RESEARCH PAPER

Assignment flows for data labeling on graphs: convergence and stability

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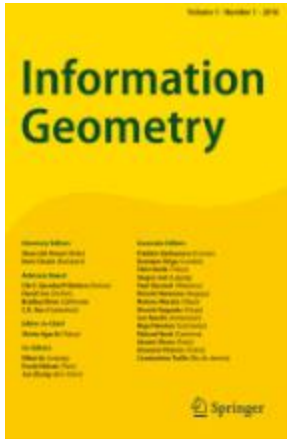
Abstract
The assignment flow recently introduced in the *J. Math. Imaging and Vision* 58/2 (2017) constitutes a high-dimensional dynamical system that evolves on a statistical product manifold and performs contextual labeling (classification) of data given in a metric space. Vertices of an underlying corresponding graph index the data points and define a system of neighborhoods. These neighborhoods together with nonnegative weight parameters define the regularization of the evolution of label assignments to data points, through geometric averaging induced by the affine e-connection of information geometry. From the point of view of evolutionary game dynamics, the assignment flow may be characterized as a large system of replicator equations that are coupled by geometric averaging. This paper establishes conditions on the weight parameters that guarantee convergence of the continuous-time assignment flow to integral assignments (labelings), up to a negligible subset of situations that will not be encountered when working with real data in practice. Furthermore, we classify attractors of the flow and quantify corresponding basins of attraction. This provides convergence guarantees for the assignment flow which are extended to the discrete-time assignment flow that results from applying a Runge–Kutta–Munthe–Kaas scheme for the numerical geometric integration of the assignment flow. Several counter-examples illustrate that violating the conditions may entail unfavorable behavior of the assignment flow regarding contextual data classification.

Keywords Assignment flow · Image and data labeling · Replicator equation · Evolutionary game dynamics · Information geometry



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Bures–Wasserstein geometry for positive-definite Hermitian matrices and their trace-one subset

Jesse van Oostrum[1]

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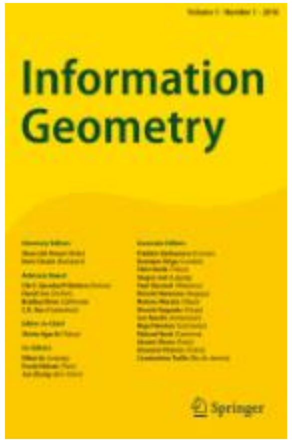
Abstract
In his classical argument, Rao derives the Riemannian distance corresponding to the Fisher metric using a mapping between the space of positive measures and Euclidean space. He obtains the Hellinger distance on the full space of measures and the Fisher distance on the subset of probability measures. In order to highlight the interplay between Fisher theory and quantum information theory, we extend this construction to the space of positive-definite Hermitian matrices using Riemannian submersions and quotient manifolds. The analog of the Hellinger distance turns out to be the Bures–Wasserstein (BW) distance, a distance measure appearing in optimal transport, quantum information, and optimisation theory. First we present an existing derivation of the Riemannian metric and geodesics associated with this distance. Subsequently, we present a novel derivation of the Riemannian distance and geodesics for this metric on the subset of trace-one matrices, analogous to the Fisher distance for probability measures.

Keywords Information geometry · Positive-definite matrices · Bures distance · Wasserstein metric · Optimal transport · Quantum information



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The information geometry of two-field functional integrals

Eric Smith[1,2,3,4]

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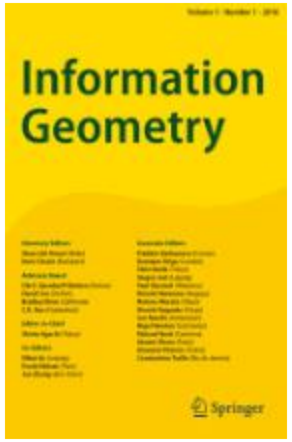
Abstract

Two-field functional integrals (2FFI) are an important class of solution methods for generating functions of dissipative processes, including discrete-state stochastic processes, dissipative dynamical systems, and decohering quantum densities. The stationary trajectories of these integrals describe a conserved current by Liouville's theorem, despite the absence of a conserved kinematic phase space current in the underlying stochastic process. We develop the information geometry of generating functions for discrete-state classical stochastic processes in the Doi-Peliti 2FFI form, and exhibit two quantities conserved along stationary trajectories. One is a Wigner function, familiar as a semiclassical density from quantum-mechanical time-dependent density-matrix methods. The second is an overlap function, between directions of variation in an underlying distribution and those in the directions of relative large-deviation probability that can be used to interrogate the distribution, and expressed as an inner product of vector fields in the Fisher information metric. To give an interpretation to the time invertibility implied by current conservation, we use generating functions to represent importance sampling protocols, and show that the conserved Fisher information is the differential of a sample volume under deformations of the nominal distribution and the likelihood ratio. We derive a pair of dual affine connections particular to Doi-Peliti theory for the way they separate the roles of the nominal distribution and likelihood ratio, distinguishing them from the standard dually-flat connection of Nagaoka and Amari defined on the importance distribution, and show that dual flatness in the affine coordinates of the coherent-state basis captures the special role played by coherent states in Doi-Peliti theory.

Keywords Information geometry · Doi-Peliti theory · Liouville's theorem · Fisher information · importance sampling · duality



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A method to construct exponential families by representation theory

Koichi Tojo[1] · Taro Yoshino[2]

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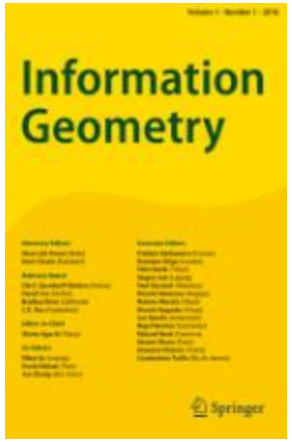
Abstract
In this paper, we give a method to construct "good" exponential families systematically by representation theory. More precisely, we consider a homogeneous space G/H as a sample space and construct an exponential family invariant under the transformation group G by using a representation of G. The method generates widely used exponential families such as normal, gamma, Bernoulli, categorical, Wishart, von Mises, Fisher–Bingham and hyperboloid distributions.

Keywords Exponential family · Representation theory · Homogeneous space · Transformation model · Harmonic exponential family



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One-dimensional exponential families with constant Hessian sectional curvature

Mathieu Molitor[1]

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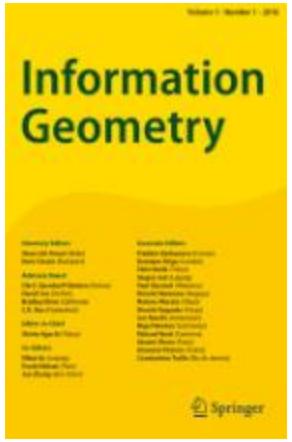

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Abstract
The concept of Hessian sectional curvature of a Hessian manifold M was introduced by Shima as a real-analogue of the holomorphic sectional curvature in Kähler geometry [15]. The former, unlike the latter, is also well-defined when the real dimension of M is 1. In this case, the Hessian sectional curvature is just a real-valued function on M. In this paper, we give a complete classification of 1-dimensional exponential families $\mathscr{E}$ defined over a finite set $\Omega = \{x_0, \ldots, x_m\}$ whose Hessian sectional curvature is constant. We observe an interesting phenomenon: if $\mathscr{E}$ has constant Hessian sectional curvature, say λ, then $\lambda = -\frac{1}{k}$ for some integer $1 \leq k \leq m$. We show that the family of Binomial distributions plays a central role in this classification.

Keywords Exponential families · Constant Hessian sectional curvature