Overview of some contributions on computational geometry on various geometric structures beyond the Euclidean structure

Frank Nielsen

Riemannian geometry

The uniqueness and circumcenter of the smallest enclosing ball on a finite point set lying on a Riemannian manifold was studied in [2].

Finsler geometry

Finsler geometry extends Riemannian geometry by considering smoothly varying Minkowski norms at tangent planes of a manifold. The forward and backward p-centers on Finsler manifolds was considered in [1].

Fisher-Rao geometry

The Fisher-Rao geometry of a parametric statistical model corresponds to the Riemannian geometry with respect to the Fisher metric. The Riemannian geodesic distance is called the Fisher-Rao distance in information geometry [9]. Approximation schemes of the Fisher-Rao distances are considered in [13, 15]. Fisher-Rao geometry of location-scale families amount to hyperbolic geometry.

Dually flat geometry

Dually flat geometry has the structures of both a Riemannian manifold with a Hessian metric and a pair of dual torsion-free affine connections. Right-angles in dual geodesic triangles in dually flat spaces are studied in [12]. The dual Voronoi diagrams in a dually flat space are dual Bregman Voronoi diagrams [3] in the dual coordinate systems. Exact and approximation of the smallest enclosing Bregman balls were studied in [29, 20]. Data structures for proximity queries on dually flat spaces are given in [25, 24]. Chernoff information is characterized on a dually flat space in [7, 8]. When the dual potential functions are not in closed-form for exponential or mixture families, Monte Carlo information-geometric structures are considered in [18]. When the Bregman generator is separable, the dually flat space amounts to Euclidean geometry [6].

Hyperbolic geometry

Bisectors in Klein ball model of hyperbolic geometry are affine hyperplanes clipped to the open ball domain [21]. Thus the Klein hyperbolic Voronoi diagram (HVD) and all its k-order Voronoi diagrams are equivalent to power diagrams clipped to the ball domain. The Klein HVD can be converted to other models of hyperbolic geometry [23] (demo: HVD https://www.youtube.com/watch?v=i9IUzNxeH4o, k-order HVD https://www.youtube.com/watch?v=sM_16XgyfhY). The hyperbolic smallest enclosing ball (SEB) in Poincaré ball model has an Euclidean shape and thus amounts to an Euclidean smallest enclosing ball. We can compute numerically the hyperbolic SEB in high dimensions in Klein model with guarantees [17]. The dual of the HVD is the hyperbolic Delaunay complex [10]. Klein Riemannian geodesics, general position and degeneracies of point sets in hyperbolic geometry are studied in [22]. Klein HVDs can be extended to Cayley-Klein HVDs [19] where the domains are ellipsoids.

Hilbert geometry and Birkhoff projective geometry

Hilbert geometry is defined on open bounded convex domain. When the domain is a ball, it amounts to Klein model of hyperbolic geometry. Hilbert geometry of the (a) simplex domain modeling the space of categorical distributions and (b) the elliptope of correlation matrices are studied in [27, 28]. Balls in Hilbert geometry with polygonal domains are investigated in [26].

For an open bounded convex domain Ω, we may define the cone $C_\Omega = \{(\lambda, \lambda\Omega), \lambda > 0\}$ by stacking all its homothets. Birkhoff geometry is a projective geometry which coincides on slices of the cone with the underlying Hilbert geometry [14].

Siegel geometry

The Siegel upper space is a generalization of the Poincaré upper plane: The set of complex square matrices with symmetric positive-definie imaginary parts [16]. The Siegel upper space can be transformed into the Siegel matrix ball which is a generalization of Poincaré ball mode of hyperbolic geometry. The Siegel-Klein geometry [11] is the Hilbert geometry of the Siegel matrix ball model.

Symmetric cone geometry

The cone of symmetric positive-definite matrices is a symmetric cone [14]. Equivariant log-extrinsic centers and Gaussian-like distributions are studied in [4].

Lightlike manifolds

The parameter space of a deep neural network can be considered as a lightlike manifold [30].

Stratifolds

The parameter space of a deep neural network can be considered as a stratifold [5].

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