Klein distance as a Hilbert distance

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Let $\mathbb{D}$ be the unit open ball of $\mathbb{R}^d$. Consider the Hilbert log cross-ratio distance on $(\mathbb{D} \in \mathbb{R}^d, \|\cdot\|_2)$:

$$\rho_H(p,q) = \frac{1}{2} \log \frac{\|p-a\| \, \|q-b\|}{\|p-b\| \, \|q-a\|}, \tag{1}$$

where a, p, q, b are colinear, i.e., a and b are the two intersection points of line (pq) with the boundary $\partial\mathbb{D}$ of the unit ball (Figure 1).

To calculate a and b, we first write the equation $ax + by + c = 0$ of the line passing through p and q:

$$\underbrace{(p_y - q_y)}_{a} x + \underbrace{(q_x - p_x)}_{b} y + \underbrace{p_y(p_x - q_x) + p_x(q_y - p_y)}_{c} = 0.$$

Then we solve a quadratic equation induced by the system:

$$\left\{ \begin{array}{lcl} x^2 + y^2 - 1 & = & 0, \\ ax + by + c & = & 0. \end{array} \right.$$

We write $y = \frac{-c-ax}{b}$ and expand this term in $x^2 + y^2 - 1 = 0$ to get the quadratic equation.

We have $\Delta = B^2 - 4AC$ with $A = 1 + \left(\frac{a}{b}\right)^2$, $B = \frac{2ac}{b^2}$, and $C = -1 + \left(\frac{c}{b}\right)^2$:

$$x_1 = \frac{-B - \sqrt{\Delta}}{2A}, \quad x_2 = \frac{-B + \sqrt{\Delta}}{2A}.$$

We find that the Hilbert distance amounts

$$\rho_K(p,q) = \operatorname{arccosh}\left(\frac{1 - p \cdot q}{\sqrt{(1 - p \cdot p)\,(1 - q \cdot q)}} \right),$$

where $x \cdot y$ is the Euclidean inner product and

$$\operatorname{arccosh}(x) = \log(x + \sqrt{x^2 - 1}).$$

Distance $\rho_K(p,q)$ is the Klein distance, the hyperbolic distance in the Klein model of hyperbolic geometry

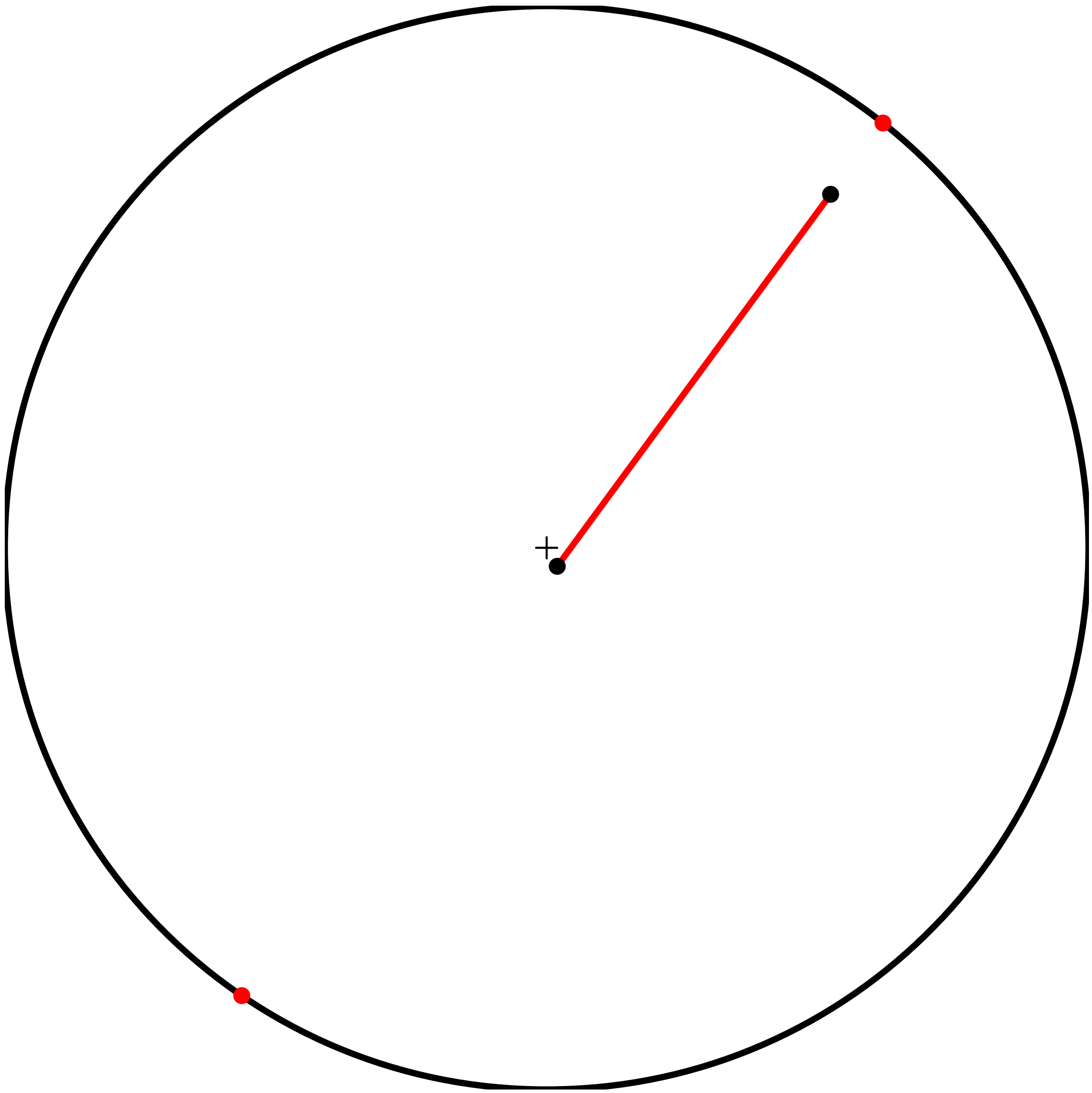

Figure 1: Klein geodesics are straight line segments. Klein distance is Hilbert distance induced by the open unit ball domain $\mathbb{D}$.