Lecture 2: Conditional Probability, Total Probability

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EEL 3850 S25

Discrete sequential model

- A sequential model is a type of experiment that has an inherent sequential character.

Example:

- Flipping a coin 3 times
- Receiving eight successive digits at a communication receiver
- Observing the value of a stock on five successive days

Discrete sequential model

- Consider the experiment where we roll a 6-sided fair die 2 times and the event E ≡ observing a 1 or 2 on either roll.
  - What is the sample space?
  - What are the subset of sample space for event E?
  - What is the probability of event E?

Discrete sequential model

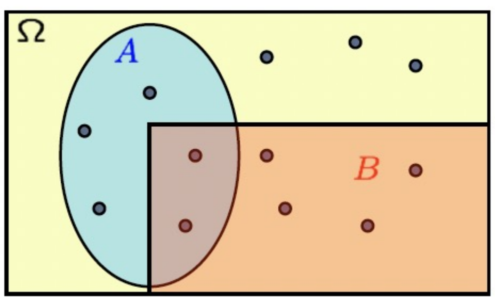
- Consider the experiment where we roll a 6-sided fair die 2 times and the event E ≡ at least one roll is 4.
  - What is the probability of event E?

Conditioning

Assume 12 equally likely events, without any information, what is the probability of event A? Event B?

P(A)=

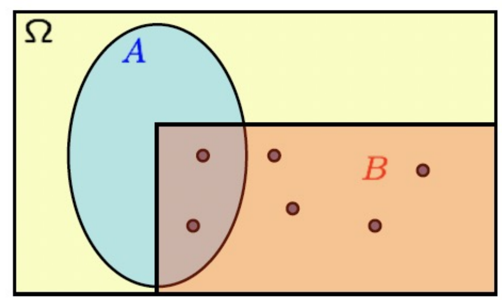
P(B)=



Received information that B occurred, what is the probability of event A given this information?

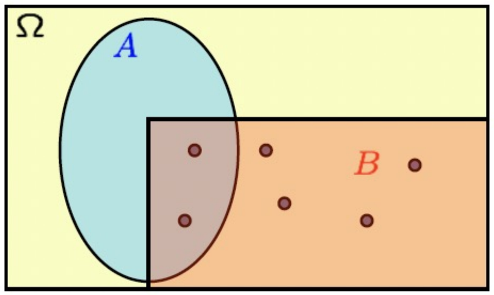
P(A|B)=

P(B|B)=



Definition of conditional probability

- Notation: The probability of A given B: P(A|B)



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

only defined when $P(B) > 0$.

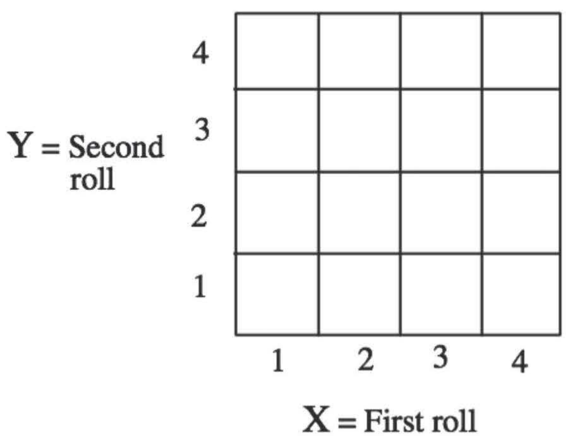
Special cases:

- $A \cap B = \emptyset$: The two events are exclusive.
- $B \subseteq A$

Example

- Consider roll of 4-sided die twice. If I tell you the minimal number of two rolls is a 2, what is the probability of that the maximal number of the two rolls is 1 (or 2, or 3, or 4)?

Determine the conditional probability $P(A|B)$ where
$A = \max(X, Y) = m$, $B = \{\min(X, Y) = 2\}$, where $m = 1, 2, 3, 4$.



Example

- A computer lab contains:
  - two computers from manufacturer A, one of which is defective
  - three computers from manufacturer B, two of which are defective

  A user sits down at a computer at random. Let the properties of the computer she sits down at be denoted by a two letter code, where the first letter is the manufacturer and the second letter is D for a defective computer or N for a non-defective computer.

Let E_A be the event that the selected computer is from manufacturer A
E_B be the event that the selected computer is from manufacturer B
E_D be the event that the selected computer is defective
Let's find

$$P(E_A) =$$

$$P(E_B) =$$

$$P(E_D) =$$

Example

- I tell you the computer is from manufacturer A. Then what is the probability that it is defective?

$$P(E_D|E_A) =$$

$$P(E_D|E_B) =$$

Example

- I tell you the computer is defective. Then what is the probability that it is from company A?

$$P(E_A|E_D) =$$

$$P(E_B|E_D) =$$

Chain Rule

- Using Conditional Probability to Decompose Events

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

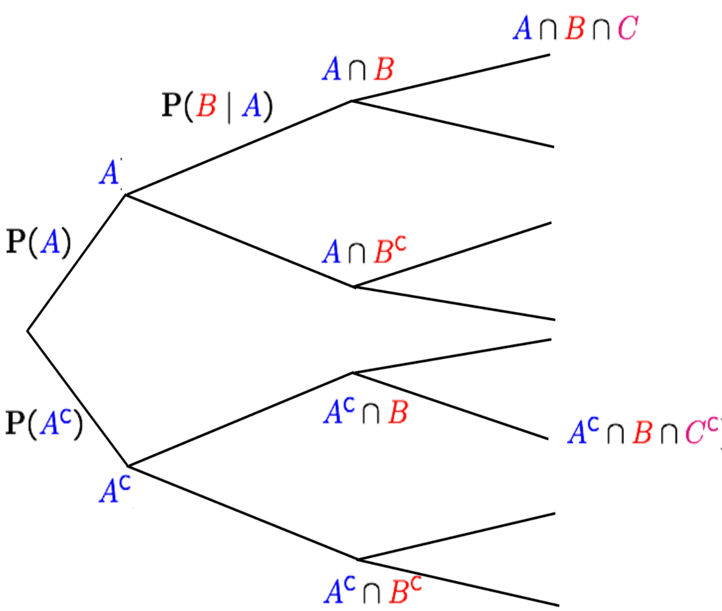
and

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(B|A)P(A)$$

Chain Rule

- Generalization



$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$
$$= P(A) \cdot \frac{P(A \cap B)}{P(A)} \cdot \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

Similarly,

$$P(A \cap B \cap C) = P(A|B \cap C)P(B|C)P(C)$$

Multiplication Rule

- Assuming that all of the conditioning events have positive probability, we have

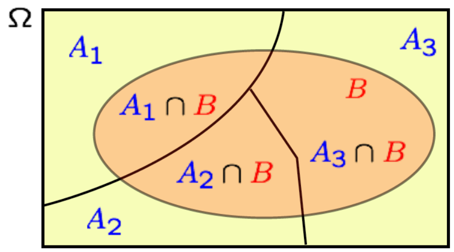
$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\dots P\left(A_n|\cap_{i=1}^{n-1} A_i\right)$$

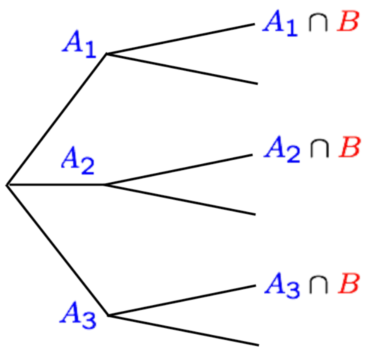
Example:

Three cards are drawn from a 52-card deck without replacement, what is the probability that none of the three cards is a heart? Recall that there is 13 hearts in a 52-card deck.

Hint: Define event $A_i = \{\text{the } i\text{-th card is not a heart}\},\ i = 1, 2, 3.$

Total Probability Theorem (the Law of Total Probability)



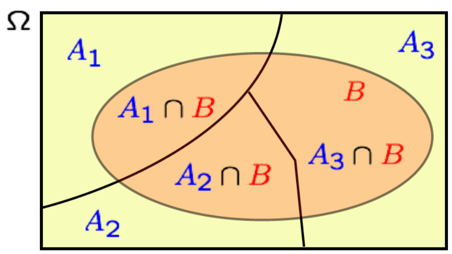


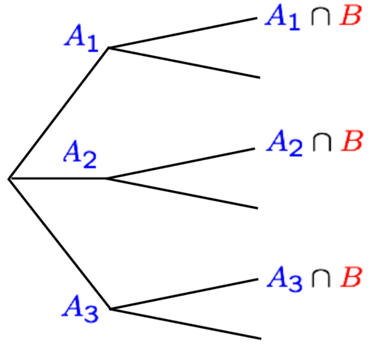
- A set of events partitioning the sample space.
  - Having $P(A_i), i = 1, 2, 3.$
  - Having $P(B|A_i),\ i = 1, 2, 3.$

$P(B)$?

Total Probability Theorem (the Law of Total Probability)

- A set of events partitioning the sample space.





- Having $P(A_i), i = 1, 2, 3.$
- Having $P(B|A_i),\ i = 1, 2, 3.$

$P(B)$?

Total Probability Law:
if the set of events $\{A_i\}$ partitions Ω, then

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

Example

- A magician has two coins, one fair and one 2-headed coin. Consider the experiment where she picks one coin at random and flips it i times.

H_i: the event that the outcome of flip i is heads. find:

- $P(H_1)$
- $P(H_1 \cap H_2)$

What is the conditioning event?

Example (exercise or in class if time permits)

- Suppose you have an urn containing **7 red and 3 blue balls.** You draw three balls at random. On each draw, if the ball is red you set it aside and if the ball is blue you put it back in the urn. What is the probability that the second draw is blue?