Problem Set of Lecture 2

1. Drawing the figure of EMPIRICAL COPULA.

Let the joint distribution of (X, Y) be $F(\cdot,\ \cdot)$, and the corresponding copula is $C(u, v)$. The marginal distribution functions of X and Y are $F_1(\cdot)$ and $F_2(\cdot)$, respectively.

Let $((X_1,\ Y_1), (X_2,\ Y_2), \ldots, (X_n,\ Y_n))$ be the random sample of the population (X, Y). The joint empirical distribution functions of (X, Y) is shown as follows:

$$F_n(x, y) = \frac{1}{n}\sum_{k=1}^{n} I\{X_k \le x, Y_k \le y\}, \forall (x, y) \in R^2 \tag{1}$$

where $I(\cdot)$ is an indicative function.

The marginal empirical distribution functions of X and Y are shown as Eq (2) and Eq (3).

$$F_{n1}(x) = \frac{1}{n}\sum_{k=1}^{n} I\{X_k \le x\} \tag{2}$$

$$F_{n2}(y) = \frac{1}{n}\sum_{k=1}^{n} I\{Y_k \le y\} \tag{3}$$

The empirical copula of (X, Y) is expressed as follows:

$$C_n(u, v) = F_n\left(F_{n1}^{-1}(u), F_{n2}^{-1}(v)\right)$$

$$0 \le u, v \le 1 \tag{4}$$

where

$$F_{n1}^{-1}(u) = \inf\{x: F_{n1}(x) \ge u\} \tag{5}$$

$$F_{n1}^{-1}(v) = \inf\{y: F_{n2}(y) \ge v\} \tag{6}$$

I calculate the daily return of the two stocks in attachment and draw the figure of the empirical copula.

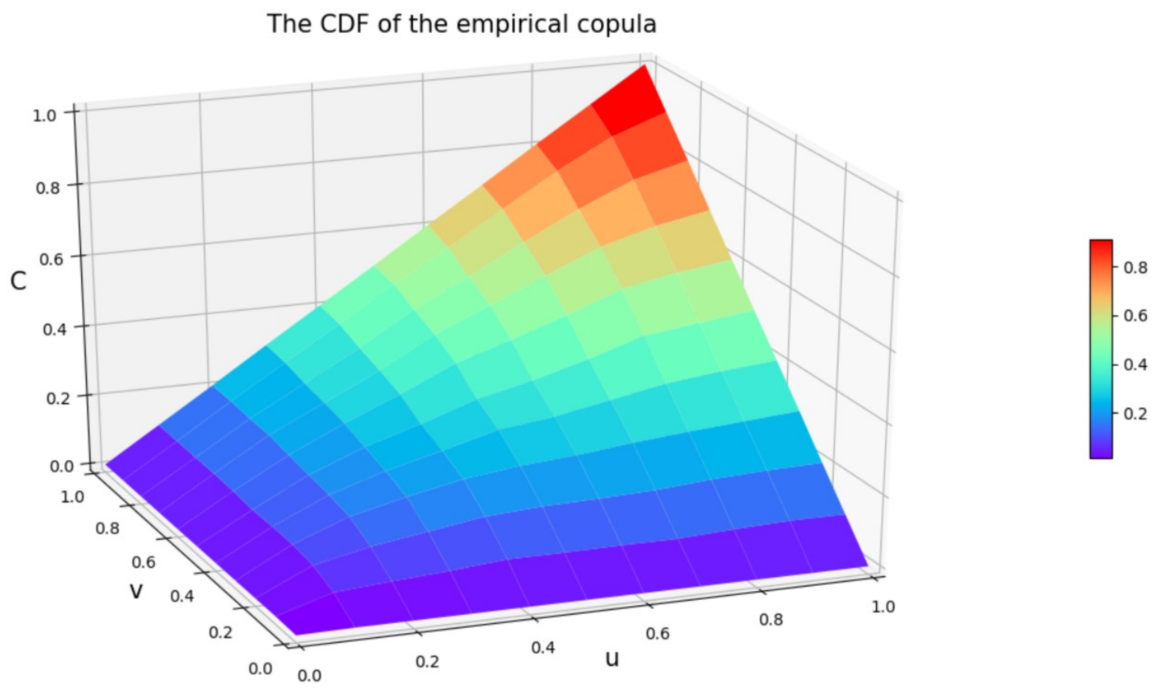

Figure 1 The cdf of the empirical copula

2. By means of REDRAW the figure shows in page 36 of Lecture 2, prove that the copula of (X_1, X_2) is the same as the copula of (C_1, X_2).

The regularized payoff of the call option of Stock 1 could be calculated by the following equation:

$$C_{1;\,\epsilon} = \frac{(X_1 - K)}{2}\left(1 + \operatorname{erf}\left(\frac{X_1 - K}{\sqrt{2\epsilon^2}}\right)\right) + \frac{\epsilon}{\sqrt{2\pi}} e^{-\frac{(X_1 - K)^2}{2\epsilon^2}} \tag{7}$$

where ϵ is a small bandwidth tending to zero. This profile is smooth, strictly increasing in X_1.

The regularized pdf of the empirical distribution reads in terms of the smooth approximation of the Dirac delta as follows:

$$f_{i_T;\,\epsilon} \equiv \frac{1}{T}\sum_{t=1}^{T} \delta_\epsilon^{(X_t)} \tag{8}$$

$$\delta_\epsilon^{(y)}(x) \equiv \frac{1}{(2\pi)^{\frac{N}{2}}\epsilon^N} e^{-\frac{1}{2\epsilon^2}(x-y)'(x-y)} \tag{9}$$

The pdf of the copula reads:

$$f_U(u_1, u_2, \cdots, u_N) = \frac{f_X\left(Q_{X_1}(u_1), Q_{X_2}(u_2), \cdots, Q_{X_N}(u_N)\right)}{f_{X_1}\left(Q_{X_1}(u_1)\right)\cdots f_{X_N}\left(Q_{X_1}(u_N)\right)} \tag{10}$$

where Q_{X_n} is the quantile of the generic n-th marginal entry of X.

The pdf of the empirical copula of (X_1, X_2) is shown as follows:

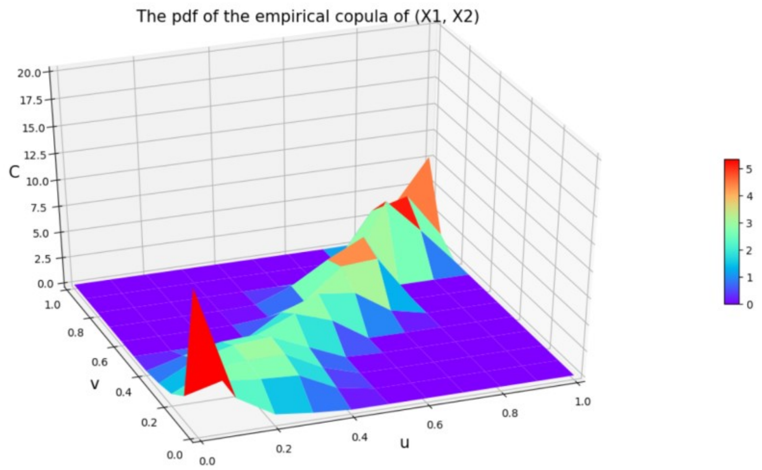

Figure 2 The pdf of the empirical copula of (X_1, X_2)

I set the strike K of the call option to 50 and draw the pdf of the empirical copula of (C_1, X_2).

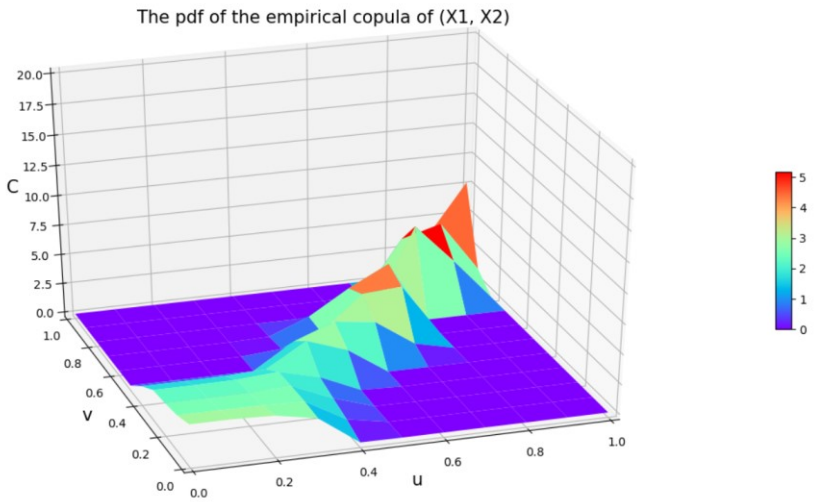

Figure 3 The pdf of the empirical copula of (C_1, X_2)

Theoretically, the pdfs of the empirical copula of (C_1, X_2) and (X_1, X_2) should be the same. From Figure 2 and Figure 3, we can observe that when $(u, v) \gg (0,0)$, the pdfs of the empirical copula of (C_1, X_2) and (X_1, X_2) are the same. However, when (u, v) approaching $(0,0)$, the two pdfs become different. **This is probably because the regularizations of $\boldsymbol{C_1}$ and the pdf of empirical distribution (Eq (7) and Eq (9)) have changed the co-monotonicity of $(\boldsymbol{C_1}, \boldsymbol{X_2})$ and $(\boldsymbol{X_1}, \boldsymbol{X_2})$.**

Appendix

```
import os
import numpy as np
import pandas as pd
import math
import matplotlib.pyplot as plt

#  导入数据

os.chdir(r"D:\量化金融风险管理\第二次作业")
df = pd.read_table("股价数据.txt", sep="\t", header=0, encoding="UTF-8", index_col=0)
stock = (np.asarray(df))
apple = (stock[:, 0]).tolist()
amd = (stock[:, 1]).tolist()

'''
如果想绘制（C1, X2）的 copula 图像，那么把文件“股价数据.txt” 中 apple 的日度股价数据按照 Eq(7) 替换为期权的日度价格即可。计算方法如下：
option = []

K = 50
w = 1

for i in apple:
    p = 0.5 * (i - K) * (1 + math.erf((i - K) / (2 * w ** 2) ** 0.5)) + w / (2 * math.pi) ** 0.5 * math.exp(
        -(i - K) ** 2 / (2 * w ** 2))
    option.append(p)
    p = 0
'''

#  定义联合经验分布函数
def cdf(x, y):
    n = len(stock)
    F = 0
    for i in stock:
        if i[0] <= x and i[1] <= y:
            F += 1
    F = F / n
    return F

#  定义边缘经验分布函数
def mcdf(x, data):
```

```
    n = len(stock)
    F = 0
    for i in data:
        if i <= x:
            F += 1
    F = F / n
    return F

# the Python code of Eq (5) and (6)
def invmcdf(u, data):
    F = 1000000
    for i in data:
        if mcdf(i, data) >= u and i < F:
            F = i
    return F

# 定义 empirical copula 的 CDF
def copula(u, v):
    C = cdf(invmcdf(u, apple), invmcdf(v, amd))
    return C

# 定义一维的 Dirac delta，用于平滑一元经验分布的 pdf
def dirac(x, x0):
    band = 3
    d = math.exp(-1 / (2 * band ** 2) * (x - x0) ** 2) / ((2 * math.pi) ** 0.5 * band)
    return d

# 定义二维的 Dirac delta，用于平滑二元经验分布的 pdf
def twodirac(x, y, x0, y0):
    band = 3
    d = math.exp(-1 / (2 * band ** 2) * ((x - x0) ** 2 + (y - y0) ** 2)) / ((2 * math.pi) ** 1 * band ** 2)
    return d

# 定义二元经验分布的 pdf
def pdf(x, y):
    f = 0
    for i in stock:
        f += twodirac(x, y, i[0], i[1])
    f = f / len(stock)
    return f

# 定义一元（边缘）经验分布的 pdf
def mpdf(x, data):
    f = 0
    for i in data:
        f += dirac(x, i)
    f = f / len(stock)
    return f
```

```python
# 定义 empirical copula 的 pdf
def pcopula(u, v):
    p = pdf(invmcdf(u, apple), invmcdf(v, amd)) / (mpdf(invmcdf(u, apple), apple) * mpdf(invmcdf(v, amd), amd))
    return p

# 作图
u = np.arange(0, 1.1, 0.1)
X, Y = np.meshgrid(u, u)

Z = []
for i in range(0, 11):
    Z.append([])
    for j in range(0, 11):
        Z[i].append(pcopula(X[i][j], Y[i][j]))

Z = np.asarray(Z)
fig = plt.figure(figsize=(20, 20))
ax = fig.add_subplot(111, projection='3d')
ax.set_xlabel('u', fontsize=15)
ax.set_ylabel('v', fontsize=15)
ax.set_zlabel('C', fontsize=15)
ax.set_xlim(0, 1)   # X 轴，横向向右方向
ax.set_ylim(0, 1)   # Y 轴,左向与 X,Z 轴互为垂直
ax.set_zlim(0, 20)   # 竖向为 Z 轴
surf = ax.plot_surface(X, Y, Z, cmap='rainbow')
fig.colorbar(surf, shrink=0.3, aspect=10)
plt.title('The pdf of the empirical copula of (X1, X2)', fontsize=15)
plt.show()
```