Reflection about the geodesic passing through two given points in the Poincaré disk model of hyperbolic geometry

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A geodesic $\gamma(l)$ is parameterized by constant speed so that $\rho(\gamma(l), \gamma(l')) = |l - l'|\, \rho(\gamma(0), \gamma(1))$. This is equivalent to saying that the geodesic $\gamma(l)$ is parameterized by arc length l.

A pregeodesic $\bar{\gamma}(t) = \bar{\gamma}(l(t))$ is a reparameterization of the geodesic such that $l(t)$ is a smooth and invertible with inverse function $t(l)$. Pregeodesics can yield simplified mathematical expressions and express equivalently the geodesic curves:

$$c_\gamma = \{\gamma(l) : l \in [0,1]\} = \{\bar{\gamma}(t) : t \in [0,1]\}.$$

In the Klein model, pregeodesics passing through two points k_1 and k_2 of the unit disk are Euclidean line segments:

$$\bar{\gamma}_K(t) = k_1 + t(k_2 - k_1),$$

with $c_{\gamma_K} = [k_1 k_2]$. The geodesic equation in Klein model has been reported in [?], i.e., the function $l(t)$ such that

$$\gamma_K(l) = \bar{\gamma}_K(l(t))$$

is given in closed-form.

To find $\Gamma_K = \{\gamma_K(l) : l \in \mathbb{R}\}$, we need to find t_m and t_M such that Γ_K is the line passing through $[k_1 k_2]$ clipped to the unit disk. That is, t_m and t_M are the two solutions of the quadratic equation:

$$\langle k_1 + t(k_2 - k_1), k_1 + t(k_2 - k_1)\rangle = 1$$

Let $\Delta = 4\langle k_1, k_2 - k_1, \rangle 2 - 4\|k_2 - k_1\|^2(\|k_1\|^2 - 1)$.

A model of a geometry is said conformal if the angles of two curves $c_1(t)$ and $c_2(t)$ intersecting at t_0 match the Euclidean angles. The Poincaré disk model is conformal but not the Klein model (except at the origin).

(%o79)
$$\begin{aligned}
&[a0 = -\frac{(b2-b1)\, b3^2 + (-b2^2+b1^2-a2^2+a1^2)\, b3 + b1\, b2^2 + (-b1^2+a3^2-a1^2)\, b2 + (a2^2-a3^2)\, b1}{(2\,a2-2\,a1)\, b3 + (2\,a1-2\,a3)\, b2 + (2\,a3-2\,a2)\, b1}, b0 = \\
&\frac{(a2-a1)\, b3^2 + (a1-a3)\, b2^2 + (a3-a2)\, b1^2 + (a2-a1)\, a3^2 + (a1^2-a2^2)\, a3 + a1\, a2^2 - a1^2\, a2}{(2\,a2-2\,a1)\, b3 + (2\,a1-2\,a3)\, b2 + (2\,a3-2\,a2)\, b1}, r0 = \mathrm{sqrt}((b2^2-2\,b1\,b2+b1^2+a2^2-2\,a1\,a2+a1^2)\, b3^4 + \\
&(-2\,b2^3+2\,b1\,b2^2+(2\,b1^2-2\,a2^2+4\,a1\,a2-2\,a1^2)\, b2-2\,b1^3+(-2\,a2^2+4\,a1\,a2-2\,a1^2)\, b1)\, b3^3+(b2^4+2\,b1\,b2^3+(-6\,b1^2+2\,a3^2+(-2\,a2-2\,a1)\, a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\, b2^2+ \\
&\left(2\,b1^3+(-4\,a3^2+(4\,a2+4\,a1)\, a3+2\,a2^2-8\,a1\,a2+2\,a1^2)\, b1\right) b2+b1^4+(2\,a3^2+(-2\,a2-2\,a1)\, a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\, b1^2+(2\,a2^2-4\,a1\,a2+2\,a1^2)\, a3^2+ \\
&(-2\,a2^3+2\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\, a3+a2^4-2\,a1\,a2^3+2\,a1^2\,a2^2-2\,a1^3\,a2+a1^4)\, b3^2+(-2\,b1\,b2^4+(2\,b1^2-2\,a3^2+4\,a1\,a3-2\,a1^2)\, b2^3+ \\
&\left(2\,b1^3+(2\,a3^2+(4\,a2-8\,a1)\, a3-4\,a2^2+4\,a1\,a2+2\,a1^2)\, b1\right) b2^2+ \\
&\left(-2\,b1^4+(2\,a3^2+(4\,a1-8\,a2)\, a3+2\,a2^2+4\,a1\,a2-4\,a1^2)\, b1^2+(-2\,a2^2+4\,a1\,a2-2\,a1^2)\, a3^2+(4\,a1\,a2^2-8\,a1^2\,a2+4\,a1^3)\, a3-2\,a1^2\,a2^2+4\,a1^3\,a2-2\,a1^4\right) b2+ \\
&(-2\,a3^2+4\,a2\,a3-2\,a2^2)\, b1^3+((-2\,a2^2+4\,a1\,a2-2\,a1^2)\, a3^2+(4\,a2^3-8\,a1\,a2^2+4\,a1^2\,a2)\, a3-2\,a2^4+4\,a1\,a2^3-2\,a1^2\,a2^2)\, b1)\, b3+(b1^2+a3^2-2\,a1\,a3+a1^2)\, b2^4+ \\
&((-2\,a3^2+4\,a1\,a3-2\,a1^2)\, b1-2\,b1^3)\, b2^3+ \\
&\left(b1^4+(2\,a3^2+(-2\,a2-2\,a1)\, a3+2\,a2^2-2\,a1\,a2+2\,a1^2)\, b1^2+a3^4+(-2\,a2-2\,a1)\, a3^3+(2\,a2^2+2\,a1\,a2+2\,a1^2)\, a3^2+(-4\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\, a3+2\,a1^2\,a2^2-2\,a1^3\,a2+a1^4\right) \\
&b2^2+((-2\,a3^2+4\,a2\,a3-2\,a2^2)\, b1^3+(-2\,a3^4+(4\,a2+4\,a1)\, a3^3+(-2\,a2^2-8\,a1\,a2-2\,a1^2)\, a3^2+(4\,a1\,a2^2+4\,a1^2\,a2)\, a3-2\,a1^2\,a2^2)\, b1)\, b2+(a3^2-2\,a2\,a3+a2^2)\, b1^4+ \\
&(a3^4+(-2\,a2-2\,a1)\, a3^3+(2\,a2^2+2\,a1\,a2+2\,a1^2)\, a3^2+(-2\,a2^3+2\,a1\,a2^2-4\,a1^2\,a2)\, a3+a2^4-2\,a1\,a2^3+2\,a1^2\,a2^2)\, b1^2+(a2^2-2\,a1\,a2+a1^2)\, a3^4+ \\
&(-2\,a2^3+2\,a1\,a2^2+2\,a1^2\,a2-2\,a1^3)\, a3^3+(a2^4+2\,a1\,a2^3-6\,a1^2\,a2^2+2\,a1^3\,a2+a1^4)\, a3^2+(-2\,a1\,a2^4+2\,a1^2\,a2^3+2\,a1^3\,a2^2-2\,a1^4\,a2)\, a3+a1^2\,a2^4-2\,a1^3\,a2^3+a1^4\,a2^2)/ \\
&((2\,a2-2\,a1)\, b3+(2\,a1-2\,a3)\, b2+(2\,a3-2\,a2)\, b1)]
\end{aligned}$$