CS5314 Randomized Algorithms

Homework 4 Suggested Solution

(Original due date was June 02, 2020)

1. Let Z be a Poisson random variable with mean μ, where $\mu \geq 1$ is an integer.

   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

   Ans. Let ρ_h denote the ratio $\Pr(Z = \mu + h)/\Pr(Z = \mu - h - 1)$, for $0 \leq h \leq \mu - 1$. We will show $\rho_h \geq 1$, thus obtaining the desired result. By definition,

   $$\rho_h = \frac{e^{-\mu}\mu^{\mu+h}/(\mu+h)!}{e^{-\mu}\mu^{\mu-h-1}/(\mu-h-1)!} = \frac{\mu^{2\mu+1}}{(\mu+h)(\mu+h-1)(\mu+h-2)\cdots(\mu-h)},$$

   which is at least 1 since $(\mu + x)(\mu - x) \leq \mu^2$ for any x.

   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

   Ans.

   $$\begin{aligned} \Pr(Z \geq \mu) &= \sum_{h=0}^{\infty} \Pr(Z = \mu + h) \geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu + h) \\ &\geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu - h - 1) = \Pr(Z < \mu). \end{aligned}$$

   Since $\Pr(Z \geq \mu) + \Pr(Z < \mu) = 1$, the above inequality implies $\Pr(Z \geq \mu) \geq 1/2$.

2. Let X be a Poisson random variable with mean μ, representing the number of criminals in a city. There are two types of criminals: For the first type, they are not too bad and are reformable. For the second type, they are flagrant. Suppose each criminal is independently reformable with probability p (so that flagrant with probability $1 - p$). Let Y and Z be random variables denoting the number of reformable criminals and flagrant criminals (respectively) in the city. Show that Y and Z are independent Poisson random variables.

   Ans. We first show that both Y and Z are Poisson random variables, and then show that they are independent. To begin with, let us analyse the value of $\Pr(Y = k)$ for a nonnegative integer k. For Y to be equal to k, exactly k out of X criminals are reformable. Thus, we have:

$$
\begin{aligned}
\Pr(Y=k) &= \sum_{n=0}^{\infty} \Pr(X=n) \times \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{n=k}^{\infty} \Pr(X=n) \times \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{n=k}^{\infty} e^{-\mu} \frac{\mu^n}{n!} \times \binom{n}{k} p^k (1-p)^{n-k} \\
&= e^{-\mu} \times \frac{\mu^k p^k}{k!} \times \sum_{n=k}^{\infty} \frac{\mu^{n-k}(1-p)^{n-k}}{(n-k)!} \\
&= e^{-\mu} \times \frac{\mu^k p^k}{k!} \times e^{(1-p)\mu} \;=\; e^{-p\mu} \frac{(p\mu)^k}{k!},
\end{aligned}
$$

which shows that Y is a Poisson random variable with parameter $p\mu$.

By symmetry, Z is a Poisson random variable with parameter $(1-p)\mu$.

It remains to show Y and Z are independent. Let us analyse $\Pr((Y=k) \cap (Z=\ell))$.

$$
\begin{aligned}
\Pr((Y=k) \cap (Z=\ell)) &= \Pr((Y=k) \cap (X=k+\ell)) \\
&= \Pr((Y=k) \mid X=k+\ell) \times \Pr(X=k+\ell) \\
&= \binom{k+\ell}{k} p^k (1-p)^{\ell} \times e^{-\mu} \frac{\mu^{k+\ell}}{(k+\ell)!} \\
&= \left(e^{-p\mu} \frac{p^k \mu^k}{k!} \right) \times \left(e^{-(1-p)\mu} \frac{(1-p)^{\ell} \mu^{\ell}}{\ell!} \right) \\
&= \Pr(Y=k) \Pr(Z=\ell);
\end{aligned}
$$

this shows that Y and Z are independent.

3. We consider another way to obtain Chernoff-like bound in the balls-and-bins setting. Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the ith bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$ be the number of empty bins.

   Let Y_i be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1-1/n)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.

   (a) Show that $\mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.

   Ans. Note that both $X_1 X_2 \cdots X_k$ and $Y_1 Y_2 \cdots Y_k$ are indicators, as they can only take on values of 0 or 1. Thus, we have:

   $$\mathrm{E}[X_1 X_2 \cdots X_k] = \Pr(X_1 X_2 \cdots X_k = 1) = (1-k/n)^n$$

   and

   $$\mathrm{E}[Y_1 Y_2 \cdots Y_k] = \Pr(Y_1 Y_2 \cdots Y_k = 1) = (1-1/n)^{kn}.$$

To complete the proof, it suffices to show that

$$1 - k/n \leq (1 - 1/n)^k.$$

We will show this by induction on k. Note that the inequality is true when $k = 1$. Suppose that the inequality is true when $k = 1, 2, \ldots, t$. Then, when $k = t + 1$,

$$1 - \frac{t+1}{n} \leq \left(1 - \frac{1}{n}\right)^t - \frac{1}{n} = \left(1 - \frac{1}{n}\right)^{t+1} + \frac{1}{n}\left(1 - \frac{1}{n}\right)^t - \frac{1}{n} \leq \left(1 - \frac{1}{n}\right)^{t+1}.$$

Thus, by principle of mathematical induction, the inequality holds for any $k = 1, 2, 3, \ldots$ This completes the proof.

(b) Show that $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.

Ans. The values of $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k}$ and $X_1 X_2 \cdots X_k$ are both 1 if all X_is are 1. Otherwise, these values are both 0. Thus, no matter what happens, these values are always the same.

(c) Show that $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$ for all $t \geq 0$.
Hint: Use the expansion for e^x and compare $\mathrm{E}[e^{tX}]$ to $\mathrm{E}[e^{tY}]$.

Ans. Using the results of (a) and (b), we see that

$$\begin{aligned}
\mathrm{E}[X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k}] &= \mathrm{E}[X_1 X_2 \cdots X_k] \\
&\leq \mathrm{E}[Y_1 Y_2 \cdots Y_k] \\
&= \mathrm{E}[Y_1^{j_1} Y_2^{j_2} \cdots Y_k^{j_k}]
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathrm{E}[X^m] &= \mathrm{E}[(X_1 + X_2 + \ldots + X_n)^m] \\
&\leq \mathrm{E}[(Y_1 + Y_2 + \ldots + Y_n)^m] \;=\; \mathrm{E}[Y^m]
\end{aligned}$$

Now, by Taylor's expansion, for all $t \geq 0$, we have

$$\begin{aligned}
\mathrm{E}[e^{tX}] &= \mathrm{E}\left[\sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\right] = \sum_{m=0}^{\infty} \mathrm{E}\left[\frac{(tX)^m}{m!}\right] \\
&= \sum_{m=0}^{\infty} \mathrm{E}\left[\frac{(tY)^m}{m!}\right] = \mathrm{E}\left[\sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\right] = \mathrm{E}[e^{tY}]
\end{aligned}$$

(d) Derive a Chernoff bound for $\Pr(X \geq (1 + \delta)\mathrm{E}[X])$.

Ans. Note that $\mathrm{E}[X] = \mathrm{E}[Y] = n(1 - 1/n)^n$, and by part (c), $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$ for any $t > 0$. Then, we have:

$$\Pr(X \geq (1 + \delta)\mathrm{E}[X]) \;\leq\; \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)\mathrm{E}[X]}} \;\leq\; \frac{\mathrm{E}[e^{tY}]}{e^{t(1+\delta)\mathrm{E}[Y]}}$$

Since Y is the sum of n independent Bermoulli trials, we can directly apply the results on Page 9 of Lecture Notes 11, so that by choosing the best $t = \ln(1+\delta)$, we get:

$$\frac{\mathrm{E}[e^{tY}]}{e^{t(1+\delta)\mathrm{E}[Y]}} = \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mathrm{E}[Y]}$$

This gives a Chernoff bound for $\Pr(X \geq (1+\delta)\mathrm{E}[X])$.

4. In the lecture, we showed that, for any nonnegative function f,

$$\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = m\right).$$

(a) Now suppose we further know that $\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})]$ is monotonically increasing in m. Show that

$$\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} \geq m\right).$$

Ans.

$$\begin{aligned}
\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})] &= \sum_k \mathrm{E}\left[f(Y_1^{(m)}, \ldots, Y_n^{(m)}) \mid \sum_{i=1}^{n} Y_i^{(m)} = k\right] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = k\right) \\
&= \sum_k \mathrm{E}[f(X_1^{(k)}, \ldots, X_n^{(k)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = k\right) \\
&\geq \sum_{k \geq m} \mathrm{E}[f(X_1^{(k)}, \ldots, X_n^{(k)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = k\right) \\
&\geq \sum_{k \geq m} \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = k\right) \\
&= \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \sum_{k \geq m} \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} = k\right) \\
&= \mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \Pr\left(\sum_{i=1}^{n} Y_i^{(m)} \geq m\right)
\end{aligned}$$

(b) Combining part (a) with the result in Question 1, show that:

$$\mathrm{E}[f(X_1^{(m)}, \ldots, X_n^{(m)})] \leq 2\,\mathrm{E}[f(Y_1^{(m)}, \ldots, Y_n^{(m)})].$$

Ans. Since $\sum_{i=1}^{n} Y_i^{(m)}$ is a Poisson random variable with parameter m, so by Q1, $\Pr\left(\sum_{i=1}^{n} Y_i^{(m)} \geq m\right) \geq 1/2$. The result of this part now follows from part (a).