

Lecture 1: Probability models and Axioms

Lecturer: Jie Fu, Ph.D.

EEL 3850 S25

Outline

- Random Experiments
- Sample space
- Events
- Probability laws
 - Axioms
 - Properties that follow from the axioms
- Probability calculation

Random Experiments

- **Random Experiments:** A **random experiment** is an experiment for which the **outcome** is **not completely predictable** to an observer based on the observer's knowledge of the system and its inputs.
- **Outcome:** An **outcome** is a non-decomposable result (or output) of a random experiment.
- A **fair** experiment means all outcomes **are equally likely**.

E.g.

	Random experiment	Outcomes	Fair experiments?
1	Flip a coin		
2	Rolling a four-sided die		
3	The measurements of the heights of college students.		

Sample space

- Let Ω be a set of possible outcomes.
 - Mutually exclusive
 - Collectively exhaustive
- Example (discrete):
 - Two sequential roll of dies

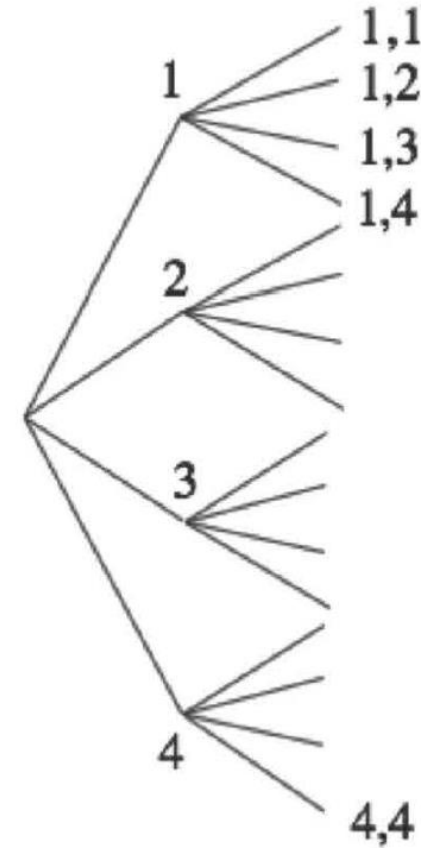
$Y = \text{Second roll}$

4				
3				
2				
1				
	1	2	3	4

$X = \text{First roll}$

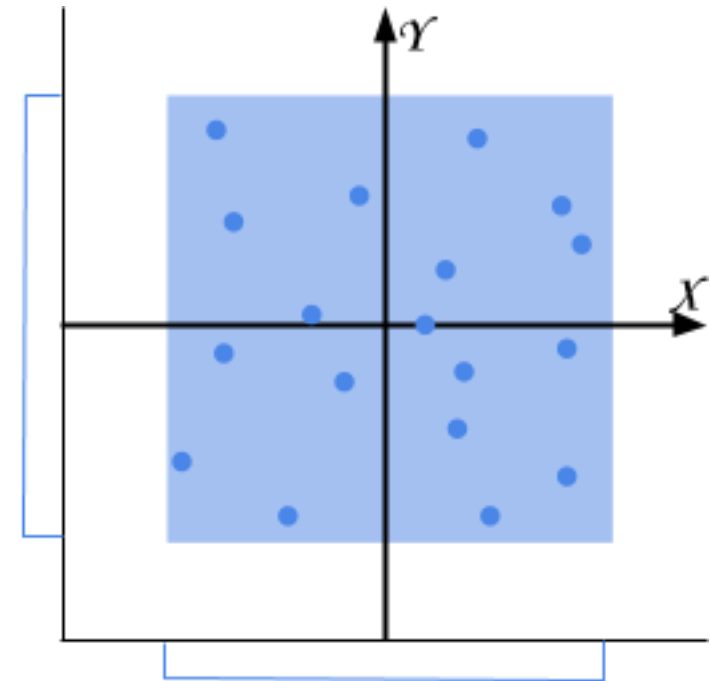
Sample space

- Let Ω be a set of possible outcomes.
 - Mutually exclusive
 - Collectively exhaustive
- Example (discrete):
 - Two sequential roll of dies:
 - Alternative/tree representation:



Sample space

- Let Ω be a set of possible outcomes.
 - Mutually exclusive
 - Collectively exhaustive
- Example (continuous):
 - The position of kids in a square playground.
(x,y) such that
$$-1 \leq x, y \leq 1$$
 - The heights of college students?



Event

- An event: A specific outcome or a set of outcomes from a random experiment.
- A subset of sample space

Simple Event: An event consisting of only one outcome.
E.g.: Rolling a die and getting a 3.

Compound Event: An event consisting of multiple outcomes.
E.g. :

- Rolling a die and getting an even number (2, 4, or 6).
- The sum of two rolls of four sided dies is below 4.
- In a sequence of 10 flips of a coin, the total number of heads is greater than or equals 5.

- In any random experiment, there is always uncertainty whether a particular event will occur or not occur.
- Probability is a measure of chance and assigned between 0 to 1.

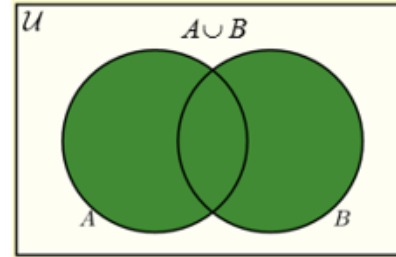
Example:

- Consider two rolls of the four-sided die, assuming each face shows up with $\frac{1}{4}$ probability.
- Let X be the value of the first roll and Y be the value of the second roll, and $Z = \min(X, Y)$, calculate
 - $P(X = 1)$
 - $P(Y = 1)$
 - $P(Z = 1)$

A quick review of set operations

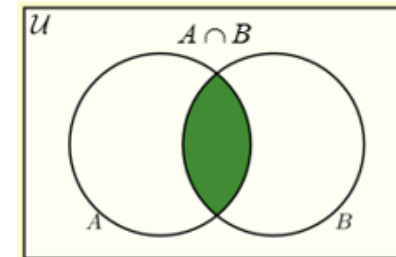
- Events in probability are subsets of a sample space,
- Set operations helps us calculate probabilities of **combined** events and understand their relationships.

Venn Diagrams



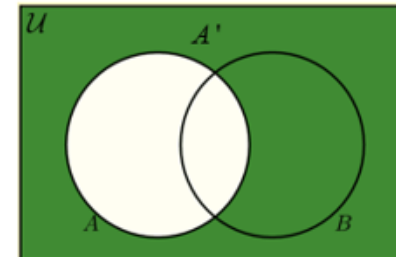
A union B

Elements that belong to either A or B or both.



A intersect B

Elements that belong to both A and B.



A complement

Elements that don't belong to A.

Set operations on events

- An event: **A subset** of sample space.

Given S a sample space, and two events A and B , then

- $A \cup B$ is the event for “either A or B or Both” - The ****union**** of A and B .
- $A \cap B$ is the event for “both A and B ” - the ****intersection**** of A and B .
- A^c is the event for “not A ”, which is the ****complement**** of A .
- $A - B = A \cap B^c$ is the event for “ A but not B ”.

Axioms of probability

- **Axioms:** a minimal set of rules that the probability P must obey.
 - $\forall A \subseteq \Omega, P(A) \geq 0.$
 - $P(\Omega) = 1$

Other axiom

- **Probability of the Empty Set:**

$$P(\emptyset) = 0$$

- **Probability of Complement:**

$$P(A^c) = 1 - P(A)$$

- **Bounded Probability:** for any event A

$$0 \leq P(A) \leq 1$$

Additivity axiom

- **Mutually exclusive events** are events that cannot occur at the same time. If one event occurs, the other cannot.
- The intersection of two mutually exclusive events is empty.
- **Formally, if event A and B are mutually exclusive, then**

$$P(A \cup B) = P(A) + P(B)$$

For n mutually exclusive events: A_1, A_2, \dots, A_n

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

Eg. The probability of getting an event number for 4-sided die:

Union of Events

- The union of two events A and B represents all outcomes that belong to either A, B, or both.
- Probability Formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Pictorially,
- E.g.: The probability of rolling a die and getting either an even number or a number greater than 3.

Intersection of Events

- The intersection of two events A and B represents all outcomes that belong to both A and B.
- Only when A and B are **independent**: The occurrence of one event does not affect the probability of the occurrence of another event.

$$P(A \cap B) = P(A) \cdot P(B)$$

- If two events are not independent, then we need to calculate the intersection without any formula.

Example: The probability of getting a 1 on the first roll and then a 4 on the second roll for a 4-sided die.

More consequence of the axioms:

- If $A \subseteq B$, then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B) \leq P(A) + P(B)$

Probability calculation

Steps

- Identify the sample space
- Specify the probability law P
- Specify the event A .
- Calculate $P(A)$.

Example

- Consider the problem of rolling a six-faced die, which is a fair die.
- Consider the event A = “even number occurs”
- What is the probability of event A ?

Consider a random experiment that has K possible outcomes

Let $N_k(n) \equiv$ the number of times the outcome is k and let the relative frequency of outcome k be

$$r_k(n) = \frac{N_k(n)}{n}$$

Probability as a Measure of Frequency of Occurrence

$$\lim_{n \rightarrow \infty} r_k(n) = p_k$$

is called the *probability of outcome k* .



Interpretation of probability as relative frequency

- In the experiments we have conducted, the relative frequencies converge to some constant values when the number of simulations increases.

“the relative frequencies converge to the probabilities”

Example

- Consider rolling a 6-sided die and let the event E = “even number occurs”.

What can we say about the number of times E is observed in n trials?

$$N_E(n) = N_2(n) + N_4(n) + N_6(n)$$

What can we say about the probability of event E ?