Jensen-Shannon symmetrization of dissimilarities

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$$D_{\mathrm{JS}}(p,q) = \frac{1}{2}\left(D_{\mathrm{KL}}\left(p : \frac{p+q}{2}\right) + D_{\mathrm{KL}}\left(q : \frac{p+q}{2}\right)\right).$$

$$D_{\mathrm{JS}}(p,q) = H\left(\frac{p+q}{2}\right) - \frac{H(p)+H(q)}{2} \geq 0$$

$$I_f(p:q) = \int p f(q/p) \mathrm{d}\mu$$

$$D^{\mathrm{JS}}(p,q) := \frac{1}{2}\left(D\left(p : \frac{p+q}{2}\right) + D\left(q : \frac{p+q}{2}\right)\right)$$

$$D = D_{\mathrm{KL}}$$

$$D_{\mathrm{JS}}(p,q) = \frac{1}{2}\left(D_{\mathrm{KL}}\left(p : \frac{p+q}{2}\right) + D_{\mathrm{KL}}\left(q : \frac{p+q}{2}\right)\right).$$

$$D = I_f$$

$$I_f^{\mathrm{JS}}(p,q) := \frac{1}{2}\left(I_f\left(p : \frac{p+q}{2}\right) + I_f\left(q : \frac{p+q}{2}\right)\right) = I_{f^{\mathrm{JS}}}(p:q)$$

$$D_{\chi^2}^{\mathrm{JS}}(p,q) = \frac{1}{2}\left(D_{\chi^2}^{\mathrm{Neyman}}\left(p : \frac{p+q}{2}\right) + D_{\chi^2}^{\mathrm{Neyman}}\left(q : \frac{p+q}{2}\right)\right) = \int \frac{(p-q)^2}{p+q} \mathrm{d}\mu = I_{f^{\mathrm{JS}}_{\mathrm{Neyman}}}(p:q)$$

$$D_{\chi^2}^{\mathrm{Neyman}}(p,q) = \int \frac{(p-q)^2}{p} d\mu$$

$$f_{\mathrm{Neyman}}(u) = (u-1)^2$$

$$f_{\mathrm{Neyman}}^{\mathrm{JS}}(u) = \frac{(u-1)^2}{u}$$

$$f_{\mathrm{KL}} \to f_{\mathrm{JS}}(u) = f_{\mathrm{KL}}^{\mathrm{JS}}(u) = -((1+u)/2)\log((1+u)/2) + (u/2)\log(u)$$

$$f^{\mathrm{JS}}(u) := \frac{1+u}{4}\left(f\left(\frac{2u}{1+u}\right) + f\left(\frac{2}{1+u}\right)\right)$$