Advanced Engineering Mathematics, by Erwin Kreyszig 10th. Ed.

Problem Set 12.3

No. 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho}$$

$$u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x, \qquad (n = 1, 2, \cdots)$$

$$\text{freqency}: f = \frac{\lambda_n}{2\pi} = \frac{cn}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\rho}}$$

$$T \to 2T. \qquad f \to \sqrt{2} f.$$

No. 2

If Assumption 3 is violated, the string can move in various planes. For large displacements and/or angles the PDE would no longer be linear and have constant coefficients. Lack of elasticity entails loss of mechanical energy by conversion into heat (damping).

If homogeneity were dropped, it is hard to see what would happen; one would first have to be specific and state in what way homogeneity is changed and perhaps support theoretical results by physical experiments.

No. 3

$$L = \pi, \quad \lambda_n = cn$$

$$u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin nx, \qquad (n = 1, 2, \cdots)$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos cnt + B_n^* \sin cnt) \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx = f(x)$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$u_t(x,0)\Big|_{t=0} = B_n^* cn \sin nx = g(x)$$

$$B^* cn = \frac{2}{\pi}\int_0^{\pi} g(x) \sin nx \, dx$$

$$B^* = \frac{2}{cn\pi}\int_0^{\pi} g(x) \sin nx \, dx$$

No. 4

No. 5

No. 6

No. 7

$$u(x,0) = kx(1-x)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos n\pi t\right) \sin n\pi x$$

$$u(x,0) = kx(1-x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

$$B_n = \frac{2}{1}\int_0^1 kx(1-x) \sin n\pi x \, dx$$

$$= \frac{4k}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{4k}{n^3\pi^3}\left[1-(-1)^n\right] \cos n\pi t \sin n\pi x$$

$$= \frac{8k}{\pi^3}\left[\cos \pi t \sin \pi x + \frac{1}{3^3}\cos 3\pi t \sin 3\pi x + \frac{1}{5^3}\cos 5\pi t \sin 5\pi x + \cdots\right.$$

No. 8

$$\frac{k}{\pi^3}\left(12\cos(\pi t)\sin(\pi x) - \frac{3}{2}\cos(2\pi t)\sin(2\pi x) + \frac{4}{9}\cos(3\pi t)\sin(3\pi x) - \frac{3}{16}\cos(4\pi t)\sin(4\pi x)\right)$$

No. 9

$$u(x,0) = f(x) = \begin{cases} \frac{1}{5}x & ; \quad 0 \le x \le 0.5 \\ \frac{1}{5} - \frac{1}{5}x ; & \quad 0.5 \le x \le 1. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos n\pi t) \sin n\pi x$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

$$B_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\left[\int_0^{0.5} \frac{1}{5}x \sin n\pi x \, dx + \int_{0.5}^{1} \left(\frac{1}{5} - \frac{1}{5}x\right) \sin n\pi x \, dx\right.$$

$$= \frac{0.4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{0.4}{n^2\pi^2}\left[1-(-1)^n\right] \cos n\pi t \sin n\pi x$$

$$= \frac{0.8}{\pi^2}\left(\cos \pi t \sin \pi x - \frac{1}{3^2}\cos 3\pi t \sin 3\pi x + \frac{1}{5^2}\cos 5\pi t \sin 5\pi x - + \cdots\right)$$

No.10

$$\frac{8}{\pi^2}\left(\frac{1}{4}\cos 2\pi t \sin 2\pi x - \frac{1}{36}\cos 6\pi t \sin 6\pi x + \frac{1}{100}\cos 10\pi t \sin 10\pi x - \cdots\right)$$

There are more graphically posed problems than in previous editions, so that CAS-using students will have to make at least *some* additional effort in solving these problems.

No.11

$$u(x,0)=\begin{cases} 0 \ ; & 0<x<\frac{1}{4} \\ x-\frac{1}{4} \ ; & \frac{1}{4}<x<\frac{1}{2} \\ \frac{3}{4}-x \ ; & \frac{1}{2}<x<\frac{3}{4} \\ 0 \ ; & \frac{3}{4}<x<0 \end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos n\pi t\right)\sin n\pi x$$

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n\sin n\pi x$$

$$B_n=\frac{2}{T}\int_0^1 f(x)\sin n\pi x\,dx$$

$$=2\left[\int_{\frac{1}{4}}^{\frac{1}{2}}\left(x-\frac{1}{4}\right)\sin n\pi x\,dx+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\frac{3}{4}-x\right)\sin n\pi x\,dx\right]$$

$$=$$

$$\therefore\ u(x,t)=\frac{2}{\pi^2}\Big[(2-\sqrt{2})\cos\pi t\sin\pi x-\frac{1}{9}(2+\sqrt{2})\cos 3\pi t\sin 3\pi x$$
$$+\frac{1}{25}(2+\sqrt{2})\cos 5\pi t\sin 5\pi x-+\cdots\Big)$$

No.12

$$\frac{\sqrt{8}}{\pi^2}\Bigg(\cos\pi t\sin\pi x+\frac{1}{9}\cos 3\pi t\sin 3\pi x-\frac{1}{25}\cos 5\pi t\sin 5\pi x$$
$$-\frac{1}{49}\cos 7\pi t\sin 7\pi x+\cdots\Bigg)$$

No.13

$$-3\,\frac{(-9/2+1/2\,\pi\,\sqrt{3})\cos(\pi t)\sin(\pi x)}{\pi^3}-3/8\,\frac{(-9/2-\pi\,\sqrt{3})\cos(2\pi t)\sin(2\pi x)}{\pi^3}$$
$$-\frac{3}{64}\frac{(-9/2-2\pi\,\sqrt{3})\cos(4\pi t)\sin(4\pi x)}{\pi^3}+\cdots$$

No.14

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

$$c^2=1,\quad L=\pi.\quad u(x,0)=0.\quad u_t(x,0)=g(x)=\begin{cases}0.01x, & 0\le x<\frac{\pi}{2}\\ 0.01(\pi-x), & \frac{\pi}{2}\le x\le\end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}B_n^*\sin nt\,\sin nx$$

$$u_t(x,0)=g(x)=\sum_{n=1}^{\infty}nB_n^*\sin nx$$

$$nB_n^*=\frac{2}{\pi}\int_0^{\pi}g(x)\sin nx\,dx$$

$$nB_n^*=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}0.01x\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}0.01(\pi-x)\sin nx\,dx\right]$$

$$B_n^*=\frac{2}{n\pi}\left[\int_0^{\frac{\pi}{2}}0.01x\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}0.01(\pi-x)\sin nx\,dx\right]$$

No.15

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$$

Let $u = F(x)G(t)$

$$F\ddot{G} = -c^2 F^{(4)} G$$

$$\frac{F^{(4)}}{F} = \frac{-\ddot{G}}{c^2 G} = \beta^4 = \text{constant}$$

$$F^{(4)} + \beta^4 F = 0 \Rightarrow F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$\ddot{G} + c^2\beta^4 G = 0 \Rightarrow G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t.$$

No.16

B.C : $u(0, t) = 0$, $u(L, t) = 0$

$u_{xx}(0, t) = 0$, $u_{xx}(L, t) = 0.$

$F(0) = 0$, $F(L) = 0$, $F''(0) = 0$, $F''(L) = 0$

$F(0) = 0 \Rightarrow A + C = 0$

$F''(0) = 0 \Rightarrow -A + C = 0$ $\quad \Rightarrow A = C = 0$

$F(L) = 0 \Rightarrow$

$F''(L) = 0 \Rightarrow$ $\quad D = 0$, $\sin\beta L = 0$, $\beta \neq 0$

$\beta L = n\pi$, $\beta = \frac{n\pi}{L}$ $\quad (n = 1, 2, \cdots)$

Let $B=1$

$$F(x)=F_n(x)=\sin\frac{n\pi}{L}x$$

I.C $\cdot$ $u_t(x,0)=0$

$$\therefore G'(0)=0=bc\beta^2 \Longrightarrow b=0.$$

$$G(t)=G_n(t)=a_n\cos c\left(\frac{n\pi}{L}\right)^2 t$$

$$u_n(x,t)=F_n(x)G_n(t)$$

$$=a_n\cos c\left(\frac{n\pi}{L}\right)^2 t\,\sin\frac{n\pi}{L}x$$

No.17

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}a_n\cos c\left(\frac{n\pi}{L}\right)^2 t\,\sin\frac{n\pi}{L}x$$

$$u(x,0)=f(x)=x(L-x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{L}x$$

$$a_n=\frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{4L^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4L^2}{n^3\pi^3}\left[1-(-1)^n\right]\cos c\left(\frac{n\pi}{L}\right)^2 t\,\sin\frac{n\pi}{L}x$$

$$=\frac{8L^2}{\pi^3}\left(\cos c\left(\frac{\pi}{L}\right)^2 t\,\sin\frac{\pi x}{L}+\frac{1}{3^3}\cos c\left(\frac{3\pi}{L}\right)^2 t\,\sin\frac{3\pi x}{L}+\cdots\right)$$

No.18

For the string the frequency of the nth mode is proportional to n, whereas for the beam it is proportional to n^2.

No.19

B.Cs :

$u(0,t)=0$, $u(L,t)=0$, $u_x(0,t)=0$, $u_x(L,t)=0$

$F(0)=F(L)=F'(0)=F'(L)=0$

$F(x)=A\cos\beta x+B\sin\beta x+C\cosh\beta x+D\sinh\beta x$

$F(0)=0 \Rightarrow A+C=0$, $C=-A$

$F'(0)=0 \Rightarrow B+D=0$, $D=-B.$

$F(L)=A\cos\beta L+B\sin\beta L-A\cosh\beta L-B\sinh\beta L=0$

$F'(L)=-A\sin\beta L+B\cos\beta L-A\sinh\beta L-B\cosh\beta L=0$

$\because A\neq 0,\ B\neq 0.$

$\therefore \underline{\cosh\beta L\cos\beta L=1}$ //

No.20

$F(0) = A + C = 0, C = -A, F'(0) = \beta(B + D) = 0, D = -B$. Then

$$F(x) = A(\cos\beta x - \cosh\beta x) + B(\sin\beta x - \sinh\beta x)$$

$$F''(L) = \beta^2[-A(\cos\beta L + \cosh\beta L) - B(\sin\beta L + \sinh\beta L)] = 0$$

$$F'''(L) = \beta^3[A(\sin\beta L - \sinh\beta L) - B(\cos\beta L + \cosh\beta L)] = 0.$$

The determinant $(\cos\beta L + \cosh\beta L)^2 + \sin^2\beta L - \sinh^2\beta L$ of this system in the unknowns A and B must be zero, and from this the result follows.

From (23) we have

$$\cos\beta L = \frac{-1}{\cosh\beta L} \approx 0$$

because $\cosh\beta L$ is very large. This gives the approximate solutions

$$\beta L \approx \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \cdots \text{(more exactly, 1.875, 4.694, 7.855, } \cdots).$$