DIVISION D

NON-HOLONOMIC GEOMETRY OF PLASTICITY AND YIELDING

GENERAL INTRODUCTION

EVEN in the case of Division C where we deal with elasticity theory, it is realized that our use of Riemannian and multidimensional concepts may create in the minds of some readers a feeling of skepticism regarding the fundamental philosophical soundness of the concepts. However, it can be argued that this issue actually contains the epistemological merit of the proposed idea. Here, in Division D, we advance a step further toward a better understanding of the non-Riemannian and non-holonomic nature of plastic phenomena, thereby clarifying the inevitability of introducing abstract-geometrical language into the theory of deformation and disruption of matter-manifolds in the hope that this may not be found too remote from accepted routine.

It must not be thought, however, that the author has placed more importance upon the sophistication of geometrical language than upon the need to grasp the physical reality connected with it, for emphatically this is not the case. Any feeling that the language overshadows the reality would create the erroneous impression that the subjects discussed belong, as it were, to a metaphysics remote from the actual physics of the study and this would be quite contrary to what the author has had in mind. It is important to remember this because much ill-informed criticism of the views expressed in this division has been directed against previously published work of a similar character, suggesting that the subject as presented is little more than a metaphysical abstraction; such, of course, is absurd.

For several years after the the first presentation of the theory of yielding stated in this division, difficulty was experienced in finding suitable words or terms to express the new viewpoint satisfactorily and also in definig the heuristic standing rightly to be assumed by the theory. The difficulty referred to was aggravated moreover, firstly by all the upheavals of post-war conditions and secondly by the fact that there is already in existence an established theory of plasticity that explains the major aspect of plasticity physics quite definitely which latter tends to increase any skepticism likely to be found of the plausibility of the new proposal which is so different, apparently, from the existing one. Handicapped in this way, the proposal herein explained has suffered various criticisms and will possibly also do so in the future before it is found ultimately acceptable. Even the author himself might not have been quite so inspired in this direction had he been well acquainted, when he started, with the recent tendencey in regard to dislocation theory allegedly supported by physicists, metallurgists and the specialists in applied mechanics.

However, efforts have been devoted to the task of diminishihg the epistemological incompleteness from which the initial formulation of the proposed theory suffered. Hence, a considerably brighter outlook has developed so that, not "Science", but only the author needs to complain of the portentous geometrical plasticity theory newly, but reluctantly, now being added to the former's possessions.

As for more details of the epistemological as well as the analytical problems of this theory, it will be more convenient for the reader to study the respective descriptions in the following treatises, where they are presented from various viewpoints. However, the diagram on the preceding page will serve for a quick understanding of the interconnexions of these papers and their relation to the contents of the other divisions, in particular to the holonomic problems of Division C.

As can be seen from this diagram and the following contents, the theory has been constructed in a certain manner by natural dialectical processes throughout its evolution, from a vague guess to a more concrete recognition. It streamed out of various sources one of which—whichever it may have been—was sufficient to lead to the result. There were mathematical, geometrical, and analytical sources as well as those that were technical, engineering and applied machanical. Our procedures were those pertaining to engineers and applied mechanical investigators rather than those belonging to mathematicians. This is clear, as we first started from a vague recognition of the possibility of there being a formal analogy—as instability phenomena—between the yielding of mild steel and the buckling of flat plates and, therefore, of assuming that the one could be translated into the other despite the difference of the dimension numbers of the respective manifolds. It is manifested also in our endeavours to extend the theory of incompatibility to cover more general ground. As by-products of these endeavours we get a summary of Riemannian geometry in terms of the elasticity model in article C-I and of non-Riemannian geometry in terms of the plasticity model in article D-I.

We are compelled to prepare for a possible misunderstanding in that our attitude may be considered as an unnecessarily positivistic violation of the established classical physical formulation. Article D-II is written in order to meet such a misunderstanding by giving an explanation based on the ontological background that definitely exists. This will be shown in the synthetic-stochastic construction of the macroscopic mechanical behaviour of microscopically non-uniform materials, and renewed essentially in the dynamical formulae in article D-VI. Media with non-uniform granular structures, studied concerning their statical behaviour in D-II and their dynamical behaviour in D-VI, are, when viewed macroscopically, possessed of non-Euclidian geometrical characteristics. Concerning the statical case, that is, yielding, all the related geometrical backgrounds are amalgamated with the physics of the situation in D-I, D-III and D-V.

However, it should be noted that further epistemological considerations are indispensable in order to reach the finally postulated formulation of the fundamental equations of yielding, namely,

The field equation of equilibrium,

and

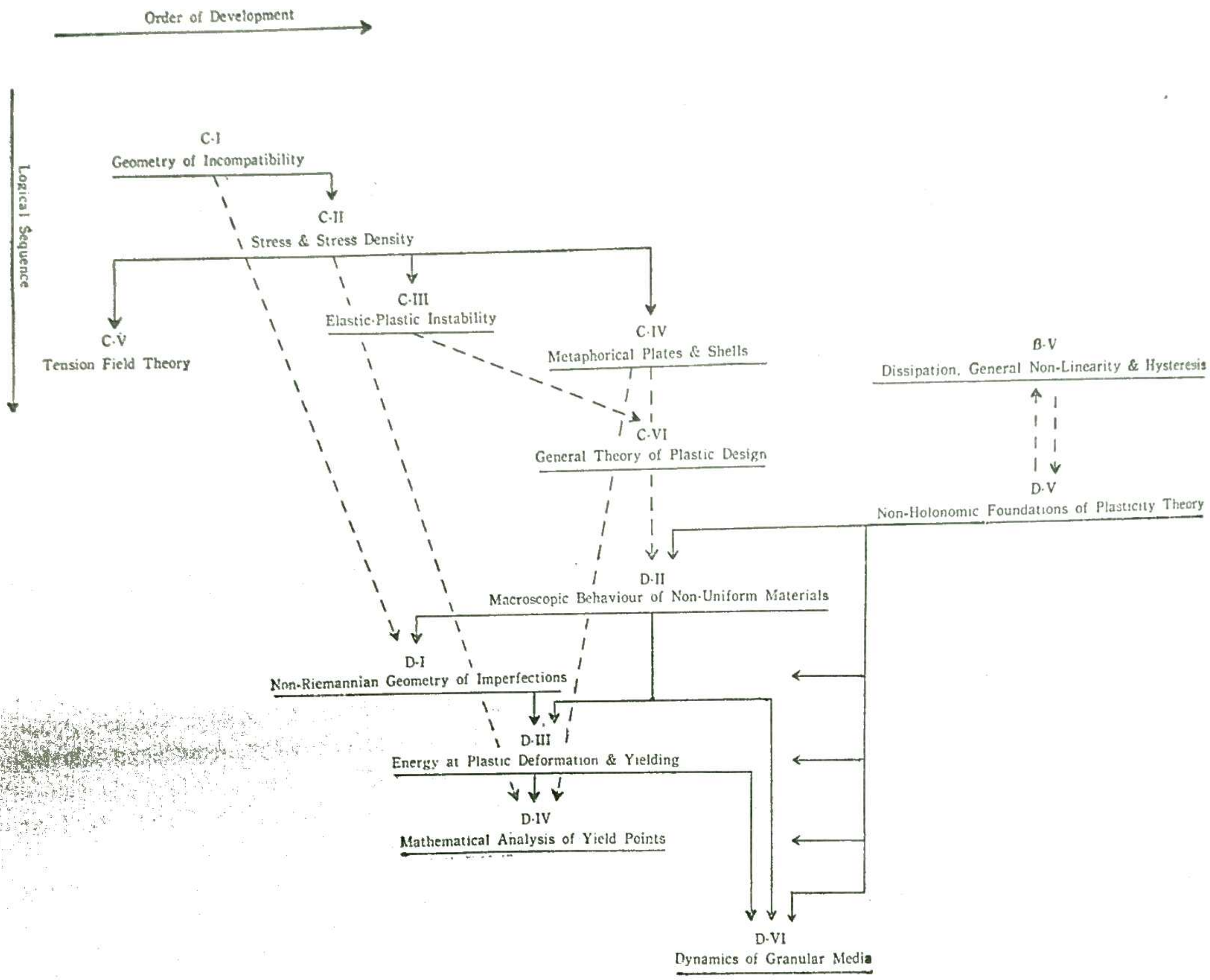
The equations of the free-boundary conditions.

It should be emphasized that these epistemological problems are entirely physical, being connected with thermo-dynamical and statistical aspects of what goes on in the matter manifold. Even article D-IV, which is intended as providing practical examples of applications of yielding, is loaded with a considerable amount of epistemology. One such remarkable but uncommon aspect is a certain quantal characteristic assumed to be associated with the matter-field. It can be interpreted in several ways statistically as well as geometrically. The granule image, such as adopted in D-VI,, is one such interpretation and will be accepted by many because of its concreteness.

To sum up, the conception of non-holonomic connexion is found to be fundamental in regard to plasticity. Quite evidently, plastic changes should be called "non-holonomic changes", treatable as the continuum version of the dislocation phenomena. No contradiction is involved between our view and that of modern dislocationists. Hence, the non-holonomic geometry and physics of matter manifolds propounded in article D-V, constitute a unified theory of dislocation in the continuum language.

CONTENTS

Interdependence of the Articles of Divisions C & D



D - I

Non-Riemannian Geometry of Imperfect Crystals from a Macroscopic Viewpoint

By

Kazuo KONDO

[Received April 1954]

INTRODUCTION

THE description of this article will make an adequate introduction to the whole Division D. It most completely reflects our endeavour to polish up the descriptions and to remove the unrefinements from our earlier proposal [1], [2], [3], [4] without losing the nuance of the original idiosyncracies with which we started on this series of investigations. This report also reflects the author's desire to convince certain sceptics of his belief that we are not going beyond the border of the world of physics in spite of the metaphorical appearance of our arguments.

By this report we would pave the way for a more elaborate investigation of the general non-holonomic image of the physics of plastic phenomena, which we shall enumerate in article D-V [5]. But the representation given there of all possible physical phenomena by means of non-holonomic spaces would be liable to make the descriptions unusually complicated and to discourage busy readers.

Hence we prefer a briefer sketch confining our present attention to the model case, namely to the theory of geometrical imperfections of nearly perfectly arranged matter. Here we mean by "perfect arrangement" almost the same thing as the perfect crystal, for matter is composed of atoms and a regular arrangement of atoms makes a perfect crystal. Hence we shall speak of crystals in this report.

However, it should be emphasized that what we relate in terms of crystal physics is to be understood as a sort of model. Similar conceptions and treatments could be extended to cover analogous phenomena of different scales. Hence the discretum language of the usual dislocations should preferably be translated into the language of the differential geometry of continua so that it may be applied more conveniently to other problems, which we shall try to accomplish in this report, at least to a certain degree.

A geometry of imperfections of crystal lattices has been extensively investigated in the modern theory of dislocations [6]. Fortunately or unfortunately, the author's approach to the problem has been from a more global aspect of such macroscopic incompatibilities as has been enunciated in C-I [7], as is natural for our purpose, and this was undertaken when he was not so well informed of the recent investigations on dislocations. Therefore he experienced certain confusion as could be expected, when he faced the excellent achievements which have been attained by dislocationists.

Our theory, which essentially includes the conception of dislocations, may also cover a wider scope. We may still expect a critical opinion based on the conviction that the present dislocation theory should be sufficiently exhaustive. But a modification or extension of what follows of the geometry of imperfections on a metric continuum and, as it were, on a larger scale, may have its own significance as the reader will no doubt be able to agree after having read the description in the following pages.

It has been pointed out that there are six primary types of crystal imperfection, namely

(a) phonons,

(b) electrons and holes,

(c) excitons,
(d) impurities or foreign atoms,
(e) vacant lattice sites and interstitial atoms,
(f) dislocations.

The first four types are of physical nature. The last two and these only are usually considered to be rather geometrical, although we can hardly refrain from doubt if they are responsible for all possible types in an essentially geometrical sense (cf. §4).

Since our first objective is to investigate the behaviour of a nearly perfect crystal with imperfections of a geometrical nature, we shall postpone the study of essentially physical imperfections for a while. So we shall aim mainly to deal with (e) and (f), of which the latter is the more important. The reader may have expected to find a continuum version of "the theory of dislacations" in the following description. But we think we capture (e) and also some influence of (d). The very existence of the imperfections of these types bears a geometrical meaning that the foreign atom should be of a different size from the surrounding atoms and that vacant lattice sites or interstitial atoms produce similar effects. Hence will arise, in relation to the affine and metric configuration for (d) as well as for (e), a question of incompatibility which our formulation in differential geometry is competent to handle.

We would draw the reader's attention to a significant view suggested by a famous geometer in his lecture on the geometry of Riemannian spaces, as much as thirty years ago. He said "The Riemannian space is for us an ensemble of small pieces of Euclidean space, lying however to a certain degree amorphously". The non-Riemannian space here employed is also an amorphous, and therefore imperfect, aggregation of Euclidean or crystalline pieces.[1] The present subject matter in this article is confined within the geometry of imperfection.

We shall investigate next in D-II [9] and D-III [10] the energy and fundamental statical condition of equilibrium of the field of plastic deformation and discuss the criterion for yielding.

We shall proceed later to the geometrical formulation of various influences of physical phenomena on plasticity other than those which we take account of here. The extension will result in the introduction of the theory of general non-holonomic space and may probably amount to the theory of fibre-bundles in the future.

We should not fail to mention that the lack of precise knowledge on the author's part of the modern theory of dislocations and crystal imperfections should have made the present scope of this investigation very limited although he has much profited from reading several expository treatises on related subjects [11], [12], [13], [14].

That a certain degree of expository as well as demonstrative material is included in this report is due to the hesitant reception of the writer's earlier publications on a related subject as he relized on occasions of contact with a number of specialists in certain fields. The general misunderstanding from which we suffered might have been avoided by the following exposition. A considerable part, especially §§ 2, 3 of this report, may be passed over by mathematically trained readers.

Chapter I

Fundamentals of the Geometry of Imperfect Crystals (Arrangements of Matter)

1. Imperfect crystal as a non-Riemannian space of metric-affine connexion

In calling a crystal perfect, we mean that the atoms form a regular pattern proper to the prescribed nature of the matter, in its strain-free or natural state as we have defined in article C-I [7]. A small piece can be cut off from an imperfect crystal and brought to its natural state in which the atoms are arranged on the respective regular position of the perfect crystal lattice. The imperfect crystal which we now investigate is an aggregation of an immense number of such small pieces of perfect crystals that cannot be connected with one another so as to form a finite lump of perfect crystal as an organic unity.

Each atom of the imperfect crystal can be marked by the three-dimensional Cartesian coordinates of the position

$$x^i, \quad i=1,2,3$$

which it occupies in real space. But the coordinates cannot in general be such integral multiples of the lattice constants as in the case of a perfect

1) See p. 90 of reference [8].

crystal. We can regard the imperfect crystal represented by these coordinates as a three-dimensional manifold.

The real metric property of an element in an imperfect crystal is different from that *natural metric* which it takes when it is cut off and developed on the perfect lattice. Hence we can associate a natural or perfect element with each point of the manifold of an imperfect crystal, as the tangent Euclidean space. Therefore, we will sometimes substitute the geometrical word "Euclidean" for the meaning of "perfect".

If we grope our way in the crystal on a sequence of such elements that can be developed as a connected natural material line on the perfect lattice, avoiding the imperfection, then a closed loop in the crystal thus followed may not generally remain closed in the development. The discrepancy of both ends of the open piece thus produced is inherent in the natural state of this material line. It is generally composed of an orthogonal transformation (rotation or reflection) combined with a translation, and the set of all discrepancies corresponding to all possible loops which connect the perfect elements in the crystal, constitutes a subgroup of the automorphism group of the perfect lattice, or in our approximate language, of the Euclidean space. This group evidently has the character of the *holonomic group* introduced by Élie Cartan [15]. Our imperfect crystal is therefore a space of connexion of Euclidean tangent elements, with a certain non-trivial holonomic group. The reader may refer to the description in article C-I [7].

A Cartesian frame can be assigned in each tangent space. It will be regarded as a moving frame. We denote it by the basis vectors

$$\mathbf{e}_\lambda, \quad \lambda=1,2,3$$

and we have

FIG. 1

$$d\mathbf{M}=\mathbf{e}_\kappa \omega^\kappa, \qquad (1)$$

$$\omega^\kappa = A^\kappa_i dx^i, \qquad (2)$$

$$d\mathbf{e}_\lambda = \mathbf{e}_\kappa \omega^\kappa{}_\lambda, \qquad (3)$$

$$\omega^\kappa{}_\lambda = \omega^\mu \Gamma^\kappa_{\mu\lambda} \qquad (4)$$

where $d\mathbf{M}$ signifies the vector drawn from x^i to x^i+dx^i in the initial imperfect state and ω^κ is the component of the corresponding vector element of the same matter in the natural state measured by the natural frame $\{\mathbf{e}_\lambda\}$, and

$$A^\kappa_i, \ \mathbf{e}_\lambda \text{ and } \Gamma^\kappa_{\mu\lambda}$$

may depend on the true coordinates $\{x^i\}$ as defined above.

Now, M is the point

$$x^i, \quad i=1,2,3$$

in the original manifold and the natural basis vectors

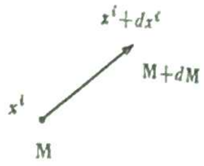


FIG. 2

$$\mathbf{e}_\kappa = \frac{d\mathbf{M}}{\omega^\kappa}, \quad \kappa=1,2,3 \qquad (1\cdot1)$$

are arbitrarily associated with it. Hence the first integrability condition, namely,

$$\frac{\partial^2 \mathbf{M}}{\partial x^j \partial x^i} = \frac{\partial^2 \mathbf{M}}{\partial x^i \partial x^j}$$

must naturally be satisfied so as to give[1)]

$$\partial_j A^\kappa_i + A^\mu_j A^\lambda_i \Gamma^\kappa_{\mu\lambda} = \partial_i A^\kappa_j + A^\mu_i A^\lambda_j \Gamma^\kappa_{\mu\lambda}$$

where

$$\partial_i = \partial/\partial x^i,$$

whence[2)]

$$\Gamma^\kappa_{[\mu\lambda]} \overset{*}{=} -A^j_\mu A^i_\lambda \partial_{[j} A^\kappa_{i]} \qquad (5)$$

---

1) Cf.:

$$\frac{\partial \mathbf{M}}{\partial x^i} = \mathbf{e}_\kappa A^\kappa_i$$

$$\frac{\partial^2 \mathbf{M}}{\partial x^j \partial x^i} = \frac{\partial \mathbf{e}_\kappa}{\partial x^j} A^\kappa_i + \mathbf{e}_\kappa \partial_j A^\kappa_i$$

$$= \mathbf{e}_\lambda \frac{\partial \omega^\lambda{}_\kappa}{\partial x^j} A^\kappa_i + \mathbf{e}_\kappa \partial_j A^\kappa_i$$

$$= \mathbf{e}_\kappa((\Gamma^\kappa_{\mu\lambda} A^\mu_j A^\lambda_i + \partial_j A^\kappa_i)$$

$$\frac{\partial^2 \mathbf{M}}{\partial x[^j \partial x^i]} = \mathbf{e}_\kappa(\Gamma^\kappa_{[\mu\lambda]} A^\mu_j A^\lambda_i + \partial_{[j} A^\kappa_{i]}).$$

2) The bracket notation [ ] is used to express "alternation" such as

$$\partial_{[j} A^\kappa_{i]} = \frac{1}{2}(\partial_j A^\kappa_i - \partial_i A^\kappa_j);$$

and | | indicates "omission from the alternation" such as

$$2\Gamma^\kappa_{[\nu|\rho|}\Gamma^\rho_{\mu]\lambda} = \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\mu\lambda} - \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\nu\lambda}$$

in equation (9).

D-I

where A_λ^i is defined by

$$A_\lambda^i A_i^\kappa = \delta_\lambda^\kappa \quad \text{and} \quad A_\lambda^j A_k^\lambda = \delta_k^j. \tag{6}$$

Because of the imperfection of the crystal, it is not always assured that the geometric object

$$\Omega_{\mu\lambda}^{\kappa} = A_\mu^j A_\lambda^i \partial_{[j} A^\kappa{}_{i]}, \tag{7}$$

called the non-holonomic object, vanishes identically at every point of the manifold. Hence a mixed tensor whose components coincide with the skew parts of the coefficients of connexion is not always a zero-tensor. We shall denote it by

$$S_{\mu\lambda}^{\cdot\cdot\kappa} \stackrel{*}{=} \Gamma_{[\mu\lambda]}^{\kappa} \stackrel{**}{=} -\Omega_{\mu\lambda}^{\kappa}. \tag{8}$$

This defines torsion in the affinely connected space of the crystal. The (i)-components of the torsion tensor are

$$S_{kj}^{\cdot\cdot i} = A_j^\mu A_k^\lambda A_\kappa^i S_{\mu\lambda}^{\cdot\cdot\kappa}.$$

But the moving frame $\{\mathbf{e}_\kappa\}$ is defined by Pfaffian relations (1) and (2), which should usually be non-holonomic and therefore cannot be determined uniquely. Hence the second integrability conditions, namely

$$\frac{\partial^2 \mathbf{e}_\lambda}{\partial x^j \partial x^i} = \frac{\partial^2 \mathbf{e}_\lambda}{\partial x^i \partial x^j},$$

are not always satisfied. We see therefore that the Riemann-Christoffel curvature tensor

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2\partial_{[\nu}\Gamma_{\mu]\lambda}^{\kappa} + 2\Gamma_{[\nu|\rho|}^{\kappa}\Gamma_{\mu]\lambda}^{\rho} + 2\Omega_{\nu\mu}^{\rho}\Gamma_{\rho\lambda}^{\kappa} \tag{9}$$

is not always a zero-tensor in our problem. Here we use the notation

$$\partial_\nu = A_\nu^i \frac{\partial}{\partial x^i} \tag{10}$$

which is recommended by J.A.Schouten [16]. All the calculations leading to (9) follow easily.[1)]

1) Cf.:

$$\begin{aligned}
\frac{\partial^2 \mathbf{e}_\lambda}{\partial x^j \partial x^i} &= \frac{\partial}{\partial x^j}\frac{\partial \mathbf{e}_\lambda}{\partial x^i} = \frac{\partial}{\partial x^j}(\mathbf{e}_\kappa A_i^\mu \Gamma_{\mu\lambda}^{\kappa}) \\
&= \mathbf{e}_\kappa A_i^\mu \frac{\partial \Gamma_{\mu\lambda}^{\kappa}}{\partial x^j} + A_j^\nu \Gamma_{\nu\kappa}^{\rho} \mathbf{e}_\rho A_i^\mu \Gamma_{\mu\lambda}^{\kappa} + \mathbf{e}_\kappa \frac{\partial A_i^\mu}{\partial x^j}\Gamma_{\mu\lambda}^{\kappa} \\
&= A_j^\nu A_i^\mu \mathbf{e}_\kappa (\partial_\nu \Gamma_{\mu\lambda}^{\kappa} + \Gamma_{\nu\rho}^{\kappa}\Gamma_{\mu\lambda}^{\rho} + A_\nu^k A_\mu^l \partial_l A_k^\rho \Gamma_{\rho\lambda}^{\kappa}).
\end{aligned}$$

We finally conclude that our manifold of the geometrically imperfect crystal can be treated as a three-dimensional space of metric-affine connexion of the Euclidean tangent space with non-zero tensors of torsion and Riemann-Christoffel curvature.

This conclusion is applicable only to problems such that certain details of the atomic structures can be treated by their mean properties. Such treatment is not to be taken as being arbitrary as the reader might at first suppose, because the majority of experiments that can readily be performed and therefore are adaptable as the basic proof of the plasticity theory, are said to be too macroscopic to be directly related to the behaviour of atoms.

2. The relation between the metric and the connexion

The principal characteristics of our non-Riemannian space are reflected in the metric differential form

$$ds^2 = g_{\kappa\lambda}\omega^\kappa\omega^\lambda, \tag{11}$$

and the coefficients of affine connexion $\Gamma_{\mu\lambda}^{\kappa}$, where

$$g_{\kappa\lambda} = \mathbf{e}_\kappa \cdot \mathbf{e}_\lambda$$

is the component of the fundamental tensor. We shall establish the fundamental relations between these quantities in what follows.

We have

$$\begin{aligned}
\partial_\nu g_{\mu\lambda} &= (\partial_\nu \mathbf{e}_\mu)\cdot\mathbf{e}_\lambda + \mathbf{e}_\mu\cdot(\partial_\nu \mathbf{e}_\lambda) \\
&= \Gamma_{\nu\mu}^{\kappa}\mathbf{e}_\kappa\cdot\mathbf{e}_\lambda + \mathbf{e}_\mu\cdot\mathbf{e}_\kappa\Gamma_{\nu\lambda}^{\kappa} \\
&= \Gamma_{\nu\mu}^{\kappa} g_{\kappa\lambda} + g_{\mu\kappa}\Gamma_{\nu\lambda}^{\kappa}
\end{aligned}$$

or writing

$$\Gamma_{\nu\lambda\mu} = \Gamma_{\nu\mu}^{\kappa} g_{\kappa\lambda},$$

$$\partial_\nu g_{\mu\lambda} = \Gamma_{\nu\lambda\mu} + \Gamma_{\nu\mu\lambda}$$

and similarly two analogous equations with cyclic permutations of the indices. These we can write as[2)]

2) Here the parenthesis ( ) is used to express mixing, | | indicating "exclusion" from the mixing.

$$\partial_\nu g_{\mu\lambda}=\Gamma_{(\nu|\lambda|\mu)}+\Gamma_{(\nu|\mu|\lambda)}+\Gamma_{[\nu|\lambda|\mu]}+\Gamma_{[\nu|\mu|\lambda]},$$
$$\partial_\lambda g_{\nu\mu}=\Gamma_{(\lambda|\mu|\nu)}+\Gamma_{(\lambda|\nu|\mu)}+\Gamma_{[\lambda|\mu|\nu]}+\Gamma_{[\lambda|\nu|\mu]},$$
$$-\partial_\mu g_{\lambda\nu}=-\Gamma_{(\mu|\nu|\lambda)}-\Gamma_{(\mu|\lambda|\nu)}-\Gamma_{[\mu|\nu|\lambda]}-\Gamma_{[\mu|\lambda|\nu]}$$

so that we get

$$\partial_\nu g_{\mu\lambda}+\partial_\lambda g_{\nu\mu}-\partial_\mu g_{\lambda\nu}=2\Gamma_{(\nu|\mu|\lambda)}-2\Gamma_{[\mu|\lambda|\nu]}+2\Gamma_{[\lambda|\nu|\mu]}$$

or

$$\Gamma_{(\nu|\mu|\lambda)}=\frac{1}{2}(\partial_\nu g_{\mu\lambda}+\partial_\lambda g_{\nu\mu}-\partial_\mu g_{\lambda\nu})+\Gamma_{[\mu|\lambda|\nu]}-\Gamma_{[\lambda|\nu|\mu]}$$

whence

$$\Gamma_{\nu\mu\lambda}=\frac{1}{2}(\partial_\nu g_{\mu\lambda}+\partial_\lambda g_{\nu\mu}-\partial_\mu g_{\lambda\nu})+\Gamma_{[\nu|\mu|\lambda]}+\Gamma_{[\mu|\lambda|\nu]}-\Gamma_{[\lambda|\nu|\mu]}.$$

Therefore we obtain

$$\Gamma_{\nu\lambda}^{\mu}=\frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\lambda}+\partial_\lambda g_{\nu\rho}-\partial_\rho g_{\lambda\nu})+\Gamma_{[\nu\lambda]}^{\mu}+g^{\mu\rho}g_{\lambda\kappa}\Gamma_{[\rho\nu]}^{\kappa}-g_{\nu\kappa}g^{\mu\rho}\Gamma_{[\lambda\rho]}^{\kappa}, \tag{12}$$

or substituting (8)

$$\Gamma_{\mu\lambda}^{\kappa}=\frac{1}{2}g^{\kappa\nu}(\partial_\mu g_{\nu\lambda}+\partial_\lambda g_{\mu\nu}-\partial_\nu g_{\mu\lambda})+S_{\mu\lambda}^{\cdot\cdot\kappa}-S_{\mu\cdot\lambda}^{\cdot\kappa}-S_{\lambda\cdot\mu}^{\cdot\kappa}. \qquad (12\cdot1)^{1)}$$

3. Dualistic viewpoint based on the Riemannian conception

The property of our space of affine connexion should be completely specified if, and only if, we can properly assign 15 parameters which can be regarded as equivalent to

(1) $\frac{n(n+1)}{2}=6$ components of the symmetric tensor $g_{\kappa\lambda}$, $\kappa,\lambda=1,\cdots,n(=3)$,

(2) $\frac{n^2(n-1)}{2}=9$ components of the torsion tensor $S_{\mu\lambda}^{\cdot\cdot\kappa}$, $\mu,\lambda,\kappa=1,\cdots,n(=3)$.

At first sight, it might appear that the six components of the fundamental tensor $\underset{E}{g_{ij}}$ of the Eulerian metric form

$$ds_E^2=\underset{E}{g_{ij}}dx^i dx^j \tag{13}$$

of the actual imperfect crystal plus the nine components of the matrix of transformation

$$(i)\rightarrow(\kappa):\quad \{A_i^\kappa\}$$

would afford a sufficient degree of freedom. But it should be pointed out that this would be a misunderstanding from which serious mischief would result.

Our non-Riemannian space is not embodied by such a Eulerian state as is characterized by the Eulerian metric (13). It is related to the natural metric (11) or

$$ds^2=\underset{N}{g_{ij}}dx^i dx^j, \tag{11·1}$$

where

$$A_i^\kappa A_j^\lambda g_{\kappa\lambda}=\underset{N}{g_{ij}}(\neq\underset{E}{g_{ij}}\,!).$$

Therefore, it is beyond doubt that this metric usually is non-Euclidean so that $\underset{N}{g_{ij}}$ cannot be equivalent to any Euclidean $\underset{E}{g_{ij}}$ in three dimensions. It is therefore not always permitted to find three-dimensional Cartesian coordinates y^1, y^2, y^3 such that

$$\underset{N}{g_{ij}}=\sum_{a=1}^{3}\frac{\partial y^a}{\partial x^i}\frac{\partial y^a}{\partial x^j},$$

although it is possible to find such z^1, z^2, z^3 that

$$\underset{E}{g_{ij}}=\sum_{a=1}^{3}\frac{\partial z^a}{\partial x^i}\frac{\partial z^a}{\partial x^j}.$$

This is implied by the similar situation which we have studied in relation to the condition of compatibility in paper C-I [7].

But it must be possible to find a set of a

1) This formula for $\Gamma_{\mu\lambda}^{\kappa}$ is a little different from the one given by Schouten and Struik in their treatise SDG, I, p 83, [16]. The terms in non-holonomic objects are dropped in ours. This is due to our definition of a torsion tensor (8), while SDG adopts

$$S_{\mu\lambda}^{\cdot\cdot\kappa}=\Gamma_{[\mu\lambda]}^{\kappa}+\Omega_{\mu\lambda}^{\kappa}\overset{h}{=}\Gamma_{[\mu\lambda]}^{\kappa}.$$

number N larger than $(n=)3$ of regular functions of x^i

$$X^A = X^A(x^i) \tag{14}$$

such that we have

$$g_{ij} = \sum_{N\ A=1}^{N} \frac{\partial X^A}{\partial x^i} \frac{\partial X^A}{\partial x^j} \tag{14·1}$$

or

$$g_{\kappa\lambda} = \sum_{N\ A=1}^{3} \partial_\kappa X^A \partial_\lambda X^A.$$

Now we can choose these $N+n^2=N+9$ parameters

$$X^A, \quad A=1, \cdots, N(>3),$$
$$A_i^\kappa, \quad \kappa, i=1, \cdots, n(=3),$$

in accordance with our convenience. Since N is greater than $n=3$, we have obtained a non-flat manifold of three dimensions, each of whose points is marked by three parameters $x^i (i=1,2,3)$, and which is immersed in a space of dimension number N larger than three. The enveloping space can be chosen arbitrarily as Euclidean.

Let an arbitrary point on this space denoted by $\mathbf{X}$ introduce a natural frame $\{\mathbf{g}_i\}$, such that

$$d\mathbf{X} = \mathbf{g}_i dx^i, \quad \left(\mathbf{g}_i = \frac{\partial X^A}{\partial x^i}\right)$$

then we have

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}.$$

It follows that we obtain

$$A_{kj}^i = \frac{1}{2}\left\{ {i \atop kj} \right\} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{kj}),$$
$$d\mathbf{g}_j = dx^k A_{kj}^i \mathbf{g}_i.$$

for the coefficients of connexion of the new set of the moving frame $\{\mathbf{g}_i\}$.

It is a well known fact that a space characterized by this kind of connexion property is a Riemannian space and in regard to the number of dimensions of the enveloping Euclidean space it is necessary and sufficient to assume

$$N = \frac{n(n+1)}{2} = 6$$

in correspondence with the number of degrees of freedom of the fundamental tensor $g_{\kappa\lambda}$, for any sort of possible Riemannian space [17].

On this basis, the object of our investigation has been translated into the geometry of a Riemannian space expressed by means of a non-holonomic moving frame (κ) which has been defined by the non holonomic transformation (2):

$$\omega^\kappa = A_i^\kappa dx^i.$$

The dual viewpoints thus interconnected and equivalent to each other are:

(i) The non-Riemannian geometry of a non-Riemannian manifold described by means of holonomic-true variables x^i, on the one hand;

(ii) The non-holonomic geometry of a Riemannian manifold associated with non-holonomic pseudo-variables ω^κ, on the other.

The former point of view is the one which we have enunciated in the preceding two sections §§ 1, 2. The latter viewpoint which we explain in this section, will later prove itself to be a convenient key-concept for the handling of certain plastic deformations. It is important that the Riemannian properties (=smooth curvature) are reflected in the holonomical subspace image that is defined by (14), and the non-Riemannian properties (=chopping into fine pieces) in the non-holonomical relation (2).

Chapter II

Classification of Imperfections

4. Topology and metric of a crystal and dislocations

Metal physicists use the terms "good" and "bad" freely, bad being synonymous with imperfect. They especially like to speak of "good or bad topology" of a crystal. Their meanings are fairly topological in the sense that good material would be a material that is perfect and regular except for elastic strains, thermal vibrations or other perturbations that leave the crystal structure clearly recognizable.

We are working on our approximate point of view of substituting a continuum, which is postulated by differential geometry, for the discrete lattice structure of a crystal. This approximation is justified from the inevitable inaccuracy in our practical observations of plastic phenomena that makes it impossible to distinguish between adjacent rational lattice points and the irrational cut defined by them. It is clear that the order relation of atoms should then be replaced by the topological invariance of a Hausdorff space. Any breach of the invariance should be compared with the badness or disorder of the crystal structure.

As it is believed that plastic deformations are geometrical changes into the bad condition of a crystal, they are, in our language of differential geometry, such deformations of a continuous matter-manifold as are non-homeomorphic. By pure mathematical reasoning we would classify the possible deformations into the following three species:

(1) Compatible deformations confined in the ordinary 3-space and preserving the topological invariance of the whole manifold;

(2) Incompatible deformations preserving the topological invariance only locally;

(3) Incompatible deformations not preserving the topological invariance even locally.

The elastic deformation belongs to the first (1) and no other can be elastic. We have seen that the whole theory of elasticity has been classically subjected to the compatibility criterion. Its violation should therefore be named non-elasticity or a kind of plasticity, because it induces a non-homeomorphism of the matter-manifold in the Euclidean 3-space.

It is true however that every material body is confined in a Euclidean 3-space. But there are the second or third kind of deformation or mapping of the material manifold that are (2) locally homeomorphic or (3) entirely non-homeomorphic. A mapping cannot be called locally homeomorphic if there is found a neighbourhood in which the mapping is not homeomorphic. If this occurs for any small element of imperfect matter in regard to its mapping on the regular lattice (in our continuum version, for any non-Riemannian manifold on the Euclidean 3-space), it gives rise to torsion as we shall explain later.

In the case of locally homeomorphic mapping, we find a matter element of a certain size, which, if cut off from the surrounding matter, can be mapped elastically onto an elementary volume of the Euclidean 3-space. If this holds only for the first approximation for an element whose diameter is $O(dx)$, the torsion only vanishes, but not the Riemann-Christoffel curvature tensor at the point in question. This situation also will be explained later. In any case, type (2) includes the Riemann-Christoffel curvature effect.

But there is another possibility, because a neighbourhood of finite size can be found in which the mapping is homeomorphic. A little study of geometry in the large would suffice to attest the existence of various spaces which are locally Euclidean but not homeomorphic to a Euclidean space in the large. The Möbius band is one such, although the corresponding conception in crystallography, the Möbius crystal is called pathological by crystal-physicists owing perhaps to the interest of their direct demands. But in relation to later chapters we must be aware of the importance of topological oddities in the large.

It is however beyond doubt that all the incompatibilities and imperfections of a geometrical character must be reflected in the behaviour of the holonomic group of the non-Riemannian manifold to which we have translated the imperfect matter. It is most interesting that we can find its counterpart in the metal-physicists' definition of dislocations by the *Burgers vector* [18]. The following definition by F. C. Frank [6], which is essentially the same as Burgers' [7] original one, is convenient for comparison with the holonomic group.

A Burgers circuit is a closed loop in a possibly bad crystal made up of atom-to-atom steps with the following properties:

1) The circuit lies entirely in a good region although it may encircle bad material;

2) The corresponding sequence of atom-to-atom steps in a perfect crystal does not necessarily close.

The closure failure of the image circuit is called "Frank's Burger vector".

It is now obvious that the Burgers-Frank circuits define the holonomic group of a space of topological connexion as defined by the atom-to-atom order of the lattice structure[1]. To the geometrically trained instinct of our colleagues, this would appear insufficient because it apparently lacks the connexion of affine and metric properties of matter elements, automatically introduced into the holonomic group of our non-Riemannian image, where two neighbouring atoms are in a sense elastometrically related and the discrepancy vector at the point of a loop

1) Cf. the foot note 9) of Cartan's original paper [15] or Veblen and Whitehead's description in their treatise [19], Chapter VIII, §7. They are suggestively general expositions, which are helpful for the understanding of the opinion proposed herein.

of matter should be not only topologically (by atom-to-atom order) but metrically (by the elastic constraint due to inter-atomic forces) connected with the discrepancies of the neighbouring circuits.

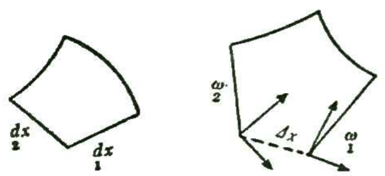
Therefore we shall undertake to modify the Burgers-Frank formulation so that the result may reflect all the non-Euclidean aspects of the possible imperfections of matter by means of the well-established language of metric differential geometry. The expected result will be very well suited to the macroscopic treatment of plastic phenomena which are sometimes not too directly related to atomic considerations, but rather intimately to the mechanical configuration of the strain field. Eventually, we shall be able to grasp the entirety of the behaviour of the imperfect crystal in relation to its atom-arrangement as well as to the incompatibility strain, whether in the small or in the large if only we take all the elements of the group of holonomy of the associated non-Riemannian manifold, into consideration.

5. Imperfections connected with the torsion and Riemann-Christoffel curvature, I

Take a quadrangle in the non-Riemannian space, the vertices of which are

$$x,\ x+\underset{1}{dx},\ x+\underset{1}{dx}+\underset{2}{dx},\ x+\underset{2}{dx}$$

respectively and consider the discrepancy transformation of the element of the holonomic group which accompanies the circuit along its periphery (Fig. 3 (b)).



(a) (b)

FIG. 3

This transformation consists of two parts, namely

(α) the translation

$$\Delta x^{\kappa}=2S_{\mu\lambda}^{\cdot\cdot\,\kappa}\underset{2}{\omega}^{\mu}\underset{1}{\omega}^{\lambda}$$

from the initial to the final end of this development (Fig. 2 (a));

(β) the orthogonal transformation of the local frames from the initial to the final configuration (Fig. 2 (b))

$$\Delta\mathbf{e}_{\lambda}=2\underset{[2\,1]}{dd}\mathbf{e}_{\lambda}=R_{\nu\mu\lambda}^{\cdot\cdot\cdot\,\kappa}\mathbf{e}_{\kappa}\underset{2}{\omega}^{\nu}\underset{1}{\omega}^{\mu}.$$

The calculations which are necessary to deduce these results easily follow from the definitions given in §1.

These can be summarized and confirm that a material vector which is

$$\boldsymbol{\xi}\ :\ \xi^{\kappa}$$

when attached at x, the initial point of circulation around this circuit, goes to

$$\bar{\bar{\boldsymbol{\xi}}}\ :\ \bar{\bar{\xi}}^{\kappa}=\xi^{\kappa}+R_{\nu\mu\lambda}^{\cdot\cdot\cdot\,\kappa}\xi^{\lambda}\underset{1}{\omega}^{\mu}\underset{2}{\omega}^{\nu}$$
$$+2S_{\mu\lambda}^{\cdot\cdot\,\kappa}\underset{1}{\omega}^{\lambda}\underset{2}{\omega}^{\mu}+O(\omega^{3}) \qquad (15)$$

when attached to the other end of the development figure. Hence the torsion term signifies the translation of the origin, and the curvature term the change of the material vector itself.

The result can be generalized for an arbitrary small circuit in the bad manifold and we obtain

$$\bar{\bar{\xi}}^{\kappa}-\xi^{\kappa}=(R_{\nu\mu\lambda}^{\cdot\cdot\cdot\,\kappa}\xi^{\lambda}+2S_{\nu\mu}^{\cdot\cdot\,\kappa})f^{\nu\mu}, \qquad (15\cdot1)$$

denoting the enscribed two-dimensional element by

$$\mathbf{f}\ :\ f^{\mu\lambda}.$$

It is obvious that the torsion tensor represents the density of dislocation per unit area of the two-dimensional element $f^{\mu\lambda}$ whose periphery forms a small Burgers circuit. Hence we conclude that the torsion tensor is a proper measure of the density of dislocation in the small.

Since the torsion tensor is defined by (7) and (8), it is essentially a rotational quantity

$$S_{ij}^{\cdot\cdot\,\kappa}\ :\ \nabla\times\mathbf{A}^{\kappa}$$

where

$$A_{i}^{\kappa}\ :\ \mathbf{A}^{\kappa}=\mathbf{i}\,A_{1}^{\kappa}+\mathbf{j}A_{2}^{\kappa}+\mathbf{k}A_{3}^{\kappa},$$

and

$$\nabla = \mathbf{i}\frac{\partial}{\partial x^1} + \mathbf{j}\frac{\partial}{\partial x^2} + \mathbf{k}\frac{\partial}{\partial x^3}.$$

Hence it is non-divergent or continuous and therefore cannot terminate within the crystal. This postulate is reflected in the non-divergent character of dislocation lines.

Now the curvature term in (15.1) indicates a possible type of defect which is distributed not only in the area of $f^{\mu\lambda}$ but also on the vector-line of ξ. This discrepancy increases in proportion to the distance along this vector from its root where it is attached to the small Burgers circuit. The vector ξ takes different configurations AB and $A'B'$ as it is attached to different ends A, A' of the developement figure of the circuit. Various types of defects are included. The most celebrated among them is the slip-plane accompanied by a dislocation line at one edge. Therefore the Riemann-Christoffel curvature is not responsible for the dislocations in the smallest, but it should be responsible for those of larger scale.

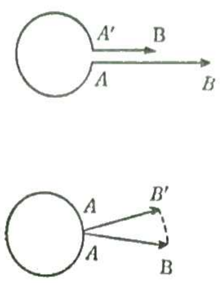


FIG. 4

We have investigated the incompatibility of the Riemannian character in C-I, which is related to the metric strain and therefore to the inter-molecular force. The curvature tensor of the affinely connected space should include metric imperfections of the same kind. The geometrical effects of the types (d) and (e) are of this kind. There may be other defects included which are metric but not necessarily atomic. For example, thermal strains can be produced as a result of plastic deformation.

Now, it is postulated by the Bianchi identity

$$R_{[\nu\mu|\lambda|,\rho]}^{\cdot\cdot\cdot\kappa} = 2S_{[\rho\nu}^{\cdot\cdot\tau}R_{\mu]\tau\lambda}^{\cdot\cdot\cdot\kappa} \qquad (16)^{1)}$$

which we obtain by calculating the absolute exterior derivative of

$$\Omega_{\lambda}^{\cdot\kappa} = \underset{1}{\omega}^{\mu}\underset{2}{\omega}^{\nu}R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}.$$

Therefore, if torsion imperfections are absent, the curvature imperfections are also subjected to a condition of continuity

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}{}_{,\rho} + R_{\rho\nu\lambda}^{\cdot\cdot\cdot\kappa}{}_{,\mu} + R_{\mu\rho\lambda}^{\cdot\cdot\cdot\kappa}{}_{,\nu} = 0. \qquad (16{\cdot}1)$$

In the general case, however, the sites of the torsion imperfections play the rôle of the sources of the curvature imperfections.

We should also think of imperfections more in the large; but we shall postpone it till later in § 7.

When torsion defects alone are considered, metric properties being discarded, the crystal-space apparently behaves as a space of distant parallelism[2], in which we can always define parallel vectors uniquely for any two distant points. We then proceed so that the change in affine (respectively: metric) properties has little to do with this observation of imperfections in the finite. Hence the crystal defects thus recognized are affine in-the-small, but topological in-the-finite.

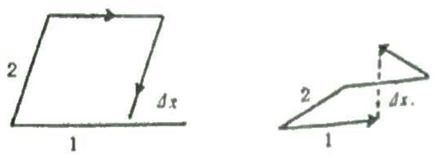
When we consider, on the contrary, the curvature incompatibility alone as we have done in C-I, discarding defects of small scale namely the torsion, the space apparently behaves as a Riemannian space. The corresponding imperfections are either affine, metric or topological in the finite.

6. Imperfections connected with the torsion and the Riemann-Christoffel curvature, II

The components of the torsion tensor are essentially classified into

(a) $S_{12}^{\cdot\cdot 1}$, $S_{12}^{\cdot\cdot 2}$; (b) $S_{12}^{\cdot\cdot 3}$.

It is evident that the former group signifies the density of the so-called "edge-dislocation" and the latter that of the "screw-dislocation" in the small (cf. Fig. 5). The classification is



(a) (b)

FGI. 5

1) See SDG I, p. 124, formula (11·60), [16]; also cf. C-I, § 11, [7]. Covariant differentiation is denoted by 'comma'.

2) Cf., for example, T.Y.Thomas' treatise [20].

connected with Pfaffian elements

$$\omega^1, \ \omega^2, \ \omega^3$$

and therefore suggests the change in affine property in the small.

More complicated are the curvature imperfections. The complications are partly due to their closer connexion with the (in-the-small) metric properties. But they are also concerned with the affine conditions in the larger scale than those in the scale of the torsion.

Considering that ξ can be taken as the vector along the slip plane from the edge and that

$$f^{[\nu\mu]} J_{\nu\mu}^{\cdot\cdot\kappa} = f^{[\nu\mu]} R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} \xi^\lambda$$

may be the discrepancy vector at the terminus of ξ, we may assign the following rôle to the indices

κ : the direction of the Burgers vector (the slip vector in particular),

λ : the direction of the vector in the slip plane,

$[\nu\mu]$: the orientation of the two-dimensional reference element enclosing the boundary of the slip-plane,

respectively. The following four types of the components can therefore be discriminated:

(a) $R_{121}^{\cdot\cdot\cdot 1}$, $R_{122}^{\cdot\cdot\cdot 2}$; (b) $R_{121}^{\cdot\cdot\cdot 3}$, $R_{122}^{\cdot\cdot\cdot 3}$

(c) $R_{123}^{\cdot\cdot\cdot 1}$; (d) $R_{123}^{\cdot\cdot\cdot 3}$.

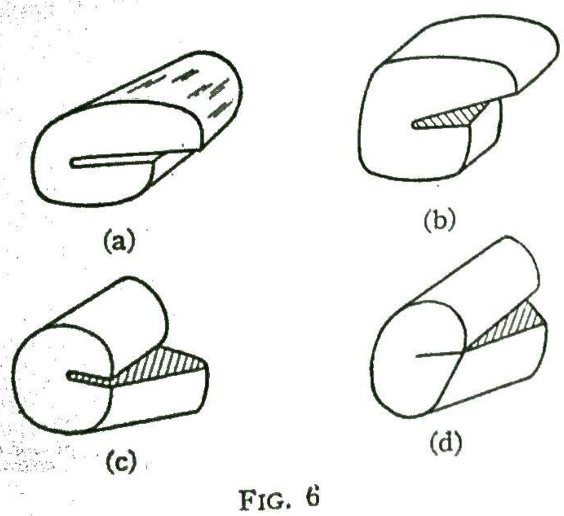


FIG. 6

It is evident that each component corresponds to a particular manner of distribution of the edge- or screw-dislocation in the respective orientation along vectors in the slip plane. The four different kinds of components can respectively signify the densities for

(a) the slip by an edge dislocation, per unit length not parallel to the edge and along the surface of slip of a dislocation,

(b) the slip by a screw dislocation per unit length, not parallel to the edge and along the surface of slip,

(c) the slip per unit length by edge dislocations linearly distributed along the axis,

(d) the slip per unit length by screw-dislocations linearly distributed along the axis, in the terminology of the usual theory of dislocations of metal-physicists. The respective configurations are sketched in Fig. 6.

Although the truth of our argument that the Riemann-Christoffel curvature tensor can reflect larger-scale dislocations against affine (and metric) conditions may be manifest from the above enunciations, a certain misunderstanding may arise. In fact, in the usual theory of dislocations, the discrepancies accompanying the Burgers circuit are defined based only on the discrete structure of a crystal. Hence they have been considered to be quantities in such a scale as to be comparable with the lattice vector which is, *a priori*, finite. We must, therefore, be careful when we undertake the translation of these quantities of finite scales into the infinitesimal conceptions of differential geometry.

The major misunderstanding which is first liable to happen, is that the curvature dislocations might be an imaginary addition from mathematics unnecessarily forged into the world of reality which should have been covered by the well-established theory of screw- and edge-dislocations. That the components of the torsion tensor can be classified into the edge and screw types may serve as a source of confusion producing an impression that the Riemann-Christoffel curvature tensor would have been something entirely out of the established realm of modern dislocation physics.

However, we should remind ourselves that we speak of the influence of the distribution of dislocations or of similar imperfections on a more global scale than in the crystal physics of lattice structures. In the more or less global observations of macroscopic plasticity, the dislocation conceptions are also measured by their average values and their discrete structures should only be reflected in the quantal aspects which we shall introduce into the behaviour of the torsion and curvature.

Furthermore, we have a certain empirical reason for not denying the intervention of the Riemann-Christoffel curvature tensor carelessly. In the terminology of physicists, the slip surface is the surface generated by lines that go through the vector line of dislocation parallel to the slip vector. Motion in the slip surface is called "glide" and motion normal to it is called "climb". It is now an established physical fact that the glide of dislocations is responsible for critical plastic deformations. It produces slip lines. This can correspond to the production of the Riemann-Christoffel curvature tensor but not necessarily of the torsion tensor. Therefore the criterion for the yielding of matter will be expressed mostly in terms of the Riemann-Christoffel curvature tensor.

7. The Euler-Schouten curvature tensor

We have pointed out two non-Euclidean differential invariants, namely the Riemann-Christoffel curvature and the torsion, as responsible for the disorder of the crystal structure. However, we have never declared that these intrinsic properties of matter elements, torsion and local curvature, are responsible for all sorts of anomalies in a wider sense than that usually preferred by dislocationists. Eventually, every non-identity element of the holonomic group, if existing, is responsible for a certain discrepancy of the whole crystal from the normal regular configuration. We shall proceed to investigate the related questions in this section.

In this connexion, Cartan's original article includes a very suggestive description in its last chapter, in reference to the possibility of a discontinuous holonomic group. He also counts a number of locally Euclidean spaces which have non-trivial holonomic groups. These mathematical anomalies indicate global non-flatnesses in spite of the locally Euclidean character. We find a most visualizable classical example in the two-dimensional case in the circular cylinder, which is undoubtedly locally flat but curved in three dimensions. Its holonomic group is not continuous but countably infinite, being composed of all possible multiples of a definite translation. It cannot be extended on a plane without cutting it off along a line. We have means mathematically to express the deviation of this sort of curved surface from a Euclidean plane which is rectilinearly imbedded in an enveloping Euclidean 3-space. They are, namely, the normal curvatures in the elementary differential geometry of surfaces. The Riemann-Christoffel or Gaussian curvature vanishes for cylindrical or conical surfaces, despite the finite normal curvature. We can expect the same for higher dimension. We shall therefore look for a certain curvature of our space of matter as responsible for its non-flatness in the large or in reference to the enveloping space, if such exists.

Conveniently for the present purpose, we have been able, in § 3, to connect our object of study with a Riemannian 3-space imbedded in a Euclidean 6-space. Its non-flatness in reference to the enveloping flat space is responsible for a measure of imperfection. The remaining part has been embodied in the non-holonomic transformation of coordinates. Since the torsion has originated from (7) as in (8), the geometrical invariants of the smooth subspace are embodied in imperfections other than the torsional, wherefore we obtain Riemannian geometry.

It is only a series of routine processes that is indispensable for the calculation of the expressions of the Riemann-Christoffel tensor R_{lkji} from (15. 1), in terms of the covariant derivatives of the Cartesian coordinates X^{A} of the enveloping space. We obtain

$$R_{lkji}=\sum_{A=1}^{N} X^{A}{}_{,i[k} X^{A}{}_{,|j|l]} \tag{17}$$

as the result. The detailed treatment concerned is carried out in the article C-IV [21] of Division C as a part of a more comprehensive problem of subspaces in Riemannian space.

The quantity $H_{jk}^{\cdot\cdot A}$ defined by

$$H_{jk}^{\cdot\cdot A}= X^{A}{}_{,kj}(=X^{A}{}_{,k,j})$$

is a multidimensional analogue of the normal curvature of the two-dimensional curved surface and is called after Euler and Schouten.

It follows that

(i) Non-vanishing Euler-Schouten curvature appears whenever there is non-vanishing Riemann-Christoffel curvature;

(ii) Non-vanishing Euler-Schouten curvature can be admitted even if the Riemann-Christoffel curvature vanishes;

(iii) If the subspace is also Euclidean, and *(i)* is a Cartesian coordinate system, we have

$$H_{ji}^{\cdot\cdot A} \overset{*}{=} \frac{\partial^2 X^A}{\partial x^j \partial x^i} = 0.$$

Hence we see that vanishing of the Euler-Schouten curvature is the necessary and sufficient criterion for the flatness of the immersed space in reference to the enveloping space, in the small as well as in the large. But this is not necessarily so for the intrinsic flatness in the small, i.e. for the vanishing of the Riemann-Christoffel curvature.

Therefore the deviation of a Riemannian manifold from a space which is flat both in the small and in the large, should be measured by the amount of the Euler-Schouten curvature. This is also sufficient for the non-flatness in the small. It also is necessary and sufficient for invariant formulation, for $H_{ji}^{\cdot\cdot A}$ always behaves as a tensor quantity in regard to the transformation of coordinates in the subspace.

Thus, finally, we conclude that all the deviations from the regular compatible configuration of a crystal have been geometrically represented by the distribution of the **Euler-Schouten curvature tensor for the curvature-like imperfection in the small as well as in the large** and by the **non-holonomic object of coordinate transformation for the torsion-like imperfections.**

REFERENCES

[1] K. Kondo, A Proposal of a New Theory concerning the Yielding of Materials based on Riemannian Geometry, I. II. *Journal of the Japan Society for Applied Mechanics.* 2, 11, 12 (1949), 123~128; 146~151.

[2] K. Kondo, On Dislocations, the Group of Holonomy and the Theory of Yielding. Ibid., 3, 17 (1950), 107~110.

[3] K. Kondo, On the Fundamental Equations of the Theory of Yielding. Ibid., 3, 20 (1950) 184~188.

[4] K. Kondo, On the Geometrical and Physical Foundations of the Theory of Yielding. *Proceedings of the 2nd Japan National Congress for Applied Mechanics, held* 1952, (published 1953), 41~47.

[5] K. Kondo, Non-Holonomic Foundations of the Theory of Plasticity and Yielding. *Memoirs,* **1**, D-V (1955), 522~562.

[6] F. C. Frank, Crystal Dislocations—Elementary Concepts and Definitions. *Philosophical Magazine,* (A) Ser. 7, **42**, 33 (1951), 809~819.

[7] K. Kondo, Geometry of Elastic Deformation and Incompatibility. *Memoirs,* **1**, C-I (1955), 361~373.

[8] É. Cartan, *Leçon sur la géometrie des espace de Riemann.* Gauthier-Villars, Paris, 1928.

[9] K. Kondo, On the Fundamental Equations of the Macroscopic Mechanical Behaviour of Microscopically Non-Uniform Materials. *Memoirs,* **1**, D-II (1955), 470~483.

[10] K. Kondo, Energy at Plastic Deformation and the Criterion for Yielding. *Memoirs,* **1**, D-III (1955), 484~494.

[11] F. Seitz and T. A. Read, Theory of the Plastic Properties of Solids, I, II, III, IV. *Journal of Applied Physics,* **12**, 2, 3, 6, 7 (1941), 100~118, 170~186, 470~486, 538~554.

[12] A. H. Cottrell, Theory of Dislocations. *Progress in Metal Physics, I.* Butterworths Scientific Publications, London, 1949.

[13] W. T. Read, *Dislocations in Crystals.* McGraw-Hill, New York, Toronto, London, 1953.

[14] W. Schockley, J. H. Hollomann, R. Manner and F. Seitz, Imperfections in Nearly Perfect Crystals. *Symposium held at Pocono Manor,* John Wiley & Sons, New York, 1950.

[15] É. Cartan, Les groupes d'holonomie des espaces généralisés. *Acta Mathematica,* 48 (1926), 1~42.

[16] J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differential Geometrie* (SDG), *I, II.* Noordhoff, Groningen-Batavia, 1935.

[17] L. F. Eisenhart, *Riemannian Geometry.* Princeton University Press, second printing, 1949.

[18] J. M. Burgers, Some Considerations on the Fields of Stress connected with Dislocations in a Regular Crystal Lattice, I, II. *Proceedings, Koninklijke Nederlandse Akademie van Wetenschappen,* **42** (1939), 293~325; 378~399.

[19] O. Veblen and J. H. C. Whitehead, *The Foundation of Differential Geometry.* Cambridge, University Press, 1932.

[20] T. Y. Thomas, *The Differential Invariants of Generalized Spaces.* Cambridge, University Press, 1934.

[21] K. Kondo, Theory of Metaphorical Plates and Shells. *Memoirs,* **1**, C-IV (1955), 403~416.