CS2336 Discrete Mathematics

Exam 1
October 30, 2017 (2 hours)

Answer all questions. Total marks = 100. For all the proofs, if it is incomplete, large portion of marks may be deducted.

1. (15%) Consider the following compound proposition:

$$[\neg q \oplus (p \wedge q)] \vee (p \to q).$$

In any way that you like, find an equivalent expression that is as short as possible. Prove that your expression is equivalent.

2. (15%) Use logical equivalences and rules of inferences to show that the following arguments are valid. Refer to the last page for some common equivalences and rules.

   - Premises: $\forall x(\neg(P(x) \vee Q(x)) \to R(x))$, $\exists x(\neg P(x) \wedge \neg Q(x))$
   - Conclusion: $\exists x R(x)$

3. (15%) Let x be an integer. Prove that if x is a multiple of 4, then x cannot be the sum of four consecutive integers.

4n. $p \to q$.

$\equiv \neg q \to \neg p$.

4. (30%) Peter is a superstitious mathematician. He thinks that the number 13 is unlucky. So, by Peter's definition, if there exists a way to write a rational number x as p/q, where p and q are integers, and both are not divisible by 13, then x is called a *lucky* number. Otherwise, x is an *unlucky* number.

$5/13$

$5+8=13$

   (a) (10%) Show that 13 is an unlucky number.
   (b) (10%) Prove or disprove: The sum of two lucky numbers is always lucky.
   (c) (10%) Prove or disprove: The product of two lucky numbers is always lucky.

$\frac{p_1}{q_1} \times \frac{p_1}{q_1}$

5. (15%) Fermat's little theorem states that for any prime number p and any integer n, the integer $n^p - n$ is always divisible by p.

   Show that Fermat's little theorem holds when $p = 3$.

6. (10%) [Adapted from R. Smullyan's book, *The Lady or The Tiger?*]

   There are three boxes A, B, and C. One box contains a diamond ring, and the other two each contains a roll of toilet paper.

   Box A is attached with a label, writing:

   "This box contains a roll of toilet paper.".

   Box B is also attached with a label, writing:

   "This box contains a diamond ring.".

   Box C is also attached with a label, writing:

   "Box B contains a roll of toilet paper."

F	T	F	F
F	F	T	F
F	F	F	T

It is known that at most one of the three labels is true. Which box contains the diamond ring? Justify your answer.

t

js

1. Identity Laws:	$p \wedge T_0 \equiv p$	$p \vee F_0 \equiv p$
2. Domination Laws:	$p \wedge F_0 \equiv F_0$	$p \vee T_0 \equiv T_0$
3. Idempotent Laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
4. Double Negation Law:	$\neg(\neg p) \equiv p$	
5. Commutative Laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
6. Associative Laws:	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
7. Distributive Laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
8. De Morgan's Laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption Laws:	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
10. Negation Laws:	$p \wedge \neg p \equiv F_0$	$p \vee \neg p \equiv T_0$
11. De Morgan's Laws with Quantifiers:	$\neg \forall x P(x) \equiv \exists x \neg P(x)$	$\neg \exists x P(x) \equiv \forall x \neg P(x)$
12. Conditional Statement Equivalences:	$p \to q \equiv \neg p \vee q$	$p \to q \equiv \neg q \to \neg p$

Figure 1: Some useful logical equivalences

1. Modus Ponens: Premises: $p, \quad p \to q$	Conclusion: q
2. Modus Tollens: Premises: $\neg q, \quad p \to q$	Conclusion: $\neg p$
3. Hypothetical Syllogism: Premises: $p \to q, \quad q \to r$	Conclusion: $p \to r$
4. Disjunctive Syllogism: Premises: $\neg p, \quad p \vee q$	Conclusion: q
5. Addition: Premise: p	Conclusion: $p \vee q$
6. Simplification: Premise: $p \wedge q$	Conclusion: p
7. Conjunction: Premises: $p, \quad q$	Conclusion: $p \wedge q$
8. Resolution: Premises: $p \vee q, \quad \neg p \vee r$	Conclusion: $q \vee r$
9. Universal Instantiation: Premise: $\forall x P(x)$	Conclusion: $P(c)$, for any c
10. Universal Generalization: Premise: $P(c)$, for any c	Conclusion: $\forall x P(x)$
11. Existential Instantiation: Premise: $\exists x P(x)$	Conclusion: $P(c)$, for some c
12. Existential Generalization: Premise: $P(c)$, for some c	Conclusion: $\exists x P(x)$

Figure 2: Some useful rules of inference

1.

p	q	$p \wedge q$	$p \to q$	$\neg q$	$\neg q \oplus (p \wedge q)$	$[\neg q \oplus (p \wedge q)] \vee (p \to q)$
T	T	T	T	F	T	T
T	F	F	F	T	T	T
F	T	F	T	F	F	T
F	F	F	T	T	T	T

$q \oplus \neg q$
T
T
T
T

$$T = q \oplus \neg q \equiv [\neg q \oplus (p \wedge q)] \vee (p \to q)$$

+15

2.

(1) $\forall x(\neg(P(x) \vee Q(x)) \to R(x))$ premise

(2) $\neg(P(c) \vee Q(c)) \Rightarrow R(c)$, for any c. Universal Instantiation of (1)

(3) $(\neg P(c) \wedge \neg Q(c)) \to R(c)$ De Morgan's Law of

(4) $\exists x(\neg P(x) \wedge \neg Q(x))$ premise

(5) $\neg P(c) \wedge \neg Q(c)$, for some c Existential Instantiation of

(6) $R(c)$, for some c. Modus Ponens of (3),(5)

(7) $\exists x.R(x)$ Existential Generalization

3. +15

prove by contrapositive proof.

assume that x is a sum of four consecutive integers.

$X = (k-1)+(k)+(k+1)+(k+2) = 4k+2$

$\therefore$ x could not be a multiple of 4

$\therefore$ If x is a multiple of 4, then x cannot be the sum of four consecutive integers

5. By direct proof.

$n^p - n$ is always divisible by p.

when $p=3$

$$n^3-n = n(n^2-1) = n(n+1)(n-1) = (n+1)\cdot n\cdot(n-1)$$

在三個連續整數中,必有一數為三的倍數

$\therefore n(n+1)(n-1) = n^3-n$ 必為3的倍數, is always divisible by 3.

得證.※

6. "at most" one of the three lables is true.

(1) if three are all false.

Box A → ring

Box B → paper

Box C → Box B is ring

contradict ∴假設錯誤

(2) if the first lable is true and the rest of them are false.

Box A → paper

Box B → paper

Box C → Box B is ring

contradict ∴假設錯誤

(3) if the second lable is true and the rest of them are false.

Box A → ring

Box B → ring

Box C → Box B is ring

it couldn't be 2 rings ∴假設錯誤

(4) if the third lable is true and the rest of them are fal.

Box A → ring

Box B → paper

Box C → Box B is paper

Correct

∴ Box A contains the diamond ring

4(a) $13 = \frac{p}{q}$, $p=13$, $q=1$

+8 p is divisible by 13. 不可以只写 existence

so 13 is an unlucky number

4(b)

Disprove

此敘述錯誤.

By Existence proof.

5 is a lucky number since $5=\frac{5}{1}$, 5 and 1 are both not divisible by 13.

8 is a lucky number since $8=\frac{8}{1}$, 8 and 1 are both not divisible by 13.

$5+8=13$, and 13 is an unlucky number (by 4(a)).

4(c). By Direct Proof

Assume that the two lucky numbers are $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$

(p_1, p_2, q_1, q_2 are all not divisible by 13, 13不是 $p_1 \cdot p_2 \cdot q_1 \cdot q_2$ 的因數).

The product of the two lucky number

$\frac{p_1 \times p_2}{q_1 \times q_2}$, 13不是 p_1、p_2、q_1、q_2 的因數, 亦不會是 $p_1 \times p_2$、$q_1 \times q_2$ 的因數, $p_1 \times p_2$、$q_1 \times q_2$ are both not divisible by 13

∴ the product of two lucky numbers is always lucky.

CS2336 Discrete Mathematics

Exam 2
December 11, 2017 (2 hours)

Answer all questions. Total marks = 100. A large portion of marks may be deducted from incomplete proofs or wrong arguments.

1. Fermat once conjectured that for $n \geq 0$, all numbers $F_n = 2^{2^n} + 1$ are primes. Indeed, the numbers

$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$

are all primes. However, in 1732, Euler showed that $F_5 = 4294967297 = 641 \times 6700417$, thus disproving Fermat's claim.

(20%) Here, you are asked to show an interesting property of F_n:

For all integer $n \geq 1$, $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$.

2. A standard chessboard contains 8×8 squares. A king controls the squares immediately adjacent to the square that it is placed, in all eight directions. See Figure 1 for an example. A king can attack a piece if it is placed on the squares it controls.

   (a) (5%) If 17 pieces of kings are placed on a chessboard, show that there must be two kings attacking each other.

   (b) (5%) If only 16 pieces of kings are placed on the board, show that it is possible that no kings are attacking any other.

   (c) (5%) If 17 pieces of kings are placed on a chessboard, show that we can find five kings such that they are not attacking any other.

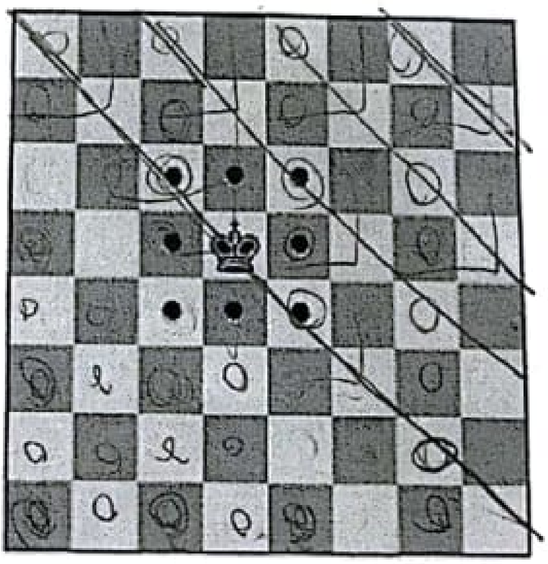
   (d) (5%) If only 16 pieces of kings are placed on a chessboard, show that it is possible that we cannot find five kings such that they are not attacking any other.

17.

3^2

8

$8/2=4$



$64/16=4$

$17/2$

Figure 1: A 8×8 chessboard and a king, with squares controlled by the king marked

P18

3. A contiguous sequence of characters in a string X is called a *substring* of X. For instance, ana is a substring of banana, but aa is not a substring of banana.

   (10%) Consider all the 5-bit binary strings. How many of them contains 11, but not 101 as its substring?

   For example, 11011 contains both 11 and 101 as its substring, 10001 does not contain 11 and 101 as its substring, while 10011 contains 11 but not 101 as its substring.

   Hint: Use a tree diagram.

4. (10%) Given that $x \geq 2, y \geq 1$, and $z \geq 0$, how many integral solutions are there for the equation

$$x + y + z = 11?$$

5. Consider the diagram in Figure 2, where each vertex represents a city, and each edge represents a one-way road.

   (a) (5%) How many ways are there to travel from A to B?

   (b) (5%) How many ways are there to travel from A to B that must pass through X?

   (c) (5%) How many ways to travel from A to B that must pass through both X and Y?

Note: For each part of this question, no explanation is needed.

5
9
12
26

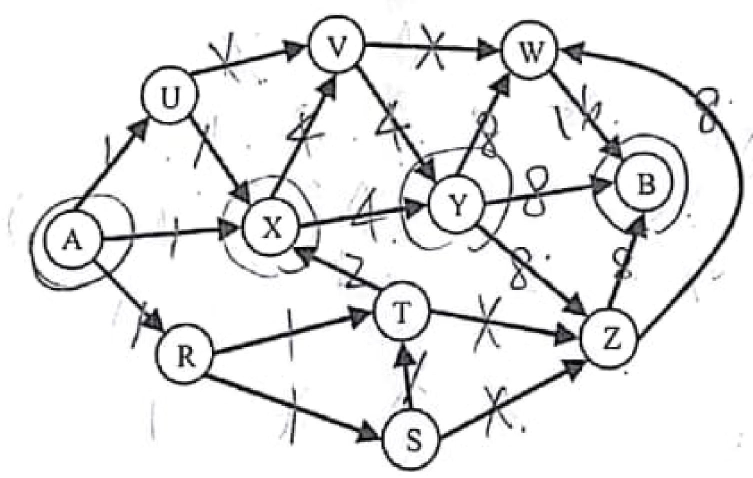


Figure 2: Diagram for Question 5

6. (15%) How many ways we can select 4 distinct integers from $\{1, 2, 3, \ldots, 100\}$, so that their sum is divisible by 3?

7. (10%) Give a combinatorial argument to show that

$$C^n_3 = C^n_1 \cdot C^n_2$$

$$\binom{2n}{3} = 2 \times \binom{n}{3} + 2n \times \binom{n}{2}.$$

Note: No marks will be given if you are not using a combinatorial argument.

1. $F_n = 2^{2^n}+1 \rightarrow F_n - 1 = 2^{2^n}$

prove $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$ $\forall n \geq 1$, n is integer

when $n=1$

$5 = F_1 = F_0 + 2 = 3+2 = 5$ 成立

Assume that

$F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2 \rightarrow F_n - 2 = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1}$

then

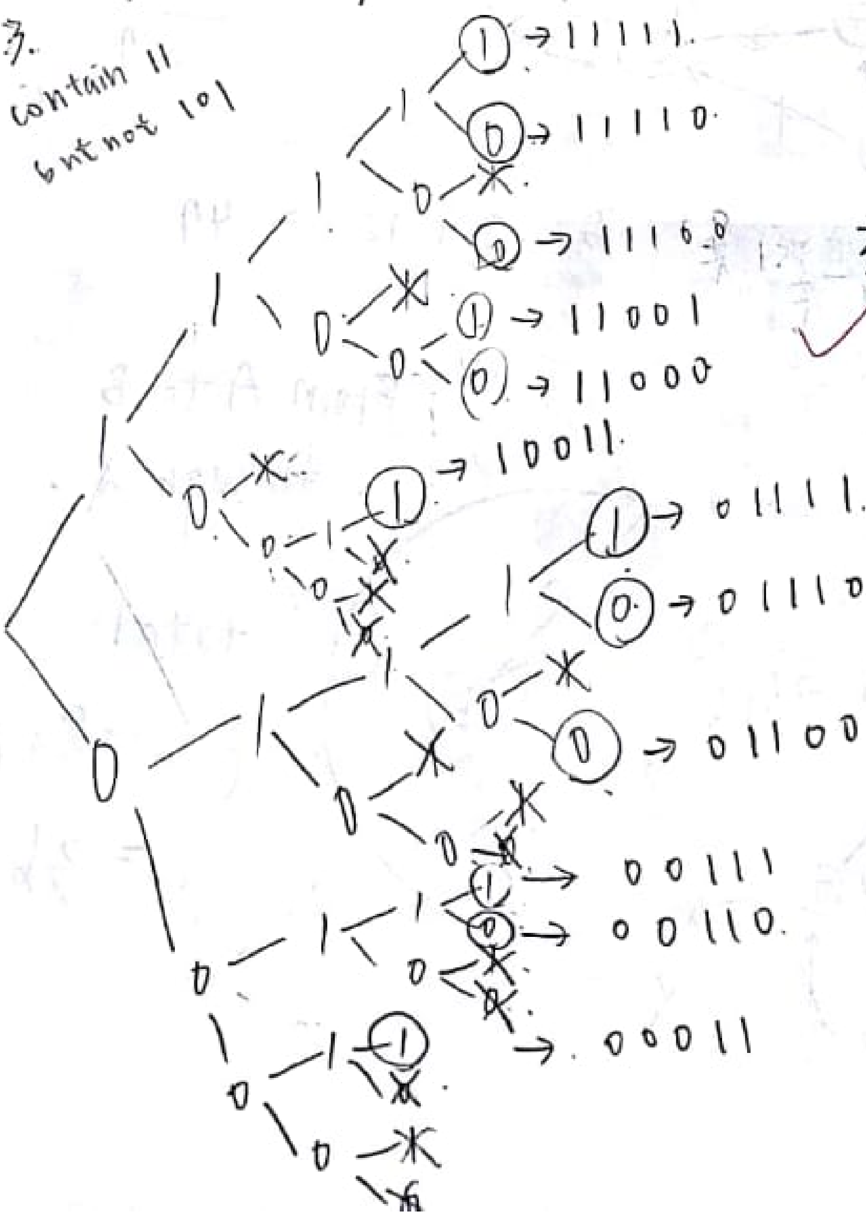
$\underline{F_0 \times F_1 \times F_2 \times \cdots \times F_n + 2} = (F_n - 2) \times F_n + 2 = F_n^2 - 2F_n + 2$

$= (F_n - 1)^2 + 1 = 2^{2^n} \cdot 2^{2^n} + 1 = 2^{2^n + 2^n} + 1 = 2^{2^{n+1}} + 1 = \underline{F_{n+1}}$.

By Mathematical Induction.

For all integer $n \geq 1$, $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$

3. contain 11 but not 101



共12組

4. $x \geq 2,\ y \geq 1,\ z \geq 0$ int solution

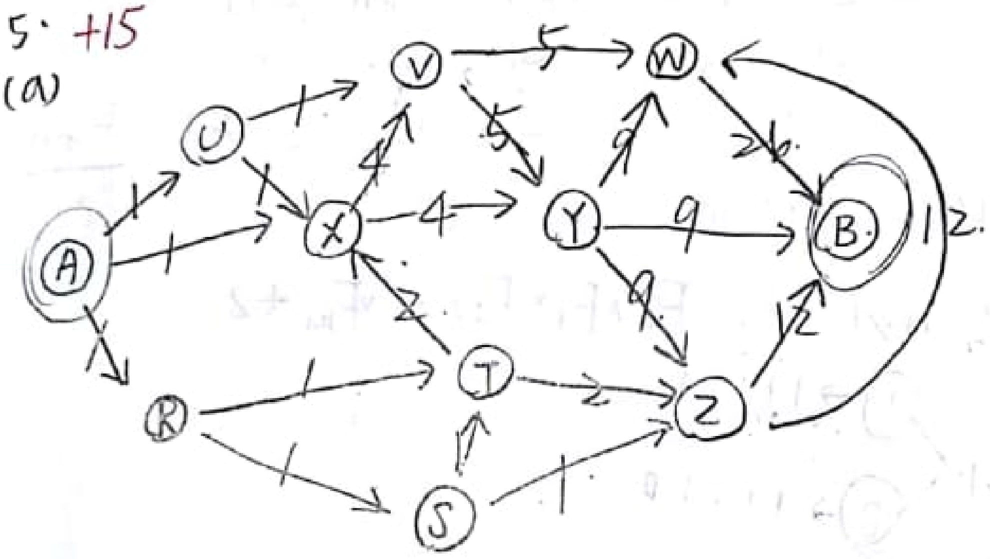
$$x+y+z=11$$

$$\underset{(2)}{\sqcup} + \underset{(1)}{\sqcup} + \sqcup = 11-3=8$$

$$=\binom{10}{8}=\binom{10}{2}=\frac{10\cdot 9}{2}=45$$

5. +15

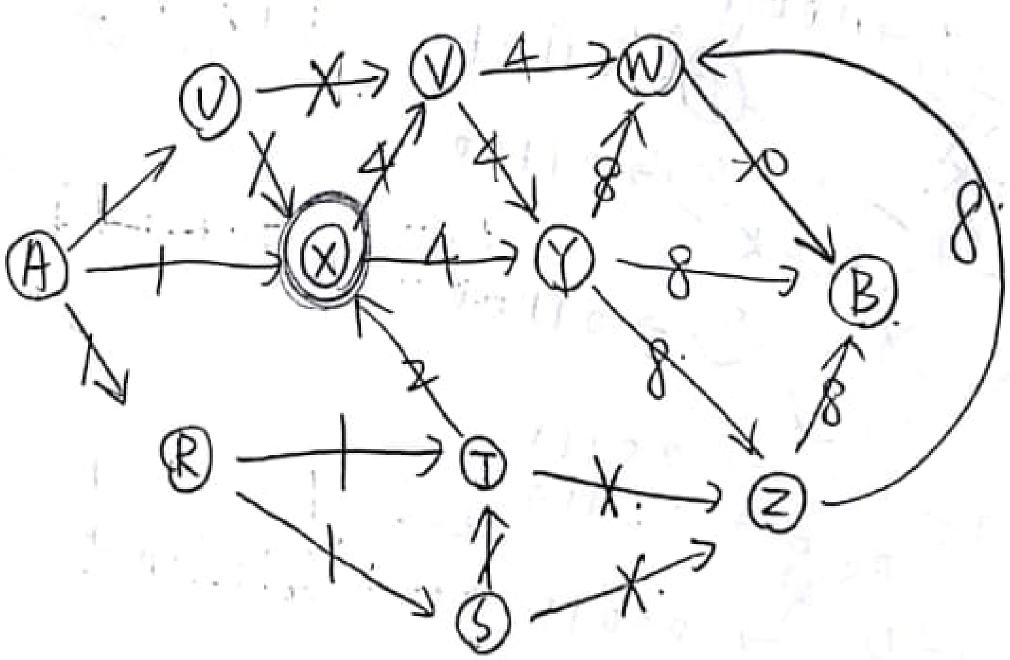
(a)



$$\begin{array}{r} 1 \\ 26 \\ 12 \\ +\ 9 \\ \hline 47 \end{array}$$

From A to B: total $26+9+12=47$.

(b)



From A to B through X:

total

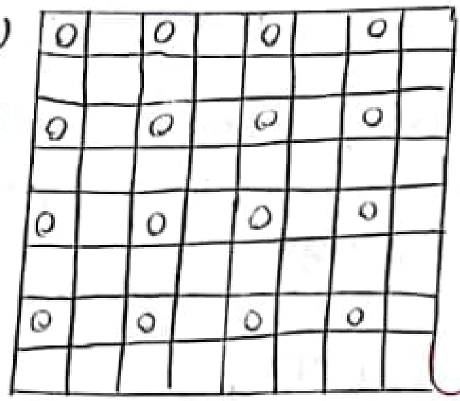
$20+8+8$

$=36$.

2. o: king

(b)

+10



← 16 pieces of kings.
no one attacking any other

(a) kings not attacking each other：左右.上下.斜的皆不相鄰.

$\frac{8}{2}=4$，最多可放4行

Not always true.

$\frac{8}{2}=4$，最多可放4列.

Not always true.

→ $4\times4=16$，最多有16個位子放king，s.t kings not attacking each other

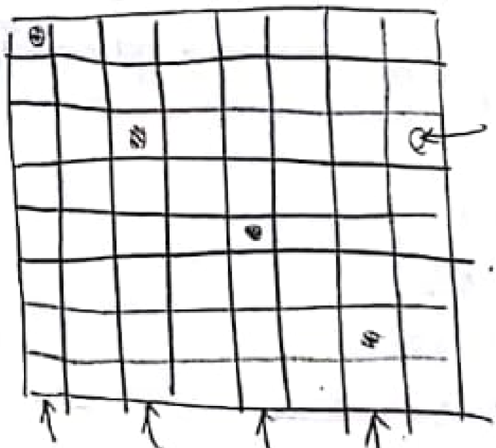
there are 17 kings. but only 16 places.

$\lceil\frac{17}{16}\rceil=2$，by pigeonhole principle：there are at least 2 kings attacking each other.

c) $\lceil\frac{17}{4}\rceil=5$ ∴ we can find five kings. s.t. they are not attacking each other.

$\frac{8}{2}$

?



剩4個選這4行的

(5)

$16+8+8=32$

6. 4 distinct int from $\{1,2,3\ldots100\}$ sum is divisible by 3.

$\%3=0$: $\{3.6.9\cdots99\}$ 33個

$\%3=1$: $\{1,4,7\cdots100\}$ 34個.

$\%3=2$: $\{2.5.8\ldots98\}$ 33個. ~~Ans:?~~

4 int sum divisible by 3 :

(i) 4 int all $\%3=0$: $\binom{33}{4}$

(ii) 3 int $\%3=1$, 1 int $\%3=0$: $\binom{34}{3}\times\binom{33}{1}$

(iii) 2 int $\%3=1$, 2 int $\%3=2$: $\binom{34}{2}\times\binom{33}{2}$

(iiii) 3 int $\%3=2$, 1 int $\%3=0$: $\binom{33}{3}\times\binom{33}{1}$

(iiiii) 1 int $\%3=1$, 1 int $\%3=2$, 2 int $\%3=0$: $\binom{34}{1}\times\binom{33}{2}\times\binom{33}{1}$

Ans: $\binom{33}{4}+\binom{34}{3}\cdot\binom{33}{1}+\binom{34}{2}\cdot\binom{33}{2}+\binom{33}{3}\cdot\binom{33}{1}+\binom{34}{1}\binom{33}{2}\binom{33}{1}$

7. $\binom{2n}{3}=2\binom{n}{3}+2n\binom{n}{2}$

左式:有2n個人,要從中任意取3個. $\left(\binom{2n}{3}\right)$ -- 左式與右式意義相同

右式:將2n個人分左.右兩組,每組n個人.

(i). 3人在同組.

從左組取3人 → $\binom{n}{3}$

或從右組取3人 → $\binom{n}{3}$

→ $2\binom{n}{3}$

(ii) 2人同組,另1人不同組

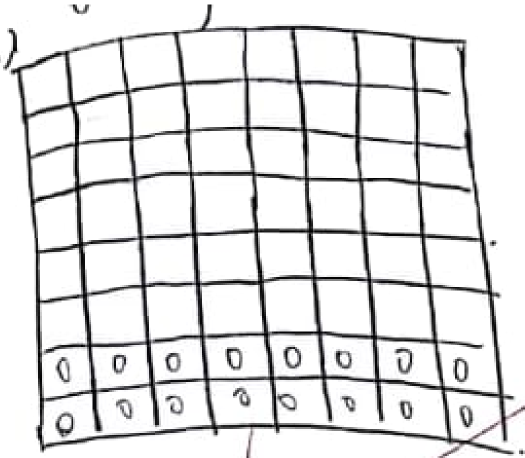
從左組取1人,右組取2人 $\binom{n}{1}\binom{n}{2}=n\cdot\binom{n}{2}$

從右組取1人,左組取2人 $\binom{n}{1}\binom{n}{2}=n\binom{n}{2}$

→ $2\cdot n\binom{n}{2}$

⇒將所有可能性加起來:加法原理: $2\binom{n}{3}+2\cdot n\binom{n}{2}$

(d)



if kings were place like the left figure ←, it is possible that we can't find 5 kings that aren't attacking each other.

CS2336 Discrete Mathematics

Exam 3 (2 hours)
January 08, 2018

Total mark = 100.

1. (10%) Show that the set

$$\{ (x, y, z) \mid x, y, z \in \mathbb{Z}^+ \}$$

is countable.

2. (20%) In each of the following functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ determine if f is one-one, or onto, or none, or both. Give brief explanations to your answer.

正整

(a) $f(x) = \lceil x/10 \rceil$

(b) $f(x) = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ x-1 & \text{if } x \text{ is even} \end{cases}$

$>k-1$ $\quad 1 \to 2$, $2 \to 1$, $3 \to 4$, $4 \to 3$ $\quad 2k-1+1 = 2k$ $\quad 2k-1$

$>k$

3. (10%) Consider the relation R on a set $\{1, 2, 3, 4\}$ as depicted in Figure 1. Find the transitive closure of R.

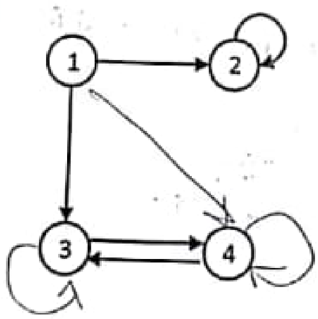


Figure 1: The relation R for Question 1.

4. (10%) Consider the Petersen graph in Figure 2. Show that we can remove three edges to make the graph bipartite.

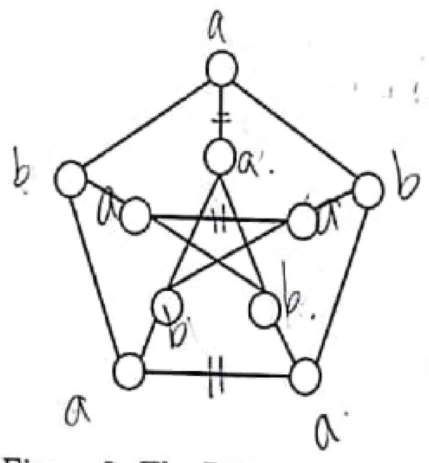
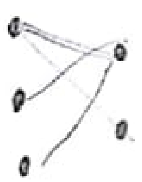


Figure 2: The Petersen graph.



5. (10%) What is the necessary and sufficient condition for the complete bipartite graph $K_{m,n}$ to contain an Euler circuit? Give a brief explanation to your answer.

2E

$V=3$
$F=2$
$E=3$

$E \geq 3V-6$ $\quad F=E+2-V \geq \frac{2}{3}E$

6. (20%) For each graph in Figure 3, prove or disprove that the graph is planar. (No marks if no proof is written down.)

$3E+6-3V \geq 2E$

$E \geq 3V-6$

$V+F=E+2$

$3F \geq 2E$

$F \geq \frac{2}{3}E$

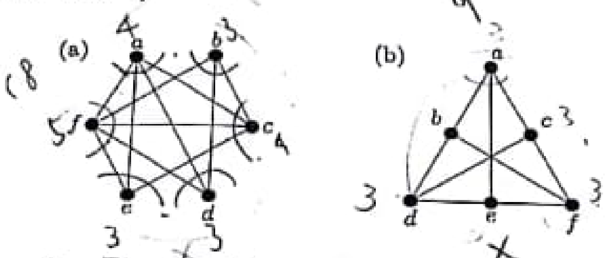


Figure 3: Planar or non-planar?

$F=18$ $\quad E=11$ $\quad V=6$.

$F=8$
$E=9$
$V=6$

$V+F \neq E+2$

7. (20%) Show that the two graphs G_1 and G_2 in Figure 4 are non-isomorphic. (No marks if no proof is written down.)

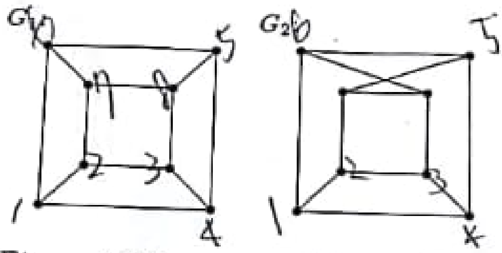


Figure 4: Non-isomorphic graphs.

Hint: Try all graph properties you have learnt. Some leads to a very simple proof.