RESEARCH PAPER

Gibbs manifolds

Dmitrii Pavlov[1] · Bernd Sturmfels[1,2] · Simon Telen[1,3]

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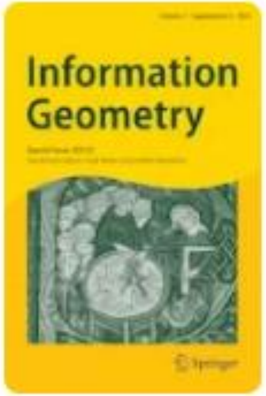

Volume 7
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Article 1
Number 122

Abstract

Gibbs manifolds are images of affine spaces of symmetric matrices under the exponential map. They arise in applications such as optimization, statistics and quantum physics, where they extend the ubiquitous role of toric geometry. The Gibbs variety is the zero locus of all polynomials that vanish on the Gibbs manifold. We compute these polynomials and show that the Gibbs variety is low-dimensional. Our theory is applied to a wide range of scenarios, including matrix pencils and quantum optimal transport.

Keywords Gibbs variety · Toric geometry · Semidefinite programming · Quantum optimal transport

RESEARCH

Dually flat structure of binary choice models

Hisatoshi Tanaka[1]

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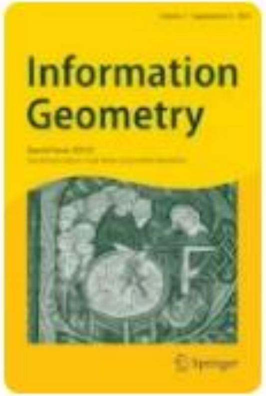
Published online: 18 May 2024

Abstract

In this study, we consider parametric binary choice models from the perspective of information geometry. The set of models is a dually flat manifold with dual connections, naturally derived from the Fisher information metric. Under the dual connections, the canonical divergence and the Kullback–Leibler divergence of the binary choice model coincide if and only if the model is a logit model.

Keywords Discrete choice models · Logit model · Single-index models · Hessian manifolds · Maximum likelihood estimation.



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A historical perspective on Schützenberger-Pinsker inequalities (extended version)

Olivier Rioul[1]

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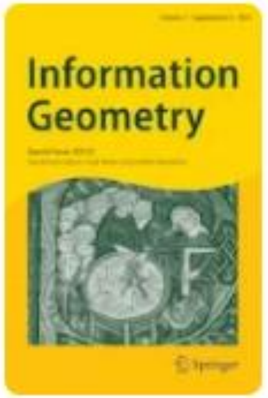
Abstract

This paper presents a tutorial overview of so-called Pinsker inequalities which establish a precise relationship between information and statistics, and whose use have become ubiquitous in many applications. According to Stigler's law of eponymy, no scientific discovery is named after its original discoverer. Pinsker's inequality is no exception: Years before the publication of Pinsker's book in 1960, the French medical doctor, geneticist, epidemiologist, and mathematician Marcel-Paul (Marco) Schützenberger, in his 1953 doctoral thesis, not only proved what is now called Pinsker's inequality (with the optimal constant that Pinsker himself did not establish) but also the optimal second-order improvement, more than a decade before Kullback's derivation of the same inequality. We review Schützenberger and Pinsker contributions as well as those of Volkonskii and Rozanov, Sakaguchi, McKean, Csiszár, Kullback, Kemperman, Vajda, Bretagnolle and Huber, Krafft and Schmitz, Toussaint, Reid and Williamson, Gilardoni, as well as the optimal derivation of Fedotov, Harremoës, and Topsøe. We also present some historical elements on the life and work of Schützenberger, and discuss an interesting problem of an erroneous constant in the Schützenberger-Pinsker inequality.

Keywords Pinsker inequality · Total variation · Kullback–Leibler divergence · Statistical distance · Mutual information · Data processing inequality



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RESEARCH

Fisher–Rao geometry of equivalent Gaussian measures on infinite-dimensional Hilbert spaces

Hà Quang Minh[1]

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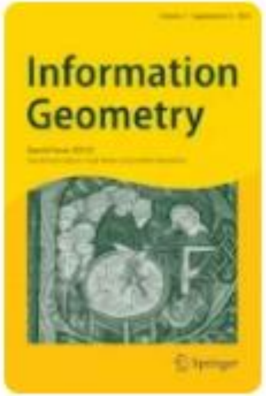
Published online: 5 June 2024

Abstract
This work presents an explicit description of the Fisher–Rao Riemannian metric on the Hilbert manifold of equivalent centered Gaussian measures on an infinite-dimensional Hilbert space. We show that the corresponding quantities from the finite-dimensional setting of Gaussian densities on Euclidean space, including the Riemannian metric, Levi–Civita connection, curvature, geodesic curve, and Riemannian distance, when properly formulated, directly generalize to this setting. Furthermore, we discuss the connection with the Riemannian geometry of positive definite unitized Hilbert–Schmidt operators on Hilbert space, which can be viewed as a regularized version of the current setting.

Keywords Fisher–Rao metric · Gaussian measures · Hilbert space · Positive Hilbert–Schmidt operators



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RESEARCH

Unveiling cellular morphology: statistical analysis using a Riemannian elastic metric in cancer cell image datasets

Wanxin Li[1] · Ashok Prasad[2] · Nina Miolane[3] · Khanh Dao Duc[4]

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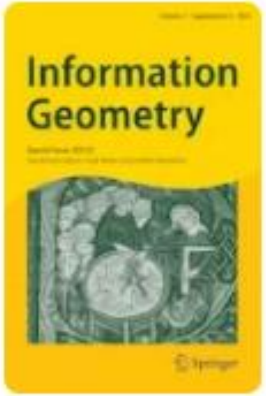
Published online: 27 August 2024

Abstract

Elastic metrics can provide a powerful tool to study the heterogeneity arising from cellular morphology. To assess their potential application (e.g. classifying cancer treated cells), we consider a specific instance of the elastic metric, the Square Root Velocity (SRV) metric and evaluate its performance against the linear metric for two datasets of osteosarcoma (bone cancer) cells including pharmacological treatments, and normal and cancerous breast cells. Our comparative statistical analysis shows superior performance of the SRV at capturing cell shape heterogeneity when comparing distance to the mean shapes, with better separation and interpretation between different cell groups. Secondly, when using multidimensional scaling (MDS) to find a low-dimensional embedding for unrescaled contours, we observe that while the linear metric better preserves original pairwise distances, the SRV yields better classification.

Keywords Elastic metric · Shape analysis · Cell morphology · Dimensionality reduction



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RESEARCH

A variational principle of minimum for Navier–Stokes equation and Bingham fluids based on the symplectic formalism

G. de Saxcé[1]

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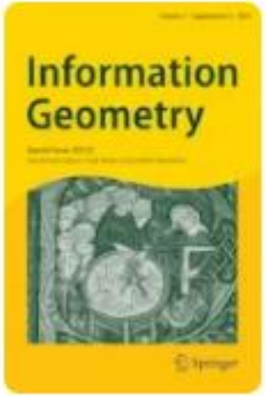
Published online: 4 November 2024

Abstract
In a previous paper, we proposed a symplectic version of Brezis-Ekeland-Nayroles principle based on the concepts of Hamiltonian inclusions and symplectic polar functions. We illustrated it by application to the standard plasticity in small deformations. The object of this work is to generalize the previous formalism to dissipative media in large deformations and Eulerian description. This aim is reached in three steps. Firstly, we develop a Lagrangian formalism for the reversible media based on the calculus of variation by jet theory. Next, we propose a corresponding Hamiltonian formalism for such media. Finally, we deduce from it a symplectic minimum principle for dissipative media and we show how to obtain a minimum principle for unstationary compressible and incompressible Navier–Stokes equation and Bingham fluids.

Keywords Dynamical dissipative systems · Hamiltonian methods · Brezis-Ekeland-Nayroles principle · Convex dissipation · Navier–Stokes equation · Bingham fluids



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RESEARCH

Cartan moving frames and the data manifolds

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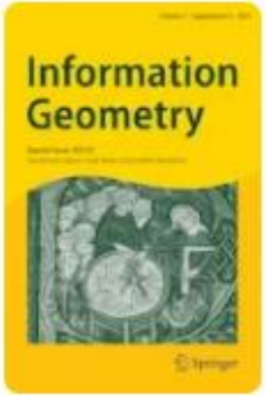
Published online: 20 November 2024



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Abstract

The purpose of this paper is to employ the language of Cartan moving frames to study the geometry of the data manifolds and its Riemannian structure, via the data information metric and its curvature at data points. Using this framework and through experiments, explanations on the response of a neural network are given by pointing out the output classes that are easily reachable from a given input. This emphasizes how the proposed mathematical relationship between the output of the network and the geometry of its inputs can be exploited as an explainable artificial intelligence tool.

Keywords Neural Networks · Data Manifolds · Moving Frames · Curvature · Explainable AI

RESEARCH

Totally geodesic submanifolds in the manifold SPD of symmetric positive-definite real matrices

Alice Barbara Tumpach[1,2,3] · Gabriel Larotonda[4,5]

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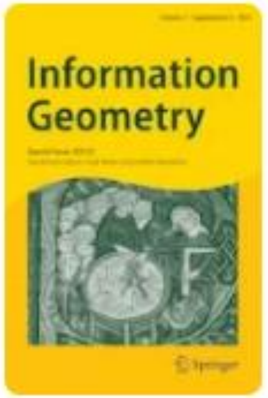
Abstract
This paper is a self-contained exposition of the geometry of symmetric positive-definite real $n \times n$ matrices $\mathrm{SPD}(n)$, including necessary and sufficent conditions for a submanifold $\mathcal{N} \subset \mathrm{SPD}(n)$ to be totally geodesic for the affine-invariant Riemannian metric. A non-linear projection $x \mapsto \pi(x)$ on a totally geodesic submanifold is defined. This projection has the minimizing property with respect to the Riemannian metric: it maps an arbitrary point $x \in \mathrm{SPD}(n)$ to the unique closest element $\pi(x)$ in the totally geodesic submanifold for the distance defined by the affine-invariant Riemannian metric. Decompositions of the space $\mathrm{SPD}(n)$ follow, as well as variants of the polar decomposition of non-singular matrices known as Mostow's decompositions. Applications to decompositions of covariant matrices are mentioned.

Keywords Covariance matrices · Reductive symmetric spaces · Decompositions of Lie groups · Symmetric positive-definite matrices



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RESEARCH

Information measures and geometry of the hyperbolic exponential families of Poincaré and hyperboloid distributions

Frank Nielsen[1] · Kazuki Okamura[2]

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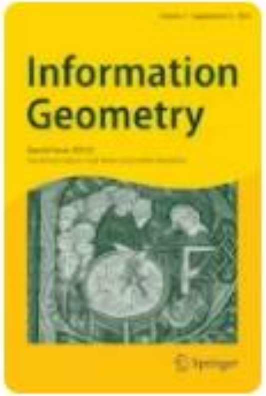
Abstract

Hyperbolic geometry has become popular in machine learning due to its capacity to embed hierarchical graph structures with low distortions for further downstream processing. It has thus become important to consider statistical models and inference methods for data sets grounded in hyperbolic spaces. In this paper, we study various information-theoretic measures and the information geometry of the Poincaré distributions and the related hyperboloid distributions, and prove that their statistical mixture models are universal density estimators of smooth densities in hyperbolic spaces. The Poincaré and the hyperboloid distributions are two types of hyperbolic probability distributions defined using different models of hyperbolic geometry. Namely, the Poincaré distributions form a triparametric bivariate exponential family whose sample space is the hyperbolic Poincaré upper-half plane and natural parameter space is the open 3D convex cone of two-by-two positive-definite matrices. The family of hyperboloid distributions form another exponential family which has sample space the forward sheet of the two-sheeted unit hyperboloid modeling hyperbolic geometry. In the first part, we prove that all Ali–Silvey–Csiszár's f-divergences between Poincaré distributions can be expressed using three canonical terms using the framework of maximal group invariance. We also show that the f-divergences between any two Poincaré distributions are asymmetric except when those distributions belong to a same leaf of a particular foliation of the parameter space. We report a closed-form formula for the Fisher information matrix, the Shannon's differential entropy and the Kullback–Leibler divergence between such distributions using the framework of exponential families. In the second part, we state the corresponding results for the exponential family of hyperboloid distributions by highlighting a parameter correspondence between the Poincaré and the hyperboloid distributions. Finally, we describe a random generator to draw variates and present two Monte Carlo methods to estimate numerically f-divergences between hyperbolic distributions.

Keywords Exponential family · Group action · Maximal invariant · Csiszár's f-divergence · Poincaré hyperbolic upper plane · Foliation · Minkowski hyperboloid sheet · Information geometry · Statistical mixture models · Statistical inference · Clustering · Expectation-maximization



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