RESEARCH PAPER

On the geometric mechanics of assignment flows for metric data labeling

Fabrizio Savarino[1] · Peter Albers[2] · Christoph Schnörr[2]

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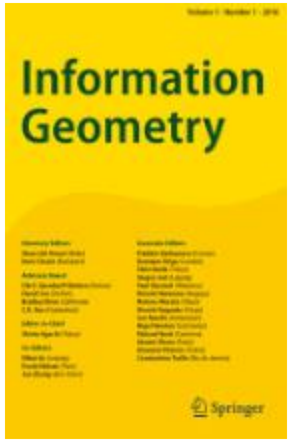
Abstract
Metric data labeling refers to the task of assigning one of multiple predefined labels to every given datapoint based on the metric distance between label and data. This assignment of labels typically takes place in a spatial or spatio-temporal context. Assignment flows are a class of dynamical models for metric data labeling that evolve on a basic statistical manifold, the so called assignment manifold, governed by a system of coupled replicator equations. In this paper we generalize the result of a recent paper for uncoupled replicator equations and adopting the viewpoint of geometric mechanics, relate assignment flows to critical points of an action functional via the associated Euler–Lagrange equation. We also show that not every assignment flow is a critical point and characterize precisely the class of coupled replicator equations fulfilling this relation, a condition that has been missing in recent related work. Finally, some consequences of this connection to Lagrangian mechanics are investigated including the fact that assignment flows are, up to initial conditions of measure zero, reparametrized geodesics of the so-called Jacobi metric.

Keywords Assignment flows · Replicator equation · Information geometry · Geometric mechanics · Metric data labeling



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RESEARCH PAPER

The generalized Pythagorean theorem on the compactifications of certain dually flat spaces via toric geometry

Hajime Fujita[1]

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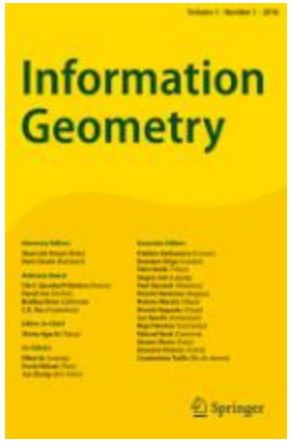
Abstract
In this paper we study dually flat spaces arising from Delzant polytopes equipped with a symplectic potential together with their corresponding toric Kähler manifolds as their torifications. We introduce a dually flat structure and the associated Bregman divergence on the boundary from the viewpoint of toric Kähler geometry. We show a continuity and a generalized Pythagorean theorem for the divergence on the boundary. We also provide a characterization for a toric Kähler manifold to become a torification of a mixture family on a finite set.

Keywords Dually flat space · Delzant polytope · Toric Kähler manifold · Bregman divergence · Pythagorean theorem



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RESEARCH PAPER

Diffusion hypercontractivity via generalized density manifold

Wuchen Li[1]

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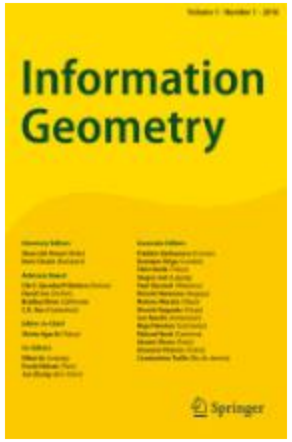
Abstract

We prove a one-parameter family of diffusion hypercontractivity from a class of drift-diffusion processes. We next derive the related log–Sobolev, Poincare, and Talagrand inequalities. The derivation is based on the calculation of Hessian operators along generalized gradient flows in Dolbeault–Nazaret–Savare metric spaces (Dolbeault et al., Calc Var Partial Differ 2:193–231, 2010). In this direction, a mean-field type Bakry–Emery iterative calculus is presented. In particular, an inequality among Pearson divergence (P), negative Sobolev metric (H^{-1}), and generalized Fisher information functional (I), named $PH^{-1}I$ inequality, is presented.

Keywords Information theory · Mean-field Bakry–Emery calculus · Generalized log–Sobolev inequality · Generalized Poincare inequality · Generalized Talagrand inequality · Generalized Yano's formula.



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RESEARCH PAPER

Information geometry of dynamics on graphs and hypergraphs

Tetsuya J. Kobayashi[1,2,3] · Dimitri Loutchko[1] · Atsushi Kamimura[1] ·
Shuhei A. Horiguchi[2] · Yuki Sughiyama[1]

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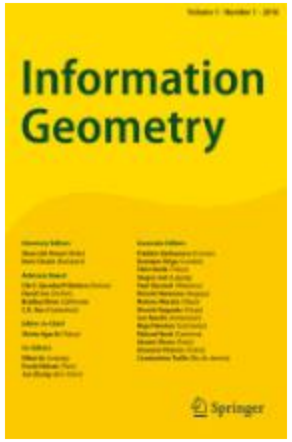
Abstract

We introduce a new information-geometric structure associated with the dynamics on discrete objects such as graphs and hypergraphs. The presented setup consists of two dually flat structures built on the vertex and edge spaces, respectively. The former is the conventional duality between density and potential, e.g., the probability density and its logarithmic form induced by a convex thermodynamic function. The latter is the duality between flux and force induced by a convex and symmetric dissipation function, which drives the dynamics of the density. These two are connected topologically by the homological algebraic relation induced by the underlying discrete objects. The generalized gradient flow in this doubly dual flat structure is an extension of the gradient flows on Riemannian manifolds, which include Markov jump processes and nonlinear chemical reaction dynamics as well as the natural gradient. The information-geometric projections on this doubly dual flat structure lead to information-geometric extensions of the Helmholtz–Hodge decomposition and the Otto structure in L^2-Wasserstein geometry. The structure can be extended to non-gradient nonequilibrium flows, from which we also obtain the induced dually flat structure on cycle spaces. This abstract but general framework can broaden the applicability of information geometry to various problems of linear and nonlinear dynamics.

Keywords Dually flat structure · Thermodynamics · Homological Algebra · Discrete calculus · Helmholtz decomposition



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On the complete integrability of gradient systems on manifold of the beta family of the first kind

Prosper Rosaire Mama Assandje[1] · Joseph Dongho[1] · Thomas Bouetou Bouetou[2]

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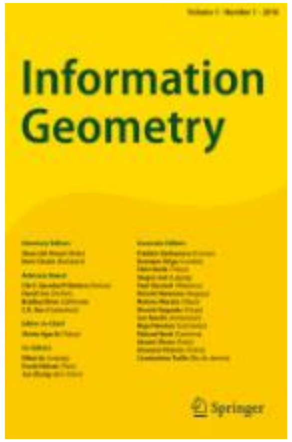
Abstract
In this article, it is shown that there exists a gradient system that is Hamiltonian and completely integrable on the beta manifold of the first kind with two parameters and whose existence depends on a potential function with duality on the manifold and which is the solution of the Legendre equation. It is shown that, by making a statistical borrowing from the gamma function, the gradient system remains Hamiltonian and completely integrable. It is shown that the gradient system constructed on the first kind of two-parameter beta manifold admits a Lax pair representation. This also makes it possible to prove its complete integrability and to determine its Hamiltonian function.

Keywords Hamiltonian system · Gradient system · Lax pair representation · Hamiltonian



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Variational representations of annealing paths: Bregman information under monotonic embedding

Rob Brekelmans[1] · Frank Nielsen[2]

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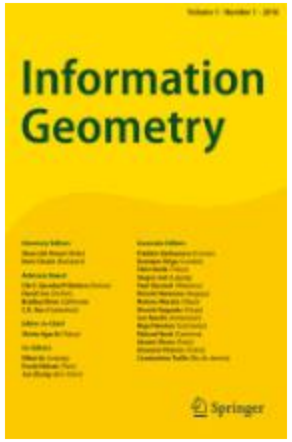
Published online: 2 February 2024

Abstract
Markov chain Monte Carlo methods for sampling from complex distributions and estimating normalization constants often simulate samples from a sequence of intermediate distributions along an *annealing path*, which bridges between a tractable initial distribution and a target density of interest. Prior works have constructed annealing paths using quasi-arithmetic means, and interpreted the resulting intermediate densities as minimizing an expected divergence to the endpoints. To analyze these variational representations of annealing paths, we extend known results showing that the arithmetic mean over arguments minimizes the expected Bregman divergence to a single representative point. In particular, we obtain an analogous result for quasi-arithmetic means, when the inputs to the Bregman divergence are transformed under a monotonic embedding function. Our analysis highlights the interplay between quasi-arithmetic means, parametric families, and divergence functionals using the rho-tau representational Bregman divergence framework, and associates common divergence functionals with intermediate densities along an annealing path.

Keywords Bregman divergence · Bregman information · Monotone embedding · Quasi-arithmetic means · Non-parametric information geometry · Gauge freedom · Annealing paths · Markov chain Monte Carlo



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RESEARCH

Statistical manifold with degenerate metric

Kaito Kayo[1]

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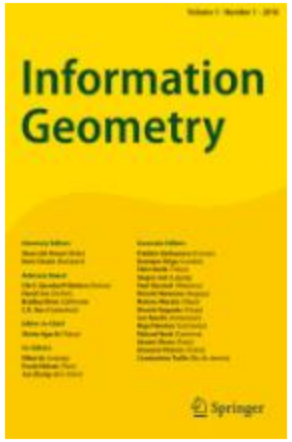
Abstract

A *statistical manifold* is a pseudo-Riemannian manifold endowed with a *Codazzi structure*. This structure plays an important role in Information Geometry and its related fields, e.g., a statistical model admits this structure with the Fisher–Rao metric. In practical application, however, the metric may be degenerate, and then this geometric structure is not fully adapted. In the present paper, for such cases, we introduce the notion of *quasi-Codazzi structure* which consists of a possibly degenerate metric (i.e., symmetric (0,2)-tensor) and a pair of coherent tangent bundles with affine connections. This is thought of as an affine differential geometry of Lagrange subbundles of para-Hermitian vector bundles and also as a submanifold theory of para-Hermitian space-form. As a special case, the quasi-Codazzi structure with flat connections coincides with the *quasi-Hessian structure* previously studied by Nakajima–Ohmoto. The relation among our quasi-Codazzi structure, quasi-Hessian structure and weak contrast functions generalizes the relation among Codazzi structure, dually flat (i.e., Hessian) structure and contrast functions.

Keywords Codazzi structure · Statistical manifold · Para-complex geometry · Dually flat structure



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Codivergences and information matrices

Alexis Derumigny[1] · Johannes Schmidt-Hieber[2]

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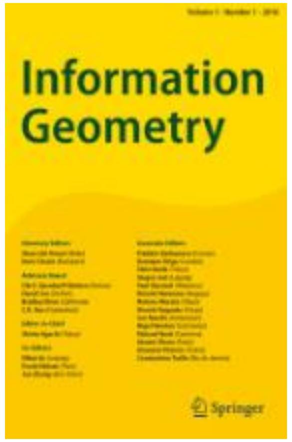
Abstract
We propose a new concept of codivergence, which quantifies the similarity between two probability measures P_1, P_2 relative to a reference probability measure P_0. In the neighborhood of the reference measure P_0, a codivergence behaves like an inner product between the measures $P_1 - P_0$ and $P_2 - P_0$. Codivergences of covariance-type and correlation-type are introduced and studied with a focus on two specific correlation-type codivergences, the χ^2-codivergence and the Hellinger codivergence. We derive explicit expressions for several common parametric families of probability distributions. For a codivergence, we introduce moreover the divergence matrix as an analogue of the Gram matrix. It is shown that the χ^2-divergence matrix satisfies a data-processing inequality.

Keywords Divergence · Chi-square divergence · Hellinger affinity · Gram matrix



Volume 7
Issue 1
Article 82