On balls in a Hilbert polygonal geometry

Frank Nielsen[1] and Laëtitia Shao[2]

1 **École Polytechnique, LIX**
Palaiseau, France
`Frank.Nielsen@acm.org`

2 **École Polytechnique**
Palaiseau, France
`Laetitia.Shao@polytechnique.edu`

Abstract

Hilbert geometry is a metric geometry that extends the hyperbolic Cayley-Klein geometry. In this video, we explain the shape of balls and their properties in a convex polygonal Hilbert geometry. First, we study the combinatorial properties of Hilbert balls, showing that the shapes of Hilbert polygonal balls depend both on the center location and on the complexity of the Hilbert domain but not on their radii. We give an explicit description of the Hilbert ball for any given center and radius. We then study the intersection of two Hilbert balls. In particular, we consider the cases of empty intersection and internal/external tangencies.

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1 Introduction: Hilbert geometry

Hilbert geometry is a projective geometry relying on the properties of the cross-ratio:

▶ **Definition 1** (Cross-ratio). For four collinear points a, b, c, d the cross ratio is defined as follows:

$$(a, b; c, d) = \frac{\|ac\|\|bd\|}{\|ad\|\|bc\|} \tag{1}$$

The cross-ratio is an invariant measure under perspective transformation:

▶ Property 1 (Projective invariance of the cross-ratio). Given four points a, b, c, d and A, B, C, D their images through a projective transformation, $(a, b; c, d) = (A, B; C, D)$. [5]

In a Hilbert geometry, the distance between two points is defined using the cross-ratio as follows:

▶ **Definition 2** (Hilbert distance). A Hilbert distance is defined in the interior of a convex bounded domain $\mathcal{C}$. Given two distinct points, a and b of the domain, the distance is defined as follows:

$$d_{HG}(a, b) = \log((a, b; A, B)) \tag{2}$$

where $(a, b; A, B)$ is the cross-ratio where A and B denote the intersection points of line (a, b) with the domain. By definition, $d_{HG}(x, x) = 0$ for all $x \in \mathcal{C}$.

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Figure 1 Left: In blue, two Hilbert balls in a circular domain. Right: In blue, three Hilbert balls in a polygonal convex domain.

▶ Property 2 (Properties of the Hilbert distance). Given two points a and b.

- The Hilbert distance is a signed distance: $d_{HG}(a,b) = -d_{HG}(b,a)$.
- $d_{HG}(a,a) = 0$ (law of the indiscernibles).
- When a is on the boundary of the convex, $\forall b \in \mathcal{C}, d_{HG}(a,b) = \infty$.
- $|d_{HG}|$ respects the triangular inequality and therefore $|d_{HG}|$ is a metric distance [1].

A key property in Hilbert geometry is that shortest-path geodesics are straight lines. The Klein disk representation of hyperbolic geometry is an example of Hilbert geometry for the unit disk (convex and smooth) domain.

In this work, we consider convex polygonal Hilbert geometries, that is, Hilbert geometries defined on a convex polygonal domain. $\mathcal{C}$ now refers to a convex polygon with s vertices: $e_1, ..e_s$. The distance between two points p and q in this domain is noted $d_{\mathcal{C}}(p,q)$. The ball of radius r and center c is denoted by $\mathcal{B}(c,r)$. The sphere is denoted by $\mathcal{S}(c,r)$. See [4] for an application of Hilbert geometry to clustering in the open probability simplex.

2 Combinatorial properties of Hilbert balls

In Klein ball hyperbolic geometry or Cayley-Klein hyperbolic geometry, the balls have the shape of (Euclidean) Mahalanobis balls with displaced centers, see [2, 3]. To contrast with this smooth shape representation of balls, let us observe that when the domain is a convex polygon, the shapes of Hilbert balls are (Euclidean) polygons.

▶ **Definition 3** (Rays). Given a center point c in the domain, line $(c, e_i), i \in [s] = \{1, \ldots, s\}$ is a ray.

▶ **Lemma 4** (Description of a Hilbert ball). *$\mathcal{B}(c,r)$ is a Euclidean polygon with at most $2s$ edges and at least s edges. Each vertex of $\mathcal{B}(c,r)$ belongs to a ray.*

Proof. We first partition the polygonal domain with s to $2s$ triangles, by tracing rays $(c, e_i), i \in [s]$. We will show that each triangle induces a linear edge of the Hilbert ball.
We consider a pair of triangles (A,B,c) and (c,C,D) such that A,c,D and B,c,C are respectively collinear. Let $P \in [A,c] \cap \mathcal{B}(c,r)$ and $O = (A,B) \cap (C,D)$, we will show that line (O,P) clipped to the triangle (A,B,c) is an edge of $\mathcal{B}(c,r)$. Let U be a point on the clipped line, and M,N the intersections points of line (Uc) with the domain such that $M \in [A,B]$ and $N \in [C,D]$. Then M,U,c,N and A,P,c,D are related by the same projective transformation. Using the invariance property of the cross-ratio, we conclude that $d_{\mathcal{C}}(c,P) = d_{\mathcal{C}}(c,U) = r$.
Thus, we proved Lemma 4. It is remarkable that depending on the position of the center, the number of triangles (and hence the complexity of the ball) varies. ◀

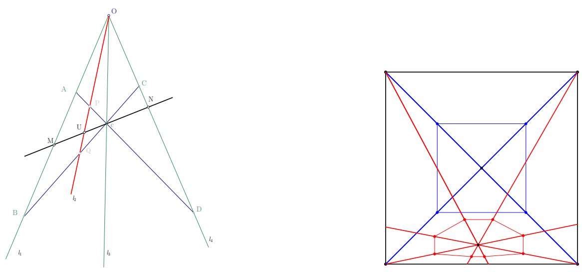

Figure 2 Left: Configuration for proof 2 (see text). Right: Varying number of rays in a square domain depending on the position of the center of the ball.

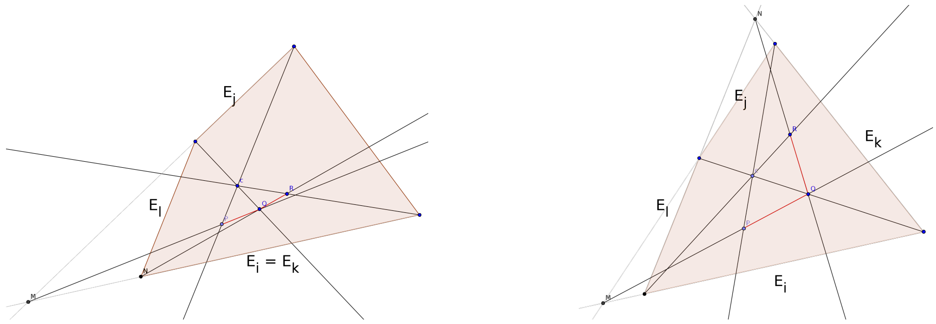

Figure 3 Left: Configuration for proof 2 when $E_i = E_k$. Right: Configuration for proof 2 when all edges are distinct.

▶ **Definition 5.** Given an edge $[P, Q]$ of a Hilbert ball that belongs to a pair of triangles (A, B, c) and (c, D, E), we say that $[P, Q]$ is induced by edges E_i and E_j of the domain, if $[A, B] \subset E_i$ and $[D, E] \subset E_j$.

▶ **Lemma 6** (Shape invariance with varying radius)**.** *For c a fixed center point, and r a varying radius, $\mathcal{B}(c, r)$ has the same number of edges.*

Proof. Let $[P, Q]$ and $[Q, R]$ be two adjacent edges of a Hilbert ball such that E_i, E_j induces $[P, Q]$ and E_k, E_l induces $[Q, R]$. We show that P, Q, R cannot be collinear. We note M the intersection points of the lines supported by E_i and E_j and N the intersection points of the lines supported by E_k and E_l. According to the previous proof, P, Q, M and Q, R, N are respectively collinear.

- If E_i, E_j, E_k, E_l are distinct edges, because $[P, Q]$ and $[Q, R]$ are adjacent, we can assume without loss of generality that E_i is adjacent to E_k and E_j is adjacent to E_l. If P, Q, R are collinear, then $E_i = E_k$ or $E_j = E_l$, which contradicts the previous assumption.
- Otherwise, we can assume that $E_i = E_k$. In this case, if P, Q, R are collinear, then they belong to line $(M, N) \subset E_i$. Which is impossible unless $r = \infty$.

Therefore, as the radius varies but stay finite, the number of edges remains constant. See Figure 3 for a visualization of the proof. For infinite radius, all balls fully cover the polygonal domain. ◀

▶ **Lemma 7** (Shape invariance in a simplex domain)**.** *In a simplex domain Δ, Hilbert polygonal balls do not change shape, and have a fixed complexity of $2s$ edges.*

Proof. It is a direct consequence of the two previous lemmas. ◀

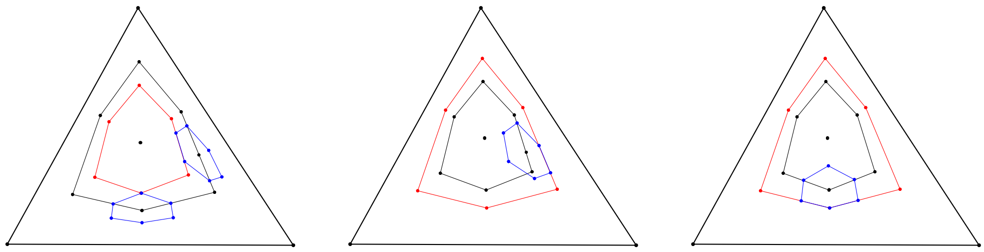

Figure 4 In a triangular domain. Left: Two cases of outer tangency. The red sphere is externally tangent to the blue spheres and share one edge with one sphere and one vertex with the other. Middle and Right: Two cases of inner tangency between the red sphere and the blue sphere. Middle: The two spheres share one edge. Right: the two spheres share two edges.

3 Intersection of Hilbert spheres

We now consider the interaction scenario of two spheres. First, let us mention a simple condition to check whether two spheres intersect or not:

▶ **Lemma 8** (Condition for empty intersection). *Given two points $c_1, c_2 \in \mathcal{C}$ and two reals $r_1, r_2 > 0$, with $r_2 >= r_1$:*

$$\mathcal{S}(c_1, r_1) \cap \mathcal{S}(c_2, r_2) \neq \emptyset \Rightarrow r_2 - r_1 \leq d_{\mathcal{C}}(c_1, c_2) \leq r_1 + r_2 \tag{3}$$

Proof. This follows from the fact that $d_{\mathcal{C}}$ respects the triangular inequality [1]. ◀

In the case of *external tangency*, *i. e.*, $d_{\mathcal{C}}(c_1, c_2) = r_1 + r_2$, if c_2 is a vertex of $\mathcal{B}(c_1, r_1 + r_2)$, the intersection of the two Hilbert spheres is reduced to a vertex. Otherwise, the two Hilbert spheres share part of an edge. In the case of *internal tangency*, *i. e.*, $d_{\mathcal{C}}(c_1, c_2) = r_2 - r_1$, if c_1 is a vertex of $\mathcal{B}(c_2, r_2 - r_1)$, the two spheres share part of two edges. Otherwise the shared part is one edge. See Figure 4 for some illustrating examples. The Java™ applet is available from `https://www.lix.polytechnique.fr/~nielsen/software.html`: The pop menu let one choose the demo to play. The online explanatory video is available at `https://www.youtube.com/watch?v=XE5x5rAK8Hk`

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