Part D. COMPLEX ANALYSIS

Major Changes

In the eighth edition, conformal mapping was distributed over several sections in the first chapter on complex analysis. It has now been given greater emphasis by consolidation of that material in a separate chapter (Chap. 17), which can be used independently of a CAS (just as any other chapter) or in part supported by the graphic capabilities of a CAS. Thus in this respect one has complete freedom.

Recent teaching experience has shown that the present arrangement seems to be preferable over that of the 8th edition.

CHAPTER 13 Complex Numbers and Functions. Complex Differentiation

SECTION 13.1. Complex Numbers and Their Geometric Representation, page 608

Purpose. To discuss the algebraic operations for complex numbers and the representation of complex numbers as points in the plane.

Main Content, Important Concepts

Complex number, real part, imaginary part, imaginary unit

The four algebraic operations in complex

Complex plane, real axis, imaginary axis

Complex conjugates

Two Suggestions on Content

1. Of course, at the expense of a small conceptual concession, one can also start immediately from (4), (5),

$$z = x + iy, \qquad i^2 = -1$$

and go on from there.

2. If students have some knowledge of complex numbers, the practical division rule (7) and perhaps (8) and (9) may be the only items to be recalled in this section. (But I personally would do more in any case.)

SOLUTIONS TO PROBLEM SET 13.1, page 612

2. Note that $z = 1 + i$ and $iz = i - 1 = -1 + i$ lie on the bisecting lines of the first and second quadrants.

4. $-12 - 14i, \quad -10 - 6i, \quad -29 + 54i, \quad 3 - 2i$

6. $z_1z_2 = 0$ if and only if

$$\text{Re}\,(z_1z_2) = x_2x_1 - y_2y_1 = 0 \qquad \text{and} \qquad \text{Im}\,(z_1z_2) = y_2x_1 + x_2y_1 = 0.$$

Let $z_2 \neq 0$, so that $x_2^2 + y_2^2 \neq 0$. Now $x_2^2 + y_2^2$ is the coefficient determinant of our homogeneous system of equations in the "unknowns" x_1 and y_1, so that this system has only the trivial solution; hence $z_1 = 0$.

8. $-1 + 17i, -7 - 17i$

9. $-21, 4$

10. $2/25, 1/8$

11. $-\dfrac{11}{16} - \dfrac{15}{4}i$, same.

12. $-\dfrac{11}{10} + \dfrac{13}{10}i, -\dfrac{11}{29} - \dfrac{13}{29}i$

13. $-29 - 14i$, same.

14. $-\dfrac{11}{10} - \dfrac{13}{10}i$, same.

15. $\dfrac{76}{61} - \dfrac{104}{61}i.$

16. $-\dfrac{y}{x^2 + y^2}, \quad -\dfrac{2xy}{(x^2 + y^2)^2}$

18. $(1 + i)^4 = -4, \quad (1 + i)^{16} = 256, \quad \text{Re}\,[(1 + i)^{16}z^2] = 256(x^2 - y^2)$

20. $\text{Im}\,(1/\bar{z}^2 = \text{Im}\,(z^2/(z\bar{z})^2) = 2xy/(x^2 + y^2)^2$

SECTION 13.2. Polar Form of Complex Numbers. Powers and Roots, page 613

Purpose. To give the student a firm grasp of the polar form, including the principal value Arg z, and its application in multiplication and division.

Main Content, Important Concepts

Absolute value $|z|$, argument θ, principal value Arg θ

Triangle inequality (6)

Multiplication and division in polar form

nth root, nth roots of unity (16)

SOLUTIONS TO PROBLEM SET 13.2, page 618

2. $2\sqrt{2}\,(\cos \tfrac{3}{4}\pi + i \sin \tfrac{3}{4}\pi)$

4. $4\,(\cos \pi + i \sin \pi)$

6. Simplification shows that the quotient equals -2, and hence the answer $2(\cos \pi + \sin \pi)$.

8. Upon simplification, the quotient equals $1 + 2i$. Hence the polar form is $\sqrt{1^2 + 2^2}\,(\cos \arctan 2 + i \sin \arctan (2))$.

9. $-\dfrac{1}{4}\pi$

10. $\pi, \quad -\pi + \arctan(\tfrac{1}{5}) = -2.944, \quad 2.944$

11. $\pm\dfrac{\pi}{6}$

12. $-3\pi/4$

13. $-1024, \pi$

14. $\pi - 0.0997 = 3.0419, \quad -\pi + 0.0997 = -3.0419$

15. $4i$

16. $3 + \sqrt{27}i$

17. $-2 - 2i$

18. $-5 + 5i$

20. **Team Project.** **(a)** Use (15).

(b) Use (10) in App. 3.1 in the form

$$\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 + \cos\theta)}, \qquad \sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(1 - \cos\theta)},$$

multiply them by $\sqrt{r}$,

$$\sqrt{r}\cos \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r + r\cos\theta)}, \qquad \sqrt{r}\sin \tfrac{1}{2}\theta = \sqrt{\tfrac{1}{2}(r - r\cos\theta)},$$

use $r\cos\theta = x$, and finally choose the sign of the Im $\sqrt{z}$ in such a way that sign $[(\text{Re}\,\sqrt{z})\,(\text{Im}\,\sqrt{z})] = \text{sign}\, y$.

(c) $\pm\sqrt{7}(1 - i), \quad \pm(4 - 5i), \quad \pm(2 + \sqrt{3}i)$

21. $\sqrt[6]{2}\left(\cos \dfrac{1}{12}k\pi - i \sin \dfrac{1}{12}k\pi\right), k = 1, 9, 17$

22. The three values are

$$\sqrt[3]{5}\,(\cos \tfrac{1}{3}\theta + i \sin \tfrac{1}{3}\theta)$$
$$\sqrt[3]{5}\,(\cos (\tfrac{1}{3}\theta + \tfrac{2}{3}\pi) + i \sin (\tfrac{1}{3}\theta + \tfrac{2}{3}\pi))$$
$$\sqrt[3]{5}\,(\cos (\tfrac{1}{3}\theta + \tfrac{4}{3}\pi) + i \sin (\tfrac{1}{3}\theta + \tfrac{4}{3}\pi))$$

where $\theta = \arctan \tfrac{4}{3}$.

23. $7, -\dfrac{7}{2} \pm \dfrac{7}{2}\sqrt{3}i$

24. $\pm(1 \pm i)$

26. $\cos \dfrac{k\pi}{4} + i \sin \dfrac{k\pi}{4}, \quad k = 0, 1, \cdots, 7$; that is, $\pm 1, \pm i, \pm(1 \pm i)/2$.

28. $3 + 2i, \quad 3 - 4i$

29. $i, 1 - i$.

30. $z^2 = \pm 18i, \quad z = \pm(3 \pm 3i)$. Each of the two factors with real coefficients is obtained if you take one solution and its complex conjugate:

$$(z - 3 - 3i)(z - 3 + 3i) = z^2 - 6z + 18$$
$$(z + 3 + 3i)(z + 3 - 3i) = z^2 + 6z + 18.$$

The product of the two factors is

$$(z^2 + 18)^2 - (6z)^2 = z^4 + 36z^2 + 324 - 36z^2 = z^4 + 324.$$

32. $|z_1 + z_2| = |1 + 5i| = \sqrt{26} = 5.10 < |z_1| + |z_2| = \sqrt{10} + \sqrt{20} = 7.63$

34. $|z| = \sqrt{x^2 + y^2} \geqq \sqrt{x^2} = |x|$, etc.

SECTION 13.3. Derivative. Analytic Function, page 619

Purpose. To define (complex) analytic functions—the class of functions complex analysis is concerned with—and the concepts needed for that definition, in particular, derivatives.

This is preceded by a collection of a few standard concepts on sets in the complex plane that we shall need from time to time in the chapters on complex analysis.

Main Content, Important Concepts

Unit circle, unit disk, open and closed disks

Domain, region

Complex function

Limit, continuity

Derivative

Analytic function

Comment on Content

The most important concept in this section is that of an **analytic function**. The other concepts resemble those of real calculus. The most important new *idea* is connected with the **limit**: the approach in infinitely many possible directions. This yields the negative result in Example 4 and—much more importantly—the **Cauchy–Riemann equations** in the next section.

SOLUTIONS TO PROBLEM SET 13.3, page 624

1. Closed disk, center $-1 + 2i$, radius $\frac{1}{4}$

2. Open unit disk without the origin $z = 0$. An old-fashioned term for this is *deleted neighborhood*; we shall not use this expression.

3. Annulus (circular ring), center $1 - 2i$, radii $\pi/2$, π

4. Open horizontal infinite strip bounded by the parallel straight lines $y = -\pi$ and π.

5. Domain bounded by straight lines in the first and fourth quadrant, that make 60° with the horizontal.

6. We obtain

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2 + y^2} < 1, \qquad x < x^2 + y^2, \qquad \tfrac{1}{4} < (x - \tfrac{1}{2})^2 + y^2,$$

hence

$$\tfrac{1}{4} < |z - \tfrac{1}{2}|^2, \qquad \tfrac{1}{2} < |z - \tfrac{1}{2}|.$$

This is the open exterior of the circle with center $\frac{1}{2}$ on the x-axis and radius $\frac{1}{2}$; hence this circle passes through the origin $z = 0$.

7. Half-plane extending from the vertical straight line $x = -1$ to the left.

8. Geometrically, we need hardly any calculation. The left side of the inequality is the distance d_1 of z from $-i$, and the right side is the distance d_2 of z from i. These distances are equal if and only if z lies on the x-axis. Sketch it. Strict inequality $d_1 > d_2$ means that z is farther away from $-i$ than from i; this is the case if and only if z lies in the upper half-plane.

Analytically, we have

$$|x + iy + i|^2 \geqq |x + iy - i|^2,$$

that is,

$$x^2 + (y + 1)^2 \geqq x^2 + (y - 1)^2$$

and thus

$$y^2 + 2y + 1 \geqq y^2 - 2y + 1,$$

so that $4y \geqq 0$ and $y \geqq 0$, the upper half-plane.

10. We obtain

$$u(x, y) = \operatorname{Re} f(z) = 5(x^2 - y^2) - 12x + 3$$

and

$$u(4, -3) = -10.$$

Similarly,

$$v(x, y) = \operatorname{Im} f(z) = 10xy - 12y + 2$$

and

$$v(4, -3) = -82.$$

11. $u(x, y) = \dfrac{x + 1}{(x + 1)^2 + y^2}$, $v(x, y) = \dfrac{-y}{(x + 1)^2 + y^2}$, $u(1, -1) = 2/5$, $v(1, -1) = 1/5$

12. We obtain

$$u(x, y) = \operatorname{Re} f(z) = \frac{x^2 - 1}{(x + 1)^2 + y^2} + \frac{y^2}{(x + 1)^2 + y^2}$$

and

$$u(0, 2) = 3/5.$$

Furthermore,

$$v(x, y) = \left(\frac{y(x + 1)}{(x + 1)^2 + y^2} - \frac{(x - 1)y}{(x + 1)^2 + y^2}\right)$$

and

$$v(0, 2) = 3/5.$$

14. Yes, because

$$(r^2 \cos 2\theta)/r = r \cos 2\theta \rightarrow 0 = f(0) \qquad \text{as } r \rightarrow 0.$$

16. No, since

$$(r^2 \sin 2\theta)/r^2 = \sin 2\theta$$

depends on θ, on the direction of approach to 0, so that it has no limit, by definition.

18. Since the differentiation rules in complex are the same as in calculus, the purpose of Probs. 18–23 is to help the student in getting accustomed to handling complex expressions.

At present, by the quotient rule,

$$f'(z) = \frac{z + i - (z - i) \cdot 1}{(z + i)^2} = \frac{2i}{(z + i)^2}.$$

Thus, $f'(i) = 2i/(2i)^2 = -i/2$.

19. $f'(z) = 3(z - 2i)^2$. Since $z - 2i = 5$, $f'(5 + 2i) = 3 \cdot 5^2 = 75$.

20. 0 since the quotient is constant, equal to $-i/2$.

22. $f'(z) = 3(iz^3 + 3z^2)^2(3iz^2 + 6z), f(2i) = 0$ because the last factor of $f'(z)$ is zero at $2i$.

23. $-3iz^2/(z - i)^4, 3i/16$

24. Team Project. (a) Use $\operatorname{Re} f(z) = [f(z) + \overline{f(z)}]/2, \operatorname{Im} f(z) = [f(z) - \overline{f(z)}]/2i$.

(b) Assume that $\lim_{z\to z_0} f(z) = l_1, \lim_{z\to z_0} f(z) = l_2, l_1 \neq l_2$. For every $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - l_j| < \epsilon \qquad \text{when} \qquad 0 < |z - z_0| < \delta_j, \qquad j = 1, 2.$$

Hence for $\epsilon = |l_1 - l_2|/2$ and $0 < |z - z_0| < \delta$, where $\delta \leqq \delta_1, \delta \leqq \delta_2$, we have

$$\begin{aligned} |l_1 - l_2| &= |[f(z) - l_2] - [f(z) - l_1]| \\ &\leqq |f(z) - l_2| + |f(z) - l_1| < 2\epsilon = |l_1 - l_2|. \end{aligned}$$

(c) By continuity, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ when $|z - a| < \delta$. Now $|z_n - a| < \delta$ for all sufficiently large n since $\lim z_n = a$. Thus $|f(z_n) - f(a)| < \epsilon$ for these n.

(d) The proof is as in calculus. We write

$$\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta.$$

Then, from the definition of a limit, it follows that for any given $\epsilon > 0$ there is a $\delta > 0$ such that $|\eta| < \epsilon$ when $|z - z_0| < \delta$. From this and the triangle inequality,

$$|f(z) - f(z_0)| = |z - z_0||f'(z_0) + \eta| \leqq |z - z_0||f'(z_0)| + |z - z_0|\epsilon,$$

which approaches 0 as $|z - z_0| \to 0$.

(e) The quotient in (4) is $\Delta x/\Delta z$, which is 0 if $\Delta x = 0$ but 1 if $\Delta y = 0$, so that it has no limit as $\Delta z \to 0$.

(f) $$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}.$$

When $z = 0$ the expression on the right approaches zero as $\Delta z \to 0$. When $z \neq 0$ and $\Delta z = \Delta x$, then $\overline{\Delta z} = \Delta x$ and that expression approaches $z + \bar{z}$. When $z \neq 0$ and $\Delta z = i\,\Delta y$, then $\overline{\Delta z} = -i\,\Delta y$ and that expression approaches $-z + \bar{z}$. This proves the statement.

SECTION 13.4. Cauchy—Riemann Equations. Laplace s Equation, page 625

Purpose. To derive and explain the most important equations in this chapter, the Cauchy–Riemann equations, a system of two PDEs, which constitute the basic criterion for analyticity.

Main Content, Important Concepts

Cauchy–Riemann equations (1)

These equations as a criterion for analyticity (Theorems 1 and 2)

Derivative in terms of partial derivatives, (4), (5) Relation of analytic functions to Laplace's equation

Harmonic function, harmonic conjugate

Comment on Content

(4), (5), and Example 3 will be needed occasionally.
The relation to Laplace's equation is basic, as mentioned in the text.

SOLUTIONS TO PROBLEM SET 13.4, page 629

2. No, $f(z) = u + iv = 0 + i|z|^2, \quad u = 0, v = |z|^2$.

3. Yes

4. No, $u_x = e^x \cos y \neq v_y = -e^x \cos y$.

6. Yes, when $z \neq 0, \pm 1, \pm i$.

7. Yes, when $z \neq 0$. Use (7).

8. No, $u = Arg\, z, v = 0$.

10. Yes, when $z \neq 0$.

11. Yes

12. No.

13. $f(z) = i(z^2 + c)$, c real

14. Yes, $f(z) = \frac{1}{2}z^2 + c$, c a real constant. Indeed using the Cauchy–Riemann equations, we obtain

$$\begin{aligned} v &= xy \\ v_y &= x = u_x \\ u &= \tfrac{1}{2}x^2 + k(y) \\ u_y &= k'(y) = -v_x = -y \end{aligned}$$

hence

$$k(y) = -\tfrac{1}{2}y^2 + c$$

so that

$$u = \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + c.$$

15. $f(z) = -1/z + c$, (c real)

16. Yes,

$$f(z) = \sin x \cosh y + i \cos x \sinh y + ic$$

because, using the Cauchy–Riemann equations, we obtain

$$\begin{aligned} u &= \sin x \cosh y \\ u_x &= \cos x \cosh y = v_y \\ v &= \cos x \sinh y + h(x) \\ v_x &= -\sin x \sinh y + h'(x) = -u_y = -\sin x \sinh y \end{aligned}$$

hence $h'(x) = 0, \quad h = c = \text{const (real)}$, so that we obtain the expression given at the beginning, which is $\sin z + ic$, as we shall see in Sec. 13.6.

17. $f(z) = z^2 - z + c$, (c real)

18. We obtain

$$f(z) = z^3 + ic = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$$

because, using the Cauchy–Riemann equations, we have

$$\begin{aligned} u &= x^3 - 3xy^2 \\ u_x &= 3x^2 - 3y^2 = v_y \\ v &= 3x^2y - y^3 + h(x) \\ v_x &= 6xy + h'(x) = -u_y = 6xy \end{aligned}$$

so that $h'(x) = 0$, $h = c = \text{const}$ (real).

19. No.

21. $a = \pi, v = e^{-\pi x} \cos \pi y$.

22. By differentiation we obtain

$$\nabla^2 u = (-a^2 + 4)u = 0,$$

so $a = \pm 2$. Hence we can take

$$u = \cos 2x \cosh 2y$$

and obtain, by using the first Cauchy–Riemann equation,

$$u_x = -2 \sin 2x \cosh 2y = v_y$$

and by integration

$$v = -\sin 2x \sinh 2y + h(x).$$

The second Cauchy–Riemann equation now gives

$$v_x = -2 \cos 2x \sinh 2y + h'(x) = -u_y = -2 \cos 2x \sinh 2y.$$

Hence $h' = 0$ and $h(x) = \text{const}$, and the *answer* is

$$v = -\sin 2x \sinh 2y + c$$

which will give the imaginary part of $\cos 2z$; see Sec. 13.6.

24. $\nabla^2 u = a^2 \cosh ax \cos y - \cosh ax \cos y = 0$, hence $a = 1$. From this and the Cauchy–Riemann equations we obtain, by differentiation,

$$\begin{aligned} u &= \cosh x \cos y \\ u_x &= \sinh x \cos y = v_y \\ v &= \sinh x \sin y + h(x) \\ v_x &= \cosh x \sin y + h'(x) \\ &= -u_y \\ &= \cosh x \sin y, \end{aligned}$$

so that $h'(x) = 0$, $h(x) = c = \text{const}$ (real), and

$$v = \sinh x \sin y + c,$$

the imaginary part of $\cosh z + ic$, as we shall see in Sec. 13.6.

30. Team Project. (a) $u = \text{const}$, $u_x = u_y = 0, v_x = v_y = 0$ by (1), $v = \text{const}$, and $f = u + iv = \text{const}$.

(b) Same idea as in **(a)**.

(c) $f' = u_x + iv_x = 0$ by (4). Hence $v_y = 0, u_y = 0$ by (1), $f = u + iv = \text{const}$.

SECTION 13.5. Exponential Function, page 630

Purpose. Sections 13.5–13.7 are devoted to the most important elementary functions in complex, which generalize the corresponding real functions, and we emphasize properties that are not apparent in real.

Basic Properties of the Exponential Function

Derivative and functional relation as in real

Euler formula, polar form of z

Periodicity with $2\pi i$, fundamental region

$e^z \neq 0$ for all z

SOLUTIONS TO PROBLEM SET 13.5, page 632

2. $e^3(\cos 4 + i \sin 4) = -13.129 - 15.201, \quad 20.086$
3. $e^{2\pi} \cdot e^{2\pi i} = e^{2\pi} = 535.4916$
4. $e^{0.6}(\cos 1.8 - i \sin 1.8) = 1.822(-0.227 - 0.974i) = -0.414 - 1.774i$
5. $-e = -2.718$
6. $e^{11\pi i/2} = e^{-\pi i/2} = -i$
7. $-e^{\sqrt{3}} = -5.652i$
8. $\sqrt[n]{r} \exp(i(\theta + 2k\pi)/n), \quad k = 0, 1, \cdots, \quad n - 1, \quad r = |z|, \theta = \operatorname{Arg} z$
9. $5e^{-\arctan(4/3)i} = 5e^{-0.9272i}$
10. $e^{\pi i/4}, \quad e^{-3\pi i/4}; \quad e^{-\pi i/4}, \quad e^{3\pi i/4}$
11. $\dfrac{3}{2} e^{\pi i}$
12. $\dfrac{1}{\sqrt{(x-1)^2 + y^2}} e^{i \arctan(y/(1-x))}$
13. $\sqrt{2} e^{-\frac{\pi}{4} i}$
14. $e^{-\pi x} \cos \pi y, \quad -e^{-\pi x} \sin \pi y$
15. $e^{-x^2+y^2} \cos 2xy, -e^{-x^2+y^2} \sin 2xy$
16. $e^{1/z} = \exp\left(\dfrac{x}{x^2+y^2}\right)\left(\cos \dfrac{y}{x^2+y^2} - i \sin \dfrac{y}{x^2+y^2}\right)$
18. **Team Project.** **(a)** $e^{1/z}$ is analytic for all $z \neq 0$. $e^{\bar{z}}$ is not analytic for any z. The last function is analytic if and only if $k = 1$.

(b) (i) $e^x \sin y = 0$, $\sin y = 0$. *Answer:* On the horizontal lines $y = \pm n\pi, n = 0, 1, \cdots$.
(ii) $e^{-x} < 1, x > 0$ (the right half-plane).
(iii) $e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i \sin y) = \overline{e^x(\cos y + i \sin y)} = \overline{e^z}$. *Answer:* All z.
(d) $f' = u_x + iv_x = f = u + iv$, hence $u_x = u, v_x = v$. By integration,

$$u = c_1(y)e^x, \qquad v = c_2(y)e^x.$$

By the first Cauchy–Riemann equation,

$$u_x = v_y = c_2' e^x, \qquad \text{thus} \qquad c_1 = c_2' \quad (' = d/dy).$$

By the second Cauchy–Riemann equation,

$$u_y = c_1' e^x = -v_x = -c_2 e^x, \qquad \text{thus} \qquad c_1' = -c_2.$$

Differentiating the last equation with respect to y, we get

$$c_1'' = -c_2' = -c_1, \qquad \text{hence} \qquad c_1 = a \cos y + b \sin y.$$

Now for $y = 0$ we must have

$$u(x, 0) = c_1(0)e^x = e^x, \qquad c_1(0) = 1, \quad a = 1,$$
$$v(x, 0) = c_2(0)e^x = 0, \qquad c_2(0) = 0.$$

Also, $b = c_1'(0) = -c_2(0) = 0$. Together $c_1(y) = \cos y$. From this,

$$c_2(y) = -c_1'(y) = \sin y.$$

This gives $f(z) = e^x(\cos y + i \sin y)$.

20. $z = \ln 5 + (\arctan \frac{3}{4} + 2n\pi)i$

21. $z = \ln 2 + (1 + 2n)\pi i$

SECTION 13.6. Trigonometric and Hyperbolic Functions, Euler s Formula, page 633

Purpose. Discussion of basic properties of trigonometric and hyperbolic functions, with emphasis on the relations between these two classes of functions as well as between them and the exponential function; here we see, on an elementary level, that investigation of special functions in complex can add substantially to their understanding.

Main Content

Definitions of $\cos z$ and $\sin z$ (1)

Euler's formula in complex (5)

Definitions of $\cosh z$ and $\sinh z$ (11)

Relations between trigonometric and hyperbolic functions

Real and imaginary parts (6) and Prob. 3

SOLUTIONS TO PROBLEM SET 13.6, page 636

2. The right side is

$$\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$= \tfrac{1}{4}(e^{z_1} + e^{-z_1})(e^{z_2} + e^{-z_2}) + \tfrac{1}{4}(e^{z_1} - e^{-z_1})(e^{z_2} - e^{-z_2}).$$

If we multiply out, then, because of the minus signs, the products $e^{z_1}e^{-z_2}$ and $e^{-z_1}e^{z_2}$ cancel in pairs. There remains, as asserted,

$$2 \cdot \tfrac{1}{4}(e^{z_1+z_2} + e^{-z_1-z_2}) = \cosh(z_1 + z_2).$$

Similarly for the other formula.

6. $i \sinh \pi/2 = 2.302i$

7. $\cosh 1 = 1.543$, $-i \sinh 1 = -1.175i$

8. $\cosh \pi = 11.59$, and

$$\cosh \pi i = \tfrac{1}{2}(e^{\pi i} + e^{-\pi i}) = \tfrac{1}{2}(-1 - 1) = -1$$

9. Both $2.033 - 3.052i$. Why?

10. $-6.548 - 7.619i, \quad -6.581 - 7.582i$

11. $i \sinh \pi/4 = 0.8688i$, both

12. We obtain

$$\cos \tfrac{1}{2}\pi i = \cosh \tfrac{1}{2}\pi = 2.509$$

and

$$\begin{aligned}\cos\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\pi i\right) &= \cos \tfrac{1}{2}\pi \cos \tfrac{1}{2}\pi i - \sin \tfrac{1}{2}\pi \sin \tfrac{1}{2}\pi i \\ &= 0 - 1 \cdot i \sinh \tfrac{1}{2}\pi \\ &= -2.301i.\end{aligned}$$

14. From (7a) we obtain

$$|\cos z|^2 = \cos^2 x + \sinh^2 y = \cos^2 x + \cosh^2 y - 1.$$

Hence $|\cos z|^2 \geqq \sinh^2 y$ from the first equality, and $|\cos z|^2 \leqq \cosh^2 y$ from the second equality because $\cos^2 x - 1 \leqq 0$. Now take square roots.

The inequality for $|\sin z|$ is obtained similarly.

16. $\cos x \sinh y = 0, \quad x = \tfrac{1}{2}\pi \pm 2n\pi, \quad \cosh y = 100, \quad \cosh y \approx \tfrac{1}{2}e^y$ for large y, $e^y \approx 200$, $y \approx 5.29832$ (agrees to 4D with the solution of $\cosh y = 100$).
Answer: $z = \tfrac{1}{2}\pi \pm 2n\pi \pm 5.29832i.$

17. $(2n + 1)\pi i/4$

18. **(a)** $\cosh x \cos y = -1$, **(b)** $\sinh x \sin y = 0$. From **(b)** we have $x = 0$ or $y = \pm n\pi$. Then $y = (2n + 1)\pi$ and $x = 0$ from **(a)**. *Answer:* $z = (2n + 1)\pi i.$

20. We obtain

$$\tan z = \frac{\sin z}{\cos z} = \frac{\sin z\, \overline{\cos z}}{|\cos z|^2}.$$

Hence the denominator is (use $\cosh^2 y = \sinh^2 y + 1$ to simplify)

$$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y.$$

Insert $\sin z$ and $\cos z$ into the numerator and multiply out. Then for the real part of the product we get

$$\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y = \sin x \cos x$$

and for the imaginary part, using $\sin^2 x + \cos^2 x = 1$, we get

$$\sin^2 x \cosh y \sinh y + \cos^2 x \cosh y \sinh y = \cosh y \sinh y.$$

SECTION 13.7. Logarithm. General Power, Principal Value, page 636

Purpose. Discussion of the complex logarithm, which extends the real logarithm $\ln x$ (defined for positive x) to an infinitely many-valued relation (3) defined for all $z \neq 0$; definition of general powers z^c.

Comment on Notation

$\ln z$ is also denoted by $\log z$, but for the engineer, who also needs logarithms $\log x$ of base 10, the notation ln is more practical; this notation is widely used in mathematics.

Important Formulas

Real and imaginary parts (1)

Relation of the principal value to the other values (3)

Relations between ln and the exponential function (4)

Functional relation in complex (5)

Derivative (6)

General power (8)

SOLUTIONS TO PROBLEM SET 13.7, page 640

5. $\ln 7 + \pi i$

6. $\frac{7}{2} \ln 2 + \pi i/4 = 2.426 + 0.7854i$

7. $\frac{7}{2} \ln 2 - \pi i/4 = 2.426 - 0.7854i$

8. $\ln \sqrt{2} \pm \pi i/4 = 0.347 \pm 0.785i$

9. $-i \arctan(4/3) = -0.927i$

10. $2.708 \pm 3.135i$. This also is a reminder that the principal value Arg z has a jump along the negative part of the real axis.

11. $1 + \pi i$

12. $\ln e = 1 \pm 2n\pi i, \quad n = 0, 1, \cdots$

14. $\ln 7 + (1 \pm 2n)\pi, \quad n = 0, 1, \cdots$

16. $\ln 5 - (\arctan \frac{3}{4} \pm 2n\pi)i = 1.609 \pm 2n\pi)i, \quad n = 0, 1, \cdots$

18. $z = e^{\frac{\pi}{2}i}$

20. $z = e^{e - \pi i} = e^e e^{\pi i} = -e^e = -15.154$

21. $e^{0.4}e^{0.2i} = e^{0.4}(\cos 0.2 + i \sin 0.2) = 1.462 + 0.2964i$

22. Using the definition, we obtain

$$
\begin{aligned}
(2i)^{2i} &= e^{2i \operatorname{Ln} 2i} \\
&= e^{2i (\ln |2i| + \pi i/2)} \\
&= e^{-\pi} e^{2i \ln 2} \\
&= e^{-\pi} e^{i \ln 4} \\
&= e^{-\pi}(\cos (\ln 4) + i \sin (\ln 4)) \\
&= 0.0079 + 0.0425i
\end{aligned}
$$

24.

$$
\begin{aligned}
e^{(1+i)\operatorname{Ln}(1-i)} &= e^{(1+i)(\ln\sqrt{2} - \pi i/4)} \\
&= \exp (\ln \sqrt{2} - \pi i/4 + i \ln \sqrt{2} + \pi/4) \\
&= \sqrt{2}e^{\pi/4}(\cos (-\tfrac{1}{4}\pi + \ln \sqrt{2}) + i \sin (-\tfrac{1}{4}\pi + \ln \sqrt{2})) \\
&= 2.8079 - 1.3179i.
\end{aligned}
$$

Note that this is the complex conjugate of the answer to Prob. 23.

26. $i^{i/2} = e^{i/2 \ln i} = e^{(i/2)\pi i/2} = e^{-\pi/4} = 0.456$

28. We obtain

$$\begin{aligned}\exp\left(\tfrac{1}{3}\,\mathrm{Ln}\,(3 + 4i)\right) &= \exp\left(\tfrac{1}{3}\left(\ln 5 + i \arctan \tfrac{4}{3}\right)\right)\\ &= \sqrt[3]{5}\left[\cos\left(\tfrac{1}{3}\arctan\tfrac{4}{3}\right) + i \sin\left(\tfrac{1}{3}\arctan\tfrac{4}{3}\right)\right]\\ &= 1.6289 + 0.5202i.\end{aligned}$$

30. Team Project. (a) $w = \arccos z$, $z = \cos w = \frac{1}{2}(e^{iw} - e^{-iw})$. Multiply by $2e^{iw}$ to get a quadratic equation in e^{iw},

$$e^{2iw} - 2ze^{iw} + 1 = 0.$$

A solution is $e^{iw} = z + \sqrt{z^2 - 1}$, and by taking logarithms we get the given formula

$$\arccos z = w = -i \ln (z + \sqrt{z^2 - 1}).$$

(b) Similarly,

$$\begin{aligned} z = \sin w &= \frac{1}{2i}(e^{iw} - e^{-iw}),\\ 2ize^{iw} &= e^{2iw} - 1,\\ e^{2iw} - 2ize^{iw} - 1 &= 0,\\ e^{iw} &= iz + \sqrt{-z^2 + 1}.\end{aligned}$$

Now take logarithms, etc.

(c) $\cosh w = \frac{1}{2}(e^{w} + e^{-w}) = z, \quad (e^w)^2 - 2ze^w + 1 = 0, \quad e^w = z + \sqrt{z^2 - 1}.$ Take logarithms.

(d) $z = \sinh w = \frac{1}{2}(e^w - e^{-w}), \quad 2ze^w = e^{2w} - 1, \quad e^w = z + \sqrt{z^2 + 1}.$ Take logarithms.

(e) $$z = \tan w = \frac{\sin w}{\cos w} = -i\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = -i\,\frac{e^{2iw} - 1}{e^{2iw} + 1},$$

$$e^{2iw} = \frac{i - z}{i + z}, \qquad w = \frac{1}{2i}\ln\frac{i - z}{i + z} = \frac{i}{2}\ln\frac{i + z}{i - z}$$

(f) This is similar to **(e)**.

SOLUTIONS TO CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS, page 641

1. $2 - 3i$

2. You obtain the complex conjugate. The absolute value.

11. $-9 + 40i$

12. $(1 - i)^{10} = [(1 - i)^2]^5 = (-2i)^5 = -32i$

13. $\dfrac{3}{25} + \dfrac{4}{25}i$

14. $\pm(1 + i)/\sqrt{2}$

15. $-i$

16. $i, \quad -i$

17. $2\sqrt{2}e^{-\frac{\pi}{4}i}$

18. $12.04e^{\pm 0.08314i}$

19. $5e^{-\pi/2i}$

20. $e^{i \arctan(0.8/0.6)} = e^{0.9273i}$

21. $\pm 5, \pm 5i$

22. $\pm 4(1 - i)$

24. $1, \quad \cos \dfrac{2\pi}{3} \pm i \sin \dfrac{2\pi}{3} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}$

25. $f(z) = iz^2/2$

26. $f(z) = -1/z$

27. $f(z) = e^{-3z}$

28. $f(z) = \cos 3z$

30. $f(z) = \sin 2z$

31. $\cos(5)\cosh(2) + i \sin(5)\sinh(2) = 1.067 - 3.478i$

32. $\ln 1 + i \operatorname{Arg}(0.6 + 0.8i) = 0 + i \arctan(0.8/0.6) = 0.927i$

33. $\dfrac{\sin(1)\cos(1)}{(\cos^2(1)) + (\sinh^2(1))} + \dfrac{i\sinh(1)\cosh(1)}{(\cos^2(1)) + (\sinh^2(1))} = 0.2718 + 1.084\mathrm{i}$

34. We obtain

$$\sinh(1 + \pi i) = \sinh 1 \cos \pi + i \cosh 1 \sin \pi = -\sinh 1 = -1.175,$$

and, totally different from this,

$$\sin(1 + \pi i) = \sin 1 \cosh \pi + i \cos 1 \sinh \pi = 9.754 + 6.240i.$$

35. $-\sinh \pi = -11.55$