CS5314 Randomized Algorithms

Exam 2
Date: June 09, 2020 (2 hours)

Part I: Basics (15%) – No explanation is needed

1. Let X be a binomial random variable with parameters n and p.

   (5%) What is the MGF for X?

2. Let Y be a Poisson random variable with parameter λ.

   (a) (5%) What is $\mathrm{E}[Y]$?

   (b) (5%) What is the MGF for Y?

Part II: Calculation and Derivation (85%)

1. Let I be an indicator with $\Pr(I = 1) = p$.

   (a) (5%) Find the moment generating function $f(t) = \mathrm{MGF}_I(t)$ for I.

   (b) (10%) Compute the functions $f'(t)$ and $f''(t)$, and obtain the values $f'(0)$ and $f''(0)$.

   (c) (5%) What is the meaning of $f''(0) - (f'(0))^2$? Explain your answer.

2. It is known that a random variable X has the following MGF:

   $$(0.2 + 0.3e^t + 0.5e^{4t})(0.4e^{2t} + 0.6e^{3t})$$

   (a) (5%) Design a random variable Y that has the same MGF as X.

   (b) (15%) Since X and Y have the same MGF, it is known that X and Y must have the same distribution. Using the result of (a), or otherwise, compute $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.

   (c) (5%) Write down $\Pr(X \geq 3)$. (No explanation is needed.)

3. Let X and Y be independent Poisson random variables with parameters 1 and 2, respectively. Let Z be the sum of them, i.e., $Z = X + Y$.

   (10%) What is the probability $\Pr(Z = 0)$? (Express your answer in the simplest form. No explanation is needed.)

4. Let Y be a Poisson random variable with parameter λ. Let x be a value such that $x < \lambda$.

   (15%) Show that

   $$\Pr(Y \leq x) \leq e^{-\lambda} \left(\frac{e\lambda}{x}\right)^x.$$

5. Let Y be a Poisson random variable with parameter λ. In the homework, we have shown that $\Pr(Y \geq \lambda) \geq 1/2$. Indeed, it is also a fact that $\Pr(Y \leq \lambda) \geq 1/2$.

   Suppose we further know that $\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})]$ is monotonically **decreasing** in m.

   (15%) Based on the above fact, show that

   $$\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \leq 2\,\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})].$$

   Note: This is not exactly the same question as in HW4, because here we assume that $\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})]$ is monotonically **decreasing** in m, not monotonically **increasing**.