Hilbert geometry of the symmetric positive-definite bicone

Application to the geometry of the extended Gaussian family

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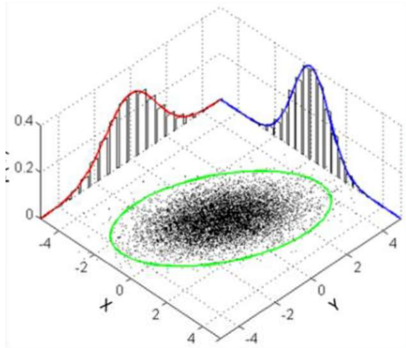
SPD/PSD cone and Gaussian family

1. Symmetric positive-definite matrix cone (SPD) is open

2. Symmetric positive semi-definite matrix cone (PSD) is closed with SPD in its interior.

3. Related to family of centered Gaussian distributions

4. Two dual parameterizations on SPD: Covariance and Precision/Information matrix:



$$p_\Sigma(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}\langle x, \Sigma^{-1}x\rangle\right) \qquad p_P(x) = \frac{\sqrt{\det(P)}}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{1}{2}\langle x, Px\rangle\right)$$

What about degeneracies of Gaussians? Extended Gaussians

TL;DR

We:

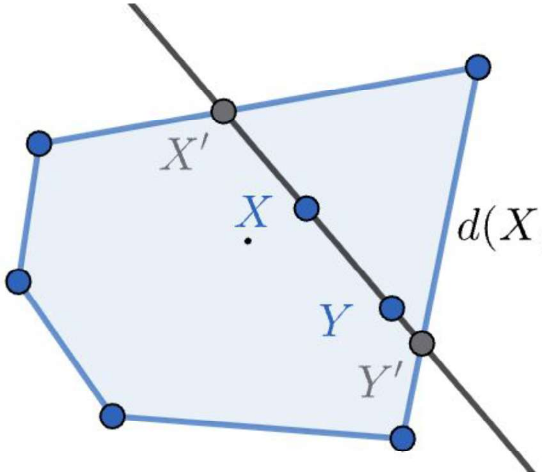
1. Find a **closed-form Hilbert metric** for the **open bicone** of real symmetric positive-definite matrices.

2. Fully characterize its **invariance properties**.

3. Outline potential applications for **extended Gaussian** distributions.

Hilbert geometry

On any open convex set **S** we can consider the Hilbert distance:

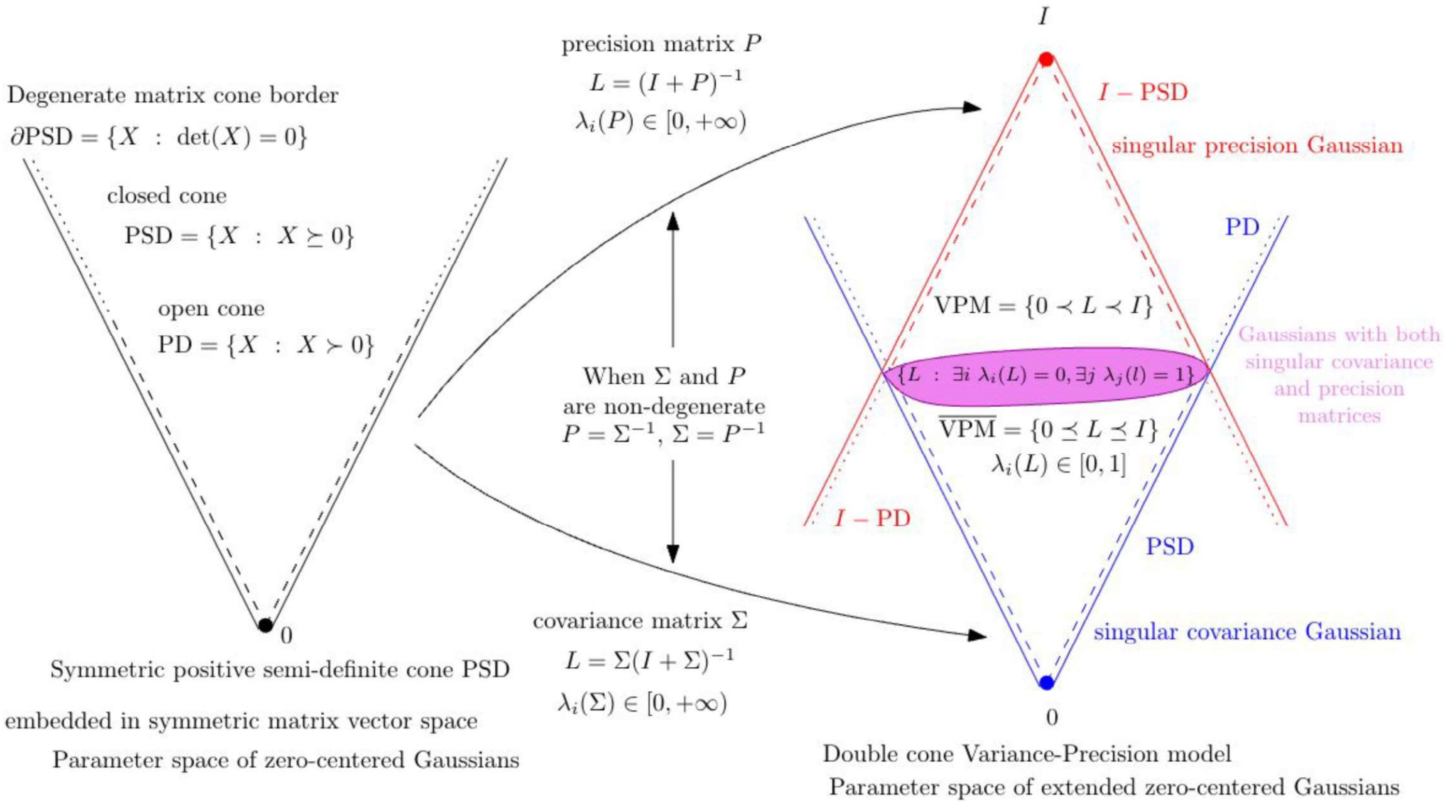


$$d(X,Y) = \log\left(\frac{|Y'X||Y'Y|}{|X'Y||X'X|}\right)$$

This always gives a proper distance function. If we take **S** to be the unit disk, we recover Beltrami–Klein model of hyperbolic geometry.

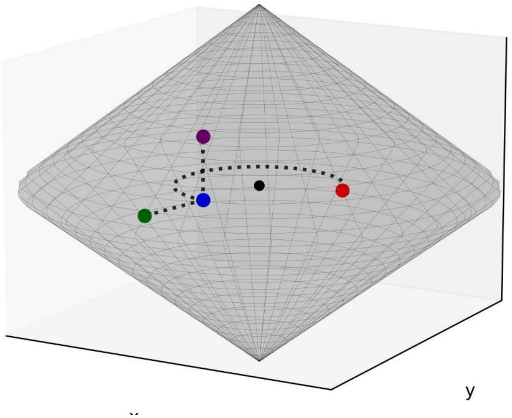
Hilbert distance gives a Finsler, but in general **not a Riemannian,** structure on **S**.

Variance-Precision model

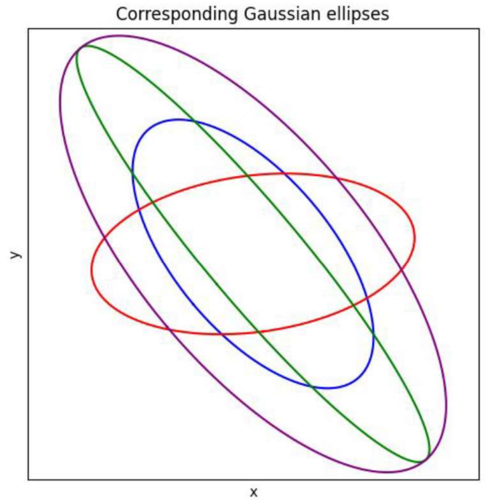


Variance-Precision model

$$
\begin{aligned}
L(\Sigma) &= \Sigma(I+\Sigma)^{-1} \\
L(P) &= (I+P)^{-1}
\end{aligned}
$$



James' parametrization of the space of 2x2 symmetric positive- definite matrices.



(See the interactive **demo**)

Reference: "The Variance Information Manifold and the Functions on It", A.T. James, 1972.

Hilbert distance for VPM

Theorem (Hilbert distance on $\mathrm{VPM}(n)$)**.** *Given two matrices* $A, B \in \mathrm{VPM}(n)$,

$$d_H(A,B) = \log \frac{\max(\lambda_{\max}, \mu_{\max})}{\min(\lambda_{\min}, \mu_{\min})}$$

where

$$\lambda_{\min} = \lambda_{\min}(B^{-1}A), \qquad \lambda_{\max} = \lambda_{\max}(B^{-1}A),$$

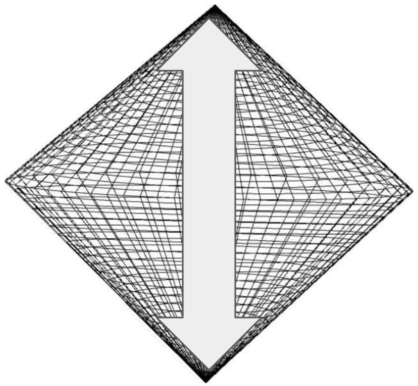
are the minimal and maximal eigenvalues of the $B^{-1}A$ *matrix, and*

$$\mu_{\min} = \lambda_{\min}((I-B)^{-1}(I-A)), \qquad \mu_{\max} = \lambda_{\max}((I-B)^{-1}(I-A)),$$

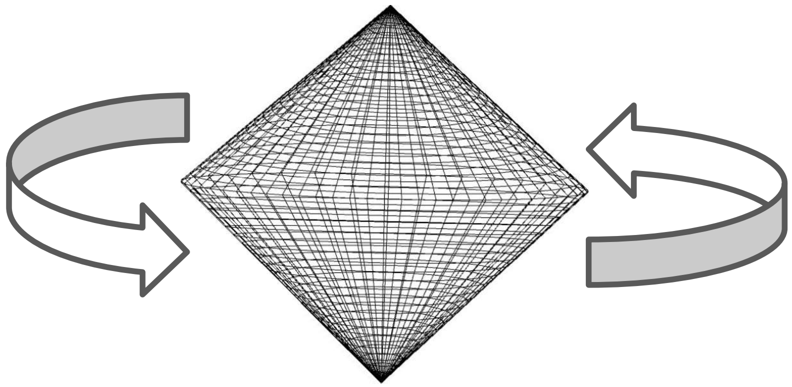
are the minimal and maximal eigenvalues of the $(I-B)^{-1}(I-A)$ *matrix.*

Classification of isometries of VPM with Hilbert distance

Theorem (Classification of VPM isometries)**.** *The group of isometries of* VPM(n) *for* $n > 1$ *is generated by conjugation by orthonormal matrices* $X \mapsto U^T XU$ *for* $U \in O(n)$ *and inversion* $X \mapsto I - X$.



Inversion



Conjugation

Comparison with Affine-Invariant Riemannian distance

Affine-Invariant Riemannian distance:

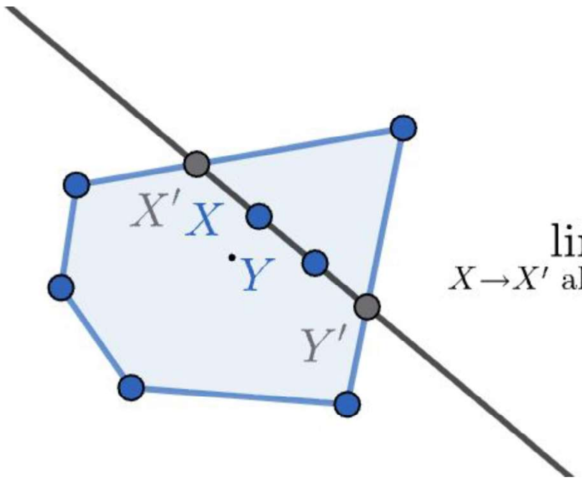
$$\rho(Q_1, Q_2) = \sqrt{\sum_{i=1}^{n} \log^2 \lambda_i(Q_1 Q_2^{-1})}.$$

Comparison of the AIRM vs Hilbert VPM distances. By $\mathrm{Mob}(Q_1, Q_2)$ we denote the Möbius transformation $\mathrm{Mob}(Q_1, Q_2) = (I - Q_1)^{-1}(I - Q_2)$.

	AIRM distance	Hilbert VPM distance
Eigenvalues:	$\{\lambda_i(Q_1 Q_2^{-1})\}_{1 \le i \le n}$	$\lambda_1(Q_1 Q_2^{-1}), \lambda_n(Q_1 Q_2^{-1})$ $\lambda_1(\mathrm{Mob}(Q_1, Q_2)), \lambda_n(\mathrm{Mob}(Q_1, Q_2))$
Invariance under a map:	$X \mapsto X^{-1}$	$X \mapsto I - X$
Invariance under congruence:	$\mathrm{GL}(n)$	$O(n)$

Extension to the boundary

Unfortunately, by default, Hilbert distance is only defined on the interior of the convex set **S**. The boundary of the closure is infinitely far away.



$$\lim_{X \to X' \text{ along } |XY|} d(X,Y) = \lim_{X \to X' \text{ along } |XY|} \log\left(\frac{|Y'X||Y'Y|}{|X'Y||X'X|}\right) = \infty$$

Comparison with Affine-Invariant Riemannian distance

One idea: we can shrink the interior. The distance remains easily computable, and its behavior is controlled.

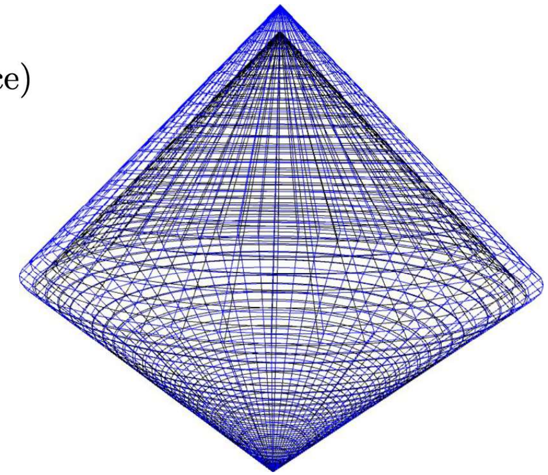
Proposition (Lower bounding Hilbert VPM distance)

$\forall \epsilon > 0, \forall S_1, S_2 \in \mathrm{VPM}(n)$

$$d_H(S_1, S_2) \geq d_{H,\epsilon}(S_1, S_2).$$

where:

$$\overline{\mathrm{VPM}}_\epsilon = \{-\epsilon\, I \preceq X \preceq (1+\epsilon)\, I\}, \quad \epsilon \geq 0.$$

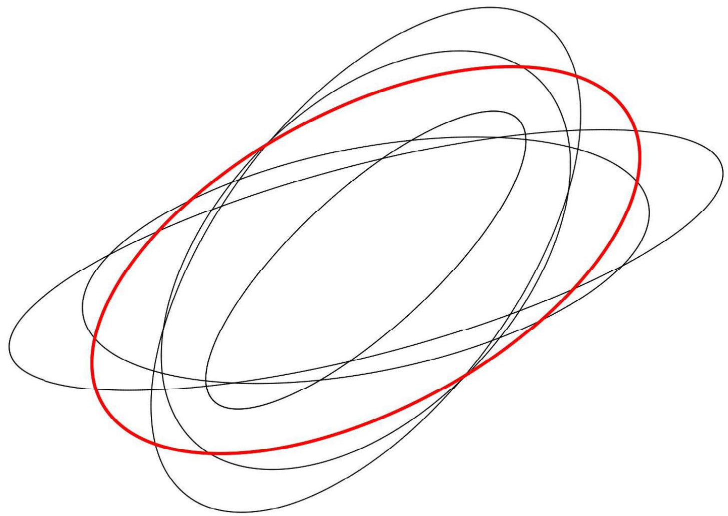


For $S_1, S_2 \in \partial\overline{\mathrm{VPM}}(n)$, $d_H(S_1, S_2) = +\infty$ but $d_{H,\epsilon}(S_1, S_2) < +\infty$.

Application: Smallest Enclosing Ball

Straight-line geodesics in Hilbert geometry allow for easy implementation of various geometric primitives.

Here, an example implementation of Badoiu and Clarkson iterative geodesic-cut algorithm for approximating Smallest Enclosing Ball.



Reference: *Optimal core-sets for balls, Mihai Bădoiu, Kenneth L. Clarkson, 2008*

Extension to non-centered Gaussians

Using Calvo-Oller embedding, we can map non-centered Gaussians into positive-definite matrices, and thus, into VPM.

$$(\mu, \Sigma) \mapsto \in \Sigma_\mu^+ = \begin{bmatrix} \Sigma + \mu\mu^\top & \mu \\ \mu^\top & 1 \end{bmatrix} \subset \mathrm{PD}(n+1).$$

Reference: *A distance between elliptical distributions based in an embedding into the Siegel group, Miquel Calvo, Josep M. Oller, 2002*