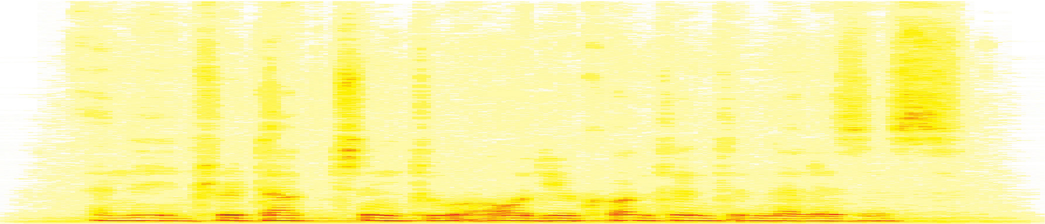
Introduction to Audio Content Analysis

Module 7.0: Dynamic Time Warping

alexander lerch



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corresponding textbook section

Chapter 7: Alignment (pp. 139–146)

- **lecture content**
  - Dynamic Time Warping (DTW): synchronization of two sequences with similar content
- **learning objectives**
  - explain the standard DTW algorithm
  - discuss disadvantages of and modifications to the standard DTW algorithm
  - implement DTW

introduction
overview

corresponding textbook section

Chapter 7: Alignment (pp. 139–146)

- **lecture content**
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  - explain the standard DTW algorithm
  - discuss disadvantages of and modifications to the standard DTW algorithm
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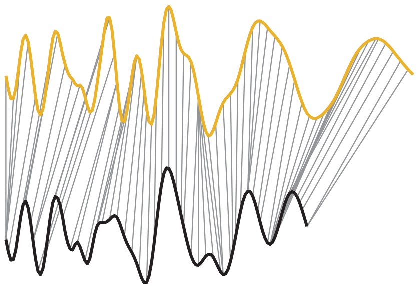
dynamic time warping

problem statement

- align/**synchronize two sequences**
  - *similar* musical content
  - *different* tempo and timing

$$A(n_\mathrm{A}) \quad n_\mathrm{A} \in [0; \mathcal{N}_\mathrm{A} - 1]$$
$$B(n_\mathrm{B}) \quad n_\mathrm{B} \in [0; \mathcal{N}_\mathrm{B} - 1]$$

$\Rightarrow$ find the alignment path that
  - minimizes the pairwise distance between sequences
  - covers the whole sequence
  - does only move forward in time



matlab source: matlab/displaySequenceAlignment.m

dynamic time warping

overview

- dynamic programming technique
- time is warped non-linearly to match sequences
- finds optimal match between two sequences given a cost function
- the overall cost indicates the overall distance between the sequences

dynamic time warping

processing steps

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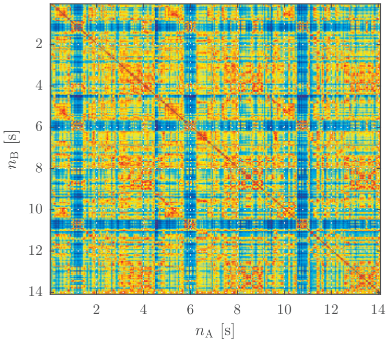
1. extract suitable features
   $\Rightarrow$ two series of feature vectors
2. compute distance matrix
   $\boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}})$
3. compute alignment path
   $\boldsymbol{p}(n_{\mathrm{P}})$ with $n_{\mathrm{P}} \in [0; \mathcal{N}_{\mathrm{P}} - 1]$
   $\Rightarrow$ minimal *overall* distance
4. (align sequences using dynamic time stretching)



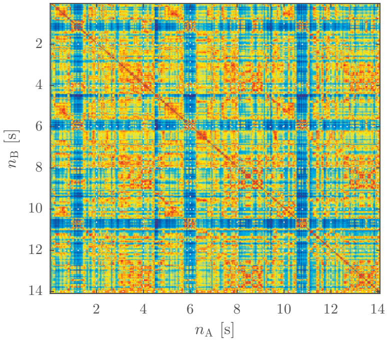
matlab source: matlab/displaySimMatrix.m

dynamic time warping

distance matrix computation

given 2 sequences of vectors, compute the distance between all pairs of observations

- given 2 sequences of vectors, compute the distance between all pairs of observations
- compute distance matrix $\boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}})$
    - example $\boldsymbol{D}_{\mathrm{AB}}(1, n_{\mathrm{B}})$ is the distance of the first vector in Seq. A to all vectors in Seq. B



matlab source: matlab/displaySimMatrix.m

dynamic time warping

path properties 1/2

- **boundaries**: covers both A, B from beginning to end

$$\begin{aligned} \boldsymbol{p}(0) &= [0, 0] \\ \boldsymbol{p}(\mathcal{N}_\mathrm{P} - 1) &= [\mathcal{N}_\mathrm{A} - 1, \mathcal{N}_\mathrm{B} - 1] \end{aligned}$$

- **causality**: only forward movement

$$n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P})} \leq n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)}$$
$$n_\mathrm{B}\big|_{\boldsymbol{p}(n_\mathrm{P})} \leq n_\mathrm{B}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)}$$

- **continuity**: no jumps

$$n_\mathrm{A}\big|_{\boldsymbol{p}(n_\mathrm{P}+1)} \leq (n_\mathrm{A} + 1)\big|_{\boldsymbol{p}(n_\mathrm{P})}$$
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dynamic time warping

path properties 1/2

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dynamic time warping

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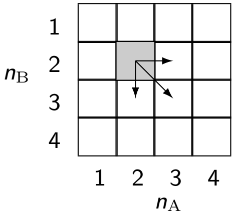
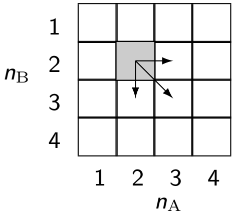
alignment

path properties 2/2



what is the minimum/maximum path length

alignment
path properties 2/2



what is the minimum/maximum path length

$$\mathcal{N}_{\mathrm{P,min}} = \max(\mathcal{N}_{\mathrm{A}}, \mathcal{N}_{\mathrm{B}})$$
$$\mathcal{N}_{\mathrm{P,max}} = \mathcal{N}_{\mathrm{A}} + \mathcal{N}_{\mathrm{B}} - 2$$

alignment
DTW: overall cost

- every path has an *overall cost*

$$\mathfrak{C}_{\mathrm{AB}}(j) = \sum_{n_{\mathrm{P}}=0}^{\mathcal{N}_{\mathrm{P}}-1} \boldsymbol{D}\left(\boldsymbol{p}_j(n_{\mathrm{P}})\right)$$

- *optimal* path minimizes the overall cost

$$\begin{aligned} \mathfrak{C}_{\mathrm{AB},min} &= \min_{\forall j}\left(\mathfrak{C}_{\mathrm{AB}}(j)\right) \\ j_{\mathrm{opt}} &= \operatorname*{argmin}_{\forall j}\left(\mathfrak{C}_{\mathrm{AB}}(j)\right) \end{aligned}$$

⇒ stay in the 'valleys' of distance matrix

how to determine the optimal path

alignment

DTW: overall cost

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$\Rightarrow$ stay in the 'valleys' of distance matrix

how to determine the optimal path

alignment

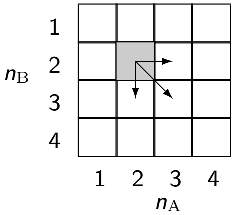
DTW: accumulated cost 1/2

accumulated cost: *cost matrix*

$$\mathbf{C}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) = \mathbf{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) + \min \begin{cases} \mathbf{C}_{\mathrm{AB}}(n_{\mathrm{A}} - 1, n_{\mathrm{B}} - 1) \\ \mathbf{C}_{\mathrm{AB}}(n_{\mathrm{A}} - 1, n_{\mathrm{B}}) \\ \mathbf{C}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}} - 1) \end{cases}$$

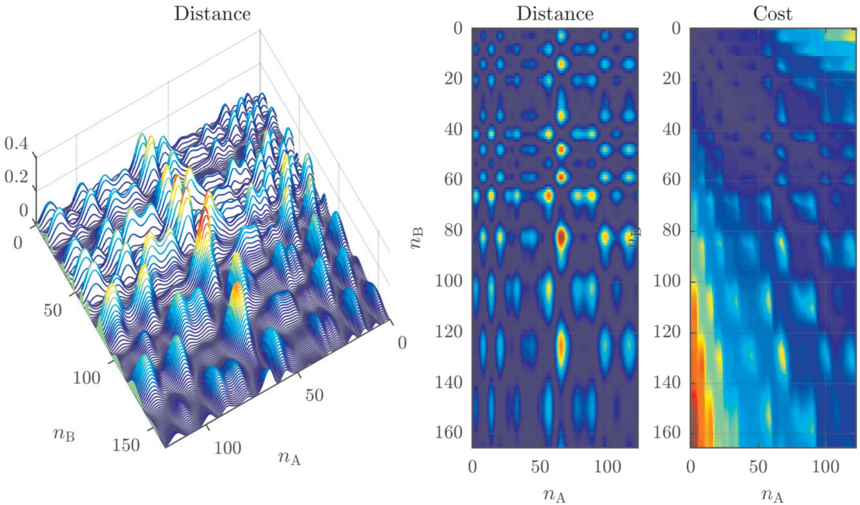
- initialization

$$\mathbf{C}_{\mathrm{AB}}(0, 0) = \mathbf{D}_{\mathrm{AB}}(0, 0)$$



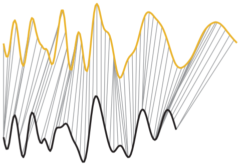
alignment

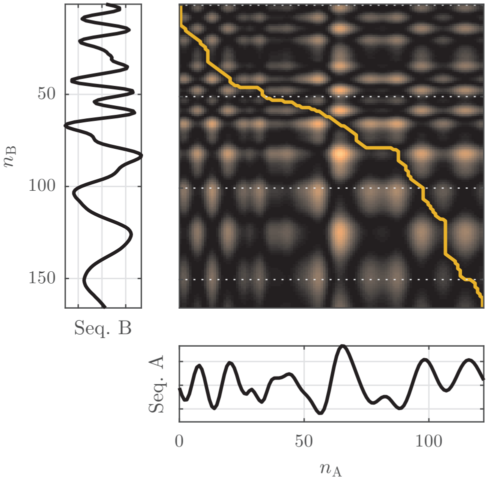
DTW: accumulated cost 2/2



matlab source: matlab/displayDtwCost.m

dynamic time warping
DTW: example





matlab source: matlab/displayDtw.m

alignment

DTW: algorithm description 1/2

- **initialization**:

$$\boldsymbol{C}_{\mathrm{AB}}(0,0) = \boldsymbol{D}_{\mathrm{AB}}(0,0),\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}},-1) = \infty,\ \boldsymbol{C}_{\mathrm{AB}}(-1,n_{\mathrm{B}}) = \infty$$

- **recursion**:

$$\begin{aligned}
\boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}) &= \boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}) + \min \begin{cases} \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}-1) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}-1) \end{cases} \\
j &= \mathrm{argmin} \begin{cases} \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}-1) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1,n_{\mathrm{B}}) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}},n_{\mathrm{B}}-1) \end{cases} \\
\Delta \boldsymbol{p}(n_{\mathrm{A}},n_{\mathrm{B}}) &= \begin{cases} [-1,-1] & \text{if } j=0 \\ [-1,0] & \text{if } j=1 \\ [0,-1] & \text{if } j=2 \end{cases}
\end{aligned}$$

alignment

DTW: algorithm description 1/2

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alignment

DTW: algorithm description 2/2

- **termination**:

$$n_\mathrm{A} = \mathcal{N}_\mathrm{A} - 1 \wedge n_\mathrm{B} = \mathcal{N}_\mathrm{B} - 1$$

- **path backtracking**:

$$\boldsymbol{p}(n_\mathrm{P}) = \boldsymbol{p}(n_\mathrm{P}+1) + \Delta\boldsymbol{p}\big(\boldsymbol{p}(n_\mathrm{P}+1)\big),\ \ n_\mathrm{P} = \mathcal{N}_\mathrm{P} - 2, \mathcal{N}_\mathrm{P} - 3, \ldots, 0$$

alignment

DTW: algorithm description 2/2

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- **termination**:

$$n_\mathrm{A} = \mathcal{N}_\mathrm{A} - 1 \wedge n_\mathrm{B} = \mathcal{N}_\mathrm{B} - 1$$

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$$\boldsymbol{p}(n_\mathrm{P}) = \boldsymbol{p}(n_\mathrm{P}+1) + \Delta\boldsymbol{p}\big(\boldsymbol{p}(n_\mathrm{P}+1)\big), \;\; n_\mathrm{P} = \mathcal{N}_\mathrm{P} - 2, \mathcal{N}_\mathrm{P} - 3, \ldots, 0$$

dynamic time warping

example

$$
\begin{aligned}
A &= [1,\ 2,\ 3,\ 0], \\
B &= [1,\ 0,\ 2,\ 3,\ 1],
\end{aligned}
$$

compute distance matrix, cost matrix, and DTW path

dynamic time warping

example

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A &= [1,\ 2,\ 3,\ 0], \\
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compute distance matrix, cost matrix, and DTW path

$$
\boldsymbol{D}_{\mathrm{AB}} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}
$$

dynamic time warping

example

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$$
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compute distance matrix, cost matrix, and DTW path

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\qquad
\boldsymbol{C}_{\mathrm{AB}} = \begin{bmatrix} 0 & \leftarrow 1 & \leftarrow 3 & \leftarrow 4 \\ \uparrow 1 & \nwarrow 2 & \nwarrow 4 & \nwarrow 3 \\ \uparrow 2 & \nwarrow 1 & \leftarrow 2 & \leftarrow 4 \\ \uparrow 4 & \uparrow 2 & \nwarrow 1 & \leftarrow 4 \\ \uparrow 4 & \uparrow 3 & \uparrow 3 & \nwarrow 2 \end{bmatrix}
$$

dynamic time warping

example

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compute distance matrix, cost matrix, and DTW path

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$$

dynamic time warping

variants

- transition weights: favor specific path directions

$$\boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) = \min \begin{cases} \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1, n_{\mathrm{B}}-1) & + \quad \lambda_{\mathrm{d}} \cdot \boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}-1, n_{\mathrm{B}}) & + \quad \lambda_{\mathrm{v}} \cdot \boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \\ \boldsymbol{C}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}-1) & + \quad \lambda_{\mathrm{h}} \cdot \boldsymbol{D}_{\mathrm{AB}}(n_{\mathrm{A}}, n_{\mathrm{B}}) \end{cases}$$

- step types

dynamic time warping

variants

- transition weights: favor specific path directions

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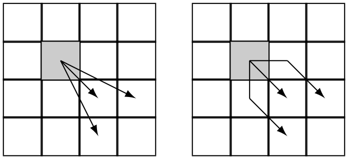

dynamic time warping
optimization

- **challenge**: distance matrix dimensions $\mathcal{N}_\mathrm{A} \cdot \mathcal{N}_\mathrm{B}$

$\Rightarrow$ DTW *inefficient* for long sequences
  - high memory requirements
  - large number of operations

optimizations:

1. maximum time and tempo deviation
2. sliding window
3. multi-scale DTW (several downsampled iterations)

dynamic time warping

optimization

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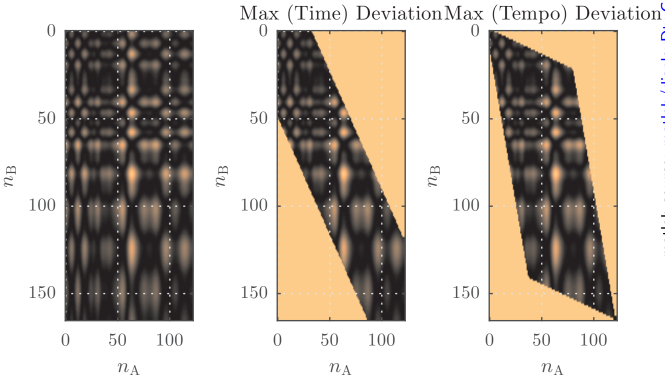

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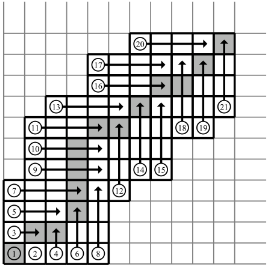

1

[1] S. Dixon and G. Widmer, "MATCH: A Music Alignment Tool Chest," in *Proceedings of the 6th International Conference on Music Information Retrieval (ISMIR)*, London, Sep. 2005.

dynamic time warping

optimization

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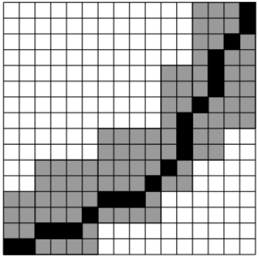

[1] M. Mller, H. Mattes, and F. Kurth, "An Efficient Multiscale Approach to Audio Synchronization," in *Proceedings of the International Society for Music Information Retrieval Conference (ISMIR)*, Victoria, 2006.

summary
lecture content

- **dynamic time warping**
  - find globally optimal alignment path between two sequences

- **processing steps**
  1. compute distance matrix
  2. compute cost matrix
  3. back-track path