The (M, N)-Bhattacharyya dissimilarity

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Let $(\mathcal{X}, \mathcal{E}, \mu)$ be a measure space with μ a positive measure. Consider the Bhattacharyya distance between two probability measures P and Q with corresponding densities p and q wrt. μ:

$$D_B[p:q] := -\log \int \sqrt{p(x)\,q(x)}\mathrm{d}\mu(x),$$

and more generally the skewed Bhattacharyya distance:

$$D_{B,\alpha}[p:q] := -\log \int p^\alpha(x)\,q^{1-\alpha}(x)\mathrm{d}\mu(x).$$

Let us prove that $D_{B,\alpha}[p:q] \geq 0$ with equality iff. $p = q$ μ-ae.:

Let us use the property thatthe weighted geometric mean is less or equal than the weighted arithmetic mean:

$$\underbrace{p^\alpha(x) q^{1-\alpha}(x)}_{:=G_\alpha(p(x):q(x))} \leq \underbrace{\alpha p(x) + (1-\alpha) q(x)}_{:=A_\alpha(p(x):q(x))},$$

with equality holding iff. $p(x) = q(x)$.

It follows that

$$\underbrace{\int \left(p^\alpha(x) q^{1-\alpha}(x)\right) \mathrm{d}\mu}_{:=\mathrm{BC}_\alpha[p:q]} \leq \underbrace{\int \left(\alpha p(x) + (1-\alpha) q(x)\right)}_{=1} \mathrm{d}\mu(x),$$

where BC_α is the skewed Bhattacharyya coefficient in $[0,1]$. Hence, we have by monotonicity of the logarithm function:

$$\log \mathrm{BC}_\alpha[p:q] \leq \log 1 = 0,$$

It follows that

$$D_{B,\alpha}[p:q] = -\log \mathrm{BC}_\alpha[p:q] \geq 0,$$

with equality iff. $p = q$ μ-ae.

However, the Bhattacharyya distance is not strictly speaking a "mathematical distance" since it does not satisfy the triangle inequality. Thus it is a misnomer and should have been better called the Bhattacharyya dissimilarity.

In general, for an inequality $\mathrm{lhs}(p:q) \leq \mathrm{rhs}(p:q)$, we may define the following inequality gap dissimilarities: $\mathrm{rhs}(p:q) - \mathrm{lhs}(p:q)$ or $-\log \frac{\mathrm{lhs}(p:q)}{\mathrm{rhs}(p:q)} \geq 0$.

Thus we could have chosen any power mean $P_{r,\alpha}(a,b) = (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}$ with $r < 1$ and $a, b \geq 0$ (and $P_{0,\alpha} = G_\alpha$, the weighted geometric mean) to define a generalized Bhattacharyya distance since

$$P_{r,\alpha}(p(x):q(x)) \leq A_\alpha(p(x):q(x)) = \alpha p(x) + (1-\alpha) q(x), r \leq 1$$

and we get

$$D_{B,\alpha}[p:q] = -\log \int P_{r,\alpha}(p(x):q(x))\mathrm{d}\mu(x) \geq 0.$$

For two comparable weighted means $M_\alpha \leq N_\alpha$, we define

$$\boxed{D_{B,\alpha}^{M,N}[p:q] := -\log \frac{\int M_{r,\alpha}(p(x):q(x))\mathrm{d}\mu(x)}{\int N_{r,\alpha}(p(x):q(x))\mathrm{d}\mu(x)} \geq 0.}$$

See

- Nielsen, Frank. Generalized Bhattacharyya and Chernoff upper bounds on Bayes error using quasi-arithmetic means. Pattern Recognition Letters 42 (2014): 25-34.
- Nielsen, Frank, Ke Sun, and Stéphane Marchand-Maillet. On Hölder projective divergences. Entropy 19.3 (2017): 122.