A Review of Variation Bayesian Gaussian Mixture Model

Naive EM

Pseudo Code

Iterate until θ converge

- E Step

  Evaluate $q(Z;\gamma) = p(Z|Y)$

- M Step

  $\arg\max_\theta \int_Z q(Z;\gamma) log\, p(Y,Z;\theta) dZ$

EM In General Form

In naive EM, the goal is to optimize

$$\arg\max_{\theta} \mathcal{L}(Y;\theta) = \arg\max_{\theta}\ log \int_Z p(Y,Z;\theta)dZ$$

With ELBO, we can derive

$$\mathcal{L}(\theta,\gamma) = \mathbb{E}_q[log(\frac{p(Y,Z;\theta)}{q(Z;\gamma)})]$$

$$= \int_Z q(Z;\gamma) log\ \frac{p(Y,Z;\theta)}{q(Z;\gamma)} dZ$$

$$= log\, p(Y;\theta) - KL[q(Z;\gamma)||p(Z|Y)]$$

$$= \mathcal{L}(Y;\theta) - KL[q(Z;\gamma)||p(Z|Y)]$$

EM In General Form

Thus

$$\mathcal{L}(\theta,\gamma) = \mathcal{L}(Y;\theta) - KL[q(Z;\gamma)||p(Z|Y)]$$

Since the KL-divergence always ≥ 0

$$\arg\max_{\theta} \mathcal{L}(Y;\theta) \geq \arg\max_{\theta,\gamma} \mathcal{L}(\theta,\gamma)$$

With KKT and Lagrange multiplier, the optimization problem can be written as

$$\arg\max_{\theta,\gamma} \mathcal{L}(\theta,\gamma) = \arg\max_{\theta,\gamma} log\ p(Y;\theta) - \beta KL[q(Z;\gamma)||p(Z|Y)]$$

EM In General Form

Pseudo Code

Iterate until θ converge

- E Step at k-th iteration

  $\gamma_{k+1} = \arg\max_{\gamma} \mathcal{L}(\theta_k, \gamma_k)$

- M Step at k-th iteration

  $\theta_{k+1} = \arg\max_{\theta} \mathcal{L}(\theta_k, \gamma_{k+1})$

Variational Bayesian Expectation Maximization(VBEM)

In VBEM, we consider an **additional prior**

$$log\, p(Y) = log \int_{Z,\theta} p(Y, Z, \theta; \lambda) dZ d\theta$$

$$= log\, \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[\frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

$$\geq \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[log\, \frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

Thus, we get the ELBO $\mathcal{L}(\phi^Z, \phi^\theta)$

$$\mathcal{L}(\phi^Z, \phi^\theta) = \mathbb{E}_{q(Z;\phi^Z)q(\theta;\phi^\theta)}[log\, \frac{p(Y, Z|\theta)p(\theta; \lambda)}{q(Z; \phi^Z)q(\theta; \phi^\theta)}]$$

Variational Bayesian Expectation Maximization(VBEM)

According to the general form of EM

$$\arg\max_{\gamma} \mathcal{L}(\theta_k, \gamma_k)$$

$$\arg\max_{\theta} \mathcal{L}(\theta_k, \gamma_{k+1})$$

We can derive

$$\frac{d}{d\phi^Z}\mathcal{L}(\phi^Z, \phi^\theta) = 0, \quad ln\, q(Z; \phi^Z) \propto \mathbb{E}_{q(\theta;\phi^\theta)}[log\, p(Y, Z, \theta)]$$

$$\frac{d}{d\phi^\theta}\mathcal{L}(\phi^Z, \phi^\theta) = 0, \quad ln\, q(\theta; \phi^\theta) \propto \mathbb{E}_{q(Z;\phi^Z)}[log\, p(Y, Z, \theta)]$$

Variational Bayesian Expectation Maximization(VBEM)

Pseudo Code

Iterate until $\mathcal{L}(\phi^Z, \phi^\theta)$ converge

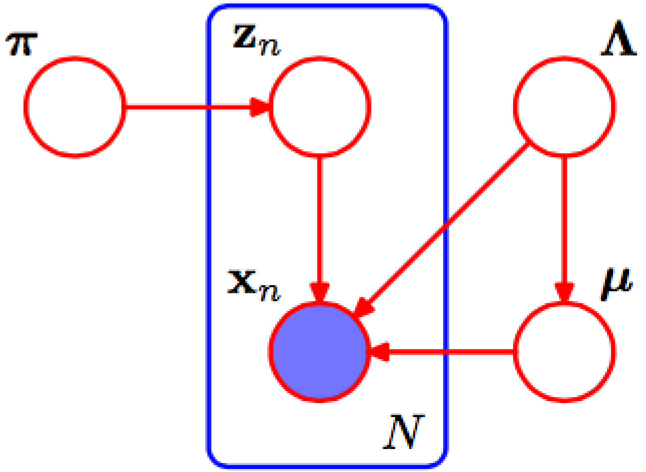
- E Step: Update the variational distribution on Z

  $q(Z;\phi^Z) \propto e^{(\mathbb{E}_{q(\theta;\phi^\theta)}[log\ p(Y,Z,\theta)])}$

- M Step: Update the variational distribution on θ

  $q(\theta;\phi^\theta) \propto e^{(\mathbb{E}_{q(Z;\phi^Z)}[log\ p(Y,Z,\theta)])}$

Variational Bayesian Gaussian Mixture Model(VB-GMM)



$$p(X, Z, \pi, \mu, \Lambda) = p(X|Z, \pi, \mu, \Lambda)p(Z|\pi)p(\pi)p(\mu|\Lambda)p(\Lambda)$$

- $p(X|Z, \pi, \mu, \Lambda)$ denotes the **Gaussian Mixture Model**
- $p(Z|\pi)$ denotes the **Latent Variables**
- $p(\pi)$ denotes the **Prior Distribution Over The Latent Variables** Z
- $p(\mu|\Lambda)p(\Lambda)$ denotes the **Priors Distribution Over The Gaussian Distribution** X