Some generalizations and perspectives on Bregman divergences

Frank Nielsen

Sony Computer Science Laboratories, Inc.

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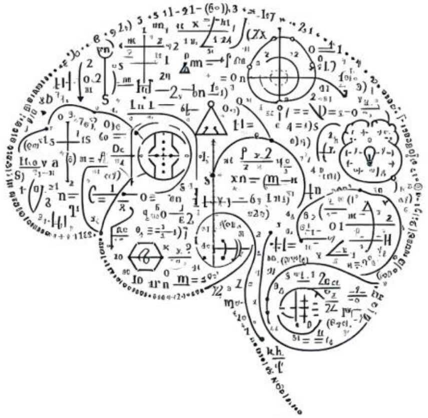
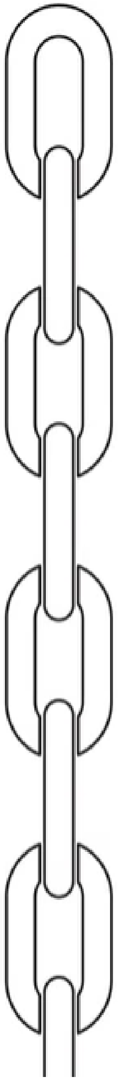


Outline: A thread of short stories



1. Quick introduction to **Bregman divergences**
2. **Duo** Bregman pseudo-divergences
3. **Curved** Bregman divergences
4. **Generalized Legendre transforms** and information geometry
5. **Generalized convexity** and Bregman divergences
6. Boolean algebra of **Bregman balls**

Bregman divergences (1960's): 1,2,3

Lev M. Bregman
(1941 - 2023)
Photo: courtesy of
Alexander Fradkov

① $F: \Theta \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ **strictly convex and smooth**

Bregman divergence $B_F : \Theta \times \text{Relative Interior}(\Theta) \rightarrow \mathbb{R}_{\geq 0}$

$$B_F(\theta_1 : \theta_2) := F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$$

Smooth measure of discrepancy, not a metric distance because it violates the triangle inequality and is asymmetric when F is not quadratic function.

② BD = **remainder of a first order Taylor expression** of $F(\theta_1)$ around θ_2:

$$F(\theta_1) = F(\theta_2) + \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle + \underbrace{B_F(\theta_1 : \theta_2)}_{\text{Taylor remainder}}$$

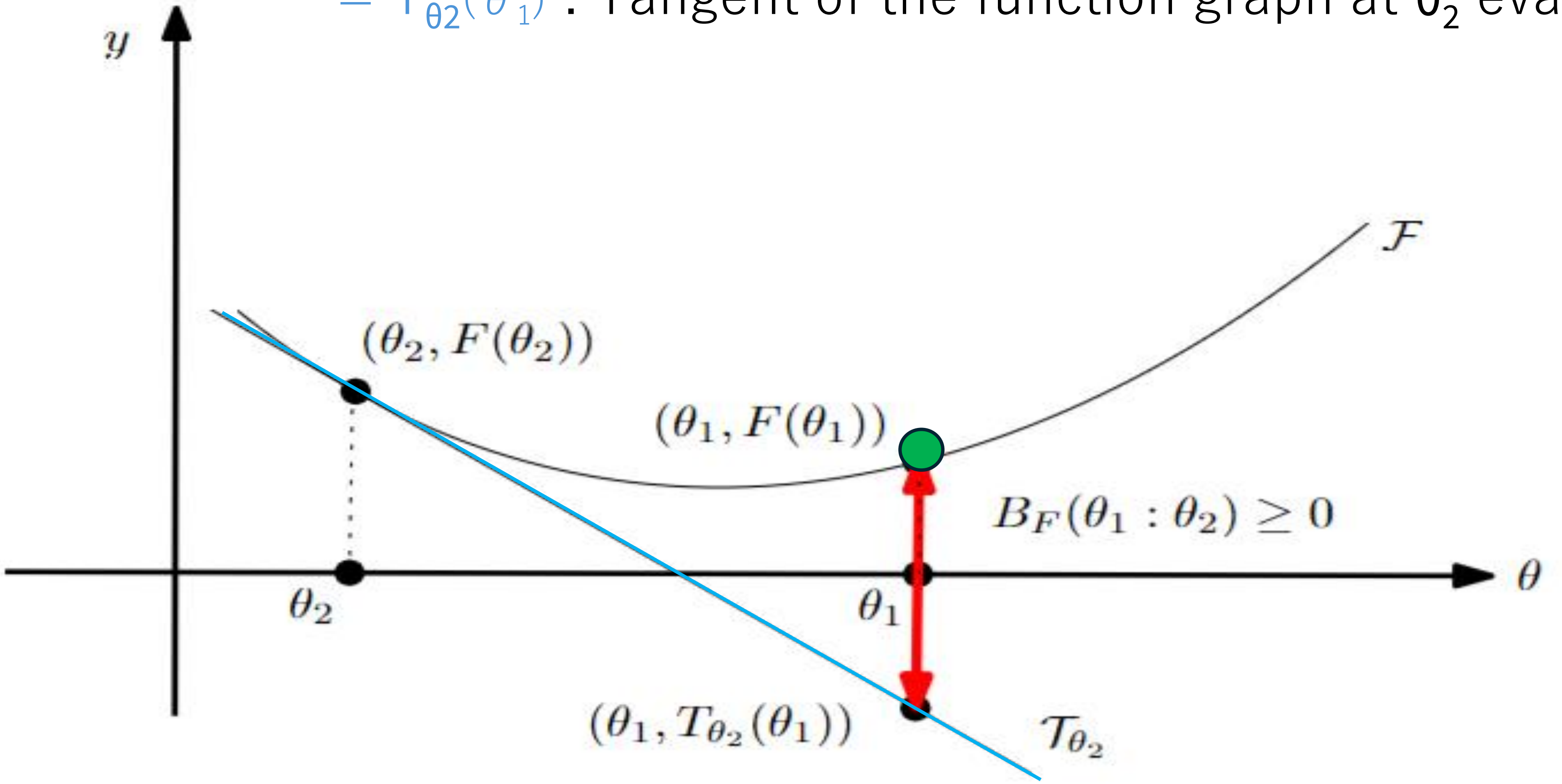
③ Example: **Lagrange remainder** (smooth C^2 generators): $\nabla^2 F$ **SPD** $\Rightarrow B_F(\theta_1 : \theta_2) \geq 0$

$B_F(\theta_1 : \theta_2) = \frac{1}{2} (\theta_2 - \theta_1)^T \nabla^2 F(\theta) (\theta_2 - \theta_1) \geq 0$, $\exists\, \theta \in [\theta_1 \theta_2]$ (squared Mahalanobis/Euclidean)

Geometric interpretation as a **vertical gap** using the graph $(\theta, F(\theta))$:

$$B_F(\theta_1 : \theta_2) = F(\theta_1) - \underbrace{(F(\theta_2) + < \theta_1 - \theta_2, \nabla F(\theta_2) >)}$$

$= T_{\theta 2}(\theta_1)$: Tangent of the function graph at θ_2 evaluated at θ_1



Relative entropy between Gaussian distributions?

- Relative entropy = **cross-entropy** minus **entropy**, a dissimilarity measure in information theory, statistics, ML, etc.

  called **Kullback-Leibler divergence** in information theory**:**

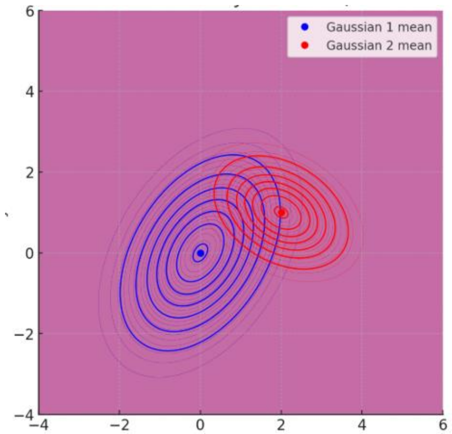
$$D_{KL}(P||Q) = \sum_i P(i) \log \frac{P(i)}{Q(i)}$$

$$D_{KL}(P||Q) = \int P(x) \log \frac{P(x)}{Q(x)} dx$$

$D_{KL}[p(x):q(x)] = \int p(x) \log (p(x)/q(x)) \, d\mu(x)$

KLD integrals ∫ is non-trivial but….

- *How to calculate the KLD between multivariate Gaussian distributions?*



$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma_p|^{1/2}} \exp\left(-\tfrac{1}{2}(x-\mu_p)^\top \Sigma_p^{-1}(x-\mu_p)\right)$$

$$q(x) = \frac{1}{(2\pi)^{d/2}|\Sigma_q|^{1/2}} \exp\left(-\tfrac{1}{2}(x-\mu_q)^\top \Sigma_q^{-1}(x-\mu_q)\right)$$

Expectation of this term?

$$\log \frac{p(x)}{q(x)} = \frac{1}{2} \log \frac{|\Sigma_q|}{|\Sigma_p|} + \frac{1}{2}(x-\mu_q)^\top \Sigma_q^{-1}(x-\mu_q) - \frac{1}{2}(x-\mu_p)^\top \Sigma_p^{-1}(x-\mu_p)$$

Bregman divergences in ML: Exponential families

- Kullback-Leibler divergence between two probability densities:

$$D_{KL}[p(x):q(x)] = \int \mathbf{p(x)\log(p(x)/q(x))\,d\mu(x)}$$

can be **difficult to calculate in closed form** because of the integral ∫

- But Kullback-Leibler divergence between two probability densities of an **exponential family** with densities $p(x|\theta) = \exp(<t(x), \theta> - F(\theta))$

amount to a **reverse Bregman divergence** $B_F^{rev}(\theta_1 : \theta_2) := B_F(\theta_2 : \theta_1)$

where F is the **cumulant function** (log partition/log Laplace function)

$$D_{KL}[p(x|\theta_1) : p(x|\theta_2)] = B_F^{rev}(\theta_1 : \theta_2) = B_F(\theta_2 : \theta_1)$$

BDs wrt CFs between parameters = reverse KLD distributions

Bypass the ∫, ∇F in BD easy to calculate! ⇒ Easy calculations of KLDs

Azoury, Katy S., and Manfred K. Warmuth. "Relative loss bounds for on-line density estimation with the exponential family of distributions." ***Machine learning*** **43 (2001)**

Kullback-Leibler divergence between non-normalized exponential family densities

- Kullback-Leibler divergence between two **positive measures**:

$$D_{KL}^{+}[q_1(x):q_2(x)] = \int \{ q_1(x) \log (q_1(x)/q_2(x)) + q_2(x)-q_1(x) \} \, d\mu(x)$$

- Exponential family density: Normalized $p(x|\theta) = \exp(\langle x, \theta \rangle - F(\theta))\, d\mu(x)$ versus **non-normalized**: $q(x|\theta) = \exp(\langle x, \theta \rangle)\, d\mu(x)$

- Hence, $p(x|\theta) = q(x|\theta)/Z(\theta)$ with **partition function** $Z(\theta)=\exp(F(\theta))$ and **cumulant function** $F(\theta)=\log Z(\theta)$. When F is convex, Z=exp(F) is **log-convex**, and log-convex functions are also convex functions:

Both F and Z are convex functions, yields Bregman divergences B_F & B_Z

- Well-known: KLD between normalized densities = reverse Bregman wrt F:

$$D_{KL}[p_{\theta 1}(x):p_{\theta 2}(x)] = B_F^{*}[\theta_1 : \theta_2] = B_F[\theta_2 : \theta_1]$$

- New: KLD between non-normalized densities = reverse **Bregman wrt Z**:

$$D_{KL}^{+}[q_{\theta 1}(x):q_{\theta 2}(x)] = B_Z^{*}[\theta_1 : \theta_2] = B_Z[\theta_2 : \theta_1]$$

2312.12849

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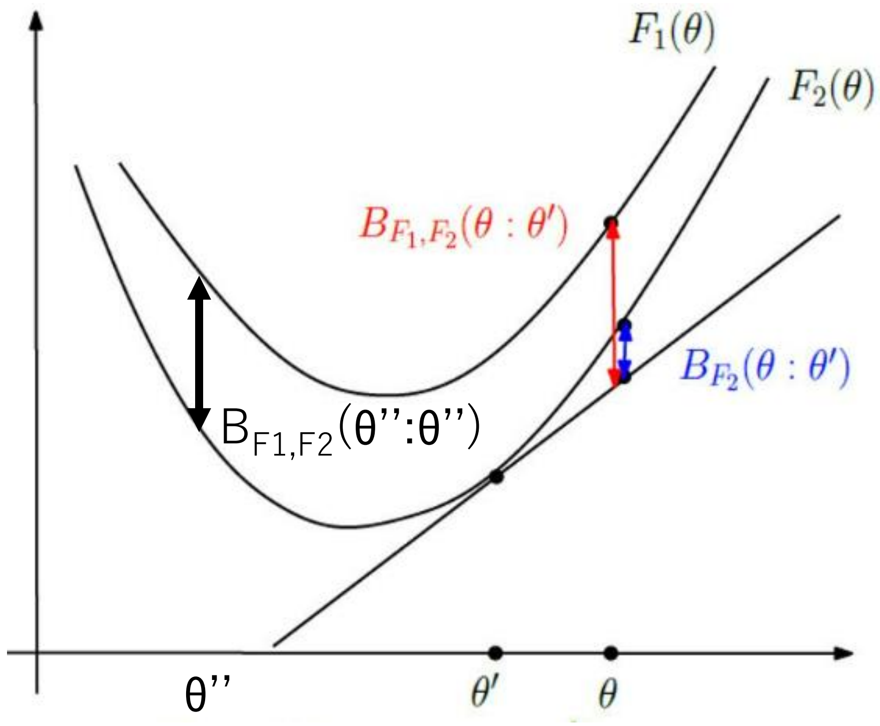
Duo Bregman divergences: Generalize BDs with a pair of generators



One generator **majorizes** the other one:

$$F_1(\theta) \geq F_2(\theta)$$

Duo Bregman pseudo-divergence:

$$B_{F_1,F_2}(\theta:\theta') = F_1(\theta) - F_2(\theta') - (\theta-\theta')^\top \nabla F_2(\theta')$$

$$\geq B_{F2}(\theta:\theta')$$

- Recover Bregman divergence when $F_1(\theta) = F_2(\theta) = F(\theta)$

$$B_F(\theta_1:\theta_2) = F(\theta_1) - F(\theta_2) - \langle \theta_1 - \theta_2, \nabla F(\theta_2) \rangle$$

- Only **pseudo-divergence** because $B_{F1,F2}(\theta'':\theta'')$ positive, not zero.

But why considering two generators?

KLD between nested exponential families amount to duo Bregman pseudo-divergences

q(x| θ)» p(x| θ)

p(x| θ) X_1

X_2

q(x| θ)

$D_{KL}[p(x):q(x)]=\int p(x) \log (p(x)/q(x))\, d\mu(x)$

$0 \log(0/0)=0$

- Consider an exponential family on support X_1:

$$p(x|\theta) = \exp(\langle x, \theta\rangle - \mathbf{F_1}(\boldsymbol{\theta}))\, d\mu(x)$$

with cumulant function $F_1(\theta)=\log \int_{X1} \exp(\langle x, \theta\rangle)\, d\mu(x)$

- Truncated exponential family with **nested supports: $X_1 \subseteq X_2$**

$$q(x|\theta) = \exp(\langle x, \theta\rangle - \mathbf{F_2}(\boldsymbol{\theta}))\, d\mu(x)$$

is an exponential family with $F_2(\theta)=\log \int_{X2} \exp(\langle x, \theta\rangle)\, d\mu(x) \geq F_1(\theta)$

- Then KLD amounts to a **reverse duo Bregman pseudo-divergence**:

$$\mathbf{D_{KL}[p(x|\theta_1) : q(x|\theta_2)] = B_{F2,F1}^{rev}(\theta_1 : \theta_2)} = B_{F2,F1}(\theta_2 : \theta_1)$$

"Statistical divergences between densities of truncated exponential families with nested supports: Duo Bregman and duo Jensen divergences." *Entropy* 24.3 (2022)

KL divergence between truncated normal densities

PDF of truncated normal on (a,b): $$p_{m,s}^{a,b}(x) = \frac{1}{\sqrt{2\pi}s\ (\Phi_{m,s}(b) - \Phi_{m,s}(a))} \exp\left(-\frac{(x-m)^2}{2s^2}\right)$$

This is an example, DO NOT READ!
For illustration purpose!

$$\Phi_{m,s}(x) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x-m}{\sqrt{2}s}\right)\right), \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t.$$

Truncated normal PDFs form an exponential family with log-normalizer :

$$F_{a,b}(m,s) = \frac{m^2}{2s^2} + \frac{1}{2}\log 2\pi s^2 + \log\left(\Phi_{m,s}(b) - \Phi_{m,s}(a)\right)$$

Moment parameters and mean & variance:

$$\mu(m,s;a,b) = m - s\,\frac{\phi(\beta)-\phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)}, \qquad \phi(x) := \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

$$\begin{aligned}\eta_1(m,s;a,b) &= E_{p_{m,s}^{a,b}}[x] = \mu(m,s;a,b),\\ \eta_2(m,s;a,b) &= E_{p_{m,s}^{a,b}}[x^2] = \sigma^2(m,s;a,b) + \mu^2(m,s;a,b).\end{aligned}$$

$$\sigma^2(m,s;a,b) = s^2\left(1 - \frac{\beta\phi(\beta)-\alpha\phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)} - \left(\frac{\phi(\beta)-\phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)}\right)^2\right),$$

Kullback-Leibler divergence between nested truncated normal distributions:

$$\begin{aligned}D_{\mathrm{KL}}[p_{m_1,s_1}^{a_1,b_1} : p_{m_2,s_2}^{a_2,b_2}] &= \frac{m_2}{2s_2^2} - \frac{m_1}{2s_1^2} + \log\frac{Z_{a_2,b_2}(m_2,s_2)}{Z_{a_1,b_1}(m_1,s_1)} - \left(\frac{m_2}{s_2^2} - \frac{m_1}{s_1^2}\right)\eta_1(m_1,s_1;a_1,b_1)\\ &\quad - \left(\frac{1}{2s_1^2} - \frac{1}{2s_2^2}\right)\eta_2(m_1,s_1;a_1,b_1)\end{aligned}$$

if nested distributions $(a_1,b_1) \subseteq (a_2,b_2)$

$D_{\mathrm{KL}}[p_{m_1,s_1}^{a_1,b_1} : p_{m_2,s_2}^{a_2,b_2}] = +\infty, (a_1,b_1) \not\subseteq (a_2,b_2)$ otherwise

Convex duality via Legendre-Fenchel transform

- Convex duality: Legendre-Fenchel transform of a convex function F:

  $F^*(\eta)=\sup_{\theta \in \Theta}\{<\theta, \eta> - F(\theta)\}$

- Problem: some *tricky functions* with gradient map ∇F domain not convex…

  Example: $h(\xi_1, \xi_2) = [(\xi_1^2/\xi_2) + \xi_1^2 + \xi_2^2]/4$ on upper plane domain $\Xi=(\xi_1, \xi_2)$

- Thus, we consider "**nice convex functions**" = **Legendre-type functions (Θ,F(θ))**

  (i) Θ open, and (ii) $\lim_{\theta \to \partial\Theta} \| \nabla F(\theta) \| = \infty$

Then we get:

❶ **reciprocal gradient maps** : $\eta = \nabla F(\theta)$, and $\theta = \nabla F^*(\eta)$ and $\nabla F^* = (\nabla F)^{-1}$

❷ conjugation yields (H,$F^*(\eta)$) of Legendre type

❸ biconjugation is an **involution**: (H,$F^*(\eta)$)* = ($H^*=\Theta$, $F^{**}=F(\theta)$)

- Convex conjugate: $F^*(\eta) = <\nabla F^{-1}(\eta), \eta> - F(\nabla F^{-1}(\eta))$ since $\eta = \nabla F(\theta)$

Fenchel-Young divergences & convex duality

- Young inequality: $F(\theta_1)+F^*(\eta_2) \geq \langle \theta_1, \eta_2 \rangle$ with equality iff $\eta_2 = \nabla F(\theta_1)$
- Build the **Fenchel-Young divergence** from inequality gap lhs-rhs ≥0

$$Y_{F,F^*}(\theta_1, \eta_2) := F(\theta_1)+F^*(\eta_2) - \langle \theta_1, \eta_2 \rangle \geq 0$$

- BD with **mixed parameterizations** θ and η : $B_F(\theta_1 : \theta_2) = Y_{F,F^*}(\theta_1, \eta_2)$
- Duality: **Four equivalent expressions** of the "Bregman divergence"

$$B_F(\theta_1 : \theta_2) = Y_{F,F^*}(\theta_1, \eta_2) = Y_{F^*,F}(\eta_2, \theta_1) = B_{F^*}(\eta_2, \eta_1)$$

- Dual BDs + Dual FYs from involution $F^{**}=F$

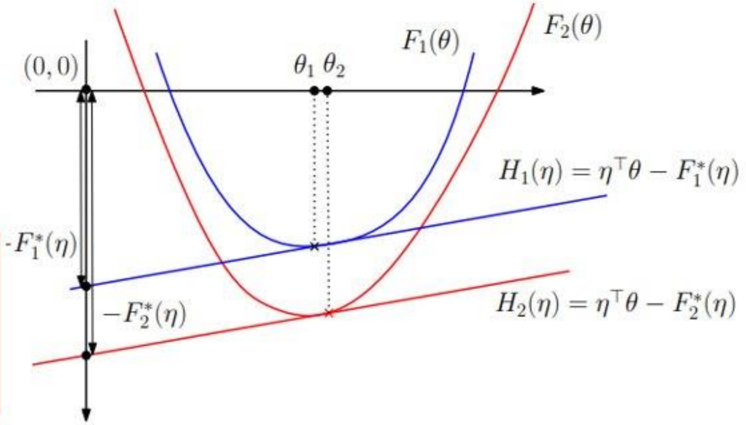
Note : $B_F(\theta_1 : \theta_2)=0 \Leftrightarrow \theta_1 = \theta_2 \Leftrightarrow \eta_1 = \eta_2$ i.e., $\nabla F(\theta_1) = \nabla F(\theta_2)$

Legendre transformation reverses majorization order

Legendre-Fenchel transformation: $F^*(\eta) := \sup_{\theta\in\Theta}\{\eta^\top\theta - F(\theta)\}$

F Legendre-type function, Moreau **biconjugation theorem**: $(F^*)^* = F$

proper+lower semi-continuous+convex

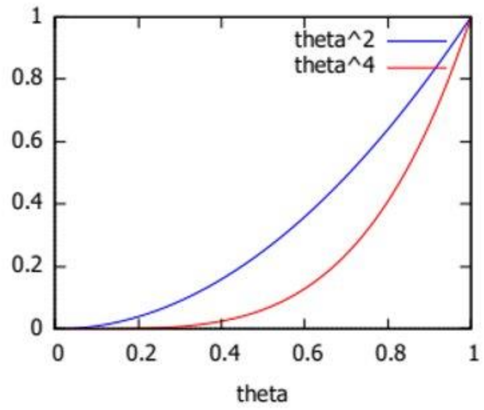


Legendre-Fenchel transform **reverses ordering**:

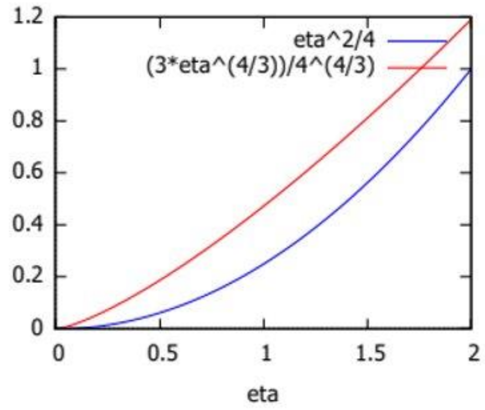
$$\boxed{\forall\theta\in\Theta,\quad F_1(\theta)\geq F_2(\theta) \Leftrightarrow \forall\eta\in H,\quad F_1^*(\eta)\leq F_2^*(\eta)}$$

Proof:

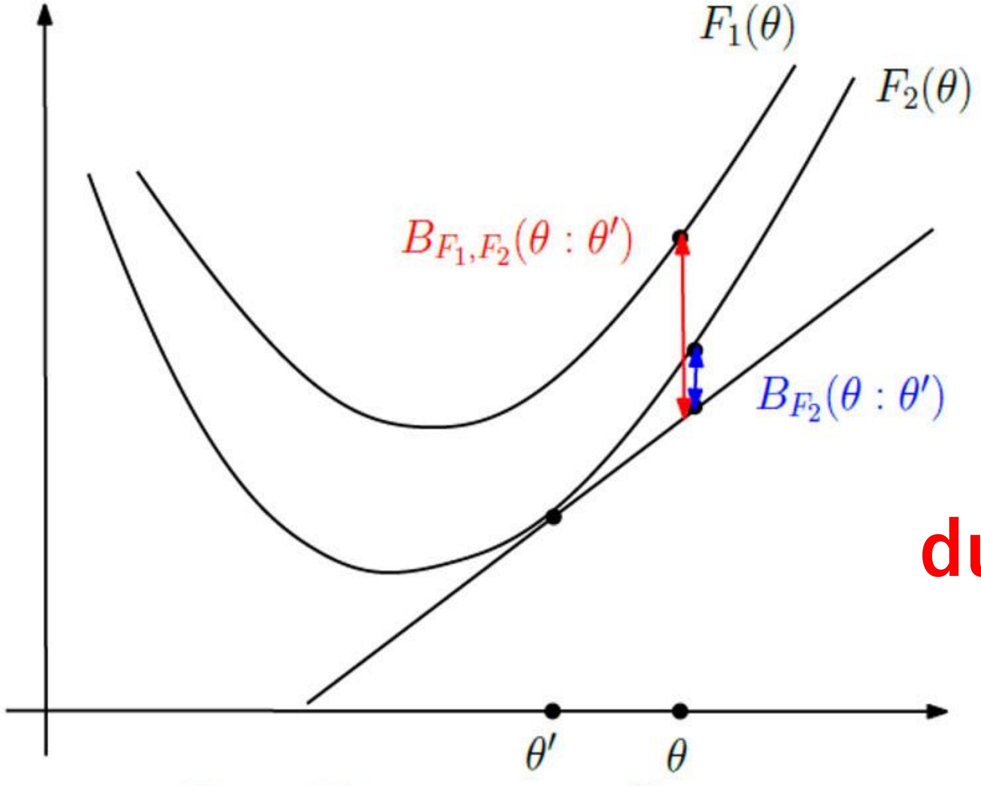
$$\begin{aligned}
F_1^*(\eta) &:= \sup_{\theta\in\Theta}\{\eta^\top\theta - F_1(\theta)\},\\
&= \eta^\top\theta_1 - F_1(\theta_1) \qquad (\text{with } \eta = \nabla F_1(\theta_1))\\
&\leq \eta^\top\theta_1 - F_2(\theta_1),\\
&\leq \sup_{\theta\in\Theta}\{\eta^\top\theta - F_2(\theta)\} =: F_2^*(\eta).
\end{aligned}$$



Convex functions $F_1(\theta)\geq F_2(\theta)$



Conjugate functions $F_1^*(\eta)\leq F_2^*(\eta)$



Duality
and
duo Fenchel-Young pseudo-divergences

Duo Bregman divergence

$$B_{F_1,F_2}(\theta:\theta') = F_1(\theta) - F_2(\theta') - (\theta-\theta')^\top \nabla F_2(\theta')$$

Duo Fenchel-Young divergence

$$Y_{F_1,F_2^*}(\theta,\eta') := F_1(\theta) + F_2^*(\eta') - \theta^\top \eta'.$$

Relationship with truncated exponential families with nested supports:

$$D_{\mathrm{KL}}[p_{\theta_1} : q_{\theta_2}] = Y_{F_2,F_1^*}(\theta_2 : \eta_1) = B_{F_2,F_1}(\theta_2 : \theta_1)$$

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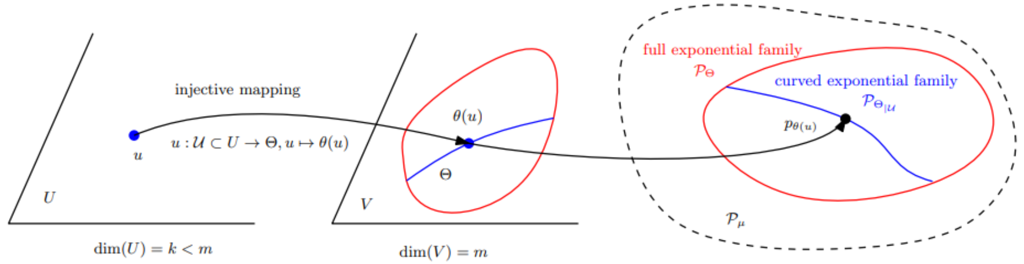
Curved Bregman divergences



Consider a domain U which maps to a subset of Θ by θ =c(u) with **dim(U) < dim(Θ)** :

$\mathbf{B_{F,c}(u_1 : u_2)}$:= $\mathbf{B_F(c(u_1):c(u_2))}$ is not Bregman when {c(u) | u ∈ U} not convex usually not a Bregman divergence unless c(.) is affine

Example: **Symmetrized Bregman divergences** (= Jeffreys-Bregman div.) are curved Bregman divergences: $S_F(\theta_1, \theta_2) = \langle \theta_1 - \theta_2, \eta_1 - \eta_2 \rangle$

$$
\begin{aligned}
S_F(\theta_1 : \theta_2) &= B_F(\theta_1 : \theta_2) + B_F(\theta_2 : \theta_1), \\
&= B_F(\theta_1 : \theta_2) + B_{F^*}(\nabla F(\theta_1) : \nabla F(\theta_2)) \\
&= \breve{B}_{F_\xi}(\xi(\theta_1) : \xi(\theta_2)),
\end{aligned}
$$

$$F^*(\eta) = \langle \theta, \eta \rangle - F(\theta) \qquad F_\xi(\theta, \eta) := F(\theta) + F^*(\eta) \qquad \xi(\theta) = (\theta, \nabla F(\theta))$$

$$\mathcal{U} = \{(\theta, \nabla F(\theta)) : \theta \in \Theta\}$$

m-dimensional submanifold in 2m-dimensional space
(usually not cvx affine space, hence not a Bregman divergence)

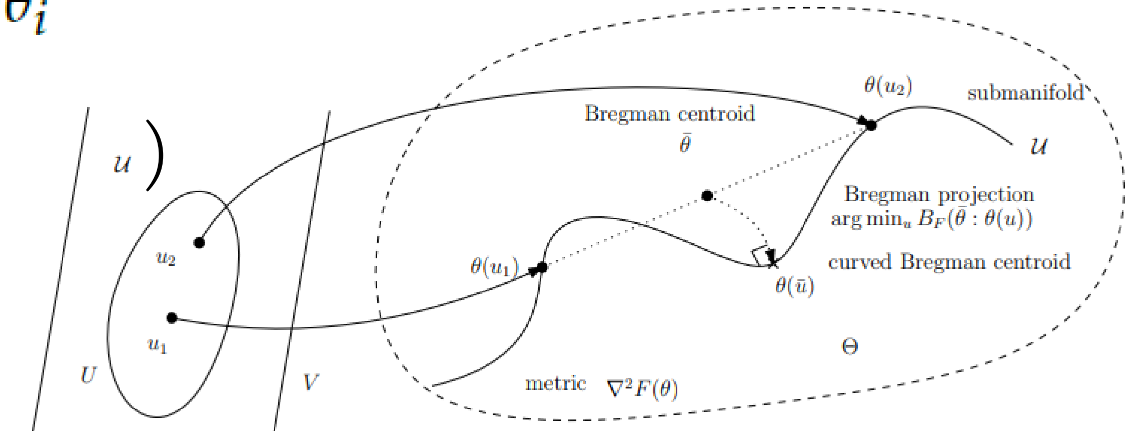
Curved Bregman centroid is the Bregman projection of the full Bregman centroid

Theorem: **Bregman projection**

$\bar{\theta} = \sum_i w_i \theta_i$

$$\arg\min_{u \in \mathcal{U}} \sum_{i=1}^{n} w_i B_F(\theta_i : \theta(u)) = \arg\min_{u \in \mathcal{U}} B_F(\bar{\theta} : \theta(u))$$

N-point opt. $\theta_i = \theta(u_i)$ **One-point opt!**



Proof.

$$\begin{aligned}
\min_{u \in \mathcal{U}} \sum_{i=1}^{n} w_i B_F(\theta_i : \theta(u)) &= \sum_{i=1}^{n} w_i (F(\theta_i) - F(\theta(u)) - \langle \theta_i - \theta(u), \nabla F(\theta(u)) \rangle), \\
&\equiv -F(\theta(u)) - \langle \bar{\theta} - \theta(u), \nabla F(\theta(u)) \rangle, \\
&\equiv F(\bar{\theta}) - F(\theta(u)) - \langle \bar{\theta} - \theta(u), \nabla F(\theta(u)) \rangle \\
&= B_F(\bar{\theta} : \theta(u)).
\end{aligned}$$

"What is... an information projection?" Notices of the AMS 65.3 (2018): 321-324.

Another example of curved Bregman divergences:

- Consider d-variate **circular complex normal distributio** $\mathcal{CN}_d(\mu_\mathbb{C}, S_\mathbb{C})$
- CNDs handled as **2d real normal distributions** $N_{2d}([\mu_\mathbb{C}]_\mathbb{R}, \frac{1}{2}[S_\mathbb{C}]_\mathbb{R})$

$$[z = a + ib]_\mathbb{R} = (a, b) \qquad [M = A + iB]_\mathbb{R} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

- Then KLD between CNDs amounts to a **curved Bregman divergence**:

$$D_{\mathrm{KL}}[p_{m_\mathbb{C}, S_\mathbb{C}} : p_{m'_\mathbb{C}, S'_\mathbb{C}}] = D_{\mathrm{KL}}[p_{\mu,\Sigma} : p_{\mu',\Sigma'}] = B_F(\theta' : \theta)$$

Natural parameters

$(\Sigma^{-1}\mu, \frac{1}{2}\Sigma^{-1})$

$\mu = [m_\mathbb{C}]_\mathbb{R}$ $\Sigma = [S_\mathbb{C}]_\mathbb{R}$ and $\mu' = [m'_\mathbb{C}]_\mathbb{R}$ $\Sigma' = [S'_\mathbb{C}]_\mathbb{R}$

- **Curved submanifold** parameter in R^{2d} x Mat(2d)

$$\mathcal{U} = \left\{ (v, M) : v \in \mathbb{R}^{2d}, M = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}, A \in \mathbb{R}^{d\times d} \succ 0, B \in \mathbb{R}^{d\times d} \succ 0 \right\}$$

2504.05654

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Generalized Legendre transforms (2009)

The concept of duality in convex analysis, and the characterization of the Legendre transform

By Shiri Artstein-Avidan and Vitali Milman*

Abstract

In the main theorem of this paper we show that any involution on the class of lower semi-continuous convex functions which is order-reversing, must be, up to linear terms, the well known Legendre transform.

Theorem 1. *Assume a transform* $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ *(defined on the whole domain* $Cvx(\mathbb{R}^n)$*) satisfies*

1. $\mathcal{T}\mathcal{T}\phi = \phi$,
2. $\phi \le \psi$ *implies* $\mathcal{T}\phi \ge \mathcal{T}\psi$.

Then, $\mathcal{T}$ *is essentially the classical Legendre transform; namely there exists a constant* $C_0 \in \mathbb{R}$, *a vector* $v_0 \in \mathbb{R}^n$, *and an invertible symmetric linear transformation* $B \in \mathrm{GL}_n$ *such that*

$$(\mathcal{T}\phi)(x) = (\mathcal{L}\phi)(Bx + v_0) + \langle x, v_0 \rangle + C_0.$$

Generalized Artstein-Avidan—Milman Legendre transforms

Definition (Generalized Legendre-Fenchel convex conjugates) *Let* $\mathcal{L}_{\lambda,E,f,g,h}$ *denote a generalized Legendre-Fenchel transform:*

$$\mathcal{L}_{\lambda,E,f,g,h}F := \mathcal{L}_P F := \lambda(\mathcal{L}F)(E\eta + f) + \langle \eta, g\rangle + h$$

for the parameter $P = (\lambda, E, f, g, h)$. where $(\mathcal{L}F)(\eta) := \sup_{\theta\in\mathbb{R}^m} \{\langle\theta,\eta\rangle - F(\theta)\}$

Affine-deformed convex functions of both argument and returned value remain convex:

$$F_P(\theta) := \lambda\, F(A\theta + b) + \langle\theta, c\rangle + d \qquad P = (\lambda, A, b, c, d) \in \mathbb{P} := \mathbb{R}_{>0} \times \mathrm{GL}(\mathbb{R}^m) \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$$

Theorem *For any* $F \in \Gamma_0$, *we have* $\mathcal{L}_P(F) := (F^*)_P = \mathcal{L}\left(F_{P^\diamond}\right)$

$$P^\diamond := \left(\lambda, \frac{1}{\lambda}A^{-1}, -\frac{1}{\lambda}A^{-1}c, -A^{-1}b, \langle b, A^{-1}c\rangle - d\right) \in \mathbb{P}$$

$$(P^\diamond)^\diamond = P$$

GLTs are LT on affinely deformed convex functions

arXiv:2507.20577

Generalized Artstein-Avidan—Milman Legendre transforms

Definition **(Generalized Legendre-Fenchel convex conjugates** **)** *Let* $\mathcal{L}_{\lambda,E,f,g,h}$ *denote a generalized Legendre-Fenchel transform:*

$$\mathcal{L}_{\lambda,E,f,g,h}F := \mathcal{L}_P F := \lambda(\mathcal{L}F)(E\eta + f) + \langle \eta, g\rangle + h$$

for the parameter $P = (\lambda, E, f, g, h)$. where $(\mathcal{L}F)(\eta) := \sup_{\theta\in\mathbb{R}^m} \{\langle\theta,\eta\rangle - F(\theta)\}$

Affine-deformed convex functions of both argument and returned value remain convex:

$F_P(\theta) := \lambda\, F(A\theta + b) + \langle\theta, c\rangle + d$ $\quad P = (\lambda, A, b, c, d) \in \mathbb{P} := \mathbb{R}_{>0} \times \mathrm{GL}(\mathbb{R}^m) \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$

Theorem *For any* $F \in \Gamma_0$*, we have* $\mathcal{L}_P(F) := (F^*)_P = \mathcal{L}\left(F_{P^\diamond}\right)$

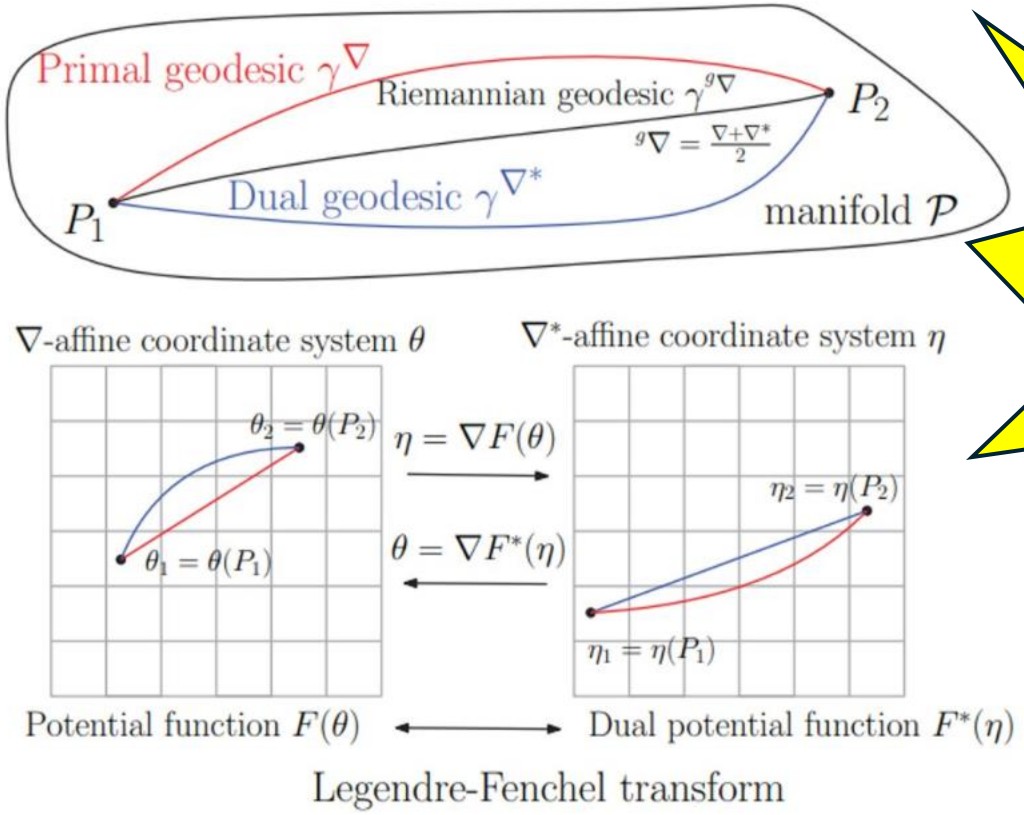
$$P^\diamond := \left(\lambda, \frac{1}{\lambda}A^{-1}, -\frac{1}{\lambda}A^{-1}c, -A^{-1}b, \langle b, A^{-1}c\rangle - d\right) \in \mathbb{P}$$

$$(P^\diamond)^\diamond = P$$

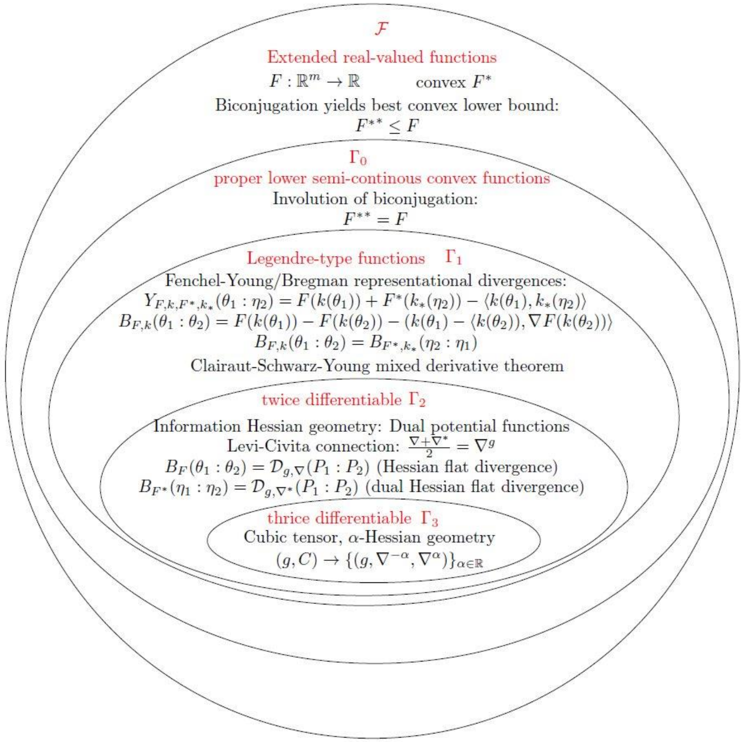
arXiv:2507.20577

GLTs can be explained from information geometry

DO NOT READ!
For illustration purpose!



Dually flat spaces, Hessian manifolds



Degrees of freedom when reconstruction dual potential functions

- Affine coordinate system up to affine transformation

$$\mathcal{L}_{\lambda,E,f,g,h}F := \mathcal{L}_P F := \lambda(\mathcal{L}F)(E\eta + f) + \langle \eta, g\rangle + h$$

- Scaling of potential functions
- Same geometric Legendre transform

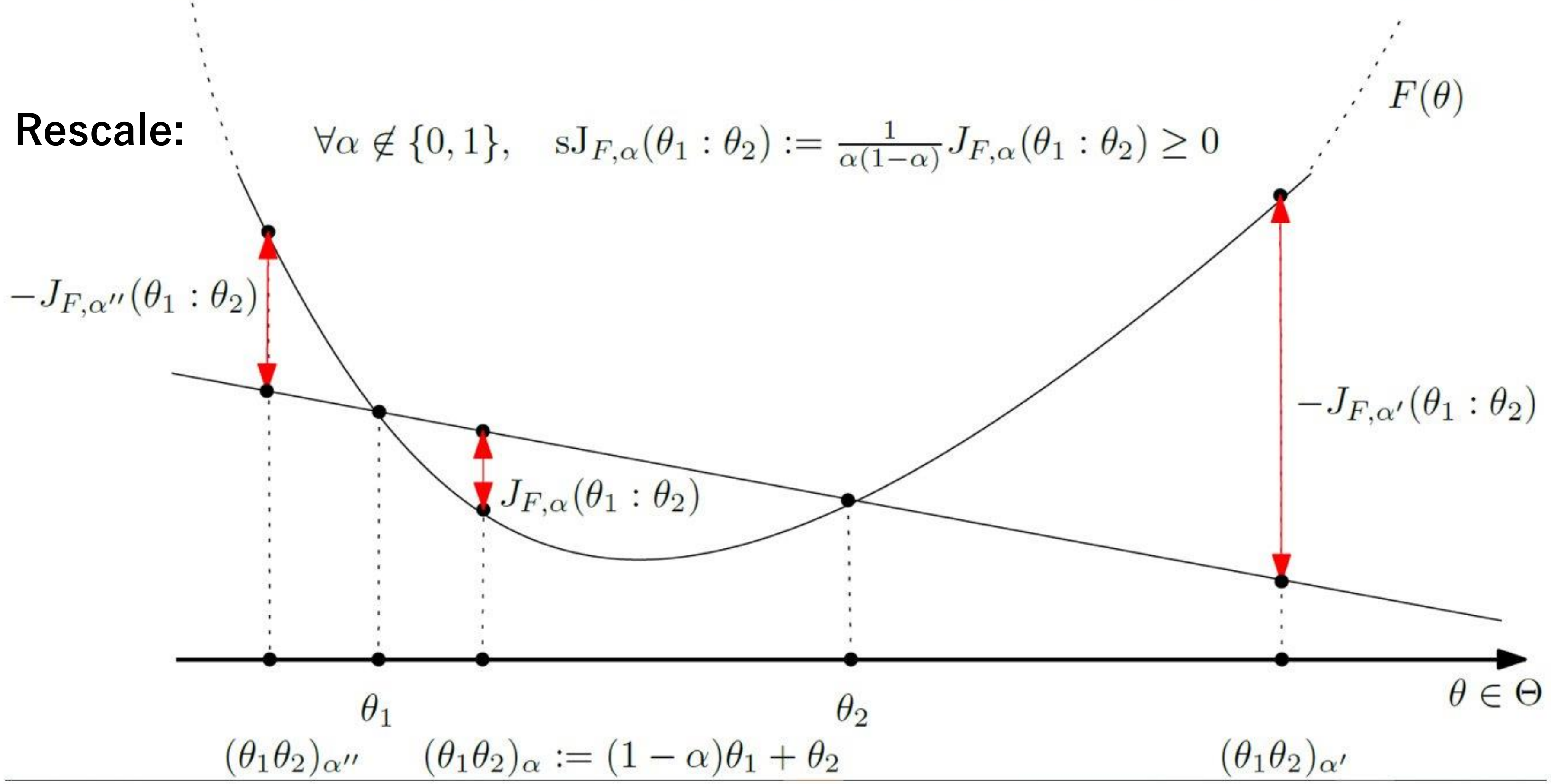
Outline

1. A quick introduction to Bregman divergences
2. Duo Bregman pseudo-divergences
3. Curved Bregman divergences
4. Generalized Legendre transforms and information geometry
5. **Generalized convexity and Bregman divergences**
6. Space of Bregman balls

Scaled skewed Jensen divergences & Bregman divergences

$$\forall \alpha \in (0,1), \quad J_{F,\alpha}(\theta_1 : \theta_2) := (1-\alpha)F(\theta_1) + \alpha F(\theta_2) - F((1-\alpha)\theta_1 + \alpha\theta_2)$$

Rescale: $\forall \alpha \notin \{0,1\}, \quad \mathrm{sJ}_{F,\alpha}(\theta_1 : \theta_2) := \frac{1}{\alpha(1-\alpha)} J_{F,\alpha}(\theta_1 : \theta_2) \geq 0$



Jensen divergences measures the vertical gap induced by a strictly convex function

$$\lim_{\alpha \to 0} \mathrm{sJ}_{F,\alpha}(\theta_1 : \theta_2) = B_F(\theta_1 : \theta_2)$$ (Bregman divergence) $$\lim_{\alpha \to 1} \mathrm{sJ}_{F,\alpha}(\theta_1 : \theta_2) = B_F(\theta_2 : \theta_1)$$ (reverse BD)

The Burbea-Rao and Bhattacharyya centroids, IEEE Transactions on Information Theory 57.8 (2011)

Comparative convexity: (M,N)-convexity

Ordinary convexity of a function: $f(tx_1+(1-t)x_2) \leq tf(x_1)+(1-t)f(x_2)$

for all t in [0,1]

- *Definition*: A function Z is **(M,N)-convex** iff for in α in [0,1]:

$$Z(M(x,y;\alpha,1-\alpha)) \leq N(Z(x),Z(y);\alpha,1-\alpha)$$

- Ordinary convexity = (A,A)-convexity wrt to arithmetic weighted mean

$$A(x,y;\alpha,1-\alpha) = \alpha x + (1-\alpha)y \qquad f(tx_1+(1-t)x_2) \leq tf(x_1)+(1-t)f(x_2)$$

for all t in [0,1]

- **Log-convexity**: **(A,G)-convexity** wrt to A/Geometric weighted means:

$$G(x,y;\alpha,1-\alpha) = x^\alpha y^{1-\alpha} \qquad f(tx_1+(1-t)x_2) \leq f(x_1)^t f(x_2)^{1-t}$$

for all t in [0,1]

Since G ≤ A, (A,G)-functions are (A,A)-convex: Log-convex functions are convex

Comparative convexity wrt quasi-arithmetic means

- **quasi-arithmetic mean** for a strictly monotone generator h(u):

$$M_h(x,y;\alpha,1-\alpha) = h^{-1}(\alpha h(x) + (1-\alpha)h(x)).$$

- Includes **power means** which are *homogeneous means*:

$$M_p(x,y;\alpha,1-\alpha) = (\alpha x^p + (1-\alpha)y^p)^{\frac{1}{p}} = M_{h_p}(x,y;\alpha,1-\alpha), \quad p \neq 0$$

$$h_p(u) = \frac{u^p - 1}{p} \qquad h_p^{-1}(u) = (1+up)^{\frac{1}{p}}$$

Include the **geometric mean** in the limit case p→0

Checking comparative convexity wrt two quasi-arithmetic means via an **ordinary convexity test**:

Proposition 6 ([1, 34]). *A function $Z(\theta)$ is strictly (M_ρ, M_τ)-convex with respect to two strictly increasing smooth functions ρ and τ if and only if the function $F = \tau \circ Z \circ \rho^{-1}$ is strictly convex.*

Generalizing Bregman divergences with (M,N)-convexity: (M,N)-Bregman divergences

- First, define **skew Jensen divergence from (M,N)-comp. convexity**:

Definition:

$$\boxed{J_{F,\alpha}^{M,N}(p:q) = N_\alpha(F(p),F(q)) - F(M_\alpha(p,q)).}$$

Non-negative for **(M,N)-convex generators** F, provided **regular means** M and N (e.g. all power means)

Definition 5 (Bregman Comparative Convexity Divergence, BCCD) *The Bregman Comparative Convexity Divergence (BCCD) is defined for a strictly (M,N)-convex function $F : I \to \mathbb{R}$ by*

$$\boxed{B_F^{M,N}(p:q) = \lim_{\alpha\to 1^-} \frac{1}{\alpha(1-\alpha)} J_{F,\alpha}^{M,N}(p:q) = \lim_{\alpha\to 1^-} \frac{1}{\alpha(1-\alpha)} \left(N_\alpha(F(p),F(q))\right) - F(M_\alpha(p,q)))} \quad (31)$$

This **definition** is by analogy to limit of scaled skewed Jensen divergences amount to forward/reverse Bregman divergences.

Generalizing Bregman divergences with quasi-arithmetic mean convexity

Theorem 1 (Quasi-arithmetic Bregman divergences, QABD) *Let $F : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued (M_ρ, M_τ)-convex function defined on an interval I for two strictly monotone and differentiable functions ρ and τ. The quasi-arithmetic Bregman divergence (QABD) induced by the comparative convexity is:*

$$\boxed{B_F^{\rho,\tau}(p:q) = \frac{\tau(F(p)) - \tau(F(q))}{\tau'(F(q))} - \frac{\rho(p)-\rho(q)}{\rho'(q)} F'(q).} \quad (45)$$

From 1st order Taylor expansion…

Amounts to a **conformal representational Bregman divergence** :

$$\boxed{B_F^{\rho,\tau}(p:q) = \frac{1}{\tau'(F(q))} B_G(\rho(p):\rho(q))}$$

With convex generator: $G(x) = \tau(F(\rho^{-1}(x)))$

Conformal factor

Remark: Conformal Bregman divergences may yield **robustness** in applications

Shape retrieval using hierarchical total Bregman soft clustering, IEEE Transactions on pattern analysis and machine intelligence (2012)

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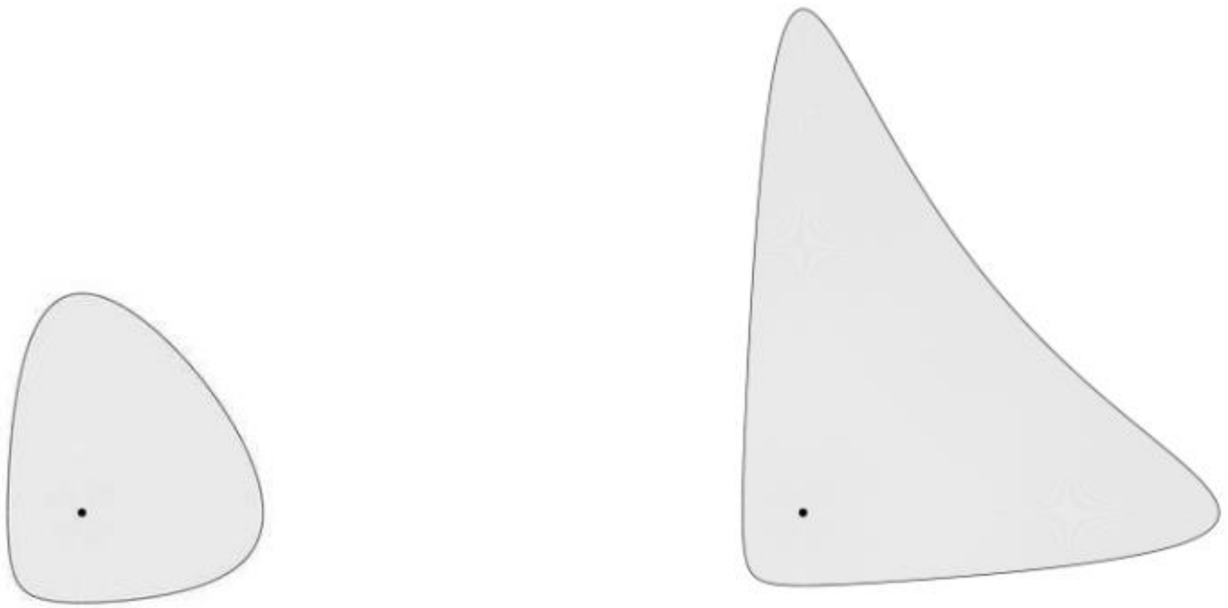
6. **Boolean algebra of Bregman balls**

Space of Bregman balls



Example:
Itakura-Saito right and left spheres

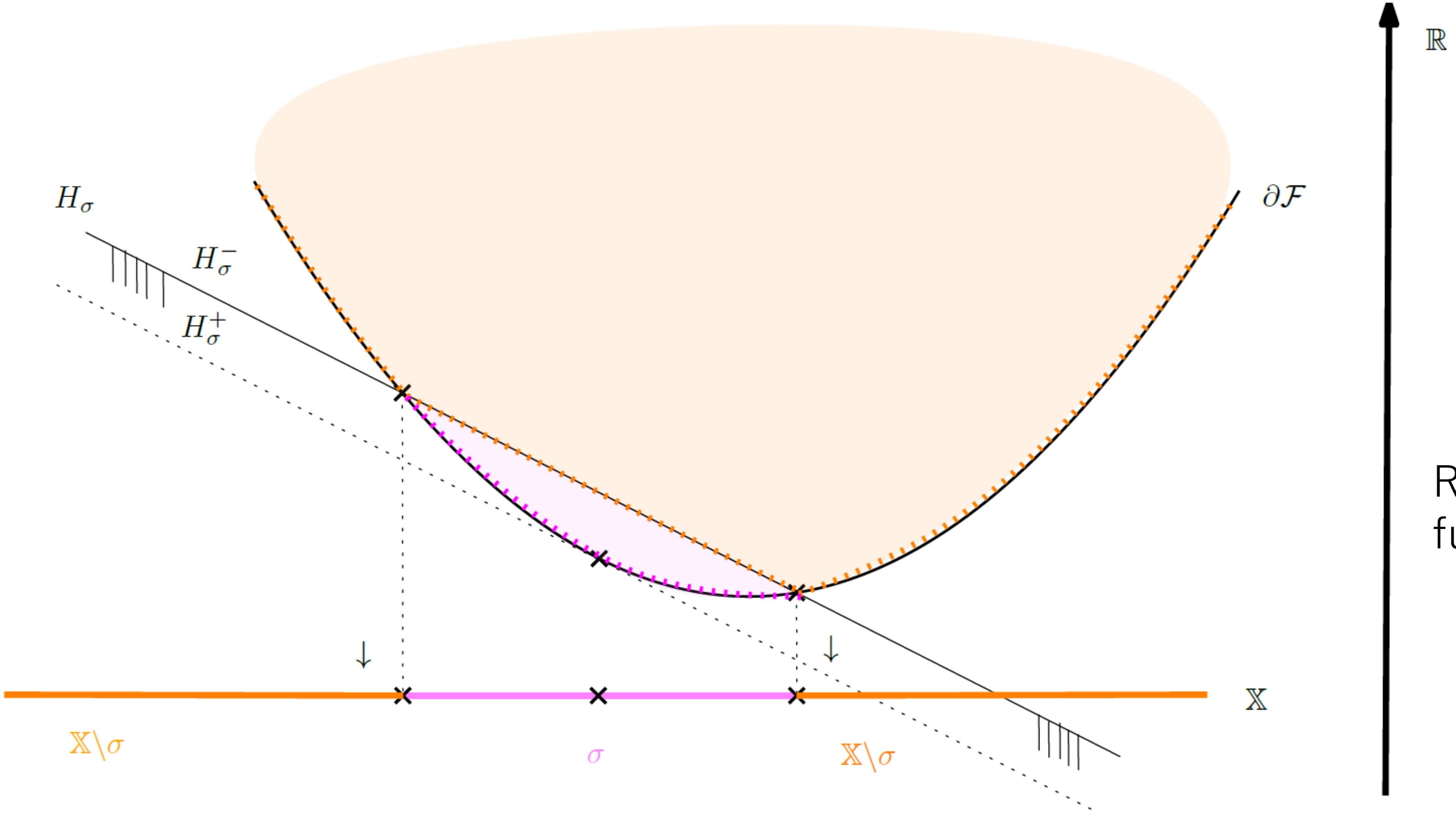
Right-sided Bregman ball: $\sigma_F(\theta, r) = \{\theta' \in \Theta : B_F(\theta' : \theta) \leq r\}$

Left-sided Bregman ball: $\sigma_F^{\star}(\theta, r) = \{\theta' \in \Theta : B_F(\theta : \theta') \leq r\}$

Application: Boolean algebra of unions & intersections of Bregman balls

Right Bregman ball and its complement

$$\mathcal{F} := \{(\theta, y \geq F(\theta)) \; : \; \theta \in \Theta \subset \mathbb{R}^m\} \subset \mathbb{R}^{m+1}$$



$\downarrow$ means vertical projection

S^c: complement of set S

To any sphere, associate a hyperplane:

$$H_{\theta,r} : y = \langle \theta' - \theta, \nabla F(\theta) \rangle + F(\theta) + r$$

Reciprocally, to a hyperplane cutting the function graph, associate a sphere

$$z = \langle \mathbf{x}, \mathbf{a} \rangle + b$$

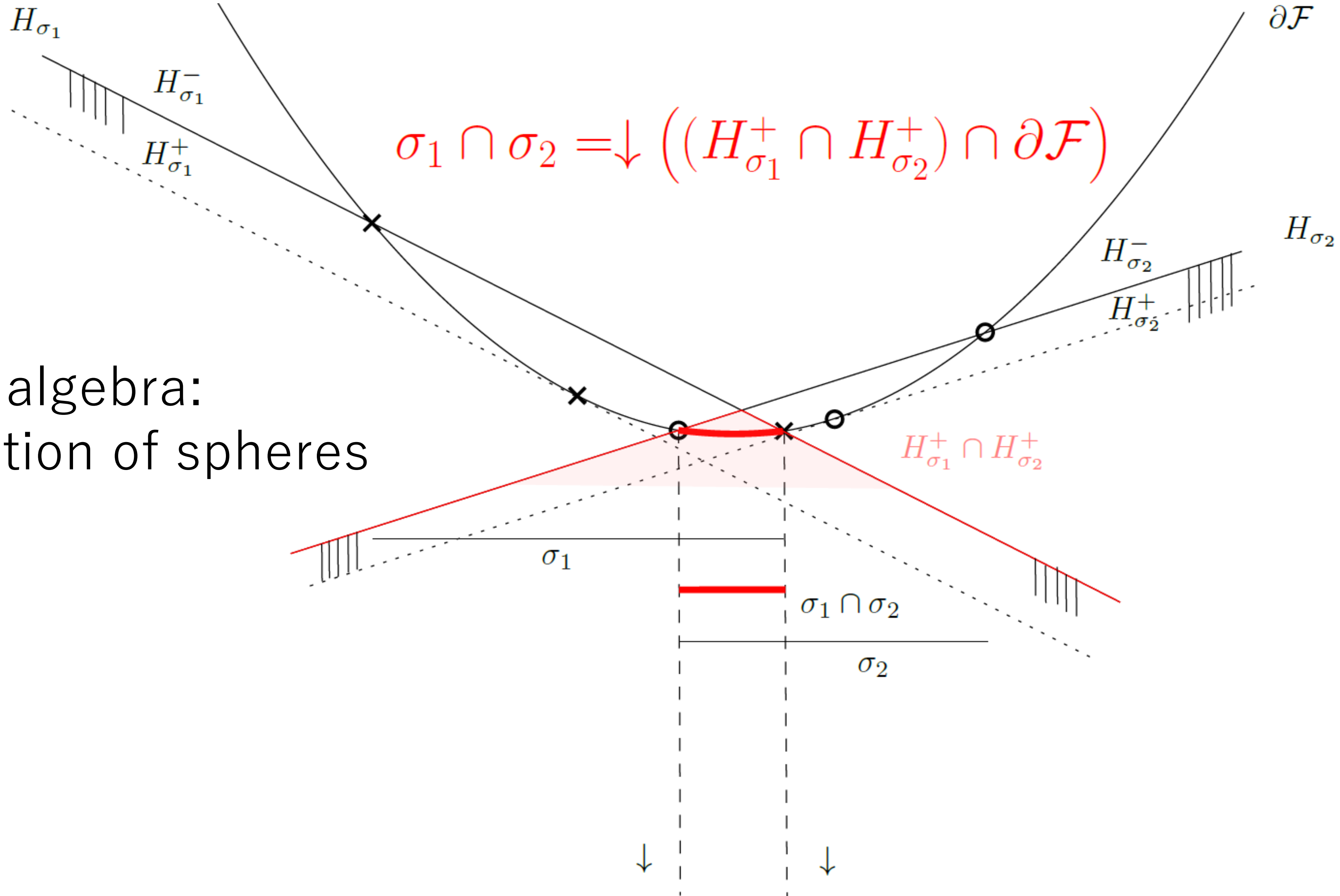
Center: $\mathbf{c} = \boldsymbol{\nabla}^{-1} F(\mathbf{a})$

Radius: $\langle \mathbf{a}, \mathbf{c} \rangle - F(\mathbf{c}) + b$

$\sigma^c = \mathbb{X}\backslash\sigma = \downarrow (H_\sigma^- \cap \partial\mathcal{F})$ $\quad \sigma = \downarrow (H_\sigma^+ \cap \partial\mathcal{F})$ $\quad \sigma^c = \mathbb{X}\backslash\sigma = \downarrow (H_\sigma^- \cap \partial\mathcal{F})$

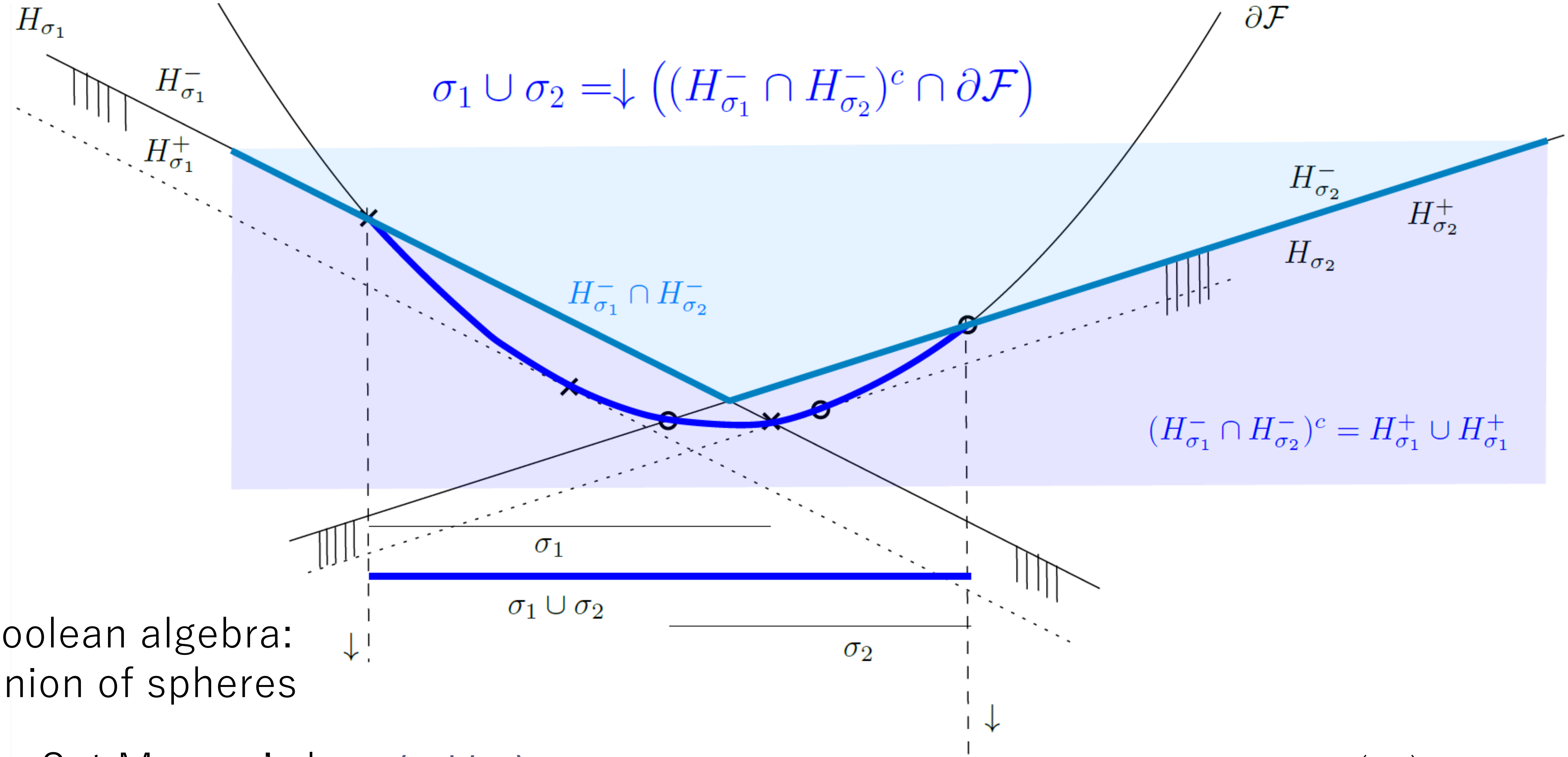
Lifting to potential Bregman generator graph & vertical projection

Intersection of two right Bregman balls



Boolean algebra:
intersection of spheres

Union of two right Bregman balls

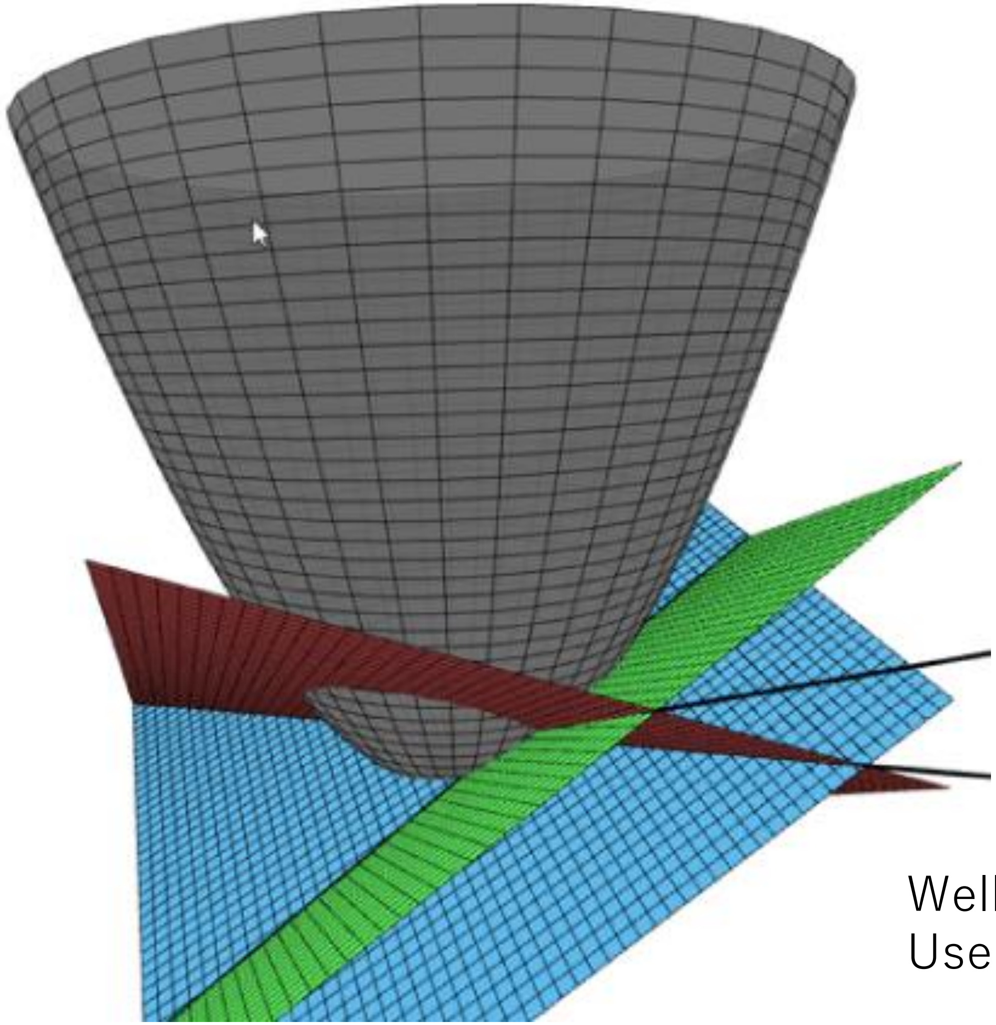


Boolean algebra:
Union of spheres

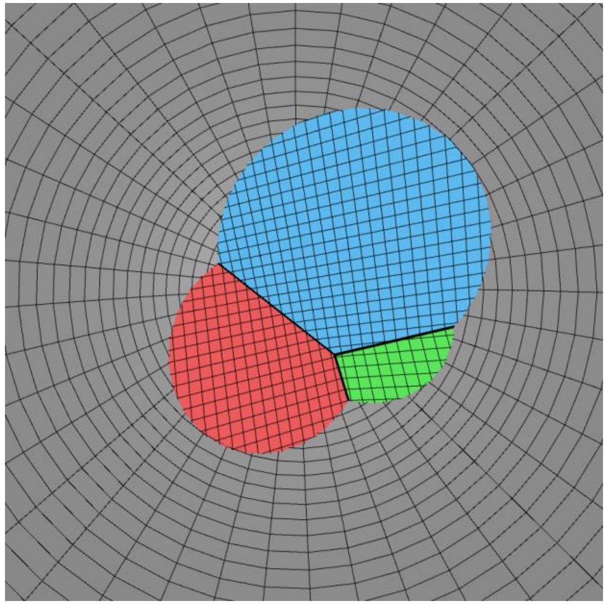
Set Morgan's law: $(A \cup B)^c = A^c \cap B^c$

Complement of halfspace $(H^+)^c = H^-$

Example: Euclidean spheres potential function: Paraboloid, L22



Top view displays the union of disks



$$B_F(\theta_1 : \theta_2) = F(\theta_1) - F(\theta_2) - < \theta_1 - \theta_2 , \nabla F(\theta_2) >$$

Well-known “paraboloid transform” in computational geometry
Used for computing (Bregman) Voronoi and dual Delaunay complex

Wrapping up

- Quick introduction to Bregman divergences

BD reconstructed for partition function Z = reverse extended KLD+

- Duo Bregman pseudo-divergences

BD with 2 majorized generators, applications to KLD between nested exp fams

- Curved Bregman divergences

Symmetrized BD or KLD between circular complex normal, projection theorem

- Generalized Legendre transforms and information geometry

Rev. order involut. transf. = affine-deformed LT = geometric LT of Hessian mfd

- Generalized convexity and Bregman divergences

BDs = limits of scaled skew Jensen div., comparative cvxity, conformal factor

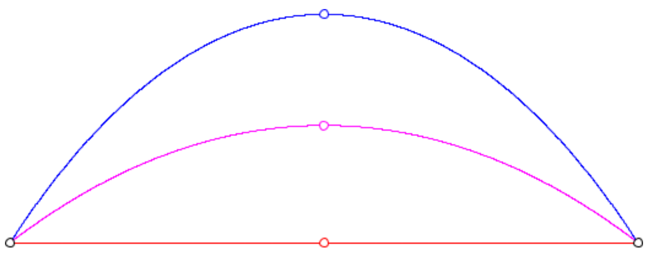
- Space of Bregman balls , Boolean algebra of Bregman balls

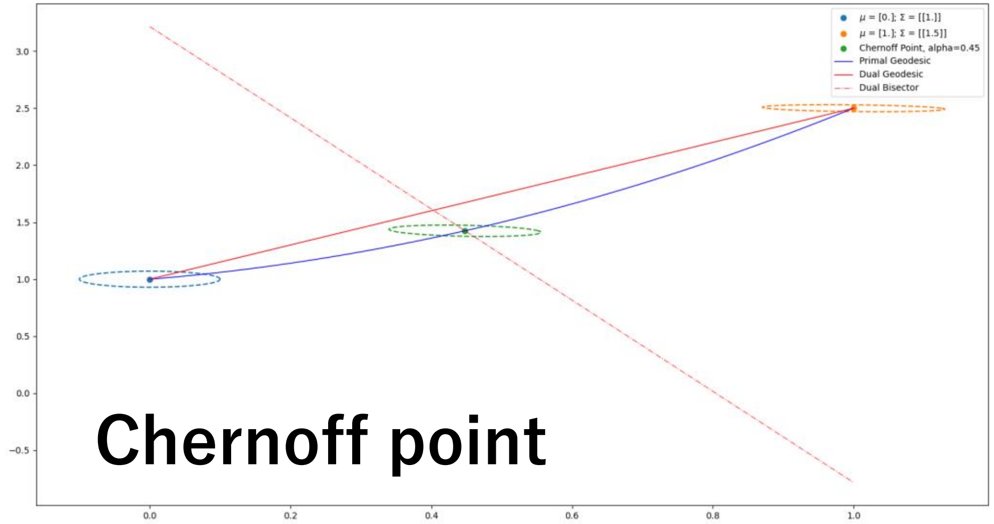
Embedding into higher dim., Boolean operations = intersections of halfspaces

A Python library for geometric computing on Bregman Manifolds

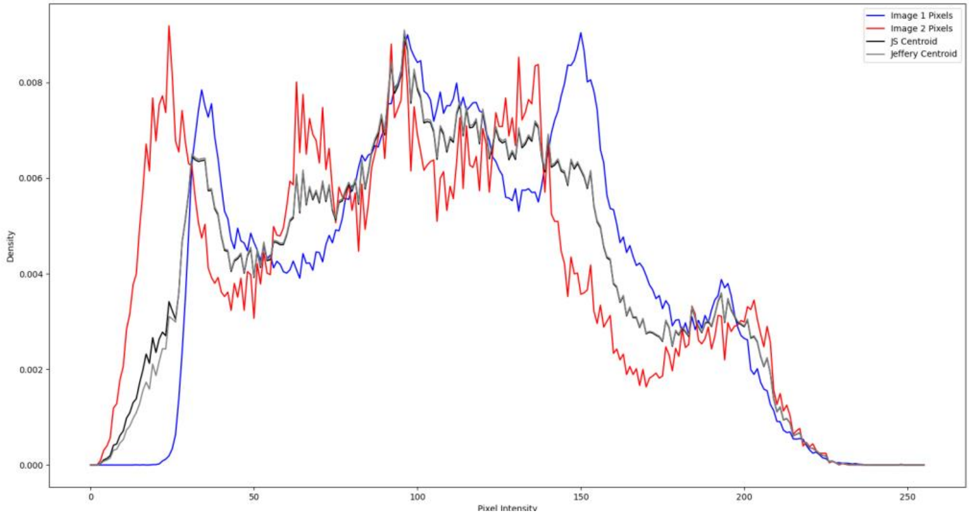
pyBregMan

https://franknielsen.github.io/pyBregMan/

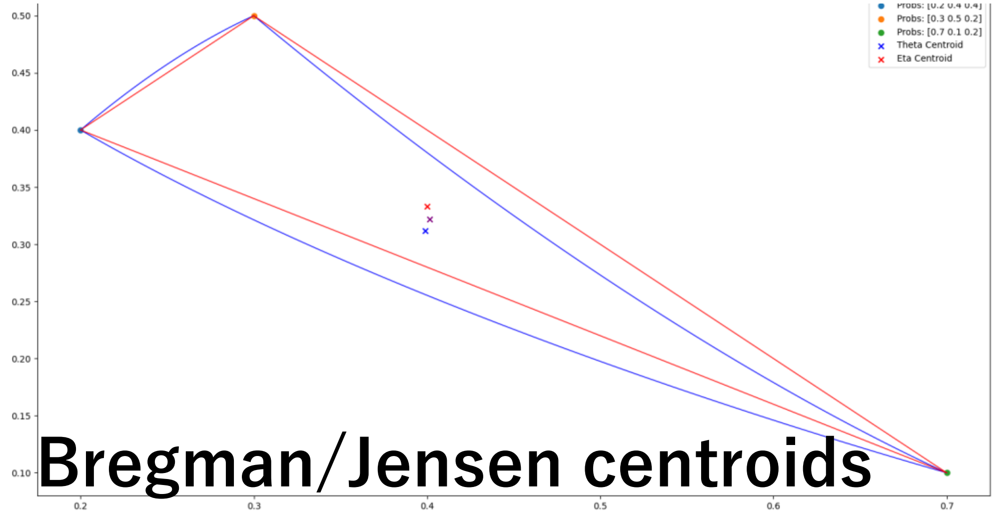




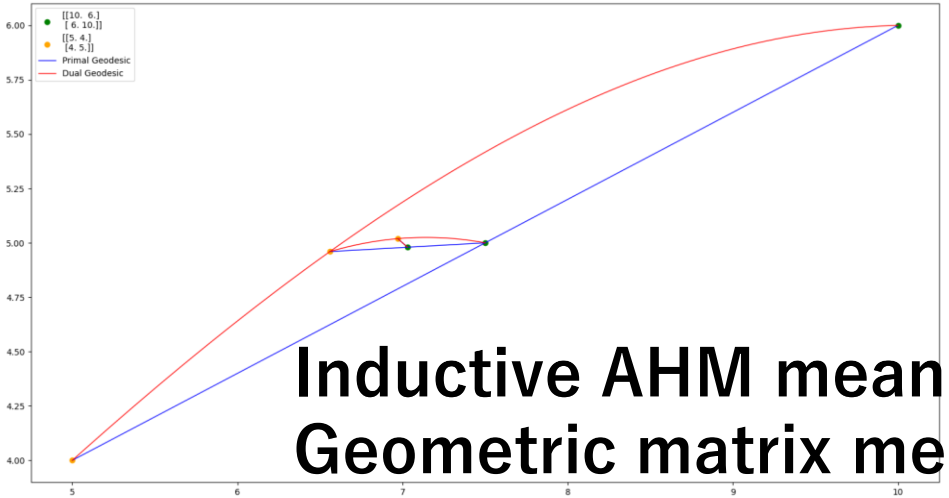
Chernoff point



Jensen-Shannon centroid



Bregman/Jensen centroids



Inductive AHM mean
Geometric matrix mean

Joint work of Frank Nielsen and Alexander Soen

Some references

- A quick introduction to Bregman divergences

Divergences Induced by the Cumulant and Partition Functions of Exponential Families and Their Deformations Induced by Comparative Convexity. Entropy 26(3): 193 (2024)

- Duo Bregman pseudo-divergences

Statistical Divergences between Densities of Truncated Exponential Families with Nested Supports: Duo Bregman and Duo Jensen Divergences. Entropy 24(3): 421 (2022)

- Curved Bregman divergences

Curved representational Bregman divergences and their applications. 2504.05654 (2025)

- Generalized Legendre transforms and information geometry

A note on the Artstein-Avidan-Milman's generalized Legendre transforms. 2507.20577 (2025)

- Generalized convexity and Bregman divergences

Generalizing Skew Jensen Divergences and Bregman Divergences With Comparative Convexity. IEEE Signal Process. Lett. 24(8): 1123-1127 (2017)

- Space of Bregman balls

Bregman Voronoi Diagrams. Discret. Comput. Geom. 44(2): 281-307 (2010)

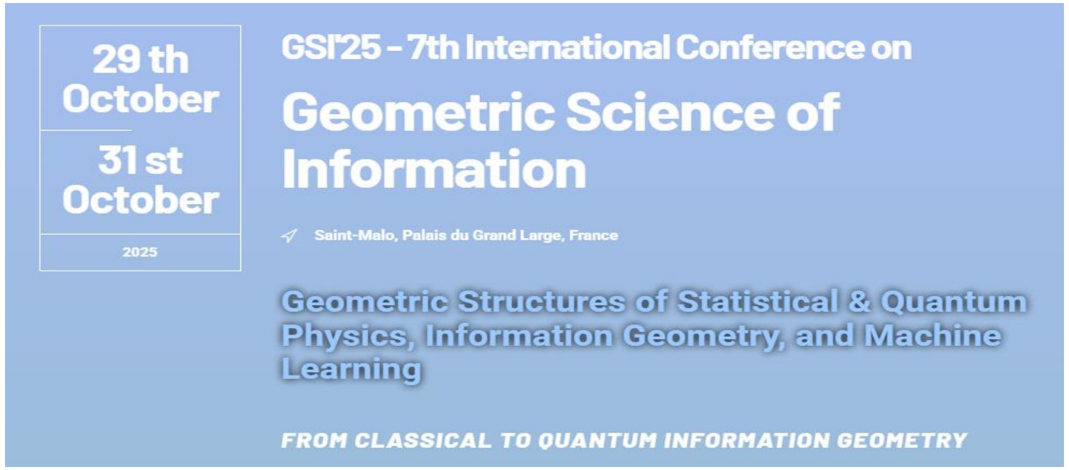
Yet further generalizations…

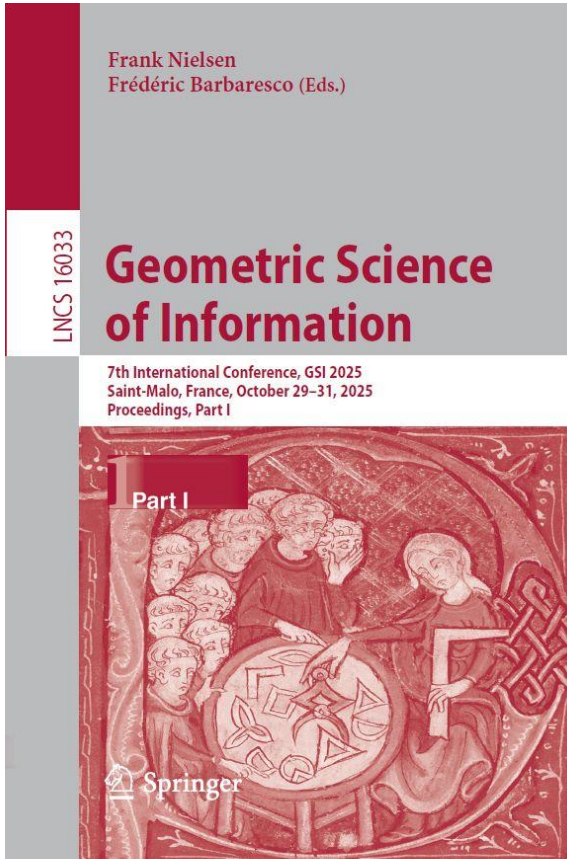
Bregman divergence

- representational Bregman divergence
- curved Bregman divergence
- total Bregman divergence
- conformal Bregman divergence
- duo pseudo-Bregman divergence
- matrix Bregman divergence
- comparative-convexity Bregman divergence
- chord Bregman divergence
- tangent Bregman divergence
- Bregman-Chernoff divergence
- Jensen-Bregman divergence
- quasi-convex Bregman divergence
- Symplectic Bregman divergence
- Symmetrized Bregman divergence

Thank you!

Geometric Science of Information conference:

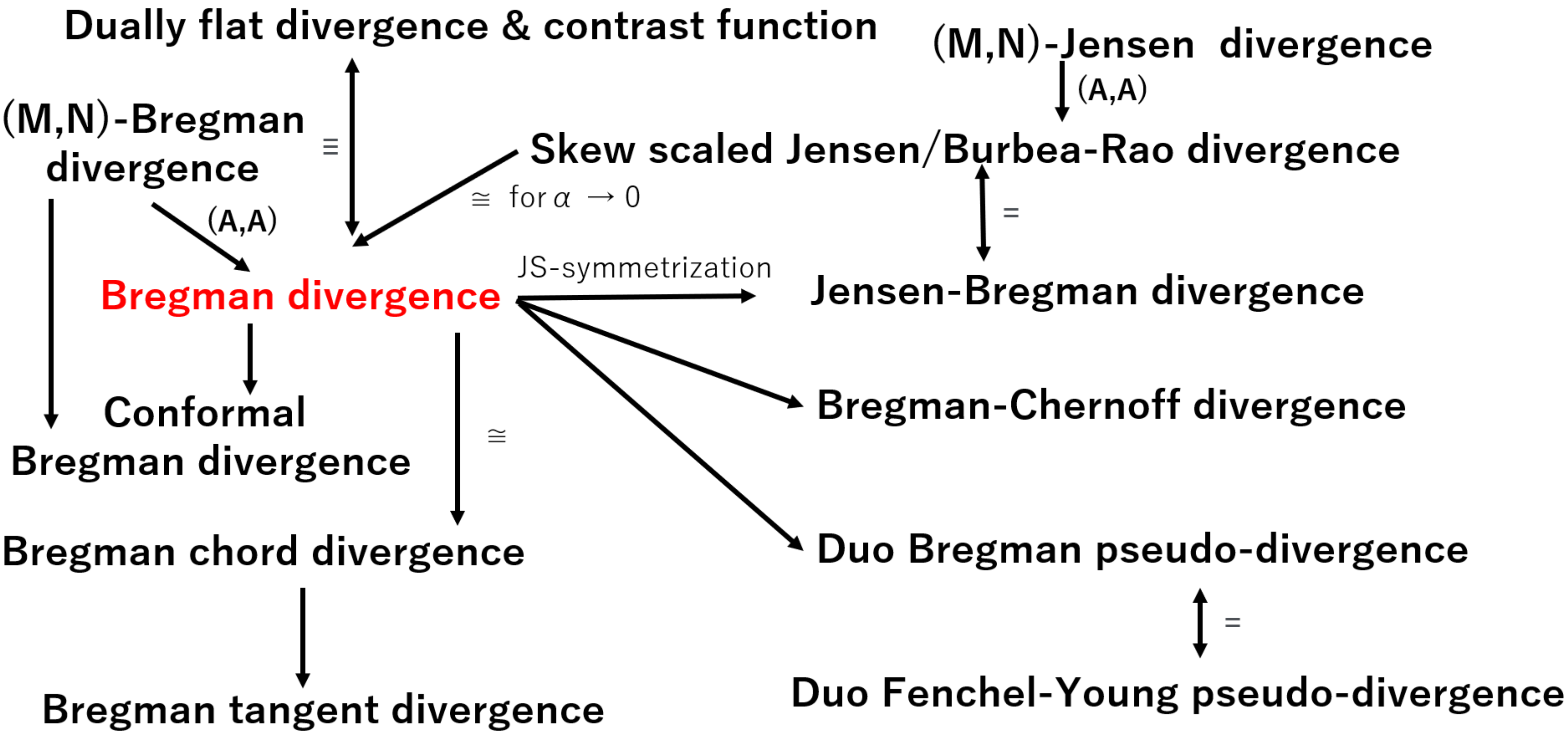




https://franknielsen.github.io/GSI/

Slides: https://franknielsen.github.io/MML25.pdf

Panorama of some generalizations of Bregman divergences



But also matrix Bregman divergence, functional Bregman divergence, etc.