On balls in a Hilbert polygonal geometry

Frank Nielsen[1] and Laëtitia Shao[2]

1 **École Polytechnique, LIX**
Palaiseau, France
`Frank.Nielsen@acm.org`
2 **École Polytechnique**
Palaiseau, France
`Laetitia.Shao@polytechnique.edu`

Abstract

Hilbert geometry is a metric geometry that extends the hyperbolic Cayley-Klein geometry. In this video, we describe the shape of balls in a polygonal Hilbert geometry and explain their main features. First, we study the combinatorial properties of Hilbert balls that have Euclidean polygonal shapes: We show that the shapes of Hilbert balls depend both on the center location and on the complexity of the Hilbert polygonal domain but not on their radii. We report an explicit construction of the Hilbert ball for any given center and radius. We then study the intersection of two Hilbert balls. In particular, we consider the cases of empty intersection and tangencies.

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1 Introduction to Hilbert geometry

Hilbert geometry is a projective geometry relying on the properties of the cross-ratio:

▶ **Definition 1** (Cross-ratio). For four collinear points a, b, c, d the cross ratio is defined as follows:

$$(a, b; c, d) = \frac{\|ac\|\|bd\|}{\|ad\|\|bc\|} \tag{1}$$

The cross-ratio is an invariant measure under perspective transformation [4]:

▶ Property 1 (Projective invariance of the cross-ratio). Given four points a, b, c, d and A, B, C, D their images through a projective transformation, we have $(a, b; c, d) = (A, B; C, D)$.

In a Hilbert geometry, the distance between two points is defined using the cross-ratio as follows:

▶ **Definition 2** (Hilbert distance). A Hilbert distance is defined in the interior of a convex bounded domain. Given two points, a and b of the domain, the distance is defined by:

$$d_{HG}(a, b) = \log((a, b; A, B)), \tag{2}$$

where $(a, b; A, B)$ is the cross-ratio with A and B denoting the intersection points of line (a, b) with the domain.

Hilbert distances generalize the hyperbolic distance and satisfy the metric properties:

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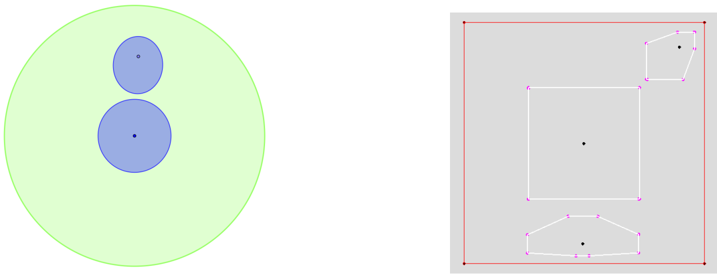

Figure 1 Left: In blue, two Hilbert balls in a circular domain. Right: In white, three Hilbert balls in a polygonal convex domain.

▶ Property 2 (Properties of the Hilbert distance). Given two points a and b:

- The Hilbert distance is a *signed distance*: $d_{HG}(a,b) = -d_{HG}(b,a)$.
- $d_{HG}(a,a) = 0$ (law of the indiscernibles).
- When a lies on the boundary of the convex domain, $d_{HG}(a,b) = +\infty$.
- $|d_{HG}|$ satisfies the triangular inequality, and therefore $|d_{HG}|$ is a metric distance [1].

A key property of Hilbert geometry is that shortest-path geodesics are (Euclidean) *straight lines*. This is computationally nice in applications [4]. The Klein disk representation of hyperbolic geometry is an example of Hilbert geometry with the convex domain being the unit disk (with smooth border).

In this work, we consider convex *polygonal* Hilbert geometries, and let $\mathcal{C}$ denote the convex polygonal domain and $e_1, \ldots, e_s$ the s vertices describing this convex polygon.
The distance between two points p and q in this domain is denoted by $d_{\mathcal{C}}(p,q)$, and the ball of radius r and center c shall be denoted by $\mathcal{B}(c,r)$.

2 Combinatorial properties of Hilbert balls

In Klein hyperbolic geometry (with $\mathcal{C}$ a unit ball) or Cayley-Klein hyperbolic geometry (with $\mathcal{C}$ a quadric [4]), the balls have the shape of (Euclidean) Mahalanobis balls with displaced centers, see [2, 3] for details. To contrast with this smooth shape representation of Hilbert balls, let us observe that when the domain is a convex polygon, the shapes of Hilbert balls are (Euclidean) convex polygons.

▶ **Lemma 3** (Description of a Hilbert ball). *$\mathcal{B}(c,r)$ has a Euclidean polygon shape with at most $2s$ edges and at least s edges. Each vertex of $\mathcal{B}(c,r)$ lies on a line $(c,e_i), i \in [s] = \{1,\ldots,s\}$.*

Proof. We first partition the polygonal domain with triangles anchored at c, by tracing all rays $(c,e_i), i \in [s]$. This yields a triangular decomposition of the domain with the number of triangles ranging from s to $2s$, depending on the center c. We will show that each triangle induces a linear edge of the Hilbert ball.
We consider a pair of triangles (A,B,c) and (c,C,D) such that A,c,D and B,c,C are respectively collinear. Let $P \in [A,c] \cap \mathcal{B}(c,r)$ and $O = (A,B) \cap (C,D)$, we will show that line (O,P) clipped to the triangle (A,B,c) is an edge of $\mathcal{B}(c,r)$. Let U be a point on the clipped line, and M,N the intersections points of line (Uc) with the domain such that $M \in [A,B]$ and $N \in [C,D]$. Then M,U,c,N and A,P,c,D are related by the same projective transformation. Using the invariance property of the cross-ratio, we conclude that $d_{\mathcal{C}}(c,P) = d_{\mathcal{C}}(c,U) = r$.
It is remarkable characteristic of Hilbert geometry that depending on the position of the center, the number of triangles (and hence the complexity of the ball) varies. ◀

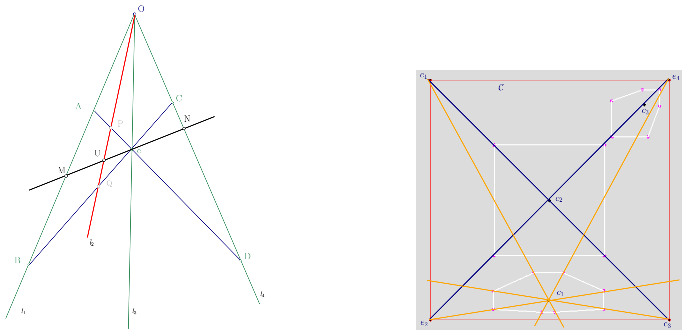

Figure 2 Left: Configuration for proof 2 (see text for details). Right: Varying number of rays in a square domain depending on the position of the center of the ball.

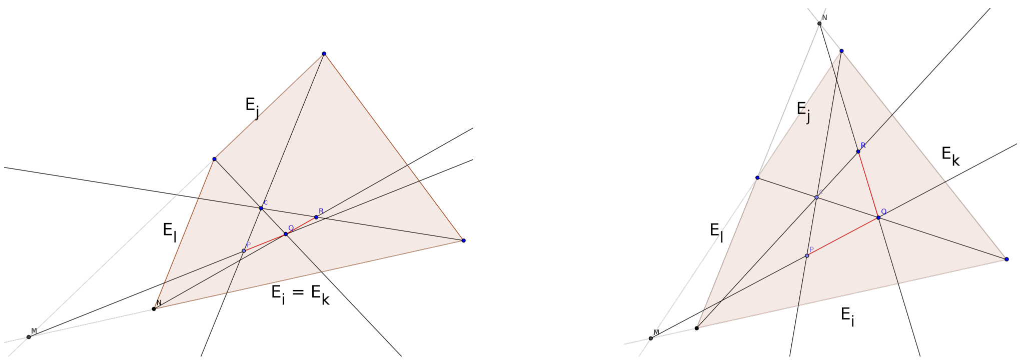

Figure 3 Left: Configuration for proof 2 when $E_i = E_k$. Right: Configuration for proof 2 when all edges are distinct.

▶ **Definition 4.** Given an edge $[P, Q]$ of a Hilbert ball that belongs to a pair of triangles (A, B, c) and (c, D, E), we say that $[P, Q]$ is induced by edges E_i and E_j of the domain, if $[A, B] \subset E_i$ and $[D, E] \subset E_j$.

▶ **Lemma 5** (Shape invariance with varying radius). *For c a fixed center point, and r a varying radius, $\mathcal{B}(c, r)$ has the same number of edges.*

Proof. Let $[P, Q]$ and $[Q, R]$ be two adjacent edges of a Hilbert ball such that E_i, E_j induces $[P, Q]$ and E_k, E_l induces $[Q, R]$. We show that P, Q, R cannot be collinear. We note M the intersection points of the lines supported by E_i and E_j and N the intersection points of the lines supported by E_k and E_l. According to the previous proof, P, Q, M and Q, R, N are respectively collinear.

- If E_i, E_j, E_k, E_l are distinct edges, because $[P, Q]$ and $[Q, R]$ are adjacent, we can assume without loss of generality that E_i is adjacent to E_k and E_j is adjacent to E_l. If P, Q, R are collinear, then $E_i = E_k$ or $E_j = E_l$, which contradicts the previous assumption.
- Otherwise, we can assume that $E_i = E_k$. In this case, if P, Q, R are collinear, then they belong to line $(M, N) \subset E_i$. This is *impossible* unless $r = \infty$.

Therefore, as the radius varies but stay finite, the number of edges remains constant. See Figure 3 for a visualization of the proof. For infinite radius, all Hilbert balls fully cover the polygonal domain. ◀

Finally, let us notice that Hilbert ball in simplices have invariant complexity:

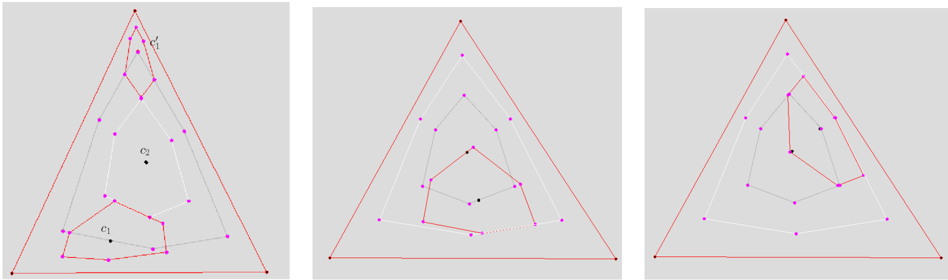

Figure 4 Left: Two cases of outer tangency. In white:$\mathcal{B}(c_2, 1)$, in grey: $\mathcal{B}(c_2, 1.5)$. $\mathcal{B}(c_1', 0.5)$ is in red and shares a vertex with $\mathcal{B}(c_2, 1)$. $\mathcal{B}(c_1, 0.5)$ is in red and shares part of an edge with $\mathcal{B}(c_2, 1)$. Middle and Right: Two cases of inner tangency between the red ball and the white ball. Middle: The two balls share one edge. Right: the two balls share two edges.

▶ **Lemma 6** (Shape invariance in a simplex domain). *In a simplex domain* Δ*, Hilbert polygonal balls do not change shape, and have a fixed complexity of* $2s$ *edges.*

Proof. It is a direct consequence of the two previous lemmata. ◀

3 Intersection of Hilbert balls

We now consider the interaction scenarii of two Hilbert balls. First, let us mention a simple condition to check whether two Hilbert balls intersect or not:

▶ **Lemma 7** (Condition for empty intersection). *Given two points* $c_1, c_2 \in \mathcal{C}$ *and two reals* $r_1, r_2 > 0$*, with* $r_2 \geq r_1$*:*

$$d_{\mathcal{C}}(c_1, c_2) < |r_1 - r_2| \text{ or } d_{\mathcal{C}}(c_1, c_2) > r_1 + r_2 \Rightarrow \mathcal{B}(c_1, r_1) \cap \mathcal{B}(c_2, r_2) = \emptyset \tag{3}$$

Proof. This follows straightforwardly from the triangular inequality [1]. ◀

In the case of *external tangency*, *i. e.*, $d_{\mathcal{C}}(c_1, c_2) = r_1 + r_2$, if c_2 is a vertex of $\mathcal{B}(c_1, r_1 + r_2)$, the intersection of the two Hilbert balls is reduced to a vertex. Otherwise, the two Hilbert Balls share part of *one* edge.
In the case of *internal tangency*, *i. e.*, $d_{\mathcal{C}}(c_1, c_2) = |r_2 - r_1|$, if c_1 is a vertex of $\mathcal{B}(c_2, r_2 - r_1)$, the two balls share part of *two* edges. Otherwise the shared part is *one* edge. See Figure 4 and video for some illustrating examples.

References

1. Curtis T McMullen. Coxeter groups, salem numbers and the Hilbert metric. *Publications mathématiques de l'IHÉS*, 95:151–183, 2002.
2. Frank Nielsen, Boris Muzellec, and Richard Nock. Classification with mixtures of curved Mahalanobis metrics. In *IEEE International Conference on Image Processing (ICIP)*, pages 241–245. IEEE, 2016.
3. Frank Nielsen, Boris Muzellec, and Richard Nock. Large margin nearest neighbor classification using curved Mahalanobis distances. *CoRR*, abs/1609.07082, 2016.
4. Jürgen Richter-Gebert. *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry.* Springer Publishing Company, Incorporated, 1st edition, 2011.