# 1 Klein geodesics (constant speed parameterized by arclengths)

The Klein pregeodesics are straight line segments clipped to the disk domain and can thus be easily parameterized by linear interpolation:

$$\Gamma(p,q) = \{(1-\alpha)p + \alpha q \ : \ \alpha \in [0,1]\}.$$

The Klein metric distance $d_K(p,q)$ between point $p$ and $q$ in the unit disk centered at the origin with curvature $-1$ is

$$d_K(p,q) = \sqrt{-\kappa}\ \mathrm{arccosh}\left(\frac{1-p^\top q}{\sqrt{(1-p^\top p)}\sqrt{(1-q^\top q)}}\right).$$

To get the matching between $(1-\alpha)p + \alpha q$ and $(1-c(\alpha))p + c(\alpha)q$, we need to solve for $\alpha$ in the equation:

$$\frac{a-b\alpha}{\sqrt{a(a-2b\alpha+c\alpha^2)}} - d(\alpha) = 0,$$

with

$$\begin{aligned}
a &= 1-p^\top p,\\
b &= p^\top(q-p),\\
c &= (q-p)^\top(q-p),\\
d(\alpha) &= \cosh(\alpha d_K(p,q))
\end{aligned}$$

Using symbolic calculations, we find the following solution:

$$c(\alpha) = \frac{ad(\alpha)\sqrt{(ac+b^2)(d(\alpha)^2-1)} + ab(1-d(\alpha)^2)}{acd(\alpha)^2+b^2}.$$

Thus we get in closed-form the Klein geodesics (albeit a large formula).
We check that we have

$$d_K(\gamma(p,q;s),\gamma(p,q;t)) = |s-t|\ d_K(p,q), \quad \forall s,t\in[0,1].$$

Extend to Cayley-Klein geometries.
Program: `KleinGeodesic.java`