A closed-form expression of geodesics in the Klein model of hyperbolic geometry

Frank Nielsen
Sony Computer Science Laboratories Inc.
Tokyo, Japan

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Hyperbolic geometry [3] is more and more often used in machine learning and computer vision, specially to embed and process hierarchical structures (e.g., [9, 8]). The five main models of hyperbolic geometry [3] are the Poincaré upper space model, the Poincaré ball model, the Beltrami hemisphere model, the Lorentz hyperboloid model, and the Klein ball model. These models yield metric spaces $(\mathbb{D}, d)$ where $\mathbb{D}$ denotes the domain of the model and $d(\cdot,\cdot)$ denotes the hyperbolic distance, a metric distance. These metric spaces are said geodesic because there exists a map $\gamma(p,q;\alpha)$ such that

$$d\left(\gamma(p,q;s),\gamma(p,q;t)\right) = |s-t|\ d(p,q),\quad \forall p,q\in\mathbb{D},\quad \forall s,t\in[0,1].$$

A closed-form expression of the geodesics in the hyperbolic Poincaré ball model can be expressed using Möbius operations in the Möbius ball gyrovector space [2, 8].

In this note, we report a closed-form expression of the geodesics in the Klein ball model of arbitrary dimension. Although the Klein ball model (K) is not conformal (except at the origin), the trace of geodesics $\Gamma_K(p,q) = \{(1-\alpha)p+\alpha q\ :\ \alpha\in[0,1]\}$ (called pregeodesics) are straight line segments making it convenient for robust geometric computing (e.g., Klein hyperbolic Voronoi diagram [5, 6]). Once a structure is computed in the Klein model, it can be converted into the other models (e.g., [7]).

Let $\mathbb{B}_n = \{x\in\mathbb{R}^n\ :\ x^\top x < 1\}$ be the n-dimensional unit open ball centered at the origin. The Klein distance $d_K(p,q)$ between point p and q in $\mathbb{B}_n$ (hyperbolic geometry with curvature $\kappa=-1$) is

$$d_K(p,q) = \operatorname{arccosh}\left(\frac{1-p^\top q}{\sqrt{(1-p^\top p)}\sqrt{(1-q^\top q)}}\right).$$

Thus we seek a parameterization

$$\gamma_K(p,q;\alpha) = (1-u(\alpha))p + u(\alpha)q \tag{1}$$

so that

$$d_K\left(\gamma_K(p,q;s),\gamma_K(p,q;t)\right) = |s-t|\ d_K(p,q).$$

In particular, when $s=0$ and $t=\alpha$, we shall have

$$d_K\left(p,(1-u(\alpha))p+u(\alpha)q\right) = \alpha d_K(p,q).$$

This latter equation amounts to solve for $u(\alpha)$ in the equation:

$$\frac{a - bu(\alpha)}{\sqrt{a(a - 2bu(\alpha) + cu(\alpha)^2)}} - d(\alpha) = 0,$$

where

$$\begin{aligned} a &= 1 - p^\top p, \\ b &= p^\top (q - p), \\ c &= (q - p)^\top (q - p), \\ d(\alpha) &= \cosh(\alpha d_K(p, q)) \end{aligned}$$

Using symbolic calculations, we find the following solution:

$$u(\alpha) = \frac{ad(\alpha)\sqrt{(ac + b^2)(d(\alpha)^2 - 1)} + ab(1 - d(\alpha)^2)}{acd(\alpha)^2 + b^2}. \tag{2}$$

Thus we get in closed-form the Klein geodesics γ_K (albeit a large formula) such that

$$d_K(\gamma(p, q; s), \gamma(p, q; t)) = |s - t| \; d_K(p, q).$$

The snippet code below implements in MAXIMA[1] the geodesics in the Klein model with a test set.

```
dKlein(p,q):=acosh((1-p.q)/(sqrt((1-p.p)*(1-q.q))));

u(p,q,alpha):= ((1-p.p)*cosh(alpha* dKlein(p,q))*sqrt(((1-p.p)*((q-p).(q-p))+(
    p.(q-p))**2)*  (cosh(alpha* dKlein(p,q))
**2-1)) +(1-p.p)* (p.(q-p))* (1-cosh(alpha* dKlein(p,q))**2))/((1-p.p)* ((q-p)
    .(q-p)) * cosh(alpha* dKlein(p,q))**2 + (p.(q-p))**2);

gammaKlein(p,q,alpha):=(1-u(p,q,alpha))*p+u(p,q,alpha)*q;

/* Test */
p: [0.5, 0.2];
q: [0.1, -0.3];

/* 1st test for Klein geodesics */
alpha:random(1.0);
alpha*dKlein(p,q);
dKlein(p,gammaKlein(p,q,alpha));

/* 2nd test for Klein geodesics */
s:random(1.0);
t:random(1.0);
dKlein(gammaKlein(p,q,s),gammaKlein(p,q,t));
abs(s-t)*dKlein(p,q);
```

[1] https://maxima.sourceforge.io/

To illustrate the use of these Klein geodesics, consider computing the smallest enclosing ball of a finite point set $\mathcal{P} = \{p_1, \ldots, p_m\}$ in hyperbolic geometry [1]. The closed-form expression of the Klein geodesic, allows to bypass the use of hyperbolic translations from/to the ball origin as this was used in [4]. The algorithm for calculating an approximation of the hyperbolic smallest enclosing ball is:

- Initialize $c_1 = p_1$
- Repeat t times: Let $c_{i+1} = \gamma_K\left(c_i, p_{f_i}, \frac{1}{i+1}\right)$ where p_{f_i} is the farthest point of $\mathcal{P}$ to c_i. That is, we have $f_i = \arg\max_{j\in\{1,\ldots,m\}} d_K(c_i, p_j)$.

The algorithm is proven to converge in [1] (i.e., $\lim_{t\rightarrow\infty} c_t = \arg\min_c \max_{i\in\{1,\ldots,m\}} d_K(p_i, c)$) since the hyperbolic geometry is a Hadamard space.

Additional material is available online at `https://franknielsen.github.io/KleinGeodesics/`

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