# The $\alpha$-representations of the Fisher Information Matrix
# — On gauge freedom of the FIM —

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The *Fisher Information Matrix* [1] (FIM) for a family of *parametric* probability models $\{p(x;\theta)\}_{\theta\in\Theta}$ (densities $p(x;\theta)$ expressed with respect to a positive base measure $\nu$) indexed by a $D$-dimensional parameter vector $\theta := (\theta^1, \dots, \theta^D)$ is historically defined by

$$I(\theta) := [I_{ij}(\theta)], \quad I_{ij}(\theta) := E_{p(x;\theta)}\left[\partial_i l(x;\theta)\partial_j l(x;\theta)\right], \tag{1}$$

where $l(x;\theta) := \log p(x;\theta)$ is the *log-likelihood function*, and $\partial_i =: \frac{\partial}{\partial\theta^i}$ (by notational convention). The FIM is a $D \times D$ positive semi-definite matrix for a $D$-order parametric family.

The FIM is a cornerstone in statistics and occurs in many places, like for example the celebrated *Cramér-Rao lower bound* [3] for an unbiased estimator $\hat{\theta}$:

$$\mathrm{Var}_{p(x;\theta)}[\hat{\theta}] \succeq I^{-1}(\theta),$$

where $\succeq$ denotes the Löwner @artial ordering of positive semi-definite matrices: $A \succeq B$ iff. $A - B \succ 0$ is positive semi-definite. Another use of the FIM is in gradient descent method using the *natural gradient* (see [6] for its use in deep learning).

Yet, it is common to encounter another equivalent expression of the FIM in the literature [3, 1]:

$$I'_{ij}(\theta) := 4\int \partial_i\sqrt{p(x;\theta)}\partial_j\sqrt{p(x;\theta)}\mathrm{d}\nu(x) \tag{2}$$

This form of the FIM is well-suited to prove that the FIM is always positive semi-definite matrix [1]: $I(\theta) \succeq 0$.

It turns out that one can define a family of equivalent representations of the FIM using the $\alpha$-*embeddings* of the parametric family. We define the $\alpha$-*representation* of densities $l^{(\alpha)}(x;\theta) := k_\alpha(p(x;\theta))$ with

$$k_\alpha(u) := \begin{cases} \frac{2}{1-\alpha}u^{\frac{1-\alpha}{2}}, & \text{if } \alpha \neq 1 \\ \log u, & \text{if } \alpha = 1. \end{cases} \tag{3}$$

The function $l^{(\alpha)}(x;\theta)$ is called the $\alpha$-*likelihood function.*

The $\alpha$-representation of the FIM (or $\alpha$-FIM for short) is

$$\boxed{I_{ij}^{(\alpha)}(\theta) := \int \partial_i l^{(\alpha)}(x;\theta)\partial_j l^{(-\alpha)}(x;\theta)\mathrm{d}\nu(x)} \tag{4}$$

In compact notation, we have $I_{ij}^{(\alpha)}(\theta) = \int \partial_i l^{(\alpha)} \partial_j l^{(-\alpha)} \mathrm{d}\nu(x)$ (this is the $\alpha$-FIM). We can expand the $\alpha$-FIM expressions as follows

$$I_{ij}^{(\alpha)}(\theta) = \begin{cases} \frac{1}{1-\alpha^2} \int \partial_i p(x;\theta)^{\frac{1-\alpha}{2}} \partial_j p(x;\theta)^{\frac{1+\alpha}{2}} \mathrm{d}\nu(x) & \text{for } \alpha \neq \pm 1 \\ \int \partial_i \log p(x;\theta) \partial_j p(x;\theta) \mathrm{d}\nu(x) & \text{for } \alpha \in \{-1, 1\} \end{cases}$$

The proof that $I_{ij}^{(\alpha)}(\theta) = I_{ij}(\theta)$ follows from the fact that

$$\partial_i l^{\alpha} = p^{-\frac{\alpha+1}{2}} \partial_i p = p^{\frac{1-\alpha}{2}} \partial_i l,$$

since $\partial_i l = \frac{\partial_i p}{p}$.

Therefore we get

$$\partial_i l^{(\alpha)} \partial_j l^{(-\alpha)} = p \partial_i l \partial_j l,$$

and $I_{ij}^{(\alpha)}(\theta) = E[\partial_i l \partial_j l] = I_{ij}(\theta)$.

Thus Eq. 1 and Eq. 2 where two examples of the $\alpha$-representation, namely the 1-representation and the 0-representation, respectively. The 1-representation of Eq. 1 is called the logarithmic representation, and the 0-representation of Eq. 2 is called the square root representation.

Note that $I_{ij}(\theta) = E[\partial_i l \partial_j l] = \int p \partial_i l \partial_j l \mathrm{d}\nu(x) = \int \partial_i p \partial_j l \mathrm{d}\nu(x) = I_{ij}^{(1)}(\theta)$ since $\partial_i l = \frac{\partial_i p}{p}$

In information geometry [1], $\{\partial_i l^{(\alpha)}\}_i$ plays the role of tangent vectors, the *$\alpha$-scores*. Geometrically speaking, the tangent plane $T_{p(x;\theta)}$ can be described using any $\alpha$-base. The statistical manifold $M = \{p(x;\theta)\}_\theta$ is imbedded into the function space $\mathbb{R}^{\mathcal{X}}$, where $\mathcal{X}$ denotes the support of the densities.

Under regular conditions [3, 1], the $\alpha$-representation of the FIM for $\alpha \neq -1$ can further be rewritten as

$$I_{ij}^{(\alpha)}(\theta) = -\frac{2}{1+\alpha} \int p(x;\theta)^{\frac{1+\alpha}{2}} \partial_i \partial_j l^{(\alpha)}(x;\theta) \mathrm{d}\nu(x). \tag{5}$$

Since we have

$$\partial_i \partial_j l^{(\alpha)}(x;\theta) = p^{\frac{1-\alpha}{2}} \left( \partial_i \partial_j l + \frac{1-\alpha}{2} \partial_i l \partial_j l \right),$$

it follows that

$$I_{ij}^{(\alpha)}(\theta) = -\frac{2}{1+\alpha} \left( -I_{ij}(\theta) + \frac{1-\alpha}{2} I_{ij} \right) = I_{ij}(\theta).$$

Notice that when $\alpha = 1$, we recover the equivalent expression of the FIM (under mild conditions)

$$I_{ij}^{(1)}(\theta) = -E[\nabla^2 \log p(x;\theta)].$$

In particular, when the family is an exponential family [5] with cumulant function $F(\theta)$, we have

$$I(\theta) = \nabla^2 F(\theta) \succ 0.$$

Similarly, the coefficients of the $\alpha$-connection can be expressed using the $\alpha$-representation as

$$\Gamma_{ij,k}^{(\alpha)} = \int \partial_i \partial_j l^{(\alpha)} \partial_k^{(-\alpha)} \mathrm{d}\nu(x).$$

The Riemannian metric tensor $g_{ij}$ (a geometric object) can be expressed in matrix form $I_{ij}^{(\alpha)}(\theta)$ using the $\alpha$-base, and this tensor is called the Fisher metric tensor.

The FIM may further be represented using the more general $(\rho, \tau)$-monotone embeddings [2]: Let $\rho$ and $\tau$ be two strictly increasing functions, and $f$ a strictly convex function such that $f'(\rho(u)) = \tau(u)$ (with $f^*$ denoting its convex conjugate). Let us write $p_\theta(x) = p(x; \theta)$. Then we have ${}^{\rho,\tau}g(\theta) = [{}^{\rho,\tau}g_{ij}(\theta)]_{ij}$ with

$$
\begin{aligned}
{}^{\rho,\tau}g_{ij}(\theta) &= \int \left(\partial_i \rho(p_\theta(x))\right) \left(\partial_j \tau(p_\theta(x))\right) \mathrm{d}\nu(x), && (6)\\
&= \int f''(\rho(p_\theta(x))) \left(\partial_i \rho(p_\theta(x))\right) \left(\partial_j \rho(p_\theta(x))\right) \mathrm{d}\nu(x), && (7)\\
&= \int (f^*)''(\tau(p_\theta(x))) \left(\partial_i \tau(p_\theta(x))\right) \left(\partial_j \tau(p_\theta(x))\right) \mathrm{d}\nu(x), && (8)\\
&= \int \frac{1}{\rho'(p_\theta(x))\tau'(p_\theta(x))} \left(\partial_i p_\theta(x)\right) \left(\partial_j p_\theta(x)\right) \mathrm{d}\nu(x). && (9)
\end{aligned}
$$

This last equation shows that there is a gauge function freedom $\Psi(u) := \frac{1}{\rho'(u)\tau'(u)}$ when calculating the FIM.

The metric tensor can be derived [4] from the $(\rho, \tau)$-divergence:

$$
D_{\rho,\tau}(p : q) = \int \left( f(\rho(p(x))) + f^*(\tau(q(x))) - \rho(p(x))\tau(q(x)) \right) \mathrm{d}\nu(x) \qquad (10)
$$

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# References


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