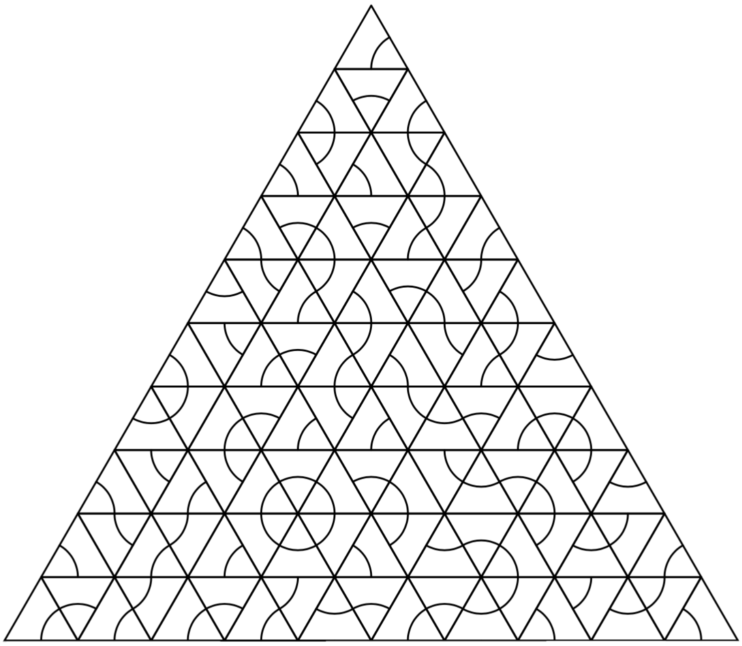
CS5314 Randomized Algorithms

Homework 2 Suggested Solution

(Homework due date was April 21, 2020)

1. An equilateral triangle is tiled with n^2 smaller congruent equilateral triangles such that there are n smaller triangles along each side of the original triangle. For each of these smaller equilateral triangles, we randomly choose a vertex v of the triangle and draw an arc with v as the center connecting the midpoints of the two sides of the triangle.

   The case when $n = 10$ is shown as follows.

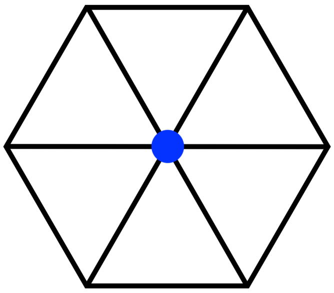
   

   Find the expected number of full circles formed, in terms of n.

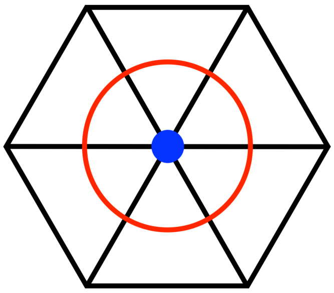
   Hint: Design appropriate indicators, and use linearity of expectation.

   Ans. $\dfrac{(n-1)(n-2)}{2 \cdot 3^6} = \dfrac{n^2 - 3n + 2}{1458}$

   Consider a vertex (the blue vertex in Figure 1.1) that has six small triangles around it. Each of these triangles has a $\frac{1}{3}$ probability of its arc being the arc of a circle centered on this vertex. Consequently, the probability that this vertex has a full circle (the red circle in Figure 1.2) around it is $\frac{1}{3^6}$.



(a) Figure 1.1



(b) Figure 1.2

The number of vertices that are surrounded by six small triangles is

$$1 + 2 + 3 + \cdots + (n-3) + (n-2) = \frac{(n-1)(n-2)}{2}$$

By linearity of expectation, the expected number of full circles is

$$\frac{1}{3^6} \cdot \frac{(n-1)(n-2)}{2} = \frac{(n-1)(n-2)}{2 \cdot 3^6} = \frac{n^2 - 3n + 2}{1458}$$

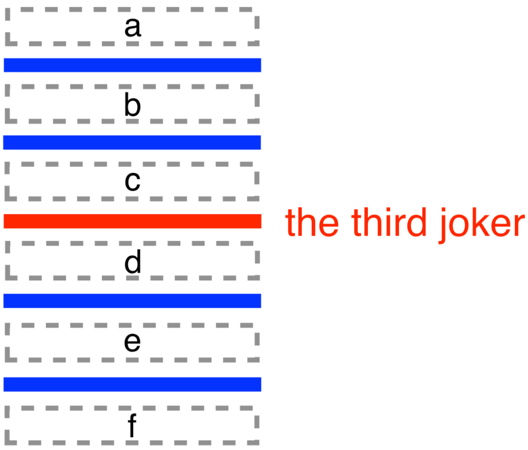
2. A deck of n playing cards, which contains five jokers, is well-shuffled. The cards are turned up one by one from the top until the third joker appears. What is the expected number of cards to be turned up?

   Hint: Let X be the expected number of cards to be turned up. What is the relationship between $\Pr(X = k)$ and $\Pr(X = n - k + 1)$?

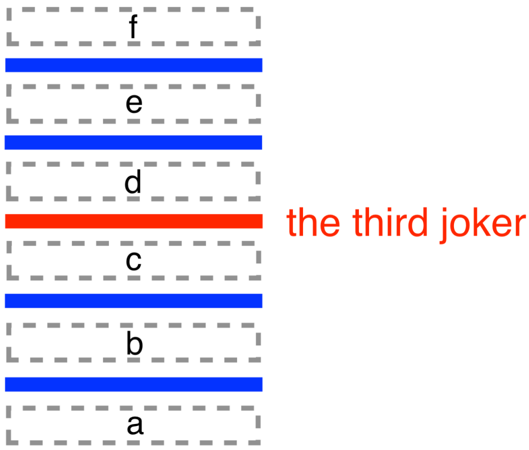
   Ans. $\frac{n+1}{2}$

   The five jokers divide the deck into six piles with a non-negative number of cards. Let a, b, c, d, e, and f be the number of cards in the first, second, third, fourth, fifth and six piles, respectively (see Figure 2.1, the blue and red lines represent the five jokers), we have

$$a + b + c + d + e + f = n - 5$$



(c) Figure 2.1



(d) Figure 2.2

Let $E(a)$, $E(b)$, $E(c)$, $E(d)$, $E(e)$, $E(f)$ denote the expected values of a, b, c, d, e, and f, we have

$$E(a) = E(b) = E(c) = E(d) = E(e) = E(f)$$

and

$$E(a) + E(b) + E(c) + E(d) + E(e) + E(f) = n - 5$$

Thus

$$E(a) = E(b) = E(c) = E(d) = E(e) = E(f) = \frac{n-5}{6}$$

Now, we could compute the expected number of cards up to and including the third jokers, which is

$$E(a) + 1 + E(b) + 1 + E(c) + 1 = \frac{n-5}{6} + 1 + \frac{n-5}{6} + 1 + \frac{n-5}{6} + 1 = \frac{n+1}{2}$$

There is an alternative reasoning based on the hint. By symmetry, the chance of seeing the third joker as the kth card from the top of the deck is equal to the chance of seeing the third joker as the kth card from the bottom of the deck. This implies

$$\Pr(X = k) = \Pr(X = n + 1 - k).$$

As a result,

$$\begin{aligned}
E[X] &= \sum_{k=1}^{n} k \Pr(X = k) \\
&= (1/2) \cdot \left(\sum_{k=1}^{n} k \Pr(X = k) + \sum_{k=1}^{n} k \Pr(X = n + 1 - k) \right) \\
&= (1/2) \cdot \sum_{k=1}^{n} (k + n + 1 - k) \Pr(X = k) \\
&= (1/2) \cdot (n + 1) \sum_{k=1}^{n} \Pr(X = k) = \frac{n+1}{2}
\end{aligned}$$

3. A lost tourist arrives at a point with 4 roads. There are no signs on the roads. The first road brings him back to the same point after 2 hours of walk. The second road leads to the city after 3 hours of walk. The third road brings him back to the same point after 4 hours of walk. The last road leads to the city after 5 hours of walk.

   Assuming that the tourist chooses a road equally likely at all times. (That is, a road may be chosen again and again.) What is the mean time until the tourist arrives to the city?

   Ans. 7.

   Let T be the time until the tourist arrives to the city. Let R_1, R_2, R_3, R_4 be the event that the tourist chooses the first, second, third and fourth road, respectively. Then, by conditional expectation formula,

$$
\begin{aligned}
E[T] &= E[T|R_1]\cdot \Pr(R_1) + E[T|R_2]\cdot \Pr(R_2) + E[T|R_3]\cdot \Pr(R_3) + E[T|R_4]\cdot \Pr(R_4) \\
&= (E[T]+2)\cdot \frac{1}{4} + 3\cdot \frac{1}{4} + (E[T]+4)\cdot \frac{1}{4} + 5\cdot \frac{1}{4} \\
&= \frac{1}{2}E[T] + \frac{7}{2}
\end{aligned}
$$

We obtain

$$E[T] = 7$$

4. We roll a fair 6-sided die over and over again.

   (a) What is the expected number of rolls until a 6 turns up twice in a row (i.e., a 6 followed by a 6)?

   (A) 24 (B) 30 (C) 36 (D) 42 (E) 48

   Ans. D.

   Let X be the number of rolls until the sequence 66 appears. All possible trails can be divided into three cases.

   - **With probability $\frac{5}{6}$ the first roll is not a 6.**

     In this case we are essentially starting over, so we expect to need another $E[X]$ rolls, for a total $E[X]+1$ rolls.

   - **With probability $\frac{1}{6}$ the first roll is a 6, and with probability $\frac{5}{6}$ the second roll is not a 6.**

     In this case we are essentially starting over, so we expect to need another $E[X]$ rolls, for a total $E[X]+2$ rolls.

   - **With probability $(\frac{1}{6})^2$ the first two rolls are both 6.**

     In this case a pair of consecutive sixes appears, and we need 2 rolls altogether.

   As mentioned above, we can write the following equation.

$$E[X] = \frac{5}{6}\cdot (E[X]+1) + \frac{1}{6}\cdot \frac{5}{6}\cdot (E[X]+2) + \left(\frac{1}{6}\right)^2 \cdot 2$$

   We obtain

$$E[X] = 42$$

There is an alternative reasoning. When we start rolling, on average, we expect to need 6 rolls until a 6 shows up. Once that happens, there is a $\frac{1}{6}$ chance that we will roll once more (a pair of consecutive sixes appears), and a $\frac{5}{6}$ chance that we will be starting over (the second roll is not a 6). As a result, we can say

$$E[X] = 6 + \frac{1}{6} \cdot 1 + \frac{5}{6} \cdot (E[X] + 1)$$

Again, we obtain

$$E[X] = 42$$

(b) What is the expected number of rolls until the sequence 65 appears (i.e., a 6 followed by a 5)?

(A) 24 (B) 30 (C) 36 (D) 42 (E) 48

Ans. C.

Let Y be the number of rolls until the sequence 65 appears. And let Z be the number of rolls until the sequence 65 appears when we start with a rolled 6. All possible trails can be divided into two cases.

- **With probability $\frac{5}{6}$ the first roll is not a 6.**

  In this case we are essentially starting over, so we expect to need another $E[Y]$ rolls, for a total $E[Y] + 1$ rolls.

- **With probability $\frac{1}{6}$ the first roll is a 6.**

  In this case there are three possibilities.

  - **With probability $\frac{4}{6}$ the second roll is not a 6 nor a 5.**

    In this case we are essentially starting over, so we expect to need another $E[Y]$ rolls.

  - **With probability $\frac{1}{6}$ the second roll is a 6.**

    In this case we expect to need another $E[Z]$ rolls.

  - **With probability $\frac{1}{6}$ the second roll is not a 5.**

    In this case the sequence 65 appears.

As mentioned above, we can write the following equation.

$$E[Y] = \frac{5}{6} \cdot (E[Y] + 1) + \frac{1}{6} \cdot (E[Z] + 1)$$

$$E[Z] = \frac{4}{6} \cdot (E[Y] + 1) + \frac{1}{6} \cdot (E[Z] + 1) + \frac{1}{6} \cdot 1$$

We obtain

$$E[Y] = 36$$

$$E[Z] = 30$$

Observe that the difference between E[Y] and E[Z] is 6, which is the expected number of rolls until the first 6 shows up.

5. Suppose that 14 boys and 6 girls line up in a row. Let N be the number of places in the row where a boy and a girl are standing next to each other.

   For example, for the row BGBBGBGBBBGBGBBBGBBB we have $N = 12$.

   The expected value of N is closest to

   (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

   Hint: Design appropriate indicators, and use linearity of expectation.

   Ans. A.

   There are 20 people (14 boys and 6 girls) line up in a row. For a pair (one boy and one girl), there are $2 \cdot 19 \cdot 18!$ ways for this pair standing next to each other. Therefore, the probability of this pair standing next to each other is $\frac{2 \cdot 19 \cdot 18!}{20!}$, which is equals to 0.1. Since there are $14 \cdot 6 = 84$ possible pairs, the expexted value of N is $84 \cdot 0.1 = 8.4$

   There is an alternative reasoning. For each girl g, the expected number of B-G pairs (g on the right, and some boy on its left) that g will form is $14/20$, as there are 20 slots that g can stand once the other people are fixed, and 14 such slots yield a B-G pair. Similarly, the expected number of G-B pairs (g on the left, and some boy on its right) that g will form is also $14/20$. By linearity of expectation, the expected number of B-G and G-B pairs that all girls will form is $(14/20) \cdot 2 \cdot 6 = 8.4$.