ is even and } n \text{ is odd; 0 otherwise.}$$

No. 7

For general m, n, $B_{mn} = 4\,\dfrac{-a(-1)^m + a(-1)^{m+n} + b(-1)^n - b(-1)^{m+n}}{n\pi^2 m}$

$B_{mn} = -\dfrac{8a}{mn\pi^2}$, when m is even and n is odd; $= -\dfrac{8}{mn\pi^2}$, when m is even and n is odd; 0 otherwise.

No. 8

$B_{mn} = 64a^2b^2/(m^3n^3\pi^6)$ if m and n are odd, $B_{mn} = 0$ otherwise

No. 9

問答或證明題，不解

No.10

The program will give you

$$\begin{aligned}
85 &= 5 \cdot 17 = 2^2 + 9^2 = 6^2 + 7^2 \\
145 &= 5 \cdot 29 = 1^1 + 12^2 = 8^2 + 9^2 \\
185 &= 5 \cdot 37 = 4^2 + 13^2 = 8^2 + 11^2 \\
221 &= 13 \cdot 17 = 5^2 + 14^2 = 10^2 + 11^2 \\
377 &= 13 \cdot 29 = 4^2 + 19^2 = 11^2 + 16^2 \\
493 &= 17 \cdot 29 = 3^2 + 22^2 = 13^2 + 18^2
\end{aligned}$$

etc.

No.11

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad c^2 = 1$$

$u = 0$, on the boundary ($x = \pi$, $y = \pi$)

$u(x, y, 0) = f(x, y)$, $u_t(x, y, 0) = 0$

$$\therefore u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \underbrace{B_{mn} \sin\frac{m\pi x}{a}}_{\cos\lambda_{mn} t} \sin\frac{n\pi y}{b}, \qquad \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$\lambda_{mn} = \pi\sqrt{\frac{m^2}{\pi^2} + \frac{n^2}{\pi^2}} = \sqrt{m^2 + n^2}.$$

$$u(x, y, 0) = f(x) = 0.1 \sin 2x \sin 4y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin mx \sin ny$$

$$B_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} (0.1 \sin 2x \sin 4y) \sin mx \sin ny \, dx\, dy$$

$$= \begin{cases} 0.1 \,; & m = 2,\ n = 4 \\ 0 \,; & m \neq 2,\ n \neq 4 \end{cases}$$

$$\lambda_{24} = \sqrt{2^2 + 4^2} = \sqrt{20}.$$

$$u(x, y, t) = 0.1 \cos\sqrt{20}\, t \sin 2x \sin 4y$$

No.12

$$u(x,y,0)=f(x)=0.01\sin x\sin y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin mx\sin ny$$

$$B_{mn}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}(0.01\sin x\sin y)\sin mx\sin ny\,dx\,dy$$

$$=\begin{cases}0.01 & ;\ m=1,\ n=1\\ 0 & ;\ m\neq1,\ n\neq1\end{cases}$$

$$\lambda_{11}=\sqrt{1^2+1^2}=\sqrt{2}$$

$$u(x,y,t)=0.01\cos\sqrt{2}t\sin x\sin y$$

No.13

$$u(x,y,0)=f(x)=0.1xy(\pi-x)(\pi-y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin mx\sin ny$$

$$B_{mn}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}0.1xy(\pi-x)(\pi-y)\sin mx\sin ny\,dx\,dy$$

$$=\begin{cases}\frac{6.4}{m^3n^3\pi^2} & ;\ (m,n=\text{odd})\\ 0 & ;\ (m,n=\text{even})\end{cases}$$

$$u(x,y,t)=\sum_{\substack{m,n=1\\ \text{odd}}}^{\infty}\sum^{\infty}\frac{6.4}{m^3n^3\pi^2}\cos\left(\sqrt{m^2+n^2}\,t\right)\sin mx\sin ny$$

No.14

問答或證明題，不解

No.15

$a=4, \quad b=2$

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} = \frac{c\pi}{4}\sqrt{m^2+4n^2}$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\cos\lambda_{mn}t)\sin\frac{m\pi}{4}x\sin\frac{n\pi}{2}y.$$

No.16

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x-x^2)(2y-y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\,dx\,dy$$

$$= \frac{1}{20}\int_0^4(4x-x^2)\sin\frac{m\pi x}{4}\,dx\int_0^2(2y-y^2)\sin\frac{n\pi y}{2}\,dy$$

$$\int_0^4(4x-x^2)\sin\frac{m\pi x}{4}\,dx = \frac{128}{m^3\pi^3}[1-(-1)^m]$$

$$= \begin{cases} \frac{256}{m^3\pi^3} & ; \ m: \text{odd} \\ 0 & ; \ m: \text{even} \end{cases}$$

$$\int_0^2(2y-y^2)\sin\frac{n\pi}{2}y\,dy = \frac{16}{n^3\pi^3}[1-(-1)^n]$$

$$= \begin{cases} \frac{32}{n^3\pi^3} & ; \ n = \text{odd} \\ 0 & ; \ n = \text{even} \end{cases}$$

No.17

$a=2,\ b=1$

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{2^2}+\frac{n^2}{1^2}}$$

$$= \frac{c\pi}{2}\sqrt{m^2+4n^2}$$

No.18

$A = ab$, $b = A/a$, so that from (9) with $m = n = 1$ by differentiating with respect to a and equating the derivative to zero, we obtain

$$\left(\frac{\lambda_{11}^2}{c^2\pi^2}\right)' = \left(\frac{1}{a^2}+\frac{1}{b^2}\right)' = \left(\frac{1}{a^2}+\frac{a^2}{A^2}\right)' = \frac{-2}{a^3}+\frac{2a}{A^2} = 0;$$

hence $a^4 = A^2$, $a^2 = A$, $b = A/a = a$.

No.19

$$u(x,y,0) = f(x,y) = \sin\frac{6\pi x}{a}\sin\frac{2\pi y}{b} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b\left(\sin\frac{6\pi x}{a}\sin\frac{2\pi y}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\cdot dy$$

$$= \begin{cases} 1 & ;\ m=6,\ n=2 \\ 0 & ;\ m\neq 6,\ n\neq 2 \end{cases}$$

$$\lambda_{62} = \pi\sqrt{\frac{6^2}{a^2}+\frac{2^2}{b^2}} = \pi\sqrt{\frac{36}{a^2}+\frac{4}{b^2}}$$

$$\therefore\ u(x,y,t) = \cos\left(\sqrt{\frac{36}{a^2}+\frac{4}{b^2}}\,\pi t\right)\sin\frac{6\pi x}{a}\sin\frac{2\pi y}{b}$$

No.20

$$\rho \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + P.$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{P}{\rho}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{P}{\rho}.$$