EE4070 Numerical Analysis

數值分析

EE/NTHU

March 2, 2020

Numerical Analysis

- Finding Neptune
  - Application of Numerical Analysis
- Calculating $\sqrt{2}$
  - Manual calculation
  - Continued fraction
  - Babylonian method
  - Comparison
- Course information
  - Time and textbooks
  - Syllabus
  - Evaluation
  - Homework and handouts
  - Suggestions

Discovery of the Planet Neptune

Planets	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Dist to Sun, AU	0.4	0.7	1	1.5	5.2	9.5	19.2	30.1
Mass, M_E	0.055	0.815	1	0.107	318	95	14	17

- Uranus was discovered by William Herschel, British, in 1781 using telescope.
  - Expected orbit calculated using widely separated observation data.
  - In 1830, the accumulated observation data showed deviation from the orbit.
  - Question raised if Newton's gravitation law still holds or if there is another planet affecting Uranus' orbit.
- John Couch Adams, British, was a undergraduate student at Cambridge University in 1843, started to calculate the Uranus' orbit assuming another planet's existence
  - 1845 he completed the calculation and asked the Royal Observatory at Greenwich to search for the hypothetical planet, but his request was not taken seriously.
- Urbain Le Verrier, French, performed similar calculation and asked Johann Gottfried Galle, head of Berlin Observatory, to confirm the prediction.
  - September 23, 1846, Neptune was found.
  - 1° from Le Verrier's prediction.
  - 12° from Adams' prediction.
- A planet was discovered due to the application of numerical analysis.

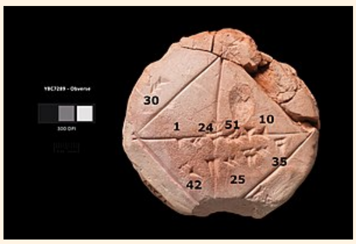
Square Root of Two, $\sqrt{2}$

- Pythagorean theorem had been known to ancient Greek and one of the implication is that the number $\sqrt{2}$ was also known very early in the history.
- For real applications, the square roots need to be calculated

- The Babylonian clay tablet ($\sim$ 1600-1800 BC) showed $\sqrt{2}$ in sexagesimal representation.
  - The estimate is

$$1 + 24 \times \frac{1}{60} + 51 \times \frac{1}{60^2} + 10 \times \frac{1}{60^3} = 1.41417$$

  - Compared to $\sqrt{2} = 1.41421$, the error is 4.68957×10^{-5}.



- It is also known that $\sqrt{2}$ is an irrational number.
- How can the value of $\sqrt{2}$ be calculated?

Finding $\sqrt{2}$ using Manual Calculation

- Step 1

	1.	
	2.	
1	1	
	1	

- Step 2

	1.	4
1	2.	
	1	00
24		96
		4

- Step 3

	1.	4	1
1	2.		
	1	00	
24		96	
		4	00
281		2	81
		1	19

- Step 4

	1.	4	1	4
1	2.			
	1	00		
24		96		
		4	00	
281		2	81	
		1	19	00
2824		1	12	96
			6	04

- The process continues until desired precision is reached.

Finding $\sqrt{2}$ using Manual Calculation, Explanation

- Note that

$$1 < \sqrt{2} < 2 \qquad \text{since} \qquad 1 < (\sqrt{2})^2 < 4.$$

- Let $x = d_0.d_1 = \sqrt{2}$, $d_0 = 1$, then taking square

$$\begin{aligned} 2 = (d_0.d_1)^2 = (d_0 + d_1 * 0.1)^2 &= d_0^2 + 0.2 d_0 d_1 + 0.01 d_1^2 \\ &= d_0^2 + 0.01(20 * d_0 + d_1) d_1 \\ d_1 &= (2 - d_0^2) \times 100/(20 * d_0 + d_1) \end{aligned}$$

- Thus a new digit is found as shown in the steps before.
- This process is essentially an iterative process.
- Due to $\sqrt{2}$ is an irrational number, the process can continue indefinitely.
    - Need to stop when the desired precision is reached.
- This calculation process is convergent.
- At each step, the error is reduced by about 0.01.

九章算術

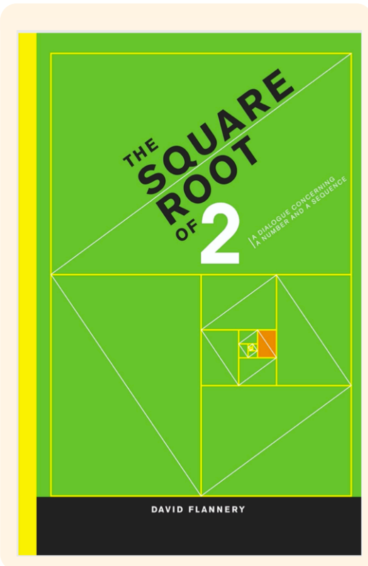
- 中國古代的數學大成，成書年代以及作者皆不可考
  - 西漢前 (西元一世紀) 前已成書
  - 第九章【勾股章】討論畢氏定理的應用，已有計算開根號的方法
  - 其方法與前述方法類似
- 北宋的賈憲 ($\sim$1100)
  - 著作【釋鎖算書】一書
  - 將開平方、立方的方法推廣到任意次方

Approximating $\sqrt{2}$ by Fractions

- David Flannery observes the following

  D. Flannery, *The square Root of 2*, Praxis Publishing, 2006.

$$
\begin{array}{ccccccccc}
1 & = & 1/1 & < & \sqrt{2} & < & 3/2 & = & 1.5 \\
1.4 & = & 7/5 & < & \sqrt{2} & < & 3/2 & = & 1.5 \\
1.4 & = & 7/5 & < & \sqrt{2} & < & 17/12 & \approx & 1.41667 \\
1.4138 & \approx & 41/29 & < & \sqrt{2} & < & 17/12 & \approx & 1.41667 \\
1.4138 & \approx & 41/29 & < & \sqrt{2} & < & 99/70 & \approx & 1.41429 \\
 & & & & \cdots & & & & \\
 & & \dfrac{n}{m} & < & \sqrt{2} & < & \dfrac{2\cdot m+n}{m+n} & &
\end{array}
$$



- The following sequence is created

$$
s_1 = \frac{n_1}{m_1} = \frac{1}{1},
$$
$$
s_k = \frac{n_k}{m_k} = \frac{2\cdot m_{k-1} + n_{k-1}}{m_{k-1} + n_{k-1}}, \qquad k > 1.
$$

Approximating $\sqrt{2}$ by Fractions, II

- More observations

k	s_k	$(\sqrt{2})^2 - s_k^2$		
1	1/1	$2-1$	$=+1$	$=+(1/1)^2$
2	3/2	$2-9/4$	$=-1/4$	$=-(1/2)^2$
3	7/5	$2-49/25$	$=+1/25$	$=+(1/5)^2$
4	17/12	$2-289/144$	$=-1/144$	$=-(1/12)^2$
5	41/29	$2-1681/841$	$=+1/841$	$=+(1/29)^2$
6	99/70	$2-9801/4900$	$=-1/4900$	$=-(1/70)^2$

- In fact, since $s_k = n_k/m_k$

$$\begin{aligned} \text{if} \quad (\sqrt{2})^2 - s_k^2 &= \pm 1/m_k^2 \\ 2 - n_k^2/m_k^2 &= \pm 1/m_k^2 \\ n_k^2 &= 2\cdot m_k^2 \mp 1 \end{aligned}$$

$$\begin{aligned} (\sqrt{2})^2 - s_{k+1}^2 &= 2 - \left(\frac{2m_k + n_k}{m_k + n_k}\right)^2 = \frac{2m_k^2 + 4m_kn_k + 2n_k^2 - 4m_k^2 - 4m_kn_k - n_k^2}{(m_k+n_k)^2} \\ &= \frac{-2m_k^2 + n_k^2}{(m_k+n_k)^2} = \frac{-2m_k^2 + 2m_k^2 \mp 1}{(m_k+n_k)^2} = \frac{\mp 1}{(m_k+n_k)^2} \end{aligned}$$

$\sqrt{2}$ is Irrational

- Any rational number can be expressed as a fraction $\dfrac{m}{n}$, where m and n are integers, and m and n have no common factors.
- Assume $\sqrt{2}$ is a rational number, then

$$\sqrt{2} = \frac{m}{n}$$

- Since $1.4 < \sqrt{2} < 1.5$, n cannot be 1 since no such integer exists.
- Also, $m > n$ due to $1.4 < \dfrac{m}{n} < 1.5$, again.
- Thus, $m > 2$.
- Note that

$$\begin{aligned} 2 &= \frac{m^2}{n^2} \\ m^2 &= 2n^2 \end{aligned}$$

Thus, m^2 is an even number and $m > 2$.

$\sqrt{2}$ is Irrational, II

- m cannot be odd. If $m = 2k + 1$ is an odd number then $m^2 = 4k^2 + 4k + 1$ is an odd number. Thus, m is an even number. And, $m = 2p$, $p > 1$ is an integer.

$$(2p)^2 = 4p^2 = 2n^2$$
$$n^2 = 2p^2$$

  We have n^2 as an even number, and $n > 1$ itself must be even.
- Thus, m and n both are even, and they have a common factor of 2. This contradicts to the assumption that m and n have no common factors.
- Therefore, $\sqrt{2}$ must be an irrational number.

$\sqrt{2}$ is Irrational, III

- A different proof by Edmund Landau (German Mathematician, 1877-1938)
- If $\sqrt{2}$ is a rational number, then there are integers m and n such that $\sqrt{2} = \dfrac{n}{m}$, m and n are relative prime and m is the smallest integer possible.
- Let $u = n - m$, and $t = m - u = 2m - n$.

$$\begin{array}{lrll}
 & 1 < & \sqrt{2} = n/m & < 3/2, \\
u = n - m, & 0 < & u/m = \sqrt{2} - 1 & < 1/2, \\
t = m - u = 2m - n, & 1/2 < & t/m = 2 - \sqrt{2} & < 1.
\end{array}$$

- Both $0 < u < m$ and $0 < t < m$.

$$\begin{aligned}
n^2 + t^2 &= (m+u)(m+u) + t^2 = m^2 + 2mu + u^2 + t^2 \\
&= m^2 + 2(u+t)u + u^2 + t^2 && (m = u + t) \\
&= m^2 + 2u^2 + 2ut + u^2 + t^2 \\
&= m^2 + 2u^2 + (u+t)^2 && \text{(merge the last 3 terms)} \\
&= m^2 + 2u^2 + m^2 && (m = u + t) \\
&= 2m^2 + 2u^2 \\
&= n^2 + 2u^2 && (\sqrt{2} = n/m)
\end{aligned}$$

- Then we have $\sqrt{2} = t/u$.
- This contradicts to that m is the smallest integer possible.

Continued Fraction Method

- A different approach to finding the value of $\sqrt{2}$
- Let $x = \sqrt{2}$, then $x^2 = 2$

$$x^2 + x = 2 + x$$
$$x(x+1) = 2 + x$$

$$\begin{aligned} x &= \frac{2+x}{x+1} = 1 + \frac{1}{1+x} \\ &= 1 + \cfrac{1}{1 + (1 + \cfrac{1}{1+x})} = 1 + \cfrac{1}{2 + \cfrac{1}{1+x}} \\ &= 1 + \cfrac{1}{2 + \cfrac{1}{1 + 1 + \cfrac{1}{1+x}}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1+x}}} \end{aligned} \tag{0.1}$$

Thus,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} \tag{0.2}$$

Continued Fraction Method, II

- A slight variation

$$x^2 = 2$$
$$x^2 - x = 2 - x$$
$$x(x-1) = 2 - x$$

$$\begin{aligned} x &= -\frac{2-x}{1-x} = -1 - \frac{1}{1-x} \\ &= -1 - \cfrac{1}{1 + (1 + \cfrac{1}{1-x})} = -1 - \cfrac{1}{2 + \cfrac{1}{1-x}} \\ &= -1 - \cfrac{1}{2 + \cfrac{1}{1 + 1 + \cfrac{1}{1-x}}} = -1 - \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1-x}}} \end{aligned}$$

Thus,

$$x = -\sqrt{2} = -1 - \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}} \tag{0.3}$$

- The negative root is derived.

Continued Fraction Method – Application

- Apply the continued fraction to find $\sqrt{2}$

$$1 \qquad = 1 \qquad (\text{error} = 0.414214)$$

$$1 + \frac{1}{2} = \frac{3}{2} = 1.5 \qquad (\text{error} = 0.0857864)$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5} = 1.4 \qquad (\text{error} = 0.0142136)$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12} = 1.41667 \qquad (\text{error} = 0.0024531)$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29} = 1.41379 \qquad (\text{error} = 0.000420459)$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}} = \frac{99}{70} = 1.41429 \qquad (\text{error} = 7.21519e-05)$$

Iterative Solution

- Note in deriving the continued fraction solution we have the following equation (0.1)

$$x = 1 + \frac{1}{1+x}$$

- An iterative solution can formed as

$$x_{n+1} = 1 + \frac{1}{1+x_n} \tag{0.4}$$

- Let $x_0 = 1$, we have the following iterations

iter	x	|error|
0	1	0.414214
1	1.5	0.0857864
2	1.4	0.0142136
3	1.41667	0.0024531
4	1.41379	0.000420459
5	1.41429	7.21519e-05
6	1.4142	1.23789e-05
7	1.41422	2.1239e-06
8	1.41421	3.64404e-07
9	1.41421	6.25218e-08
10	1.41421	1.0727e-08
11	1.41421	1.84047e-09
12	1.41421	3.15775e-10
13	1.41421	5.41782e-11

Continued Fraction and Iterative solution

- Note that the continued fraction and the iterative solution methods have exactly the same solution and errors
  - Both methods converge to $\sqrt{2}$.
- To see convergence behavior, let

$$e_n = \sqrt{2} - x_n, \qquad e_{n+1} = \sqrt{2} - x_{n+1}.$$

Then

$$\begin{aligned} e_{n+1} &= \sqrt{2} - x_{n+1} = \sqrt{2} - 1 - \frac{1}{1+x_n} = \sqrt{2} - 1 - \frac{1}{1+\sqrt{2}-e_n} \\ &= \frac{(\sqrt{2}-1)(1+\sqrt{2}-e_n) - 1}{1+\sqrt{2}-e_n} = \frac{2-1-(\sqrt{2}-1)e_n - 1}{1+\sqrt{2}-e_n} \\ &= -\frac{\sqrt{2}-1}{1+\sqrt{2}-e_n} e_n \approx -\frac{\sqrt{2}-1}{\sqrt{2}+1} e_n \approx -0.171573 e_n \end{aligned} \tag{0.5}$$

where we assume $e_n \ll 1 + \sqrt{2}$.

- Thus, $|e_{n+1}| < |e_n|$ and the iterative method is convergent to $\sqrt{2}$.

Babylonian Method

- Let x be an approximation of $\sqrt{2}$, or

$$\sqrt{2} = x + e$$

where e is a small error, that is $|e| \ll |x|$. Then

$$2 = (x+e)^2 = x^2 + 2xe + e^2 \tag{0.6}$$

$$e(2x+e) = 2 - x^2 \tag{0.7}$$

$$e = \frac{2-x^2}{2x+e} \approx \frac{2-x^2}{2x} \tag{0.8}$$

This is due to the assumption $|e| \ll |x|$.

$$\sqrt{2} = x + e = x + \frac{2-x^2}{2x} = \frac{2+x^2}{2x} = \frac{\frac{2}{x}+x}{2} \tag{0.9}$$

This leads to the recursion formula:

$$x_{n+1} = \frac{\frac{2}{x_n}+x_n}{2}. \tag{0.10}$$

As $n \to \infty$, $x_n \to \sqrt{2}$.

Babylonian Method, II

- Example of iterations using Babylonian method

$$x_{n+1} = \frac{\frac{2}{x_n} + x_n}{2}$$

Babylonian Method Iterations

iter	x_n	$\lvert e \rvert$
0	1	0.414214
1	1.5	0.0857864
2	1.41667	0.0024531
3	1.41422	2.1239e-06
4	1.41421	1.59472e-12
5	1.41421	2.22045e-16

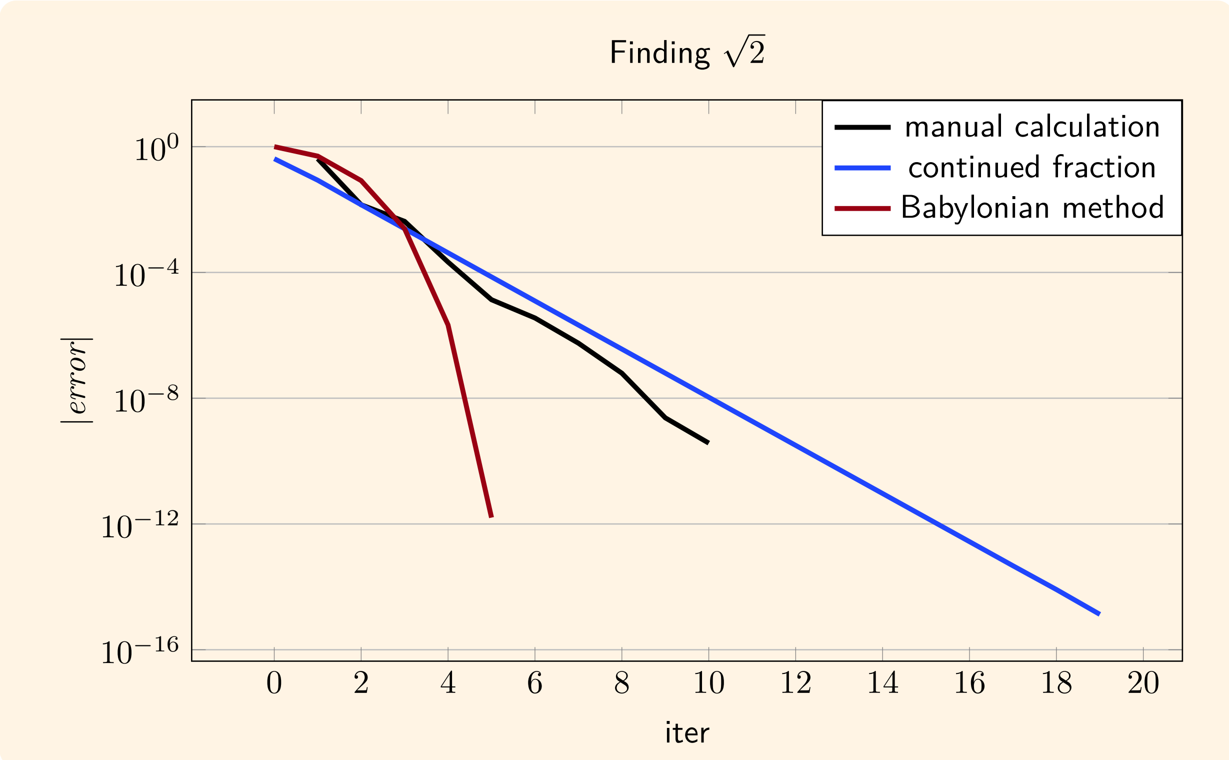
- To find e_{n+1}, we note $e_n = \sqrt{2} - x_n$, or $x_n = \sqrt{2} - e_n$.

$$e_{n+1} = \sqrt{2} - x_{n+1} = \sqrt{2} - \frac{\frac{2}{x_n} + x_n}{2} = \sqrt{2} - \frac{2 + x_n^2}{2x_n}$$

$$= -\frac{2 - 2\sqrt{2}x_n + x_n^2}{2x_n} = -\frac{(\sqrt{2} - x_n)^2}{2(\sqrt{2} - e_n)} = -\frac{e_n^2}{2(\sqrt{2} - e_n)} \approx -\frac{1}{2\sqrt{2}} e_n^2.$$

Thus, the Babylonian method converges quadratically as e_n^2.

Comparisons



Algorithms to Find $\sqrt{2}$

- Three methods to find $\sqrt{2}$
  - Manual calculation
    - Can be carried out using simple arithmetic operations
    - Linear convergence with small coefficient
  - Continued fraction method: two approaches
    - Continued fraction: developed early in history
    - Iterative approach: simpler to implement as a computer program
    - Linear convergence
  - Babylonian method
    - Straightforward to implement as a computer program
    - Quadratic convergence
- This is an example of numerical analysis
  - Finding a numerical solution with a prescribed accuracy
  - Explore mathematical properties to find algorithms
    - Convergent to the right answer
    - Efficiency in computation time and memory requirements

Numerical Analysis – Course Information

- Class time: M3,M4,R4: lectures and discussions.
- Class room: 台達館 208.
- Text book
  - *Numerical Mathematics*, by A. Quarteroni, R. Sacco, F. Saleri, 2nd edition, Springer, 2007.
- Reference books
  - *Introduction to Numerical Analysis*, J. Stoer, R. Bulirsch, 3rd edition, Springer, 2002.
  - *Introduction to Numerical Analysis*, E.B. Hildebrand, 2nd edition, Dover, 1987.
  - *Numerical Recipes in C*, W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, Cambridge University Press, 1988.
- TA: to be announced
- Office hours: Wednesday 10 - 11:30 AM.
  - Or by appointment (michang@ee.nthu.edu.tw).

Numerical Analysis – Syllabus

Unit 0. Introduction
- 0.0. Course information
- 0.1. Introduction
- 0.2. C++ review
- 0.3. Vector and matrix classes

Unit 1. Linear system solutions
- 1.1 Direct solution method
- 1.2 Special matrices
- 1.3 Error bounds
- 1.4 Applications

Unit 2. Errors
- 2.1 Error analysis and data fitting

Unit 3. Iterative solutions
- 3.1 Linear iterative methods
- 3.2 Conjugate gradient method

Unit 4. Eigenvalues
- 4.1 Power method
- 4.2 QR method

Unit 5. Interpolations
- 5.1 Polynomial interpolations.
- 5.2 Spline interpolations.

Unit 6. Integrations
- 6.1 Quadrature Formulas.
- 6.2 Special Integrals.

Unit 7. Nonlinear system solutions
- 7.1 Nonlinear equation solutions
- 7.2 Roots of Polynomials
- 7.3 Nonlinear system solutions

Unit 8. Ordinary differential equations
- 8.1 One step methods.
- 8.2 Multistep methods.
- 8.3 Solution stability.
- 8.4 Variable step methods.

Unit 9. Partial differential equations
- 9.1 Finite difference approach.
- 9.2 Finite element approach.

Evaluation

- Evaluation

Category	% each	Number	Total
Homework	4	12	48
Midterm	16	2	32
Final	20	1	20
Absence	-1	-	-

- Homework:
  - Could be a significant loading,
  - C++ programming and report writing.
- Mid-term exams:
  - Apr. 27,
  - May 25,
  - Machine tests at EECS 406
- Final exam:
  - Jun. 29,
  - Machine test at EECS 406

Homework

- Homework is designed for you to practice what you have learned in class.
- Grading criteria:
  - Ontime submission (20%),
    - Due on 11:59 PM of the day specified on the announcement.
  - Solution correctness (50%),
  - Program and report writing (30%),
    - Legibility and efficiency,
    - Clearness and logic,
    - Solution approach and comments.
- Download and submit on EE workstations.
- Discussions with classmates encouraged but no plagiarism.
  - Write your own programs.

Handouts and Homework

- Class handouts can be found on EE workstation.
  - Download (ftp) through daisy (140.114.24.31).
  - Directory: `~ee4070/notes`
    - `lec00.pdf`,
    - `lec10.pdf`,
    - `lec21.pdf`, ...
- Homework can be found in each homework directory.
  - `~ee4070/hw01`,
  - `~ee4070/hw02`, ...
- Homework should be turned in on EE workstations.
- Submission command:

```
$ ~ee4070/bin/submit hw01 hw01.cpp hw01a.pdf
```

- To check homework or exam grades:

```
$ ~ee4070/bin/score
```

A Few Suggestions

- Be an active learner.
  - *"Stay Hungry. Stay Foolish."*
  - Make it a life long habit.
- Ask questions.
  - It is an important tool.
  - Make the most out of the time you spent.
- Practice makes it perfect.
  - *"The devil is in the details."*
  - You really learn the subject if you can put it in use.
  - Apply to your study, research or work.
  - Understand the assumptions and limitations.
- Discover the beauty in the theories.
  - Simplification of the powerful theories.
  - Generalization to practical problems.