Lecture 1: Probability models and Axioms

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EEL 3850 S25

Outline

- Random Experiments
- Sample space
- Events
- Probability laws
  - Axioms
  - Properties that follow from the axioms
- Probability calculation

Random Experiments

- **Random Experiments:** A random experiment is an experiment for which the outcome is not completely predictable to an observer based on the observer's knowledge of the system and its inputs.

- **Outcome**: An outcome is a non-decomposable result (or output) of a random experiment.

- A **fair** experiment means all outcomes are equally likely.

E.g.

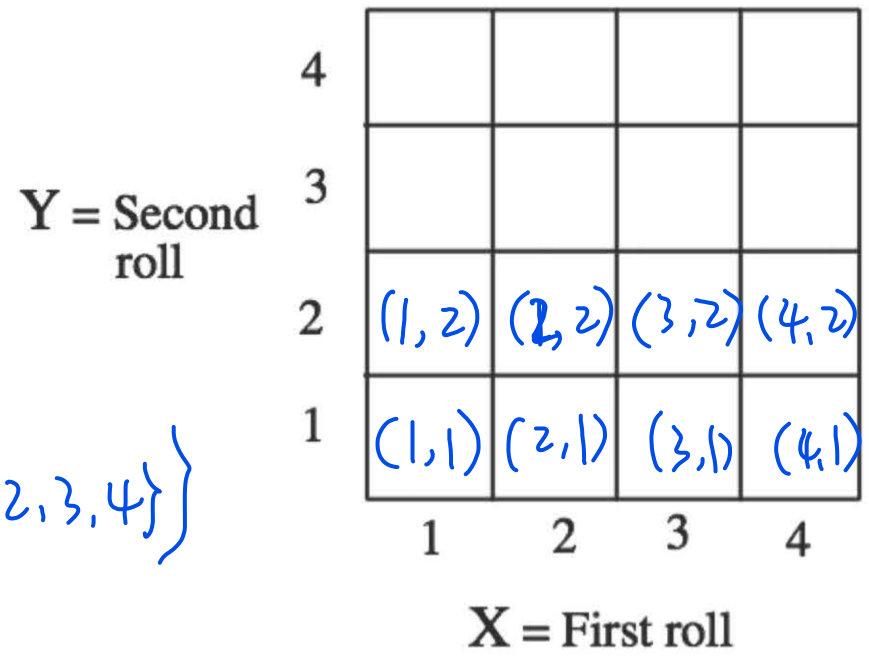
	Random experiment	Outcomes	Fair experiments?
1	Flip a coin	{'H', 'T'}	✓ Fair coin
2	Rolling a four-sided die	{1, 2, 3, 4}	✓ fair die
3	The measurements of the heights of college students.	[0, 8.5 ft]	X

Sample space

- Let Ω be a set of possible outcomes.
  - Mutually exclusive
  - Collectively exhaustive

$\{'H', 'T'\} = \Omega$

- Example (discrete):
  - Two sequential roll of dies



(outcome of roll 1, outcome of roll 2)

$$\Omega = \{(R_1, R_2) \mid R_1 \in \{1, 2, 3, 4\}, R_2 \in \{1, 2, 3, 4\}\}$$

Sample space

- Let Ω be a set of possible outcomes.
  - Mutually exclusive
  - Collectively exhaustive

- Example (discrete):
  - Two sequential roll of dies:
    - Alternative/tree representation:

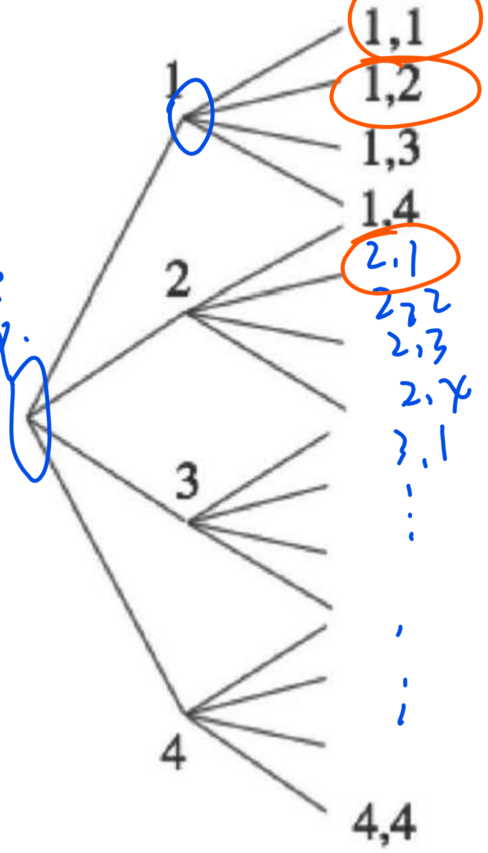
K > 4

2nd exp.

Event (below 4)

= {(1,1), (1,2), (2,1)}

1st exp.



leaves of tree ⇒ Sample space

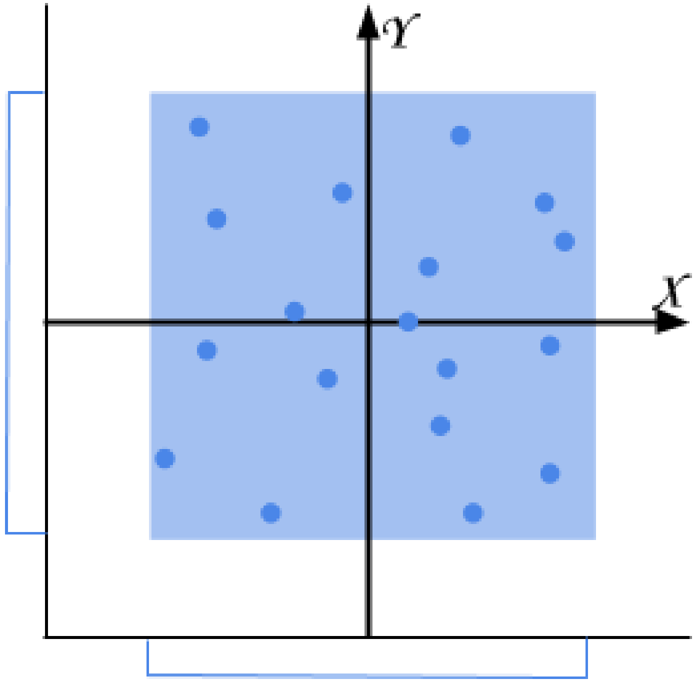
Sample space

- Let Ω be a set of possible outcomes.
  - Mutually exclusive
  - Collectively exhaustive

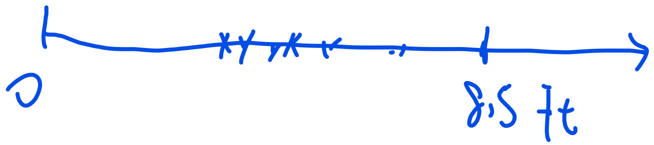
- Example (continuous):
  - The position of kids in a square playground.

(x,y) such that

$$-1 \leq x, y \leq 1$$

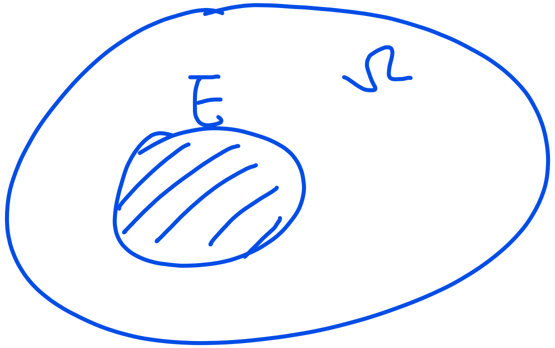


  - The heights of college students?



Event

- An event: A specific outcome or a set of outcomes from a random experiment.
- A subset of sample space



Simple Event: An event consisting of only one outcome.
E.g.: Rolling a die and getting a 3.

1, 2, 3, 4 Ω

Compound Event: An event consisting of multiple outcomes.
E.g. :

6-side

- Rolling a die and getting an even number (2, 4, or 6).
- The sum of two rolls of four sized dies is below 4.
- In a sequence of 10 flips of a coin, the total number of heads is greater than or equals 5.

1, 2, 3, 4, 5 Ω

(H, H, H, H, T, H, H, T T, T)

(T H H H H H T T T T)

Probability

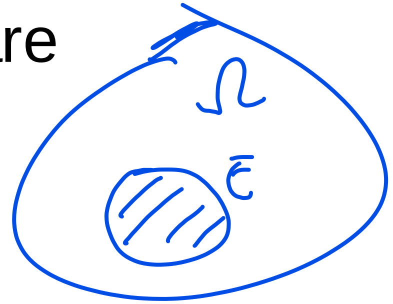
- In any random experiment, there is always uncertainty whether a particular event will occur or not occur.
- Probability is a measure of chance and assigned between 0 to 1.

Example:

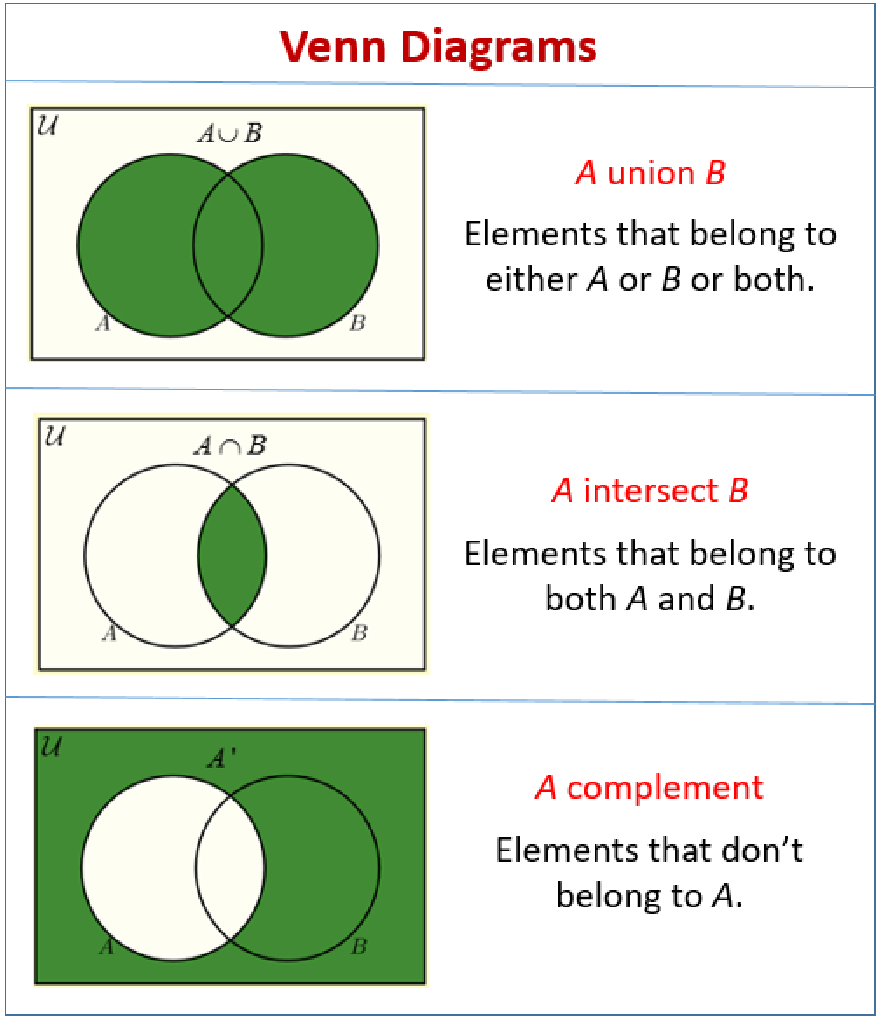
- Consider two rolls of the four-sided die, assuming each face shows up with ¼ probability.
- Let X be the value of the first roll and Y be the value of the second roll, and Z= min(X,Y), calculate
  - $P(X = 1) = \frac{1}{4}$
  - $P(Y = 1) = \frac{1}{4}$
  - $P(Z = 1) = P(X=1 \vee Y=1)$

A quick review of set operations

- Events in probability are subsets of a sample space,



- Set operations helps us calculate probabilities of combined events and understand their relationships.



Set operations on events

logic operator:

$\vee$ or

$\wedge$ and

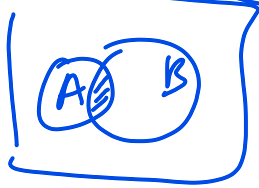
- An event: A subset of sample space.

Given S a sample space, and two events A and B, then

- $A \cup B$ is the event for "either A or B or Both" - The **union** of A and B.

  $P(Z = \min(X, Y) = 1) = P(X=1 \overset{\text{"or"}}{\vee} Y=1)$

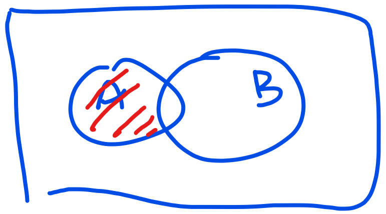
- $A \cap B$ is the event for "both A and B" - the **intersection** of A and B.



- A^c is the event for "not A", which is the **complement** of A.

  A not occur

- $A - B = A \cap B^c$ is the event for "A but not B".



Axioms of probability

- **Axioms**: a minimal set of rules that the probability P must obey.

for all subset A of Ω, (for all events)

- $\forall A \subseteq \Omega,\ P(A) \geq 0.$
- $P(\Omega) = 1$

Other axiom

- **Probability of the Empty Set**:

$$P(\emptyset) = 0$$

proof: $\underbrace{P(\Omega)}$ = 1

$P(\Omega \cup \phi) = 1 \Rightarrow P(\Omega) = 1 \Rightarrow P(\phi) = 0$

- **Probability of Complement**:

$$P(A^c) = 1 - P(A)$$

$A^c \cup A = \Omega$

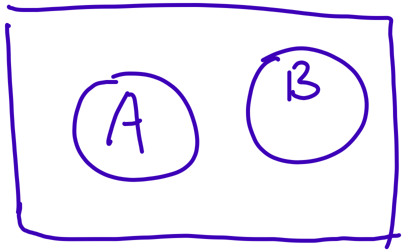
$P(A^c) = P(\Omega) - P(A) = 1 - P(A)$

- **Bounded Probability**: for any event A

$$0 \leq P(A) \leq 1$$

Additivity axiom

- **Mutually exclusive events** are events that cannot occur at the same time. If one event occurs, the other cannot.
- The intersection of two mutually exclusive events is empty.



- **Formally, if event A and B are mutually exclusive, then**

$$P(A \cup B) = P(A) + P(B)$$

For n mutually exclusive events: $A_1, A_2, \ldots, A_n$

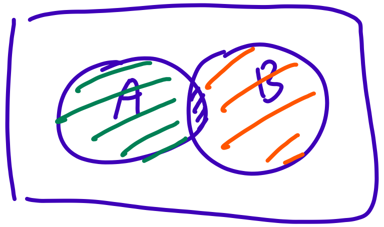
$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Eg. The probability of getting an even~~t~~ number for 4-sided die:

$P(\text{even number}) = P(N_2 \vee N_4) = P(N_2) + P(N_4) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Union of Events

- The union of two events A and B represents all outcomes that belong to either A, B, or both.
- Probability Formula:



$$P(A \cup B) = \underbrace{P(A)} + \underbrace{P(B)} - P(A \cap B)$$

$X=1 \vee Y=1$

$G + B + 0 + B - B = G + 0 + B$

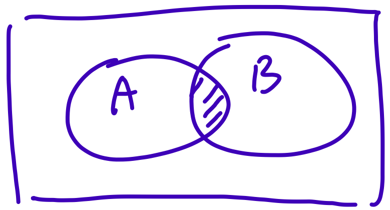
- Pictorially,
- E.g.: The probability of rolling a die and getting either an even number or a number greater than 3.

$P(\text{Even} \vee \geq 3) = P(\text{Even}) + P(\geq 3) - P(\text{even} \wedge \geq 3)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$

Intersection of Events

- The intersection of two events A and B represents all outcomes that belong to both A and B.



- Only when A and B are independent: The occurrence of one event does not affect the probability of the occurrence of another event.

$$P(A \cap B) = P(A) \cdot P(B)$$

- If two events are not independent, then we need to calculate the intersection without any formula. (Bayes thm)

Example: The probability of getting a 1 on the first roll and then a 4 on the second roll for a 4-sided die.

$$P(X=1 \wedge Y=4) = P(X=1)\,P(Y=4) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

More consequence of the axioms:

- If $A \subseteq B$, then $P(A) \leq P(B)$.

- $\cancel{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$

- $\underbrace{P(A \cup B)} \leq P(A) + P(B)$

$$P(A) + P(B) - \underbrace{P(A \cap B)}_{\geq 0} \leq P(A) + P(B)$$

Probability calculation

Steps

- Identify the sample space
- Specify the probability law P
- Specify the event A.
- Calculate P(A).

Example

- Consider the problem of rolling a six-faced die, which is a fair die.
- Consider the event A =“even number occurs”
- What is the probability of event A?

Consider a random experiment that has K possible outcomes

n: number of experiments to perform.

1, 2, 3
4, 5, 6

Let $N_k(n) \equiv$ the number of times the outcome is k and let the relative frequency of outcome k be

$$r_k(n) = \frac{N_k(n)}{n}$$

Probability as a Measure of Frequency of Occurrence

$$\lim_{n \to \infty} r_k(n) = p_k$$

is called the *probability of outcome* k.

Jupiter notebook

Interpretation of probability as relative frequency

- In the experiments we have conducted, the relative frequencies converge to some constant values when the number of simulations increases.

"the relative frequencies converge to the probabilities"

Example

- Consider rolling a 6-sided die and let the event E= “even number occurs”.

What can we say about the number of times E is observed in n trials?

$$N_E(n) = N_2(n) + N_4(n) + N_6(n)$$

What can we say about the probability of event E?