Division C

Geometry of Deformations and Stresses

GENERAL INTRODUCTION

THE indispensability of the introduction of non-Euclidean concepts originally occurred to two different groups of engineer-tensorists quite independently. No intimate mutual influence seems to have taken place between the separate developments in their respective fields. While electrical engineers have been applauding Gabriel Kron's success, another essential feature of the applied tensor calculus remained somewhat less popular.

To electrical engineers applied tensor analysis as well as applied matrix theory may seem to relate only to the theory of electrical machines and networks. Stress and strain, however, are quoted in numerous books of classical field-physics as the most typical examples of the use of tensors in physics. If, as seems to be the case, the latter have not attracted so much attention as have the former, such is partly due to the fact that a sudden advance such as Kron's electrodynamics did not take place in the evolution of the classical treatment of force fields in the theory of continuous media. On account of the very gradual recognition accorded to it, the Riemannian character of the incompatibility and residual stresses of elastic bodies failed to draw anything like such general attention of mathematical enginneers as did the non-Riemannian unification of the theory of electrical machinery.

However, the applied mechanics problem of deformable bodies is more directly a "Geometry" than is any unified consideration of electrical and mechanical machinery. The former is the study of the deformable manifold itself with a few mechanical properties, while the latter is indirectly connected with multidimensional terminology and the related geometry by virtue of the situation that the dynamical system usually has many degrees of freedom.

We can trace back the first recognition of the "Compatibility Problem" to the classical statement by B. de St. Venant[1)] in his edition of Navier's Leçons published in 1864. It has been pointed out in many recent tensorial treatises on elasticity that the compatibility criterion is related to a Riemannian concept, in that it can be formulated as the criterion of vanishing of the Riemann-Christoffel curvature tensor of the matter manifold. Unfortunately, however, little has been said of the importance of going over to "Incompatibility Mechanics" rather than of remaining within the limit of compatibility. Hence the standard treatises on elasticity still linger in the Euclidean world; and the essay to peep into the essential significance of Riemannian notions in elasticity theory has been delayed or become sophisticated either with the mere pursuits of easily followable analysis by pure theorists or with a somewhat superstitious belief in the utility of approximate considerations by practical people.

It is a pity that elasticity theory was not the first cospicuous application of Riemannian geometry, the practical value of which was partly overshadowed by the seemingly mystifying theory of relativity. Yet our study of Riemannian elasticity, stimulated by the practical need connected with the thermal treatment of materials, leads to the conclusion that the elasticity theory is Riemannian geometry itself or *vice versa*. Thus Riemannian geometry and elastisity theory are combined as a unified whole in article C-I which, taken in conjunction with the description of non-Riemannian geometry in the first article

1) See B. de Saint-Venant's Appendix 3 to the third edition of Navier's "Résumé des leçons données a l'ecole des ponts et chaussées sur l'application du la mécanique a l'établissements des constructions et des machines," Paris, 1864.

D-I of the next division, gives the practical as well as theoretical basis of the principal aspect of this division.

Very often disagreement has been observed concerning the definifion of engineering concepts connected with stresses and strains. We emphasize that geometry is the most useful means of throwing light on the confusion. As the result of a long continued debate, the issue is sometimes reduced to the simple problem of looking up an appropriate geometrical concept and the disagreement is usually resolved into a difference of standpoints and of reference frames. The confusion in regard to the concepts of stress and stress-density is one such problem. Article C-II gives the answer to it, C-III, and C-IV, too.

The theories of finite deformation and the related definitions of stress have long been subject to a confusion of this sort. The reader will be convinced, from our exposition in C-II, of how often engineers' language has ignored the difference between the Lagrangian and Eulerian standpoints, and how amply the geometrical terminology supplies the means for clarifying the ambiguity. An article concerned with the same subject was initially planned for inclusion here. But the auther prefers to postpone its publication in order to geometrize the employed language more thoroughly.

Necking is an important and interesting phenomenon connected largely with the concept of stress-density. It is taken up as the central subject in paper C-III, where similar instability phenomena are also discussed.

As a generalization or modification of the tensor-density concept, we develop in C-IV the description of a notion of partial density which serves as the basis of the shell concept. Since a membrane is the limiting case of a shell, it is natural for membrane geometry to be included in C-II, in advance of C-IV.

There have been several reasons why the Tension-Field Theory, which is a special problem of memebrane geometry, deserved Japanese scientists' particular attention. (See, Introduction, C-V.) Now we have the pleasure of summarizing the results reported by four investigators in article C-V.

Some topics taken up in this division may at first sight seem to be classical and out of place, being either too well-established or too specialized. We should therefore like to emphasize the authors' full conviction of the necessity of reestablishing the fundamental formulae of classical subjects, such as shell theory in C-IV from a unified geometrical point of view. It will also enable us, as in C-II and C-IV, to extend the classical shell and membrane concept to the multidimensional analogues relate to other physical phenomena, thus supplying a possible route to the plastitcity theory of Division D.

However, an article on plastic design, namely C-VI, is considered to be suitable for inclusion in this division, because its geometrical background is holonomic rather than non-holonomic. This article presents one of the recent theoretical investigations, the practical importance of which is often unduly ignored owing perthaps to the apparently abstract appearance of the mathematical instruments inevitably applied.

Because the topics here taken up are mostly classical, there exist a considerable literature on these subjectes. What is more, many of the books even use certain tensorial geometrical terminologies. Yet the yoke of the Euclidean world-picture has been far too heavy in them to give satisfaction to the authors of this volume.

Therefore, the description in this division will serve as a geometrically unified treatment of classical elasticity from an advanced point of view on the one hand, and on the other, as a preparatory step to the more obscure but promising field for the future investigation of plasticity and non-Riemannian aspects of the mechanics of deformable bodies in Division D.

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Geometry of Elastic Deformation and Incompatibility

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INTRODUCTION

IT is generally agreed that problems of elasticity present interesting examples of tensor calculus and we know several treatises[1)] in which considerable attention is turned in that direction. In spite of this, it seems to be not without use to add a further comment on the close connexion which exists between the fundamental conceptions of elasticity and Riemannian geometry. This might have escaped the attention of a considerable number of tensorists, as the well-established classical treatises on elasticity are too persuasive. Recognition of this fact may have been prevented by the hasty conclusion that non-Euclidean concepts of Riemannian geometry have little to do with the real world of practical engineers.

Although the remarkable success of the general relativity theory impressed the importance of tensor calculus and Riemannian geometry on public opinion, it was unfortunate that it gave a metaphorical appearance to Riemannian expression, banishing it for a time from the attention of engineers. But the reader, after having studied the following analysis, will agree with us in the opinion that it is strange that the first practical field of application was not the theory of elasticity, especially of residual strains.

Recently the importance of the phenomena of internal stresses and strains has been emphasized in engineering circles in regard to the problems of welding and other thermal treatments as these phenomena are beyond the usual scope of the so-called "compatibility". The conception of incompatibility now claims an important place in our attention. This topic which is the major subject of the present paper has the appearance of an illustrative exposition of fundamental conceptions of Riemannian geometry.

Hence this article is a good place for laying the basis of a Riemannian manner of thinking It may be useful training for the study of other topics expounded in this volume. The following exposition will therefore be so arranged as to serve if need be as a text-book. We shall also aim to have it serve as a preliminary remark to the other topics of Division C, as well as to more intricate problems of plastic deformations in Division D.

In regard to the tensor notation and the general purport of the exposition we follow Élie Cartan's method for the most part, but sometimes other conventional notations are also adopted. In general the problem of notation and indices is not of much trouble here, because the major purport of this article is not to give analytical formulae for calculation, but rather to rationalize and classify the basic ideas. Cartan's method seems most suitable for this.

1) For example, see references [1] to [7]. But these are in no way exhaustive. We cite only those which have certain relations to the subject of the present treatise. There is in particular a considerable amount of literature on the tensor algebra of elastic constants especially on anisotropic media, to which we need not refer at present.

CHAPTER I

FUNDAMENTAL DEFINITIONS

1. Strains and elasticity

Even in a classical treatise on elasticity [8], it is stated that all kinds of matter, however apparently continuous, are ultimately granular in structure, being composed of very minute *material particles or molecules.*

The molecules exert upon one another certain mutual forces to which the *cohesiveness of matter* is due. It is also stated that the molecules are liable to be influenced by external "impressed" or "applied" forces, such as gravitation and other natural forces of attraction and repulsion.

It has also been recommended that a state of matter be called "the natural state". This term "natural" does not imply that matter is ever found in this state under natural conditions, but that in this state, and in this only, it may be supposed to be isolated from all co-existing matter, so that all the phenomena it presents depend only on its individual nature. In other cases it is expected to be under constraint by the cohesive properties.

Little has been known with certainty, up to very recent times, of the nature of intermolecular forces. (It appears, however, that we are justified in assuming their sphere of action to be exceedingly limited.) But according to the definition, it is unlikely that the natural state can be always realized for a finite volume of matter, because none else than an isolated small element of matter can be regarded entirely free of constraint from the surroundings owing to the cohesive intermolecular forces.

As an apparently continuous lump of matter cannot, in general, be in the natural state, we will name the deviation of the state of each element of matter from the supposed natural state "the strain" of that element. It must be therefore only for special cases that we can take away the strain from all parts of a given mass of matter by deforming it in ordinary three-dimensional Euclidean space, without any change of the order or the arrangement of its molecules. We shall name that part of the strain which cannot be taken away by any continuous deformation in the three-dimensional Euclidean space "the proper or internal strain" of the material element.

The natural configuration of the molecules (and therefore the natural form and volume) must by definition be unique for each element of a material body (at the same temperature), irrespective of the fact that it can or cannot be realized under the constraint of the surroundings in the ordinary space. We can therefore substitute a more technical term "the unstrained state" for the natural state.

We shall now speak of Elasticity and Stresses. A classical definition of elasticity is that *the application of external force is always required to produce strain.* It is necessary to define elasticity in terms of relative finite deformations. If there is any one-to-one correspondence between the applied force and the difference of the states of strain of the material body with and without its application, we call it "elastic". It appears that an uninterrupted exertion of *a definite external force* is required to maintain the body in a given state of strain when there is no proper strain. Of course there are certain exceptions such as are pointed out by Almansi [9], as is observed, for example, in the case of the inversion of an elastic hemi-spherical shell. The inversed hemi-spherical shell can present "some residual strain" even after the removal of the external force. But this strain can be taken away by a continuous deformation and cannot be called proper or internal in our sense.

Now let us consider an isolated material element which is so sufficiently small that it can be easily brought to its natural state, if it is cut off from the constraint of the surrounding matter. If it has elasticity, it must be unstrained when it is cut off, for it is then subjected to no external force from the surroundings. Or "Elasticity" can be defined as the character of the element, owing to which it returns to its natural state when freed.

2. Deformations of the material manifold

The object of our study is the behaviour of the aggregation of material points of molecules which constitute an elastic body. We shall name such an aggregation an "elastic material manifold" using geometrical terminology. The totality of mutual relations of material particles is called the "configuration" of the (material) manifold and its change is the so-called "deformation".

If a finite lump of elastic matter is realized in the ordinary space, each of its material points occupy a point in the space and so can be specified by three components x^1, x^2, x^3 of a rectangular Cartesian frame. The square of the distance of two adjacent material points $\mathbf{M}$ and $\mathbf{M}'$, whose coordinates are respectively

$$x^1, x^2, x^3 \quad \text{and} \quad x^1+dx^1, x^2+dx^2, x^3+dx^3,$$

is expressed by the sum of the squared differentials

$$ds^2_{\underset{E}{}}=(dx^1)^2+(dx^2)^2+(dx^3)^2. \qquad \text{(I)}$$

The material line-element

$$d\mathbf{M}=\mathbf{M}'-\mathbf{M}$$

cannot in general be unstrained at this state, which we call the "Eulerian or realized configuration". The quadratic differential form ds^2_E gives an Eulerian or real metric of the element $d\mathbf{M}$. If this is cut off from its surroundings, its length will usually change as it takes the configuration of the natural state. Denoting its length in this unstrained state by ds_N, we may assign a quadratic differential form

$$ds^2_N=\underset{N}{g_{ij}}dx^i dx^j \qquad \text{(II)}$$

as defining its natural metric property. Since the natural state is defined to be unique, its configuration can be taken as a proper invariant of the material manifold in spite of its various Eulerian states by possible deformations. (In view of the invariance, we may call this a "Lagrangian configuration").

If an elastic body can be unstrained in ordinary space, the corresponding Cartesian coordinates can be regarded as natural parameters of the material point. Hence each material point is naturally denoted by these three parameters and the manifold is of necessity three-dimensional. If, on the contrary, there is a proper strain, the three-dimensional Cartesian coordinates of any particular Eulerian state cannot be regarded as natural (although it may be taken as Lagrangian in the usual sense). We are even uncertain as to whether the natural state is altogether three-dimensional. We can suggest a possibility of a further parameter in relation to a thermo-elastic deformation, pointing out that a sort of proper strain called the "thermal strain" is introduced by certain distributions of temperature, which we may regard as the fourth coordinate. But we can often eliminate the fourth and further coordinates. For example, the temperature may not be independent. It may depend on the other coordinates.

In any case, three Cartesian coordinates may suffice as Lagrangian parameters of the material points, if we are not in a position to discuss thermal strains and the like. Our material manifold can therefore be regarded as three-dimensional, but not in general as Euclidean. It is true however that there is a certain case in which the metric differential form of the natural state (II) can be reduced to a sum of squares, even if there is a proper strain. In spite of such a special cases and of various three-dimensional Eulerian configurations, the invariance of the natural metric (II) is to be postulated. Hence a Eulerian configuration corresponds to a set of Eulerian coordinates. **The elastic deformation in the real space is reduced to no other than a transformation of Eulerian coordinates under invariance of the natural metric.**

Moreover all the transformations of Eulerian coordinates are holonomic, that is, they can be expressed by a set of continuous analytic functions

$$x^\kappa=f^\kappa(x^i), \quad i, \kappa=1, 2, 3 \qquad (1)$$

where the initial coordinate system is denoted by (i) and the new system by (κ), while the kernel letter x is kept the same, indicating that the respective coordinates are related to the same material point. Thus the theory of elasticity is reduced to the geometry of a three-dimensional Riemannian space which deals with the invariance of the metric defined by (II) under any holonomic transformation of coordinates (1).

Hence our elastic manifold is a Riemannian space, the elastic deformation being no other than a holonomic transformation of coordinates.

3. Metric properties of the strain

All the metric properties of a material element $d\mathbf{M}$ are reflected in the fundamental forms (I) and (II) respectively for its Eulerian-real and natural states. Hence all the metric properties in the small of the strain are given by

$$ds^2_E-ds^2_N=(\underset{E}{g_{\kappa\lambda}}-\underset{N}{g_{\kappa\lambda}})dx^\kappa dx^\lambda. \qquad (2)$$

We write

$$\varepsilon_{\kappa\lambda}=\frac{1}{2}(\underset{E}{g_{\kappa\lambda}}-\underset{N}{g_{\kappa\lambda}}) \qquad (3)$$

or especially

$$\varepsilon_{ij}=\frac{1}{2}(\underset{E}{\delta_{ij}}-\underset{N}{g_{ij}}), \qquad (3{\cdot}1)$$

and we say that the symmetric tensor $\varepsilon_{\kappa\lambda}$ of the second order measures the metric properties of the strain of the elastic medium at the point (x^κ). The use of the factor $\frac{1}{2}$ is due to the fact that this relation involves the difference of the square of ds_E and ds_N and will prove later to be in conformity with the usual definition of strains for the case in which the deformation is small.

A parallel statement has appeared in a recent treatise by Murnaghan [7]. Ours originated independently and aims to prepare especially for the investigation of internal strains.

CHAPTER II

COMPATIBILITY

Any deformation of a material body is called compatible in so far as it behaves in a three-dimensional Euclidean space without any change of the connexion relation of any adjacent elements. Since our readers must be familiar with the compatibility relations in the infinitesimal theory of elasticity, which were originally given by Saint-Venant,[1] it seems useless to repeat the elementary exposition. We shall undertake the corresponding mathematical representation for the case of finite deformation.

4. Natural frames

Since it makes no difference, we shall not at first impose any restriction on the number of dimensions except that they be finite. Let a set of curvilinear coordinates in Euclidean space of n dimensions be

$$u^1, \cdots\cdots\cdots, u^n,$$

and let the vector line-element connecting its point $\mathbf{M}(u)$ to any nearby point $\mathbf{M}(u+du)$ at an infinitesimal distance from $\mathbf{M}$ be denoted by

$$d\mathbf{M}=\mathbf{M}'-\mathbf{M}=\mathbf{e}_\lambda du^\lambda, \qquad (4)$$

then

$$\mathbf{e}_\lambda=\frac{\partial \mathbf{M}}{\partial u^\lambda}, \quad \lambda=1, \cdots, n \qquad (4\cdot1)$$

is a set of vectors, whose inner products

$$\mathbf{e}_\lambda \cdot \mathbf{e}_\mu = g_{\lambda\mu} (=g_{\mu\lambda}) \qquad (5)$$

stand in the metric differential quadratic form

$$ds^2=g_{\kappa\lambda}du^\kappa du^\lambda, \quad d\mathbf{M}^2=ds^2 \qquad (6)$$

of the underlying Euclidean space, at the point $\mathbf{M}$. This set of vectors forms an n-leg or the fundamental measuring vectors at $\mathbf{M}$ and is called the "natural frame" according to Élie Cartan [10]. It is easy to see that $\mathbf{e}_\lambda$ coincides with the velocity along the λ-th coordinate curve if we measure u^λ by the length of time with which we sweep on this line.

The natural frame is determined by the reference point and the curvilinear coordinates which we adopt. Hence there is only a difference of the order $O(du)$ between the frame

$$\mathbf{e}_1, \cdots, \mathbf{e}_n \quad \text{at} \quad \mathbf{M}$$

and the frame

$$\mathbf{e}_1', \cdots, \mathbf{e}_n' \quad \text{at the neighbouring point } \mathbf{M}'.$$

We can therefore establish a set of Pfaffian forms

$$\omega_\lambda^\kappa=\Gamma_{\lambda\mu}^\kappa du^\mu, \quad \kappa, \lambda, \mu=1, \cdots, n \qquad (7)$$

so that the fundamental vectors $\mathbf{e}_\lambda'$ of the measuring frame at $\mathbf{M}'$ have the components

$$\omega_\lambda^1, \omega_\lambda^2, \cdots, \omega_\lambda^{\lambda-1}, \ 1+\omega_\lambda^\lambda, \omega_\lambda^{\lambda+1}, \cdots, \omega_\lambda^n$$

when they are measured by the natural frame $\{\mathbf{e}_\lambda\}$ at $\mathbf{M}$. The coefficients $\Gamma_{\lambda\mu}^\kappa$'s of the Pfaffians vary as functions of u^κ's. The distribution of the natural frames in space is reflected in the expression of the differentials

$$d\mathbf{e}_\lambda=\omega_\lambda^\kappa \mathbf{e}_\kappa=\Gamma_{\lambda\mu}^\kappa du^\mu \mathbf{e}_\kappa, \qquad (8)$$

which can be readily obtained from the definition of ω_λ^κ's. The coefficients $\Gamma_{\lambda\mu}^\kappa$'s thus connect the neighbouring frames and can be named "the coefficients of connexion", as is usual in differential geometry.

We assume $\mathbf{M}$ and $\mathbf{e}_\lambda$ to be continuous and differentiable with respect to u^κ or the Pfaffian

1) B. de Saint-Venant gave the relations between strain-components in an appendix to his edition of Navier's "Résumé des leçons sur l'applications de la mécanique", Paris, 1864.

relations

$$dM = e_\lambda du^\lambda, \tag{4}$$

$$de_\lambda = \omega_\lambda^\kappa e_\kappa \tag{8}$$

to be integrable. The first integrability condition, namely of (4), meaning the holonomy of the coordinates u^κ, postulates the equality of

$$\frac{\partial^2 M}{\partial u^\lambda \partial u^\mu} = \frac{\partial}{\partial u^\lambda}\left(\frac{\partial M}{\partial u^\mu}\right) = \frac{\partial e_\mu}{\partial u^\lambda} = \Gamma_{\mu\lambda}^{\kappa} e_\kappa$$

and

$$\frac{\partial^2 M}{\partial u^\mu \partial u^\lambda} = \frac{\partial}{\partial u^\mu}\left(\frac{\partial M}{\partial u^\lambda}\right) = \frac{\partial e_\lambda}{\partial u^\mu} = \Gamma_{\lambda\mu}^{\kappa} e_\kappa$$

whence the symmetry

$$\Gamma_{\mu\lambda}^{\kappa} = \Gamma_{\lambda\mu}^{\kappa} \tag{9}$$

of the coefficients of connexion in regard to the lower indices.

Similarly the second integrability condition, namely of (8), postulates

$$\frac{\partial^2 e_\lambda}{\partial u^\nu \partial u^\mu} = \frac{\partial^2 e_\lambda}{\partial u^\mu \partial u^\nu}$$

or

$$\left(\frac{\partial \Gamma_{\lambda\mu}^{\kappa}}{\partial u^\nu} + \Gamma_{\lambda\mu}^{\rho} \Gamma_{\rho\nu}^{\kappa}\right) e_\kappa = \left(\frac{\partial \Gamma_{\lambda\nu}^{\kappa}}{\partial u^\mu} + \Gamma_{\lambda\nu}^{\rho} \Gamma_{\rho\mu}^{\kappa}\right) e_\kappa,$$

whence

$$\frac{\partial \Gamma_{\lambda\mu}^{\kappa}}{\partial u^\nu} - \frac{\partial \Gamma_{\lambda\nu}^{\kappa}}{\partial u^\mu} - \left(\Gamma_{\lambda\nu}^{\rho}\Gamma_{\rho\mu}^{\kappa} - \Gamma_{\lambda\mu}^{\rho}\Gamma_{\rho\nu}^{\kappa}\right) = 0. \tag{10}$$

We will denote the left-hand member of (10) by

$$R_{\lambda\mu\nu}^{\kappa}$$

in the following.

5. The Christoffel three-index symbols

Since ds^2 as defined in (6) is an invariant under transformation of coordinates, $g_{\kappa\lambda}$ is a fundamental covariant tensor, to which we can define a contravariant components $g^{\kappa\lambda}$ by

$$g^{\kappa\lambda} g_{\lambda\mu} = \delta_\mu^\kappa, \quad g_{\lambda\mu} g^{\mu\kappa} = \delta_\lambda^\kappa. \tag{11}$$

We can associate covariant components with any (contravariant) vector v^κ which has upper indices, lowering the indices by inner multiplication with $g_{\lambda\kappa}$

$$v_\lambda = g_{\lambda\kappa} v^\kappa,$$

and *vice versa*

$$v^\kappa = g^{\kappa\lambda} v_\lambda.$$

The Pfaffian forms ω_λ^κ in (8) can be regarded as contravariant components of the vector de_λ. Its covariant components are given by

$$\omega_{\lambda\mu} = \omega_\lambda^\kappa g_{\kappa\mu}.$$

But from (5) and (8) we obtain

$$\begin{aligned} dg_{\lambda\mu} &= de_\lambda \cdot e_\mu + e_\lambda \cdot de_\mu \\ &= \omega_\lambda^\kappa e_\kappa \cdot e_\mu + e_\lambda \cdot (\omega_\mu^\kappa e_\kappa) \\ &= \omega_\lambda^\kappa g_{\kappa\mu} + \omega_\mu^\kappa g_{\lambda\kappa} \end{aligned}$$

by differentiation and substitution. These $n(n+1)/2$ Pfaffian relations

$$\omega_{\lambda\mu} + \omega_{\mu\lambda} = dg_{\lambda\mu}$$

are equivalent to $n^2(n+1)/2$ partial differential equations

$$\Gamma_{\lambda\nu}^{\kappa} g_{\mu\kappa} + \Gamma_{\mu\nu}^{\kappa} g_{\lambda\kappa} = \frac{\partial g_{\lambda\mu}}{\partial u^\nu}. \tag{12}$$

Writing

$$g_{\mu\kappa} \Gamma_{\lambda\nu}^{\kappa} = \Gamma_{\lambda\nu\mu}$$

and considering (9), we get

$$\Gamma_{\lambda\nu\mu} + \Gamma_{\mu\nu\lambda} = \frac{\partial g_{\lambda\mu}}{\partial u^\nu}$$

and two similar equations with the cyclic change of indices. Solving them algebraically, we obtain

$$\left.\begin{aligned} \Gamma_{\mu\nu\lambda} &= \frac{1}{2}\left(\frac{\partial g_{\nu\lambda}}{\partial u^\mu} + \frac{\partial g_{\lambda\mu}}{\partial u^\nu} - \frac{\partial g_{\mu\nu}}{\partial u^\lambda}\right) = [\mu\nu, \lambda], \\ \Gamma_{\lambda\mu}^{\kappa} &= g^{\kappa\nu}[\lambda\mu, \nu] = \left\{ {\kappa \atop \lambda\mu} \right\}. \end{aligned}\right\} \tag{13}$$

The symbols defined by (13) are called the Christoffel three-index symbols of the first and second kinds respectively. They embody the relations between the holonomic distribution of the natural frames and the metric properties of the underlying (Euclidean) space.

The total number of different Christoffel symbols of the first or second kind is equal to the product of n—the number of possible κ's—with $(n+n^2)/2$—the number of possible pairs $(\lambda\mu)$'s,

$$n^2(n+1)/2,$$

which is necessary and just sufficient in order to solve the simultaneous equations (12).

6. Tensor fields and absolute differentiation

Any vector field of a Euclidean n-space is expressed by

$$\mathbf{X}=X^\kappa \mathbf{e}_\kappa,$$

using natural frames, where X^κ is its (contravariant) component. It follows that

$$d\mathbf{X}=dX^\kappa \mathbf{e}_\kappa+X^\kappa d\mathbf{e}_\kappa=(dX^\kappa+X^\lambda\omega_\lambda^\kappa)\mathbf{e}_\kappa.$$

Hence the set of Pfaffian expressions

$$DX^\kappa=dX^\kappa+X^\lambda\omega_\lambda^\kappa \tag{14}$$

is a proper measure of the absolute difference of the vector field at the adjacent point $\mathbf{M}(u)$ and $\mathbf{M}'(u+du)$, in regard to a Euclidean frame $\{\mathbf{e}_\lambda\}$ which we have established at $\mathbf{M}$ as the natural frame.

It is obvious that we have

$$DX^\kappa=0$$

if, and only if, the field is uniform. We name therefore such DX^κ as defined by (14) the "absolute differential" of the vector (components) X^κ. If it vanishes at a certain point but irrespective of the curvilinear mesh of coordinates, that means the parallelism in its neighbourhood.

The inner product of two vector fields $\mathbf{X}$ and $\mathbf{Y}$ forms a field of scalar

$$\mathbf{X}\cdot\mathbf{Y}=X_\kappa Y^\kappa=g_{\kappa\lambda}X^\lambda Y^\kappa.$$

Its differential, namely,

$$\begin{aligned}d(X_\kappa Y^\kappa)&=X_\kappa dY^\kappa+(dX_\kappa)Y^\kappa\\&=X_\kappa(dY^\kappa+\omega_\lambda^\kappa Y^\lambda)+(dX_\kappa-\omega_\kappa^\lambda X_\lambda)Y^\kappa,\end{aligned}$$

being also a scalar, defines the absolute differential for the covariant components,

$$DX_\lambda=dX_\lambda-\omega_\lambda^\kappa X_\kappa. \tag{14·1}$$

If we postulate this, we should write

$$\begin{aligned}d(X_\lambda Y^\lambda)&=D(X_\lambda Y^\lambda)\\&=X_\lambda DY^\lambda+(DX_\lambda)Y^\lambda.\end{aligned}$$

A tensor field of an arbitrary order is defined as a quantity which forms a field of scalar by inner multiplication with so great a number of vectors as the indices, i.e. if

$$a^{\kappa_1\cdots\kappa_p}{}_{\lambda_1\ldots\lambda_q}X_{\kappa_1}\cdots X_{\kappa_p}Y^{\lambda_1}\cdots Y^{\lambda_q}$$

be a scalar when X's and Y's are vectors,

$$a^{\kappa_1\cdots\kappa_p}{}_{\lambda_1\ldots\lambda_q}$$

is a mixed tensor of order $p+q$ contravariant in $\kappa_1, \cdots, \kappa_p$ and covariant in $\lambda_1, \cdots, \lambda_q$. By differentiation we obtain the corresponding expression for the absolute or covariant differentials

$$\begin{aligned}Da^{\kappa_1\ldots\kappa_p}{}_{\lambda_1\cdots\lambda_q}&=da^{\kappa_1\ldots\kappa_p}{}_{\lambda_1\cdots\lambda_q}\\&+\sum_{r=1}^{p}a^{\kappa_1\ldots\kappa_{r-1}\mu\kappa_{r+1}\ldots\kappa_p}{}_{\lambda_1\ldots\lambda_q}\omega_\mu^{\kappa_r}\\&-\sum_{s=1}^{q}a^{\kappa_1\ldots\kappa_p}{}_{\lambda_1\cdots\lambda_{s-1}\mu\lambda_{s+1}\cdots\lambda_q}\omega_{\lambda_s}^{\mu},\end{aligned} \tag{15}$$

similarly as above, in virtue of the relation such as

$$\begin{aligned}D(a_\lambda^\kappa X_\kappa Y^\lambda)&=d(a_\lambda^\kappa X_\kappa Y^\lambda)\\&=(da_\lambda^\kappa)X_\kappa Y^\lambda+a_\lambda^\kappa(dX_\kappa)Y^\lambda+a_\lambda^\kappa X_\kappa(dY^\lambda)\\&=[da_\lambda^\kappa-a_\mu^\kappa\omega_\lambda^\mu+a_\lambda^\mu\omega_\mu^\kappa]X_\kappa Y^\lambda\\&\quad+a_\lambda^\kappa(DX_\kappa)Y^\lambda+a_\lambda^\kappa X_\kappa(DY^\lambda).\end{aligned}$$

The tensor field is characterized by the law of transformation for the change of coordinates:

$$a^{\kappa'_1\cdots}{}_{\lambda'_1\cdots}=\left(\frac{\partial u^{\kappa'_1}}{\partial u^{\kappa_1}}\cdots\frac{\partial u^{\lambda_1}}{\partial u^{\lambda'_1}}\cdots\right)a^{\kappa_1\cdots}{}_{\lambda_1\cdots}. \tag{16}$$

A vector is the tensor of order one. A scalar is of order zero.

The simplest example of a contravariant vector is the set of differentials du^κ of the coordinate variables. That of a covariant vector is the gradient $\partial\varphi/\partial x^\kappa$ of a scalar φ. Any inner and non-inner product of tensors is a tensor. There is a simple but important theorem that is called the "quotient law". It states that an indexed quantity which produces a tensor by inner or non-inner multiplication with a tensor is a tensor. The set of coefficients $g_{\kappa\lambda}$ of the metric quadratic differential form (6) constitutes a tensor of order two, for it is inner-multiplied with a tensor $du^\kappa du^\lambda$ to form a scalar ds^2.

Ordinary partial derivatives of a tensor field cannot make a tensor field except it be a scalar. But we can testify that the so-called "absolute or covariant derivatives"

$$a^{\kappa 1\cdots}{}_{\lambda 1\cdots,\mu}\equiv\frac{Da^{\kappa 1\cdots}{}_{\lambda 1\cdots}}{du^\mu}$$

or

$$X^\kappa_{,\mu}\equiv\frac{DX^\kappa}{du^\mu}=\frac{dX^\kappa+\omega^\kappa_\lambda X^\lambda}{du^\mu}$$
$$=\frac{\partial X^\kappa}{\partial u^\mu}+X^\lambda\Gamma^\kappa_{\lambda\mu}$$

make a tensor field of the same order higher by one than the original one.

The coefficients of connexion $\Gamma^\kappa_{\lambda\mu}$, or the Christoffel symbols of the second kind in particular for the present case make an important geometric object in connexion with Absolute Calculus thus obtained. But they themselves do not make a tensor as they obey a transformation law different from that of a tensor, i.e.

$$\Gamma^{\lambda'}_{\mu'\nu'}\frac{\partial u^\lambda}{\partial u^{\lambda'}}=\Gamma^{\lambda}_{\mu\nu}\frac{\partial u^\mu}{\partial u^{\mu'}}\frac{\partial u^\nu}{\partial u^{\nu'}}+\frac{\partial^2 u^\lambda}{\partial u^{\mu'}\partial u^{\nu'}}. \quad (17)$$

The relations (17) sugest a tensor

$$S^\kappa_{\lambda\mu}=\frac{1}{2}\left(\Gamma^\kappa_{\lambda\mu}-\Gamma^\kappa_{\mu\lambda}\right).$$

But it should vanish for the present case by virtue of the symmetry in the lower indices λ and μ, i.e., by (9). For a space where we have such connexions that (9) does not hold, the non-zero tensor $S^\kappa_{\lambda\mu}$ exists and is called the "torsion tensor" and the space is said to have torsion.

The tensor field of the next lowest order which we can establish from the derivatives of the coefficients of connexion is found to be

$$R^\kappa_{\lambda\mu\nu}=\frac{\partial\Gamma^\kappa_{\lambda\mu}}{\partial u^\nu}-\frac{\partial\Gamma^\kappa_{\lambda\nu}}{\partial u^\mu}-\left(\Gamma^\rho_{\lambda\nu}\Gamma^\kappa_{\rho\mu}-\Gamma^\rho_{\lambda\mu}\Gamma^\kappa_{\rho\nu}\right) \quad (18)$$

which appeared above. We name this tensor of the fourth order the Riemann-Christoffel curvature tensor. But it vanishes identically in the case of any Euclidean space, as we have seen.

7. Compatibility relations

When the natural configuration is compatible, the Lagrangian parameters x^i's which stand in the fundamental form (II) form a set of curvilinear coordinates, attached holonomically to a mesh-line system of material lines, which are usually curved according to the impressed deformations. Since the parameters can be so chosen that they coincide with the Cartesian coordinates of a particular Eulerian state, the essential characters of the deformation from, or to, the natural configuration must be reflected in the curvilinear properties of this system of Lagrangian parameters.

It is especially important that any choice of curvilinear coordinates should not violate the basic condition (10). Translated into the language of deformation, this means that any Eulerian state cannot be free from a condition which is expressed with the same form as (10) if, and only if, the deformation from, or to, the natural state is compatible.

In view of these considerations, we see that the compatibility condition can be expressed by the vanishing of all the components of the tensor R^h_{ijk} in relation to the fundamental tensor of the Lagrangian metric such as appear in (II), for which the coefficients of connexion of the natural frames are the Christoffel symbols of the second kind

$$\left\{{i \atop jk}\right\}_N=\frac{1}{2}\underset{N}{g}^{il}\left(\frac{\partial \underset{N}{g}_{kl}}{\partial x^j}+\frac{\partial \underset{N}{g}_{lj}}{\partial x^k}-\frac{\partial \underset{N}{g}_{jk}}{\partial x^l}\right)$$

of the natural metric. Hence we obtain

$$\underset{N}{R}{}^i_{jkl}\equiv\frac{\partial\left\{{i \atop jk}\right\}_N}{\partial x^l}-\frac{\partial\left\{{i \atop jl}\right\}_N}{\partial x^k}+\left(\left\{{h \atop jk}\right\}_N\left\{{i \atop hl}\right\}_N\right.$$
$$\left.-\left\{{h \atop jl}\right\}_N\left\{{i \atop hk}\right\}_N\right)=0 \quad (10\cdot 1)$$

for the required equations of the condition of compatibility.[1)] We shall omit the capital index

1) Cf. M. Kotani, Mechanics of Continua (in Japanese, Renzoku-Buttai no Rikigaku), Iwanami, Tokyo, 1934, [8] and F.D. Murnaghan, Finite Deformation of an Elastic Solid. John Wiley & Sons, New York, Chapman & Hall, London, 1951, [7].

N, in so far as there is no fear of confusion.

By inner multiplication with g_{hi}, we can lower the indices i in (10·1) and obtain

$$R_{hijk}=\frac{\partial[ij,h]}{\partial x^k}-\frac{\partial[ik,h]}{\partial x^j}-g^{ml}([ik,m][lj,h]-[ij,m][lk,h])=0. \tag{10·2}$$

But the Christoffel symbols of the first kind consist of linear combinations of the second derivatives of g_{Nij}, hence of $2\varepsilon_{ij}$, only as it is defined in § 3. Hence the general equations of compatibility of the element of an elastic body can be described in terms of the strain components as defined above and they are eventually the equations (10·1).

The compatibility equations for finite deformation which are essentially the same as (10) have appeared several times in the literature. We wish especially to mention an excellent treatise on elasticity by Kotani. Murnaghan treated the same problem in his recent publication.

8. Equations for small deformations

We have

$$\frac{\partial[ij,h]}{\partial x^k}=\frac{1}{2}\left(\frac{\partial^2 g_{jh}}{\partial x^k\partial x^i}+\frac{\partial^2 g_{hi}}{\partial x^k\partial x^j}-\frac{\partial^2 g_{ij}}{\partial x^k\partial x^h}\right),$$

$$\frac{\partial[ik,h]}{\partial x^j}=\frac{1}{2}\left(\frac{\partial^2 g_{kh}}{\partial x^j\partial x^i}+\frac{\partial^2 g_{hi}}{\partial x^j\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^j\partial x^h}\right).$$

Substituting these relations in (10·2), we get

$$R_{hijk}\equiv\frac{1}{2}\left(\frac{\partial^2 g_{jh}}{\partial x^k\partial x^i}+\frac{\partial^2 g_{ik}}{\partial x^j\partial x^h}-\frac{\partial^2 g_{ij}}{\partial x^k\partial x^h}-\frac{\partial^2 g_{kh}}{\partial x^j\partial x^i}\right)+g^{ml}([ij,m][lk,h]-[ik,m][lj,h])=0,$$

or in terms of the strain components

$$\frac{\partial^2\varepsilon_{jh}}{\partial x^k\partial x^i}+\frac{\partial^2\varepsilon_{ik}}{\partial x^j\partial x^h}-\frac{\partial^2\varepsilon_{ij}}{\partial x^k\partial x^h}-\frac{\partial^2\varepsilon_{kh}}{\partial x^j\partial x^i}+O(\varepsilon^2)=0. \tag{19}$$

When the deformation is to give rise only to a small amount of strain, we can neglect the small terms $O(\varepsilon^2)$ in (19), obtaining six linear equations of the second order:

$$\frac{\partial^2\varepsilon_{hk}}{\partial x^i\partial x^j}+\frac{\partial^2\varepsilon_{ij}}{\partial x^h\partial x^k}=\frac{\partial^2\varepsilon_{hj}}{\partial x^i\partial x^k}+\frac{\partial^2\varepsilon_{ik}}{\partial x^h\partial x^j}. \tag{20}$$

These equations are compatible with each other if, and only if,

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x^j}+\frac{\partial v_j}{\partial x^i}\right). \tag{21}$$

But the original meaning of the coordinates x^i is connected with rectangular Cartesian axes in the Eulerian configuration as defined in § 3. Hence any set of functions v_1, v_2, v_3 which stand in (21) can be regarded as components of a real displacement from which this strain has originated. It is obvious that our criterion of compatibility for finite deformation that has been enunciated in this chapter includes the standard formulation for small strains, adopted in the classical treatises on elasticity.[1)]

CHAPTER III

INCOMPATIBILITY

Our definitions which have been adopted in Chapter I point to the recognition that there are certain kinds of strain which cannot be removed without change in the connexion relation of the matter in Euclidean space. The criterion of compatibility should no longer hold for the strain components which appear in such an incompatible formulation. Not all the components of the tensor defined by (18) can vanish in this case, i.e., the Riemann-Christoffel tensor R^l_{ijk} is not necessarily a zero tensor. The (quantity represented by the) tensor

$$R^l_{ijk} \quad \text{or} \quad R_{hijk}$$

is therefore a proper measure of the "incompatibility" of the material element.

Our problem of incompatibility in the small can therefore be reduced to a geometry in which the fundamental metric property as defined by the quadratic differential form (II) is invariant under all possible transformations of the Eulerian coordinates of the matter and it cannot be assured that all the components of R^i_{jkl} vanish everywhere in the matter.

Since the present problem dose not include any change of connexion relation, the allowable

1) See, for instance, A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity. Cambridge. University Press, 2nd ed., 1906, p. 49, [11].

transformations of coordinates are holonomic and can be written in such an analytical form as in (1). The theory of invariance of the quadratic differential form under transformations of this nature constitutes "Riemannian Geometry". Hence our theory of incompatibility can be framed in the scope of three-dimensional Riemannian geometry.

9. Tangent spaces

In accordance with the basic standpoint of our problem, we now consider the case in which an element of volume can be brought to the natural state when cut off from the surroundings in a sufficiently small size. A mesh of the ordinary sort of curvilinear coordinates can be established within such a small volume cut off in three-dimensional Euclidean space. The condition of integrability can also be established within this small volume. Let us investigate this situation more closely.

Any two points in this element of volume cannot be more apart than its major diameter, which must also be sufficiently small. Hence the criterion of integrability for (4), namely

$$\frac{\partial^2 \mathbf{M}}{\partial x^\kappa \partial x^\lambda}=\frac{\partial^2 \mathbf{M}}{\partial x^\lambda \partial x^\kappa} \tag{22}$$

is satisfied throughout its interior. But the integrability criterion for (8) cannot be regarded as a problem confined in this small region, for it is connected with the configuration of the point on the material vector corresponding to $\mathbf{e}_\lambda$. If we take $\mathbf{M}$ at the boundary of the elementary volume, in which the first integrability (22) holds, one or another of the measuring vectors may finds itself outside this region, where the integrability may no longer hold. Hence there is no assurance of uniqueness for the configuration of the point say P on $\mathbf{e}_\lambda$ of Fig. 1. It may be different according as we reach it along one or another of the possible paths of integration, that is, of the connected chains of matter in the natural state produced by cutting them off from the given Eulerian configuration.

Therefore, we cannot always define the second derivatives of $\mathbf{e}_\lambda$ uniquely, for all the points within any small volume in the natural state.

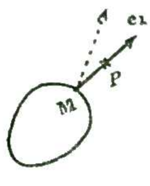

FIG. 1

But its first derivatives $\partial\mathbf{e}_\lambda/\partial u^\mu$ and $\mathbf{e}_\lambda$ itself must be determined as a unique function of the Lagrangian coordinates. It follows that we have a uniquely determined distribution of the fundamental tensor

$$g_{\kappa\lambda}=\mathbf{e}_\kappa\cdot\mathbf{e}_\lambda$$

and of the Christoffel symbols

$$[\lambda\mu,\kappa]=\frac{1}{2}\left(\frac{\partial g_{\mu\kappa}}{\partial u^\lambda}+\frac{\partial g_{\kappa\lambda}}{\partial v^\mu}-\frac{\partial g_{\lambda\mu}}{\partial u^\kappa}\right),$$

$$\left\{{\kappa \atop \lambda\mu}\right\}=g^{\kappa\rho}[\lambda\mu,\rho],$$

whence also the coefficients of connexion $\Gamma^\kappa_{\lambda\mu}$.

But it dose not follow that the Riemann-Christoffel curvature tensor $R^\nu_{\lambda\mu\nu}$, which we can formally construct out of these coefficients by (18), can be uniquely defined for every material point, because it depends on the second derivatives of $g_{\kappa\lambda}$. This forms a characteristic property of a Riemannian manifold.

The linear geometrical properties of any sufficiently small volume of matter in the natural state can be regarded as Euclidean apart from small errors of the order $O((dx)^2)$, or less than that, when its diameter is $O(dx)$. The Euclidean space thus substituted for the element of a Riemannian manifold is called the "tangent space" of a given point. Hence a Riemannian space is an aggregation of an infinite number of tangent Euclidean spaces with a certain connexion property.

10. Conservation of incompatibility

A most important property of the tensor concept is that a tensor can be a zero-tensor only if there is an admissible system of coordinates for which all its components vanish identically. This has importance also for a practical problem, as follows:

By the procedure which is defined for an Euclidiean space, absolute differentiation can be defined uniquely in each tangent space. That is to say, we can obtain the covariant derivative at the origin of each tangent frame of a Riemannian manifold in so far as the coefficients of connexion are defined uniquely at each point, chosen as the origin of the tangent frame, of the manifold for the given system of coordinates. The field of the covariant derivative thus obtained forms a tensor field of the order higher by one.

But the covariant derivative of the fundamental tensor is a zero tensor because of (12). The

Riemann-Christoffel curvature tensor is the simplest tensor of the lowest order which can be constructed by derivatives of the components of the fundamental tensor. Since it is a tensor, it represents an invariant characteristic of the Riemann metric, in spite of any choice of co-ordinate variables. That it is a tensor can be easily testified to as follows:

Take a closed parallelogram in an elementary region of a Riemannian n-space and let its adjacent sides be represented by differentials $\underset{1}{dx^\kappa}$ and $\underset{2}{dx^\kappa}$ of the coordinate variables (see Fig. 2). Take the tangent space at a vertex A and project each natural element AB, BC, CD, DA consecutively and continuously on the same Euclidean frame, we then get the projection of the natural frames at A, B, C, D consecutively one after another and the final position of the frame at A after we have described the whole circuit will not generally coincide with the initial position.

This discrepancy should be equal to

$$\underset{2}{d}\underset{1}{d}\mathbf{e}_\lambda-\underset{1}{d}\underset{2}{d}\mathbf{e}_\lambda$$

$$=\underset{2}{d}(\underset{1}{\omega}{}^\kappa_\lambda\mathbf{e}_\kappa)-\underset{1}{d}(\underset{2}{\omega}{}^\kappa_\lambda\mathbf{e}_\kappa)$$

$$=(\underset{2}{d}\underset{1}{\omega}{}^\kappa_\lambda-\underset{1}{d}\underset{2}{\omega}{}^\kappa_\lambda)\mathbf{e}_\kappa$$

$$+(\underset{1}{\omega}{}^\kappa_\lambda\underset{2}{\omega}{}^\mu_\kappa-\underset{2}{\omega}{}^\kappa_\lambda\underset{1}{\omega}{}^\mu_\kappa)\mathbf{e}_\mu$$

$$=2(\underset{[21]}{d\omega}{}^\kappa_\lambda+\underset{[1\ \ 2]}{\omega^\mu_\lambda\omega^\kappa_\mu})\mathbf{e}_\kappa,$$

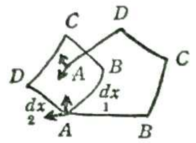

FIG. 2

where we use the bracket notation for alternation as follows:

$$\left.\begin{aligned}2\underset{[21]}{d\omega}&=\underset{2}{d}\underset{1}{\omega}-\underset{1}{d}\underset{2}{\omega},\\ 2\underset{[1\ \ 2]}{\omega^\mu_\lambda\omega^\kappa_\mu}&=\underset{1}{\omega}{}^\mu_\lambda\underset{2}{\omega}{}^\kappa_\mu-\underset{2}{\omega}{}^\mu_\lambda\underset{1}{\omega}{}^\kappa_\mu.\end{aligned}\right\}\tag{23}$$

(See reference [12].)

It is evident that the coefficients

$$2\Omega^\lambda_\kappa=2\underset{[21]}{d\omega}{}^\lambda_\kappa+2\underset{[1\ \ 2]}{\omega^\mu_\kappa\omega^\lambda_\mu}\tag{24}$$

should be contravariant components of a vector in the tangent Euclidean space at A, with respect to λ, while κ behaves dualistically as covariant index. Hence Ω^λ_κ is a mixed tensor of the second order.

A tensor of the fourth order can be introduced by

$$\Omega^\kappa_\lambda=R^\kappa_{\lambda\mu\nu}dx^\mu dx^\nu\tag{25}$$

by virtue of the quotient law and we can readily see from (24) that this is the Riemann-Christoffel curvature as defined above.

Now there is established one-to-one correspondency in the small between Riemann-metric properties and proper in the small strains of a material element. The Riemann-Christoffel tensor is therefore a proper measure of the non-Euclidean or *incompatible* property ofthe element of the elastic manifold. By elastic we mean "homeomorphic and preserving the natural metric" in regard to deformations.

This testifies that the incompatibility once formed in an element of matter cannot be taken away by any elastic deformation in an Euclidean space, because an elastic deformation is no other than a mere transformation of coordinates. This law of conservation of incompatibility is essentially an extension of a theorem which has been proved by S. Moriguti [13] for the case when the deformation is small. His method makes an interesting analogue of the one usually adopted to obtain the conservation law of vorticity and circulation in a perfect fluid. But we are assured by the above that the theorem holds for any finite continuous deformations, without any further analytical explanation.

11. Bianchi identities

The conception of exterior products is convenient for the following description [14]. Let us write

$$[A^iB^j]=\begin{vmatrix}A^i & A^j\\ B^i & B^j\end{vmatrix}=A^iB^j-B^iA^j$$

and especially for two Pfaffians

$$\underset{1}{\varpi}=a_i\underset{1}{dx^i},\quad \underset{2}{\varpi}=b_i\underset{2}{dx^i},$$

$$[\underset{1}{\varpi}\underset{2}{\varpi}]=a_ib_j[\underset{1}{dx^i}\underset{2}{dx^j}]$$

$$=a_ib_j\begin{vmatrix}\underset{1}{dx^i} & \underset{1}{dx^j}\\ \underset{2}{dx^i} & \underset{2}{dx^j}\end{vmatrix}=a_{[i}b_{j]}(2dx^idx^j).$$

If we have further

$$\underset{1}{\varpi}=c_idx^i,\quad \Omega=C_{ij}\underset{2}{dx^i}\underset{3}{dx^j},$$

then

$$[\varpi\Omega]=c_iC_{jk}[\underset{1}{dx^i}[\underset{2}{dx^j}\underset{3}{dx^k}]]$$

$$=c_iC_{jk}\begin{vmatrix} \underset{1}{dx^i} & \underset{1}{dx^j} & \underset{1}{dx^k} \\ \underset{2}{dx^i} & \underset{2}{dx^j} & \underset{2}{dx^k} \\ \underset{3}{dx^i} & \underset{3}{dx^j} & \underset{3}{dx^k} \end{vmatrix}$$

$$=c_{[i}C_{jk]}(3\,!\,\underset{1}{dx^i}\underset{2}{dx^j}\underset{3}{dx^k}).$$

We have identically

$$[d\Omega^\kappa_\lambda]=-[\Omega^\mu_\lambda\omega^\kappa_\mu]+[\omega^\mu_\lambda\Omega^\kappa_\mu]$$

or in terms of absolute differentials

$$[D\Omega^\kappa_\lambda]=\left[dx^\alpha\frac{D}{dx^\alpha}(R^\kappa_{\lambda\beta\gamma}dx^\beta dxx^\gamma)\right]$$

$$=R^\kappa_{\lambda\beta\gamma,\alpha}\begin{vmatrix} \underset{1}{dx^\alpha} & \underset{1}{dx^\beta} & \underset{1}{dx^\gamma} \\ \underset{2}{dx^\alpha} & \underset{2}{dx^\beta} & \underset{2}{dx^\gamma} \\ \underset{3}{dx^\alpha} & \underset{3}{dx^\beta} & \underset{3}{dx^\gamma} \end{vmatrix}=0$$

or

$$3R^\kappa_{\lambda[\beta\gamma,\alpha]}=R^\kappa_{\lambda\beta\gamma,\alpha}+R^\kappa_{\lambda\gamma\alpha,\beta}+R^\kappa_{\lambda\alpha\beta,\gamma}=0 \quad (26)$$

up to the tri-linear factor $dx^\alpha dx^\beta dx^\gamma\neq 0$, where $R^\kappa_{\lambda\beta\gamma,\alpha}$ is the covariant derivative with respect to x^α.

The relations (26) are called the "Bianchi identities". They hold for any Eulerian coordinates, in particular for a rectangular coordinates of a particular Eulerian configuration. In that case, if the space is three-dimensional, the cyclic indices

$$[\beta\gamma,\alpha]:\ [23,1],[31,2],[12,3]$$

may be regarded as equivalent to the divergent indices

$$(i,i):\ (1,1),(2,2),(3,3)$$

whence we write (26) in a non-divergence form

$$C^i_{,i}\overset{*}{=}0,\quad i=1,2,3 \qquad (26\cdot 1)$$

where

$$C^i\overset{*}{=}R^\kappa_{\lambda\beta\gamma},$$

i.e. a Cartesian vector C^i can be identified with the bi-vector in the following arrangement of a skew matrix:

$$\mathbf{C}:\begin{bmatrix} 0 & R^\kappa_{\lambda 12} & -R^\kappa_{\lambda 31} \\ -R^\kappa_{\lambda 12} & 0 & R^\kappa_{\lambda 23} \\ R^\kappa_{\lambda 31} & -R^\kappa_{\lambda 23} & 0 \end{bmatrix},$$

The asterisk means that the relations hold for the particular coordinate system adopted, where, it being rectangular Cartesian, we need not descriminate covariance and contravariance.

The non-divergence condition signifies a continuity of the field of C^i therefore of the distribution of incompatibility $R^\kappa_{\lambda\beta\gamma}$ in the ordinary 3-space. Therefore it is not permitted that an incompatibility zone terminate in a finite region without forming a closed cycle. This statical property also is analogous to the corresponding kinematical property of the continuity of vorticity.

12. Some general considerations connected with space characteristics

Although a finite lump of matter cannot be elastically brought to the natural state when it has an internal strain, a slender piece which is cut off from the main body can always be brought to the natural state without violating the mutual relation prescribed by the metric and connexions along the length. The prescribed condition is reflected on, and only on, the metric tensor and the coefficients of connexion which can be the same over the whole cross section of the piece if it is cut slender enough.

This slender matter is free of strain as it is in the natural state and is measured on the Euclidean scale. The natural frame at each point is projected on the frame of a certain point arbitrarily chosen.

Any tensor field is uniform if, and only if, its absolute differential vanishes. A uniform vector field is "parallel" in the usual sense, as the parallelism in the Euclidean sense is demonstrated by the vanishing of the absolute derivative of the respective field. Vector fields at different points of the natural line are compared only after the parallel translation of one of them to the point of the other. This process induces a parallel translation in the corresponding Riemannian state along the matter line but the result is different according to the path of translation. This is called the "parallelism of

Levi-Civita" in recognition of the initiator of this conception.

Hence, by parallel translation along a closed path in a Riemannian space, we obtain a different setting of a vector field from the initial one. The corresponding physical meaning is as follows. When a closed thin piece of matter is taken out of a mass of matter, freed from the surroundings and cut off at an arbitrary point so as to form a non-closed piece, both ends separate and take a different setting, if internal strain is present.

Such an effect appears even for the smallest circuit as we have seen in § 9. But this is for the orientation of the frame only. No dislocation occurs for a mere positional property in this case. This is due to the fact that the directional dislocation is proportional to the Riemann-Christoffel curvature tensor which is not a zero-tensor while the other one should have been proportional to the torsion tensor which is always zero for a Riemannian space.

In general for a finite circuit we have a finite dislocation either for the position or for the direction.

Here we hit upon an important conception called the "group of holonomy", which has been propounded by Élie Cartan [15].

There are an infinite number of material circuits through a material point. Each of these circuits presents a proper dislocation when it is developed in the natural state, cut off at the initial point. The dislocation is properly represented by a homeomorphic transformation between the tangent Euclidean spaces attached to the ends of the developed material line, which maps the one onto the other and *vise versa* (Fig. 3). Any consecutive material circuits through a point compose a new circuit for which the dislocation is the resultant of the dislocations of the component circuits (Fig. 4). Correspondingly a resultant mapping of the tangent spaces will occur.

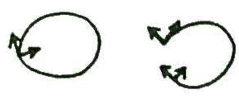

FIG. 3

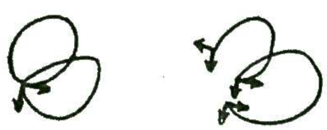

FIG. 4

We have an important theorem propounded by Cartan on the homogeneity of the holonomic group. Take any pair of points A and B in a connected Riemannian manifold. Take an arbitrary closed circuit ACA through A, we then have a corresponding dislocation. This circuit can be prolonged to the point B by such a double line as is shown in Fig. 5(a), so as to form a circuit $BACAB$ through B. We can map the tangent of B to A

FIG. 5

by means of the Levi-Civita parallelism, which transformation we shall denote by T. This transformation, as it is a geometric transformation, is independent of the measuring frames arbitrarily chosen, hence it makes no difference whether the segment AB is measured on different positions in the development, as the upper and lower branch in Fig. 5(b). Further let the dislocation mapping of the circuit ACA be denoted by C, then the dislocation mapping of the resultant circuit $BACAB$ should be equal to TCT^{-1}. This relation holds for every circuit through A, establishing one-to-one correspondence between the sets of circuits through A and B. Hence the group of holonomy for B is isomorphic with

$$TgT^{-1}$$

if g is the holonomic group for A. Since A and B are arbitrary points, the same similarity can be established for all points of a connected Riemannian space. They are equivalent as group of geometrical transformations, whence we can say, we have obtained the homogeneity.

It is obvious that the non-Euclidean properties of a Riemannian space is entirely reproduced in its holonomic group. The holonomic group of a Euclidean space consists of an identity alone. But there are Riemannian spaces which are locally Euclidean but characterized by holonomic groups provided with non-identity elements [10]. For example, a circular cylindrical surface is a locally Euclidean two-dimensional space but its holonomic group contains a finite transformation and its integral multiples. A circular conical surface is also locally Euclidean but its holonomic group contains a finite rotation around the vertex and its integral multiples. There are locally Euclidean spaces of greater numbers of dimensions with non-trivial holonomic groups.

Such topological properties of spaces, which are major subjects of study in geometry in the large, should bear a close relation to engineering problems, because they can be regarded as a proper measure of *incompatibility in the large*. They have been studied in relation to the problems of stresses and strains of multiply connected regions. Klein's classical investigation has pointed out an importance of the Betti number [16]. The general as well as special problems relating to the discontinuous solution for multiply connected regions in two- or three-dimensional elastic systems have been investigated by a number of later authors. A general treatise on the fundamental formulation of the topological problems in the large of the elastic field will be published in a forthcomming report by Y. Yamamoto.

The conservation, under Eulerian deformations, of the dislocation adjoint to any particular loop of matter can be established, because it is defined in regard to the natural state invariant under any transformation of Eulerian coordinates. Moriguti's analytical proof relates especially to this problem. In the present paper it is reduced simply to a problem of definition.

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