Jean-Louis Koszul and the elementary structures of Information Geometry

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Abstract. This paper is an admiration exercise of Jean-Louis Koszul works on homogeneous bounded domains that have appeared over time as elementary structures of Information Geometry. Koszul has introduced fundamental tools to characterize the geometry of these sharp convex cones, as Koszul-Vinberg characteristic Function, Koszul Forms, and affine representation of Lie Algebra and Lie Group. The 2^{nd} Koszul form is an extension of classical Fisher metric. Koszul theory of hessian structures could be considered as main foundation of Information Geometry.

Keywords: Koszul-Vinberg Characteristic Function, Koszul Forms, Affine representation of Lie Algebra and Lie Group, Homogeneous Bounded Domains

1 Preamble

In this article, we go tribute to a part of Professor Jean-Louis Koszul work and basic contributions of this great algebraist and geometer in the field of Information Geometry, which have many applications in the field of applied mathematics, and in the emerging field of Artificial Intelligence where the most powerful algorithms are based on the natural gradient of the information geometry deduced from the Fisher matrix, as Yann Ollivier recently showed [112-113].

Strong of the French mathematical tradition, and the teachings of his master Elie Cartan, Jean-Louis Koszul was a real avant-garde, if we take the definition given by Clausewitz «*An avant-garde is a group of units intended to move in front of the army to: explore the terrain to avoid surprises, quickly occupy the strong positions of the battlefield (high points), screen and contain the enemy the time the army can deploy*". Indeed, Jean-Louis Koszul was a pioneer and cleared many areas of mathematics, described in the book “Selected papers of JL Koszul” [3]. What we will expose here is therefore only one part of his work which concerns the study of homogeneous bounded domains, on the basis of Elie Cartan's earlier work on symmetric bounded domains. In a letter from André Weil to Henri Cartan, linked in the proceedings of the conference "*Elie Cartan and today's mathematics*” in 1984, it says "*As to the symmetrical spaces, and more particularly to the symmetric bounded domains at the birth of which you contributed, I have kept alive the memory of the satisfaction I felt in finding*

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some incarnations in Siegel from his first works on quadratic forms. and later to convince Siegel of the value of your father's ideas on the subject". At this 1984 conference, two disciples of Elie Cartan gave a conference, Jean-Louis Koszul [5] and Jean-Marie Souriau.

Fig. 1. (on the left) Jean-Louis Koszul student at ULM ENS in 1940, (on the right) Jean-Louis Koszul at GSI'13 conference at the École des Mines of Paris August 2013

In the book "*Selected papers of JL Koszul*" [3], Koszul summarizes the work we are going to detail in the following : "*It is with the problem of the determination of the homogeneous bounded domains posed by E. Cartan around 1935 that are related [my papers]. The idea of approaching the question through invariant Hermitian forms already appears explicitly in Cartan. This leads to an algebraic approach which constitutes the essence of Cartan's work and which, with the Lie J-algebras, was pushed much further by the Russian School* [19-32]. *It is the work of Piatetski Shapiro on the Siegel domains, then those of E.B. Vinberg on the homogeneous cones that led me to the study of the affine transformation groups of the locally flat manifolds and in particular to the convexity criteria related to invariant forms.".* In particular, J.L. Koszul found one of the sources of his inspiration in this last sentence of Elie Cartan 's 1935 article [2]:

"It is clear that if one could demonstrate that all homogeneous domains whose form $\Phi = \sum_{i,j} \frac{\partial^2 \log K\left(z, z^*\right)}{\partial z_i \partial z_j^*} dz_i dz_j^*$ *is positive definite are symmetric, the whole theory of homogeneous bounded domains would be elucidated. This is a problem of Hermitian geometry certainly very interesting*". It was not until 1953 that the classification of

non-Riemannian symmetric spaces has been achieved by Marcel Berger [61]. The work of Koszul has also been extended and deepened by one of his student Jacques Vey in [16] and [17]. Jacques Vey has transposed the notion of hyperbolicity, developed by W. Kaup for Riemann surfaces, into the category of differentiable manifolds with flat linear connection (locally flat manifolds), which makes it possible to completely characterize the locally flat manifolds admitting as universal covering a convex open sharp cone of R^n , which had been studied by Koszul in [11]. The links between Koszul's work and those of Ernest B. Vinberg [19-26] were recently developed at the conference "*Transformation groups 2017*" in Moscow dedicated to the 80^{th} anniversary of Professor EB Vinberg, in the Dmitri Alekseevsky talk on “*Vinberg's theory of homogeneous convex cones: developments and applications*" [18]. Koszul and Vinberg are actually associated with the concept of Koszul-Vinberg's characteristic function on convex cones, which we will develop later in the paper. In the framework of this study, Koszul introduced the so-called “*Koszul forms*” and a canonical metric given by the Hessian of the opposite of the logarithm of this Koszul-Vinberg characteristic function, from which we will see the links with Fisher's metric in Information Geometry, the scope of which it generalizes.

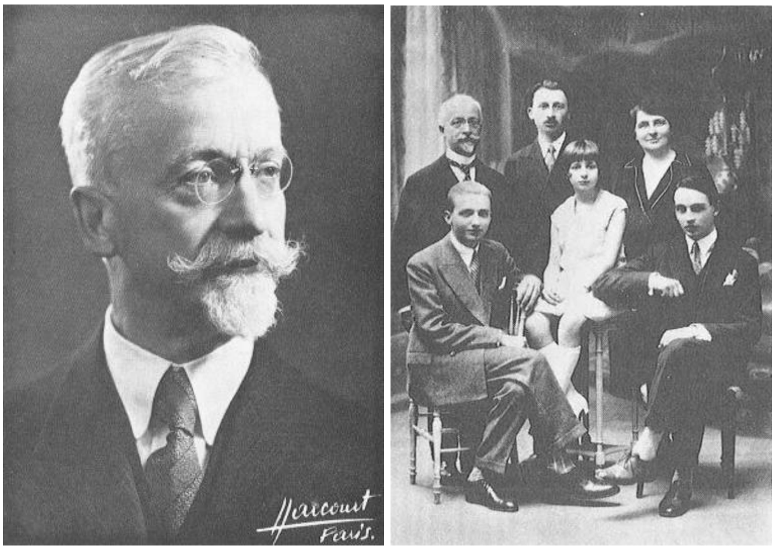

Fig. 2. (on the left) Professor Elie Cartan, (to the right) the Cartan family

Professor Koszul's main papers, which form the elementary structures of information geometry, are as follows:

- « ***Sur la forme hermitienne canonique des espaces homogènes complexes*** » [7] of 1955 : Koszul considers the Hermitian structure of a homogeneous *G/B* manifold (*G* related Lie group and *B* a closed subgroup of *G*, associated , up to a constant factor, to the single invariant *G* , and to the invariant complex structure by the operations of *G)* . Koszul says "*The interest of this form* *for the determination of homogeneous bounded domains has been emphasized by E. Cartan: a necessary condition for G/B to be a bounded domain is indeed that this form is positive definite*". Koszul calculates this canonical form from infinitesimal data Lie algebra of *G*, the sub-algebra corresponding to *B* and an endomorphism algebra defining the invariant complex structure of *G/B.* The results obtained by Koszul proved that the homogeneous bounded domains whose group of automorphisms is semi-simple are bounded symmetric domains in the sense of Elie Cartan. Koszul also refers to André Lichnerowicz's work on Kählerian homogeneous spaces [37]. Koszul introduces a left invariant form of degree 1 on *G*: $\Psi(X) = Tr_{g/b}\left[ad(JX) - J.ad(X)\right] \quad \forall X \in g$ with J an endomorphism of the Lie algebra space and the trace $Tr_{g/b}[.]$ corresponding to that of the endomorphism *g/b.* The Kähler form of the canonical Hermitian form is given by the differential of $-\frac{1}{4}\Psi(X)$ of this form of degree 1.

- ***« Exposés sur les espaces homogènes symétriques »*** [8] of 1959 is a Lecture written as part of a seminar held in September and October 1958 at the University of Sao Paulo, which details the determination of homogeneous bounded domains. He returns to [7] and shows that any symmetric bounded domain is a direct product of irreducible symmetric bounded domains, determined by Elie Cartan (4 classes corresponding to classical groups and 2 exceptional domains). For the study of irreducible symmetric bounded domains, Koszul refers to Elie Cartan, Carl-Ludwig Siegel and Loo-Keng Hua. Koszul illustrates the subject with two particular cases, the half-plane of Poincaré and the half-space of Siegel, and shows that with its trace formula of endomorphism *g/b*, he finds that the canonical Kähler hermitian form and the associated metrics are the same as those introduced by Henri Poincaré and Carl-Ludwig Siegel [35] (who introduced them as invariant metric under action of the automorphisms of these spaces).
- ***« Domaines bornées homogènes et orbites de groupes de transformations affines »*** [9] of 1961 is written by Koszul at the Institute for Advanced Study at Princeton during a stay funded by the National Science Foundation. On a complex homogeneous space, an invariant volume defines with the complex structure the canonical invariant Hermitian form introduced in [7]. If the homogeneous space is holomorphically isomorphic to a bounded domain of a space C^n, this Hermitian form is positive definite because it coincides with the Bergmann metric of the domain. Koszul demonstrates in this article the reciprocal of this proposition for a class of complex homogeneous spaces. This class consists of some open orbits of complex affine transformation groups and contains all homogeneous bounded domains. Koszul addresses again the problem of knowing if a complex homogeneous space, whose canonical Hermitian form is positive definite is isomorphic to a bounded domain, but via the study of the invariant bilinear form defined on a real homo-

geneous space by an invariant volume and an invariant flat connection. Koszul demonstrates that if this bilinear form is positive definite then the homogeneous space with its flat connection is isomorphic to a convex open domain containing no straight line in a real vector space and extends it to the initial problem for the complex homogeneous spaces obtained in defining a complex structure in the variety of vectors of a real homogeneous space provided with an invariant flat connection. It is in this article that Koszul uses the affine representation of Lie groups and algebras. By studying the open orbits of the affine representations, he introduced an affine representation of G, written $(\mathbf{f},\mathbf{q})$, and the following equation setting f the linear representation of the Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$ and q the restriction to $\mathfrak{g}$ and the differential of $\mathbf{q}$ (f and q are differential respectively of $\mathbf{f}$ and $\mathbf{q}$):

$$f(X)q(Y)-f(Y)q(X)=q\left(\left[X,Y\right]\right)\quad \forall X,Y\in\mathfrak{g}$$

with $f:\mathfrak{g}\to gl(E)$ and $q:\mathfrak{g}\mapsto E$

- ***« Ouverts convexes homogènes des espaces affines »*** [10] of 1962. Koszul is interested in this paper by the structure of the convex open non-degenerate Ω (with no straight line) and homogeneous (the group of affine transformations of *E* leaving stable Ω operates transitively in Ω) in a real affine space of finite dimension. Koszul demonstrates that they are all deduced from non-degenerate and homogeneous convex open cones built in [9]. He uses for this the properties of the group of affine transformations leaving stable a non-degenerate convex open domain and an homogeneous domain.
- ***« Variétés localement plates et convexité »*** [11] of 1965. Koszul established the following theorem: let *M be* a locally related differentiable manifold. If the universal covering of *M* is isomorphic as a flat manifold with a convex open domain containing no straight line in a real affine space, then there exists on *M* a closed differential form α such that $D\alpha$ *(D* linear covariant derivative of zero torsion) is defined positive in all respects and which is invariant under every automorphism of *M*. If *G* is a group of automorphisms of *M* such that *G\M* is quasi-compact and if there exists on *M* a closed 1-differential form α invariant by *G* and such that $D\alpha$ is definite positive at any point, then the universal covering of *M* is isomorphic as a flat manifold with a convex open domain that does not contain a straight line in a real affine space.
- ***« Lectures on Groups of Transformations »*** [12] of 1965. This is lecture notes given by Koszul at Bombay "Tata Institute of Fundamental Research " on transformation groups. In particular in Chapter 6, Koszul studies discrete linear groups acting on convex open cones in vector spaces based on the work of C.L. Siegel (work on quadratic forms [34]). Koszul uses what we will call in the following Koszul-Vinberg characteristic function on convex sharp cone.

- ***« Déformations des variétés localement plates »*** [13] of 1968. Koszul provides other proofs of theorems introduced in [11]. Koszul considers related differentiable manifolds of dimension *n* and *TM* the fibered space of *M* vectors. The linear connections on *M* constitute a subspace of the space of the differentiable applications of the *TM*x*TM* fiber *product* in the space *T(TM)* of the *TM* vectors. Any locally flat connection *D* (the curvature and the torsion are zero) defines a locally flat connection on the covering of *M*, and is hyperbolic when universal covering of *M*, with this connection, is isomorphic to a sharp convex open domain (without straight lines) in R^n. Koszul shows that, if *M* is a compact manifold, for a locally flat connection on *M to* be hyperbolic, it is necessary and sufficient that there exists a closed differential form of degree 1 on *M* whose covariant differential is positive definite.
- ***« Trajectoires Convexes de Groupes Affines Unimodulaires »*** [14] in 1970 . Koszul demonstrates that a convex sharp open domain in R^n that admits a unimodular transitive group of affine automorphisms is an auto-dual cone. This is a more geometric demonstration of the results shown by Ernest Vinberg [25] on the automorphisms of convex cones.

The elementary geometric structures discovered by Jean-Louis Koszul are the foundations of Information Geometry. These links were first established by Professor Hirohiko Shima [42-47]. These links were particularly crystallized in Shima book 2007 "*The Geometry of Hessian Structures*" [48], which is dedicated to Professor Koszul. The origin of this work followed the visit of Koszul in Japan in 1964, on a mission for the French government. Koszul taught lectures on the theory of flat manifolds at Osaka University. Hirohiko Shima was then a student and attended these lectures with the teachers Matsushima and Murakami. This lecture was at the origin of the notion of Hessian structures and the beginning of the works of Hirohiko Shima. Henri Cartan noted concerning Koszul's ties with Japan, "*Koszul has attracted eminent mathematicians from abroad to Strasbourg, in Grenoble. I would like to mention in particular the links he has established with representatives of the Japanese School of Differential Geometry*". Shima's book [48] is a systematic introduction to the theory of Hessian structures (provided by a pair of a flat connection *D* and an Hessian metric *g).* Koszul studied flat manifolds with a closed 1-form α , such that $D\alpha$ be positive definite, where $D\alpha$ is a hessian metric . However, not all Hessian metrics are globally of the form $g = D\alpha$. Shima introduces the notion of Codazzi structure for a pair (*D*,*g*), with *D* a torsion-free connection, checking the equations of Codazzi $(D_X g)(Y,Z) = (D_Y g)(X,Z)$. A Hessian structure is a Codazzi structure for which connection *D* is flat. This is an extension of Riemannian geometry. It is then possible to define a connection D' and a structure of dual Codazzi *(D',g)* with $D' = \nabla - D$ with ∇ the connection of Levi-Civita. For a hessian structure (D,g) with $g = Dd\varphi$, the dual Codazzi structure (D',g) is also a Hessian structure and $g = D'd\varphi'$, where φ' is the Legendre transform of φ: $\varphi' = \sum_i x^i \frac{\partial \varphi}{\partial x^i} - \varphi$. Shima observed that Information Geometry framework could be introduced by dual connections, and linked with Fréchet, Rao and Chentsov works [58]. A hessian structure *(D, g)* is of Koszul type, if there is a closed 1-form ω as $g = D\omega$. Using *D* and the volume element of *g*, Koszul introduced a 2nd form, which plays a similar role to the Ricci tensor for a Kählerian metric. Let υ be the volume element of g, we define a closed 1-form α such that $D_X\upsilon = \alpha(X)\upsilon$ and a symmetric bilinear form $\gamma = D\alpha$. α and γ forms are called 1st and 2nd form of Koszul for Hessian structure (D, g). We can consider the forms associated with the Hessian dual structure (D', g) by $\alpha' = -\alpha$ and $\gamma' = \gamma - 2\nabla\alpha$. In the case of a homogeneous regular convex cone Ω, the Koszul forms α and γ for the canonical Hessian structure $(D, g = Dd\psi)$ are given by $\alpha = d\log\psi$ and $\gamma = g$. The volume element υ determined by g is invariant under the action of the group of automorphisms G of Ω.

Jean-Louis Koszul attended the 1 st GSI "*Geometric Science of Information*" in August 2013 at the Ecole des Mines in Paris, where he followed Hirohiko Shima talk, given for his honor on the topic "*Geometry of Hessian Structures* " [49]. In the photo below, we can see from left to right, Jean-Louis Koszul, Hirohiko Shima and Michel Nguiffo Boyom. Professor Michel Boyom has extensively studied and developed, at the University of Montpellier, Koszul models [93-100] in relation to symplectic flat affine manifolds and to the cohomology of Koszul-Vinberg algebras (KV Cohomology) and with his PhD student Byande [91-92] links with Information Geometry. See also André Lichnerowicz's work on the topic of homogeneous Kähler manifolds [111].

Fig. 3. From left to right, Jean-Louis Koszul, Hirohiko Shima and Michel Nguiffo Boyom at GSI'13 conference at the École des Mines of Paris in August 2013

2 Biographical reminder of Jean-Louis Koszul scientific life

Jean Louis André Stanislas Koszul born in Strasbourg in 1921, is the son a family of four (with three older sisters, Marie Andrée, Antoinette and Jeanne). He is the son of André Koszul (born in Roubaix on November 19th 1878, professor in Strasbourg university), and Marie Fontaine (born in Lyon on June 19th 1887), who was a friend of Henri Cartan's mother. Henri Cartan writes on this subject "*My mother in her youth, had been a close friend of the one who was to become the mother of Jean-Louis Koszul*"[4]. His paternal parents were Julien Stanislas Koszul and Hélène Ludivine Rosalie Marie Salomé. He attended high school in Fustel-de-Coulanges in Strasbourg and the Faculty of Science in Strasbourg and in Paris. He entered ENS Ulm in the class of 1940 and defended his thesis with Henri Cartan. Henri Cartan notes "*This promotion included other mathematicians like Belgodère or Godement, and also physicists and some chemists, like Marc Julia and Raimond Castaing*"[4]. Jean-Louis Koszul married on July 17th 1948 with Denise Reyss-Brion, student of ENS Sèvres, entered in 1941. They have three children Michel (married to Christine Duchemin), Anne (wife of Stanislas Crouzier) and Bertrand. He then taught in Strasbourg and was appointed Associate Professor at the University of Strasbourg in 1949. He was promoted to professor in 1956. He became a member of Bourbaki with 2nd generation, Dixmier, R. Godement, S. Eilenberg, P. Samuel, J. P. Serre and L. Schwartz. Henri Cartan remark in [4] "*In the vehement discussions within Bourbaki, Koszul was not one of those who spoke loudly; but we learned to listen to him because we knew that if he opened his mouth he had something to say*". He was promoted to Senior Lecturer at the University of Grenoble in 1963. He became an honorary professor at the Joseph Fourier University [6]. In Grenoble, he practiced mountaineering and was a member of the French Alpine Club. Koszul was elected correspondent at the Academy of Sciences on January 28th 1980. The following year, he was elected to the Academy of São Paulo. Jean-Louis Koszul was the cousin of the composer Henri Dutilleux, whose common grandfather was Julien Koszul (1844-1927), pupil of Camille Saint-Saëns, friend of Gabriel Fauré, and professor of Albert Roussel. The most famous works of his grandfather Julien Koszul were replayed recently at IHES by Bertrand Maury, following a presentation that developed the elements described in this paper by the author. The most remarkable works of Julien Koszul, the grandfather of Jean-Louis Koszul are: Quo Vadis for chorus of men with 5 voices, Pié Jesus in si m, Pieces for piano with two hands and 1 piece for piano with 4 Hands and Melodies of 1872 and 1879. Jean-Louis Koszul died on January 12th 2018, at the age of 97.

As early as 1947, Jean-Louis Koszul published three articles in CRAS of the Academy of Sciences, on the Betti number of a simple compact Lie group , on cohomology rings , generalizing ideas of Jean-Leray, and finally on the homology of homogeneous spaces. In 1987, an International Symposium on Geometry was held in Grenoble in honor of Jean-Louis Koszul, whose proceedings are published in the Annales of the Fourier Institute, Volume 37, No. 4. This conference began with a presentation by Henri Cartan , who noted in his article in honor of Koszul the mention he made when passing the aggregation of Koszul [4]: "*Distinguished Spirit; he is* *successful in his problems. Should beware, orally, of overly systematic trends. a little less subtle complications, baroque ideas, a little more common sense and balance would be desirable".* Commenting on the fact that Koszul is addressing him for his thesis, Henri Cartan writes *"Why did he turn to guide him (so-called) ? Is it because he found inspiration in Elie Cartan's work on the topology of Lie groups ? Perhaps he was surprised to note that mathematical knowledge is not necessarily transmitted by descent. In any case it was he who made me better know what my father had brought to the theory*" [4]. On the work of Koszul algebrisation, Henri Cartan notes "*it was Koszul who was the first to give a precise algebraic formalization of the situation studied by Leray in his 1946 publication, which became the theory of the spectral sequence. It took a good deal of insight to unravel what lay behind Leray's study. In this respect, Koszul's Note in the July 1947 CRAS is of historical significance*."[4]. From June 26th to July 2nd 1947, CNRS, received an International conference in Paris, on "Algebraic Topology". This was Leray's first postwar international diffusion of ideas. Koszul writes about this lecture "*I can still see Leray putting his chalk at the end of his talk by saying (modestly?) that he definitely did not understand anything about Algebraic Topology*". In writing his lectures at the Collège de France, Leray adopted the algebraic presentation of the spectral suite elaborated by Koszul. As early as 1950, J.P. Serre used the term "Leray-Koszul". Speaking of Leray, Koszul wrote "*around 1955 I remember asking him what had put him on the path of what he called the ring of homology of a representation in his Notes to the CRAS of 1946. His answer was Künneth's theorem ; I could not find out more.*". The sheaf theory, introduced by Jean-Leray, followed in 1947, at the same time as the spectral sequences.

In 1950, Koszul published an important 62-pages book entitled Homology and Cohomology of Lie Algebras in which he studied the links between homology and cohomology (with real coefficients) of a compact connected Lie group and purely algebraic problems of Lie algebra. Koszul then gave a lecture in São Paulo on the topic " *sheaves and cohomology*". The superb lecture notes were published in 1957 and dealt with the cohomology of Čech with coefficients in a sheaf. In the autumn of 1958, he again organized a series of seminars in São Paulo, this time on symmetric spaces [8]. R. Bott commented on these seminars "*very pleasant. The pace is fast, and the considerable material is covered elegantly. In addition to the more or less standard theorems on symmetric spaces, the author discusses the geometry of geodesics, Bergmann's metrics, and finally studies the bounded domains with many details*.". In the mid-1960s, Koszul taught at the Tata Institute in Bombay on transformation groups [12] and on fiber bundles and differential geometry. The second lecture dealt with the theory of connections and the lecture notes were published in 1965. In 1986 he published "*Introduction to symplectic geometry*"[15] following a Chinese course in China (with the agreement of Jean-Louis Koszul given in 2017, this lecture given at the University of Nanjing will be translated into English by Springer and will be published in 2018). This book takes up and develops works of Jean-Marie Souriau [107-108] on homogeneous symplectic manifolds and the affine representation of algebras and Lie groups in geometric mechanics (another fundamental source of structures of Information Geometry extended on homogeneous varieties [101-105]). Chuan Yu Ma writes in a review, on this latest book in Chinese, that "*This work coincided with developments in the field of analytical mechanics. Many new ideas have also been derived using a wide variety of notions of modern algebra, differential geometry, Lie groups, functional analysis, differentiable manifolds, and representation theory. [Koszul's book] emphasizes the differential-geometric and topological properties of symplectic manifolds. It gives a modern treatment of the subject that is useful for beginners as well as for experts*".

In 1994, in [3], a comment by Koszul explains the problems he was preoccupied with when he invented what is now called the "Koszul complex". This was introduced to define a theory of cohomology for Lie algebras and proved to be a general structure useful in homological algebra.

3 Koszul-Vinberg Characteristic Function, Koszul forms and Maximum Entropy Density

Through the study of the geometry of bounded homogeneous domains initiated by Elie Cartan [1-2], Jean-Louis Koszul discovered that the elementary structures are associated with Hessian manifolds and sharp convex cones [7-14]. In 1935, Elie Cartan proved in [2] that the symmetric homogeneous irreducible bounded domains could be reduced to 6 classes, 4 canonical models and 2 exceptional cases. Ilya Piatetski-Shapiro [27-31], after Luogeng Hua [36], extended Siegel's description [34-35] to other symmetric spaces, and showed by a counterexample that Elie Cartan's conjecture, that all transitive domains are symmetrical, was false. At the same time, Ernest B. Vinberg [19-26] worked on the theory of homogeneous convex cones and the construction of Siegel domains [34-35]. More recently, the classical complex symmetric spaces were studied by F. Berezin [38][77] in the context of quantification . In parallel, O.S. Rothaus [33] and Piatetski-Shapiro [27-31] with Karpelevitch, explored the underlying geometry of these complexes homogeneous fields, and more particularly the fibration areas on the components of the border. In Italy, we note the work of E. Vessentini [39] and U. Sampieri [40-41]. The Siegel domains, which fit into these classes of structures, nowadays play an important role in the processing of radar spatio-temporal signals and, more broadly, in learning from structured covariance matrices.

Jean-Louis Koszul and Ernest B. Vinberg have introduced an affinely invariant hessian metric on a sharp convex cone Ω through a function, called characteristic function ψ. In the following Ω is a sharp convex cone in a vector space E of finite size on R (a convex cone is sharp is there is no straight lines). In dual space E^* of E, Ω^* is the set of linear strictly positive forms on $\overline{\Omega}-\{0\}$. Ω^*, dual cone of Ω, is also a sharp convex cone. If $\xi \in \Omega^*$, then intersection $\Omega \cap \{x \in E / \langle x, \xi \rangle = 1\}$ is bounded. $G = Aut(\Omega)$ is the group of linear transformation from E *that preserves* Ω

(group of automorphisms). $G = Aut(\Omega)$ acts on Ω^* such that, $\forall g \in G = Aut(\Omega), \forall \xi \in E^*$ then $\tilde{g}.\xi = \xi \circ g^{-1}$. Koszul introduce an integral, of Laplace kind, on sharp dual convex cone, as :

Koszul-Vinberg Characteristic definition:

Let $d\xi$ Lebesque measure on E^*, following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega \qquad (1)$$

with Ω^* the dual cone, is analytical function on Ω, with $\psi_\Omega(x) \in \left]0, +\infty\right[$, called Koszul-Vinberg characteristic function of cone Ω.

Nota: the logarithm of the characteristic function is called « barrier function » for convex optimization algorithms. Yurii Nesterov and Arkadii Nemirovskii [114] have proved in modern theory of « *interior point* », using function $\Theta_\Omega(x) = \log\left(vol_n\left\{s \in \Omega^* / \langle s, x \rangle \leq 1\right\}\right)$, that all convex cones in R^n have a self-dual barrier, linked with Koszul characteristic function.

Koszul-Vinberg Characteristic function has the following properties:

- Bergman kernel of $\Omega + iR^{n+1}$ is written $K_\Omega(\mathrm{Re}(z))$ up to a constant. K_Ω is defined by integral:

$$K_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} \psi_{\Omega^*}(\xi)^{-1} d\xi \qquad (2)$$

- ψ_Ω is an analytical function defined in the interior of Ω and $\psi_\Omega(x) \to +\infty$ when $x \to \partial\Omega$. If $g \in Aut(\Omega)$ then $\psi_\Omega(gx) = |\det g|^{-1} \psi_\Omega(x)$ and as $tI \in G = Aut(\Omega)$ for all $t > 0$, we have:

$$\psi_\Omega(tx) = \psi_\Omega(x) / t^n \qquad (3)$$

- ψ_Ω is strictly log convex, such that $\phi_\Omega(x) = \log(\psi_\Omega(x))$ is strictly convex.

From this characteristic function, Koszul introduced 2 formes:

1st Koszul form α : Differential 1-form

$$\alpha = d\phi_\Omega = d\log\psi_\Omega = d\psi_\Omega / \psi_\Omega \qquad (4)$$

is invariant with respect to all automorphisms $G = Aut(\Omega)$ of Ω. If $x \in \Omega$ and $u \in E$ then

$$\langle \alpha_x, u \rangle = -\int_{\Omega^*} \langle \xi, u \rangle . e^{-\langle \xi, x \rangle} d\xi \text{ and } \alpha_x \in -\Omega^* \tag{5}$$

and

2nd Koszul form γ : Differential symmetric 2-form

$$\gamma = D\alpha = Dd \log \psi_\Omega \tag{6}$$

Is a bilinear symmetric positive definite form invariant with respect to the action of $G = Aut(\Omega)$ and $D\alpha > 0$

Positivity is given by Schwarz inequality and :

$$Dd \log \psi_\Omega (u, v) = \int_{\Omega^*} \langle \xi, u \rangle \langle \xi, v \rangle e^{-\langle \xi, u \rangle} d\xi \tag{7}$$

Koszul has proved that from this 2nd form, we can introduce an invariant Riemannian metric with respect to the action of cone automorphisms:

Koszul Metric: $D\alpha$ defines a Riemannian invariant structure by $Aut(\Omega)$, and the Riemannian metric is given by :

$$g = Dd \log \psi_\Omega \tag{8}$$

$$\left(Dd \log \psi(x)\right)(u) = \frac{1}{\psi(u)^2} \left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi \right)^2 \right] > 0 \tag{9}$$

with $F(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle}$ and $G(\xi) = e^{-\frac{1}{2}\langle x, \xi \rangle} \langle u, \xi \rangle$

We can prove the positivity with Schwarz inequality, $d \log \psi = \frac{d\psi}{\psi}$ and $Dd \log \psi = \frac{Dd\psi}{\psi} - \left(\frac{d\psi}{\psi} \right)^2$ where $\left(d\psi(x)\right)(u) = -\int_{\Omega^*} e^{-\langle x, \xi \rangle} \langle u, \xi \rangle d\xi$ and $\left(Dd\psi(x)\right)(u) = \int_{\Omega^*} e^{-\langle x, \xi \rangle} \langle u, \xi \rangle^2 d\xi$.

Koszul uses this diffeomorphism to define dual coordinates :

$$x^* = -\alpha_x = -d \log \psi_\Omega (x) \tag{10}$$

with $\langle df(x), u \rangle = D_u f(x) = \left. \frac{d}{dt} \right|_{t=0} f(x + tu)$. When the cone Ω is symmetric, the map $x \mapsto x^* = -\alpha_x$ is a bijection and an isometry with only one fixed point (the manifold is a symmetric Riemannian space given by its isometry):

$(x^*)^* = x$, $\left\langle x, x^* \right\rangle = n$ et $\psi_\Omega(x)\psi_{\Omega^*}(x^*) = cste$ (11)

x^* is characterized by $x^* = \arg\min\left\{\psi(y) / y \in \Omega^*, \left\langle x, y \right\rangle = n\right\}$ and x^* is the gravity center of the transverse cut $\left\{y \in \Omega^*, \left\langle x, y \right\rangle = n\right\}$ of Ω^* :

$$x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi$$

$$\text{and} \quad \left\langle -x^*, h \right\rangle = d_h \log \psi_\Omega(x) = -\int_{\Omega^*} \left\langle \xi, h \right\rangle e^{-\langle \xi, x\rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \qquad (12)$$

In [78-81], Misha Gromov was interested by these structures. If we set $\Phi(x) = -\log \psi_\Omega(x)$, Gromov has observed that $x^* = d\Phi(x)$ is an injection where the image closure is equal to the convex envelop of the support and the volume of this envelop is the n-dimensionnel volume defined by the integral of hessian determinant of this function, $\Phi(x)$, where the map $\Phi \mapsto M(\Phi) = \int_\Omega \det\left(Hess(\Phi(x))\right).dx$ obeys a non-trivial inequality given by Brunn-Minkowsky :

$$\left[M(\Phi_1 + \Phi_2)\right]^{1/n} \geq \left[M(\Phi_1)\right]^{1/n} + \left[M(\Phi_2)\right]^{1/n} \qquad (13)$$

These relations appear also in statistical physics. As the physicist Jean-Marie Souriau [101-107] did, it is indeed possible to define the concept of Shannon's Entropy via the Lengendre transform associated with the opposite of the logarithm of this characteristic function of Koszul-Vinberg. Taking up the seminal ideas of François Massieu [63-66] in Thermodynamics (classmate of the Corps des Mines, it is François Massieu who influenced Henri Poincaré [62] who introduced the characteristic function in Probability, with a Laplace transform, and not a Fourier transform as did then Paul Levy), which were recently developed by Roger Balian in Quantum Physics [67-76], replacing Shannon Entropy Entropy by von Neumann one. We will also note the work of Jean-Leray on the extensions of the Laplace transform in [50]. Starting from the characteristic function of Koszul-Vinberg, it is thus possible to introduce an entropy of Koszul defined as the Legendre transform of this function, which is the opposite of the logarithm of the characteristic function of Koszul-Vinberg (a logarithm lies the characteristic function of Massieu and the characteristic function of Koszul or Poincaré). Starting from the Koszul function, its Legendre transform gives a dual potential function in the dual coordinate system. x^* :

$$\Phi^*(x^*) = \left\langle x, x^* \right\rangle - \Phi(x) \text{ with } x^* = D_x \Phi \text{ and } x = D_{x^*}\Phi^* \text{ where } \Phi(x) = -\log \psi_\Omega(x) \qquad (14)$$

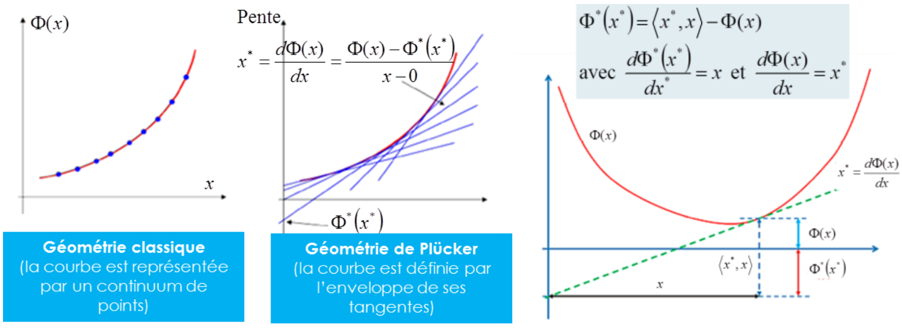

Fig. 4. Legendre Transform and Plücker Geometry

Concerning the Legendre transform [51], Darboux gives in his book an interpretation of Chasles: "*What comes back according to a remark of M. Chasles, to replace the surface with its polar reciprocal with respect to a paraboloid*". We have the same reference to polar reciprocal in "*Lessons on the calculus of variations*" by Jacques Hadamard, taken by M.E. Vessiot, which uses the "*figuratrice*", as polar reciprocal of the "*figurative*".

It is possible to express this Legendre transform only from the dual coordinate system x^*, using that $x = D_{x^*}\Phi^*$. We then obtain the Clairaut equation:

$$\Phi^*(x^*) = \left\langle (D_x\Phi)^{-1}(x^*), x^* \right\rangle - \Phi\left[(D_x\Phi)^{-1}(x^*)\right] \quad \forall x^* \in \{D_x\Phi(x) / x \in \Omega\} \tag{15}$$

This equation was discovered by Maurice Fréchet in his 1943 paper [55], in which he introduced for the first time the bound on the variance of any statistical estimator via the Fisher matrix, wrongly attributed merely Cramer and Rao [56]. Fréchet was looking for "distinguished densities" [106], densities whose covariance matrix of the estimator of these parameters reaches this bound. Fréchet there showed that these densities were expressed while using this characteristic function $\Phi(x)$, and that these densities belong to the exponential densities family.

$$(55) \qquad \mu = \theta\,\mu' - \psi(\mu')$$

c'est-à-dire une équation de Clairaut. La solution $\mu' =$ constante réduirait $f(x, \theta)$, d'après (48) à une fonction indépendante de θ, cas où le problème n'aurait plus de sens. μ est donc donné par la solution singulière de (55), qui est unique et s'obtient en éliminant s entre $\mu = \theta\, s - \psi(s)$ et $\theta = \psi'(s)$ ou encore entre

Fig. 5. Legendre-Clairait equation in 1943 Fréchet paper

Apparently, this discovery by Fréchet dates from winter 39, because Fréchet writes at the bottom of the page [55] "*The content of this dissertation formed part of our mathematical statistics Lecture at the Institut Henri Poincaré during the winter of 1939 -1940. It is one of the chapters of the second edition (in preparation) of our ' Lessons in Mathematical Statistics', the first of which is 'Introduction: Preliminary Lecture on the Probability Calculation' (119 pages in quarto, typed in) has just been published at the University Documentation Center , Tournaments and Constans. Paris'.*". More recently Muriel Casalis [87-88], the PhD student of Gérard Letac [86], was interested in the exponential families and their invariance with respect to the affine group.

To make the link between the characteristic function of Koszul-Vinberg and Entropy of Shannon, we will detail the formulas of Koszul in the following developments. Using the fact that $-\left\langle \xi, x \right\rangle = \log e^{-\langle \xi, x \rangle}$, we can write:

$$-\left\langle x^*, x \right\rangle = \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \,/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \tag{16}$$

and then developing the Legendre transform to make appear the density of maximum entropy in $\Phi^*(x^*)$, and also the Shannon entropy :

$$\Phi^*(x^*) = \left\langle x, x^* \right\rangle - \Phi(x) = -\int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \,/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi + \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\left(\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \right) . \log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . e^{-\langle \xi, x \rangle} d\xi \right] / \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right]$$

$$\Phi^*(x^*) = \left[\log \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi . \left(\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right) - \int_{\Omega^*} \log e^{-\langle \xi, x \rangle} . \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi \right]$$

with $\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} d\xi = 1$

$$\Phi^*(x^*) = \left[-\int_{\Omega^*} \frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} . \log \left(\frac{e^{-\langle \xi, x \rangle}}{\int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi} \right) d\xi \right] \tag{17}$$

In this last equation, $p_x(\xi)=e^{-\langle \xi,x\rangle} / \int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi$ plays the role of maximum entropy density as introduced by Jaynes [52-54] (also called, Gibbs density in Thermodynamics). We call the associated entropy, Koszul Entropy:

$$\Phi^* = -\int\limits_{\Omega^*} p_x(\xi)\log p_x(\xi) d\xi \tag{18}$$

with $p_x(\xi)=e^{-\langle \xi,x\rangle} / \int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi = e^{-\langle x,\xi\rangle - \log \int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi} = e^{-\langle x,\xi\rangle + \Phi(x)}$ and $x^* = \int\limits_{\Omega^*} \xi . p_x(\xi) d\xi$ (19)

This Koszul density $p_x(\xi)=\dfrac{e^{-\langle \xi,x\rangle}}{\int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi}$ help us to develop the log likelihood :

$$\log p_x(\xi) = -\langle x,\xi\rangle - \log \int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi = -\langle x,\xi\rangle + \Phi(x) \tag{20}$$

and deduce from the expectation:

$$E_\xi\left[-\log p_x(\xi)\right] = \left\langle x,x^*\right\rangle - \Phi(x) \tag{21}$$

We also obtain the equation about normalization:

$$\begin{aligned} &\Phi(x) = -\log \int\limits_{\Omega^*} e^{-\langle \xi,x\rangle} d\xi = -\log \int\limits_{\Omega^*} e^{-\left[\Phi^*(\xi)+\Phi(x)\right]} d\xi = \Phi(x) - \log \int\limits_{\Omega^*} e^{-\Phi^*(\xi)} d\xi \\ &\Rightarrow \int\limits_{\Omega^*} e^{-\Phi^*(\xi)} d\xi = 1 \end{aligned} \tag{22}$$

But we have to make appear the variable x^* in $\Phi^*(x^*)$. We have then to write:

$$\begin{aligned} &\log p_x(\xi) = \log e^{-\langle x,\xi\rangle + \Phi(x)} = \log e^{-\Phi^*(\xi)} = -\Phi^*(\xi) \\ &\Rightarrow \Phi^* = -\int\limits_{\Omega^*} p_x(\xi)\log p_x(\xi) d\xi = \int\limits_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*(x^*) \end{aligned} \tag{23}$$

Last equality is true, if we have:

$$\int\limits_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi = \Phi^*\left(\int\limits_{\Omega^*} \xi . p_x(\xi) d\xi \right) \text{ with } x^* = \int\limits_{\Omega^*} \xi . p_x(\xi) d\xi \tag{24}$$

This last relation is associated to classical Jensen inequality. Equality is obtained for Maximum Entropy density for $x^* = D_x\Phi$ [60]:

$$\begin{aligned} &\text{Legendre-Moreau Transform: } \Phi^*(x^*) = \underset{x}{Sup}\left[\left\langle x, x^* \right\rangle - \Phi(x) \right] \\ &\Rightarrow \begin{cases} \Phi^*(x^*) \geq \left\langle x, x^* \right\rangle - \Phi(x) \\ \Phi^*(x^*) \geq \int\limits_{\Omega^*} \Phi^*(\xi) p_x(\xi) d\xi \end{cases} \Rightarrow \begin{cases} \Phi^*(x^*) \geq E\left[\Phi^*(\xi) \right] \\ \text{equality if } x^* = \dfrac{d\Phi}{dx} \end{cases} \end{aligned} \tag{25}$$

We obtain for the maximum entropy density, the equality:

$$E\left[\Phi^*(\xi)\right] = \Phi^*\left(E[\xi]\right) \text{ , } \xi \in \Omega^* \tag{26}$$

To make the link between this Koszul model and maximum entropy density [82-84] introduced by Jaynes [52-54], we use previous notation and we look for the density $p_x(\xi)$ that is the solution to this maximum entropy variational problem. Find the density that maximizes the Shannon entropy with constraint on normalization and the knowledge of first moments:

$$\underset{p_x(.)}{Max}\left[-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi\right] \text{ tel que } \begin{cases} \int_{\Omega^*} p_x(\xi)d\xi = 1 \\ \int_{\Omega^*} \xi.p_x(\xi)d\xi = x^* \end{cases} \tag{27}$$

If we consider the density $q_x(\xi) = e^{-\langle \xi,x\rangle} / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi}$ such that:

$$\begin{cases} \int_{\Omega^*} q_x(\xi).d\xi = \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi / \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi = 1 \\ \log q_x(\xi) = \log e^{-\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle \xi,x\rangle}d\xi} = -\langle x,\xi\rangle - \log \int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi \end{cases} \tag{28}$$

By using the inequality $\log x \geq \left(1 - x^{-1}\right)$ with equality if $x = 1$, we can then write that:

$$-\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \leq -\int_{\Omega^*} p_x(\xi)\left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right)d\xi \tag{29}$$

We develop the right term of the equation:

$$\int_{\Omega^*} p_x(\xi)\left(1 - \frac{q_x(\xi)}{p_x(\xi)}\right)d\xi = \int_{\Omega^*} p_x(\xi)d\xi - \int_{\Omega^*} q_x(\xi)d\xi = 0 \tag{30}$$

knowing that $\int_{\Omega^*} p_x(\xi)d\xi = \int_{\Omega^*} q_x(\xi)d\xi = 1$, we can deduce that:

$$-\int_{\Omega^*} p_x(\xi)\log\frac{p_x(\xi)}{q_x(\xi)}d\xi \leq 0 \Rightarrow -\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq -\int_{\Omega^*} p_x(\xi)\log q_x(\xi)d\xi \tag{31}$$

We have then to develop the right term by using previous expression of $q_x(\xi)$:

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq -\int_{\Omega^*} p_x(\xi)\left[-\langle x,\xi\rangle - \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi\right]d\xi \tag{32}$$

$$-\int_{\Omega^*} p_x(\xi)\log p_x(\xi)d\xi \leq \left\langle x, \int_{\Omega^*} \xi.p_x(\xi)d\xi\right\rangle + \log\int_{\Omega^*} e^{-\langle x,\xi\rangle}d\xi \tag{33}$$

If we use that $x^* = \int\limits_{\Omega^*} \xi . p_x(\xi) d\xi$ et $\Phi(x) = -\log \int\limits_{\Omega^*} e^{-\langle x,\xi \rangle} d\xi$, then we obtain that the density $q_x(\xi) = e^{-\langle \xi,x \rangle} / \int\limits_{\Omega^*} e^{-\langle \xi,x \rangle} d\xi = e^{-\langle x,\xi \rangle - \log \int\limits_{\Omega^*} e^{-\langle \xi,x \rangle} d\xi}$ is the maximum entropy density constrained by $\int\limits_{\Omega^*} p_x(\xi) d\xi = 1$ et $\int\limits_{\Omega^*} \xi . p_x(\xi) d\xi = x^*$:

$$- \int\limits_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \le \langle x, x^* \rangle - \Phi(x) \tag{34}$$

$$- \int\limits_{\Omega^*} p_x(\xi) \log p_x(\xi) d\xi \le \Phi^*(x^*) \tag{35}$$

In the following, we will write $x^* = \hat{\xi}$, to give to this variable his link with momentum $\hat{\xi} = \int\limits_{\Omega^*} \xi . p_{\hat{\xi}}(\xi) d\xi$. To express the density with respect to 1st moment as variable, we have to inverse $\hat{\xi} = \Theta(x) = \dfrac{d\Phi(x)}{dx}$, by writting $x = \Theta^{-1}(\hat{\xi})$ the inverse function (given by Legendre transform):

$$p_{\hat{\xi}}(\xi) = \frac{e^{-\langle \xi, \Theta^{-1}(\hat{\xi}) \rangle}}{\int\limits_{\Omega^*} e^{-\langle \xi, \Theta^{-1}(\hat{\xi}) \rangle} d\xi} \text{ with } \hat{\xi} = \int\limits_{\Omega^*} \xi . p_{\hat{\xi}}(\xi) d\xi \text{ and } \Phi(x) = -\log \int\limits_{\Omega^*} e^{-\langle x,\xi \rangle} d\xi \tag{36}$$

We find finally the Maximum entropy density parametrized by 1st moment $\hat{\xi}$.

4 Links between Koszul-Vinberg Characteristic Function, Koszul forms and Information Geometry

Koszul Hessian Geometry Structures is the key tool to define elementary structures of Information Geometry, that appear as one particular case of more general framework studied by Koszul. In the Koszul-Vinberg Characteristic function $\psi_\Omega(x) = \int\limits_{\Omega^*} e^{-\langle x,\xi \rangle} d\xi, \ \forall x \in \Omega$ where Ω is a sharp convex cone and Ω^* its dual cone, the duality bracket <.,.> has to be defined. We will introduce it by using Cartan-Killing form $\langle x,y \rangle = -B(x, \theta(y))$ with $B(.,.)$ killing form and $\theta(.)$ Cartan involution. The inner product is then invariant with respect to automorphims of cone Ω. Koszul-Vinberg characteristic function could be developed as [57]:

$$\psi_\Omega(x + \lambda u) = \psi_\Omega(x) - \lambda \langle x^*, u \rangle + \frac{\lambda^2}{2} \langle K(x)u, u \rangle + ... \tag{37}$$

with $x^* = \dfrac{d\Phi(x)}{dx}$, $\Phi(x) = -\log \psi_\Omega(x)$ and $K(x) = \dfrac{d^2\Phi(x)}{dx^2}$

In the following developments, we will write β, previous variable written x, because in thermodynamics, this variable corresponds to the Planck temperature, classically $\beta = \frac{1}{T}$. The variable β will be the dual variable of $\hat{\xi}$.

$$p_{\hat{\xi}}(\xi) = \frac{e^{-\langle \Theta^{-1}(\hat{\xi}),\xi \rangle}}{\int_{\Omega^*} e^{-\langle \Theta^{-1}(\hat{\xi}),\xi \rangle}.d\xi} \quad \hat{\xi} = \Theta(\beta) = \frac{\partial \Phi(\beta)}{\partial \beta} \ \text{ with } \ \Phi(\beta) = -\log \psi_\Omega(\beta)$$
$$\psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta,\xi \rangle} d\xi \quad , \quad S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi) \log p_{\hat{\xi}}(\xi).d\xi \ \text{ and } \beta = \Theta^{-1}(\hat{\xi}) \tag{38}$$
$$S(\hat{\xi}) = \langle \hat{\xi}, \beta \rangle - \Phi(\beta)$$

Inversion of the function $\Theta(.)$ is given by $\beta = \Theta^{-1}(\hat{\xi})$ is achieved by Legendre transform using relation between Entropy $S(\hat{\xi})$ and the function $\Phi(\beta)$ (opposite of the logarithm of the Koszul-Vinberg characteristic function):

$$S(\hat{\xi}) = \langle \beta, \hat{\xi} \rangle - \Phi(\beta) \ \text{ with } \ \Phi(\beta) = -\log \int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi \quad \forall \beta \in \Omega \ \text{ and } \ \forall \xi, \hat{\xi} \in \Omega^* \tag{39}$$

We will prove that the 2nd Koszul form $-\frac{\partial^2 \Phi(\beta)}{\partial \beta^2}$ is linked with Fisher Metric of Information Geometry :

$$I(\beta) = -E\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2} \right] \tag{40}$$

To compute the Fisher metric $I(\beta)$, we use the following relations between variable

$$\begin{cases} \log p_{\hat{\xi}}(\xi) = -\langle \xi, \beta \rangle + \Phi(\beta) \\ S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi).\log p_{\hat{\xi}}(\xi).d\xi = -E\left[\log p_{\hat{\xi}}(\xi) \right] \end{cases} \tag{41}$$
$$\Rightarrow S(\hat{\xi}) = \langle E[\xi], \beta \rangle - \Phi(\beta) = \langle \hat{\xi}, \beta \rangle - \Phi(\beta)$$

We can observe that the logarithm of the density is affine with respect to the variable β, and that the Fisher matrix is given by the hessian. We can then deduce that the Fisher Metric is given by the hessian.

$$I(\beta) = -E\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2} \right] = -E\left[\frac{\partial^2 \left(-\langle \xi, \beta \rangle + \Phi(\beta) \right)}{\partial \beta^2} \right] = -\frac{\partial^2 \Phi(\beta)}{\partial \beta^2} = \frac{\partial^2 \log \Psi_\Omega(\beta)}{\partial \beta^2} \tag{42}$$

We can also identify the Fisher metric as a variance:

$$\log \Psi_\Omega(\beta) = \log \int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi \Rightarrow \frac{\partial \log \Psi_\Omega(\beta)}{\partial \beta} = -\frac{1}{\int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi} \int_{\Omega^*} \xi.e^{-\langle \xi,\beta \rangle} d\xi \tag{43}$$

$$\frac{\partial^2 \log \Psi_\Omega(\beta)}{\partial \beta^2} = -\frac{1}{\left(\int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi\right)^2} \left[-\int_{\Omega^*} \xi^2 . e^{-\langle \xi,\beta \rangle} d\xi . \int_{\Omega^*} e^{-\langle \xi,x \rangle} d\xi + \left(\int_{\Omega^*} \xi . e^{-\langle \xi,\beta \rangle} d\xi \right)^2 \right] \quad (44)$$

$$\frac{\partial^2 \log \Psi_\Omega(\beta)}{\partial \beta^2} = \int_{\Omega^*} \xi^2 . \frac{e^{-\langle \xi,\beta \rangle}}{\int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi} d\xi - \left(\int_{\Omega^*} \xi . \frac{e^{-\langle \xi,\beta \rangle}}{\int_{\Omega^*} e^{-\langle \xi,\beta \rangle} d\xi} d\xi \right)^2 \quad (45)$$

$$= \int_{\Omega^*} \xi^2 . p_\beta(\xi) d\xi - \left(\int_{\Omega^*} \xi . p_\beta(\xi) d\xi \right)^2$$

$$I(\beta) = -E_\xi \left[\frac{\partial^2 \log p_\beta(\xi)}{\partial \beta^2} \right] = \frac{\partial^2 \log \psi_\Omega(\beta)}{\partial \beta^2} = E_\xi\left[\xi^2\right] - E_\xi\left[\xi\right]^2 = Var(\xi) \quad (46)$$

In 1977, Crouzeix [59][85] has identified the following relation between both hessian of entropy and characteristic function $\frac{\partial^2 \Phi}{\partial \beta^2} = \left[\frac{\partial^2 S}{\partial \hat{\xi}^2} \right]^{-1}$ giving a relation between the dual metrics in their dual coordinate systems. The metric could be given by Fisher metric or given by the hessian of Entropy S :

$$ds_g^2 = d\beta^T I(\beta) d\beta = \sum_{ij} g_{ij} d\beta_i d\beta_j \quad \text{avec} \quad g_{ij} = \left[I(\beta)\right]_{ij} \quad (47)$$

Thanks to Crouzeix relation [59][85], we observe that 2 geodesic distances given by hessian of dual potential functions in dual coordinates systems that are equal:

$$ds_h^2 = d\hat{\xi}^T \left[\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2} \right] d\hat{\xi} = \sum_{ij} h_{ij} d\hat{\xi}_i d\hat{\xi}_j \quad \text{with} \quad h_{ij} = \left[\frac{\partial^2 S(\hat{\xi})}{\partial \hat{\xi}^2} \right]_{ij} \quad (48)$$

$$ds_h^2 = ds_g^2 \quad (49)$$

One can ask oneself the question of what is the most natural product of duality. This question has been treated by Elie Cartan in his thesis in 1894, by introducing a form called Cartan-Killing form, a symmetric bilinear form naturally associated with any Lie algebra. This form of Cartan-Killing is defined via the endomorphism ad_x of Lie algebra g via the Lie bracket :

$$ad_x(y) = [x,y] \quad (50)$$

The trace of the composition of these 2 endomorphisms defines this bilinear form by:

$$B(x,y) = Tr\left(ad_x ad_y\right) \quad (51)$$

The Cartan-Killing form is symmetric:

$$B(x,y) = B(y,x) \quad (52)$$

and verify associativity property:

$$B([x,y],z) = B(x,[y,z]) \quad (53)$$

given by:

$$B([x,y],z) = Tr\left(ad_{[x,y]}ad_z\right) = Tr\left(\left[ad_x, ad_y\right]ad_z\right)$$
$$= Tr\left(ad_x\left[ad_y, ad_z\right]\right) = B\left(x,[y,z]\right) \tag{54}$$

Elie Cartan proved that if g is a simple Lie algebra (the form of Killing is non-degenerate) then any symmetric bilinear form is a scalar multiple of the form Cartan-Killing. The Cartan-Killing form is invariant under the action of automorphisms $\sigma \in Aut(g)$ of the algebra g:

$$B(\sigma(x),\sigma(y)) = B(x,y) \tag{55}$$

This invariance is deduced from:

$$\begin{cases} \sigma[x,y] = [\sigma(x),\sigma(y)] \\ z = \sigma(y) \end{cases} \Rightarrow \sigma\left[x,\sigma^{-1}(z)\right] = [\sigma(x),z] \tag{56}$$

by writting $ad_{\sigma(x)} = \sigma \circ ad_x \circ \sigma^{-1}$

Then, we can write:

$$B(\sigma(x),\sigma(y)) = Tr\left(ad_{\sigma(x)}ad_{\sigma(y)}\right) = Tr\left(\sigma \circ ad_x ad_y \circ \sigma^{-1}\right) = Tr\left(ad_x ad_y\right) = B(x,y) \tag{57}$$

Cartan has introduced this natural inner product that is invariant by the automorphisms of the Lie algebra , from this Cartan-Killing form:

$$\langle x,y\rangle = -B(x,\theta(y)) \tag{58}$$

with $\theta \in g$ the Cartan involution (an involution on the Lie algebra g is an automorphism θ such that the square is equal to identity).

We summarize all these relations of information geometry from the characteristic function of Koszul-Vinberg, and the duality given via the Cartan-Killing form, as described in the figure below:

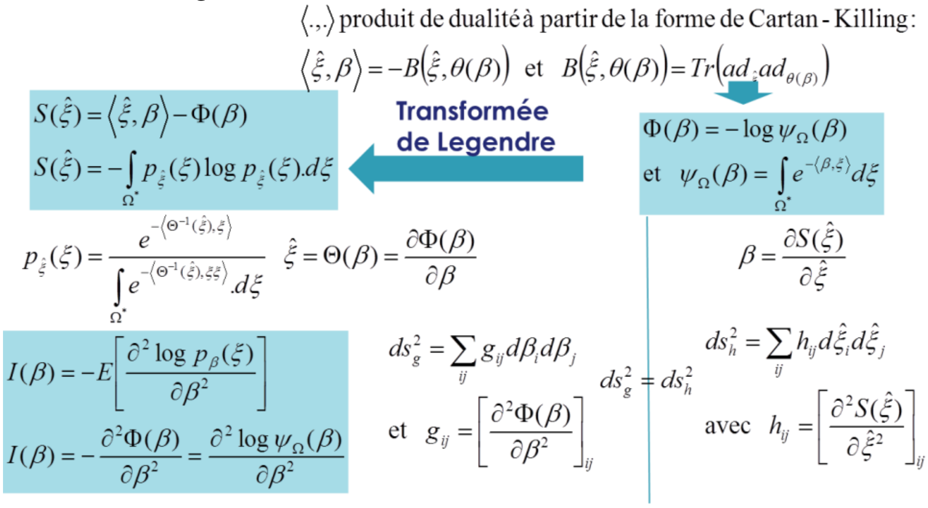

Fig. 6. Relations between Cartan-Killing Form, Koszul-Vinberg Characteristic Function, Potentials and Dual Coordinates, and Metrics of Information Geometry

Thanks to the expression of the characteristic function of Koszul-Vinberg and the Cartan-Killing form, one can express the maximum Entropy density in very general way. For example, by applying these formulas to the cone Ω (self-dual : $\Omega^* = \Omega$) symmetric positive definite matrices $Sym^+(n)$, Cartan-Killing form gives us the duality of product :

$$\langle \eta, \xi \rangle = Tr\left(\eta^T \xi\right), \quad \forall \eta, \xi \in Sym^+(n) = \left\{ \xi / \xi^T = \xi, \xi > 0 \right\} \tag{59}$$

The maximum entropy density is given by:

$$\psi_\Omega(\beta) = \int_{\Omega^*} e^{-\langle \beta, \xi \rangle} d\xi = \det\left(\beta\right)^{-\frac{n+1}{2}} \psi_\Omega(I_d)$$
$$\text{and} \quad \hat{\xi} = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\partial(-\log \psi_\Omega(\beta))}{\partial \beta} = \frac{n+1}{2} \beta^{-1} \tag{60}$$

From which, we can deduce the final expression:

$$p_{\hat{\xi}}(\xi) = e^{-\left\langle \Theta^{-1}(\hat{\xi}), \xi \right\rangle + \Phi\left(\Theta^{-1}(\hat{\xi})\right)} = \psi_\Omega\left(I_d\right) . \left[\det\left(\alpha \hat{\xi}^{-1}\right) \right] . e^{-Tr\left(\alpha \hat{\xi}^{-1} \xi\right)}$$
$$\text{with} \quad \alpha = \frac{n+1}{2} \tag{61}$$

In the case of multivariate Gaussian densities, as noted by Souriau [107-108], the classical Gibbs expression can be rewritten by modifying the coordinate system and defining a new duality product [101-106]. The multivariate Gaussian density is classically written with the following coordinate system (m, R), with m the mean vector, and R the covariance matrix of the vector z :

$$p_{\hat{\xi}}(\xi) = \frac{1}{\left(2\pi\right)^{n/2} \det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)} \quad \text{with} \quad \begin{cases} m = E(z) \\ R = E\left[\left(z-m\right)\left(z-m\right)^T \right] \end{cases} \tag{62}$$

By developing the term in the exponential:

$$\begin{aligned} \frac{1}{2}(z-m)^T R^{-1}(z-m) &= \frac{1}{2}\left[z^T R^{-1} z - m^T R^{-1} z - z^T R^{-1} m + m^T R^{-1} m \right] \\ &= \frac{1}{2} z^T R^{-1} z - m^T R^{-1} z + \frac{1}{2} m^T R^{-1} m \end{aligned} \tag{63}$$

We can writte this density as a Gibbs density by introducing a new duality braket between $\left(z, zz^T\right)$ and $\left(-R^{-1}m, \frac{1}{2} R^{-1}\right)$:

$$p_{\hat{\xi}}(\xi)=\frac{1}{\left(2\pi\right)^{n/2}\det(R)^{1/2}e^{\frac{1}{2}m^TR^{-1}m}}e^{-\left[-m^TR^{-1}z+\frac{1}{2}z^TR^{-1}z\right]}=\frac{1}{Z}e^{-\langle\xi,\beta\rangle}$$

$$\xi=\begin{bmatrix} z \\ zz^T\end{bmatrix} \text{ et } \beta=\begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1}\end{bmatrix}=\begin{bmatrix} a \\ H\end{bmatrix} \tag{64}$$

with $\langle\xi,\beta\rangle=a^Tz+z^THz=Tr\left[za^T+H^Tzz^T\right]$

We can then writte the density in Koszul form:

$$p_{\hat{\xi}}(\xi)=\frac{1}{\int\limits_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}e^{-\langle\xi,\beta\rangle}=\frac{1}{\mathrm{Z}}e^{-\langle\xi,\beta\rangle}$$

with $\log\left(Z\right)=n\log(2\pi)+\frac{1}{2}\log\det(R)+\frac{1}{2}m^TR^{-1}m$

$$\xi=\begin{bmatrix} z \\ zz^T\end{bmatrix},\ \hat{\xi}=E\left[\xi\right]=\begin{bmatrix} E\left[z\right] \\ E\left[zz^T\right]\end{bmatrix}=\begin{bmatrix} m \\ R+mm^T\end{bmatrix},\ \beta=\begin{bmatrix} a \\ H\end{bmatrix}=\begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1}\end{bmatrix} \tag{65}$$

with $\langle\xi,\beta\rangle=Tr\left[za^T+H^Tzz^T\right]$

$$R=E\left[\left(z-m\right)\left(z-m\right)^T\right]=E\left[zz^T-mz^T-zm^T+mm^T\right]=E\left[zz^T\right]-mm^T$$

We can then compute the Koszul-Vinberg characteristic function whose opposite of the logarithm provides the potential function:

$$\psi_\Omega(\beta)=\int\limits_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi$$

$$\text{and}\quad \Phi(\beta)=-\log\psi_\Omega(\beta)=\frac{1}{2}\left[-Tr\left[H^{-1}aa^T\right]+\log\left[(2)^n\det H\right]-n\log\left(2\pi\right)\right] \tag{66}$$

That verify the following relation given by Koszul and linked with 1st Koszul form:

$$\frac{\partial\Phi(\beta)}{\partial\beta}=\frac{\partial\left[-\log\psi_\Omega(\beta)\right]}{\partial\beta}=\int\limits_{\Omega^*}\xi\frac{e^{-\langle\xi,\beta\rangle}}{\int\limits_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}.d\xi=\int\limits_{\Omega^*}\xi.p_{\hat{\xi}}(\xi).d\xi=\hat{\xi}$$

$$\frac{\partial\Phi(\beta)}{\partial\beta}=\begin{bmatrix} \frac{\partial\Phi(\beta)}{\partial a} \\ \frac{\partial\Phi(\beta)}{\partial H}\end{bmatrix}=\begin{bmatrix} m \\ R+mm^T\end{bmatrix}=\hat{\xi} \tag{67}$$

The 2nd dual potential is given by the Legendre transform of $\Phi(\beta)$:

$$S(\hat{\xi})=\left\langle\hat{\xi},\beta\right\rangle\text{-}\Phi\left(\beta\right)\quad\text{with}\quad\frac{\partial\Phi(\beta)}{\partial\beta}=\hat{\xi}\quad\text{and}\quad\frac{\partial S(\hat{\xi})}{\partial\hat{\xi}}=\beta$$

$$S\left(\hat{\xi}\right)=-\int\limits_{\Omega^*}\frac{e^{-\langle\xi,\beta\rangle}}{\int\limits_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}\log\frac{e^{-\langle\xi,\beta\rangle}}{\int\limits_{\Omega^*}e^{-\langle\xi,\beta\rangle}.d\xi}.d\xi=-\int\limits_{\Omega^*}p_{\hat{\xi}}(\xi)\log p_{\hat{\xi}}(\xi).d\xi \tag{68}$$

that is the Shannon Entropy:

$$S(\hat{\xi}) = -\int_{\Omega^*} p_{\hat{\xi}}(\xi) \log p_{\hat{\xi}}(\xi).d\xi$$
$$= \frac{1}{2}\left[\log(2)^n \det\left[H^{-1}\right] + n\log\left(2\pi.e\right)\right] = \frac{1}{2}\left[\log\det\left[R\right] + n\log\left(2\pi.e\right)\right] \quad (69)$$

The Fisher metric of Information Geometry is given by the hessian of the opposite of the logarithm of the Koszul-Vinberg characteristic function:

$$ds_g^2 = d\beta^T I(\beta) d\beta = \sum_{ij} g_{ij} d\beta_i d\beta_j$$
$$\text{with} \quad g_{ij} = \left[I(\beta)\right]_{ij} \ \text{ and } \ I(\beta) = -E_\xi\left[\frac{\partial^2 \log p_\beta(\xi)}{\partial\beta^2}\right] = \frac{\partial^2 \log\psi_\Omega(\beta)}{\partial\beta^2} \quad (70)$$

Then, for the multivariate gaussian density, we have the following Fisher metric :

$$ds^2 = \sum_{ij} g_{ij} d\theta_i d\theta_j = dm^T R^{-1} dm + \frac{1}{2} Tr\left[\left(R^{-1}dR\right)^2\right] \quad (71)$$

Geodesic equations are given by Euler-Lagrange equations:

$$\sum_{i=1}^{n} g_{ik}\ddot{\theta}_i + \sum_{i,j=1}^{n} \Gamma_{ijk}\dot{\theta}_i\dot{\theta}_j = 0 \quad , \quad k = 1,...,n$$
$$\text{with} \quad \Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial\theta_i} + \frac{\partial g_{jk}}{\partial\theta_j} + \frac{\partial g_{ij}}{\partial\theta_k}\right] \quad (72)$$

that can be reduced to the equations:

$$\begin{cases} \ddot{R} + \dot{m}\dot{m}^T - \dot{R}R^{-1}\dot{R} = 0 \\ \ddot{m} - \dot{R}R^{-1}\dot{m} = 0 \end{cases} \quad (73)$$

We can use a result of Souriau [107] that the component of « moment map » are constants (geometrization of Emmy Noether theorem), to identify the following constants [104]:

$$\frac{d\Pi_R}{dt} = \begin{bmatrix} \dfrac{d\left(R^{-1}\dot{R} + R^{-1}\dot{m}m^T\right)}{dt} & \dfrac{d\left(R^{-1}\dot{m}\right)}{dt} \\ 0 & 0 \end{bmatrix} = 0$$
$$\Rightarrow \begin{cases} R^{-1}\dot{R} + R^{-1}\dot{m}m^T = B = cste \\ R^{-1}\dot{m} = b = cste \end{cases} \quad (74)$$

with Π_R the moment map introduced by Souriau [107]. This moment map could be computed if we consider the following Lie group acting in case of gaussian densities:

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} R^{1/2}X + m \\ 1 \end{bmatrix} \quad , \quad \begin{cases} (m,R) \in R^n \times Sym^+(n) \\ M = \begin{bmatrix} R^{1/2} & m \\ 0 & 1 \end{bmatrix} \in G_{aff} \end{cases} \quad (75)$$
$$X \approx \aleph(0,I) \rightarrow Y \approx \aleph(m,R)$$

$R^{1/2}$ is given by Cholesky decomposition of R. $R^{1/2}$ is the group of triangular matrix with positive elements on the diagonal. Euler-Poincaré equations, reduced equations from Euler-Lagrange equations:

$$\begin{cases} \dot{m} = Rb \\ \dot{R} = R\left(B - bm^T\right) \end{cases} \tag{76}$$

Geodesic distance between multivariate gaussian density is then obtained by "*geodesic shooting*" method that will provide iteratively the final solution from the tangent vector at the first point:

$$\left(R^{-1}(0)\dot{m}(0), R^{-1}(0)\left(\dot{R}(0) + \dot{m}(0)m(0)^T\right)\right) = (b, B) \in R^n \times Sym^+(n) \tag{77}$$

and then deduce the distance :

$$d = \sqrt{\dot{m}(0)^T R^{-1}(0)\dot{m}(0) + \frac{1}{2}Tr\left[\left(R^{-1}(0)\dot{R}(0)\right)^2\right]} \tag{78}$$

Geodesic shooting is obtained by using equations established by Eriksen [89-90] for "*exponential map*" using the following change of variables:

$$\begin{cases} \Delta(t) = R^{-1}(t) \\ \delta(t) = R^{-1}(t)m(t) \end{cases} \Rightarrow \begin{cases} \dot{\Delta} = -B\Delta + bm^T \\ \dot{\delta} = -B\delta + \left(1 + \delta^T \Delta^{-1}\delta\right)b \\ \Delta(0) = I_p, \delta(0) = 0 \end{cases} \text{ with } \begin{cases} \dot{\Delta}(0) = -B \\ \dot{\delta}(0) = b \end{cases} \tag{79}$$

The method based on geodesic shooting consists in iteratively approaching the solution by geodesic shooting in direction $\left(\dot{\delta}(0), \dot{\Delta}(0)\right)$, using the following exponential map :

$$\Lambda(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \begin{pmatrix} \Delta & \delta & \Phi \\ \delta^T & \varepsilon & \gamma^T \\ \Phi^T & \gamma & \Gamma \end{pmatrix} \tag{80}$$

$$\text{with} \quad A = \begin{pmatrix} -B & b & 0 \\ b^T & 0 & -b^T \\ 0 & -b & B \end{pmatrix}$$

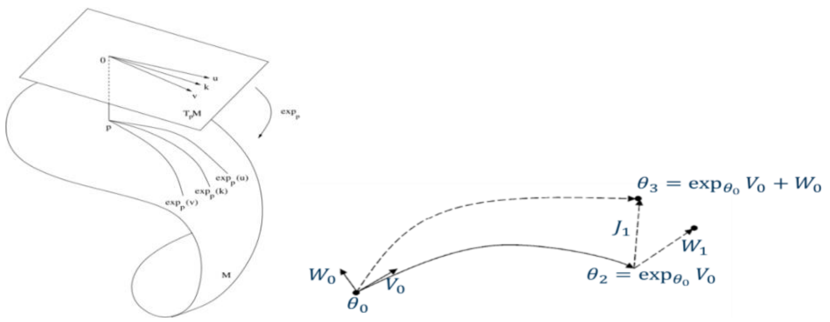

Fig. 7. Principle of Geodesic Shooting in the direction of the initial vector V0 at the origin and correction by W0

The principle of geodesic shooting is the following. We consider one geodesic χ between θ_0 and θ_1 with an initial tangent vector V from origin, and assume that V is modified by W, with respect to $V+W$. Variation of final point θ_1 could be obtained by Jacobi vector field $J(0)=0$ and $\dot{J}(0)=W$:

$$J(t)=\frac{d}{d\alpha}\exp_{\theta_0}\left(t(V+\alpha W)\right)\Big|_{\alpha=0} \tag{81}$$

5 Koszul's study of homogeneous bounded domains and affine representations of Lie groups and algebras

Jean-Louis Koszul [7-14] and his student Jacques Vey [16-17] introduced new theorems with more general extension than previous results:

Koszul theorem [14] : Let Ω be a sharp convex open in an affine space of E of finite dimension on R. If a unimodular Lie group of affine transformations operates transitively on Ω, Ω is a cone.

Koszul-Vey Theorem [17]: Let M a hessian connected manifold associated with the hessian metric g. Assume that M has a closed 1-form α such that $D\alpha = g$ and that there is a group G of affine automorphisms of M preserving α, then :

- If M/G is almost compact, then the manifold, universal covering of M, is affinely isomorphic to a convex domain of an affine space containing no straight line.
- If M/G is compact, then Ω is a sharp convex cone.

Jean-Louis Koszul developed his theory, studying the homogeneous domains, in particular the homogeneous symmetric bounded domains of Siegel, which we note DS [34-35]. He has proved that there is a subgroup *G* in the group of complex affine automorphisms of these domains (Iwasawa subgroup), so that *G* acts on DS in a merely transitive way. The Lie algebra g of *G* has a structure which is an algebraic translation of the Kähler structure DS.

Koszul considered on *G/B* an invariant complex structure tensor *I*. All the invariant volumes on *G/B*, equal up to a constant factor, define with the complex structure the same invariant Hermitian form on *G/B*, called Hermitian canonical form, denoted *h*. Let *E* be a differentiable fiber space of base *M* and let *p* be the projection of *E* on *M*, such that $p^*\left(\left(pX\right).f\right)=X.\left(p^*f\right)$. The projection $p:E\to M$ defines an injective homomorphism p^* of the space of differential forms of M in the space of the differential forms of *E* such that for any form α of degree *n* on *M* and any sequence of n projectable vectors fields, we have $p^*\left(\alpha\left(pX_1,pX_2,...,pX_n\right)\right)=\left(p^*\alpha\right)\left(X_1,X_2,...,X_n\right)$.

Let I be the tensor of an almost complex structure on the basis M, there exists on E a tensor J of type *(1,1)* and only one which possesses the following properties $p(JX) = I(pX)$ and $J^2X = -X \mod h$, $X \in g$ for any vector field X on E. Let G be a connected Lie group and B a closed subgroup of G, we note g the Lie algebra left invariant vector fields on G and b sub-algebra of g corresponding to B. The canonical mapping of G on G/B is denoted p (defining E as before). We assume that there exists on G/B an invariant volume by G, which consist in assuming that, for all $s \in B$, the automorphism $X \to Xs$ of g defines by passing to the quotient an automorsphism of determinant *1* in *g/b* . Let r be the dimension of G/B and $(X_i)_{1\le i\le m}$ a base of g such that $X_i \in b$, for $r \le i \le m$. Let $(\xi_i)_{1\le i\le m}$ the base of the space of differential forms of degree *1* left invariant on G such that $\xi_i(X_j) = \delta_{ij}$. If ω is an invariant volume on G/B, then $\Omega = p^*\omega$ is equal, up to a constant factor, to $\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_r$. We will assume the base (X_j) chosen so that this factor is equal to *1*, let $\Omega = \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_r$. For any vector field that can be projected X on G, we have:

$$p^*(div(pX))\Omega = p^*((div(pX))\omega) = p^*((pX)\omega) = X\Omega = \sum_{j=1}^{r} \xi_j([X_j, X])\Omega \quad (82)$$

$$p^*(div(pX)) = \sum_{j=1}^{r} \xi_j([X_j, X]) \quad (83)$$

These elements being defined, Koszul calculates the Hermitian canonical form of G/B, denoted h, more particularly $\eta = p^*h$ on G. Let X and Y both right invariant vector fields on G. They are projectable and the fields pX and pY are conformal vector fields on G/B such that $div(pX) = div(pY) = 0$, because the volume and the complex structure of G/B are invariant under G. As a result, if κ is the Kähler form of h and if $\alpha = p^*\kappa$, then:

$$4\alpha(X,Y) = 4p^*(\kappa(pX, pY)) = p^*div(I[pX, pY]) \quad (84)$$

and as $p(J[X,Y]) = I[pX, pY]$, we obtain :

$$4\alpha(X,Y) = p^*div(J[X,Y]) = \sum_{i=1}^{2n} \xi_i([X_i, J[X,Y]]) \quad (85)$$

X and Y are two left invariant vectors fields on G. X' and Y' right invariant vectors fields coinciding with X and Y at the point e, neutral element of G. If $T = [X', Y']$ is tight invariant vectors fields which coincide with $-[X,Y]$ on e, then:

$$[X, JT] = J[X,[X,Y]] - [X, J[X,Y]] \text{ at point e} \quad (86)$$

At point e, we have the equality:

$$4\alpha(X,Y) = \sum_{i=1}^{2n} \xi_i([J[X,Y], X_i] - J[[X,Y], X_i]) \quad (87)$$

As the form α is invariant on the left by *G*, this equality is verified for all points. For any endomorphism Θ of the space *g* such that $\Theta b \subset b$, we denote by $Tr_b\Theta$ the trace of the restriction of Θ to *b* and by $Tr_{g/b}\Theta$ the trace of the endomorphism of *g/b* deduced from Θ by passage to the quotient, with $Tr\Theta = Tr_b\Theta + Tr_{g/b}\Theta$. We have :

$$Tr_{g/b}\Theta = \sum_{i=1}^{2n} \xi_i\left(\Theta X_i\right) \quad (88)$$

Whatever $X \in g$ and $s \in B$, we have $J\left(Xs\right) - \left(JX\right)s \in b$. If $ad(Y)$ is the endomorphism of *g* defined by $ad(Y).Z = [Y,Z]$, we have $\left(Jad\left(Y\right) - ad\left(Y\right)J\right)g \subset b$ for all $Y \in b$. We can deduce, for all $X \in g$, the endomorphism $ad\left(JX\right) - Jad(X)$ leaves steady the subspace *b*. Koszul defines a linear form Ψ on the space *g* by defining:

$$\Psi(X) = Tr_{g/b}\left(ad\left(JX\right) - Jad(X)\right) \quad , \quad \forall X \in g \quad (89)$$

Koszul has finally obtained the following fundamental theorem :

Theorem of Koszul [7]:
The Kähler form of the Hermitian canonical form has for image by p^* the differential of the form $-\frac{1}{4}\Psi(X) = -\frac{1}{4}Tr_{g/b}\left(ad\left(JX\right) - Jad(X)\right) \quad , \quad \forall X \in g$

Koszul note that the form Ψ is independent of the choice of the tensor *J*. It is determined by the invariant complex structure of *G/B*. The form Ψ is right invariant by B. For all $s \in B$, note the endomorphism $r(s): X \to Xs$ of g . Since $J(Xs) = (JX)s \bmod b$ and that $Tr_{g/b}ad(Y) = 0$, we have:

$$\Psi(Xs) = Tr_{g/b}\left(ad\left(\left(JX\right)s\right) - Jad(Xs)\right) \quad , \quad \forall X \in g, \forall Y \in b \quad (90)$$

$$\Psi(Xs) = Tr_{g/b}\left(r(s)ad(JX)r(s)^{-1} - Jr(s)ad(X)r(s)^{-1}\right) \quad (91)$$

$$\Psi(Xs) = \Psi(X) + Tr_{g/b}\left(\left(J - r(s)^{-1}Jr(s)\right)ad(X)\right) \quad , \quad \forall X \in g, s \in B \quad (92)$$

As $\left(J - r(s)^{-1}Jr(s)\right)$ maps *g* in *b*, we get $\Psi(Xs) = \Psi(X)$. The form Ψ is not zero on *b*. This is not the image by p^* of a differential form of *G/B*. However, the right invariance of Ψ on *B* is translated, infinitesimally by the relation:

$$\Psi\left(\left[b,g\right]\right) = (0) \quad (93)$$

Koszul proved that the canonical hermitian form *h* of a homogeneous Kähler manifold *G/B* has the following expression:

$$\eta(X,Y)=\frac{1}{2}\Psi([JX,Y])$$
$$\text{with}\quad \begin{cases}\Psi([X,Y])=\Psi([JX,JY]) \\ \eta([JX,JY])=\eta(X,Y)\end{cases}\quad \forall X,Y\in g \tag{94}$$

To do, the link with the first chapters, we can summarize the main result of Koszul that there is an integrable structure almost complex J on g, and for $l\in g^*$ defined by a positive J -invariant inner product on g:

$$\langle X,Y\rangle_l=\langle [JX,Y],l\rangle \tag{95}$$

Koszul has proposed as admissible form, $l\in g^*$, the form ξ:

$$\Psi(X)=\langle X,\xi\rangle=Tr[ad(JX)-J.ad(X)]\quad \forall X\in g \tag{96}$$

Koszul proved that $\langle X,Y\rangle_\xi$ coincides, up to a positive multiplicative constant, with the real part of the Hermitian inner product obtained by the Bergman metric of symmetric homogeneous bounded domains DS by identifying g with the tangent space of DS . The 1st Koszul form is then given by:

$$\alpha=-\frac{1}{4}d\Psi(X) \tag{97}$$

We can illustrate this structure for the simplest example of DS, the Poincaré upper half-plane $V=\{z=x+iy\,/\,y>0\}$ which is isomorphic to the open $zz^*<1$, which is a bounded domain. The group *G* of transformations $z\to az+b$ with *a* and *b* real values with $a>0$ is simply transitive in *V*. We identify *G* and *V* by the application passing from $s\in G$ an element to the image $i=\sqrt{-1}$ by s .

Let's define vector fields $X=y\frac{d}{dx}$ and $Y=y\frac{d}{dy}$ which generate the vector space of left invariant vectors fields on *G*, and *J* an almost complex structure on *V* defined by $JX=Y$. As $[X,Y]=-Y$ and $ad(Y).Z=[Y,Z]$ then:

$$\begin{cases}Tr[ad(JX)-Jad(X)]=2 \\ Tr[ad(JY)-Jad(Y)]=0\end{cases} \tag{98}$$

The Koszul forms and the Koszul metric are respectively given by :

$$\Psi(X)=2\frac{dx}{y}\Rightarrow\alpha=-\frac{1}{4}d\Psi=-\frac{1}{2}\frac{dx\wedge dy}{y^2}\Rightarrow ds^2=\frac{dx^2+dy^2}{2y^2} \tag{99}$$

We note that $\alpha=-\frac{1}{4}d\Psi(X)$ is indeed the Kähler form of Poincaré's metric, which is invariant by the automorphisms of the upper half-plane.

The following example concerns $V=\{Z=X+iY/X,Y\in Sym(p),\ Y>0\}$ the upper half-space of Siegel (which is the most natural extension of the Poincaré half-plane) with:

$$\begin{cases} SZ=(AZ+B)D^{-1} \\ A^T D=I, B^T D=D^T B \end{cases} \text{ with } S=\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \text{ and } J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad (100)$$

We can then compute Koszul forms and the metric:

$$\Psi(dX+idY)=\frac{3p+1}{2}Tr\left(Y^{-1}dX\right)$$
$$\Rightarrow \begin{cases} \alpha=-\frac{1}{4}d\Psi=\frac{3p+1}{8}Tr\left(Y^{-1}dZ\wedge Y^{-1}d\bar{Z}\right) \\ ds^2=\frac{(3p+1)}{8}Tr\left(Y^{-1}dZY^{-1}d\bar{Z}\right) \end{cases} \qquad (101)$$

We recover Carl-Ludwig Siegel metric for the upper half space.
More recent development on Kähler manifolds are described in [109] et [110].

Koszul studied symmetric homogeneous spaces and defines the relation between invariant flat affine connections and the affine representations of Lie algebras and invariant Hessian metrics characterized by affine representations of Lie algebras. Koszul provides a correspondence between symmetric homogeneous spaces with invariant Hessian structures using affine representations of Lie algebras, and proves that a symmetric homogeneous space simply connected with an invariant Hessian structure is a direct product of a Euclidean space and of a homogeneous dual-cone. Let *G* be a connected Lie group and *G/K* a homogeneous space over which *G* acts effectively. Koszul gives a bijective correspondence between all planar *G* -invariantes connections on *G/K* and all of a certain class of affine representations of the Lie algebra of *G*. The main theorem of Koszul is:

Koszul's theorem: Let *G/K* be a homogeneous space of a connected Lie group *G* and be $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of *G* and *K*, assuming that *G/K* has G-invariant connection, then admits an affine representation *(f, q)* on the vector space *E*. Conversely, assume that *G* is simply connected and has an affine representation, then *G/K* admits a flat G-invariant connection.

In the foregoing, the basic tool studied by Koszul is the affine representation of Lie algebra and Lie group. To study these structures, Koszul introduced the following developments.

Let Ω a convex domain on R^n without any straight lines, and an associated convex cone $V(\Omega)=\left\{(\lambda x, x)\in R^n\times R / x\in\Omega, \lambda\in R^+\right\}$, then there exist an affine embedding:

$$\ell : x\in\Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega) \tag{102}$$

If we consider η the group of homomorphism of $A(n,R)$ in $GL(n+1,R)$ given by:

$$s\in A(n,R)\mapsto \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1,R) \tag{103}$$

and the affine representation of Lie algebra:

$$\begin{bmatrix} f & q \\ 0 & 0 \end{bmatrix} \tag{104}$$

with $A(n,R)$ the group of all affine representations of R^n. We have $\eta(G(\Omega))\subset G(V(\Omega))$ and the pair (η,ℓ) of homomorphism $\eta : G(\Omega)\to G(V(\Omega))$ and the application $\ell : \Omega\to V(\Omega)$ is equivariant.

If we observe Koszul affine representations of Lie algebra and Lie group, we have to consider G a convex Lie group and E a real or complex vector space of finite size, Koszul has introduced an affine representation of G in E such that:

$$\begin{aligned} &E\to E \\ &a\mapsto sa \quad \forall s\in G \end{aligned} \tag{105}$$

is an affine representation. We set $A(E)$ the set of all affine transformation of a real vector space E, a Lie group called affine representation group of E. The set $GL(E)$ of all regular linear represnetation of E, a sub-group of $A(E)$.

We define a linear representation of G in $GL(E)$:

$$\begin{aligned} &\mathbf{f} : G\to GL(E) \\ &\quad s\mapsto \mathbf{f}(s)a = sa - so \quad \forall a\in E \end{aligned} \tag{106}$$

and a map from G to E:

$$\begin{aligned} &\mathbf{q} : G\to E \\ &\quad s\mapsto \mathbf{q}(s) = so \quad \forall s\in G \end{aligned} \tag{107}$$

then, we have $\forall s,t\in G$:

$$\mathbf{f}(s)\mathbf{q}(t)+\mathbf{q}(s)=\mathbf{q}(st) \tag{108}$$

deduced from $\mathbf{f}(s)\mathbf{q}(t)+\mathbf{q}(s)=s\mathbf{q}(t)-so+so=s\mathbf{q}(t)=sto=\mathbf{q}(st)$.

Inversely, if a map $\mathbf{q}$ from G to E and a linear representation $\mathbf{f}$ from G to $GL(E)$ verifying previous equation, then we can define an affine representation from G in E, written by $(\mathbf{f},\mathbf{q})$:

$$\mathrm{A}ff(s): a \mapsto sa = \mathbf{f}(s)a + \mathbf{q}(s) \quad \forall s \in G, \forall a \in E \tag{109}$$

The condition $\mathbf{f}(s)\mathbf{q}(t)+\mathbf{q}(s)=\mathbf{q}(st)$ is equal to the request that the following mapping is an homomorphism:

$$\mathrm{A}ff : s \in G \mapsto \mathrm{A}ff(s) \in A(E) \tag{110}$$

We write f the affine representation of Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$ and q the restriction to $\mathfrak{g}$ to the differential of $\mathbf{q}$ (f and q differential of $\mathbf{f}$ and $\mathbf{q}$ respectivelly), Koszul proved the following equation:

$$f(X)q(Y) - f(Y)q(X) = q([X,Y]) \quad \forall X,Y \in \mathfrak{g} \tag{111}$$

with $f : \mathfrak{g} \to gl(E)$ and $q : \mathfrak{g} \mapsto E$

where $gl(E)$ the set of all linear endomorphisms of E, Lie algebra of $GL(E)$.

We use the assumption that:

$$q(Ad_s Y) = \left.\frac{d\mathbf{q}(s.e^{tY}.s^{-1})}{dt}\right|_{t=0} = \mathbf{f}(s)f(Y)\mathbf{q}(s^{-1}) + \mathbf{f}(s)q(Y) \tag{112}$$

We then obtain:

$$q([X,Y]) = \left.\frac{d\mathbf{q}(Ad_{e^{tX}}Y)}{dt}\right|_{t=0} = f(X)q(Y)\mathbf{q}(e) + \mathbf{f}(e)f(Y)(-q(X)) + f(X)q(Y) \tag{113}$$

where e is neutral element of G. Since $\mathbf{f}(e)$ is identity map and $\mathbf{q}(e)=0$, we have the equality:

$$f(X)q(Y) - f(Y)q(X) = q([X,Y]) \tag{114}$$

A pair (f,q) of linear representation of f of a Lie algebra $\mathfrak{g}$ on E and a linear map q from $\mathfrak{g}$ in E is an affine representation of $\mathfrak{g}$ in E, if it satisfy:

$$f(X)q(Y) - f(Y)q(X) = q([X,Y]) \tag{115}$$

Inversely, if we assume that $\mathfrak{g}$ has an affine representation (f,q) on E, by using the coordinate systems $\{x^1,...,x^n\}$ on E, we can express the affine map $v \mapsto f(X)v + q(Y)$ by a matrix representation of size $(n+1)\times(n+1)$:

$$aff(X) = \begin{bmatrix} f(X) & q(X) \\ 0 & 0 \end{bmatrix} \tag{116}$$

where $f(X)$ is a matrix of size $n \times n$ and $q(X)$ a vector of size n.

$X \mapsto aff(X)$ is an injective homomorphism of Lie algebra $\mathfrak{g}$ in Lie algebra of matrices $(n+1)\times(n+1)$, $gl(n+1,R)$:

$$\left|\begin{array}{l} \mathfrak{g} \to gl(n+1,R) \\ X \mapsto aff(X) \end{array}\right. \tag{117}$$

If we note $\mathfrak{g}_{aff} = aff(\mathfrak{g})$, we write G_{aff} linear Lie sub-group of $GL(n+1,R)$ generated by $\mathfrak{g}_{aff}$. One element of $s \in G_{aff}$ could be expressed by:

$$Aff(s) = \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \quad (118)$$

Let M_{aff} the orbit of G_{aff} from the origin o, then $M_{aff} = \mathbf{q}(G_{aff}) = G_{aff} / K_{aff}$ where $K_{aff} = \{ s \in G_{aff} / \mathbf{q}(s) = 0 \} = Ker(\mathbf{q})$.

We can give as example the following case. Let Ω a convex domain in R^n without any straight line, we define the cone $V(\Omega)$ in $R^{n+1} = R^n \times R$ by $V(\Omega) = \{ (\lambda x, x) \in R^n \times R / x \in \Omega, \lambda \in R^+ \}$. Then, there is an affine embedding:

$$\ell : x \in \Omega \mapsto \begin{bmatrix} x \\ 1 \end{bmatrix} \in V(\Omega) \quad (119)$$

If we consider η the group of homomorphisms of $A(n,R)$ in $GL(n+1,R)$ given by:

$$s \in A(n,R) \mapsto \begin{bmatrix} \mathbf{f}(s) & \mathbf{q}(s) \\ 0 & 1 \end{bmatrix} \in GL(n+1,R) \quad (120)$$

with $A(n,R)$ the group of all affine transformations in R^n. We have $\eta(G(\Omega)) \subset G(V(\Omega))$ and the pair (η, ℓ) of homomorphism $\eta : G(\Omega) \to G(V(\Omega))$ and the map $\ell : \Omega \to V(\Omega)$ are equivariant:

$$\ell \circ s = \eta(s) \circ \ell \text{ and } d\ell \circ s = \eta(s) \circ d\ell \quad (121)$$

6 Conclusion

The community of "Geometric Science of Information" (GSI) has lost a mathematician of great value, which informed his views by the depth of his knowledge of the elementary structures of hessian geometry and bounded homogeneous domains. His modesty was inversely proportional to his talent. Professor Koszul built in over 60 years of mathematical career, in the silence of his passions, an immense work, which makes him one of the great mathematicians of the XXth century, whose importance will only affirm with the time. In this troubled time and rapid transformation of society and science, the example of Professor Koszul must be regarded as a model for future generations, to avoid them the trap of fleeting glories and recognitions too fast acquired. The work of Professor Koszul is also a proof of fidelity to his masters and in the first place to Prof. Elie Cartan, who inspired him throughout his life. Henri Cartan writes on this subject "*I do not forget the homage he paid to Elie Cartan's work in Differential Geometry during the celebration, in Bucharest, in 1969, of the centenary of his birth. It is not a coincidence that this centenary was also celebrated in Grenoble the same year. As always, Koszul spoke with the discretion and tact that we know him, and that we love so much at home*". I will conclude by quoting Jorge Luis Borges "Forgetfulness and memory are also inventive" (Brodie's report). Our generation and previous one have forgotten or misunderstood the depth of the work of Jean-Louis Koszul and Elie Cartan on the study of bounded homogeneous domains. It is our responsibility to correct this omission, and to make it the new inspiration for the Geometric Science of Information. I will conclude by asking you to listen to the last interview of Jean-Louis Koszul for 50th birthday of Joseph Fourier Institute [6], especially when Koszul is interested by "*conifers and cedars trees planted by Claude Chabauty*", or by the "*pretty catalpa tree*" which was at the Fourier Institute and destroyed by wind, "*the tree with parentheses*" he says, to which he seemed to be sentimentally attached. He also regrets that the Institute did not use the 1% artistic fund for the art mosaic project in the library. In this Koszul family of mathematicians, musicians, and Scientifics, there was a constant recollection of "beauty" and "truth". Our society no longer cares about timeless "beauty". We have then to extase ourself with Jean-Louis Koszul by observing beautiful "*Catalpa tree*" with "*Parenthese Mushroom*", before there is no longer people to contemplate them.

"*I am studying! I am only the subject of the verb study. To think I do not dare. Before thinking, we must study* "- Gaston Bachelard, the flame of a candle

Fig. 8. (on the left) Jean-Louis Koszul at Grenoble in December 1993, (on the right) last interview of Jean-Louis Koszul in 2016 for 50th birthday of Institut Joseph Fourier in Grenoble

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