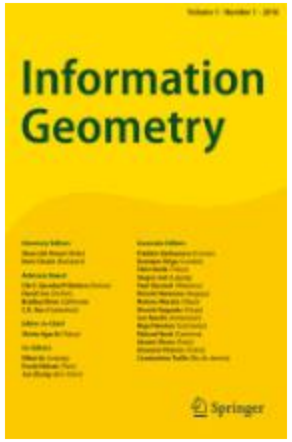
RESEARCH PAPER

Self-similar solutions to the Hesse flow

Shun Maeta[1]

Received: 26 January 2021 / Revised: 27 May 2021 / Accepted: 9 August 2021 /
Published online: 17 August 2021



Volume 4
Issue 2
Article 36

Abstract
We define a Hesse soliton, that is, a self-similar solution to the Hesse flow on Hessian manifolds. On information geometry, the e-connection is important, which does not coincide with the Levi–Civita one. Therefore, it is interesting to consider a Hessian manifold with a flat connection which does not coincide with the Levi–Civita one. We call it a proper Hessian manifold. In this paper, we show that any compact proper Hesse soliton is expanding and any non-trivial compact gradient Hesse soliton is proper. Furthermore, we show that the dual space of a Hesse–Einstein manifold can be understood as a Hesse soliton.

Keywords Hesse flow · Hesse solitons · Hessian manifolds · Information geometry

RESEARCH PAPER

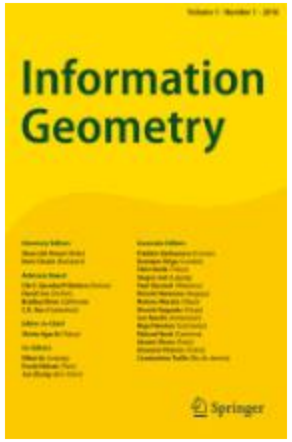
Riemannian barycentres of Gibbs distributions: new results on concentration and convexity in compact symmetric spaces

Salem Said[1] · Jonathan H. Manton[2]

Received: 3 June 2019 / Revised: 17 August 2021 / Accepted: 23 August 2021 /
Published online: 1 September 2021

Abstract
The Riemannian barycentre (or Fréchet mean) is the workhorse of data analysis for data taking values in Riemannian manifolds. The Riemannian barycentre of a probability distribution P on a Riemannian manifold M is a possible generalisation of the concept of expected value, at least when the barycentre is unique. Knowing when the barycentre of P is unique is of fundamental importance for its interpretation and computation. Existing results can only guarantee this uniqueness by assuming P is supported inside a convex geodesic ball $B(x^*, \delta) \subset M$. This assumption is overly restrictive since many distributions have support equal to M yet are sufficiently concentrated within a convex geodesic ball that they nevertheless have a unique barycentre. This paper studies the concentration of Gibbs distributions on Riemannian manifolds and gives conditions for the barycentre to be unique. Specifically, consider the Gibbs distribution $P = P_T$ with unnormalised density $\exp(-U/T)$ for some potential $U : M \to \mathbb{R}$ and some temperature $T > 0$. If M is a simply connected compact Riemannian symmetric space, and U has a unique global minimum at x^*, then for each $\delta < \frac{1}{2} r_{cx}$ (r_{cx} the convexity radius of M), there exists a critical temperature T_δ such that $T < T_\delta$ implies P_T has a unique Riemannian barycentre $\bar{x}_T$ and this $\bar{x}_T$ belongs to the geodesic ball $B(x^*, \delta)$. Moreover, if U is invariant by geodesic symmetry about x^*, then $\bar{x}_T = x^*$. Remarkably, this conclusion does not require the potential U to be smooth and therefore serves as the foundation of a new general algorithm for black-box optimisation. This algorithm is briefly illustrated with two numerical experiments.

Keywords Gibbs distribution · Riemannian barycentre · Wasserstein distance · Symmetric space · Black-box optimisation



Volume 4
Issue 2
Article 37

RESEARCH PAPER

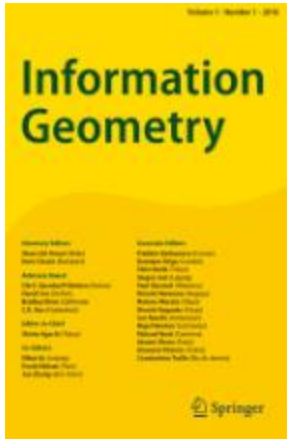
The Bonnet theorem for statistical manifolds

Taiji Marugame[1,2]

Received: 18 March 2021 / Revised: 18 July 2021 / Accepted: 16 September 2021 /
Published online: 29 September 2021

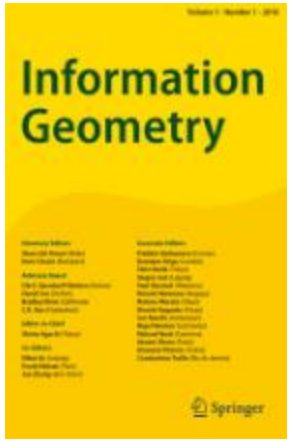
Abstract
We prove the Bonnet theorem for statistical manifolds, which states that if a statistical manifold admits tensors satisfying the Gauss–Codazzi–Ricci equations, then it is locally embeddable to a flat statistical manifold (or a Hessian manifold). The proof is based on the notion of statistical embedding to the product of a vector space and its dual space introduced by Lauritzen. As another application of Lauritzen's embedding, we show that a statistical manifold admitting an affine embedding of codimension 1 or 2 is locally embeddable to a flat statistical manifold of the same codimension.

Keywords Statistical manifolds · Hessian manifolds · The Gauss–Codazzi–Ricci equations · The Bonnet theorem



Volume 4
Issue 2
Article 38

RESEARCH PAPER



Volume 4
Issue 2
Article 39

The Banach manifold of measures and the Lagrange multipliers of statistical mechanics

S. Dostoglou[1] · A. M. Hughes[1]

Received: 26 March 2021 / Revised: 6 September 2021 / Accepted: 18 October 2021 /
Published online: 3 November 2021

Abstract
Using a Banach manifold structure on the space of finite positive measures it is shown that all critical points of the Gibbs/information entropy are grand canonical equilibria when the constraints are scalar, and local equilibria when the constraints are integrable functions. This provides a rigorous derivation of equilibrium and local equilibrium Gibbs measures via Lagrange multipliers.

Keywords Gibbs ensembles · Lagrange multipliers · Local equilibrium · Banach manifold

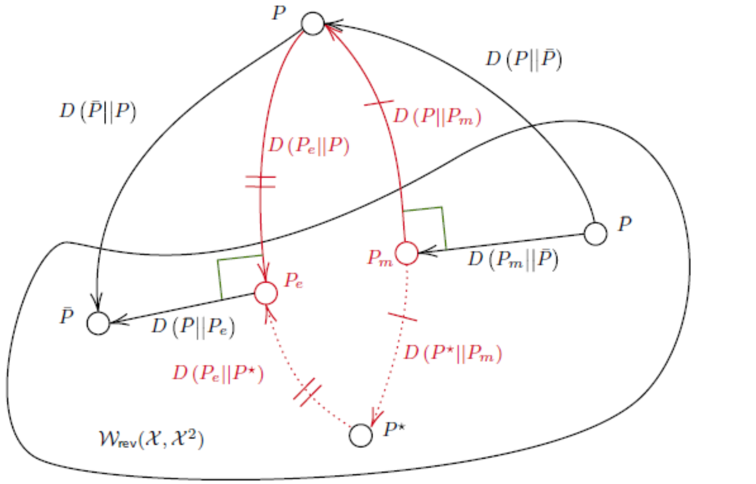


Fig. 1 Information projections P_e and P_m of P onto $\mathcal{W}_{\mathrm{rev}}(\mathcal{X}, \mathcal{X}^2)$ in the full support case ($\mathcal{E} = \mathcal{X}^2$) (Theorem 7), Pythagorean identities (Theorem 7), and the bisection property (Proposition 2)

RESEARCH PAPER

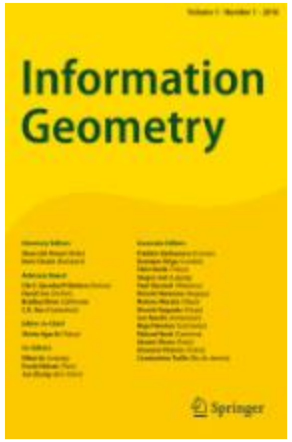
Information Geometry of Reversible Markov Chains

Geoffrey Wolfer[1] · Shun Watanabe[1]

Received: 11 June 2021 / Revised: 1 November 2021 / Accepted: 2 November 2021 /
Published online: 22 November 2021

Abstract
We analyze the information geometric structure of time reversibility for parametric families of irreducible transition kernels of Markov chains. We define and characterize reversible exponential families of Markov kernels, and show that irreducible and reversible Markov kernels form both a mixture family and, perhaps surprisingly, an exponential family in the set of all stochastic kernels. We propose a parametrization of the entire manifold of reversible kernels, and inspect reversible geodesics. We define information projections onto the reversible manifold, and derive closed-form expressions for the e-projection and m-projection, along with Pythagorean identities with respect to information divergence, leading to some new notion of reversiblization of Markov kernels. We show the family of edge measures pertaining to irreducible and reversible kernels also forms an exponential family among distributions over pairs. We further explore geometric properties of the reversible family, by comparing them with other remarkable families of stochastic matrices. Finally, we show that reversible kernels are, in a sense we define, the minimal exponential family generated by the m-family of symmetric kernels, and the smallest mixture family that comprises the e-family of memoryless kernels.

Keywords Irreducible Markov chain · Reversible Markov chain · Exponential family · Mixture family · Information projection



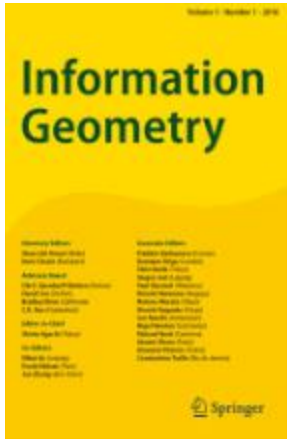
Volume 4
Issue 2
Article 40

RESEARCH PAPER

Transport information Bregman divergences

Wuchen Li[1]

Received: 29 January 2021 / Revised: 3 November 2021 / Accepted: 3 November 2021 /
Published online: 15 November 2021



Volume 4
Issue 2
Article 41

Abstract
We study Bregman divergences in probability density space embedded with the L^2–Wasserstein metric. Several properties and dualities of transport Bregman divergences are provided. In particular, we derive the transport Kullback–Leibler (KL) divergence by a Bregman divergence of negative Boltzmann–Shannon entropy in L^2–Wasserstein space. We also derive analytical formulas and generalizations of transport KL divergence for one-dimensional probability densities and Gaussian families.

Keywords Transport Bregman divergence · Transport KL divergence · Transport Jensen- Shannon divergence

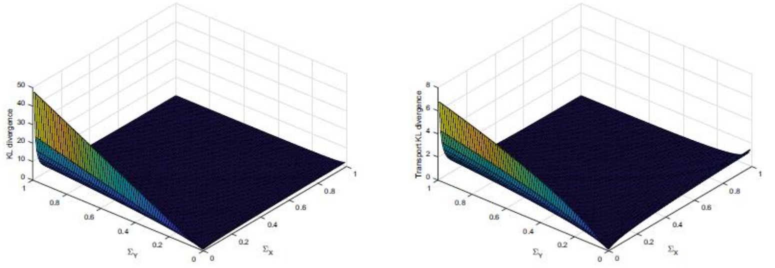


Fig. 1 A comparison between KL divergence and transport KL divergence for one dimensional Gaussian distributions. Left represents the KL divergence. Right represents the transport KL divergence