Geometric Sciences of Information: Random musings

Frank Nielsen
Frank.Nielsen@acm.org

Last updated, May 12, 2022

f-divergences between location-scale families

- Multivariate location-scale family with standard density $p(x)$: $p_{\mu,\Sigma}(x) := |\Sigma|^{-1/2}\, p\left(\Sigma^{-1/2}(x-\mu)\right), x \in \mathbb{R}^d$, $p(x) = \tilde{p}(\|x^2\|)$ (include multivariate normal distributions, multivariate Cauchy distributions)
- f-divergence between $P, Q \ll \mu$ for strictly convex generator $f \in C^2$ with $f(1) = 0$: $I_f(P:Q) := \int_{\mathcal{X}} p\, f\left(\frac{q}{p}\right) \mathrm{d}\mu$ (include KL divergence)
- $I_f\left(p_{\mu_1,\Sigma} : p_{\mu_2,\Sigma}\right) = h_f\left(\Delta_\Sigma^2(\mu_1,\mu_2)\right)$ where $\Delta_\Sigma^2(\mu_1,\mu_2) := (\mu_2-\mu_1)^\top \Sigma^{-1}(\mu_2-\mu_1)$ for strictly increasing h_f (Mahalanobis distance Δ_Σ)

  Preserve relative comparisons:

  $I_f(p_{\mu_1,\Sigma} : p_{\mu_2,\Sigma}) < I_f(p_{\mu_1',\Sigma} : p_{\mu_2',\Sigma}) \Leftrightarrow \Delta_\Sigma^2(\mu_1,\mu_2) < \Delta_\Sigma^2(\mu_1',\mu_2')$

  $\Rightarrow$ Voronoi diagrams, minimum enclosing ball, k-center clustering **independent** of f
- Since $\Delta_\Sigma^2(\mu_1,\mu_2) = \Delta_1(0, \Delta_\Sigma^2(\mu_1,\mu_2))$, multivariate f-div. amounts to univariate f-div ($\rightarrow$ fast Monte Carlo estimations: eg., Jensen-Shannon div.): $I_f[p_{\mu_1,\Sigma}, p_{\mu_2,\Sigma}] = I_f\left[p_{0,1}, p_{\Delta_\Sigma(\mu_1,\mu_2),1}\right]$
- Spectral matrix f-divergences for multivariate scale families: $I_f[p_{\mu,\Sigma_1} : p_{\mu,\Sigma_2}] = E_f(|1-\lambda_1|, \ldots, |1-\lambda_d|)$, where $E_f(\cdot)$ is a d-variate totally symmetric function and $\lambda_i \in \mathrm{sp}(\Sigma_2\Sigma_1^{-1})$. [arXiv:2204.10952]