# Lecture 3: Bayes Theorem and Inference

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# Outline

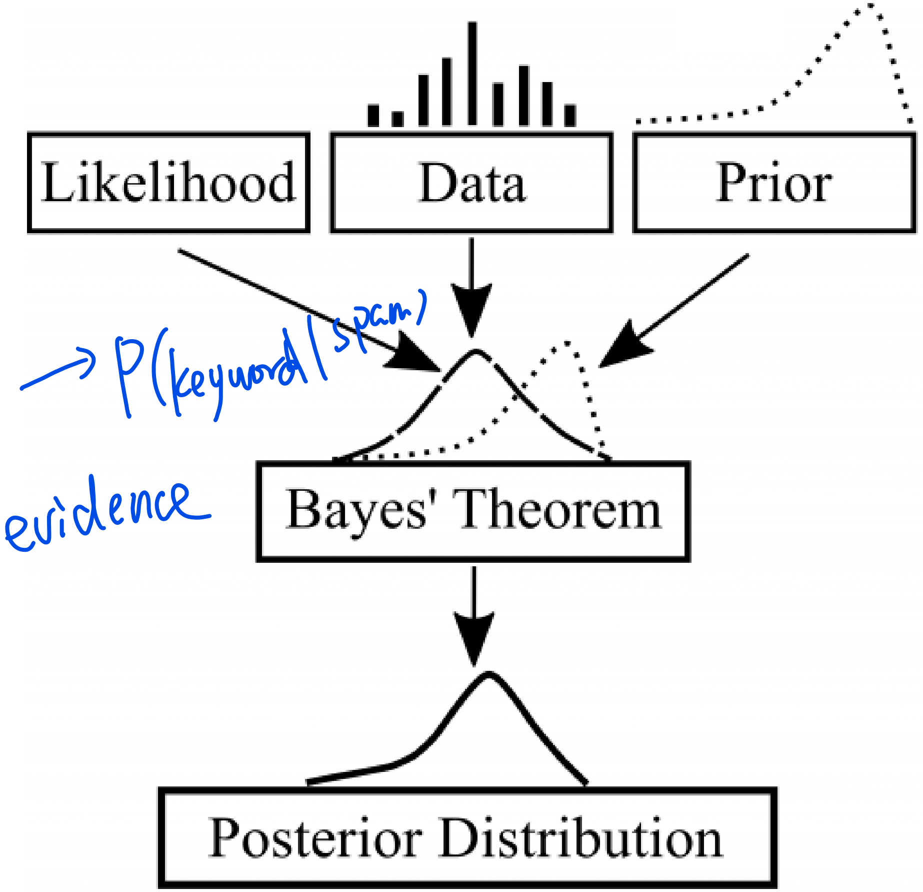
- Bayes Theorem
- Statistical Independence

# Related applications

- **Disease Diagnosis**: • Detecting cancer based on medical imaging results. • Estimating the probability of infection based on symptoms and diagnostic test outcomes.
- **Spam Filtering**: Bayesian spam filters calculate the probability of an email being spam based on keywords and patterns. $P(\text{spam} \mid \text{keywords})$
- **Bayesian Neural Networks**: Incorporating uncertainty in predictions by treating weights as distributions instead of fixed values.
- **Sensor Fusion**: Combining data from multiple sensors (e.g., GPS and IMU) to improve state estimation.

evidence

$P(\text{keyword} \mid \text{spam})$



# A motivating problem

- A magician has two coins, one fair and one 2-headed coin. Consider the experiment where she picks one coin at random and flips it once, the output is a head, what is the probability that the coin is a fair coin?

- Initial belief $P(A_i)$ for possible cause of an observed event $B$.

  e.g. Both coins are equally likely —- a prior knowledge:

  $= A_1$: F. $A_2$: 2H

- probability of the observation under each $A_i$:

  e.g. Probability of a head under each coin.

$$A_i \xrightarrow[\mathbf{P}(B \mid A_i)]{\text{model}} B \rightarrow$$

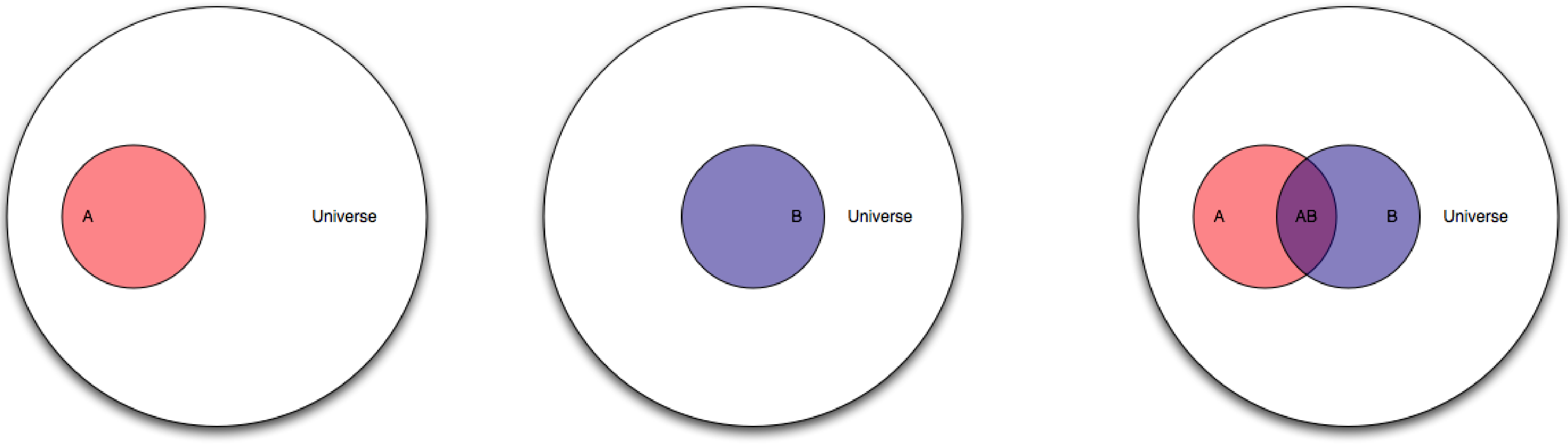
observed event "Heads"

- Draw conclusion about the cause given the observed event.

  e.g. infer if the coin is fair or biased.

$$B \xrightarrow[\mathbf{P}(A_i \mid B)]{\text{inference}} A_i$$

# Venn diagram visualization




$$A_i \xrightarrow[\mathbf{P}(B \mid A_i)]{\text{model}} B$$

# Bayes Theorem

Consider two events $A$ and $B$, by the chain rule:

$$P(A \cap B) = P(A|B)P(B)$$

and

$$P(B \cap A) = P(B|A)P(A)$$

Note that

observed

$$P(A \cap B) = P(B \cap A)$$

$$\underbrace{P(A|B)}_{\text{infer model form observation:}} \underbrace{P(B)}_{\text{event B occurs.}} = \underbrace{P(B|A)}_{\text{prediction based on model: A}} \underbrace{P(A)}_{\text{prior prob of selecting the model A.}}$$

# Bayes Theorem

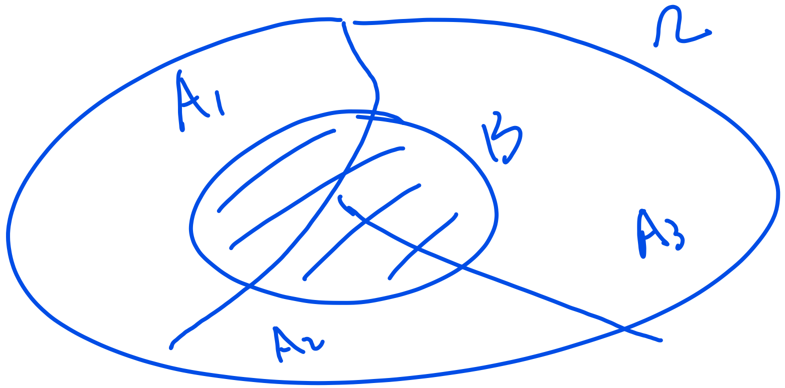
If the set of events $\{A_i\}_{i=1}^n$ partitions the sample space $\Omega$, and assuming $P(A_i) > 0$, for all $i$. Then, for any event $B$ such that $P(B) > 0$, we have

infer

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

where $P(B)$ can be computed using the Law of Total Probability,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)$$




$$P(A_i|B) \propto P(B|A_i)P(A_2)$$

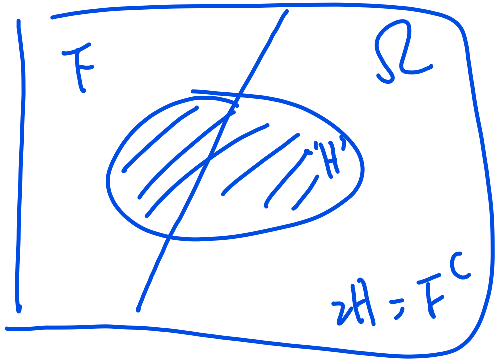
"proportional to"

# Example

$P(F) = \frac{1}{2} \xrightarrow{\text{observed } H} P(F|H) = \frac{1}{3}$

initial belief — current belief.

- A magician has two coins, one fair and one 2-headed coin. Consider the experiment where she picks one coin at random and flips it once, the output is a head, what is the probability that the coin is a fair coin?




$\{A_1 \dots A_n\}$ partition

observed event B :

$$P(F|H) = \frac{P(H|F)\,P(F)}{P(H)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$$

$A_1$

inference

$B \longrightarrow A_i$

$P(A_i|B)$

$$P(H) = P(H|F)\,P(F) + P(H|2H)\,P(2H)$$

$$= \frac{3}{4}$$

? if observed another head:

$$P(F \mid 2\text{nd head}) = \frac{P(H|F) \cdot \hat{P}(F)}{\hat{P}(H)} = \frac{\frac{1}{2} \times \hat{p}(F)}{\hat{P}(H)}$$

# Example

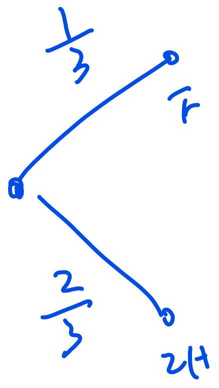
$$P(H_2 | H_1) = P(H_2 | F)\, P_1(F) + P(H_2 | 2H)\, P_1(2H)$$

belief after 1 heads.

$$= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{5}{6}.$$

- A magician has two coins, one fair and one 2-headed coin. Consider the experiment where she picks one coin at random and flips it once, the output is a head, what is the probability that the next second flip is a head?

$$P(F \mid 2\text{nd Head}) = \frac{\frac{1}{2} \times \frac{1}{3}}{\hat{P}(H)} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times 1} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{4}{6}} = \frac{1}{5}$$

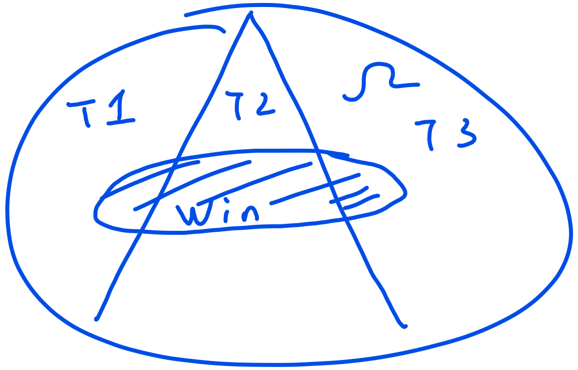



| | | | |
|---|---|---|---|
| t: | 0 | 1 | 2 |
| | 0 | Head | Head |
| $P_t(F)$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{5}$ |

# Some terminologies

- $P(A_i|B)$ as the **posterior probability** of event $A_i$ given the information
- $P(A_i)$ as the **prior probability**
- $P(B|A_i)$ as the **likelihood**
- $P(B)$ as the **evidence/effect probability**

# Example




- Three types of players.
  - Type 1: 50%
  - Type 2: 25%
  - Type 3: 25%
- You winning probability with these players:
  - Against type 1: 0.3.
  - Against type 2: 0.4.
  - Against type 3: 0.5.
- Now you play a game with a randomly chosen player.
- *Question*: What's your winning probability?

$$P(W) = P(W|T_1)P(T_1) + P(W|T_2)P(T_2) + P(W|T_3)P(T_3)$$

$$= 0.3\times 0.5 + 0.4\times 0.25 + 0.5\times 0.25.$$

observed event. $W \xrightarrow{\text{infer}}$ Type 1.

prior: Types. Win. against type. $\xrightarrow[\text{win}]{\text{infer}}$ ? $P(T_1|W)$

$$P(T_1|W) = \frac{P(W|T_1) \cdot P(T_1)}{P(W)} = \frac{0.3 \times 0.5}{P(W)}$$

posterior

? $P(T_1|W) + P(T_2|W) + P(T_3|W)$

# Example

- Suppose that you win. What is the probability that you had an opponent of type 1?

# Example

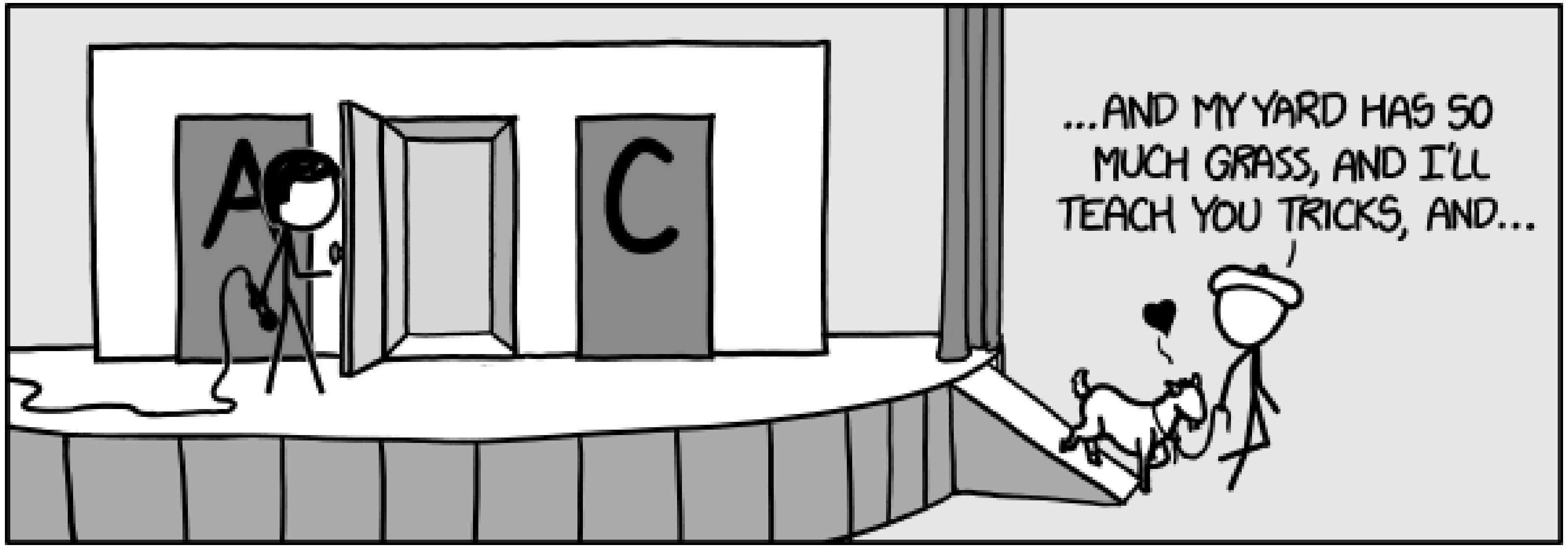
## Example: Diagnosis

- A random person drawn from a certain population has probability 0.001 of having a certain disease.
- The test satisfies
  - $\Pr[\text{test positive} \mid \text{disease}] = 0.95$
  - $\Pr[\text{test negative} \mid \text{no disease}] = 0.95$
- *Question*: Given that the person just tested positive, what is the probability of having the disease?

# Monty Hall Problem

- Suppose you're on a game show, and you're given the choice of three doors:
- behind one door is a car
- behind the other doors are goats

You pick a door, and the host, who knows what's behind the doors, opens another door, which he knows has a goat. The host then offers you the option to switch doors. Does it matter if you switch?



# Monty hall problem

- Let $W_i$ be the event of winning a car on the i-th choice.
- Consider three strategies:
  - Never switch.
  - Always switch.
  - Flip a coin, if heads, switch, if tail, no switch.

# Independence

In general, for two events $A$ and $B$, when $P(A|B) = P(A)$, we say that $A$ is statistically independent (s.i.) of $B$, since the probabilities are not affected by knowledge of $B$ having occurred.

* By the chain rule, if $A$ is independent of B:

$$P(A \cap B) = P(A|B)P(B) =$$

Events $A$ and $B$ are **statistically independent (s.i.)** if and only if (iff)

$$P(A \cap B) = P(A)P(B)$$

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# Independence

- If A is independent of B, then B is also independent of A.
- Why?

If $A$ and $B$ are s.i. events, then the following pairs of events are also s.i.:
* $A$ and $\overline{B}$
* $\overline{A}$ and $B$
* $\overline{A}$ and $\overline{B}$

# Conditional independence

given an event $C$, the events $A$ and $B$ are called *conditionally independent* if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

Consider two independent coin tosses. Let

$$H_1 := \{\text{ 1st toss is a head}\}$$

$$H_2 := \{\text{ 2nd toss is a head}\}$$

$$D := \{\text{ two tosses have different results}\}$$

Compare $P(H_1 \cap H_2|D)$ and $P(H_1 \cap H_2)$

# Example