Unit 0.1 Introduction

Numerical Analysis

EE/NTHU

Mar. 2, 2020

Numerical Analysis

- Mathematics are important in our study, research and work.
  - Mathematical formulation provides clear and precise description of the problem.
    - Quantified or analytically.
  - Analytic solutions have been studied throughout your lives.
  - Few problems have closed form solutions.
  - How do you solve mathematical problems to get numerical solutions?
- Numerical method is the science and art of computation (Tom. Apostol)
  - Explore the mathematical properties of the problems.
  - Develop algorithms to solve the problem effectively.
  - Find approximated solutions if more efficient.
  - Overcome computer inexactness.
- Numerical methods have been used extensively in our daily lives already.
  - List your examples.

Topics to Be Studied

- In this course, we will study the following topics in numerical analysis.
  - Linear system solutions,
  - Errors,
  - Iterative solution methods,
  - Eigenvalues and eigenvectors,
  - Interpolations,
  - Integrations,
  - Nonlinear system solutions,
  - Ordinary differential equations,
  - Partial differential equations.

Linear System Solutions

- You know how to solve this problem.

$$\begin{array}{rrl} 2x_1 & +x_2 & = 3 \\ x_1 & +2x_2 & = 3 \end{array}$$

- How about this?

$$\begin{array}{rrrrrrrl} 70x_1 & +60x_2 & +50x_3 & +40x_4 & +30x_5 & +20x_6 & +10x_7 & = 1 \\ 60x_1 & +70x_2 & +60x_3 & +50x_4 & +40x_5 & +30x_6 & +20x_7 & = 2 \\ 50x_1 & +60x_2 & +70x_3 & +60x_4 & +50x_5 & +40x_6 & +30x_7 & = 3 \\ 40x_1 & +50x_2 & +60x_3 & +70x_4 & +60x_5 & +50x_6 & +40x_7 & = 4 \\ 30x_1 & +40x_2 & +50x_3 & +60x_4 & +70x_5 & +60x_6 & +50x_7 & = 3 \\ 20x_1 & +30x_2 & +40x_3 & +50x_4 & +60x_5 & +70x_6 & +60x_7 & = 2 \\ 19x_1 & +29x_2 & +39x_3 & +49x_4 & +59x_5 & +69x_6 & +59x_7 & = 1 \end{array}$$

- What if we have 1,000 or 1,000,000 variables?
  - Can we solve it efficiently?

Errors

- We know computer numbers, floating numbers, are inexact.
  - Experimental data have inherent inaccuracy also.
  - How do these errors affect our solutions?
  - Can the algorithm be rewritten for smaller errors?

$$\begin{bmatrix} 70 & 60 & 50 & 40 & 30 & 20 & 10 \\ 60 & 70 & 60 & 50 & 40 & 30 & 20 \\ 50 & 60 & 70 & 60 & 50 & 40 & 30 \\ 40 & 50 & 60 & 70 & 60 & 50 & 40 \\ 30 & 40 & 50 & 60 & 70 & 60 & 50 \\ 20 & 30 & 40 & 50 & 60 & 70 & 60 \\ 19 & 29 & 39 & 49 & 59 & 69 & 59 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

$$\mathbf{x}^T = [\, 0.45 \;\; 0 \;\; 0 \;\; 0.1 \;\; 0 \;\; -3.9 \;\; 4.35]$$

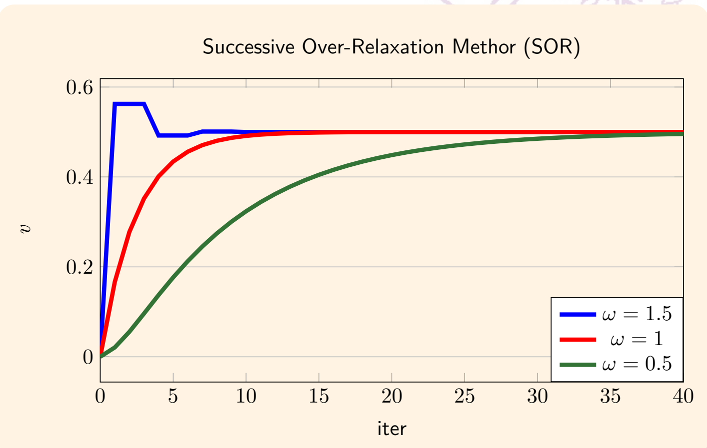
$$\begin{bmatrix} 70 & 60 & 50 & 40 & 30 & 20 & 10 \\ 60 & 70 & 60 & 50 & 40 & 30 & 20 \\ 50 & 60 & 70 & 60 & 50 & 40 & 30 \\ 40 & 50 & 60 & 70 & 60 & 50 & 40 \\ 30 & 40 & 50 & 60 & 70 & 60 & 50 \\ 20 & 30 & 40 & 50 & 60 & 70 & 60 \\ 19 & 29 & 39 & 49 & 59 & 69 & 59 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 3 \\ 2 \\ 1.1 \end{bmatrix}.$$

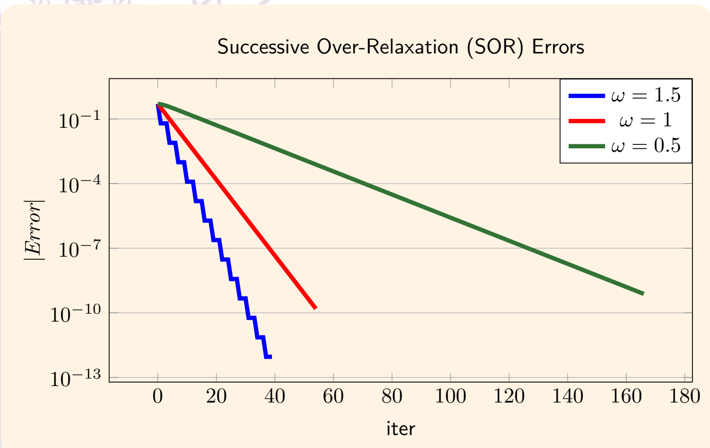
$$\mathbf{x}^T = [\, 0.4 \;\; 0 \;\; 0 \;\; 0.1 \;\; 0 \;\; -3.5 \;\; 3.9]$$

- A small error can cause significant changes in solutions.

Iterative Solution Methods

- Solving a linear system with n variables takes $\mathcal{O}(n^3)$ CPU time.
- For large n the solution time can be too long.
- Iterative solution methods find an approximate solution with less amount of time.
- Matrix may need to possess some special properties to be applicable.
- More general solutions have been developed, but not covered in this course.





Eigenvalues and Eigenvectors

- You know how to find the eigenvalues and eigenvectors for

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- How about this?

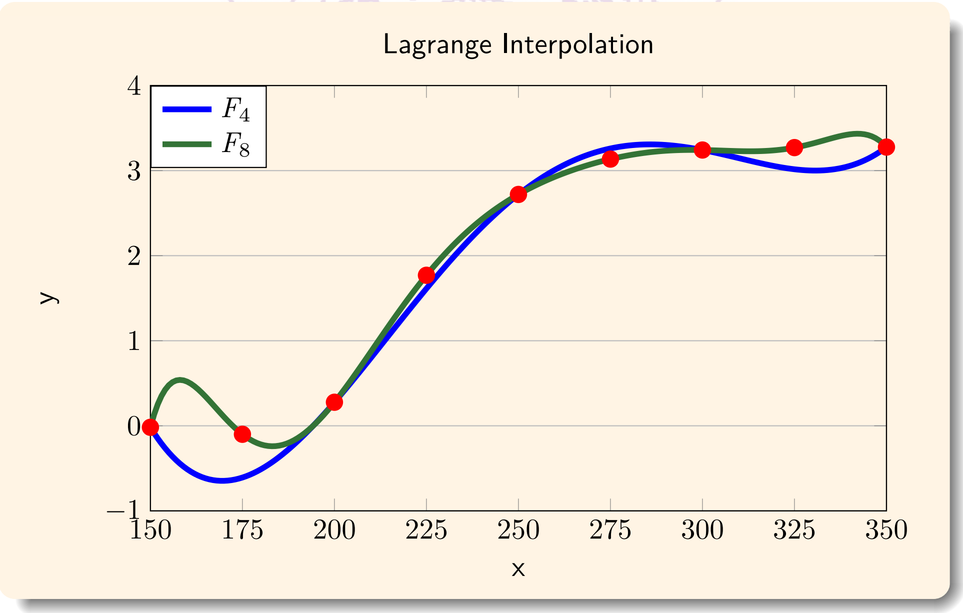
$$\begin{bmatrix} 70 & 60 & 50 & 40 & 30 & 20 & 10 \\ 60 & 70 & 60 & 50 & 40 & 30 & 20 \\ 50 & 60 & 70 & 60 & 50 & 40 & 30 \\ 40 & 50 & 60 & 70 & 60 & 50 & 40 \\ 30 & 40 & 50 & 60 & 70 & 60 & 50 \\ 20 & 30 & 40 & 50 & 60 & 70 & 60 \\ 19 & 29 & 39 & 49 & 59 & 69 & 59 \end{bmatrix}$$

$$\lambda^T = [\ 339.07\ \ 96.29\ \ 20.67\ \ 10.89\ \ 6.80\ \ 5.41\ \ -0.12]$$

- What if we have a $1,000 \times 1,000$ or $1,000,000 \times 1,000,000$ matrix?
  - Can we solve it efficiently?

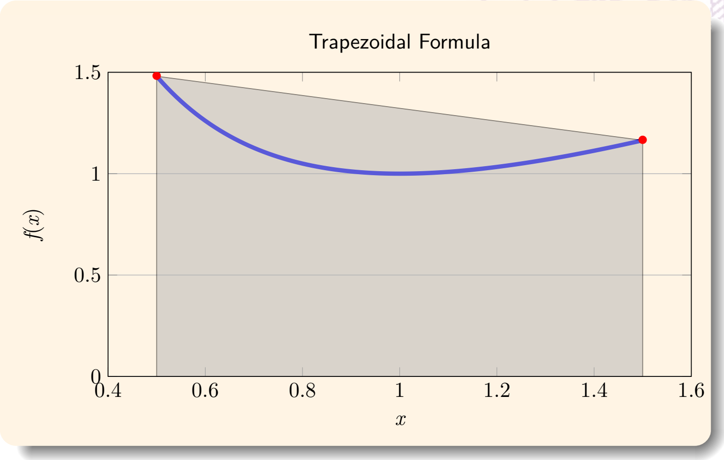
Interpolations

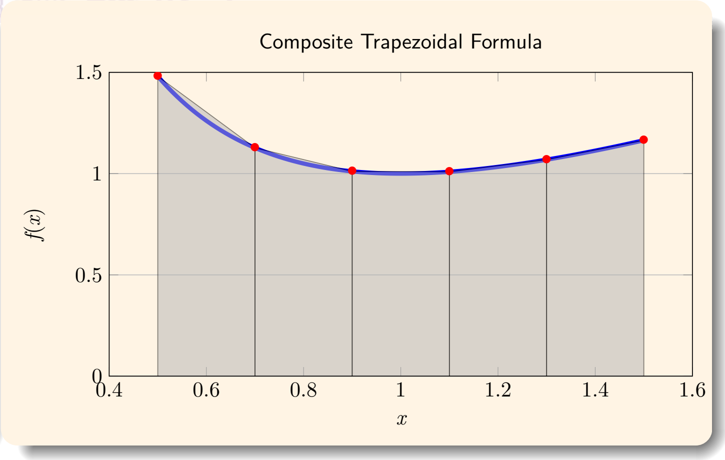
- Given a set of data, find a function to represent these data points.
  - Polynomial interpolations,
  - Rational interpolations (not covered),
  - Spline interpolations.



Integrations

- Integral equations have few closed form solutions.
- Numerical integration has been studied and adopted early.





Nonlinear System Solutions

- Example of nonlinear function

$$f(x) = \log^2 x - x + 0.9$$

  - What is the value of x such that $f(x) = 0$?
  - $f(x)$ can be any nonlinear function.
- What are the roots of

$$x^7 + 6x^5 + 3x^4 + 10x^3 + 25x^2 + 100 = 0.$$

- It has been proven that no closed form solution exists for polynomial of order greater than 5.
- Numerical methods are usually necessary to find the solutions.

Ordinary Differential Equations

- Differential equations have been used extensively in science and engineering fields.

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= f(t, x(t)), \qquad t \geq 0, \\ x(0) &= x_0. \end{aligned}$$

- Finding the solution quickly and accurately is the key of this study.
- Electric circuits with capacitors or inductors, or both, need to be solved using this method if time domain solution is needed.
- Again, solution is usually an approximation.
  - Limitations and accuracy should be well versed to get acceptable solutions.
- In many applications, $f(t, x(t))$ is nonlinear. Thus, solving ODE needs to apply all techniques learned in this course.

Partial Differential Equations

- Example of PDE

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\rho}{\epsilon}$$

- Boundary conditions are needed to get a set of unique solution
  - Boundary value problem (BVP)
- PDE are common in Physics
  - A few different forms with different characteristics
- Solution methods
  - Finite difference approach
    - Adoption to numerical analysis is straightforward
  - Finite element approach
    - Can be more efficient for general boundary conditions

Practice and Programming

- We will analyze different numerical analysis topics and develop algorithms or methods to solving these problems.
- These algorithms or methods can be implemented using different tools.
- Some tools are primitive and need more efforts.
- Some with built-in functions and are very easy to use.
- In this class, we will adopt C++ to practice detailed implementations.
- Basic functions will be developed such that you know all the basics.
- Using operator overloading, our program will be very easy to understand and minimize potential errors.
- C++ class and operator overloading will be reviewed next.
- Since all the algorithms will be discussed, the programming difficulty level is actually quite low.
- You should enjoy our homework every week.