RESEARCH PAPER

# Exponential arcs in the manifold of vector states on a $\sigma$-finite von Neumann algebra

Jan Naudts[1]

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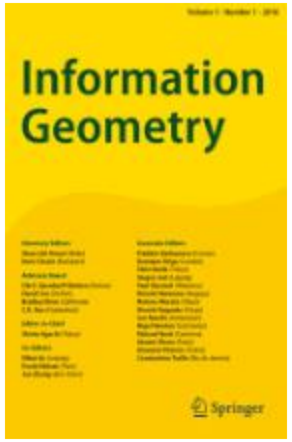
Published online: 5 January 2022


## Abstract

This paper introduces the notion of exponential arcs in Hilbert spaces and of exponential arcs connecting vector states on a sigma-finite von Neumann algebra in its standard representation. Results from Tomita–Takesaki theory form an essential ingredient. Starting point is a non-commutative Radon–Nikodym theorem that involves positive operators affiliated with the commutant algebra. It is shown that exponential arcs are differentiable and that parts of an exponential arc are again exponential arcs. Special cases of probability theory and of quantum probability are used to illustrate the approach.

**Keywords** Exponential arc · Exponential family · Tomita–Takesaki theory · Information geometry · Probability theory · Quantum probability

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RESEARCH PAPER

# On a constant curvature statistical manifold

Shimpei Kobayashi[1] · Yu Ohno[1]

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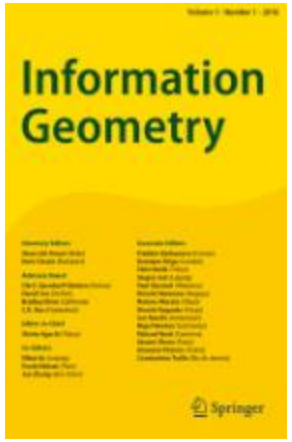


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**Abstract**
We will show that a statistical manifold $(M, g, \nabla)$ has a constant curvature if and only if it is a projectively flat conjugate symmetric manifold, that is, the affine connection $\nabla$ is projectively flat and the curvatures satisfies $R = R^*$, where $R^*$ is the curvature of the dual connection $\nabla^*$. Moreover, we will show that properly convex structures on a projectively flat compact manifold induces constant curvature $-1$ statistical structures and vice versa.

**Keywords** Statistical manifolds · constant curvatures · Conjugate symmetries · Projective flatness · Properly convex structures

SURVEY PAPER

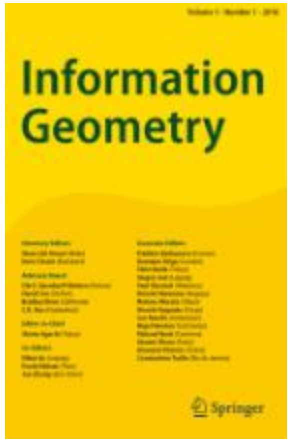
# When optimal transport meets information geometry

Gabriel Khan[1] · Jun Zhang[2]

Received: 22 March 2022 / Revised: 16 May 2022 / Accepted: 19 May 2022 /
Published online: 30 June 2022


## Abstract
Information geometry and optimal transport are two distinct geometric frameworks for modeling families of probability measures. During the recent years, there has been a surge of research endeavors that cut across these two areas and explore their links and interactions. This paper is intended to provide an (incomplete) survey of these works, including entropy-regularized transport, divergence functions arising from $c$-duality, density manifolds and transport information geometry, the para-Kähler and Kähler geometries underlying optimal transport and the regularity theory for its solutions. Some outstanding questions that would be of interest to audience of both these two disciplines are posed. Our piece also serves as an introduction to the Special Issue on Optimal Transport of the journal *Information Geometry*.

**Keywords** Entropy-regulated transport · $c$-duality and divergence function · Kähler and para-Kähler geometries

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RESEARCH PAPER

# Optimal transportation plans with escort entropy regularization

Takashi Kurose[1] · Shintaro Yoshizawa[2] · Shun-ichi Amari[3,4]

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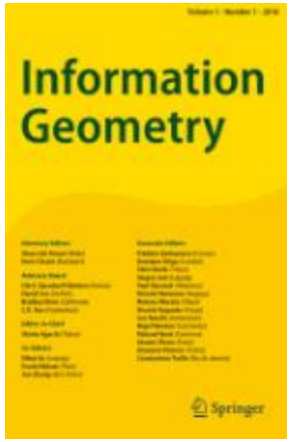
Published online: 9 November 2021


**Abstract**
The entropy-regularized Wasserstein transportation problem is useful for solving lots of problems in various applications. We study the deformed escort entropy regularization including the $q$-escort regularization and prove that the family of the inverse escort distributions of the optimal transportation plans forms a deformed exponential family, which has dually flat information-geometric structure. This elucidates the role of the escort transformation and its inverse in the theory of deformed exponential families. We further prove that the regularized cost function gives the dual potential of the flat manifold. We derive a new divergence function between two probability distributions in a probability simplex and a related Riemannian metric based on the regularized cost.

**Keywords** Wasserstein geometry · Deformed exponential family · Generalized entropy regularization · Dually flat manifold

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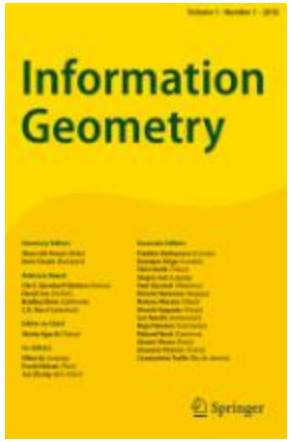


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# Optimal transport problems regularized by generic convex functions: a geometric and algorithmic approach

Daiji Tsutsui[1]

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## Abstract

In order to circumvent the difficulties in solving numerically the discrete optimal transport problem, in which one minimizes the linear target function $P \mapsto \langle C, P\rangle := \sum_{i,j} C_{ij} P_{ij}$, Cuturi introduced a variant of the problem in which the target function is altered by a convex one $\Phi(P) = \langle C, P\rangle - \lambda\mathcal{H}(P)$, where $\mathcal{H}$ is the Shannon entropy and $\lambda$ is a positive constant. We herein generalize their formulation to a target function of the form $\Phi(P) = \langle C, P\rangle + \lambda f(P)$, where $f$ is a generic strictly convex smooth function. We also propose an iterative method for finding a numerical solution, and clarify that the proposed method is particularly efficient when $f(P) = \frac{1}{2}\|P\|^2$.

**Keywords** Information geometry · Discrete optimal transport · Entropic regularization · Convex optimization · Wasserstein barycenter

RESEARCH PAPER

# Pseudo-Riemannian geometry encodes information geometry in optimal transport

Ting-Kam Leonard Wong[1] · Jiaowen Yang[2]

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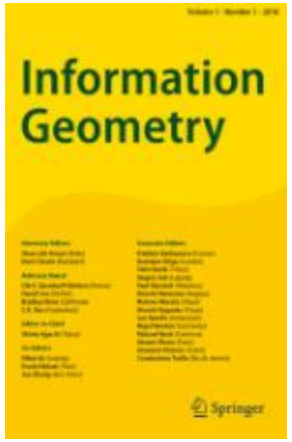
Published online: 30 July 2021


**Abstract**
Optimal transport and information geometry both study geometric structures on spaces of probability distributions. Optimal transport characterizes the cost-minimizing movement from one distribution to another, while information geometry originates from coordinate invariant properties of statistical inference. Their relations and applications in statistics and machine learning have started to gain more attention. In this paper we give a new differential-geometric relation between the two fields. Namely, the pseudo-Riemannian framework of Kim and McCann, which provides a geometric perspective on the fundamental Ma–Trudinger–Wang (MTW) condition in the regularity theory of optimal transport maps, encodes the dualistic structure of statistical manifold. This general relation is described using the framework of $c$-divergence under which divergences are defined by optimal transport maps. As a by-product, we obtain a new information-geometric interpretation of the MTW tensor on the graph of the transport map. This relation sheds light on old and new aspects of information geometry. The dually flat geometry of Bregman divergence corresponds to the quadratic cost and the pseudo-Euclidean space, and the logarithmic $L^{(\alpha)}$-divergence introduced by Pal and the first author has constant sectional curvature in a sense to be made precise. In these cases we give a geometric interpretation of the information-geometric curvature in terms of the divergence between a primal-dual pair of geodesics.

**Keywords** Optimal transport · Information geometry · Pseudo-Riemannian geometry · $c$-Divergence · Logarithmic divergence · Bregman divergence · Ma–Trudinger–Wang tensor

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RESEARCH PAPER

# Transport information geometry: Riemannian calculus on probability simplex

Wuchen Li[1]

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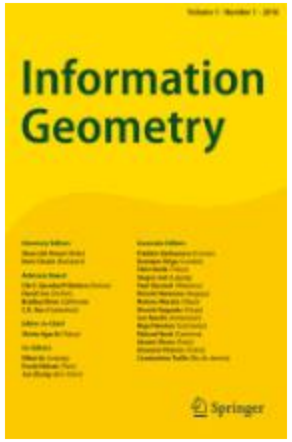
Published online: 8 November 2021


**Abstract**
We formulate the Riemannian calculus of the probability set embedded with $L^2$-Wasserstein metric. This is an initial work of transport information geometry. Our investigation starts with the probability simplex (probability manifold) supported on vertices of a finite graph. The main idea is to embed the probability manifold as a submanifold of the positive measure space with a weighted graph Laplacian operator. By this viewpoint, we establish torsion–free Christoffel symbols, Levi–Civita connections, curvature tensors and volume forms in the probability manifold by Euclidean coordinates. As a consequence, the Jacobi equation, Laplace-Beltrami, Hessian operators and diffusion processes on the probability manifold are derived. These geometric computations are also provided in the infinite-dimensional density space (density manifold) supported on a finite-dimensional manifold. In particular, we present an identity connecting among Baker–Émery $\Gamma_2$ operator (carré du champ itéré), Fisher–Rao metric and optimal transport metric. Several examples are demonstrated.

**Keywords** Optimal transport · Information geometry · Probability manifold · Linear weighted Laplacian · Graph

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RESEARCH PAPER

# Kantorovich distance on finite metric spaces: Arens–Eells norm and CUT norms

Luigi Montrucchio[1] · Giovanni Pistone[2]

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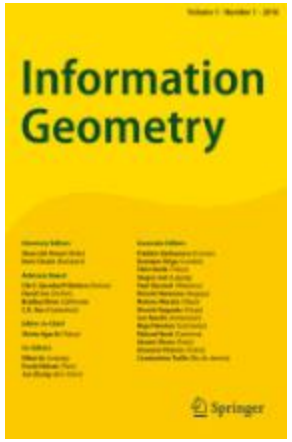
Published online: 1 July 2021


## Abstract

We study the possible closed-form representations of the K-distance arising in the Kantorovich transport problems on finite metric spaces. Weighted graphs, $\ell_1$-embeddable metrics, and the related CUT norms receive special attention. Following an in-depth analysis of weighted trees and the tree-like spaces, we treat general metric spaces through their spanning trees.

**Keywords** Optimal transport · Kantorovich distance · Arens–Eells space · Finite metric space · Tree · Tree-like space · Spanning tree · Cut metric · $\ell_1$-Embeddable space · Quotient map

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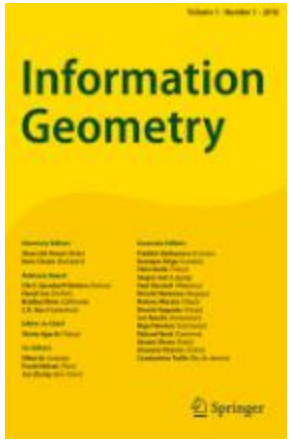
# Tropical optimal transport and Wasserstein distances

Wonjun Lee[1] · Wuchen Li[2] · Bo Lin[3] · Anthea Monod[4]

Received: 7 January 2020 / Revised: 18 May 2021 / Accepted: 20 May 2021 / Published online: 7 June 2021


**Abstract**
We study the problem of optimal transport in tropical geometry and define the Wasserstein-$p$ distances in the continuous metric measure space setting of the tropical projective torus. We specify the tropical metric—a combinatorial metric that has been used to study of the tropical geometric space of phylogenetic trees—as the ground metric and study the cases of $p = 1, 2$ in detail. The case of $p = 1$ gives an efficient computation of the infinitely-many geodesics on the tropical projective torus, while the case of $p = 2$ gives a form for Fréchet means and a general inner product structure. Our results also provide theoretical foundations for geometric insight a statistical framework in a tropical geometric setting. We construct explicit algorithms for the computation of the tropical Wasserstein-1 and 2 distances and prove their convergence. Our results provide the first study of the Wasserstein distances and optimal transport in tropical geometry. Several numerical examples are provided.

**Keywords** Optimal transport · Tropical geometry · Tropical metric · Tropical projective torus · Wasserstein distances

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# Entropy-regularized 2-Wasserstein distance between Gaussian measures

Anton Mallasto[1] · Augusto Gerolin[2] · Hà Quang Minh[3]

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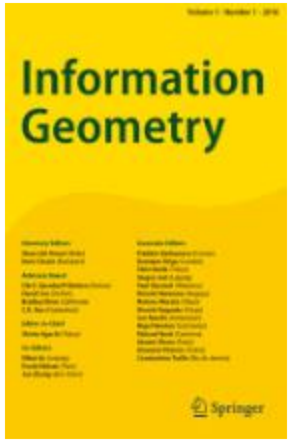
Published online: 16 August 2021


## Abstract
Gaussian distributions are plentiful in applications dealing in uncertainty quantification and diffusivity. They furthermore stand as important special cases for frameworks providing geometries for probability measures, as the resulting geometry on Gaussians is often expressible in closed-form under the frameworks. In this work, we study the Gaussian geometry under the entropy-regularized 2-Wasserstein distance, by providing closed-form solutions for the distance and interpolations between elements. Furthermore, we provide a fixed-point characterization of a population barycenter when restricted to the manifold of Gaussians, which allows computations through the fixed-point iteration algorithm. As a consequence, the results yield closed-form expressions for the 2-Sinkhorn divergence. As the geometries change by varying the regularization magnitude, we study the limiting cases of vanishing and infinite magnitudes, reconfirming well-known results on the limits of the Sinkhorn divergence. Finally, we illustrate the resulting geometries with a numerical study.

**Keywords** Sinkhorn divergences · Multivariate Gaussian measures · Optimal transportation theory

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# Coordinate-wise transformation of probability distributions to achieve a Stein-type identity

Tomonari Sei[1]

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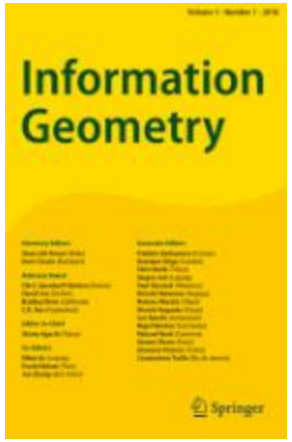
Published online: 28 June 2021


**Abstract**
It is shown that for any given multi-dimensional probability distribution with regularity conditions, there exists a unique coordinate-wise transformation such that the transformed distribution satisfies a Stein-type identity. A sufficient condition for the existence is referred to as copositivity of distributions. The proof is based on an energy minimization problem over a totally geodesic subset of the Wasserstein space. The result is considered as an alternative to Sklar's theorem regarding copulas, and is also interpreted as a generalization of a diagonal scaling theorem. The Stein-type identity is applied to a rating problem of multivariate data. A numerical procedure for piece-wise uniform densities is provided. Some open problems are also discussed.

**Keywords** Copositive distribution · Copula · Energy minimization · Optimal transportation · Stein-type distribution · Wasserstein space

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