RESEARCH PAPER

# Alpha-Beta Log-Determinant Divergences Between Positive Definite Trace Class Operators

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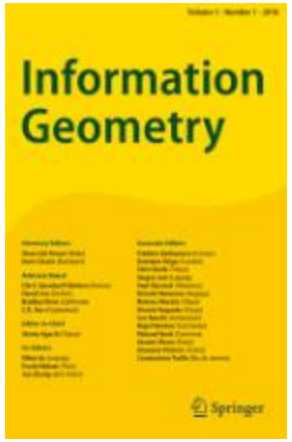
## Abstract

This work presents a parametrized family of divergences, namely Alpha-Beta Log-Determinant (Log-Det) divergences, between positive definite unitized trace class operators on a Hilbert space. This is a generalization of the Alpha-Beta Log-Determinant divergences between symmetric, positive definite matrices to the infinite-dimensional setting. The family of Alpha-Beta Log-Det divergences is highly general and contains many divergences as special cases, including the recently formulated infinite-dimensional affine-invariant Riemannian distance and the infinite-dimensional Alpha Log-Det divergences between positive definite unitized trace class operators. In particular, it includes a parametrized family of metrics between positive definite trace class operators, with the affine-invariant Riemannian distance and the square root of the symmetric Stein divergence being special cases. For the Alpha-Beta Log-Det divergences between covariance operators on a Reproducing Kernel Hilbert Space (RKHS), we obtain closed form formulas via the corresponding Gram matrices.


**Keywords** Positive definite operators · Trace class operators · Infinite-dimensional Log-Determinant divergences · Alpha-Beta divergences · Affine-invariant Riemannian distance · Stein divergence · Extended trace · Extended Fredholm determinant · Reproducing kernel Hilbert spaces · Covariance operators

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# The geometry of recombination

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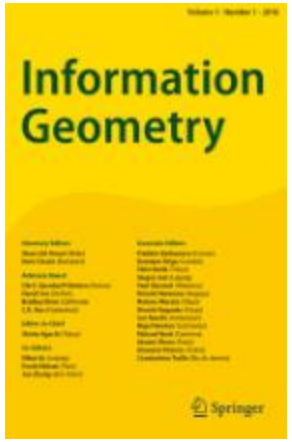
**Abstract**
With the tools of information geometry, we can express relations between marginals of a joint distribution in geometric terms. We develop this framework in the context of population genetics and use this to interpret the famous Ohta–Kimura formula (cf. Ohta and Kimura in Genet Res 13(01):47–55, 1969) and discuss its generalizations for linkage equilibria in Wright–Fisher models with recombination with several loci. The state space associated with the Ohta–Kimura model is simply a Riemannian manifold of constant positive curvature. Furthermore, the equilibria states for recombination can be interpreted geometrically as a product of spheres. In the case of only 2 loci, we also derive the behavior of the mutual information between these two loci.

**Keywords** Wright–Fisher model · Random genetic drift · Recombination · Linkage · Fisher information metric · Equilibrium states · Compositionality

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# Modelling election dynamics and the impact of disinformation

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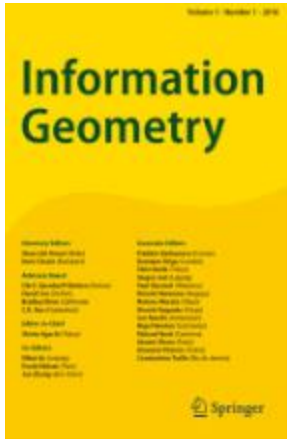
**Abstract**
Complex dynamical systems driven by the unravelling of information can be modelled effectively by treating the underlying flow of information as the model input. Complicated dynamical behaviour of the system is then derived as an output. Such an information-based approach is in sharp contrast to the conventional mathematical modelling of information-driven systems whereby one attempts to come up with essentially *ad hoc* models for the outputs. Here, dynamics of electoral competition is modelled by the specification of the flow of information relevant to election. The seemingly random evolution of the election poll statistics are then derived as model outputs, which in turn are used to study election prediction, impact of disinformation, and the optimal strategy for information management in an election campaign.

**Keywords** Information-based modelling · Electoral competition · Election prediction · Information control · Fake news

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# Manifolds of classical probability distributions and quantum density operators in infinite dimensions

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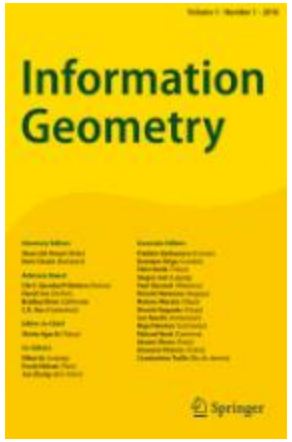
## Abstract

The manifold structure of subsets of classical probability distributions and quantum density operators in infinite dimensions is investigated in the context of $C^*$-algebras and actions of Banach-Lie groups. Specificaly, classical probability distributions and quantum density operators may be both described as states (in the functional analytic sense) on a given $C^*$-algebra $\mathscr{A}$ which is Abelian for Classical states, and non-Abelian for Quantum states. In this contribution, the space of states $\mathscr{S}$ of a possibly infinite-dimensional, unital $C^*$-algebra $\mathscr{A}$ is partitioned into the disjoint union of the orbits of an action of the group $\mathscr{G}$ of invertible elements of $\mathscr{A}$. Then, we prove that the orbits through density operators on an infinite-dimensional, separable Hilbert space $\mathcal{H}$ are smooth, homogeneous Banach manifolds of $\mathscr{G} = \mathcal{GL}(\mathcal{H})$, and, when $\mathscr{A}$ admits a faithful tracial state $\tau$ like it happens in the Classical case when we consider probability distributions with full support, we prove that the orbit through $\tau$ is a smooth, homogeneous Banach manifold for $\mathscr{G}$.

**Keywords** Probability distributions · Quantum states · $C^*$-algebras · Banach manifolds · Homogeneous spaces

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# Spectral dimensionality reduction for Bregman information


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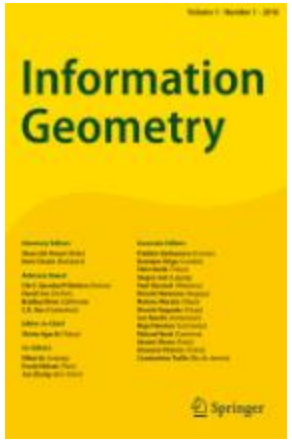



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## Abstract

A spectral dimensionality reduction in dually flat spaces is formulated. On the basis of the expansion with respect to the affine coordinate, it is shown that some inner products compose Bregman divergence. Since the centroids are determined in two ways, depending on coordinate used for centering, Bregman information is also evaluated in two ways. In both cases, Bregman information is shown to be represented as a weighted sum of the eigenvalues of two inner product matrices. For given dissimilarities, the problem which finds the values of the inner product matrices is described as a semidefinite programming. As an example, numerical calculations are also presented.



**Keywords** Dimensionality reduction · Information geometry · Bregman divergence · Dually flat space

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# A class of non-parametric statistical manifolds modelled on Sobolev space

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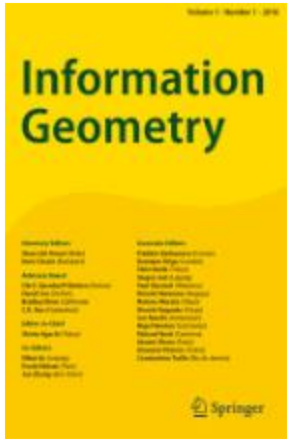
**Abstract**
We construct a family of non-parametric (infinite-dimensional) manifolds of finite measures on $\mathbb{R}^d$, each containing a smoothly embedded submanifold of *probability* measures. The manifolds are modelled on a variety of weighted Sobolev spaces, including Hilbert–Sobolev spaces and mixed-norm spaces, and support the Fisher–Rao metric as a weak Riemannian metric. Densities are expressed in terms of a *deformed exponential* function having linear growth. Unusually for the Sobolev context, and as a consequence of its linear growth, this "lifts" to a nonlinear superposition (Nemytskii) operator that acts continuously on a particular class of mixed-norm model spaces, and on the fixed norm space $W^{2,1}$; i.e. it maps each of these spaces continuously into itself. In contrast with non-parametric exponential manifolds, the density itself belongs to the model space, and the range of the chart is the whole of this space. Some of the results make essential use of a log-Sobolev embedding theorem, which also sharpens existing results concerning the regularity of statistical divergences on the manifolds. Applications to the stochastic partial differential equations of nonlinear filtering (and hence to the Fokker–Planck equation) are outlined.

**Keywords** Banach manifold · Fisher–Rao metric · Fokker–Planck equation · Log-Sobolev Embedding · Non-parametric statistics · Sobolev space

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