What is Quantum Information Geometry?

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In Quantum Information Geometry the notion of a statistical manifold is generalized to that of a quantum statistical manifold. A related domain of research is that of Quantum Information Theory which concentrates on the theory behind quantum computing.

A *quantum state* is determined by a wave function ψ, which is a normalized element of a Hilbert space $\mathscr{H}$. In a statistical context the quantum state is determined by a *density matrix* or a *density operator* ρ. This is a positive trace-class operator the trace of which equals 1. The *quantum expectation value* of a bounded operator B on $\mathscr{H}$ is usually denoted $\langle B\rangle$. Given an orthonormal diagonalizing basis $(\psi_n)_n$ one can write

$$\langle B\rangle = \operatorname{Tr}\rho B = \sum_i p_i(B\psi_n,\psi_n)$$

where p_i are the eigenvalues of ρ. Because the eigenvalues are non-negative and add up to 1 one can make the interpretation that with probability p_i the quantum system is in the state determined by the wave function ψ_i. The novel aspect of *quantum statistics* is that the quantum expectation values depend not only on the probabilities p_i but also on the basis of eigenvectors of the density operator ρ.

The obvious models of Quantum Information Geometry belong to the *quantum exponential family*, this is the exponential family of non-degenerate density matrices of dimension N-by-N. See for instance Chapter 7 of [11]. In Statistical Physics the states of a model belonging to the quantum exponential family are known as quantum Gibbs distributions. They depend on a number of thermodynamic parameters such as the inverse temperature β or a chemical potential μ. The importance of these quantum models for different branches of Physics cannot be overestimated.

A model belonging to the quantum exponential family is described by a parameterized density operator ρ_θ, $\theta\in\mathbb{R}^n$, of the form

$$\rho_\theta = \exp\left(\theta^i E_i - \alpha(\theta)\right)$$

with Hermitian N-by-N matrices E_i and with the normalization function $\phi(\theta)$ given by

$$\phi(\theta) = \log\ \operatorname{Tr}\exp\left(\theta^i E_i\right).$$

The latter acts as a potential function from which one can derive Amari's dually flat geometry [11]. A short calculation gives

$$\frac{\partial\phi}{\partial\theta^p} = \eta_p \quad \text{with} \quad \eta_p = \operatorname{Tr}\rho_\theta E_p = \langle E_p\rangle.$$

These η_p are the dual coordinates, dual to the θ^p.

The directional derivatives $\partial\rho_\theta/\partial\theta^i$ of the density matrices span the tangent spaces of the manifold of quantum states. Eguchi's method [5] can be used to define the inner product of pairs of tangent vectors starting from Umegaki's relative entropy/divergence [1]. The result is known as Bogoliubov's metric [8].

Geodesics of the e-connection are affine in the parameters θ. If the affine combination $(1-t)\rho_\theta + t\rho_\zeta$ lies in the manifold then it is a geodesic for the dual connection, which is called the m-connection. IN the classical context Chentsov [4] gave a characterization of the Fisher metric as the unique metric which is invariant under Markov type transformations [18]. On the other hand, quantum measurements are modeled by completely-positive trace-preserving maps acting on the manifold of density matrices. Petz [6, 10] gave a complete characterization of the class of metrics which are monotone w.r.t. these maps. Grasselli and Streater [12] then showed that the Bogoliubov metric is the unique element of this class with the property that the e- and m-connections are each other dual w.r.t. this metric.

The parameter-free approach to Information Geometry was initiated by Pistone and Sempi [9]. A non-commutative generalization is studied for instance in [13, 14, 15, 16]. These papers use the C^*-algebraic formulation of quantum mechanics because it clarifies the link between classical (i.e. non-quantum) and quantum statistics.

Areas of further research include the following.

The definition of the quantum exponential family, as given above, is not the only possibility. It is argued in [19, 20] that the definition is highly non-unique because of the non-uniqueness [3] of the Radon–Nikodym derivative in a non-commutative context. The latter result relies on the theory of the modular operator also known as Tomita-Takesaki Theory [2, 7].

Technical difficulties show up for families of density operators on an infinite-dimensional Hilbert space. The action of the group of invertible elements of a C^*-algebra $\mathfrak{A}$ on the set of states of $\mathfrak{A}$ induces a partition into a disjoint union of orbits each of which is a Banach manifold [17]. Exponential arcs in a manifold of quantum states are studied in [19, 20]. These exponential arcs are candidates for being geodesics of the quantum statistical manifold.

Almost unexplored up to now is the possible impact of Quantum Information Geometry on some of the specific models well-known in Quantum Statistical Physics. An example in this direction is the study of scalar curvature in the transverse Ising chain [21].

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