INFORMATION GEOMETRY AND THE EFFECTIVE FIELD THEORY OF FLOCKING

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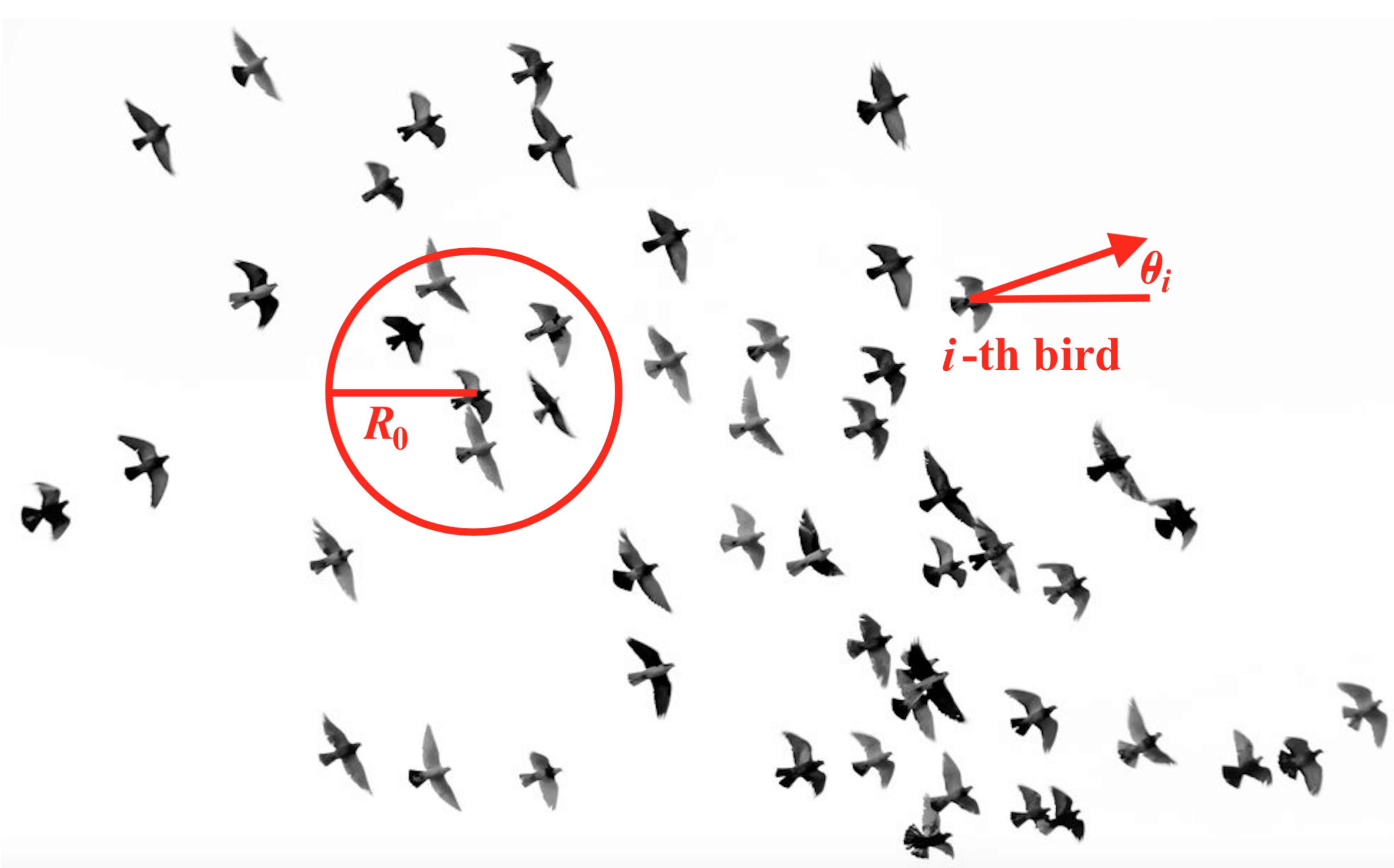
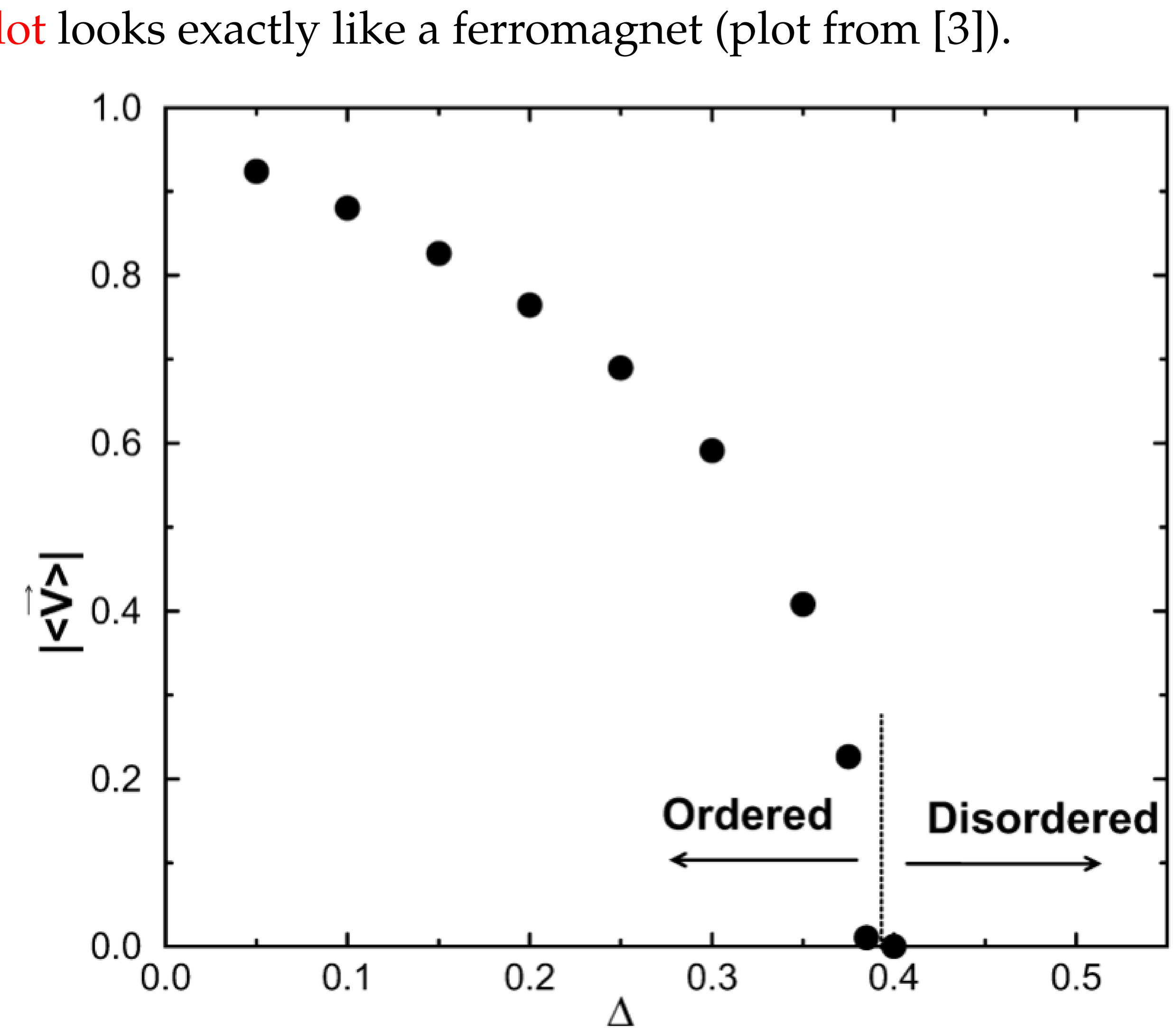
Introduction

- Some flocking behavior (self-driven collective motion) can be modeled as a non-equilibrium ferromagnet [1].
- Flocking behavior is also described by non-relativistic hydrodynamics[2, 3].
- The theory of non-relativistic perfect fluids was recently developed [4]. This can be applied to flocking [5].
- I propose to study these models using information geometry.
- An action principle for these theories would be useful for this purpose.

Basic Model of Flocking



- Birds are initially randomly distributed within a flock of linear size L.
- Each bird moves at the same constant speed v_0.
- The angle at $t+1$ is the average at time t of the angles of the birds within the radius $R_0 \ll L$ plus a Gaussian "error" noise of strength Δ.
- Update position at time step t to $t+1$ according to the angle θ at $t+1$.
- Without position updating, this models an equilibrium ferromagnet with average velocity and noise strength $\sim$ magnetization and temperature.
- Phase plot looks exactly like a ferromagnet (plot from [3]).



- But, unlike ferromagnets, the ordered phase exists even in $d = 2$.
- Birds circumvent the Mermin-Wagner theorem by moving, so that the flock is out of equilibrium.

Non-relativistic Hydrodynamics

- (Homogeneous) background medium breaks boost-invariance.
- Assume rotation- and translation-invariance and bird conservation.
- Describe the system in terms of coarse-grained quantities.
- $\rho(\vec{r}, t)$ = bird density and $\vec{v}(\vec{r}, t)$ = bird velocity.
- Evolution equations:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\mathcal{D}_t \vec{v} = (\alpha - \beta v^2)\vec{v} - \vec{\nabla} P + D_0 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) + D_1 \nabla^2 \vec{v} + D_2 (\vec{v} \cdot \vec{\nabla})^2 \vec{v} + \vec{f}, \tag{2}$$

$$\mathcal{D}_t \vec{v} \equiv \partial_t \vec{v} + \lambda_1 (\vec{v} \cdot \vec{\nabla}) \vec{v} + \lambda_2 (\vec{\nabla} \cdot \vec{v}) \vec{v} + \lambda_3 \vec{\nabla} v^2 \quad \text{(convective derivative)},$$

$$P = P(\rho) = \textstyle\sum_{n=1}^{\infty} \sigma_n (\rho - \rho_0)^n \quad \text{(pressure)},$$

$$D_i = \text{diffusion coefficients},$$

$$\langle f_i(\vec{r}, t) f_j(\vec{r}', t') \rangle = \Delta \delta_{ij} \delta(t - t') \delta^{(d)}(\vec{r} - \vec{r}') \quad \text{(random "error" force)}$$

- Scaling exponents (perpendicular and parallel to net flock motion):

$$(\vec{x}_\perp, x_{||}, t, \vec{v}_\perp) \to (b\vec{x}_\perp, b^\zeta x_{||}, b^z t, b^\chi \vec{v}_\perp). \tag{3}$$

- Renormalization group analysis of the exponents for $d \leq 4$ (and $\lambda_2 = 0$):

$$\zeta = \frac{d+1}{5}, \qquad z = \frac{2(d+1)}{5}, \qquad \chi = \frac{3-2d}{5}. \tag{4}$$

- The flock has long-ranged order if $\chi < 0$, or $d > 1.5$.
- In $d = 2$, λ_2 can be set to 0 since it is equivalent to λ_1.
- In $d > 4$, information flow is diffusive. In $d \leq 4$, it is convective.

Perfect Fluids

- Without boost-invariance, a system can transfer momentum to its environment. So, there is an extra $\vec{v} \cdot d\vec{P}$ term in the first law of thermodynamics.
- This leads to a generalized Navier-Stokes equation like eqn. (2).
- Imposing a barotropic condition of the form $P = w\rho$, and expanding ρ as $\rho = \rho_0 - av^2$ relates the parameters λ_i, α, β, and D_i.
- For example, $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -aw/\rho_0$, and $\alpha = -\mathcal{D}_t \ln \rho_0$.
- We are exploring a possible action principle for these systems [5].

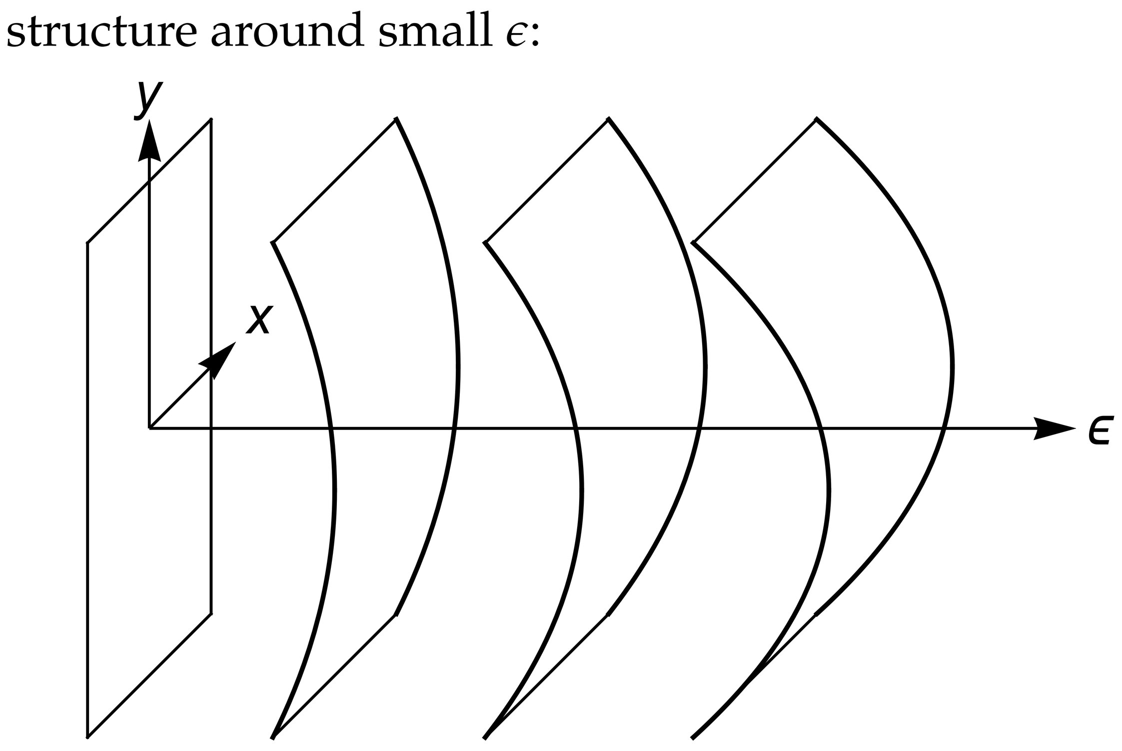
Information Geometry

- It would be nice to find the Fisher metric on the space of states of a flock.
- Does the Fisher metric capture the diffusive-convective transition at $d = 4$?
- We can find the Fisher metric on the states of barotropic perfect fluids or on their couplings if we have an action principle.
- Example: Ideal Gas of Lifshitz particles with dispersion $E = ap^z$.
  States are parametrized by P, T and $\vec{v}$ (fluid velocity).
  Coordinates and Fisher metric:

$$x = \ln \frac{P}{T} \qquad y = \sqrt{\frac{d}{z}} \ln \frac{a}{T} \qquad \vec{\epsilon} = \sqrt{\frac{\Gamma(\frac{d+2}{z})}{d\,\Gamma(\frac{d}{z})}} \frac{\vec{v}/T}{(a/T)^{2/z}} \tag{5}$$

$$ds^2 = dx^2 + \left(1 + \frac{z-2}{zd} \epsilon^2 e^{2y/\sqrt{zd}}\right) dy^2 + e^{2y/\sqrt{zd}} d\vec{\epsilon} \cdot d\vec{\epsilon} \tag{6}$$

Foliation structure around small ϵ:



Interestingly, the leaves are exactly flat when $z = 2$.

References

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