

Lecture 1: Probability models and Axioms

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EEL 3850 S25

- Random Experiments
- Sample space
- Events
- Probability laws
 - Axioms
 - Properties that follow from the axioms
- Probability calculation

Random Experiments

- **Random Experiments:** A **random experiment** is an experiment for which the **outcome** is **not completely predictable** to an observer based on the observer's knowledge of the system and its inputs.
- **Outcome:** An **outcome** is a non-decomposable result (or output) of a random experiment.
- A **fair** experiment means all outcomes **are equally likely**.

E.g.

	Random experiment	Outcomes	Fair experiments?
1	Flip a coin	$\{ 'H', 'T' \}$	✓ Fair coin
2	Rolling a four-sided die	$\{ 1, 2, 3, 4 \}$	✓ Fair die
3	The measurements of the heights of college students.	$[0, 8.5 \text{ ft}]$	X

Sample space

- Let Ω be a set of possible outcomes.

- Mutually exclusive

- Collectively exhaustive

$$\{ 'H' , 'T' \} = \Omega$$

- Example (discrete):

- Two sequential roll of dies

(outcome of roll 1, outcome of roll 2)

$$\Omega = \{ (R_1, R_2) \mid R_1 \in \{1, 2, 3, 4\}, R_2 \in \{1, 2, 3, 4\} \}$$

Y = Second
roll

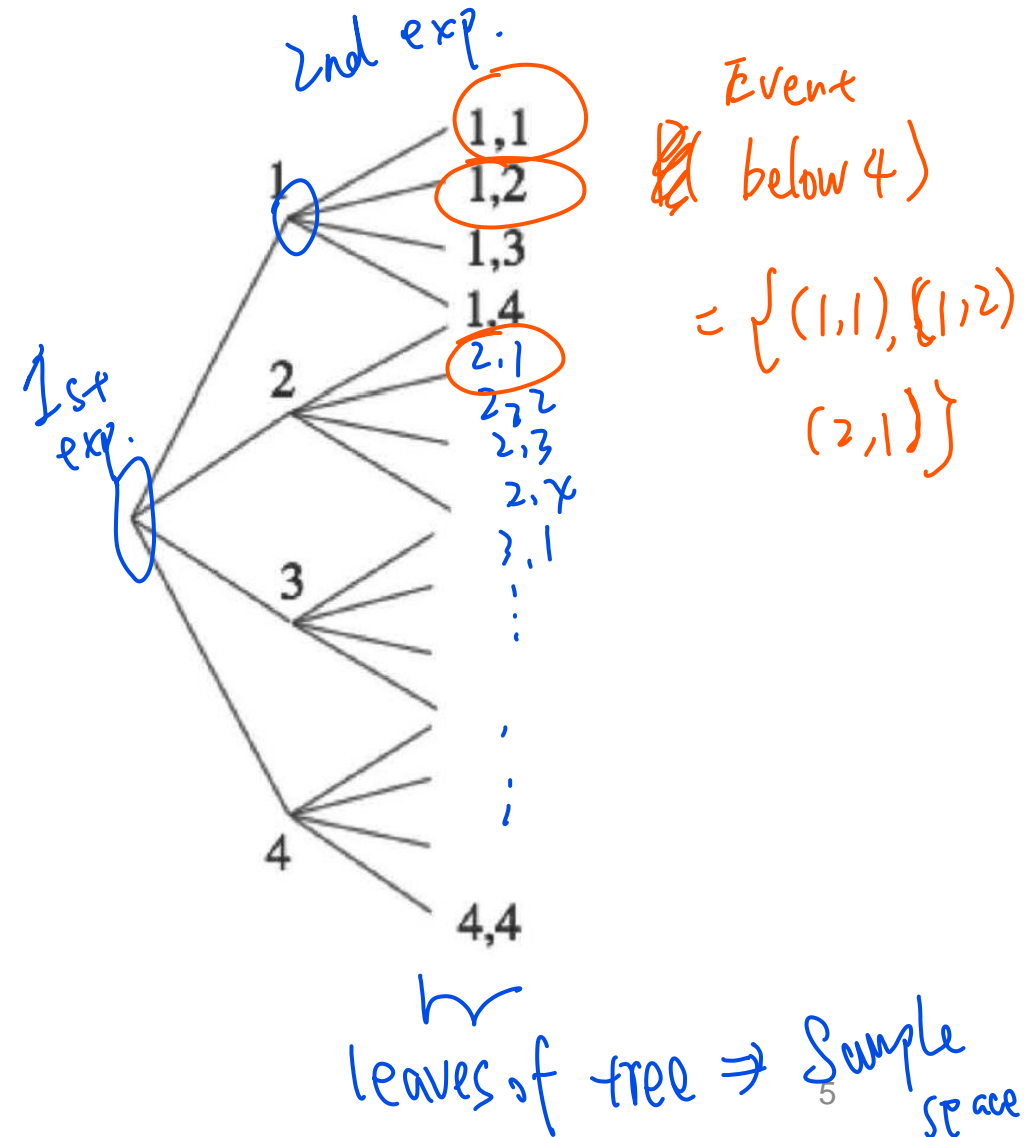
4				
3				
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	1	2	3	4

X = First roll

Sample space

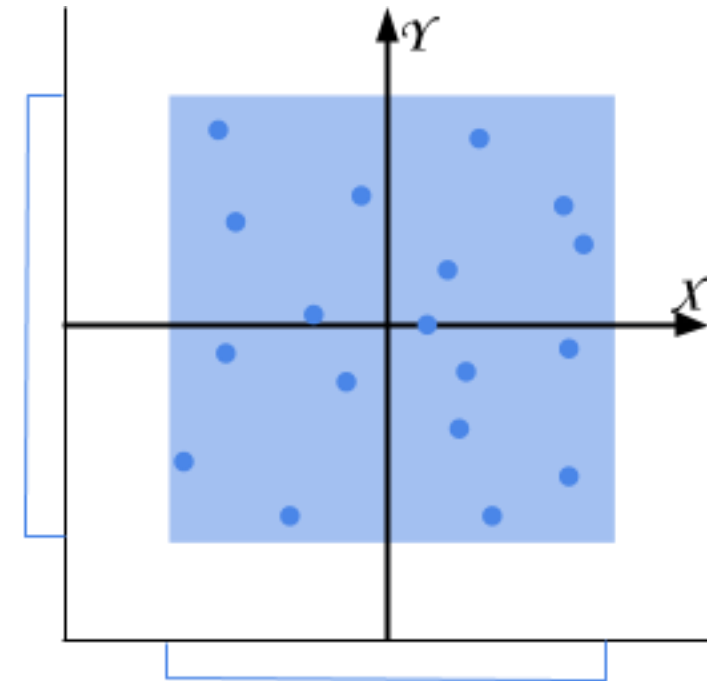
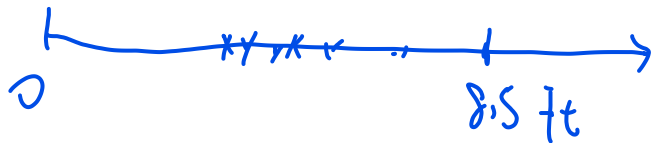
- Let Ω be a set of possible outcomes.
 - Mutually exclusive
 - Collectively exhaustive
- Example (discrete):
 - Two sequential roll of dies:

Alternative/tree representation:
 $K = 4$



Sample space

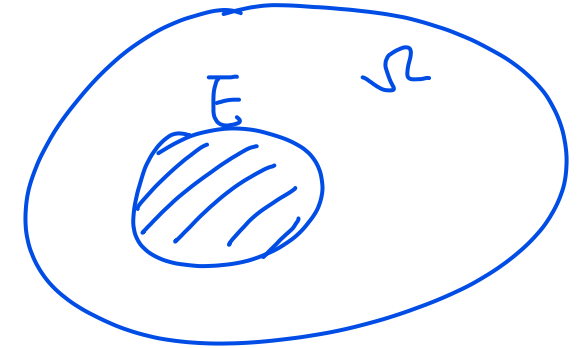
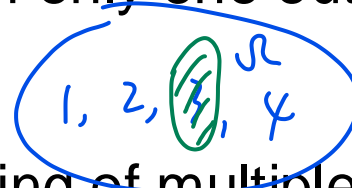
- Let Ω be a set of possible outcomes.
 - Mutually exclusive
 - Collectively exhaustive
- Example (continuous):
 - The position of kids in a square playground.
(x,y) such that $-1 \leq x, y \leq 1$
 - The heights of college students?



Event

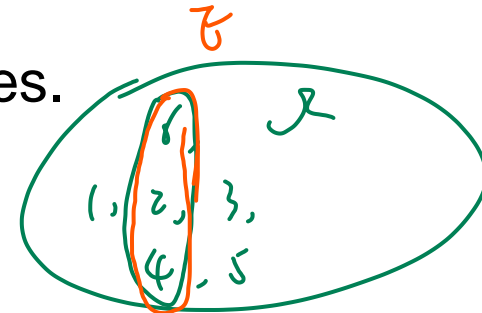
- An event: A specific outcome or a set of outcomes from a random experiment.
- A subset of sample space

Simple Event: An event consisting of only one outcome.
E.g.: Rolling a die and getting a 3.



Compound Event: An event consisting of multiple outcomes.
E.g.:

- Rolling a ^{6-side} die and getting an even number (2, 4, or 6).
- The sum of two rolls of four sided dies is below 4.
- In a sequence of 10 flips of a coin, the total number of heads is greater than or equals 5.



(H, H, H, H, T, H, H, T, T, T)

(T, H, H, H, H, H, T, T, T, T)

- In any random experiment, there is always uncertainty whether a particular event will occur or not occur.
- Probability is a measure of chance and assigned between 0 to 1.

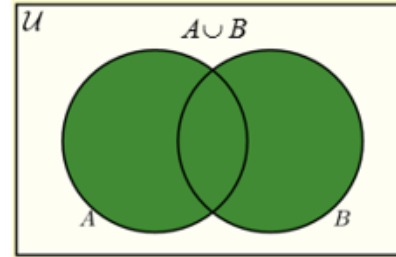
Example:

- Consider two rolls of the four-sided die, assuming each face shows up with $\frac{1}{4}$ probability.
- Let X be the value of the first roll and Y be the value of the second roll, and $Z = \min(X, Y)$, calculate
 - $P(X = 1)$
 - $P(Y = 1)$
 - $P(Z = 1)$

A quick review of set operations

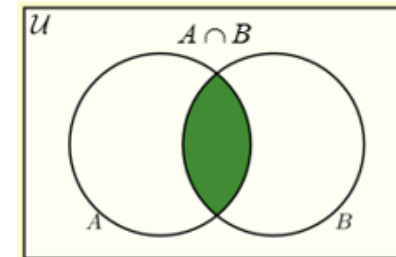
- Events in probability are subsets of a sample space,
- Set operations helps us calculate probabilities of **combined** events and understand their relationships.

Venn Diagrams



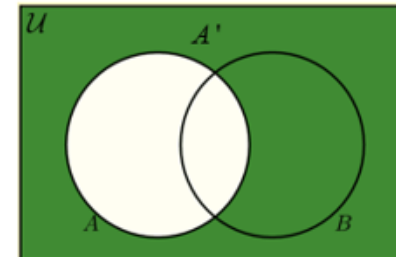
A union B

Elements that belong to either A or B or both.



A intersect B

Elements that belong to both A and B.



A complement

Elements that don't belong to A.

Set operations on events

- An event: **A subset** of sample space.

Given S a sample space, and two events A and B , then

- $A \cup B$ is the event for “either A or B or Both” - The ****union**** of A and B .
- $A \cap B$ is the event for “both A and B ” - the ****intersection**** of A and B .
- A^c is the event for “not A ”, which is the ****complement**** of A .
- $A - B = A \cap B^c$ is the event for “ A but not B ”.

Axioms of probability

- **Axioms:** a minimal set of rules that the probability P must obey.
 - $\forall A \subseteq \Omega, P(A) \geq 0.$
 - $P(\Omega) = 1$

Other axiom

- **Probability of the Empty Set:**

$$P(\emptyset) = 0$$

- **Probability of Complement:**

$$P(A^c) = 1 - P(A)$$

- **Bounded Probability:** for any event A

$$0 \leq P(A) \leq 1$$

Additivity axiom

- **Mutually exclusive events** are events that cannot occur at the same time. If one event occurs, the other cannot.
- The intersection of two mutually exclusive events is empty.
- **Formally, if event A and B are mutually exclusive, then**

$$P(A \cup B) = P(A) + P(B)$$

For n mutually exclusive events: A_1, A_2, \dots, A_n

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

Eg. The probability of getting an event number for 4-sided die:

Union of Events

- The union of two events A and B represents all outcomes that belong to either A, B, or both.
- Probability Formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Pictorially,
- E.g.: The probability of rolling a die and getting either an even number or a number greater than 3.

Intersection of Events

- The intersection of two events A and B represents all outcomes that belong to both A and B.
- Only when A and B are **independent**: The occurrence of one event does not affect the probability of the occurrence of another event.

$$P(A \cap B) = P(A) \cdot P(B)$$

- If two events are not independent, then we need to calculate the intersection without any formula.

Example: The probability of getting a 1 on the first roll and then a 4 on the second roll for a 4-sided die.

More consequence of the axioms:

- If $A \subseteq B$, then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B) \leq P(A) + P(B)$

Probability calculation

Steps

- Identify the sample space
- Specify the probability law P
- Specify the event A .
- Calculate $P(A)$.

Example

- Consider the problem of rolling a six-faced die, which is a fair die.
- Consider the event A = “even number occurs”
- What is the probability of event A ?

Consider a random experiment that has K possible outcomes

Let $N_k(n) \equiv$ the number of times the outcome is k and let the relative frequency of outcome k be

$$r_k(n) = \frac{N_k(n)}{n}$$

Probability as a Measure of Frequency of Occurrence

$$\lim_{n \rightarrow \infty} r_k(n) = p_k$$

is called the *probability of outcome k* .



Interpretation of probability as relative frequency

- In the experiments we have conducted, the relative frequencies converge to some constant values when the number of simulations increases.

“the relative frequencies converge to the probabilities”

Example

- Consider rolling a 6-sided die and let the event E = “even number occurs”.

What can we say about the number of times E is observed in n trials?

$$N_E(n) = N_2(n) + N_4(n) + N_6(n)$$

What can we say about the probability of event E ?