AN EFFICIENT CLOSED FORM METHOD FOR ROTATION FITTING OF POINT CLOUD DATA

F. NAETS
Department of Mechanical Engineering, Celestijnenlaan 300B, B-3001 Heverlee (Leuven), Belgium

1. INTRODUCTION

In a wide range of applications, it is important to transform a set of points back as close as possible to an initial reference. This typically requires two steps:

- translation to a common center;
- rotation to minimize the remaining displacement.

Moreover, in many applications one also needs the derivatives of these transformations with respect to the relevant degrees-of-freedom.

Many approaches exist to tackle finding the rotational transformation:

- rotation by three or four predefined points
- optimize full rotation matrix - multiple methods (e.g. [1], but this methods suffers from a singularity in an undistorted unrotated configuration)
- ...

In this work I present an approach which uses the rotation matrix based on Euler parameters (quaternions) to obtain the optimal rotational transformation. It can be shown that this optimization problem reduces to finding the largest eigenvalue and corresponding eigenvector of a four-by-four matrix. This also makes it particularly straightforward to obtain the derivatives of the rotation with respect to the point degrees-of-freedom.

2. REFERENCE TRANSLATION

In this work we propose to determine the reference translation $\mathbf{r} \in \mathbb{R}^3$ simply as the average displacement of the different points:

$$\mathbf{r} = \frac{1}{2}\sum_{i=1}^{n_p} \mathbf{x}_i - \mathbf{x}_i^0, \tag{1}$$

with $\mathbf{x}_i \in \mathbb{R}^3$ the current coordinates of a point i, $\mathbf{x}_i^0 \in \mathbb{R}^3$ the reference coordinates of a point i, and n_p the number of points. $\mathbf{R}$ is the desired rotation matrix and $\mathbf{I}$ is a 3×3 unity matrix.

3. OPTIMAL ROTATION DERIVATION

In this work we aim to solve the optimization problem:

$$min_{\mathbf{R}\in\mathbb{R}^{3\times 3}} \sum_{i=1}^{n_p} |\mathbf{R}\mathbf{x}_i - \mathbf{x}_i^0|_2 \qquad \text{s.t. } \mathbf{R}^T\mathbf{R} = \mathbf{I}, \tag{2}$$

with $\mathbf{x}_i \in \mathbb{R}^3$ the current coordinates of a point i[1], $\mathbf{x}_i^0 \in \mathbb{R}^3$ the reference coordinates of a point i, and n_p the number of points. In the first subsection I discuss how this optimization can be further simplified by accounting for the orthogonality of the rotation matrix. In section 3.2 I then introduce the Euler parameters

[1]We assume these coordinates are already corrected for the reference translation

to describe the rotation and how this allows to reduce the optimization to a simple eigenvalue problem. Finally the calculation of the rotational derivatives with respect to the point coordinates is considered in section 3.3.

3.1. **Rotation optimization function.** The goal function from Eq. (2) can be expanded into:

$$\sum_{i=1}^{n_p} |\mathbf{R}\mathbf{x}_i - \mathbf{x}_i^0|_2 = \sum_{i=1}^{n_p} \left(\mathbf{R}\mathbf{x}_i - \mathbf{x}_i^0\right)^T \left(\mathbf{R}\mathbf{x}_i - \mathbf{x}_i^0\right) \tag{3}$$

$$= \sum_{i=1}^{n_p} \mathbf{x}_i^T \mathbf{R}^T \mathbf{R}\mathbf{x}_i - 2\left(\mathbf{x}_i^0\right)^T \mathbf{R}\mathbf{x}_i + \left(\mathbf{x}_i^0\right)^T \mathbf{x}_i^0 \tag{4}$$

$$= \sum_{i=1}^{n_p} -2\left(\mathbf{x}_i^0\right)^T \mathbf{R}\mathbf{x}_i + \mathbf{x}_i^T \mathbf{x}_i + \left(\mathbf{x}_i^0\right)^T \mathbf{x}_i^0. \tag{5}$$

In order to obtain we exploited the fact that the rotation matrix $\mathbf{R}$ has to be orthonormal. From the perspective of finding the optimal rotation, we can omit the terms independent of the rotation matrix in this goal function without influencing the result. We can therefore rewrite Eq. (2) as:

$$min_{\mathbf{R}\in\mathbb{R}^{3\times3}} \sum_{i=1}^{n_p} -2\left(\mathbf{x}_i^0\right)^T \mathbf{R}\mathbf{x}_i \qquad \text{s.t. } \mathbf{R}^T\mathbf{R} = \mathbf{I}. \tag{6}$$

However, the fact that this problem is linear in the rotation matrix and redudantly constrained makes it relatively challenging to solve. We therefore introduce the use of Euler parameters to describe the rotation in the following section.

3.2. **Euler parameters in rotation optimization.** Euler parameters $\mathbf{p} \in \mathbb{R}^4$ are a type of quaternion. This implies that $\mathbf{p}^T\mathbf{p} = 1$. For convenience we write:

$$\mathbf{p} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \tag{7}$$

and the rotation matrix as a function of these Euler parameters can be written as:

$$\mathbf{R}(\mathbf{p}) = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{bmatrix}, \tag{8}$$

which can also be written as:

$$\mathbf{R}(\mathbf{p}) = \sum_{k=1}^{4}\sum_{l=1}^{4} \frac{\partial^2\mathbf{R}}{\partial\mathbf{p}_k\partial\mathbf{p}_l}\mathbf{p}_k\mathbf{p}_l = \sum_{k=1}^{4}\sum_{l=1}^{4} \mathbf{S}_{kl}\mathbf{p}_k\mathbf{p}_l, \tag{9}$$

with $\mathbf{S}_{kl}$ a set of constant three-by-three matrices. With this rotational parameterization, the optimization problem from Eq. (6) can be rewritten as:

$$min_{\mathbf{p}\in\mathbb{R}^4} \sum_{i=1}^{n_p} -2\left(\mathbf{x}_i^0\right)^T \mathbf{R}(\mathbf{p})\mathbf{x}_i \qquad \text{s.t. } \mathbf{p}^T\mathbf{p} = 1. \tag{10}$$

With the rotation matrix as defined in Eq. (8) this becomes a quadratic minimization problem with a single constraint.

We convert this problem into a single minimization problem through the addition of a Lagrange multiplier λ:

$$min_{\mathbf{p}\in\mathbb{R}^4,\lambda\in\mathbb{R}} \sum_{i=1}^{n_p} \left(-2\left(\mathbf{x}_i^0\right)^T \mathbf{R}(\mathbf{p})\mathbf{x}_i\right) - \left(\mathbf{p}^T\mathbf{p} - 1\right)\lambda. \tag{11}$$

We find the minimizers of this problem by solving for the KKT conditions:

$$\begin{aligned}\sum_{i=1}^{n_p}\left(-2\left(\mathbf{x}_i^0\right)^T\frac{\partial\mathbf{R}(\mathbf{p})}{\partial\mathbf{p}_k}\mathbf{x}_i\right)+2\mathbf{p}_k\lambda &= 0 \qquad (12)\\ \mathbf{p}^T\mathbf{p} &= 1. \qquad (13)\end{aligned}$$

which is further simplified into:

$$\begin{aligned}\sum_{i=1}^{n_p}\left(\left(\mathbf{x}_i^0\right)^T\frac{\partial\mathbf{R}(\mathbf{p})}{\partial\mathbf{p}_k}\mathbf{x}_i\right)-\mathbf{p}_k\lambda &= 0 \qquad (14)\\ \mathbf{p}^T\mathbf{p} &= 1. \qquad (15)\end{aligned}$$

Together with Eq. (9), Eq. (14) can be rewritten as:

$$\mathbf{A}\mathbf{p}-\mathbf{p}\lambda=\mathbf{0}, \qquad (16)$$

where

$$\mathbf{A}_{kl}=\sum_{i=1}^{n_p}\left(\mathbf{x}_i^0\right)^T\mathbf{S}_{kl}\mathbf{R}(\mathbf{p})\mathbf{x}_i. \qquad (17)$$

It is now easy to see that Eq. (16) together with the normalization constraints on $\mathbf{p}$ is a simple four-by-four eigenvalue problem. There are four potential solutions for $\mathbf{p}$ with corresponding Lagrange multipliers. Next we will consider which of these solution is the one to choose as the global minimizer. Using the $\mathbf{A}$ matrix, we can reformulate Eq. (10) as:

$$min_{\mathbf{p}\in\mathbb{R}^4}-2\mathbf{p}^T\mathbf{A}\mathbf{p} \qquad \text{s.t. } \mathbf{p}^T\mathbf{p}=1, \qquad (18)$$

and by introducing the eigenvalue problem this becomes

$$min_{\mathbf{p}\in\mathbb{R}^4}-2\mathbf{p}^T\lambda\mathbf{p} \qquad \text{s.t. } \mathbf{p}^T\mathbf{p}=1, \qquad (19)$$

such that the eigenvalue solution with the largest positive eigenvalue is the global minimizer of the problem and therefore the desired solution.

The desired rotation matrix can then be easily determined based on the optimal Euler parameters with Eq. (8).

3.3. **Derivatives of rotation matrix.** In many applications (e.g. determining internal forces in a large motion problem) the derivatives of the rotation matrix with respect to the degrees-of-freedom associated to the point coordinates are required as well. As the problem is now formulated as an eigenvalue problem, this becomes a trivial exercise.

The derivative of an eigenmode and eigenvalue for a given variable v is obtained from solving:

$$\begin{bmatrix}\mathbf{A}-\mathbf{I}\lambda & -\mathbf{p}\\ \mathbf{p}^T & 0\end{bmatrix}\begin{bmatrix}\frac{\partial\mathbf{p}}{\partial v}\\ \frac{\partial\lambda}{\partial v}\end{bmatrix}=\begin{bmatrix}\left(-\frac{\partial\mathbf{A}}{\partial v}+\lambda\mathbf{I}\right)\mathbf{p}\\ 0\end{bmatrix}, \qquad (20)$$

where it is interesting to see that in Eq. (17) is a linear function of the point coordinates $\mathbf{x}$, such that $\frac{\partial A}{\partial x}$ is a constant matrix. Higher order derivatives are straightforward to compute by further differentiating Eq. (20).

4. Validation

In this section we show some results for randomly generated point clouds. The files to generate these results can be found on GitHub [2].

4.1. **Undistorted configuration.** As a first example we apply the proposed approach on a rotated point cloud on which no distortion is applied.

4.2. **Distorted configuration.** In this second example we add random noise to the point locations in order to verify how the rotation reconstruction varies with increasing distortion.

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REFERENCES

[1] www.people.csail.mit.edu/bkph/articles/Nearest_Orthonormal_Matrix.pdf. 2017

[2] Frank Naets - GitHub, github.com/FrankNaets/OptimalRotation. 2017