

Generalizing Bregman divergences: Old and new directions

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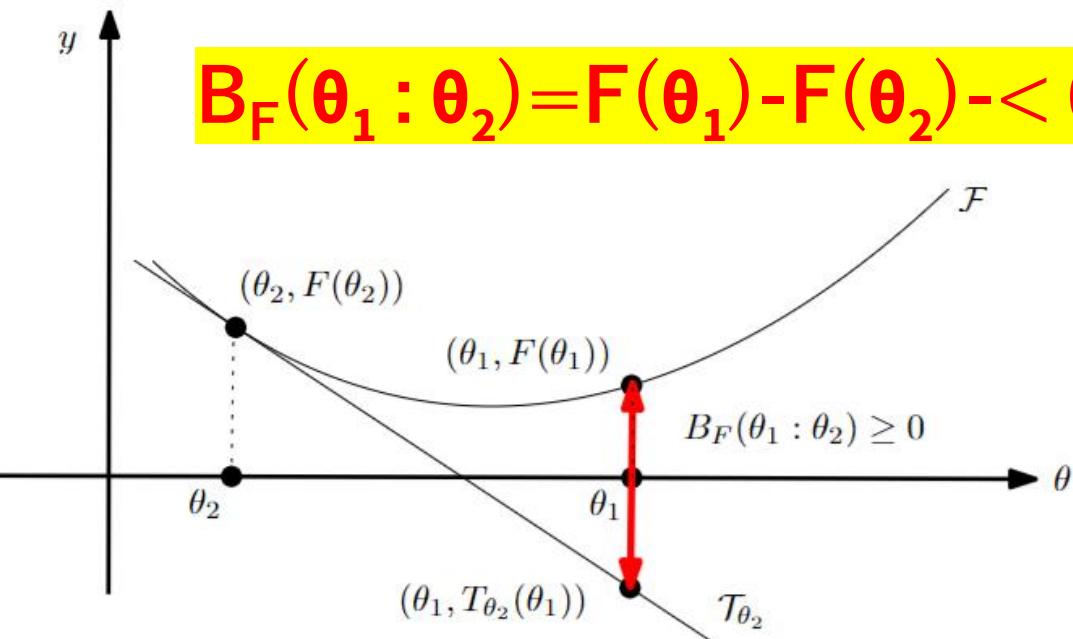
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Bregman divergence (1960's)

- Let $F: \Theta \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ be a strictly convex and smooth real-valued function on a Hilbert space $\langle \cdot, \cdot \rangle$

Bregman divergence $B_F: \Theta \times \text{Int}(\Theta) \rightarrow \mathbb{R}$



Popular in geometry, information theory, signal/sound processing!



Lev M. Bregman
(1941 - 2023)
Photo: courtesy of
Alexander Fradkov

- Unify squared Euclidean divergence with Kullback-Leibler divergence $F(\theta) = \sum_i \theta_i \log(\theta_i)$ and Itakura-Saito divergence $F(\theta) = \sum_i -\log(\theta_i)$.
- The L22, KLD and ISD belong to a single family of **β -divergences**, learn β

Bregman divergences in machine learning (ML)

- Kullback-Leibler divergence between two probability densities:

$$D_{KL}[p(x) : q(x)] = \int p(x) \log(p(x)/q(x)) d\mu(x)$$

difficult to calculate in closed form because of the integral $\int \dots$

- But the Kullback-Leibler divergence between two probability densities of an exponential family like Gaussians, Poisson, Dirichlet, Gamma/Beta, Wishart

$$p_\lambda(x) \propto \tilde{p}_\lambda(x) = \exp(\langle \theta(\lambda), t(x) \rangle) h(x) \quad p(x|\theta) \propto \exp(\langle x, \theta \rangle)$$

amount to a **reverse Bregman divergence** $B_F^*(\theta_1 : \theta_2) := B_F(\theta_2 : \theta_1)$

$$D_{KL}[p(x|\theta_1) : p(x|\theta_2)] = B_F^*(\theta_1 : \theta_2) = B_F(\theta_2 : \theta_1) \Rightarrow \text{Easy calculations}$$

Bypass the $\int, \nabla F$ easy!

- Notice divergence between parameters B_F vs divergence between functions KL

Azoury, Katy S., and Manfred K. Warmuth. "Relative loss bounds for on-line density estimation with the exponential family of distributions." *Machine learning* 43 (2001)

Tractability: Monte Carlo Bregman divergences

- In some cases, **integral Bregman generators** are not available in closed-form or **computationally intractable**. Examples:
 $F(\theta) = \log \int \exp(\langle \text{Polynomial}(x), \theta \rangle) d\mu(x)$ or $F(\theta) = \log \sum \exp(\langle x, \theta \rangle)$
- By MC importance sampling of the integral, get with high probability a convex function approximating the generator. Perform algorithms on this randomized Bregman manifolds to get **consistent algorithms**.

$$G(\eta) = -h(m(x; \eta)) = \int_{x \in \mathcal{X}} m(x; \eta) \log m(x; \eta) d\mu(x).$$



$$G(\eta) \simeq \tilde{G}_S(\eta) := \frac{1}{m} \sum_{i=1}^m \frac{1}{q(x_i)} m(x_i; \eta) \log m(x_i; \eta).$$

Stochastic Bregman generator

Monte Carlo information-geometric structures

Geometric Structures of Information (2019): 69-103.

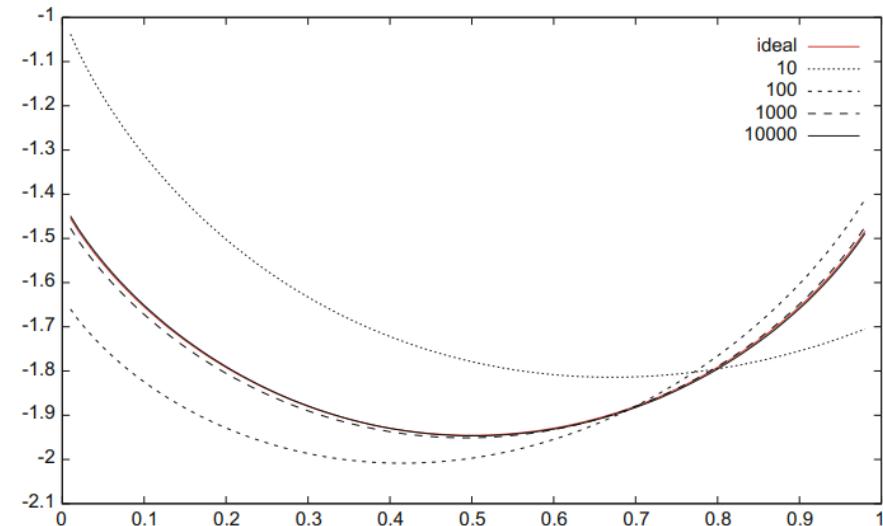


Fig. 2 A series $G_S(\eta)$ of Bregman Monte Carlo Mixture Family generators (for $m = |\mathcal{S}| \in \{10, 100, 1000, 10000\}$) approximating the untractable ideal negentropy generator $G(\eta) = -h(m(x; \eta))$ (red) of a mixture family with prescribed Gaussian distributions $m(x; \eta) = (1 - \eta)p(x; 0, 3) + \eta p(x; 2, 1)$ for the proposal distribution $q(x) = m(x; \frac{1}{2})$

Convex duality via Legendre-Fenchel transform

- Legendre-Fenchel transform of a convex function F :

$$F^*(\eta) = \sup_{\theta \in \Theta} \{ \langle \theta, \eta \rangle - F(\theta) \}$$

- Consider nice convex functions = **Legendre-type functions** $(\Theta, F(\theta))$:
(i) Θ open, and (ii) $\lim_{\theta \rightarrow \partial \Theta} \| \nabla F(\theta) \| = \infty$

Then we get:

- reciprocal gradient maps $\eta = \nabla F(\theta)$ and $\theta = \nabla F^*(\eta)$, $\nabla F^* = (\nabla F)^{-1}$
- conjugation yields $(H, F^*(\eta))$ of Legendre type
- biconjugation is an involution: $(H, F^*(\eta))^* = (H^* = \Theta, F^{**} = F(\theta))$

- Convex conjugate: $F^*(\eta) = \langle \nabla F^{-1}(\eta), \eta \rangle - F(\nabla F^{-1}(\eta))$ since $\eta = \nabla F(\theta)$

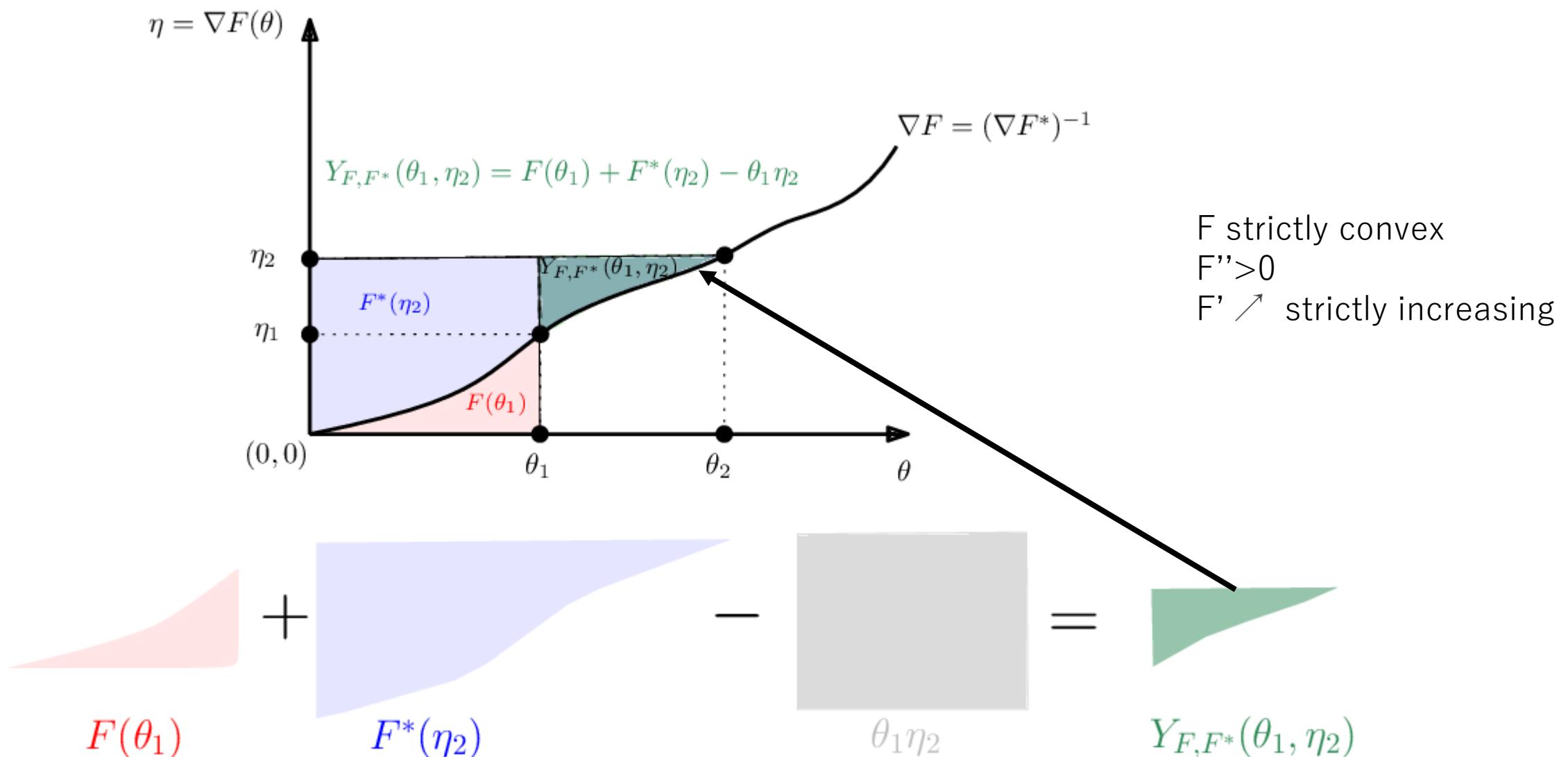
Fenchel-Young divergence: Mixed parameterization

- Young inequality: $F(\theta_1) + F^*(\eta_2) \geq \langle \theta_1, \eta_2 \rangle$ with equality when $\eta_2 = \nabla F(\theta_1)$
- Build the Fenchel-Young divergence from the inequality: lhs-rhs ≥ 0

$$Y_{F, F^*}(\theta_1, \eta_2) = F(\theta_1) + F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle \geq 0$$

- **Mixed parameterizations** θ and η : $B_F(\theta_1 : \theta_2) = Y_{F, F^*}(\theta_1, \eta_2)$

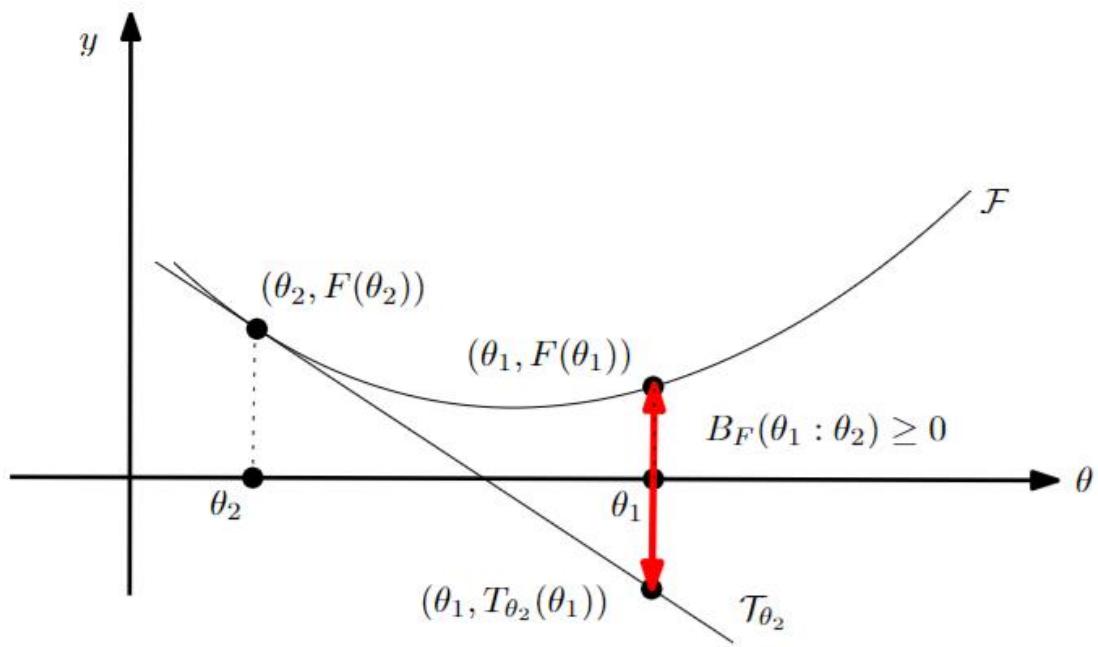
Visual interpretation of Fenchel-Young divergences



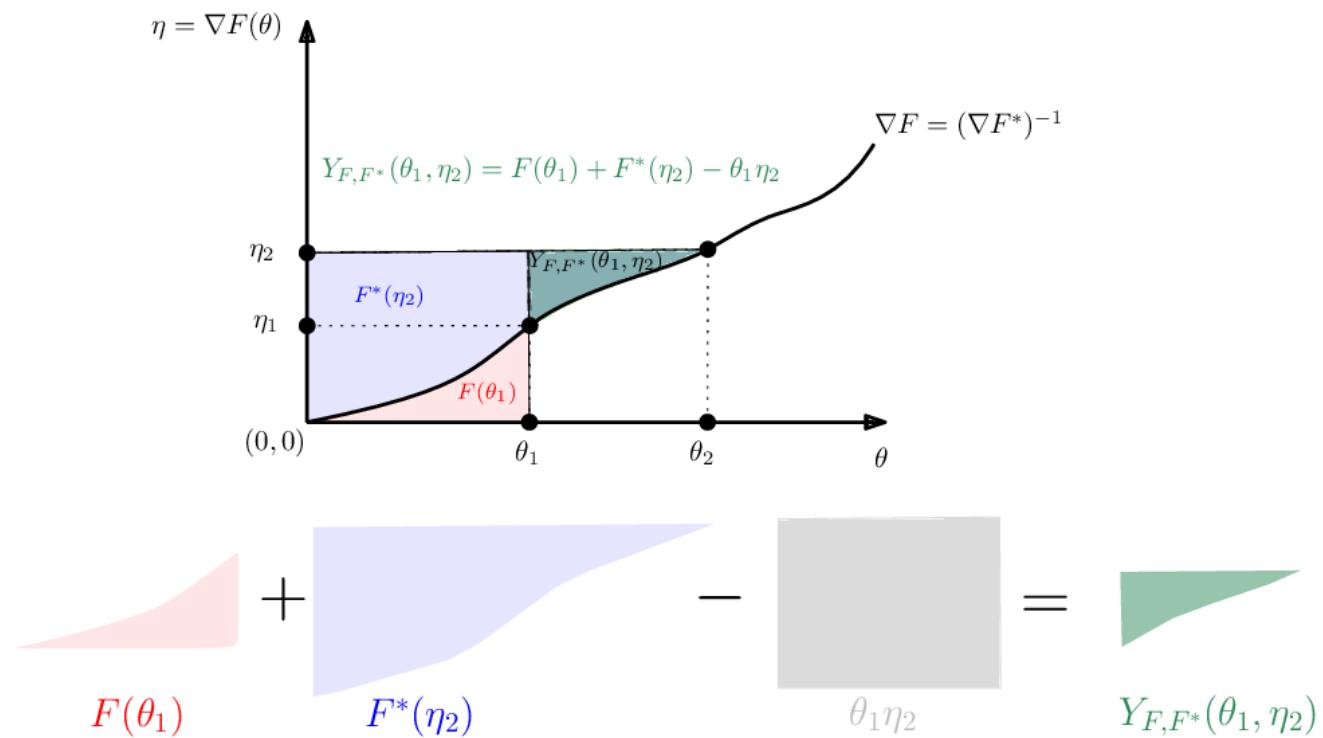
Bregman divergence vs Fenchel-Young divergence

Same parameterization $B_F(\theta_1 : \theta_2) = Y_{F, F^*}(\theta_1, \eta_2)$ mixed parameterization

F strictly convex and differentiable



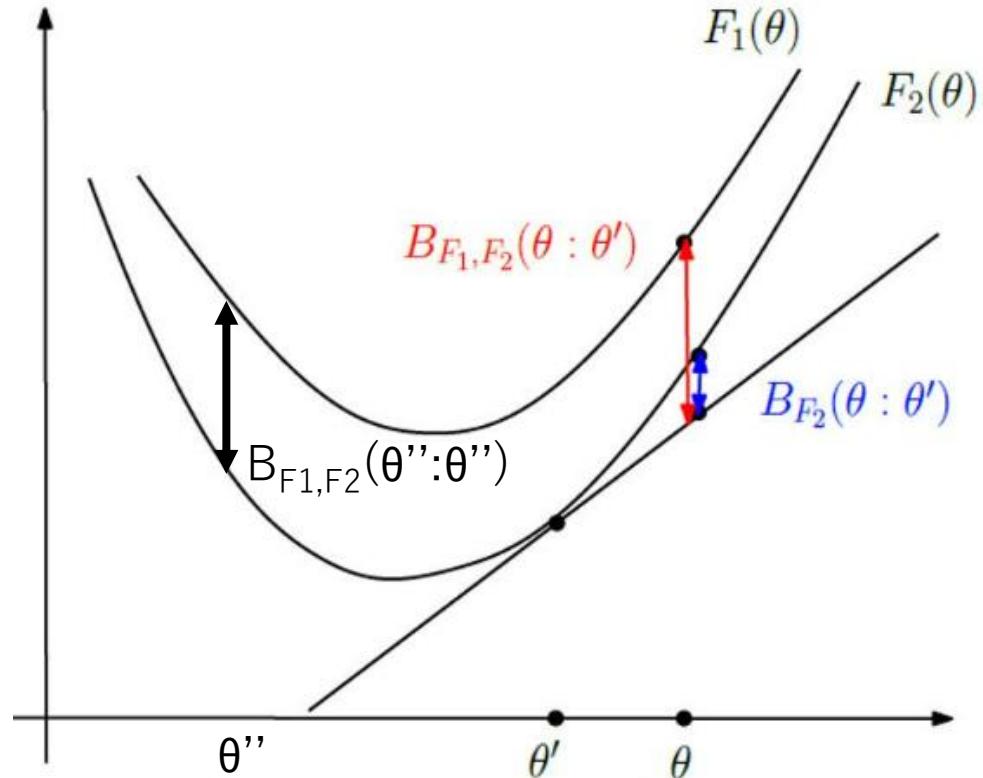
$F' \nearrow$ strictly increasing



$$B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$$

$$Y_{F, F^*}(\theta_1, \eta_2) = F(\theta_1) + F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle$$

Duo Bregman divergences: Generalize BDs with a pair of generators



One generator majorizes the other one:

$$F_1(\theta) \geq F_2(\theta)$$

Then

$$\begin{aligned} B_{F1,F2}(\theta : \theta') &= F_1(\theta) - F_2(\theta') - (\theta - \theta')^\top \nabla F_2(\theta') \\ &\geq B_F(\theta : \theta') \end{aligned}$$

- Recover Bregman divergence when $F_1(\theta) = F_2(\theta) = F(\theta)$
 $B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$
- Only **pseudo-divergence** because $B_{F1,F2}(\theta'':\theta'')$ may be positive, not zero

KLD between nested exponential families amount to duo Bregman divergences

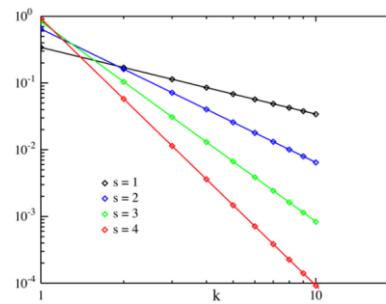
$$\frac{\int_{X_1} \frac{q(x|\theta) \gg p(x|\theta)}{p(x|\theta)} d\mu(x)}{\int_{X_2} q(x|\theta) d\mu(x)}$$

- Consider an exponential family on support X_1 : $D_{KL}[p(x):q(x)] = \int p(x) \log(p(x)/q(x)) d\mu(x)$
 $p(x|\theta) = \exp(\langle x, \theta \rangle - F_1(\theta)) d\mu(x)$ $0 \log(0/0) = 0$
with cumulant function $F_1(\theta) = \log \int_{X_1} \exp(\langle x, \theta \rangle) d\mu(x)$
- Another exponential family with **nested supports: $X_1 \subseteq X_2$**
 $q(x|\theta) = \exp(\langle x, \theta \rangle - F_2(\theta)) d\mu(x)$
is an exponential family with $F_2(\theta) = \log \int_{X_2} \exp(\langle x, \theta \rangle) d\mu(x) \geq F_1(\theta)$
- Then KLD amounts to a reverse duo Bregman pseudo-divergence:

$$D_{KL}[p(x|\theta_1) : q(x|\theta_2)] = B_{F2F1}^*(\theta_1; \theta_2) = B_{F2F1}(\theta_2; \theta_1)$$

"Statistical divergences between densities of truncated exponential families with nested supports: Duo Bregman and duo Jensen divergences." *Entropy* 24.3 (2022)

Application of duo Bregman divergences: Clustering distributions with different supports



- Consider n truncated densities $p(x|\theta_i) = \exp(\langle x, \theta_i \rangle - F_i(\theta_i)) d\mu(x)$ with potentially different supports $X_i \subseteq X$: e.g., **Zipf's distributions**

$$f(k; N, s) = \frac{1}{H_{N,s}} \frac{1}{k^s} \quad H_{N,s} = \sum_{k=1}^N \frac{1}{k^s}$$

- Cluster those distributions using **full support X prototypes**: e.g., **Zeta distributions**

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad \text{This ensures that } q(x|\lambda_j) \gg p(x|\theta_i)$$

- Objective is a la “k-means”: minimize $\sum_i D_{KL}[p(x|\theta_i) : \{q(x|\lambda_j) : j \text{ in } \{1, \dots, k\}\}]$
- Duo Bregman k-means** algorithm as an extension of Bregman k-means
- Example: Cluster Zipf's distributions from word frequencies in a collection of translations of a renown book. Find similarities of word frequency in languages

Duo Fenchel-Young divergences

- Young inequality: $F(\theta_1) + F^*(\eta_2) \geq \langle \theta_1, \eta_2 \rangle$ with equality when $\eta_2 = \nabla F(\theta_1)$
- Build the Fenchel-Young divergence from the inequality: lhs-rhs ≥ 0

$$Y_{F, F^*}(\theta_1, \eta_2) = F(\theta_1) + F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle \geq 0$$



- Legendre transform reverses majorization order:

$$F_1(\theta) \geq F_2(\theta) \Leftrightarrow F_1^*(\eta) \leq F_2^*(\eta)$$

- **Duo Fenchel-Young divergence:**

$$Y_{F_1, F_2^*}(\theta, \eta') := F_1(\theta) + F_2^*(\eta') - \theta^\top \eta',$$

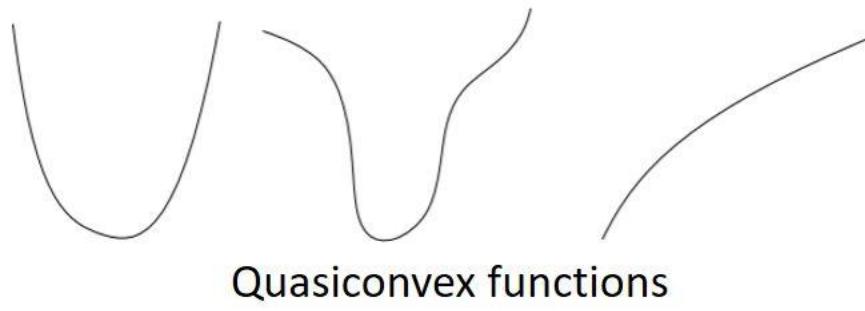
$$\geq F_1(\theta) + F_1^*(\eta') - \theta^\top \eta' = Y_{F_1, F_1^*}(\theta, \eta') \geq 0$$

Quasiconvex Jensen and Bregman divergences

- **Quasiconvex Jensen divergence** for a generator Q and α in $(0,1)$:

$${}^{\text{qcvx}} J_Q^\alpha(\theta : \theta') := \max\{Q(\theta), Q(\theta')\} - Q((1 - \alpha)\theta + \alpha\theta')).$$

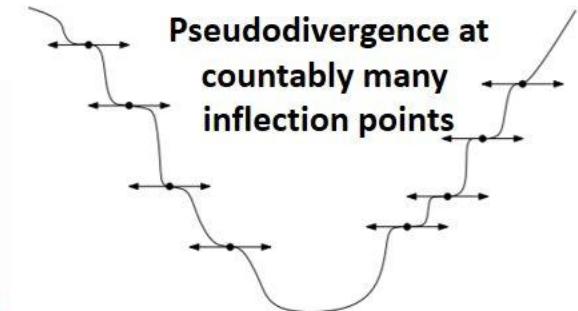
Quasiconvex Bregman pseudodivergence (not separable divergence):



$$\begin{aligned} {}^{\text{qcvx}} B_Q(\theta : \theta') &:= \lim_{\alpha \rightarrow 1} \frac{1}{\alpha(1 - \alpha)} {}^{\text{qcvx}} J_Q^\alpha(\theta : \theta'), \\ &= \begin{cases} -(\theta - \theta')^\top \nabla Q(\theta') & \text{if } Q(\theta) \leq Q(\theta') \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

- **δ -quasiconvex Bregman divergence** for $\delta > 0$:

$${}^{\text{qcvx}} B_Q^\delta(\theta, \theta') := \begin{cases} \frac{1}{\delta} (Q(\theta' + \delta(\theta' - \theta)) - Q(\theta')) & \text{if } Q(\theta') \geq Q(\theta) \\ +\infty & \text{otherwise} \end{cases}$$



- Qcvx Bregman pseudodivergence related to **Kullback-Leibler divergence** for distributions with **nested supports**

$$\text{KL}[p : q] = \begin{cases} \int_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} d\mu(x), & \text{supp}(p) \subset \text{supp}(q), \\ +\infty, & \text{supp}(p) \not\subset \text{supp}(q). \end{cases}$$

Biduality reference/representation

Legendre-Fenchel transformation * induces two functions:

- ① a convex conjugate function $F^*(\cdot) : \eta \rightarrow F^*(\eta)$
- ② a gradient map: $\eta = \nabla F(\cdot) : \theta \rightarrow \eta(\theta) = \nabla F(\theta)$

This yields the interplay of two dualities:

- ① **Reference duality** of divergences: $D^*(p_1:p_2) := D(p_2: p_1)$, $(D^*)^* = D$
- ② **Representation duality**: $\theta^* = \nabla F(\theta)$, and $(\theta^*)^* = \theta$

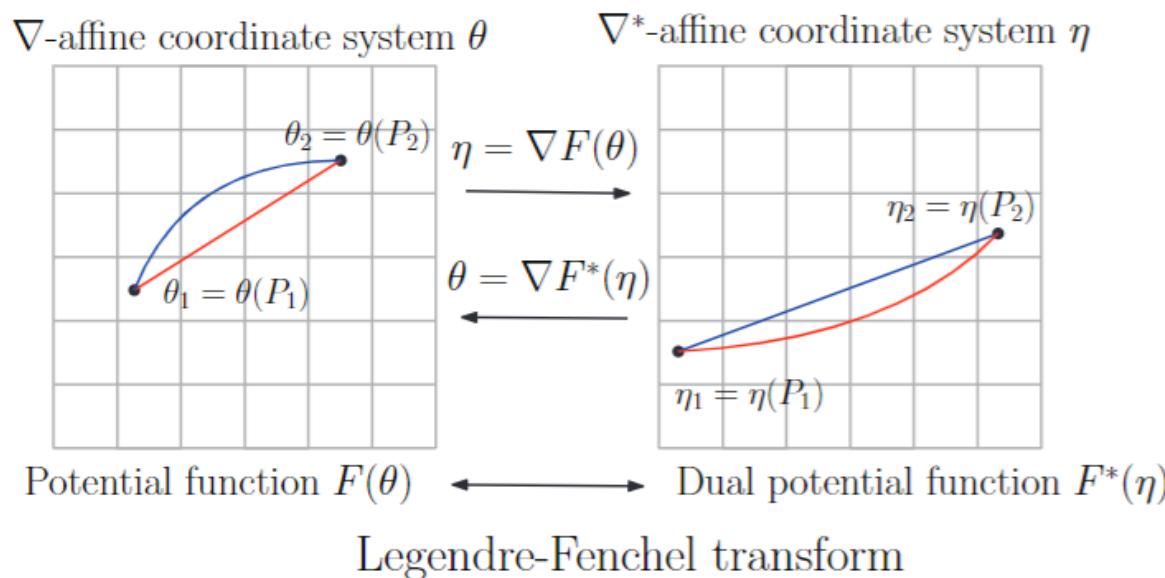
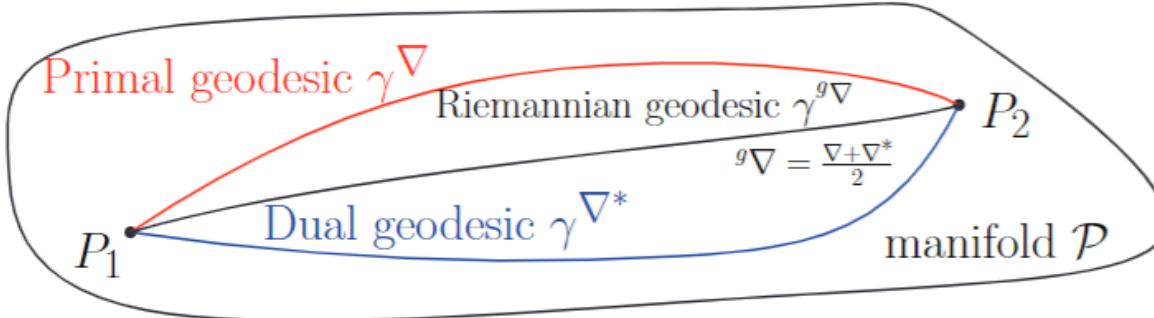
Fenchel-Young divergence demonstrates the biduality interaction:

$$(Y_F)^*(\theta_1: \theta_2^*) = Y_F(\theta_2^*: \theta_1) = Y_{F^*}(\theta_2^*: \theta_1)$$

Functional divergences on densities as pointwise scalar BD can use monotone (ρ, τ) -embedding dualities

Bregman manifolds: Convex conjugates (F, F^*) yield dual flat connections

$$(M, F \rightarrow g(\theta) = \nabla^2 F(\theta), F \rightarrow \nabla, F^* \rightarrow \nabla^*)$$



- A connection ∇ is **flat** if there exists a coordinate system θ such that all Christoffel symbols vanish: $\Gamma(\theta) = 0$.

- θ is called **∇ -affine coordinate system**
- **∇ -geodesic** solves as **line segments**

$$\cancel{\frac{d^2 \theta_k}{dt^2} + \sum_{i=1}^p \sum_{j=1}^p \Gamma_{ij}^k \frac{d\theta_i}{dt} \frac{d\theta_j}{dt} = 0}$$

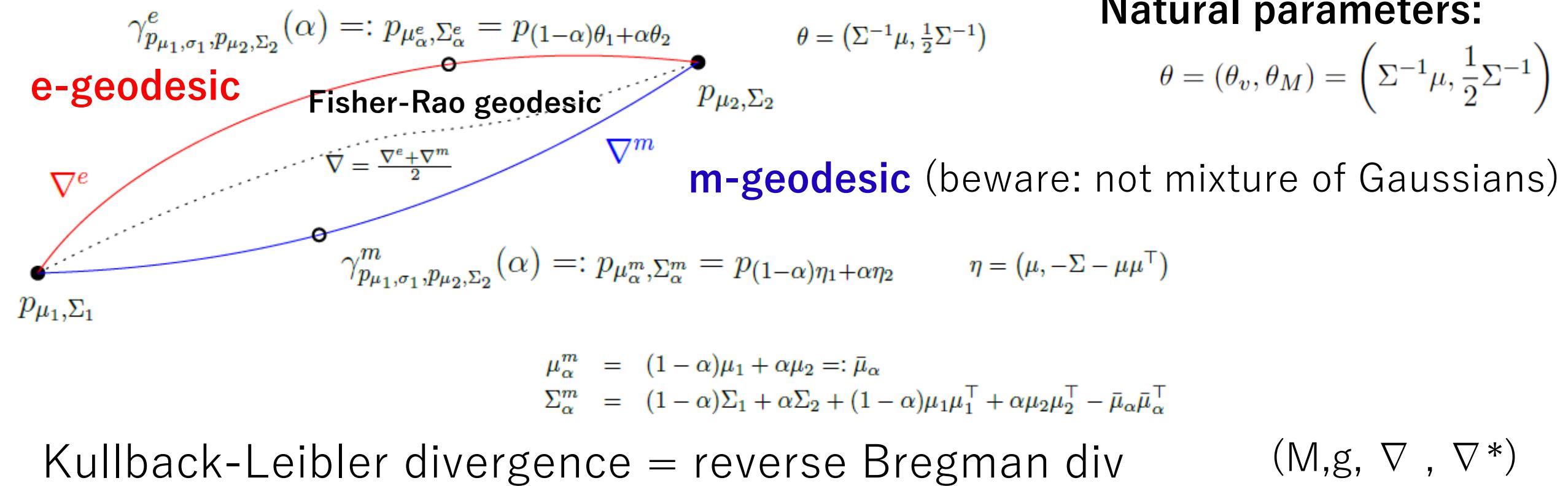
Example: Bregman manifold of multivariate normals

Cumulant function, convex:

$$F_\theta(\theta) = \frac{1}{2} \left(d \log \pi - \log |\theta_M| + \frac{1}{2} \theta_v^\top \theta_M^{-1} \theta_v \right)$$

Natural parameters:

$$\theta = (\theta_v, \theta_M) = \left(\Sigma^{-1} \mu, \frac{1}{2} \Sigma^{-1} \right)$$



Kullback-Leibler divergence = reverse Bregman div

(M,g, ∇ , ∇^*)

$$\frac{1}{2} \left(\text{tr}(\Sigma_2^{-1} \Sigma_1) - \log \frac{\det(\Sigma_2)}{\det(\Sigma_1)} - d + (\mu_2 - \mu_1)^\top \Sigma_2^{-1} (\mu_2 - \mu_1) \right)$$

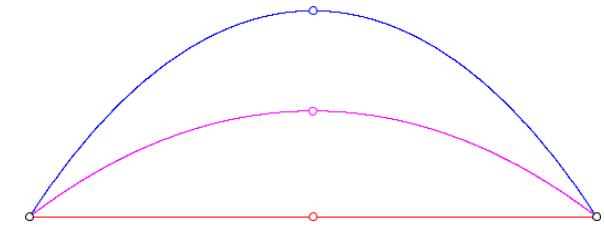
Geometric Jensen-Shannon divergence, 2019

Bregman manifolds and Bregman divergences

- Any Legendre-type function $(\theta, F(\theta))$ generates DFS (M, g, ∇, ∇^*) where $F(\theta)$ defines flat connection ∇ via Christoffel symbols $\Gamma(\theta) = 0$, and $F^*(\eta)$ defines flat connection ∇^* via Christoffel symbols $\Gamma^*(\eta) = 0$
- **Duality in information geometry:** $(\nabla + \nabla^*)/2$ is Levi-Civita connection ∇^g

For example:

- The **cumulant functions** of exponential families $F(\theta) = \log \int \exp(\langle x, \theta \rangle) d\mu(x)$. In that case, the Bregman divergence amounts to a reverse Kullback-Leibler divergence
- The **partition functions** $Z(\theta) = \int \exp(\langle x, \theta \rangle) d\mu(x) = \exp(F(\theta))$ is log-convex and log-convex functions are convex. Hence, we can build a Bregman manifold from $Z(\theta)$ too!
- **Question: What is the reconstructed statistical divergence from Bregman divergence B_Z ?**



Kullback-Leibler divergence between non-normalized exponential family densities

- Kullback-Leibler divergence between two positive measures:

$$D_{KL}^+[p_1(x):p_2(x)] = \int \{ p_1(x) \log(p_1(x)/p_2(x)) + p_2(x) - p_1(x) \} d\mu(x)$$

- Exponential family density:

- Normalized: $p(x|\theta) = \exp(\langle x, \theta \rangle - F(\theta)) d\mu(x)$
- Non-normalized: $q(x|\theta) = \exp(\langle x, \theta \rangle) d\mu(x)$

- Hence, $p(x|\theta) = q(x|\theta)/Z(\theta)$ with partition function $Z(\theta) = \exp(F(\theta))$ and cumulant function $F(\theta) = \log Z(\theta)$
- F is convex and Z is log-convex, and log-convex functions are convex functions.

- KLD between normalized densities = reverse Bregman wrt F :

$$D_{KL}[p_{\theta_1}(x):p_{\theta_2}(x)] = B_F^*[\theta_1: \theta_2] = B_F[\theta_2: \theta_1]$$

- KLD between non-normalized densities = reverse Bregman wrt Z :

$$D_{KL}^+[p_{\theta_1}(x):p_{\theta_2}(x)] = B_Z^*[\theta_1: \theta_2] = B_Z[\theta_2: \theta_1]$$

Bregman divergences and Jensen divergences

Cumulant functions/Partition functions

$$F(\theta) = \log Z(\theta) \Leftrightarrow Z(\theta) = \exp(F(\theta)) \quad Z(\theta) = \int \tilde{p}_\theta(x) d\mu(x)$$

$$\textcircled{1} \quad B_Z(\theta_1 : \theta_2) = Z(\theta_1) - Z(\theta_2) - \langle \theta_1 - \theta_2, \nabla Z(\theta_2) \rangle \geq 0,$$

$$\textcircled{2} \quad B_{\log Z}(\theta_1 : \theta_2) = \log\left(\frac{Z(\theta_1)}{Z(\theta_2)}\right) - \left\langle \theta_1 - \theta_2, \frac{\nabla Z(\theta_2)}{Z(\theta_2)} \right\rangle \geq 0,$$

And furthermore, we can define **skewed Jensen divergences** from the convex generators:

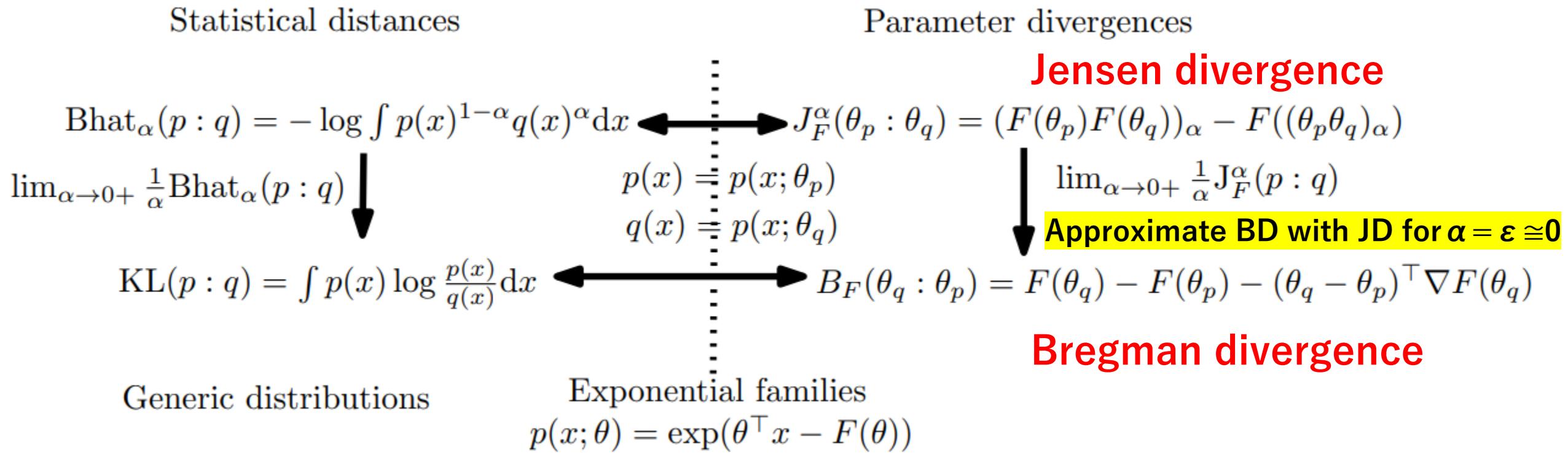
$$\textcircled{1} \quad J_{Z,\alpha}(\theta_1 : \theta_2) = \alpha Z(\theta_1) + (1 - \alpha)Z(\theta_2) - Z(\alpha\theta_1 + (1 - \alpha)\theta_2) \geq 0,$$

$$\textcircled{2} \quad J_{\log Z,\alpha}(\theta_1 : \theta_2) = \log \frac{Z(\theta_1)^\alpha Z(\theta_2)^{1-\alpha}}{Z(\alpha\theta_1 + (1 - \alpha)\theta_2)} \geq 0.$$

Including the **symmetric Jensen divergence** when $\alpha=1/2$:

$$J_F(\theta_1, \theta_2) = J_{F,\frac{1}{2}}(\theta_1 : \theta_2) = \frac{F(\theta_1) + F(\theta_2)}{2} - F\left(\frac{\theta_1 + \theta_2}{2}\right)$$

KLD/ α -Bhattacharyya \Leftrightarrow Bregman/Jensen divergences when considering exponential families, F cumulant function



Question: What are the reconstructed statistical divergences from Jensen-Bregman divergence J_Z / B_Z ?
 Z : partition function

Zhang, Divergence function, duality, and convex analysis, *Neural computation* 16.1 (2004)

N + Boltz. "The Burbea-Rao and Bhattacharyya centroids." *IEEE Transactions on Information Theory* (2011)

Bregman divergences corresponding to partition functions

Question: What is the reconstructed statistical divergence from Bregman divergence B_Z ?

$$J_{Z,\alpha}^s(\theta_1 : \theta_2) = \begin{cases} \frac{1}{\alpha(1-\alpha)} J_{Z,\alpha}(\theta_1 : \theta_2), & \alpha \in \setminus\{0,1\}, \\ B_Z(\theta_1 : \theta_2), & \alpha = 0, \\ 4 J_Z(\theta_1, \theta_2), & \alpha = \frac{1}{2}, \\ B_Z^*(\theta_1 : \theta_2) = B_Z(\theta_2 : \theta_1), & \alpha = 1. \end{cases} \quad \longleftrightarrow \quad D_\alpha(\tilde{p} : \tilde{q}) = \begin{cases} \frac{1}{\alpha(1-\alpha)} \int (\alpha \tilde{p} + (1-\alpha)\tilde{q} - \tilde{p}^\alpha \tilde{q}^{1-\alpha}) d\mu, & \alpha \notin \{0,1\} \\ D_{\text{KL}}^*(\tilde{p} : \tilde{q}) = D_{\text{KL}}(\tilde{q} : \tilde{p}) & \alpha = 0, \\ 4 D_H^2(\tilde{p}, \tilde{q}) & \alpha = \frac{1}{2}, \\ D_{\text{KL}}(\tilde{p} : \tilde{q}) & \alpha = 1. \end{cases}$$

Scaled skewed Jensen divergence
for partition function Z



Amari α -divergences extended to
positive measures

So B_Z corresponds to the reverse **extended Kullback-Leibler divergence**:

$$\begin{aligned} D_{\text{KL}}(\tilde{p} : \tilde{q}) &= H^\times(\tilde{p} : \tilde{q}) - H(\tilde{p}), \\ &= \int \left(\tilde{p} \log \frac{\tilde{p}}{\tilde{q}} + \tilde{q} - \tilde{p} \right) d\mu \end{aligned}$$

Comparative convexity: (M,N)-convexity

Ordinary convexity of a function: $f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$
for all t in $[0,1]$

- Definition: A function Z is **(M,N)-convex** iff for α in $[0,1]$:

$$Z(M(x, y; \alpha, 1 - \alpha)) \leq N(Z(x), Z(y); \alpha, 1 - \alpha)$$

- Ordinary convexity = (A,A)-convexity wrt to arithmetic weighted mean

$$A(x, y; \alpha, 1 - \alpha) = \alpha x + (1 - \alpha)y \quad f(tx_1 + (1 - t)x_2) \leq tf(x_1) + (1 - t)f(x_2)$$

for all t in $[0,1]$

- **Log-convexity: (A,G)-convexity** wrt to A/geometric weighted means:

$$G(x, y; \alpha, 1 - \alpha) = x^\alpha y^{1-\alpha} \quad f(tx_1 + (1 - t)x_2) \leq f(x_1)^t f(x_2)^{1-t}$$

for all t in $[0,1]$

Comparative convexity wrt quasi-arithmetic means

- **quasi-arithmetic mean** for a strictly monotone generator $h(u)$:

$$M_h(x, y; \alpha, 1 - \alpha) = h^{-1}(\alpha h(x) + (1 - \alpha)h(y)).$$

- Includes **power means** which are *homogeneous means*:

$$M_p(x, y; \alpha, 1 - \alpha) = (\alpha x^p + (1 - \alpha)y^p)^{\frac{1}{p}} = M_{h_p}(x, y; \alpha, 1 - \alpha), \quad p \neq 0$$

$$h_p(u) = \frac{u^p - 1}{p} \quad h_p^{-1}(u) = (1 + up)^{\frac{1}{p}}$$

Include the **geometric mean** in the limit case $p \rightarrow 0$

Proposition 6 ([1, 34]). A function $Z(\theta)$ is strictly (M_ρ, M_τ) -convex with respect to two strictly increasing smooth functions ρ and τ if and only if the function $F = \tau \circ Z \circ \rho^{-1}$ is strictly convex.

Generalizing Bregman divergences with (M,N)-convexity: (M,N)-Bregman divergences

- Skew Jensen divergence from (M,N) comparative convexity:

Definition:

$$J_{F,\alpha}^{M,N}(p : q) = N_\alpha(F(p), F(q)) - F(M_\alpha(p, q)).$$

Non-negative for **(M,N)-convex generators** F , provided regular means M and N (e.g. power means)

Definition 5 (Bregman Comparative Convexity Divergence, BCCD) *The Bregman Comparative Convexity Divergence (BCCD) is defined for a strictly (M, N) -convex function $F : I \rightarrow \mathbb{R}$ by*

$$B_F^{M,N}(p : q) = \lim_{\alpha \rightarrow 1^-} \frac{1}{\alpha(1-\alpha)} J_{F,\alpha}^{M,N}(p : q) = \lim_{\alpha \rightarrow 1^-} \frac{1}{\alpha(1-\alpha)} (N_\alpha(F(p), F(q))) - F(M_\alpha(p, q))) \quad (31)$$

By analogy to limit of skewed Jensen divergences amount to forward/reverse Bregman divergences.

Generalizing Bregman divergences with quasi-arithmetic mean convexity

Theorem 1 (Quasi-arithmetic Bregman divergences, QABD) *Let $F : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued (M_ρ, M_τ) -convex function defined on an interval I for two strictly monotone and differentiable functions ρ and τ . The quasi-arithmetic Bregman divergence (QABD) induced by the comparative convexity is:*

$$B_F^{\rho, \tau}(p : q) = \frac{\tau(F(p)) - \tau(F(q))}{\tau'(F(q))} - \frac{\rho(p) - \rho(q)}{\rho'(q)} F'(q). \quad (45)$$

Amounts to a **conformal Bregman divergence** on monotonic representations:

$$B_F^{\rho, \tau}(p : q) = \frac{1}{\tau'(F(q))} B_G(\rho(p) : \rho(q))$$

Conformal factor

With generator:
 $G(x) = \tau(F(\rho^{-1}(x)))$

Remark: Conformal Bregman divergences may yield **robustness** in applications

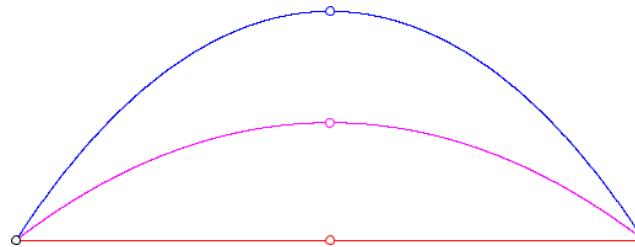
(M,N) -convexity for convex-preserving deformations

- Recall that for exponential families, we had two convex functions:

Convex $F(\theta) = \log Z(\theta) \leftrightarrow$ Log-convex $Z(\theta) = \exp(F(\theta))$, hence convex
Cumulant function \leftrightarrow Partition function

Proposition 6 ([1, 34]). *A function $Z(\theta)$ is strictly (M_ρ, M_τ) -convex with respect to two strictly increasing smooth functions ρ and τ if and only if the function $F = \tau \circ Z \circ \rho^{-1}$ is strictly convex.*

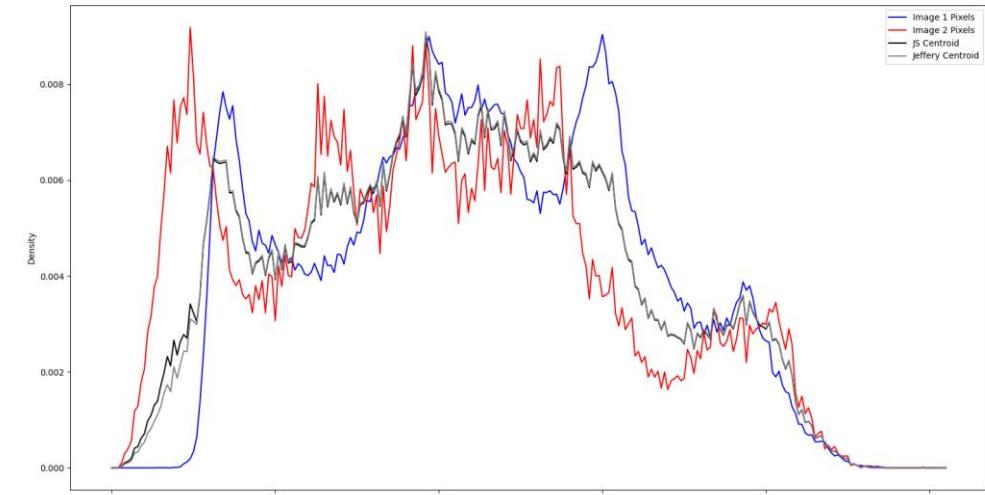
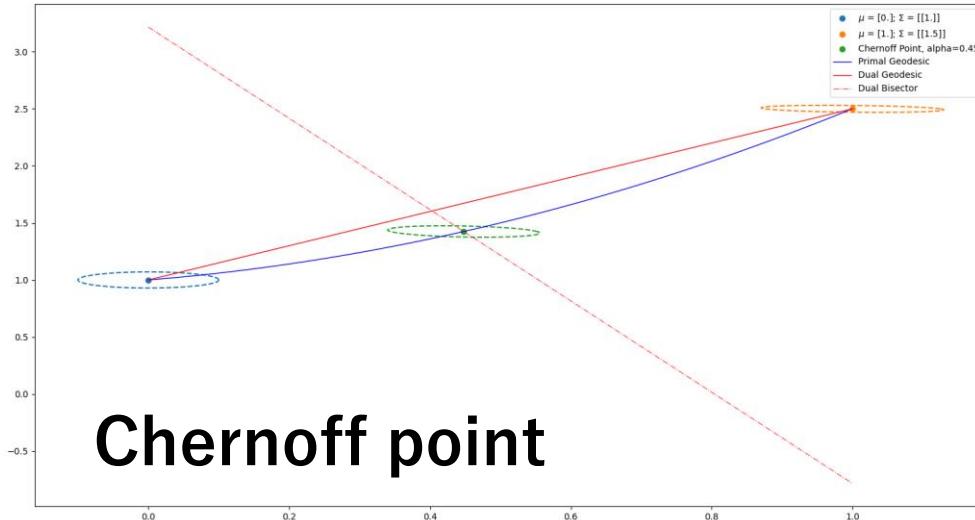
- We may thus **deform F with (ρ, τ) -quasi-arithmetic means** and seek the (ρ, τ) -convex function which also remains ordinary convex
- For those convex-preserving (ρ, τ) -deformations, we get new Bregman divergences and Bregman manifolds to play with!



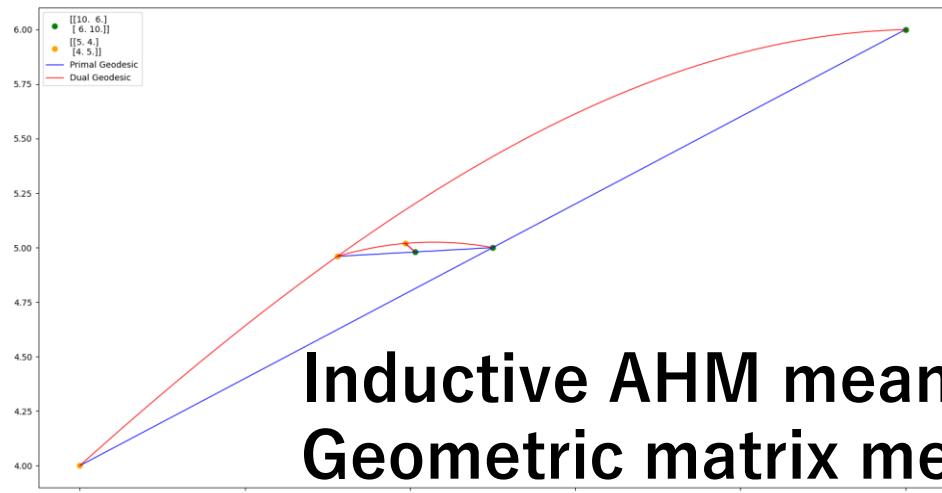
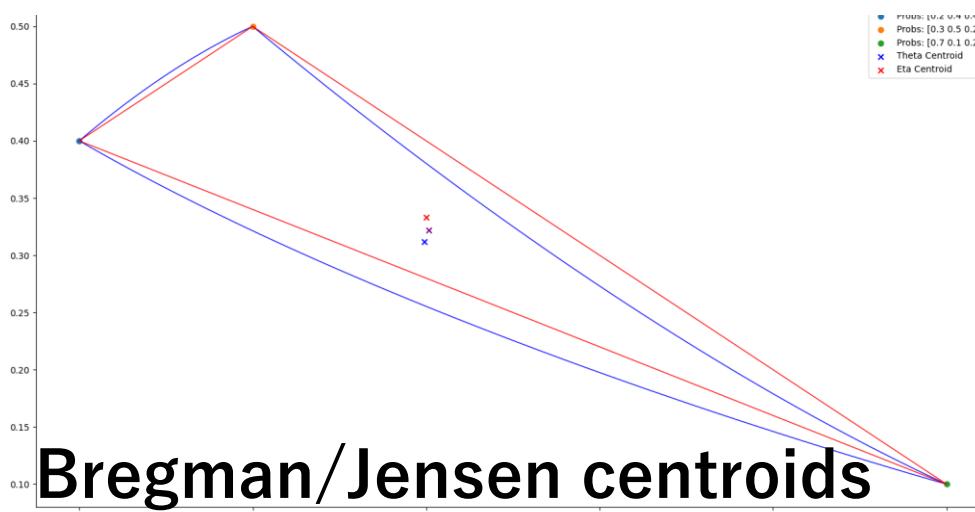
A Python library for geometric computing on Bregman Manifolds

pyBregMan

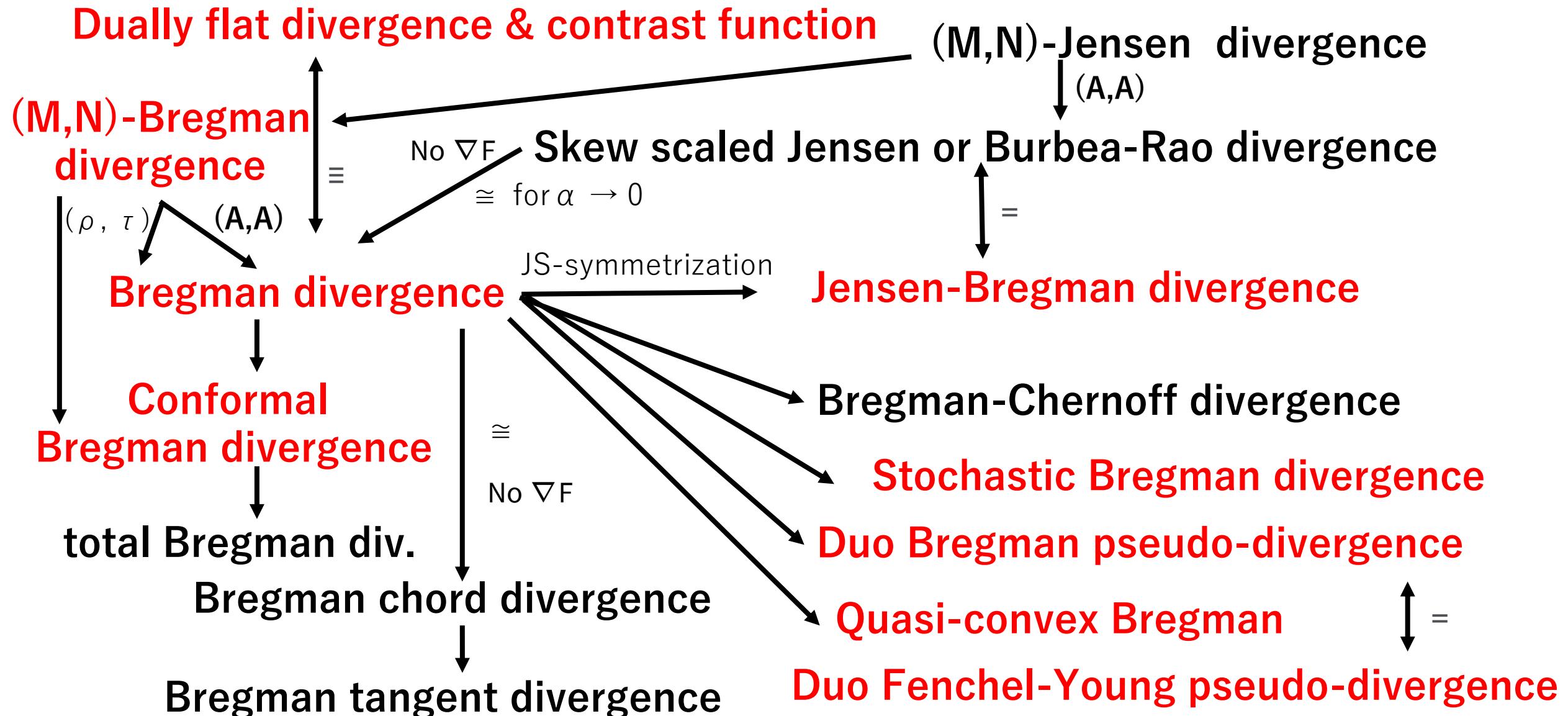
<https://franknielsen.github.io/pyBregMan/>



Jensen-Shannon centroid



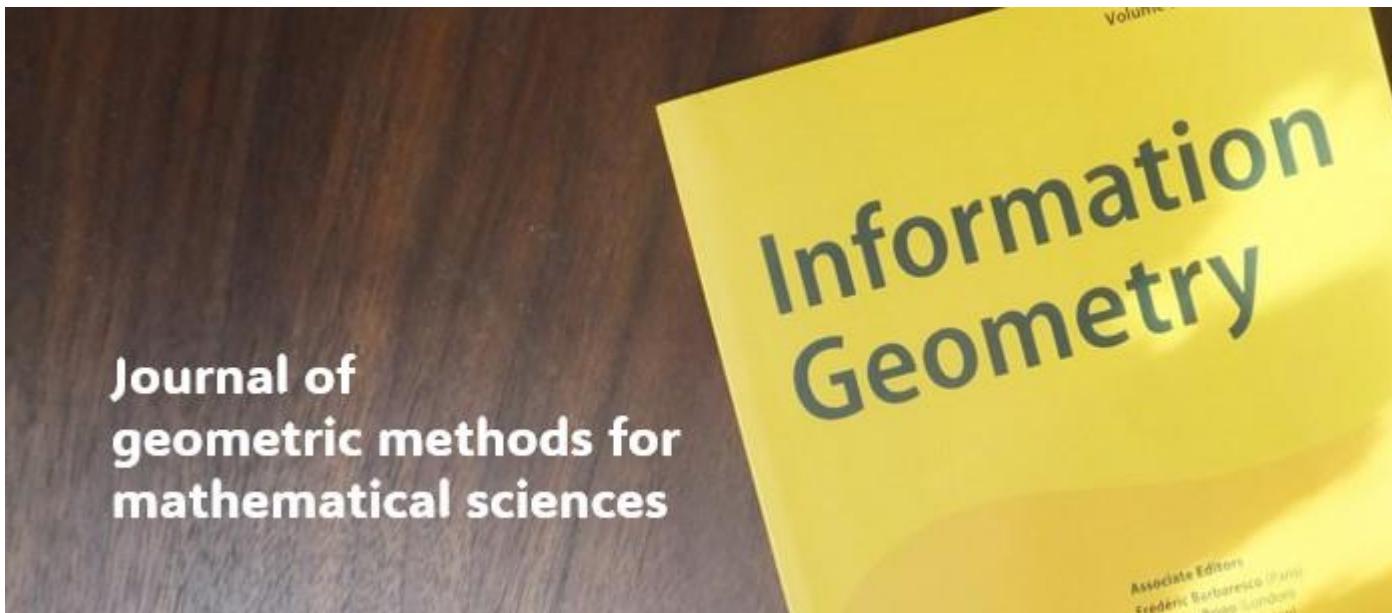
Some generalizations of Bregman divergences



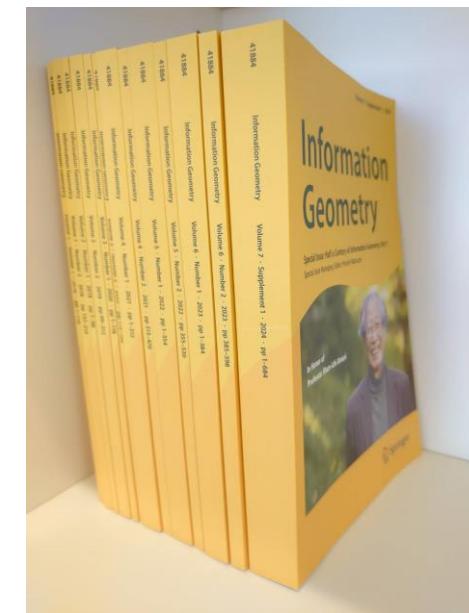
Thank you!

Many thanks to all my inspiring collaborators.

Special thanks to Richard Nock, Ke Sun, Ehsan Amid, and Alexander Soen



<https://link.springer.com/journal/41884>



<https://franknielsen.github.io/>



Back

MS: Nonstandard Dualities, Associated Dynamics, and Algorithms. Slot 1/2. (MS-24)

18/07/2024 17:30 - 18/07/2024 19:30 - ROOM 203

19h00

Generalizing Bregman divergences: Old and new directions

Speaker

Nielsen Frank

Sony Computer Science Laboratories Inc, Tokyo, Japan

Presentation:

25 minutes

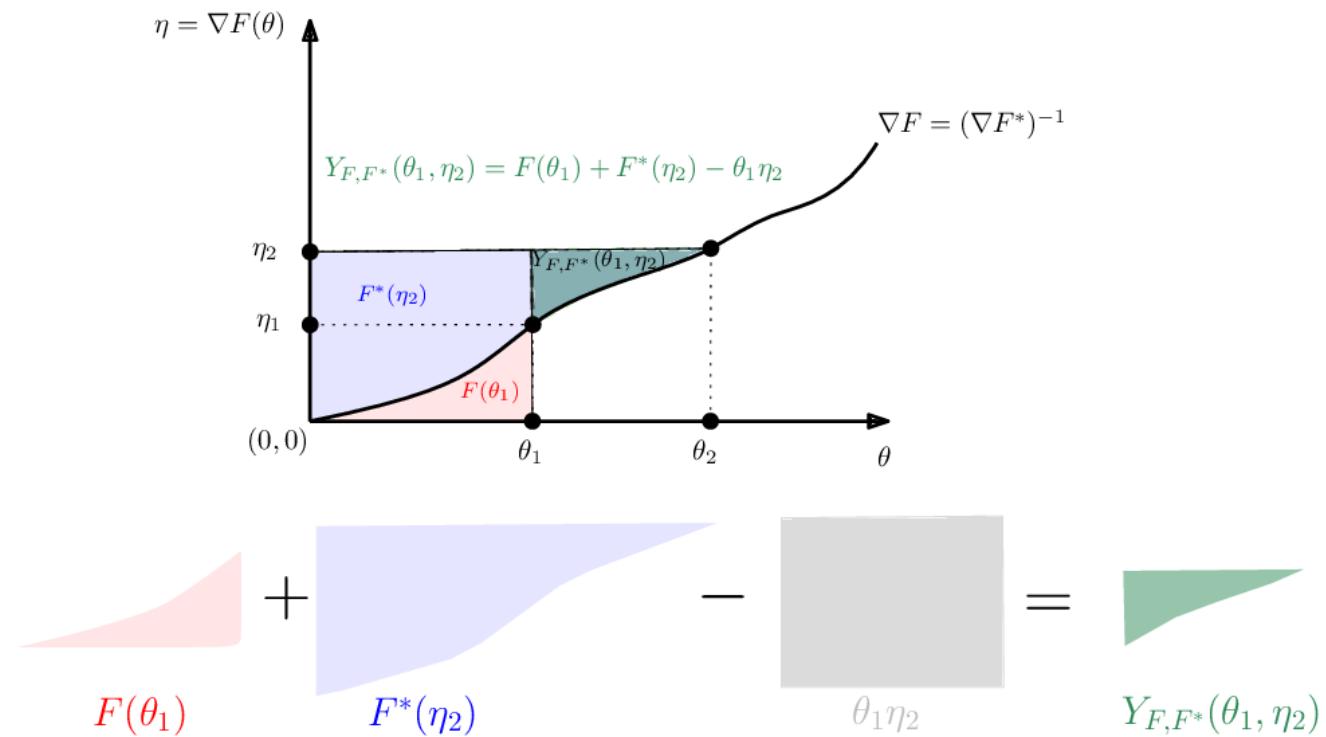
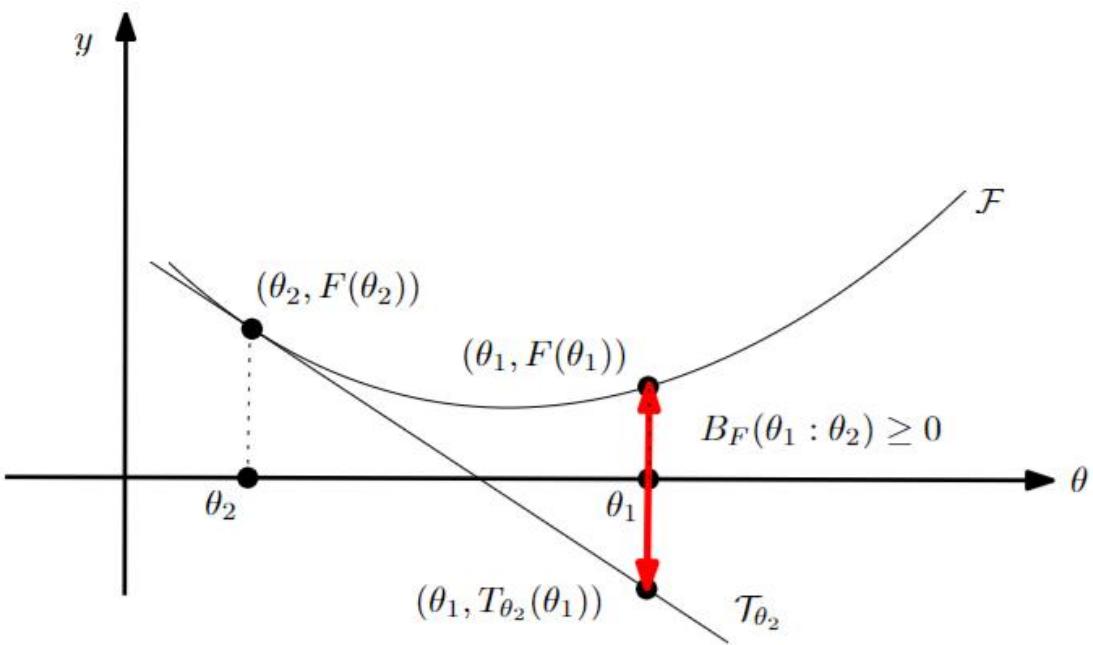
Discussion:

5 minutes

Abstract 

Bregman divergence vs Fenchel-Young divergence

Same parameterization $B_F(\theta_1 : \theta_2) = Y_{F,F^*}(\theta_1, \eta_2)$ mixed parameterization



$$B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$$

$$Y_{F,F^*}(\theta_1, \eta_2) = F(\theta_1) + F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle$$

Proof: KLD non-normalized EFs = BD wrt Z

$$\begin{aligned} D_{KL}^+[q_{\theta_1}(x):q_{\theta_2}(x)] &= \int \{ q_{\theta_1}(x) \log(q_{\theta_1}(x)/q_{\theta_2}(x)) + q_{\theta_2}(x) - q_{\theta_1}(x) \} d\mu(x) \\ &= Z(\theta_2) - Z(\theta_1) + E_{q_{\theta_1}}[<\theta_1 - \theta_2, x>] \end{aligned}$$

Remember “moment parameter”: $E_{p_{\theta}}[x] = \eta = \nabla F(\theta_1)$

$$\text{But } E_{q_{\theta_1}}[x] = Z(\theta_1) \quad E_{p_{\theta_1}}[x] = Z(\theta_1) \quad \eta_1 = Z(\theta_1) \quad \nabla F(\theta_1)$$

Since $F(\theta) = \log Z(\theta)$, we have $\nabla F(\theta) = \nabla Z(\theta)/Z(\theta)$

$$\text{Hence, } E_{q_{\theta_1}}[x] = Z(\theta_1) \quad \nabla F(\theta_1) = Z(\theta_1) \quad \nabla Z(\theta_1)/Z(\theta_1) = \nabla Z(\theta_1)$$

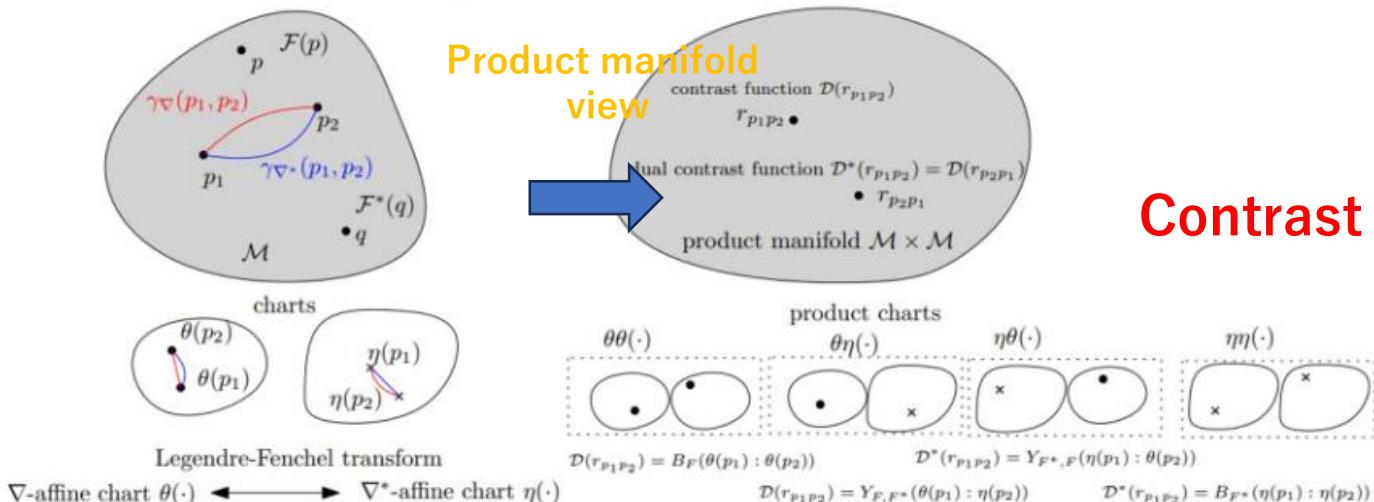
$$D_{KL}^+[q_{\theta_1}(x):q_{\theta_2}(x)] = Z(\theta_2) - Z(\theta_1) - <\theta_2 - \theta_1, \nabla Z(\theta_1)> = B_Z(\theta_2: \theta_1)$$

Bregman manifolds and contrast functions

- A strictly convex and smooth Legendre-type function induces a **dually flat space** also called a global Hessian manifold in diff. geo.

Dually flat divergence $D_{\nabla, \nabla^*}(p : q) = B_F(\theta(p) : \theta(q)) = Y_{F, F^*}(\theta(p) : \eta(q))$

$$\begin{aligned}\mathcal{D}(r_{pq}) &= B_F(\theta(p) : \theta(q)) = Y_{F, F^*}(\theta(p) : \eta(q)), \\ &= \mathcal{D}^*(r_{qp}) = B_{F^*}(\eta(q) : \eta(p)) = Y_{F^*, F}(\eta q : \theta(p))\end{aligned}$$



**Divergence =
Contrast function on product manifold**

- Reciprocally, a dually flat space induces a class of equivalent pair of Legendre-type functions with dual Bregman/Fenchel-Young divergences. Bregman divergences = canonical divergences of dually flat spaces

Symmetrized Bregman divergences are not Bregman divergences except for quadratic distances

- **Symmetrized Bregman divergence:**

$$S_F(\theta_1; \theta_2) := B_F(\theta_1 : \theta_2) + B_F(\theta_2 : \theta_1) = (\theta_2 - \theta_1)^\top (\eta_2 - \eta_1) = S_{F^*}(\eta_1; \eta_2)$$

- We may double the dimension, and write:

$$S_F(\theta_1, \theta_2) = B_{\hat{F}}(\theta_1^\uparrow : \theta_2^\uparrow) \quad \xi = \begin{bmatrix} \theta \\ \eta \end{bmatrix} \quad \hat{F}(\xi) = F(\theta) + F^*(\eta) \quad \theta^\uparrow = \begin{bmatrix} \theta \\ \nabla F(\theta) \end{bmatrix}$$

- parameter space Θ^\uparrow is **not convex** in general !

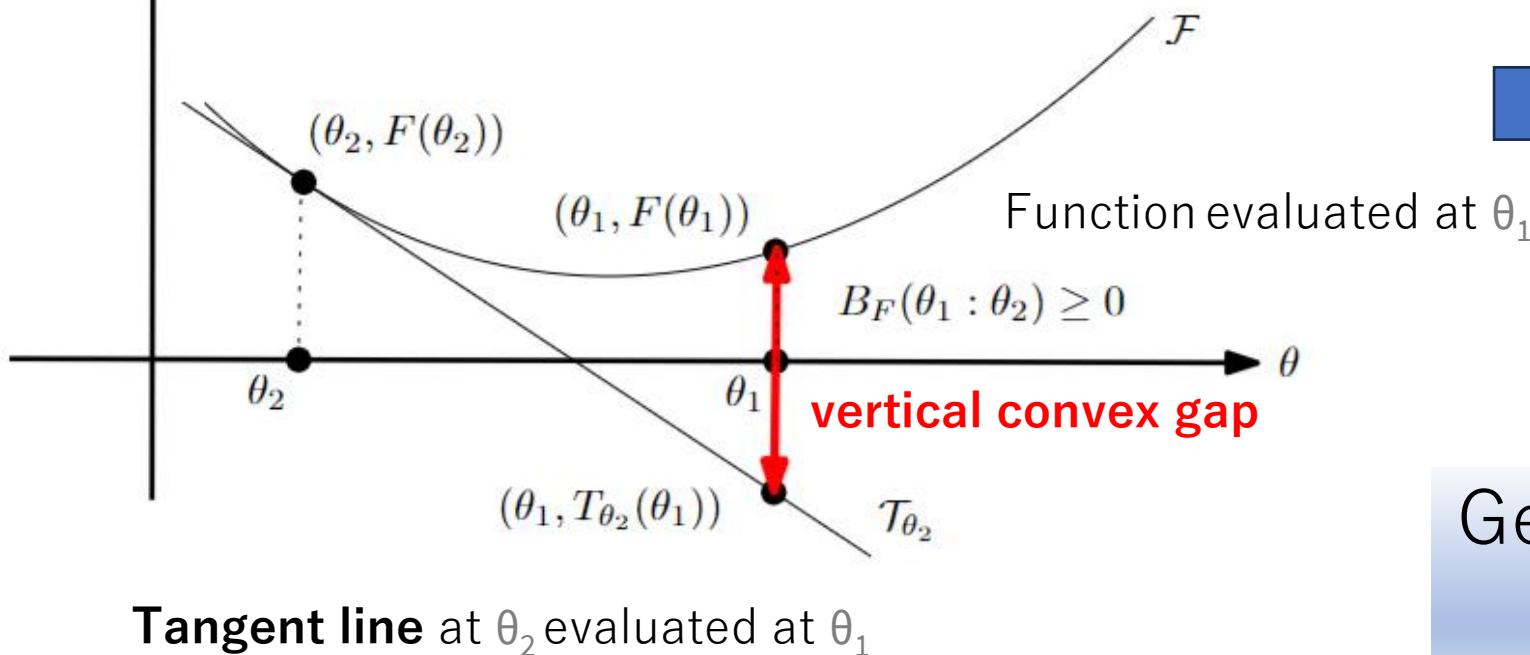
$$\Theta^\uparrow = \left\{ \theta^\uparrow = \begin{bmatrix} \theta \\ \nabla F(\theta) \end{bmatrix} : \theta \in \Theta \right\} \subset \Xi \quad \Xi = \left\{ \xi = \begin{bmatrix} \theta \\ \eta \end{bmatrix} : (\theta, \eta) \in \Theta \times H \right\}$$

- Except for generalized quadratic distances (Mahalanobis), SBDs are not BDs. SBDs are **curved Bregman divergences**. Update $B_F: \theta \times \text{Int}(\theta)$ to $\theta \times \text{RellInt}(\theta)$
- Bregman divergence restricted to a linear subspace is **sub-dimensional Bregman divergence** : For example, extended KLD vs KLD on simplex Δ

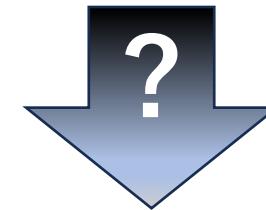
How can we measure other convexity gaps in graphs?

$$B_F(\theta_1 : \theta_2) = F(\theta_1) - (F(\theta_2) + \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle)$$

$$B_F(\theta_1 : \theta_2) = F(\theta_1) - T_{\theta_2}(\theta_1)$$



Bregman divergence
as a **vertical convex gap**



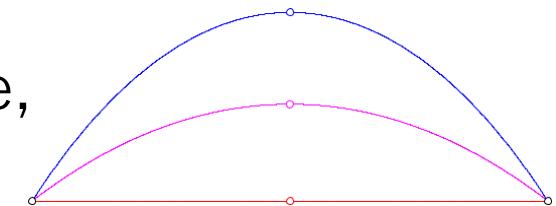
Generalize BDs by measuring
various convex gaps?

Univariate: $B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - (\theta_1 - \theta_2) F'(\theta_2)$

Fisher-Rao geodesics for d-variate normals

When $d > 1$, some sectional curvatures of (M, g_{Fisher}) are positive,
Fisher-Rao is not Hadamard manifold.

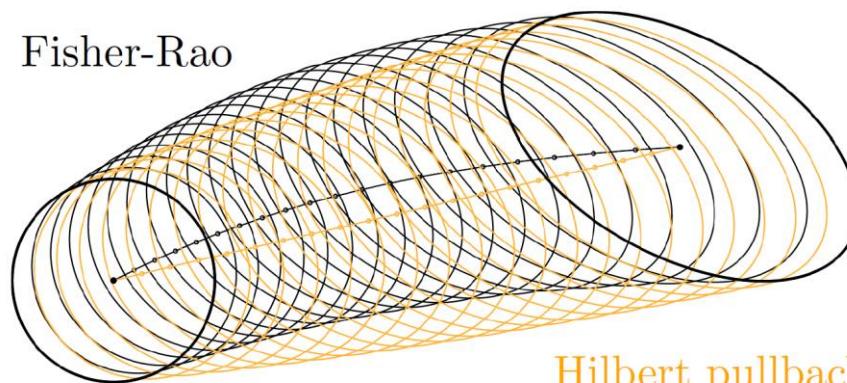
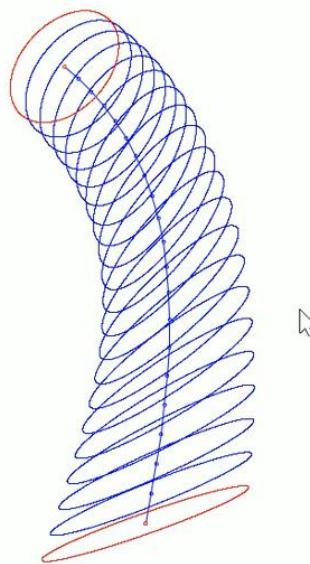
But centered normal submanifold is Hadamard



∇^* -geodesic (m-geodesic)

∇^g -geodesic (length minimizing geodesic)

∇ -geodesic (e-geodesic)



Embed in SPD
of dimension
 $2d+1$, submersion

Geodesic in closed form same-mean centered normal:

$$\gamma_{\mathcal{P}}(P, Q; t) = P^{\frac{1}{2}} \left(P^{-\frac{1}{2}} Q^{\frac{1}{2}} P^{-\frac{1}{2}} \right)^t P^{\frac{1}{2}}, \quad t \in [0, 1]$$

$$t=1/2 \text{ yields matrix geometric mean } G(X, Y) = X^{\frac{1}{2}} (X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})^{\frac{1}{2}} X^{\frac{1}{2}}$$

Kobayashi, Geodesics of multivariate normal distributions and a Toda lattice type Lax pair, Physica Scripta 98.11 (2023)
Fisher-Rao and pullback Hilbert cone distances on the multivariate Gaussian manifold with applications to simplification and
quantization of mixtures, ICML TAG 2023.

Inductive matrix arithmetic-harmonic mean (AHM)

- Consider the cone of symmetric positive-definite matrices (SPD cone), and extend the AHM to SPD matrices:

[Nakamura 2001]

$$A_{t+1} = \frac{A_t + H_t}{2} = A(A_t, H_t) \quad \leftarrow \text{arithmetic mean}$$

$$H_{t+1} = 2(A_t^{-1} + H_t^{-1})^{-1} = H(A_t, H_t) \quad \leftarrow \text{harmonic mean}$$

- Sequence with $A_0=X$ and $H_0=Y$ converge quadratically to **matrix geometric mean**:

$$\text{AHM}(X, Y) = \lim_{t \rightarrow +\infty} A_t = \lim_{t \rightarrow +\infty} H_t.$$

$$\text{AHM}(X, Y) = X^{\frac{1}{2}} (X^{-\frac{1}{2}} Y X^{-\frac{1}{2}})^{\frac{1}{2}} X^{\frac{1}{2}} = G(X, Y)$$

which is also the **Riemannian center of mass** wrt the trace metric:

$$G(X, Y) = \arg \min_{M \in \mathbb{P}(d)} \frac{1}{2} \rho^2(X, M) + \frac{1}{2} \rho^2(Y, M). \quad \rho(P_1, P_2) = \sqrt{\sum_{i=1}^d \log^2 \lambda_i (P_1^{-\frac{1}{2}} P_2 P_1^{-\frac{1}{2}})} \quad \text{Riemannian distance}$$

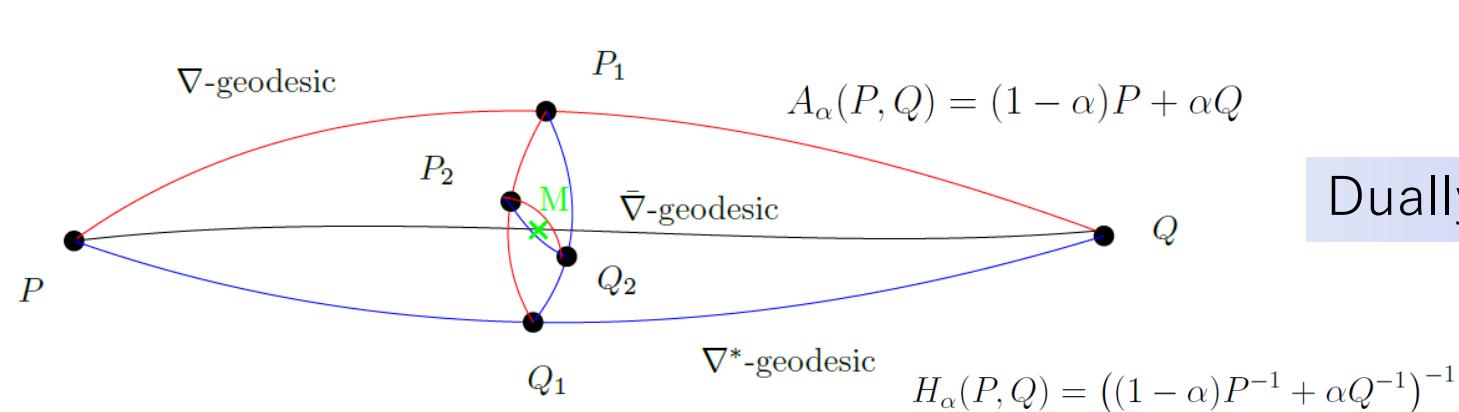
Geometric interpretation of the AHM matrix mean

Repeat:

$$A_{t+1} = \frac{A_t + H_t}{2} = A(A_t, H_t) \quad P_{t+1} = \gamma \left(P_t, Q_t : \frac{1}{2} \right)$$

$$H_{t+1} = 2(A_t^{-1} + H_t^{-1})^{-1} = H(A_t, H_t) \quad Q_{t+1} = \gamma^* \left(P_t, Q_t : \frac{1}{2} \right)$$

(SPD, g^G , ∇^A , ∇^H) is a dually flat space, ∇^G is Levi-Civita connection



$$G_\alpha(P, Q) = P^{\frac{1}{2}} \left(P^{-\frac{1}{2}} Q P^{-\frac{1}{2}} \right)^\alpha P^{\frac{1}{2}}$$

Dually flat space (SPD, g^G , ∇^A , ∇^H)

Primal geodesic midpoint is the arithmetic center wrt Euclidean metric
 Dual geodesic midpoint = harmonic center wrt an isometric Eucl. metric
 Levi-Civita geodesic midpoint is geometric Karcher mean

Here, all 3 connections are metric connections

$$g_P^A(X, Y) = \text{tr}(X^\top Y)$$

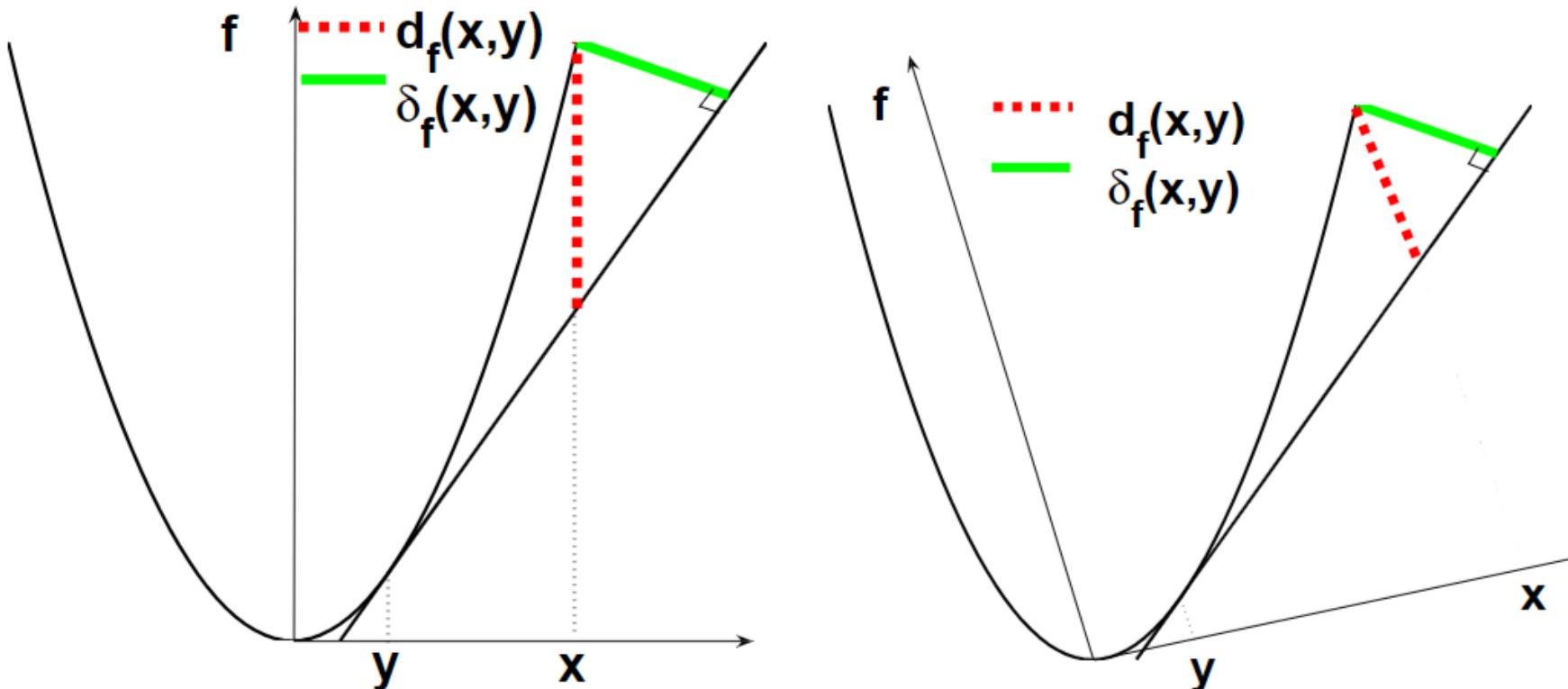
$$g_P^H(X, Y) = \text{tr}(P^{-2} X P^{-2} Y)$$

$$g_P^G(X, Y) = \text{tr}(P^{-1} X P^{-1} Y)$$

[Nakamura 2001]

Total Bregman divergences

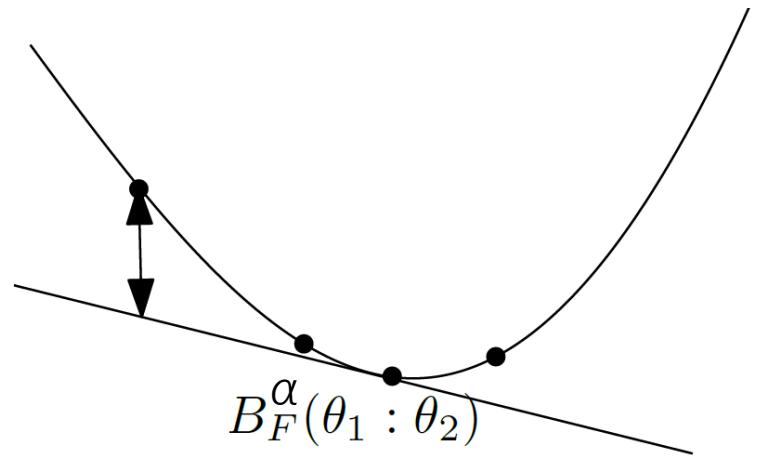
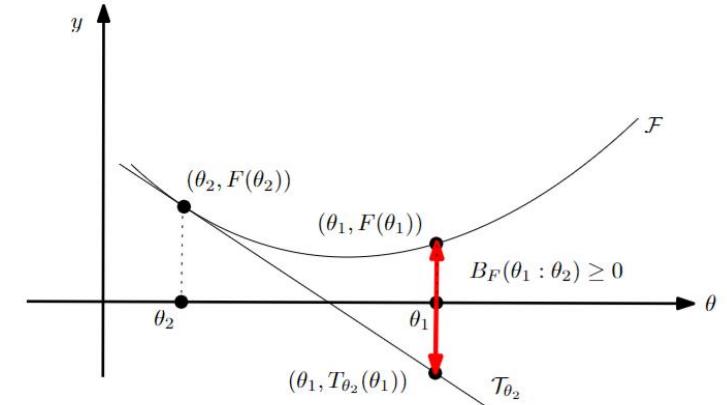
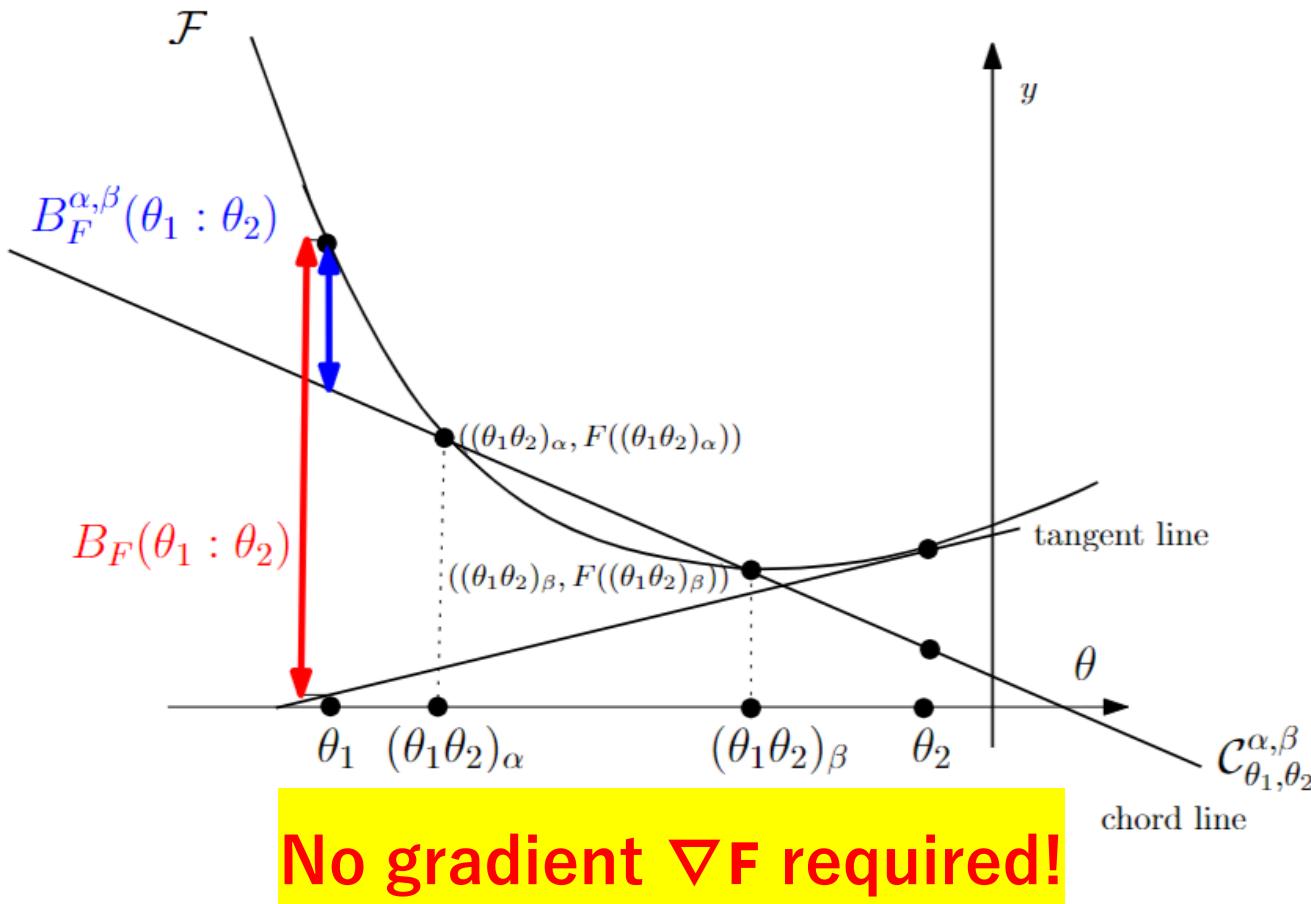
$$\delta_f(x, y) = \frac{f(x) - f(y) - \langle x - y, \nabla f(y) \rangle}{\sqrt{1 + \|\nabla f(y)\|^2}}$$



- Invariant to rotation
- Robust clustering with respect outliers
- Conformal Bregman divergence

$$\begin{aligned} tB(p : q) &= \frac{B(p : q)}{\sqrt{1 + \langle \nabla F(q), \nabla F(q) \rangle}} = \rho_B(q) B(p : q) \\ \rho_B(q) &= \frac{1}{\sqrt{1 + \langle \nabla F(q), \nabla F(q) \rangle}}. \end{aligned}$$

Bregman-type divergences measuring convexity gaps



When $\alpha \rightarrow \beta$, **Bregman chord divergences** \rightarrow **Bregman tangent divergences**
or **Bregman secant divergences**

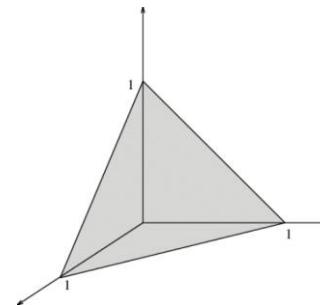
Information geometry in action! (1/2)

- Set of categorical distributions form a **mixture family \mathbf{M}** , a Bregman manifold for the negentropy generator

$$\mathcal{M} = \left\{ m_\theta(x) = \sum_{i=1}^D \theta_i \delta(x - x_i) + \left(1 - \sum_{i=1}^D \theta_i\right) \delta(x - x_0) \right\}$$

$$F(\theta) = -h(m_\theta) = \sum_{i=1}^D \theta_i \log \theta_i + \left(1 - \sum_{i=1}^D \theta_i\right) \log \left(1 - \sum_{i=1}^D \theta_i\right).$$

A mixture family is closed under mixture operations



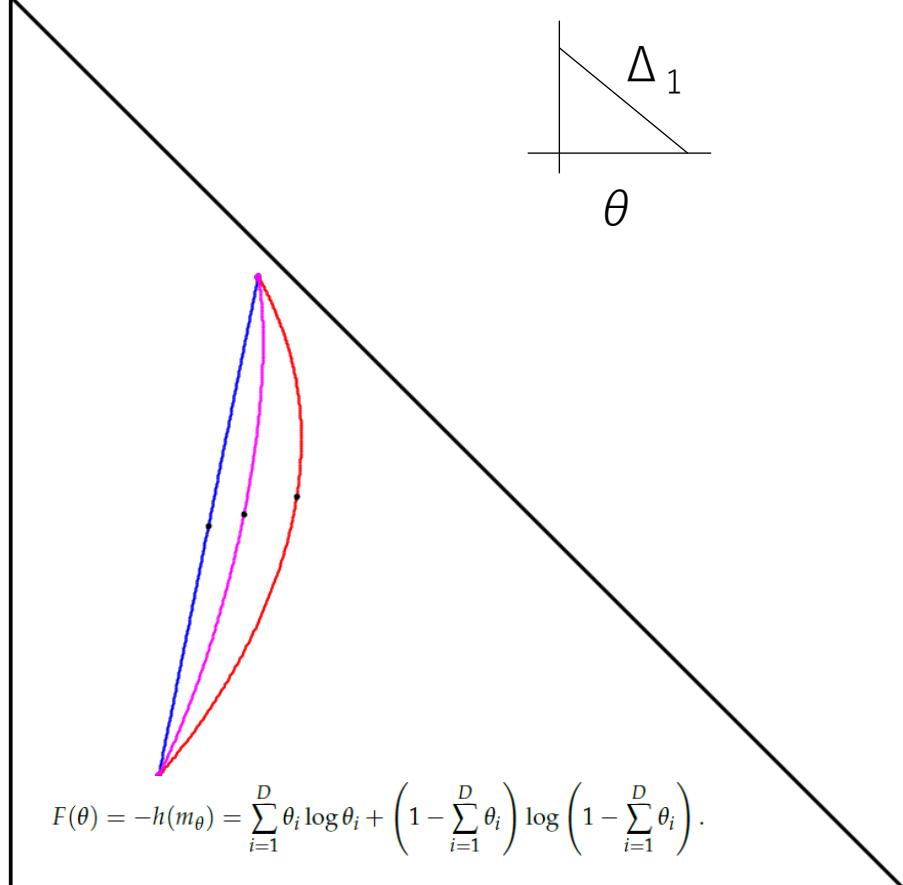
- Given a set of n discrete distributions (categorical distributions, normalized histograms), calculate its **Jensen-Shannon centroid**

$$\text{JS}(p, q) := \frac{1}{2} \left(\text{KL}\left(p : \frac{p+q}{2}\right) + \text{KL}\left(q : \frac{p+q}{2}\right) \right)$$

$$\begin{aligned} \text{JS}(p, q) &= h\left(\frac{p+q}{2}\right) - \frac{h(p) + h(q)}{2} \\ h(p) &= -\int p \log p d\mu \end{aligned}$$

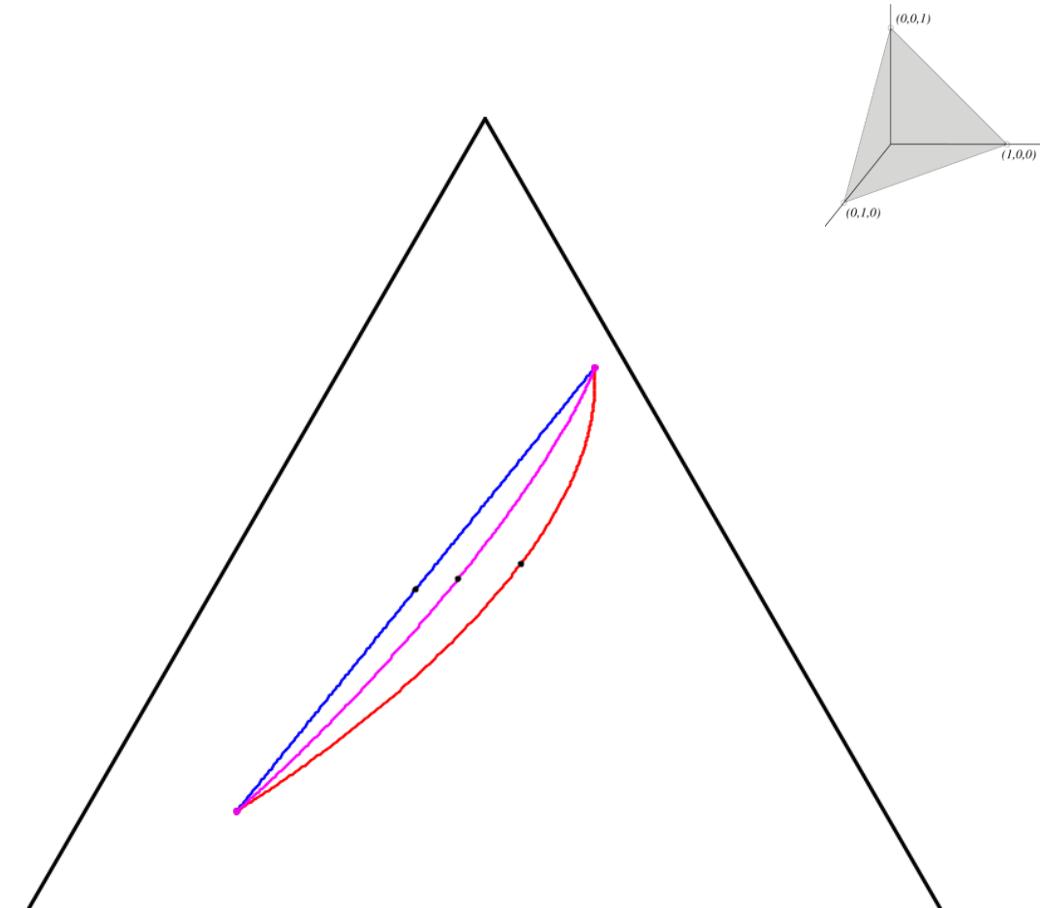
Dual geodesics and Fisher-Rao geodesics on the categorical manifold

Mixture parameter space

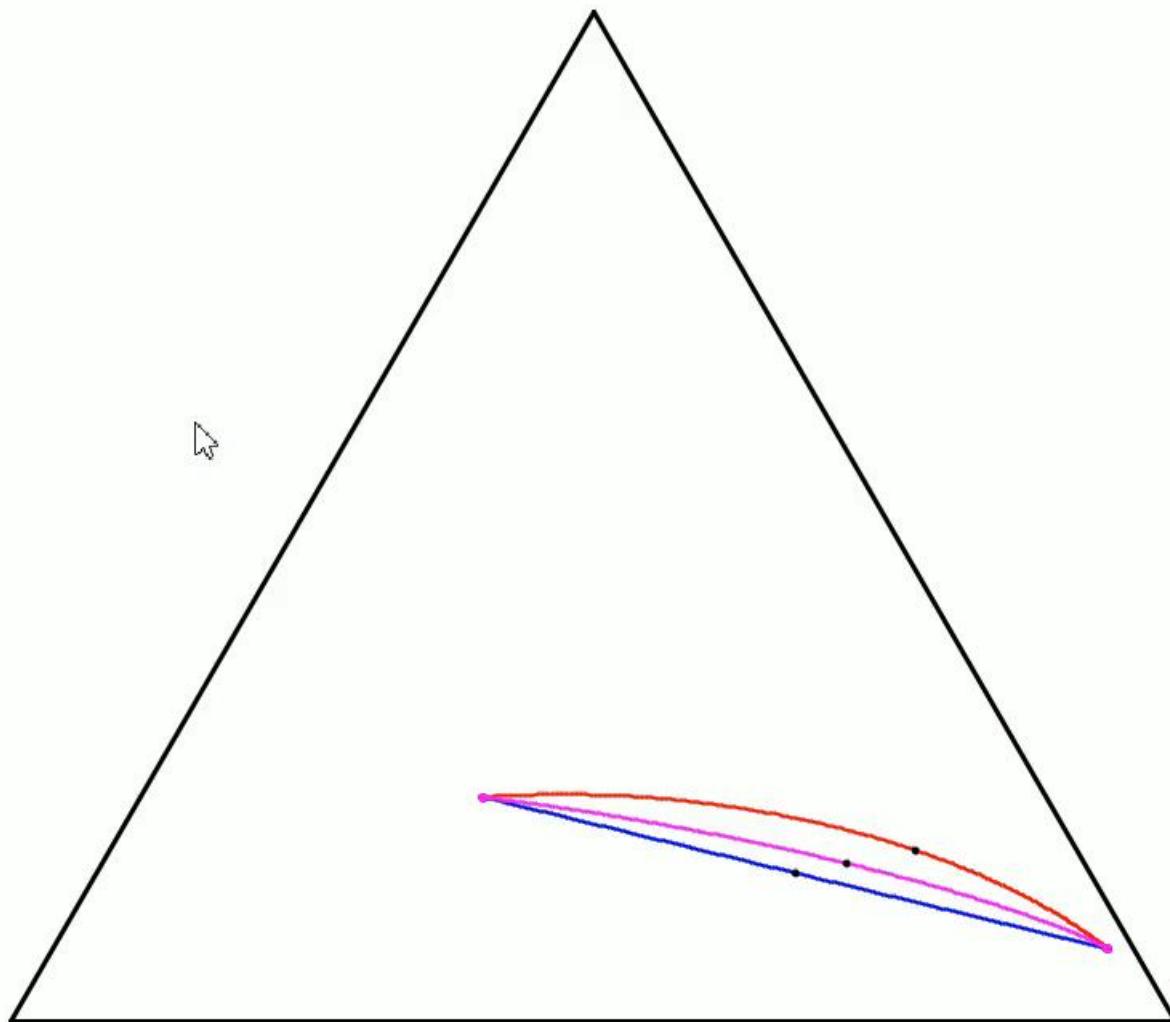


Exponential ∇ -geodesic
Mixture ∇^* -geodesic
Fisher-Rao ∇^g -geodesic (Levi-Civita)

Probability simplex/Categorical manifold

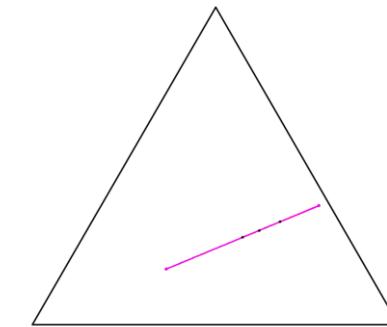


Probability simplex/Categorical manifold

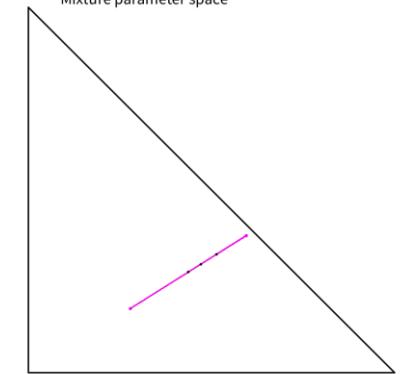


Geodesics coincide when passing through a simplex vertex but not the midpoints

Probability simplex/Categorical manifold



Mixture parameter space



Exponential ∇ -geodesic

Mixture ∇^* -geodesic

Fisher-Rao ∇^g -geodesic
(Levi-Civita)

Jensen-Shannon centroid for mixtures

- Jensen-Shannon divergence between two mixtures amounts to a Jensen divergence: $\text{JS}(p_1, p_2) = J_F(\theta_1, \theta_2)$ for $p_1 = m_{\theta_1}$ and $p_2 = m_{\theta_2}$, where

$$J_F(\theta_1 : \theta_2) = \frac{F(\theta_1) + F(\theta_2)}{2} - F\left(\frac{\theta_1 + \theta_2}{2}\right).$$

- Task: Given a set of discrete distributions (categorical distributions, normalized histograms), calculate its Jensen-Shannon centroid:

$$\min_p \sum_i \text{JS}(p_i, p),$$

$$\min_\theta \sum_i J_F(\theta_i, \theta),$$

$$\min_\theta \sum_i \frac{F(\theta_i) + F(\theta)}{2} - F\left(\frac{\theta_i + \theta}{2}\right),$$

$$\equiv \min_\theta \frac{1}{2}F(\theta) - \frac{1}{n} \sum_i F\left(\frac{\theta_i + \theta}{2}\right) := E(\theta)$$

Need to minimize a difference of convex functions
DCA or **ConCave Convex algorithm!**

Jensen-Bregman divergence

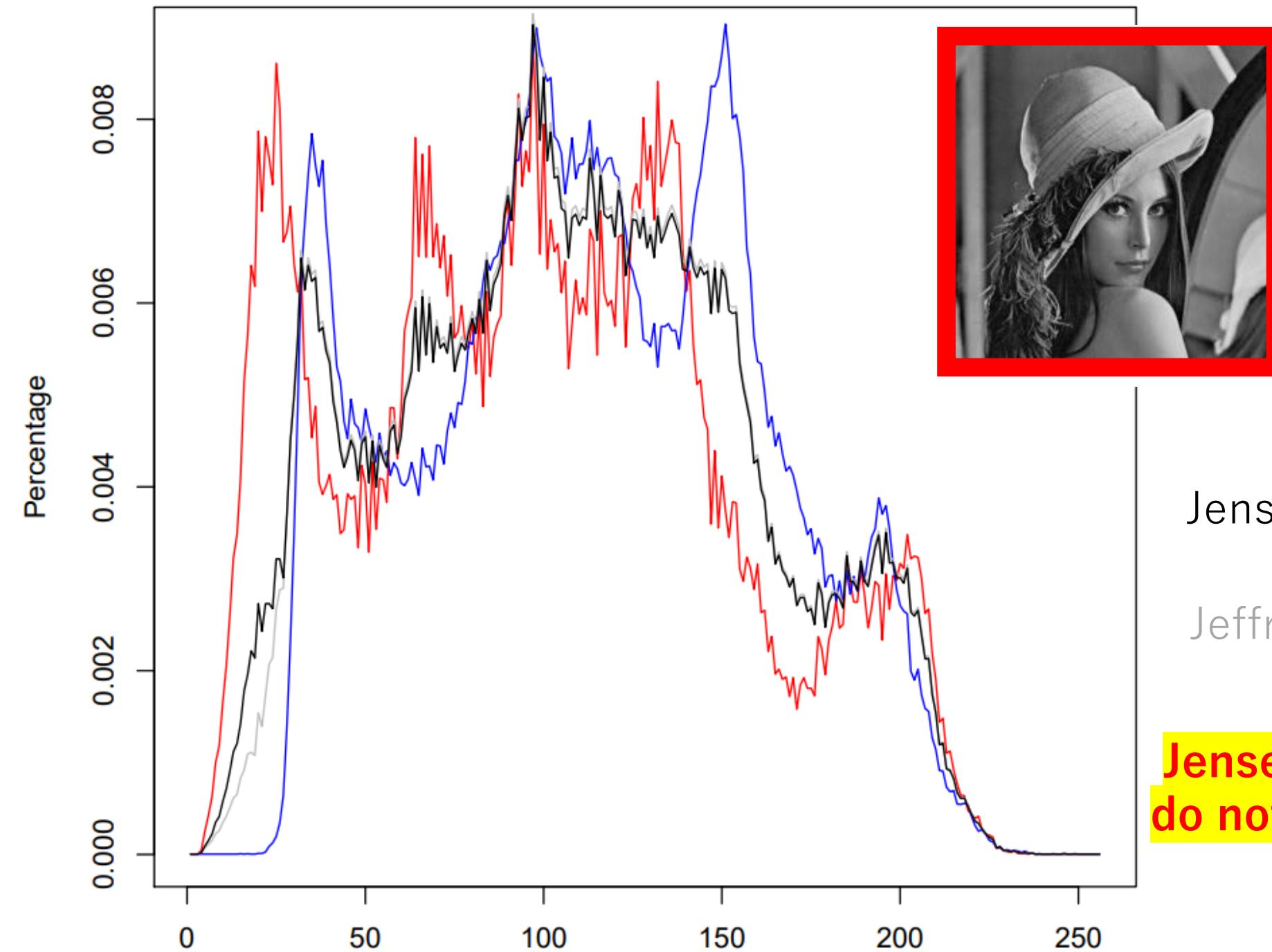
$$\begin{aligned} & \min_p \sum_i \text{JS}(p_i, p), \\ & \min_{\theta} \sum_i J_F(\theta_i, \theta), \\ & \min_{\theta} \sum_i \frac{F(\theta_i) + F(\theta)}{2} - F\left(\frac{\theta_i + \theta}{2}\right) \end{aligned}$$

- Jensen-Bregman divergence is Jensen-Shannon symmetrization of Bregman divergence:

$$\begin{aligned} \text{JB}_F(\theta : \theta') &:= \frac{1}{2} \left(B_F \left(\theta : \frac{\theta + \theta'}{2} \right) + B_F \left(\theta' : \frac{\theta + \theta'}{2} \right) \right) \\ &= \frac{F(\theta) + F(\theta')}{2} - F\left(\frac{\theta + \theta'}{2}\right) =: J_F(\theta : \theta') \end{aligned}$$

amounts to a Jensen divergence (also called Burbea-Rao divergence).

- Jensen-Shannon centroid of a mixture family = Jensen-Bregman centroid = Jensen centroid



Jensen–Shannon centroid

Jeffreys SKL centroid

**Jensen–Shannon centroid
do not require same support**

Bregman divergence: Taylor remainder

$$B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$$

- BDs are non-negative because = remainder of Taylor 1st order expansion:

$$F(\theta_1) = F(\theta_2) + \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle + \text{Remainder}_F(\theta_1 : \theta_2)$$

BD = exact Lagrange remainder = a quadratic distance:

$$\text{LagrangeRemainder}_F(\theta_1 : \theta_2) = \frac{1}{2} (\theta_1 - \theta_2)^T \nabla^2 F(\xi) (\theta_1 - \theta_2) = B_F(\theta_1 : \theta_2)$$

for some $\xi \in [\theta_1 \ \theta_2]$

- BDs are *never metrics* and
- *BDs are only symmetric for generalized quadratic distance* (proof: $\nabla^2 F$ constant)

$$F_Q(\theta) = \frac{1}{2} \theta^T Q \theta, \text{ where } Q \text{ is symmetric positive-definite matrix}$$

$Q=I$, squared Euclidean distance

Information geometry in action! (2/2)

- The **Chernoff information** between two distributions is defined by

$$D_C[P, Q] := \max_{\alpha \in (0,1)} -\log \rho_\alpha[P : Q]$$

$$\rho_\alpha[P : Q] := \int p^\alpha q^{1-\alpha} d\mu = \rho_{1-\alpha}[Q : P].$$

- Chernoff information is the maximal **skew Bhattacharrya distance** (not metric!):

$$D_{B,\alpha}[p : q] := -\log \rho_\alpha[P : Q] = D_{B,1-\alpha}[q : p],$$

- α -Bhattacharrya distances related to **Rnyi α -divergences**:

$$D_{R,\alpha}[P : Q] = \frac{1}{\alpha - 1} \log \int p^\alpha q^{1-\alpha} d\mu = \frac{1}{1-\alpha} D_{B,\alpha}[P : Q].$$

$$D_{B,\alpha}[P : Q] = (1 - \alpha) D_{R,\alpha}[P : Q]$$

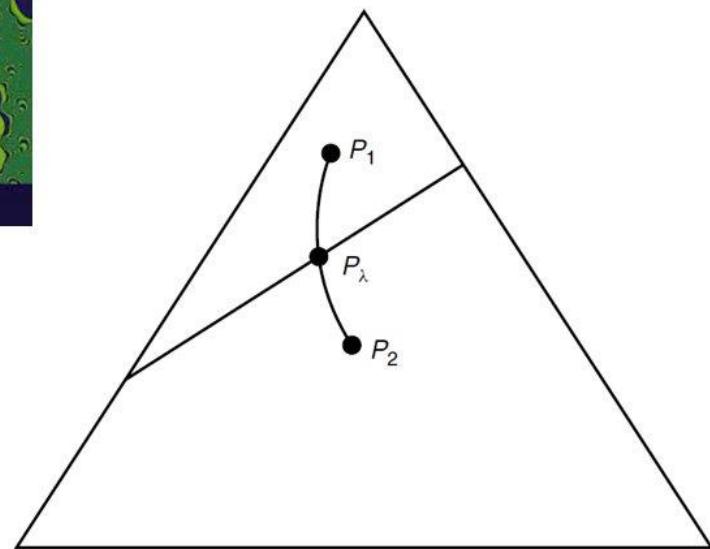
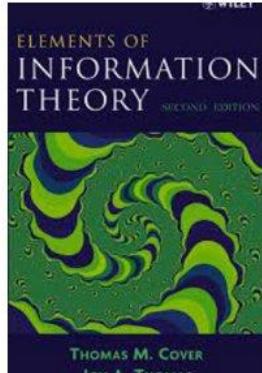
- CI is often used in **Bayesian hypothesis testing & information fusion**

- An information-geometric characterization of Chernoff information, *IEEE Signal Processing Letters* (2013)
- Revisiting Chernoff information with likelihood ratio exponential families, *Entropy* 24.10 (2022)
- Julier, An empirical study into the use of Chernoff information for robust, distributed fusion of Gaussian mixture models , *IEEE Information Fusion* 2006.

Chernoff information: A geometric characterization

$$C(P_1, P_2) \triangleq - \min_{0 \leq \lambda \leq 1} \log \left(\sum_x P_1^\lambda(x) P_2^{1-\lambda}(x) \right)$$

$$= D(P_{\lambda^*} || P_1) = D(P_{\lambda^*} || P_2)$$



$$P_\lambda = \frac{P_1^\lambda(x) P_2^{1-\lambda}(x)}{\sum_{a \in \mathcal{X}} P_1^\lambda(a) P_2^{1-\lambda}(a)}$$

Probability simplex

Generalized
to
exponential family
manifold

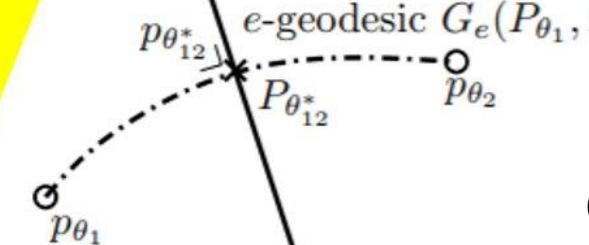
$$C(P, Q) = - \log \min_{\alpha \in (0,1)} \int p^\alpha(x) q^{1-\alpha}(x) d\nu(x).$$

$$C(P_{\theta_1} : P_{\theta_2}) = B(\theta_1 : \theta_{12}^{(\alpha^*)}) = B(\theta_2 : \theta_{12}^{(\alpha^*)})$$

$$P^* = P_{\theta_{12}^*} = \mathbf{G}_e(P_1, P_2) \cap \mathbf{Bi}_m(P_1, P_2)$$

η -coordinate system

m -bisector
 $\mathbf{Bi}_m(P_{\theta_1}, P_{\theta_2})$



$$C(\theta_1 : \theta_2) = B(\theta_1 : \theta_{12}^*)$$

$$p(x | \theta) \propto \exp(\langle x, \theta \rangle)$$

**Example:
Gaussian manifold**

Exponential family manifold

Chernoff-Bregman divergence

- Chernoff information: $C(P_{\theta_1} : P_{\theta_2}) = B(\theta_1 : \theta_{12}^{(\alpha^*)}) = B(\theta_2 : \theta_{12}^{(\alpha^*)})$
- Chernoff-Bregman is another way to symmetrize a Bregman divergence by maximizing the skew Jensen divergence:

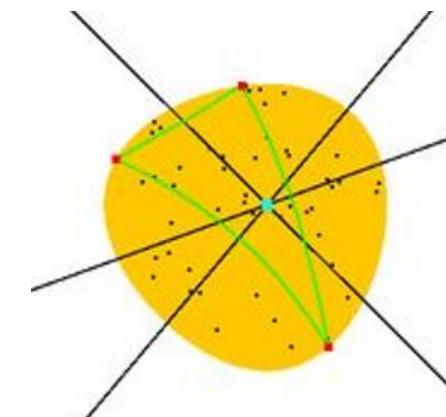
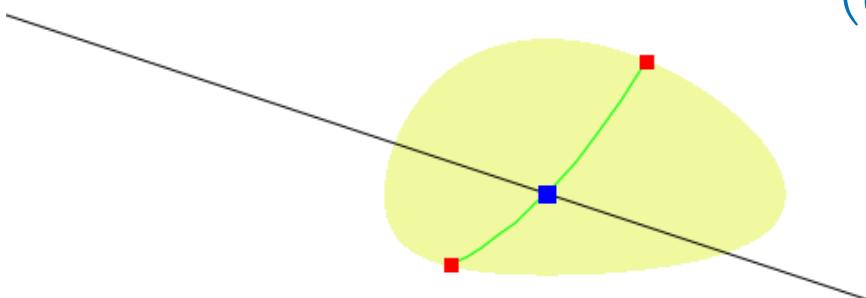
$$C_F(\theta_1, \theta_2) = \max_{\alpha \in (0,1)} J_{F,\alpha}(\theta_1 : \theta_2)$$

$$C_F(\theta_1, \theta_2) = \min_{\theta} \{B_F(\theta_1 : \theta), B_F(\theta_2 : \theta)\}$$

- This amounts to

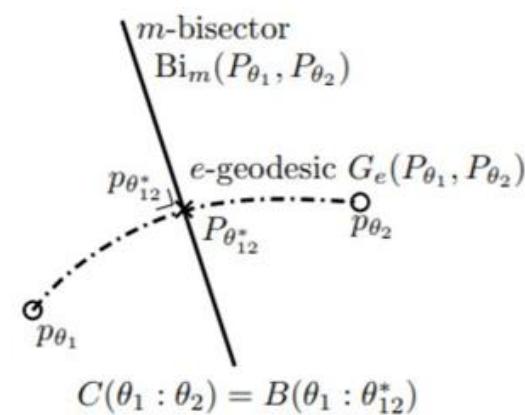
- Chernoff-Bregman divergence is the radius of a right-sided Bregman ball of two points

Chernoff point r^*
(eKLD)



Exact smallest enclosing
Bregman ball of n points

Fitting the smallest enclosing Bregman ball, ECML'05

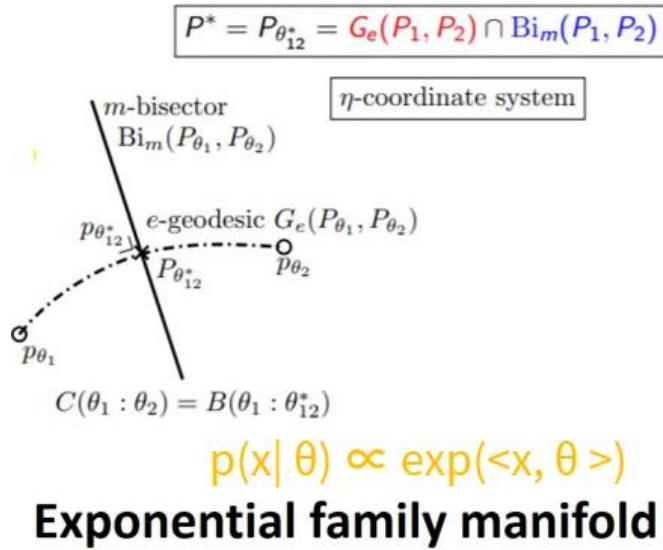


Categorical distributions: Both an exponential and a mixture family!

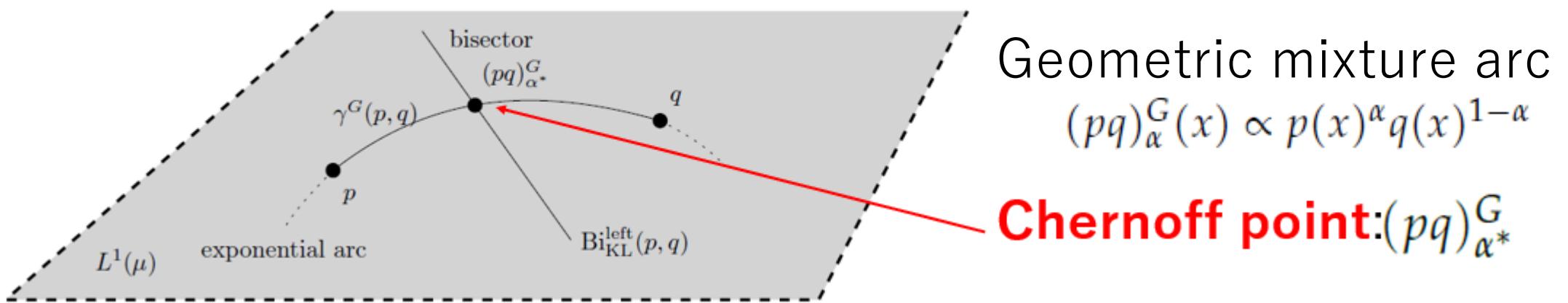
	Exponential Family	*	Mixture Family
pdf	$p_\theta(x) = \prod_{i=1}^d p_i^{t_i(x)}$, $p_i = \Pr(x = e_i)$, $t_i(x) \in \{0, 1\}$, $\sum_{i=1}^d t_i(x) = 1$		$m_\theta(x) = \sum_{i=1}^d p_i \delta_{e_i}(x)$
primal θ	$\theta_i = \log \frac{p_i}{p_d}$		$\theta_i = p_i$
$F(\theta)$	$\log(1 + \sum_{i=1}^D \exp(\theta_i))$		$\theta_i \log \theta_i + (1 - \sum_{i=1}^D \theta_i) \log(1 - \sum_{i=1}^D \theta_i)$
dual $\eta = \nabla F(\theta)$	$\frac{e^{\theta_i}}{1 + \sum_{j=1}^D \exp(\theta_j)}$		$\log \frac{\theta_i}{1 - \sum_{j=1}^D \theta_j}$
primal $\theta = \nabla F^*(\eta)$	$\log \frac{\eta_i}{1 - \sum_{j=1}^D \eta_j}$		$\frac{e^{\theta_i}}{1 + \sum_{j=1}^D \exp(\theta_j)}$
$F^*(\eta)$	$\sum_{i=1}^D \eta_i \log \eta_i + (1 - \sum_{j=1}^D \eta_j) \log(1 - \sum_{j=1}^D \eta_j)$		$\log(1 + \sum_{i=1}^D \exp(\eta_i))$
Bregman divergence	$B_F(\theta : \theta') = \text{KL}^*(p_\theta : p_{\theta'})$ $= \text{KL}(p_{\theta'} : p_\theta)$		$B_F(\theta : \theta') = \text{KL}(m_\theta : m_{\theta'})$

Dual of categorical exponential family is categorical mixture family,
and vice versa

Revisiting Chernoff information/Point



Consider p and q arbitrary probability densities:



Geometric mixture arc is a 1D likelihood ratio exponential family

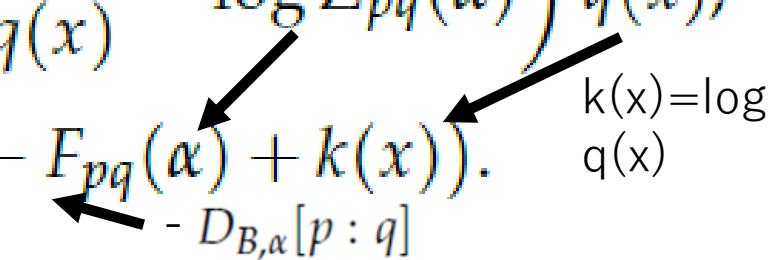
- Geometric mixture Bhattacharyya / exponential arc) $(pq)_{\alpha}^G(x) \propto p(x)^{\alpha}q(x)^{1-\alpha}$
between two densities p, q of Lebesgue Banach space $L_1(\mu)$
- Set of **geometric mixtures**:
with **normalization factor**:

$$\mathcal{E}_{pq} := \left\{ (pq)_{\alpha}^G(x) := \frac{p(x)^{\alpha}q(x)^{1-\alpha}}{Z_{pq}(\alpha)} : \alpha \in \Theta \right\}$$

$$Z_{pq}(\alpha) = \int_{\mathcal{X}} p(x)^{\alpha}q(x)^{1-\alpha} d\mu(x) = \underline{\rho_{\alpha}[p : q]}$$
- geometric mixture interpreted as a **1D exponential family**: LREF

$$(pq)_{\alpha}^G(x) = \exp \left(\alpha \log \frac{p(x)}{q(x)} - \log Z_{pq}(\alpha) \right) q(x),$$

$$=: \exp(\alpha t(x) - F_{pq}(\alpha) + k(x)).$$



Natural parameter space:

$$\Theta := \{\alpha \in \mathbb{R} : Z_{pq}(\alpha) < \infty\}.$$

LREFs: EF cumulant function is always analytic

- Cumulant function of EF is **strictly convex**

- Cumulant function is neg-Bhattacharvva distance:

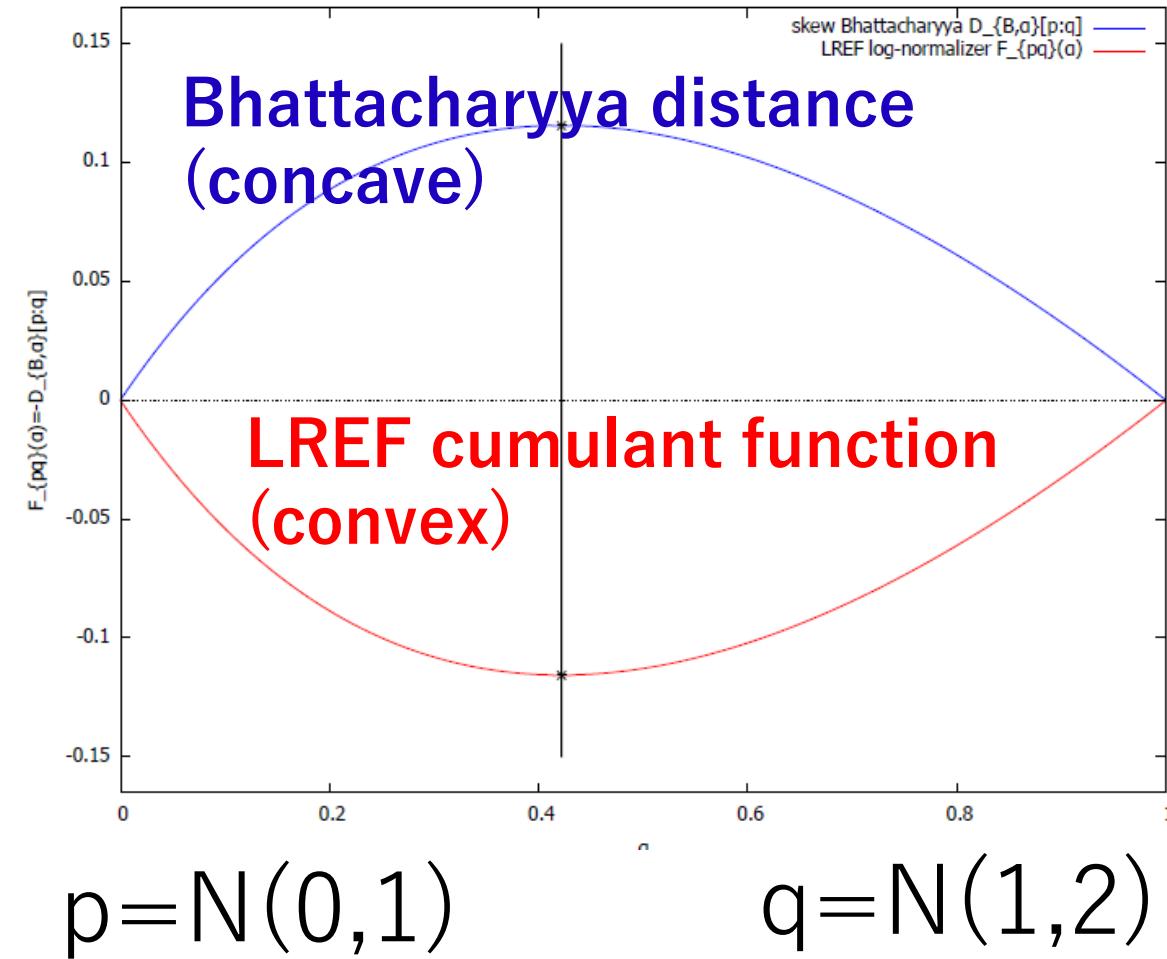
$$F_{pq}(\alpha) = \log Z_{pq}(\alpha) = -D_{B,\alpha}[p : q] < 0$$

⇒ Bhattacharyya. distance is **strictly concave**

- Theorem:

Chernoff exponent exists and is unique

$$D_C[p, q] = D_{B,\alpha^*(p:q)}(p : q) = D_{B,\alpha^*(q:p)}(q : p) = D_C[q, p].$$



$$(pq)_\alpha^G(x) \propto p(x)^\alpha q(x)^{1-\alpha}$$

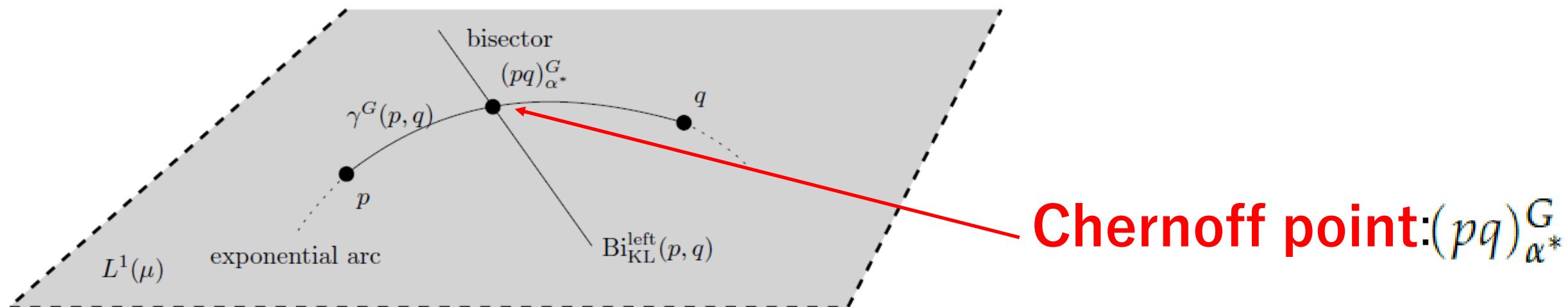
Geometric interpretation for densities on $L_1(\mu)$

Proposition (Geometric characterization of the Chernoff information). *On the vector space $L^1(\mu)$, the Chernoff information distribution is the unique distribution*

$$(pq)_{\alpha^*}^G = \gamma^G(p, q) \cap \text{Bi}_{\text{KL}}^{\text{left}}(p, q).$$

Left KL Voronoi bisector: $\text{Bi}_{\text{KL}}^{\text{left}}(p, q) := \left\{ r \in L^1(\mu) : D_{\text{KL}}[r : p] = D_{\text{KL}}[r : q] \right\}$.

Geodesic = exponential arc: $\gamma^G(p, q) := \left\{ (pq)_{\alpha}^G : \alpha \in [0, 1] \right\}$



Outline

- Bregman divergences with a single generator and a pair of generators
- Bregman manifolds
 - Normalized/unnormalized exponential families and Bregman divergences
 - Fisher-Rao distance and arithmetic-harmonic mean on the SPD manifold
- Further information geometry in action!
 - Jensen-Shannon centroid on a mixture family manifold
 - Chernoff information and 1D exponential family geometric arc manifolds
- **Bregman divergence with respect to comparative convexity**

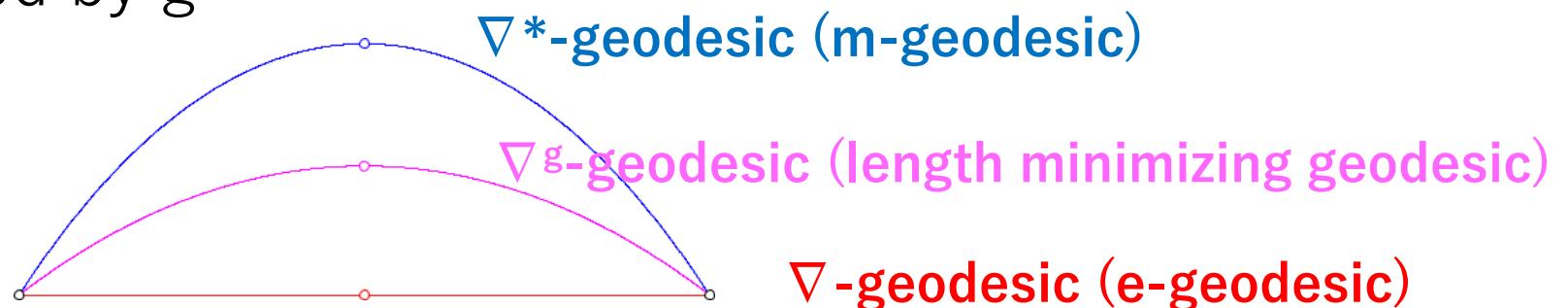
Summary

- Bregman divergences: arbitrarily well approximated by **Bregman chord divergences** or **skewed Jensen divergences** without using gradient ∇F
- **Jensen-Shannon centroid** on **mixture family manifold** using ConCave-Convex procedure
- Chernoff information on **exponential family manifold** using exact geometric characterization ``**Chernoff point**'' = unique intersection of primal geodesic with dual bisector
- Define Bregman divergences with respect to (M,N) -convexity:
 (M,N) -Bregman divergences as **conformal Bregman divergences**
Convex-preserving (ρ, τ) -deformations
- **Duality:** biduality reference/representation from convex duality

Bregman manifolds have Hessian metrics

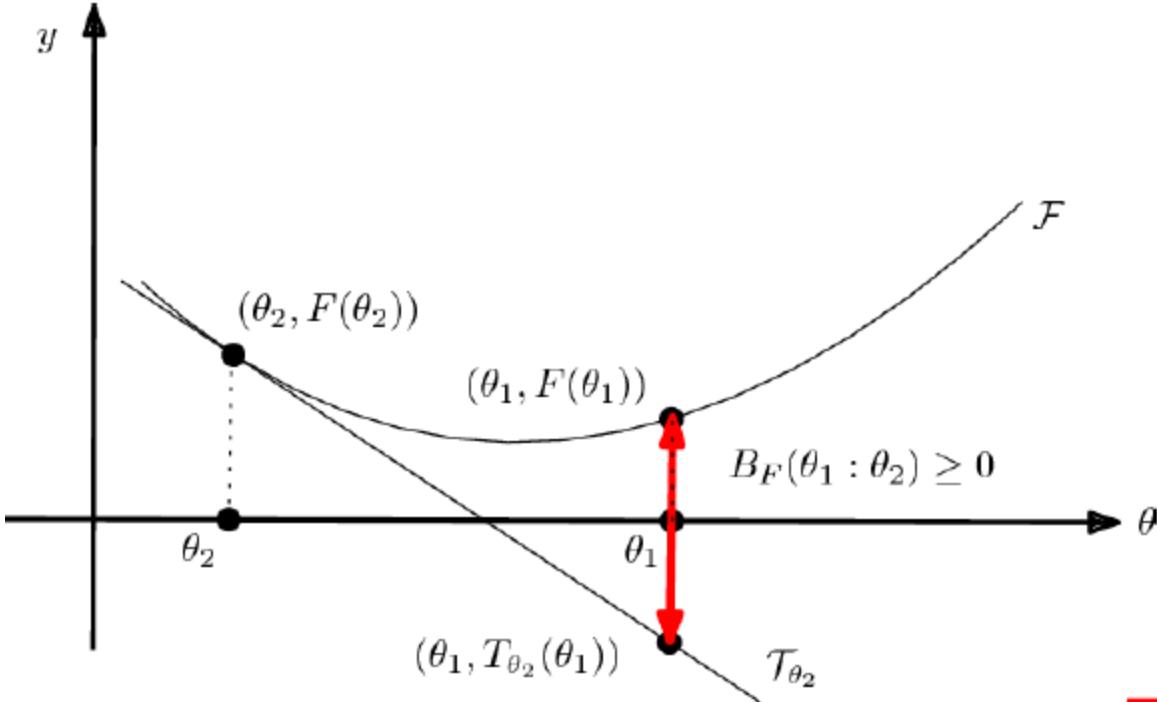
- The metric g of a Bregman manifold (M, g, ∇, ∇^*) is **Hessian**:
$$g(\theta) = \nabla^2 F(\theta)$$
 and $g(\eta) = \nabla^2 F^*(\eta)$
Hessian $\nabla^2 = \nabla \nabla^\top$
- The dual basis $e(p)$ and $e^*(p)$ in tangent planes T_p are reciprocal:
$$g(e_i, e^{*j}) = \delta_{ij}.$$

Crouzeix identity: $\nabla^2 F(\theta) \nabla^2 F^*(\eta(\theta)) = \nabla^2 F(\theta(\eta)) \nabla^2 F^*(\eta) = I$
- Riemannian manifold (M, g) is not flat with respect to the Levi-Civita connection ∇^g induced by g



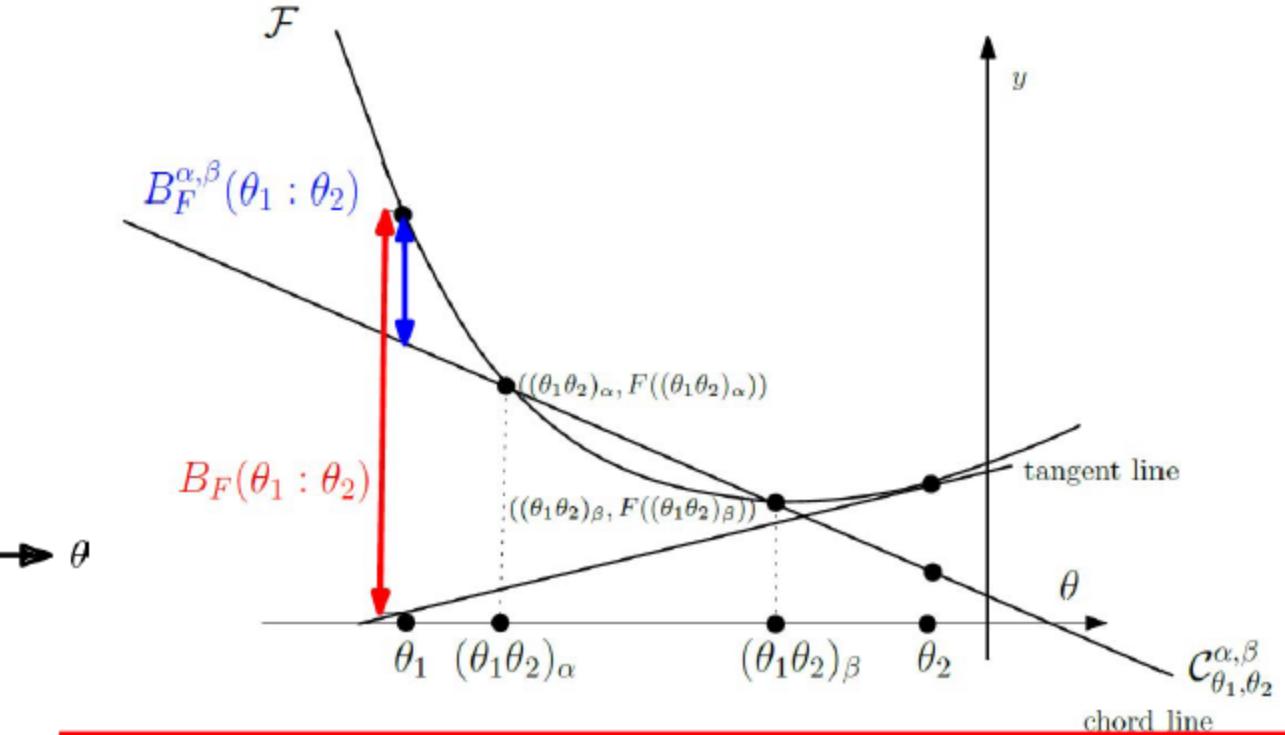
Scalar Bregman chord divergences

Idea: *Get rid of the gradient* in the Bregman formula (yet approximate BD)



$$B_F(\theta_1 : \theta_2) = F(\theta_1) - T_{\theta_2}(\theta_1)$$

$$T_\theta(\omega) := F(\theta) + (\omega - \theta)F'(\theta)$$



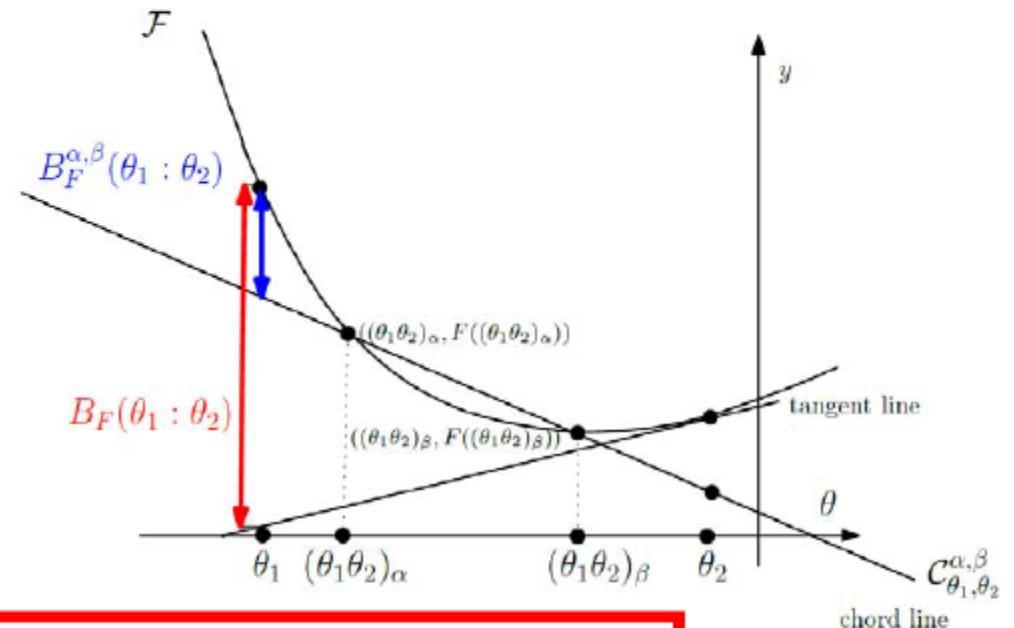
$$B_F^{\alpha,\beta}(\theta_1 : \theta_2) := F(\theta_1) - C_F^{(\theta_1\theta_2)_\alpha, (\theta_1\theta_2)_\beta}(\theta_1)$$

$$\rightarrow B_F^{\alpha,\beta}(\theta_1 : \theta_2) \leq B_F(\theta_1 : \theta_2)$$

Linear interpolation (LERP): $(pq)_\lambda := (1 - \lambda)p + \lambda q$

Scalar Bregman chord divergences

$$B_F^{\alpha,\beta}(\theta_1 : \theta_2) := F(\theta_1) - C_F^{(\theta_1\theta_2)_\alpha, (\theta_1\theta_2)_\beta}(\theta_1)$$



$$\begin{aligned} B_F^{\alpha,\beta}(\theta_1 : \theta_2) &:= F(\theta_1) - \Delta_F^{\alpha,\beta}(\theta_1, \theta_2)(\theta_1 - (\theta_1\theta_2)_\alpha) - F((\theta_1\theta_2)_\alpha), \\ &= F(\theta_1) - F((\theta_1\theta_2)_\alpha) + \frac{\alpha \{F((\theta_1\theta_2)_\alpha) - F((\theta_1\theta_2)_\beta)\}}{\beta - \alpha} \end{aligned}$$

where the *slope of the chord* is

$$\Delta_F^{\alpha,\beta}(\theta_1, \theta_2) := \frac{F((\theta_1\theta_2)_\alpha) - F((\theta_1\theta_2)_\beta)}{(\theta_1\theta_2)_\alpha - (\theta_1\theta_2)_\beta}$$

Scalar Bregman chord divergences: Properties

- Symmetry:

$$B_F^{\alpha,\beta}(\theta_1 : \theta_2) = B_F^{\beta,\alpha}(\theta_1 : \theta_2)$$

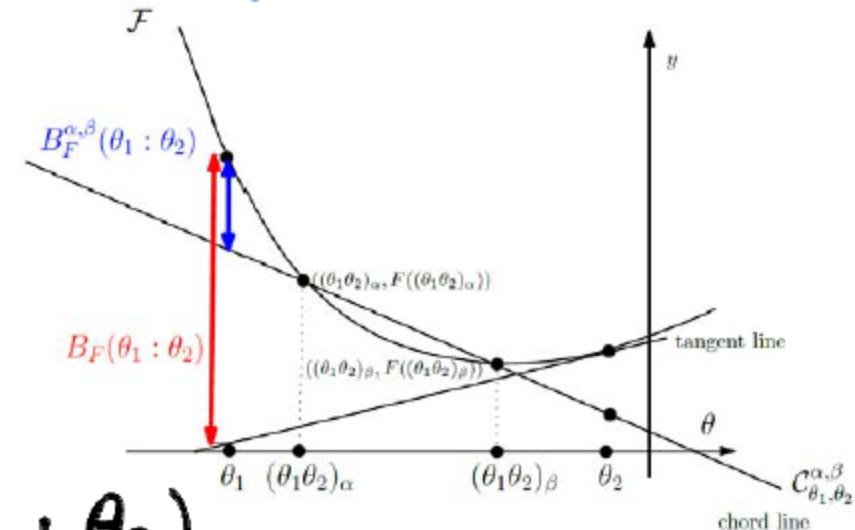
- Generalization of BD:

$$\lim_{\alpha \rightarrow 0, \beta \rightarrow 1} B_F^{\alpha,\beta}(\theta_1 : \theta_2) = B_F(\theta_1 : \theta_2).$$

- Subfamily of **Bregman tangent divergences**:

$$B_F^\alpha(\theta_1 : \theta_2) = \lim_{\beta \rightarrow \alpha} B_F^{\alpha,\beta}(\theta_1 : \theta_2) \leq B_F(\theta_1 : \theta_2)$$

$$\begin{aligned} B_F^\alpha(\theta_1 : \theta_2) &:= F(\theta_1) - F((\theta_1 \theta_2)_\alpha) - (\theta_1 - (\theta_1 \theta_2)_\alpha)^\top \nabla F((\theta_1 \theta_2)_\alpha), \\ &= F(\theta_1) - F((\theta_1 \theta_2)_\alpha) - \alpha(\theta_1 - \theta_2)^\top \nabla F((\theta_1 \theta_2)_\alpha), \end{aligned}$$



Summary

- Bregman divergences induce dually flat spaces for **any** Legendre-type C^3 strictly convex generator
- When the generator is an integral from statistical models, we can **reconstruct a statistical divergence**:
 - Reverse KLD from cumulant function of exponential families, rev ext KLD from partition function
 - KLD from negentropy of mixture families
- Jensen-Shannon centroid on **mixture family manifold** using concave-convex algorithm
- Chernoff information on **exponential family manifold** using exact geometric characterization ``Chernoff point'' = unique intersection of primal geodesic with dual bisector
- Define Bregman divergences with respect to **(M,N)-convexity**:
(M,N)- Bregman divergences as conformal Bregman divergences
- **Duality and conjugacies** (convex duality, reference/representation biduality, connection duality) are at the heart of information geometry!

Jensen-Shannon centroid of categorical distributions

Input: A set $\{p_i = (p_i^1, \dots, p_i^d)\}_{i \in [n]}$ of n categorical distributions belonging to the $(d - 1)$ -dimensional probability simplex Δ_{d-1}

Input: T : The number of CCCP iterations

Output: An approximation ${}^{(T)}\bar{p}$ of the Jensen–Shannon centroid \bar{p} minimizing $\sum_i D_{JS}(c, p_i)$

/* Convert the categorical distributions to their natural parameters by dropping the last coordinate

$\theta_i^j = p_i^j$ for $j \in \{1, \dots, d - 1\}$;

/* Initialize the JS centroid

$t \leftarrow 0$;

${}^{(0)}\bar{\theta} = \frac{1}{n} \sum_{i=1}^n \theta_i$;

/* Convert the initial natural parameter of the JS centroid to a categorical distribution

${}^{(0)}\bar{p}^j = {}^{(0)}\bar{\theta}^j$ for $j \in \{1, \dots, d - 1\}$;

${}^{(0)}\bar{p}^d = 1 - \sum_{i=1}^d {}^{(0)}\bar{p}^j$;

/* Perform the ConCave-Convex Procedure (CCCP)

while $t \leq T$ do

/* Use $\nabla F(\theta) = \left[\log \frac{\theta_i}{1 - \sum_{j=1}^d \theta_j} \right]_i$ and $\nabla F^{-1}(\eta) = \frac{1}{1 + \sum_{j=1}^d \exp(\eta_j)} [\exp(\eta_i)]_i$

$${}^{(t+1)}\theta = (\nabla F)^{-1} \left(\frac{1}{n} \sum_i \nabla F \left(\frac{\theta_i + {}^{(t)}\theta}{2} \right) \right);$$

$t \leftarrow t + 1$;

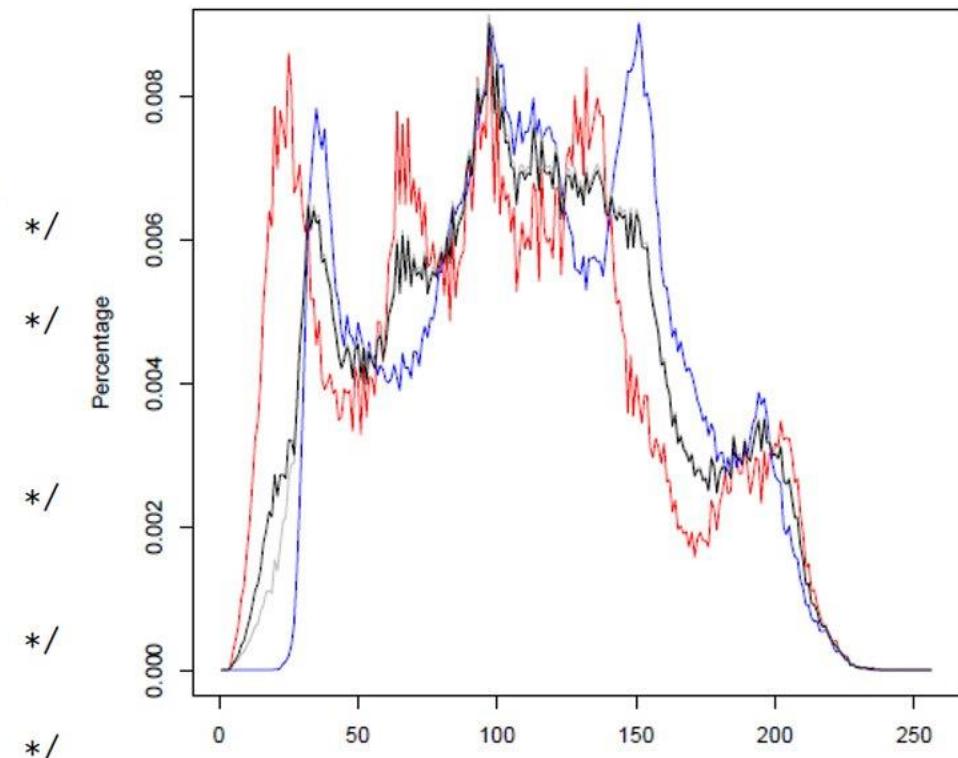
end

/* Convert back the natural parameter to the categorical distribution of the approximated Jensen–Shannon centroid

${}^{(T)}\bar{p}^j = {}^{(T)}\bar{\theta}^j$ for $j \in \{1, \dots, d - 1\}$;

${}^{(T)}\bar{p}^d = 1 - \sum_{i=1}^d {}^{(T)}\bar{p}^j$;

return ${}^{(T)}\bar{p}$;



- Use the fact that the set of categorical distributions is a **mixture family** in information geometry

JSD centroid = Jensen centroid

Multivariate Bregman divergence as families of univariate Bregman divergences

- A d-variate function $F(\theta)$ can be equivalently handled as a **family** of 1D convex functions: $\{F_{\theta_1, \theta_2}(\alpha) = F((1-\alpha)\theta_1 + \alpha\theta_2)\}$
- A d-variate BD can be written as an equivalent 1D scalar BD:

Directional derivative $\nabla_{\theta_2 - \theta_1} F_{\theta_1, \theta_2}(u) = (\mathbf{F}_{\theta, \theta})'(\mathbf{u})$

$$\lim_{\epsilon \rightarrow 0} \frac{F(\theta_1 + (\epsilon + u)(\theta_2 - \theta_1)) - F(\theta_1 + u(\theta_2 - \theta_1))}{\epsilon}$$
$$= (\theta_2 - \theta_1)^\top \nabla F(\theta_1 + u(\theta_2 - \theta_1)).$$

Hence, write BD as equivalent scalar BDs:

$$B_F(\theta_1 : \theta_2) := B_{F_{\theta_1, \theta_2}}(0 : 1)$$

write a BD wrt to anchor points as a **sub-dimensional Bregman divergence**