Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 11

The Lebesgue Theory

Exercise 11.1 If $f \geq 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. *Hint:* Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \cup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.

Solution. The assertion in the hint is immediate. If $\mu(A) = 0$, then $\mu(E_n) = 0$ also, since $E_n \subseteq A$. Conversely, letting $F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$, we have $F_n \subset E_n$, $F_m \cap F_n = \emptyset$ if $m \neq n$, and $\cup F_n = \cup E_n = A$. Hence if $\mu(E_n) = 0$, then $\mu(F_n) = 0$ also, and therefore $\mu(A) = 0$ by the countable additivity of μ.

Given the hint, the solution is immediate, since A is the subset of E on which $f(x) > 0$. If $\mu(F_n) > 0$ for any n, then $\int_E f\,d\mu \geq \int_{F_n} f\,d\mu \geq \mu(F_n)/n > 0$.

Exercise 11.2 If $\int_A f\,d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.

Solution. The hypothesis applies in particular if A is the set on which $f(x) > 0$. Since $\chi_A f \geq 0$, the preceding exercise shows that $\mu(A) = 0$. Likewise, taking B as the set on which $-f(x) > 0$, we find that $\mu(B) = 0$. Hence $f(x) = 0$ for almost every x.

Exercise 11.3 If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Solution. This set can be written as

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \bigcap_{l=m}^{\infty} \{x : |f_k(x) - f_l(x)| < \frac{1}{n}\}.$$

For this set is the set of x such that for every n there exists m such that $|f_k(x) - f_l(x)| < 1/n$ for all $k \geq m$, $l \geq m$. That is precisely the Cauchy criterion for convergence.

Exercise 11.4 If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathcal{L}(\mu)$ on E.

Solution. This follows immediately from the dominated convergence theorem and the fact that $|g(x)| \leq M$ for some constant M. (Take $f_n(x) = g_n(x)f(x)$ for all n, where $g_n(x)$ is a sequence of simple functions converging to $g(x)$ almost everywhere. We can assume $|g_n(x)| \leq M$ and let the dominating function be $M|f(x)|$.)

Exercise 11.5 Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$
$$f_{2k}(x) = g(x) \quad (0 \leq x \leq 1),$$
$$f_{2k+1}(x) = g(1-x) \quad (0 \leq x \leq 1).$$

Show that

$$\liminf_{n\to\infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x)\, dx = \frac{1}{2}.$$

[Compare with (77).]

Solution. Since for each $x \in [0, \frac{1}{2}]$ we have $f_{2k}(x) = 0$ for all k, it follows that the inferior limit at such an x is zero. The same is true for $x \in [\frac{1}{2}, 1]$, since $f_{2k+1}(x) = 0$ for all these x. The value of the integral is immediate, since each $f_n(x)$ is a step function.

The point of this exercise is that strict inequality can easily occur in Fatou's Lemma.

Exercise 11.6 Let

$$f_n(x) = \begin{cases} \dfrac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \to 0$ uniformly on R^1, but

$$\int_{-\infty}^{\infty} f_n\, dx = 2 \quad (n = 1, 2, 3, \ldots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{R^1}$.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

Solution. The uniform convergence to zero is obvious, since $0 \le f_n(x) \le 1/n$ for all x and all n.

Again, since $f_n(x)$ is a step function, the value of the integral is immediate.

Exercise 11.7 Find a necessary and sufficient condition that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. *Hint:* Consider Example 11.6(*b*) and Theorem 11.33.

A bounded function f belongs to $\mathcal{R}(\alpha)$ on $[a, b]$ if and only if the following two conditions hold:

(*i*) f is right-continuous wherever α is not right-continuous and left-continuous wherever α is not left-continuous (that is, one of f and α is right-continuous at each point and one is left-continuous);

(*ii*) the set of points where α is continuous and f is not continuous is a set of zero α-variation. That is, this set has μ_α-measure zero, where μ_α is the regular Borel measure generated by the function α, as in Example 11.6(*b*).

To prove this fact, all we have to do is copy the proof of Theorem 11.33, *mutatis mutandis*, specifically, replacing dx by $d\alpha$ and Δx by $\Delta\alpha$ at every stage. It will follow as a corollary of the proof that if $f \in \mathcal{R}(\alpha)$, then $f \in \mathcal{L}(\mu_\alpha)$ and

$$\int_{[a,b]} f \, d\mu_\alpha = \mathcal{R} \int_a^b f(x) \, d\alpha(x).$$

In modifying the proof we need to clear out just one case in order to make the changes run smoothly. To that end, we note that if f and α both have a one-sided discontinuity from the same side and at the same point, it is impossible for f to belong to $\mathcal{R}(\alpha)$. Indeed, suppose p is a common right-sided discontinuity of both f and α. For any partition P we have $x_k \le p < x_{k+1}$ for some index k, and then

$$U(P, f, \alpha) - L(P, f, \alpha) \ge o \cdot (\alpha(p+) - \alpha(p)) > 0,$$

where o is the limit of $\sup\limits_{p \le x < p+h} f(x) - \inf\limits_{p \le x < p+h} f(x)$ as $h \downarrow 0$. (The function $f(x)$ is right-continuous at p if and only if $o = 0$.)

Note that if f and α are discontinuous from opposite sides at a point, it is quite possible that $f \in \mathcal{R}(\alpha)$. For example, let $f(x) = \chi_{[0,1/2]}(x)$ and $\alpha(x) = \chi_{[1/2,\infty)}(x)$. Then for any partition P of $[0, 1]$ containing $1/2$, we have $x_k = \frac{1}{2}$ for some k, and

$$U(P, f, \alpha) - L(P, f, \alpha) = 1 - 1 = 0.$$

(It is for this reason that I define $\mathcal{R}(\alpha)$ differently in my courses. I require that for each $\varepsilon > 0$ there must exist $\delta > 0$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for *all* partitions P such that $\max\limits_{1 \le k \le n} (x_k - x_{k-1}) < \delta$. When this is done, conditions (*i*) and (*ii*) are no longer sufficient for f to belong to $\mathcal{R}(\alpha)$, and the theory is somewhat simpler. Except for special considerations at discontinuities of α, however, the results are the same in both theories.)

We now suppose that condition (i) holds and prove the necessity of condition (ii). To avoid having to single out the endpoints in what follows, we simply extend α outside the interval $[a,b]$ by specifying $\alpha(x) = \alpha(b)$ for $x > b$ and $\alpha(x) = \alpha(a)$ for $x < a$.

Suppose that f is in $\mathcal{R}(\alpha)$. Let $\{P_k\}$ be a sequence of partitions such that P_{k+1} is a refinement of P_k, the distance between adjacent points of P_k is less than $\frac{1}{k}$, P_k contains all points x at which $\alpha(x+) - \alpha(x-) > \frac{1}{k}$, and

$$U(P_k, f, \alpha) \to \mathcal{R}\overline{\int} f\,d\alpha, \quad L(P_k, f, \alpha) \to \mathcal{R}\underline{\int} f\,d\alpha.$$

We note that every discontinuity of α belongs to some partition P_k. Assume P_k consists of the points $a = x_{k,0} < x_{k,1} < \cdots < x_{k,n_k} = b$.

As in the proof of Theorem 11.33, we define $U_k(x) = M_i$ and $L_k(x) = m_i$ for $x_{k,i-1} < x \le x_{k,i}$, $1 \le i \le n_k$. For definiteness we define $U_k(a) = M_1$ and $L_k(a) = m_1$. Then by definition of the upper and lower sume, the definition of $\mu_\alpha((a,b])$, and the definition of the integral of a simple function,

$$\begin{aligned}
\int_{[a,b]} U_k\,d\mu_\alpha &= U(P_k, f, \alpha) + M_1\big(\alpha(a+) - \alpha(a)\big), \\
\int_{[a,b]} L_k\,d\mu_\alpha &= L(P_k, f, \alpha) + m_1\big(\alpha(a+) - \alpha(a)\big).
\end{aligned}$$

By condition (i), either $M_1 - m_1 \to 0$ as $k \to \infty$ or $\alpha(a+) = \alpha(a)$. It then follows that the monotonic sequences L_k and U_k have limits L and U that are measurable, and either

$$\int_{[a,b]} L\,d\mu_\alpha = \mathcal{R}\underline{\int} f\,d\alpha, \quad \int_{[a,b]} U\,d\mu_\alpha = \mathcal{R}\overline{\int} f\,d\alpha,$$

(when $\alpha(a+) = \alpha(a)$) or

$$\begin{aligned}
\int_{[a,b]} L\,d\mu_\alpha &= \mathcal{R}\underline{\int} f\,d\alpha + f(a)\big(\alpha(a+) - \alpha(a)\big), \\
\int_{[a,b]} U\,d\mu_\alpha &= \mathcal{R}\overline{\int} f\,d\alpha + f(a)\big(\alpha(a+) - \alpha(a)\big),
\end{aligned}$$

(when $\alpha(a+) > \alpha(a)$).

If these two integrals are the same, it follows that $U(x) = L(x)$ almost everywhere with respect to the measure μ_α. If x is not a point of any partition P_k and $U(x) = L(x)$, then f is continuous at x. As for points of the partition, they are either points of discontinuity of α or points x such that $\mu_\alpha(\{x\}) = 0$. Since there are only countably many points in all the partitions, the partition points x for which $\mu_\alpha(\{x\}) = 0$ form a set of measure zero. Thus the set of discontinuities of $f(x)$ can be written as the union $A \cup B$, where A consists of points where f is continuous from only one side and α is discontinuous from that side (these points are all among the points of partition), and B consists of the points of discontinuity of f where α is continuous. We have just shown that B is of zero α-variation, as claimed.

Conversely, if f satisfies these two conditions, we note that $U(x) = L(x)$ at all points where $f(x)$ is continuous. Hence if the discontinuities of $f(x)$ other than one-sided discontinuities at points where α is continuous from the side on which f is discontinuous form a set of zero α-variation, then $U(P_k, f, \alpha) - L(P_k, f, \alpha) \to 0$.

Exercise 11.8 If $f \in \mathcal{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)\,dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Solution. We know by Theorem 6.20 that $F'(x) = f(x)$ at every point where $f(x)$ is continuous. Theorem 11.33 shows that if f is Riemann-integrable on $[a, b]$, then it is continuous almost everywhere. Hence the result follows.

Exercise 11.9 Prove that the function F given by (96) is continuous on $[a, b]$.

Solution. The function F is the one in the preceding exercise. Its continuity follows from the dominated convergence theorem, taking $|f|$ as the "dominating" function and letting $f_n = \chi_{[a,x_n]}f$ where $\{x_n\}$ is any sequence of numbers converging to x, so that f_n converges pointwise to $\chi_{[a,x]}f$ except possibly at the point x, which is a set of measure 0. The dominated convergence theorem then guarantees that $F(x_n) \to F(x)$. Since the sequence $\{x_n\}$ is arbitrary, it follows that F is continuous at x, as in Theorem 4.2.

Exercise 11.10 If $\mu(X) < +\infty$ and $f \in \mathcal{L}^2(\mu)$ on X, prove that $f \in \mathcal{L}(\mu)$ on X. If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f \in \mathcal{L}^2$ on R^1, but $f \notin \mathcal{L}$ on R^1.

Solution. This follows from Theorems 11.27 and 11.29, if we let $A = \{x : |f(x)| \le 1\}$ and $B = \{x : |f(x)| > 1\}$. We can then write

$$|f| \le \chi_A + \chi_B \cdot |f|^2.$$

and χ_A is integrable by Theorem 11.23(a).

As for the counterexample, we have

$$|f|^2 \leq \chi_{[-1,1]} + \chi_{[1,\infty)} \cdot \frac{1}{x^2},$$

which implies that $f \in \mathcal{L}^2$, and

$$f(x) \geq \chi_{[0,n]}(x)\frac{1}{1+x},$$

so that

$$\int f\,dx \geq \int_0^n \frac{1}{1+x}\,dx = \ln(1+n) \to \infty.$$

Hence $f \notin \mathcal{L}$.

Exercise 11.11 If $f, g \in \mathcal{L}(\mu)$ on X, define the distance between f and g by

$$\int_X |f-g|\,d\mu.$$

Prove that $\mathcal{L}(\mu)$ is a complete metric space.

Solution. We have to regard functions equal almost everywhere as the same function. Given that, it does follow that if $d(f,g) = 0$, then $f = g$. The fact that $d(f,g) = d(g,f)$ is immediate from the definition and the triangle inequality follows from simply integrating the triangle inequality for the values of the functions. Hence $\mathcal{L}$ is a metric space.

To prove that it is complete, we merely repeat the reasoning of Theorem 11.42, replacing $\mathcal{L}^2$ by $\mathcal{L}$ and taking the function $g(x)$ to be identically equal to 1. When this is done, every step in the proof of Theorem 11.42 follows for $\mathcal{L}$.

Exercise 11.12 Suppose

(*a*) $|f(x,y) \leq 1$ if $0 \leq x \leq 1$, $0 \leq y \leq 1$,

(*b*) for fixed x, $f(x,y)$ is a continuous function of y,

(*c*) for fixed y, $f(x,y)$ is a continuous function of x.

Put

$$g(x) = \int_0^1 f(x,y)\,dy \quad (0 \leq x \leq 1).$$

Is g continuous?

Solution. Yes, $g(x)$ is continuous. Let $x_n \to x$. Then by (*c*), $f(x_n,y) \to f(x,y)$ for each $y \in [0,1]$, in particular for almost every y. Since $|f(x_n,y)| \leq 1$ for all x_n and y by assumption (*a*), and the set $[0,1]$ has finite measure, it follows from the dominated convergence theorem that $g(x_n) \to g(x)$.

Note that property (*b*) was used only to guarantee that $g(x)$ is actually defined. Thus the word *continuous* could be replaced by *integrable* in this condition.

Exercise 11.13 Consider the functions

$$f_n(x) = \sin nx \quad (n = 1, 2, 3, \ldots, -\pi \le x \le \pi)$$

as points of $\mathcal{L}^2$. Prove that the set of these points is closed and bounded, but not compact.

Solution. We compute by brute force that

$$\|f_m - f_n\|^2 = \begin{cases} 0, & \text{if } m = n, \\ 2\pi, & \text{if } m \neq n. \end{cases}$$

Further, it is easy to see that $\|f_n\|^2 = \pi$. Hence the set $\{f_n\}$ is bounded and has no limit points. (The $\sqrt{\frac{\pi}{2}}$-neighborhood of any point contains at most one point of this set.) Having no limit points, it contains all of its limit points and is therefore closed. Being infinite, if it were compact, it *would* have a limit point. Therefore it is not compact.

Exercise 11.14 Prove that a complex function f is measurable if and only if $f^{-1}(V)$ is measurable for every open set V in the plane.

Solution. By definition $f = u + iv$, where u and v are real-valued, is measurable if and only if u and v are.

Suppose f is measurable (that is, u and v are measurable). Let V be any open set in the plane and $(x, y) \in V$. Then there exists $\delta > 0$ such that the square $S(x, y) = (x - \delta, x + \delta) \times (y - \delta, y + \delta)$ is contained in V. The union of these open squares is all of V, and there is a countable set of points $(x_n, y_n) \in V$ such that $\bigcup_{n=1}^{\infty} S(x_n, y_n) = V$. (This is proved by appealing to Exercise 23 of Chapter 2.) But then

$$f^{-1}(V) = \bigcup_{n=1}^{\infty} f^{-1}(S(x_n, y_n)) = \bigcup_{n=1}^{\infty} u^{-1}(x_n - \delta, x_n + \delta) \cap v^{-1}(y_n - \delta, y_n + \delta).$$

It follows that $f^{-1}(V)$ is measurable.

Conversely if $f^{-1}(V)$ is measurable for every open set in the plane , then in particular this set is measurable if $V = (a, b) \times R^1$ (where $f^{-1}(V) = u^{-1}((a, b))$) or $V = R^1 \times (a, b)$ (where $f^{-1}(V) = v^{-1}((a, b))$), and hence both u and v are measurable. By definition, that means that f is measurable.

Exercise 11.15 Let $\mathcal{R}$ be the ring of all elementary subsets of $(0, 1]$. If $0 < a \le b \le 1$, define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on $\mathcal{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

Solution. In brief, since an elementary set A is a finite disjoint union of intervals, $\phi(A)$ is the sum of the lengths of those intervals if 0 is not the endpoint of any interval in A and 1 larger than the sum of the lengths of the intervals if 0 is one of the endpoints. In particular $\phi(A) < 1$ if A is a closed set, since 0 cannot be the endpoint of any closed set that is a finite union of intervals in $(0, 1]$.

(This alternate definition is independent of the particular way in which the set A is represented as a finite disjoint union of intervals, since if $A = \bigcup_{i=1}^{m} I_i = \bigcup_{j=1}^{n} J_j$, where each of the collections $\{I_i\}$ and $\{J_j\}$ is a set of pairwise disjoint intervals, one can easily verify that

$$|I_i| = \sum_{j=1}^{n} |I_i \cap J_j|, \quad |J_j| = \sum_{i=1}^{m} |I_i \cap J_j|,$$

so that $\sum_{i=1}^{m} |I_i| = \sum_{j=1}^{n} |J_j| = \sum_{i,j} |I_i \cap J_j|$. HEre $|I|$ is the length of the interval I.)

If two elementary sets A and B are disjoint, at most one of them can have the point 0 as the endpoint of one of its intervals. Then $\phi(A \cup B)$ is the sum of the lengths of the intervals in $A \cup B$ if neither set contains an interval having 0 as the endpoint, and 1 larger than this sum if one of them does contain an interval with 0 as endpoint. In either case $\phi(A \cup B) = \phi(A) + \phi(B)$ when $A \cap B = \emptyset$. Thus the function ϕ is additive.

The function ϕ is not regular, however, since there is no closed subset of $(0, c]$ that can approximate $(0, c]$ if $c < 1$. For $\phi((0, c]) = 1 + c$, but $\phi(A) \leq 1$ if A is closed.

The function ϕ also cannot be extended to a countably additive set function on a σ-ring, since

$$(0, \frac{1}{2}] = \bigcup_{n=1}^{\infty} (\frac{1}{2^{n+1}}, \frac{1}{2^n}],$$

and

$$\phi((0, \frac{1}{2}]) = \frac{3}{2}, \quad \sum_{n=1}^{\infty} \phi((\frac{1}{2^{n+1}}, \frac{1}{2^n}]) = \frac{1}{2}.$$

Exercise 11.16 Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint:* For every $A \subset E$,

$$\int_A \sin n_k x \, dx = 0,$$

and

$$2\int_A (\sin n_k x)^2\,dx = \int_A (1-\cos 2n_k x)\,dx \to m(A) \text{ as } k\to\infty.$$

Solution. The two statements in the hint follow from the Riemann–Lebesgue lemma (or from Bessel's inequality applied to the Fourier series of χ_A, if you wish). Let $f(x)$ be the limit of $\sin n_k x$ on the set E. Then, since termwise integration is justified by the dominated convergence theorem, we have

$$\int_A [(f(x))^2 - \frac{1}{2}]\,dx = 0,$$

for all A. Hence, by Exercise 2 above, $f(x) = \pm\frac{1}{\sqrt{2}}$ almost everywhere on E. If we let A be the set of points of E at which $f(x) = \frac{1}{\sqrt{2}}$, we find that $\int_A f(x)\,dx = 0$, and so by Exercise 1, $f(x) = 0$ almost everywhere on A. Since in fact $f(x) \neq 0$ on A, it follows that A has measure 0. Similarly the set where $f(x) = -\frac{1}{\sqrt{2}}$ has measure 0.

Exercise 11.17 Suppose $E \subset (-\pi,\pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin nx \geq \delta$ for all $x \in E$.

Solution. For any integer with this property we have

$$\int_E \sin nx\,dx \geq \delta\mu(E),$$

and the Bessel inequality implies that this inequality can hold for only a finite number of n. (The integral is the imaginary part of the Fourier coefficient of the $\mathcal{L}^2$-function χ_E.)

Exercise 11.18 Suppose $f \in \mathcal{L}^2(\mu)$, $g \in L^2(\mu)$. Prove that

$$\left|\int f\bar{g}\,d\mu\right|^2 = \int |f|^2\,d\mu \int |g|^2\,d\mu$$

if and only if there is a constant c such that $g(x) = cf(x)$ almost everywhere. (Compare Theorem 11.35.)

Solution. There is a slight mistake in the statement of the problem, since equality certainly holds if $f(x)$ is identically zero, whether $g(x)$ equals zero or not. We must either assume that $f(x)$ is not identically zero, or allow the possibility that $f(x) = cg(x)$.

Equality can hold if $g(x) = 0$ almost everywhere, and in that case $c = 0$ in the relation $g(x) = cf(x)$. Hence assume now that $\int |g|^2\,d\mu > 0$. The inequality

$$0 \le \int (|f| + \lambda|g|)^2\,d\mu,$$

which holds for real values of λ, is equivalent to the inequality

$$-2\lambda \int |fg|\,d\mu \le \int |f|^2\,d\mu + \lambda^2 \int |g|^2\,d\mu.$$

In this inequality take $\lambda = -\sqrt{\dfrac{\int |f|^2\,d\mu}{\int |g|^2\,d\mu}}$. The result is

$$2\frac{\int |fg|\,d\mu \sqrt{\int |f|^2\,d\mu}}{\sqrt{\int |g|^2\,d\mu}} \le 2\int |f|^2\,d\mu,$$

which is equivalent to

$$\left[\int |fg|\,d\mu\right]^2 \le \int |f|^2\,d\mu \int |g|^2\,d\mu.$$

Hence the equality in the problem can hold only if equality holds in this last equality, which, since it implies that

$$\int \left(|f| + \lambda|g|\right)^2 d\mu = 0,$$

implies that $|f| = -\lambda|g|$ almost everywhere. In particular f vanishes almost everywhere that g vanishes. In addition, the equality in the hypothesis of the problem requires that

$$\left|\int f\overline{g}\,d\mu\right| = \int |fg|\,d\mu.$$

If both sides of this last equality are zero, then at almost every point either $f(x) = 0$ or $g(x) = 0$. Since $|f| = -\lambda|g|$, it then follows that in fact either both functions vanish identically, a case we have already discussed, or $\lambda = 0$, in which case only f vanishes identically. In either case we do have the kind of linear dependence specified in the amended statement of the problem.

Hence assume that neither side of this equality is zero. Let ω be the complex number

$$\omega = \frac{\overline{\int f\overline{g}\,d\mu}}{|\int f\overline{g}\,d\mu|},$$

so that $|\omega| = 1$. We note that

$$\int \omega f\overline{g}\,d\mu = \omega \int f\overline{g}\,d\mu = \left|\int f\overline{g}\,c\mu\right| \le \int |f\overline{g}|\,d\mu = \int |\omega f\overline{g}|\,d\mu.$$

This means that the real parts of the two integrals on the extremes here are equal, and the imaginary parts of both are zero. Taking just the real parts, since $\mathrm{Re}\,(\omega f\overline{g}) \leq |\omega f\overline{g}|$, this implies that the real part of $\omega f\overline{g}$ is equal to $|fg| = -\lambda g\overline{g}$ almost everywhere, and therefore that the imaginary part is zero almost everywhere. But then, almost everywhere where g does not vanish, we can cancel $\overline{g}$ from the equality, getting $f = -\lambda\overline{\omega}g$ wherever g does not vanish. Since this equality also holds almost everywhere where g does vanish, we are done.