

Lecture notes: Discrete Random Variables

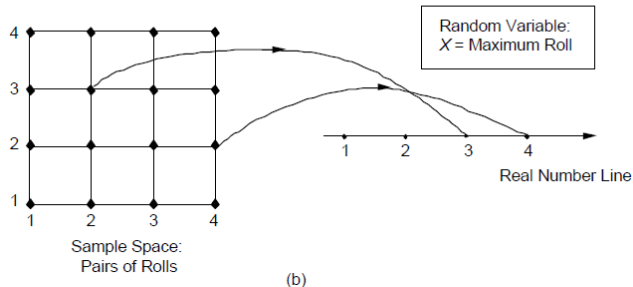
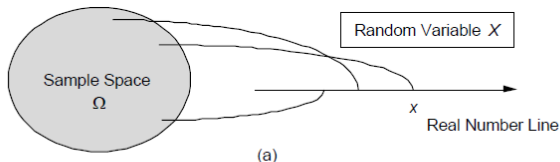
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- ▶ Random variables.
- ▶ Discrete Random variables.
- ▶ Probability mass function.
- ▶ Cumulative distribution function.
- ▶ Three important discrete RVs.

Random variable

A random variable (r.v.) associates a value (a number) to **every possible outcome**.
e.g. In a dice roll, the outcome $\{1, 2, 3, 4, 5, 6\}$ can be directly assigned to the random variable X , where $X = 1, 2, \dots, 6$.



- ▶ We can have several random variables defined on the same sample space.
- ▶ A function of one or several random variables' is also a random variable.

E.g. Let X, Y be two R.V., a function $X + Y$ is also a RV.

Let a be a constant, a function $a + X$ is also a RV.

A discrete random variable has an associated **probability mass function (PMF)**, which gives the probability of each numerical value that the random variable can take.

$$p_X : \mathcal{X} \rightarrow [0, 1]$$

where \mathcal{X} is all possible values X can take.

Notation:

$$p_X(x) = P(\{X = x\})$$

Properties: $p_X(x) \geq 0$, $\sum_x P_X(x) = 1$.

Example



Two independent tosses of a fair coin:

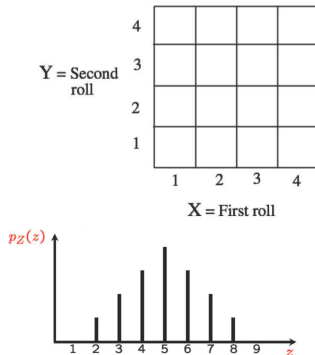
Define discrete RV X : The total number of heads.

The PMF of X is

Example

Two rolls of a 4-sided die:

$$Z = X + Y$$



Calculation of the PMF of a Random Variable For each possible value z of Z :

1. Collect all the possible outcomes that give rise to the event $\{Z = z\}$.
2. Add their probabilities to obtain $p_Z(z)$.

If (Ω, \mathcal{F}, P) is a probability space with X a real discrete RV on Ω , the **Cumulative Distribution Function (CDF)** is denoted as $F_X(x)$ and provides the probability $P(X \leq x)$. In particular, for every x we have

$$F_X(x) = P(X \leq x) = \sum_{k \leq x} p_X(k)$$

Loosely speaking, the CDF $F_X(x)$ “accumulates” probability “up to” the value x .

Example

I toss a coin twice. Let X be the number of observed heads. Find the CDF of X .
Recall the PMF of X :

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0, 2 \\ \frac{1}{2} & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To find $F_X(x)$

- ▶ $x < 0$:
- ▶ $x = 0$:
- ▶ $x = 1$:
- ▶ $x = 2$:
- ▶ $x > 2$:

Next, we will introduce some important discrete RV.

- ▶ Bernoulli Random Variable
- ▶ Binomial Random Variable.
- ▶ Geometric Random Variable.

e.g. Medical treatment: Two outcomes in the sample space: $x = 1$ for “success” and $x = 0$ for “failure”.

Bernoulli RV: Models a trial that results in success/failure, Heads/Tails, etc.

- ▶ A **Bernoulli RV** X takes two values 0 and 1.
- ▶ The PMF for a Bernoulli RV X is defined by

$$p_X(x) = P(X = x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \\ 0, & \text{o.w.} \end{cases}$$

$$X \sim \text{Bernoulli}(p)$$

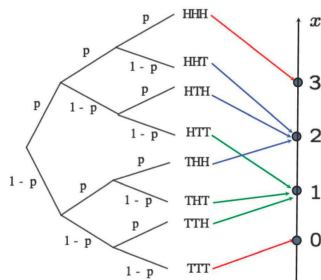
Examples/applications:

- ▶ The coin flip is either heads (1) or tail (0).
- ▶ whether a bit is 0 or 1, whether a bit is in error, whether a component has failed, whether something has been detected.
- ▶ The state of a telephone at a given time that can be either free (0) or busy (1).
- ▶ A person who can be either healthy or sick with a certain disease.

The Binomial Random Variable

consider the following experiment:

- ▶ A biased coin is tossed n times.
- ▶ Each toss is independently of prior tosses: Head with probability p ; Tail with probability $1 - p$.
- ▶ The number X of heads up is a binomial random variable, refer to as the **Binomial RV with parameters n and p** .



$$X \sim \text{Binomial}(n, p)$$

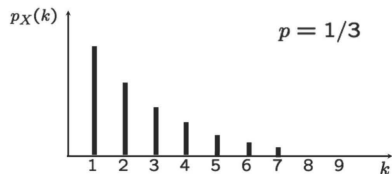
$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \dots, n \\ 0, & \text{o.w.} \end{cases}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the different combinations of x objects chosen from n objects.

A pharmaceutical company is testing a new medication to see if it is effective in treating a certain disease. The company wants to know the probability that the medication will work for at least 80% of patients. To test this, they give the medication to 20 patients and observe whether it works or not.

- ▶ Based on the observation that the medicine worked for 5 out of 20 patients, do you think you believe the company's claim that the effective rate is 80%?

- ▶ Experiment: infinitely many independent tosses of a coin $P(\text{Heads}) = p$.
- ▶ Sample space: Set of infinite sequences of H and T.
- ▶ Random variable X : number of tosses until the first Heads.
- ▶ Models of: waiting times; number of trials until a success.



decreases as a **geometric progression** with parameter $1 - p$.

$$p_X(k) = (1 - p)^{k-1} p$$

what is the probability of no heads ever?

A customer calls a tech support line, and each agent has a 20% chance of successfully resolving the issue. The customer may need to speak to multiple agents before the problem is fixed. Assuming all agents are not sharing the knowledge/experience learned from talking with the customer, how many agents the customer need to talk to before his issue is solved with probability $> 95\%$?

- ▶ Bernoulli: A single trial with two possible outcomes: success (1) with probability p and failure (0) with probability $1 - p$.
- ▶ Binomial: The number of successes in n independent Bernoulli trials, each with success probability p .
- ▶ Geometric: The number of trials until the first success in repeated independent Bernoulli trials with success probability p .