Review report on the paper entitled

"Conformal flattening on the probability simplex and its applications to Voronoi partitions and centroids"

by Atsumi Oharal

The authors studied an affine immersion of the probability simplex $\mathcal{S}_n$. He showed that $\mathcal{S}_n$ is realized as a 1-conformally flat statistical manifold in $\boldsymbol{R}^n$. As applications, the author studied Voronoi partitions with respect to divergences (Theorem 1), and weighted centroids with respect to the escort probability distributions (Theorem 2).

Many parts of this paper are the same as those commonly known in information geometry. So the main contributions of this paper are above applications.

The paper is well-written, however the reviewer has several questions to the author.

Comments and questions to the author

- The immersion f is defined coordinatewisely. Please give comments why the immersion is independent from the choice of coordinate systems.

  In addition, why the pair $\{f, \xi\}$ is an affine immersion? That is, for a non-singular matrix $A \in GL(n, \boldsymbol{R})$, and a vector $\boldsymbol{b} \in \boldsymbol{R}^n$, define

  $$\tilde{f} = Af + \boldsymbol{b}, \quad \tilde{\xi} = A\xi.$$

  Then the pair $\{\tilde{f}, \tilde{\xi}\}$ is also an affine immersion, and two affine immersions $\{f, \xi\}$ and $\{\tilde{f}, \tilde{\xi}\}$ induce the same geometric structures for M However, assumptions $1 \sim 4$ in pp.2 $\sim$ pp.3 may not satisfy the conditions above. If it is a coordinatewise definition, please give such a remark.

- The author showed that the image $f(\mathcal{S}^n)$ is expressed as a level surface

  $$\Psi(x) := \sum_{i=1}^{n+1} E(x^i) = 1.$$

  Why this expression is invariant under affine transformations? It seems that the condition $\sum E(x^i) = 1$ is not an affine invariant. If it is a coordinatewise definition, please give such a remark.

- In Section 3.2, the author studied weighted centroids. However they are weighted means of distributions $\{P_i\}$

  $$\begin{aligned} P_i(c^{(\rho)}) &= \frac{1}{\sum \omega_\lambda Z(p_\lambda)} \sum \omega_\lambda Z(p_\lambda) P_i(p_\lambda) = \frac{1}{\sum \tilde{\omega}_\lambda} \sum \tilde{\omega}_\lambda P_i(p_\lambda) \\ &\quad (\ \tilde{\omega}_\lambda = \omega_\lambda Z(p_\lambda)\ ), \end{aligned}$$

  since $\{\omega_\lambda\}$ are arbitrary numbers. Does the distribution $\tilde{\omega}_\lambda / \sum \tilde{\omega}_\lambda$ have any meaning?