

# The harmonic mean of two independent Cauchy distributions is a Cauchy distribution

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Consider  $C_1 \sim \text{Cauchy}(l_1, s_1)$  and  $C_2 \sim \text{Cauchy}(l_2, s_2)$  two independent Cauchy distributions. Then their harmonic mean

$$C_{12} = \frac{1}{\frac{1}{2} \frac{1}{C_1} + \frac{1}{2} \frac{1}{C_2}} = \frac{2 C_1 C_2}{C_1 + C_2}$$

follows a Cauchy distribution. The proof is based on the following properties of Cauchy distributions:

- Let  $C \sim \text{Cauchy}(l, s)$  then  $\frac{1}{C} \sim \text{Cauchy}\left(\frac{l}{l^2+s^2}, \frac{s}{l^2+s^2}\right)$ .
- Let  $C \sim \text{Cauchy}(l, s)$  then  $\lambda C \sim \text{Cauchy}(l, \lambda C)$ .
- Let  $C_1 \sim \text{Cauchy}(l_1, s_1)$  and  $C_2 \sim \text{Cauchy}(l_2, s_2)$  be two independent Cauchy distributions. Then  $C_1 + C_2 \sim \text{Cauchy}(l_1 + l_2, s_1 + s_2)$ .

It follows that  $C_{12} \sim \text{Cauchy}(l_{12}, s_{12})$  with

$$l_{12} = 2 \frac{(l_1 s_2^2 + l_2 s_1^2 + l_1 l_2^2 + l_1^2 l_2)}{(l_1 + l_2)^2 + (s_1 + s_2)^2}, \quad s_{12} = 2 \frac{(s_1 s_2^2 + (s_1^2 + l_1^2) s_2 + l_2^2 s_1)}{(l_1 + l_2)^2 + (s_1 + s_2)^2}.$$

The following code below in R illustrates the result:

```
# install.packages("univariateML")
library("univariateML")
n <- 100000
l1 <- 1.5
s1 <- 1
l2 <- 2
s2 <- 3
x1 <- rcauchy(n, l1, s1)
x2 <- rcauchy(n, l2, s2)
h12 <- 2*x1*x2/(x1+x2)
mlcauchy(h12)
#l12
2*(l1*s2*s2+l2*s1*s1+l1*l2*l2+l1*l1*l2)/((s1+s2)*(s1+s2)+(l1+l2)*(l1+l2))
#s12
2*(s1*s2*s2+(s1*s1+l1*l1)*s2+l2*l2*s1)/((s1+s2)*(s1+s2)+(l1+l2)*(l1+l2))

%bibliographyNGDBib.bib
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