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# CONTENTS

# ON THE INTEGRALS $\int_0^x J_k(x)dx$ AND $\int_0^x J_{-k}(x)dx$

By

H. Sircar

(*Received October 24, 1944*)

P. Humbert (1935) has, with the help of symbolic calculus of Heaviside, considered the properties of Fresnel's integrals. With the usual techniques, we propose to record below some results (including some analogues of Humbert's results) for the more general pair of integrals $\int_0^x J_k(x)dx$ and $\int_0^x J_{-k}(x)dx$ ($k > 0$, but $\neq$ an integer) denoted by $S_k(x)$ and $S_{-k}(x)$ respectively. When $k$ is zero or a positive integer, since $J_{-k}(x) = (-)^k J_k(x)$ and therefore $S_{-k} = \delta S_k$, $\delta = 1$ or $-1$, according as $k$ is even (including zero) or odd, we shall denote the corresponding integrals by $C_k(x)$. We shall assume that the relevant rules and images are known and therefore avoid the insertion, even of those that would be employed here. Frequent application of Lerch's Theorem (1903) has been made in what follows in the derivation of an equality from an operation.

1. We immediately find from Carson's rule that

(*i*)
$$S_k = \int_0^x J_k(x)dx \doteqdot \frac{[\sqrt{(p^2+1)}-p]^k}{\sqrt{(p^2+1)}},$$

$$S_{-k} = \int_0^x J_{-k}(x)dx \doteqdot \frac{[\sqrt{(p^2+1)}-p]^{-k}}{\sqrt{(p^2+1)}} = \frac{[\sqrt{(p^2+1)}+p]^k}{\sqrt{(p^2+1)}}.$$

Again
$$S_k(x) \doteqdot \frac{1}{p}\frac{[\sqrt{(p^2+1)}-p]^m}{\sqrt{(p^2+1)}}\cdot p[\sqrt{(p^2+1)}-p]^{k-m}, \quad k-m>0,$$

$$\doteqdot (k-m)\int_0^x \frac{J_{k-m}(y)}{y} S_m(x-y)\, dy.$$

Thus
$$S_k(x) = (k-m)\int_0^x \frac{J_{k-m}(y)}{y} S_m(x-y)\, dy, \quad k-m>0.$$

(*ii*) Next consider

$$\sum_{n=0}^{\infty} C_n(x) \doteqdot \frac{1}{\sqrt{(p^2+1)}}\sum_{n=0}^{\infty}[\sqrt{(p^2+1)}-p]^n$$

$$= \frac{1}{\sqrt{(p^2+1)}}\cdot\frac{1}{1+p-\sqrt{(p^2+1)}}, \quad \left([\sqrt{(p^2+1)}-p] = \frac{1}{\sqrt{(p^2+1)}+p} < 1, \quad p \geq 0\right)$$

$$= \frac{1+p+\sqrt{(p^2+1)}}{2p\sqrt{(p^2+1)}} = \frac{1}{2}\left[\frac{1}{p}\left(\frac{1}{p}+1\right)\frac{p}{\sqrt{(p^2+1)}}+\frac{1}{p}\right] \doteqdot \frac{1}{2}\left[\int_0^x J_0(y)(1+x-y)dy + x\right],$$

whence
$$\sum_{n=0}^{\infty} C_n(x) = \frac{1}{2}\left[x + \int_0^x J_0(y)(1+x-y)dy\right].$$

(*iii*) We have

$$\int_0^x S_k(x-y)\, S_k(y)dy \doteqdot \frac{1}{p}\,\frac{[\sqrt{(p^2+1)}-p]^{2k}}{p^2+1} = \frac{1}{p}\left[\frac{1}{p} - \frac{p}{p^2+1}\right]p[\sqrt{(p^2+1)}-p]^{2k}$$
$$\doteqdot 2k\int_0^x \frac{J_{2k}(y)}{y}\,[x-y-\sin(x-y)]dy, \quad k>0.$$

Thus
$$\int_0^x S_k(x-y)\, S_k(y)dy = 2k\int_0^x \frac{J_{2k}(y)}{y}\,[x-y-\sin(x-y)]dy, \quad k>0.$$

(*iv*) It is not difficult to prove the equality

$$\int_0^x J_s(y)\, S_k(x-y)\, dy = \int_0^x J_{s_1}(y)\, S_{k_1}(x-y)\, dy, \quad s_1+k_1 = s+k.$$

Differentiation yields

$$\int_0^x J_s(y)\, J_k(x-y)dy = \int_0^x J_{s_1}(y)\, J_{k_1}(x-y)dy, \quad s_1+k_1 = s+k,$$

which can also be obtained directly.

2. Application of Dr. Pol's and of S. Goldstein's formulae (of which the former is a special case of the latter) yields the two following results. The advantage of the application resides in the fact that the integration of complicated functions can be made to depend on known and simpler functions.

(*i*) With $0<k<1$,

$$\int_0^\infty \frac{S_{-k}(x)-S_k(x)}{x}\,dx = \int_0^\infty \frac{[\sqrt{(x^2+1)}-x]^{-k} - [\sqrt{(x^2+1)}-x]^k}{x\sqrt{(x^2+1)}}\,dx,$$

the integrand on the right has the value $2k$ when $x\to 0$, and $\mu$-test (with $\mu = 2-k$, $0<k<1$) shows that the infinite integral is convergent. Putting $x = \sinh\theta$, this reduces to the well-known integral

$$2\int_0^\infty \frac{\sinh k\theta}{\sinh\theta}\,d\theta = \pi\,\frac{\sin k\pi}{1+\cos k\pi} = \pi\tan\tfrac{1}{2}k\pi, \quad 0<k<1.$$

$k = \frac{1}{2}$ corresponds to Humbert's result.

(*ii*) With

$$S_k \doteqdot \frac{[\sqrt{(p^2+1)}-p]^k}{\sqrt{(p^2+1)}}, \quad \frac{x^s}{\Gamma(1+s)} \doteqdot p^{-s},$$

$s$, a positive integer and $k \geq s-1$,

$$I = \int_0^\infty x^{-s+1}\, S_k(x)dx = \frac{1}{\Gamma(1+s)}\int_0^\infty x^{s-1}\,\frac{[\sqrt{(x^2+1)}-x]^k}{\sqrt{(x^2+1)}}\,dx.$$

$\mu$-test with $\mu = k-s+2$ readily shows that the integral is convergent.

$$I = \frac{1}{s!}\int_0^\infty e^{-k\theta}\sinh^{s-1}\theta\, d\theta, \quad (x=\sinh\theta),$$

$$=\frac{1}{2^{s-1}s!}\sum_{r=0}^{s-1}(-)^r C_r^{s-1}\int_0^\infty e^{-(k+2r+1-s)\theta}\,d\theta = \frac{1}{2^{s-1}s!}\sum_{r=0}^{s-1}(-)^r C_r^{s-1}\frac{1}{k+2r+1-s}.$$

In particular

$$\text{for } s=1,\ I=\frac{1}{k},\ k>0; \quad \text{for } s=2,\ I=\frac{1}{2}\frac{1}{k^2-1},\ k\geq 1.$$

3. With $k+k' = k_1+k_1' = s =$ a positive integer,

$$\int_0^x S_k(x-y)\,S_{k'}(y)dy \doteqdot \frac{1}{p}\ \frac{[\sqrt{(p^2+1)}-p]^s}{p^2+1},$$

and

$$\int_0^x S_{-k_1}(x-y)\,S_{-k_1'}(y)dy \doteqdot \frac{1}{p}\frac{[\sqrt{(p^2+1)}+p]^s}{p^2+1},$$

we have, writing $a=\sqrt{(p^2+1)}+p$, $b=\sqrt{(p^2+1)}-p$, $a-b=2p$, $ab=1$,

$$\int_0^x [S_{-k_1}(x-y)\,S_{-k_1'}(y)-S_k(x-y)S_{k'}(y)]dy \doteqdot \frac{2}{p^2+1}\ \frac{a^s-b^s}{a-b} = \frac{2}{p^2+1}\ \sum_{r=0}^{s-1} a^{s-r-1}\,b^r,$$

$$\doteqdot 2\sum_{r=0}^{s-1}(-)^{s-r-1}\int_0^x J_r(y)C_{s-r-1}(x-y)dy. \quad (C_{-r}=(-)^r C_r).$$

Whence*, when $s$ is odd,

$$\int_0^x [S_{-k_1}(x-y)\,S_{-k_1'}(y)-S_k(x-y)\,S_{k'}(y)]dy$$

$$=2\int_0^x [J_0(y)C_{s-1}(x-y)-J_1(y)C_{s-2}(x-y)+\cdots-J_{s-2}(y)C_1(x-y)+J_{s-1}(y)C_0(x-y)]dy.$$

Differentiation gives

$$\int_0^x [J_{-k_1}(x-y)S_{-k_1'}(y)-J_k(x-y)S_{k'}(y)]dy$$

$$=2\int_0^x [J_0(y)J_{s-1}(x-y)-J_1(y)J_{s-2}(x-y)+\cdots-J_{s-2}(y)J_1(x-y)+J_{s-1}(y)J_0(x-y)]dy,$$

which can also be derived directly by the above process.

* If $k=k_1=k_1'=k'=\frac{1}{2}$ with $s=1$, the integral is equal to $4\sin^2 x/2$, which corresponds to Humbert's results.

Again, with $k+k' = k_1+k_1' = s =$ an odd positive integer,

$$\int_0^x [S_{-k_1}(x-y)S_{-k_1'}(y)+S_k(x-y)S_{k'}(y)]dy$$

$$\doteqdot \frac{2}{p\sqrt{(p^2+1)}}\frac{a^s+b^s}{a+b} = \frac{2}{p\sqrt{(p^2+1)}}\sum_{r=0}^{s-1}(-)^r a^{s-r-1}b^r = 2\sum_{r=0}^{s-1}(-)^r\frac{b^{-s+2r+1}}{p\sqrt{(p^2+1)}}, \quad (\because \quad ab=1),$$

$$\doteqdot 2\sum_{r=0}^{s-1}(-)^r\int_0^x C_{2r-s+1}(x)dx = 4\sum_{r=1}^{(s-1)/2}(-)^{r-1}\int_0^x C_{s-2r+1}(x)dx+2(-)^{(s-1)/2}\int_0^x C_0(x)dx.$$

Therefore,

$$\int_0^x [S_{-k_1}(x-y)S_{-k_1'}(y)+S_k(x-y)S_{k'}(y)]dy = 4\sum_{r=1}^{(s-1)/2}(-)^{r-1}\int_0^x C_{s-2r+1}(x)dx+2(-)^{(s-1)/2}\int_0^x C_0(x)dx.$$

On differentiation,

$$\int_0^x [J_{-k_1}(x-y)S_{-k_1'}(y)+J_k(x-y)S_{k'}(y)]dy = 4\sum_{r=1}^{(s-1)/2}(-)^{r-1}C_{s-2r+1}(x)+2(-)^{(s-1)/2}C_0(x).$$

Similar results may be obtained by combining $S_k$ and $S_{-k_1}$ with $k-k_1 =$ a positive integer, even or odd according to need.

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# ON FEJÉR'S CALCULATION OF THE LEBESGUE CONSTANTS

By

Lee Lorch

(*Communicated by Prof. N. R. Sen—Received January 25, 1945*)

1. *Introduction.* In this note, Fejér's calculation (1910) of the Lebesgue constants $\{L_n\}$ is altered and made elementary. The gamma function, of which he made essential use, is not required in this determination and, hence can also be eliminated from the calculation of the Lebesgue constants for Borel summability (Lorch, 1944)*, placing that too on an elementary level. The constant term appears here in a different guise.

Using methods essentially different from Fejér's, other authors have obtained more complete results. Szegö (1921). for instance, has shown in a very neat fashion that these constants are completely monotonic and has stated the entire asymptotic expansion.

Lebesgue (1906, p. 86) introduced these constants, and established their divergence to prove the existence of a continuous function whose Fourier series diverges at a point.

2. *Preliminary results.* Some preliminary formulas are needed for the calculation. As usual, $(a)$ denotes the fractional part and $[a]$ the integral part of $a$.

$$\int_0^{\pi x}\left\{\frac{2}{\pi}-|\sin t|\right\}dt = 2(x)+\cos\pi(x)-1 = O(1). \tag{2.1}$$

*Proof:*

$$\int_0^{\pi x}\left\{\frac{2}{\pi}-|\sin t|\right\}dt = \int_0^{\pi[x]}+\int_{\pi[x]}^{\pi x} = \int_0^{\pi(x)} = 2(x)+\cos\pi(x)-1.$$

$$\int_0^{x}\int_0^{y}\left\{\frac{2}{\pi}-|\sin t|\right\}dtdy = \sin\pi(x/\pi)-\pi(x/\pi)+\pi(x/\pi)^2 = O(1). \tag{2.2}$$

*Proof:*

$$\int_0^x\int_0^y = \pi\int_0^{x/\pi}\int_0^{\pi y} = \pi\int_0^{x/\pi}\{2(y)+\cos\pi(y)-1\}dy = \pi\int_0^{[x/\pi]}+\pi\int_{[x/\pi]}^{x/\pi}$$

$$= \pi\int_0^{[x/\pi]}+\pi\int_0^{(x/\pi)} = \pi\sum_{n=0}^{[x/\pi]-1}\int_n^{n+1}+\pi\int_0^{(x/\pi)} = 0+\pi\int_0^{(x/\pi)}$$

$$= \sin\pi(x/\pi)-\pi(x/\pi)+\pi(x/\pi)^2.$$

Lemma 2.1. *If $f(t)$ is differentiable and $N$ is an integer, then*

$$\int_0^{\pi/2} f(t)\,|\sin Nt|\,dt = \frac{2}{\pi}\int_0^{\pi/2} f(t)\,dt+o(1/N). \tag{2.3}$$

*Proof:*

$$|\sin Nt| = \frac{2}{\pi}-\frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos 2jNt}{4j^2-1} \tag{2.4}$$

and the series converges uniformly and absolutely.

* Hereafter referred to as (A).

Hence, integrating by parts,

$$N\int_0^{\pi/2} f(t)\left\{\frac{2}{\pi}-|\sin Nt|\right\}dt = f(\tfrac{1}{2}\pi)\int_0^{\pi N/2}\left\{\frac{2}{\pi}-|\sin t|\right\}dt$$
$$-\frac{2}{\pi}\sum_{j=1}^{\infty}\frac{1}{j(4j^2-1)}\int_0^{\pi/2} f'(t)\sin 2jNt\,dt.$$

From (2.1) it follows that the first integral on the right vanishes. The infinite series converges uniformly and hence may be evaluated termwise. The lemma thus follows from the Riemann-Lebesgue lemma.

In the application, $f(t)=(1/\sin t)-(1/t)$ and $N=2n+1$. In connection therewith the evaluation below, due to Gronwall (1912), is used:

$$\frac{4}{\pi^2}\int_0^{\pi/2}\left(\frac{1}{\sin t}-\frac{1}{t}\right)dt = \frac{4}{\pi^2}\log\frac{4}{\pi}. \tag{2.5}$$

*Remark*: Lemma (2.1), with a remainder $o(1)$, is found in Fejér (1910) for Riemann-integrable $f(t)$ but is proved there in an entirely different manner. The result can be generalized in various directions (*cf.* A, §5).

3. *Calculation of the Lebesgue constants.* Denote the $n$th Lebesgue constant by $L_n$ and let $N=2n+1$. Then

$$L_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{|\sin Nt|}{\sin t}\,dt, \tag{3.1}$$

and

$$L_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{|\sin Nt|}{t}\,dt+\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1}{\sin t}-\frac{1}{t}\right)|\sin Nt|\,dt$$
$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{|\sin Nt|}{t}\,dt+\frac{4}{\pi^2}\log(4/\pi)+o(1/n)$$
$$= \frac{2}{\pi}\int_{1/N}^{\pi/2}\frac{|\sin Nt|}{t}\,dt+\frac{2}{\pi}\int_0^{1/N}\frac{|\sin Nt|}{t}\,dt+\frac{4}{\pi^2}\log(4/\pi)+o(1/n).$$

$$L_n = \frac{2}{\pi}\int_1^{\pi N/2}\frac{|\sin t|}{t}\,dt+\frac{2}{\pi}\int_0^1\frac{|\sin t|}{t}\,dt+\frac{4}{\pi^2}\log(4/\pi)+o(1/n). \tag{3.2}$$

Define

$$d_n = \frac{2}{\pi}\int_1^{\pi N/2}\frac{1}{t}\left\{\frac{2}{\pi}-|\sin t|\right\}dt = \frac{4}{\pi^2}\log N+\frac{4}{\pi^2}\log(\pi/2)-\frac{2}{\pi}\int_1^{\pi N/2}\frac{|\sin t|}{t}\,dt \tag{3.3}$$

Integrating twice by parts, using (2.2), and noting the existence of the infinite integrals, yields

$$\tfrac{1}{2}\pi d_n = \frac{2}{\pi N}\int_0^{\pi N/2}\left\{\frac{2}{\pi}-|\sin t|\right\}dt-\int_0^1\left\{\frac{2}{\pi}-|\sin t|\right\}dt$$
$$+\int_1^{\pi N/2} x^{-2}\int_0^x\left\{\frac{2}{\pi}-|\sin t|\right\}dt\,dx$$

$$= -\int_0^1 \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + \int_1^{\pi N/2} x^{-2} \int_0^x \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dx$$

$$= -\int_0^1 \left\{ \frac{2}{\pi} - |\sin t| \right\} dt - \int_0^1 \int_0^y \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dy$$

$$+ 4\pi^{-2}N^{-2} \int_0^{\pi N/2} \int_0^y \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dy + 2\int_1^{\pi N/2} x^{-3} \int_0^x \int_0^y \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dy\, dx$$

$$= -\int_0^1 \left\{ \frac{2}{\pi} - |\sin t| \right\} dt - \int_0^1 \int_0^y \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dy$$

$$+ 2\int_1^{\infty} x^{-3} \int_0^x \int_0^y \left\{ \frac{2}{\pi} - |\sin t| \right\} dt\, dy\, dx + O(1/n^2)$$

$$= \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + O(1/n^2).$$

Thus

$$d_n = \frac{2}{\pi} \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + O(1/n^2). \tag{3.4}$$

Substituting (3.4) in (3.3) and the result in (3.2) gives the desired formula for $L_n$:

$$L_n = \frac{4}{\pi^2} \log(2n+1) + \frac{4}{\pi^2} \log 2 + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + o(1/n). \tag{3.5}$$

This can be re-written in a form more suitable for comparison with Fejér's by recalling

$$\log(2n+1) = \log n + \log 2 + \frac{1}{2n} + O(1/n^2). \tag{3.6}$$

Hence

$$L_n = \frac{4}{\pi^2} \log n + \frac{8}{\pi^2} \log 2 + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + \frac{2}{\pi^2} \frac{1}{n} + o(1/n). \tag{3.7}$$

4. *Comparison of the constant terms.* Fejér's representation of the constant term, as modified by Gronwall, is

$$(4/\pi^2) \log(4/\pi) + 2\int_0^1 \log \Gamma(t) \cos \pi t\, dt. \tag{4.1}$$

Equating this to the constant term in (3.7) gives

$$\int_0^1 \log \Gamma(t) \cos \pi t\, dt + \frac{1}{\pi} \int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt = \frac{2}{\pi^2} \log \pi + \frac{1}{\pi} \int_0^1 \frac{\sin t}{t} dt, \tag{4.2}$$

where the right side is composed of tabulated functions.

Watson's (1930) constant term, $(4/\pi^2)c_0$, is equal to the constant term in (3.5). He calculated $c_0$ to be 2·441323813694835, *i.e.*, to fifteen decimal places. Using the first nine places gives the value 1·270358... to the constant term of (3.7) and

$$\int_1^{\infty} \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt = -·16684074..., \tag{4.3}$$

and
$$\int_0^1 \log \Gamma(t) \cos \pi t \, dt = \cdot 68622546\ldots \tag{4.4}$$

Further, since $\Gamma(t+1) = t\Gamma(t)$,
$$\int_0^1 \log \Gamma(t) \cos \pi t \, dt = \int_0^1 \log \Gamma(t+1) \cos \pi t \, dt - \int_0^1 \log t \cos \pi t \, dt$$
$$= -\int_1^2 \log \Gamma(t) \cos \pi t \, dt + \frac{1}{\pi} \int_0^\pi \frac{\sin t}{t} dt.$$

Hence
$$\int_0^2 \log \Gamma(t) \cos \pi t \, dt = \frac{1}{\pi} \int_0^\pi \frac{\sin t}{t} \, dt = \cdot 58936\ldots . \tag{4.5}$$

The right hand integral is tabulated as $\mathrm{Si}\,\pi$. Thus
$$\int_1^2 \log \Gamma(t) \cos \pi t \, dt = \cdot 00314\ldots . \tag{4.6}$$

*Remark*: (4.4) shows that the relation (Milne-Thomson, 1933, p. 270)
$$4n \int_0^1 \log \Gamma(t) \cos 2\pi n t \, dt = 1, \quad n = 1, 2, \ldots, \tag{4.7}$$
cannot be extended to $n = \frac{1}{2}$.

5. *Application to the Borel-Lebesgue constants*. The results of §3 here (extended to the continuous case with the help of the bracket function) can replace the remarks of §4 of (A). Doing so eliminates the gamma function from the calculation of the Borel Lebesgue constants $L_B(x)$. The result of (A) then becomes
$$L_B(x) = \frac{2}{\pi^2} \log x - \frac{2}{\pi^2} C + \frac{2}{\pi^2} \log 2 + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^\infty \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt + O(1/x^{\frac{1}{2}}), \tag{5.1}$$
where $C$ is Euler's constant.

The numerical value of the constant term is $0{\cdot}782002\ldots$ .

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# ON THE DIFFERENTIABILITY OF MONOTONE FUNCTIONS

By

P. D. Shukla

(*Communicated by Mr. R. C. Bose—Received January 31, 1945*)

1. The object of this paper* is to study a bounded, monotone and continuous function of $x$, investigate conditions for its differentiability at a point, and find the value of the differential co-efficient whenever it exists.

If $P(x)$ is any monotone function of $x$, then it is either monotone non-decreasing or monotone non-increasing. There is, therefore, no loss of generality in taking $P(x)$ to be a monotone non-decreasing function of $x$; because similar differentiability results can be proved to be true also for a monotone non-increasing function of $x$, say $Q(x)$, by considering $-Q(x)$ instead of $P(x)$. Also for the sake of simplicity, we study differentiability at the point $x=0$ and assume $P(0)=0$; for if $P(0)=k$ where $k$ is a non-zero finite number, then instead of $P(x)$ we can consider another function $R(x)=P(x)-k$, which is a monotone non-decreasing function of $x$ with $R(0)=0$. We shall thus confine our attention to the function $P(x)$ which is any continuous and monotone non-decreasing function of $x$ with $P(0)=0$.

Further, $P(0)$ being zero, we have

$$P'(0)=\lim_{x\to 0}\frac{P(x)}{x}.$$

Consequently, the study of $P'(0)$ is the study of $\lim\limits_{x\to 0}[P(x)/x]$. But instead of studying the existence of $P'(0)$, it may be considered enough to study $P'_+(0)$ alone†, *i.e.*, $\lim\limits_{x\to 0^+}[P(x)/x]$ alone §. For

(*i*) if $P(x)$ is an odd function of $x$, then

$$\lim_{x\to 0^+}\frac{P(x)}{x}=\lim_{x\to 0^-}\frac{P(x)}{x},$$

*i.e.*, if $\lim\limits_{x\to 0^+}[P(x)/x]$ exists, then $P'(0)$ also exists and

$$P'(0)=\lim_{x\to 0^+}\frac{P(x)}{x};$$

(*ii*) if $P(x)$ is an even function of $x$, then

$$\lim_{x\to 0^+}\frac{P(x)}{x}=-\lim_{x\to 0^-}\frac{P(x)}{x},$$

*i.e.*, if $\lim\limits_{x\to 0^+}[P(x)/x]$ exists and equals zero, then $P'(0)$ also exists and equals zero; and

* I take this opportunity of expressing my sincere thanks to Prof. A. N. Singh for his suggestions and help.

† After Titchmarsh (1939) $P'_+(x)$ denotes the right hand derivative of $P(x)$. Similarly $P'_-(x)$.

§ $x\to 0^+$ means $x$ tending towards zero from the right. Similarly $x\to 0^-$ means $x$ tending towards zero from the left.

if $\lim_{x\to 0^+} [P(x)/x]$ exists and is not equal to zero, then $P'(0)$ can not exist, but both the derivatives $P'_+(0)$ and $P'_-(0)$ exist, and $P'_+(0) = -P'_-(0)$;

(*iii*) if $P(x)$ is neither an even nor an odd function of $x$, then the existence of $\lim_{x\to 0^+} [P(x)/x]$ means the existence of $P'_+(0)$, which may really be of interest in such a case. If, however, the existence of $P'(0)$ is required, then $\lim_{x\to 0^-} [P(x)/x]$ also can be considered in the same away as $\lim_{x\to 0^+} [P(x)/x]$;

(*iv*) if $\lim_{x\to 0^+} [P(x)/x]$ does not exist, then $P'_+(0)$ does not exist, and *a fortiori* $P'(0)$.

Consequently, instead of investigating the existence of $P'(0)$, we shall investigate the existence of $P'_+(0)$, and express our theorems in the same terms.

*Necessary and Sufficient Condition*

We establish in two different forms the necessary and sufficient condition for the differentiability of $P(x)$ at $x = 0$. These are given below in §2 and §3.

2. THEOREM. *If $P(x)$ is a monotone non-decreasing and continuous function of $x$ in any interval $(0, \alpha)$ with $P(0) = 0$, then the necessary and sufficient condition for the existence of $P'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{P(x_r)}{x_r} \text{ exists,} \tag{1}$$

*where $\{x_r\}$ is a sequence of positive values of $x$ having the limit zero, provided*

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1. \tag{2}$$

*Necessity.* That (1) is necessary for the existence of $P'_+(0)$ is obvious. For, if $P'_+(0)$ exists, then we must have (1) satisfied for all sequences $\{x_r\}$ of positive values of $x$ in which $x_r \to 0$ as $r \to \infty$; hence *a fortiori* for the given sequence as well.

*Sufficiency.* Suppose a sequence $\{x_r\}$ satisfying (2) has been found such that from (1),

$$\lim_{r\to\infty} \frac{P(x_r)}{x_r} = k. \tag{3}$$

Consider then any $x > 0$. There must obviously be some value of $r$, say $n$, such that

$$x_{n+1} \leq x \leq x_n.$$

Also, because of the monotone character of $P(x)$, we have for $(x_{n+1}, x_n)$,

$$\frac{P(x_{n+1})}{x_n} \leq \frac{P(x)}{x} < \frac{P(x_n)}{x_{n+1}}. \tag{4}$$

But

$$\frac{P(x_{n+1})}{x_n} = \frac{P(x_{n+1})}{x_{n+1}} \cdot \frac{x_{n+1}}{x_n}.$$

Hence by (2) and (3) we get

$$\lim_{n\to\infty} \frac{P(x_{n+1})}{x_n} = k.$$

Similarly

$$\lim_{n\to\infty} \frac{P(x_n)}{x_{n+1}} = k.$$

Hence by (4),

$$\lim_{x\to 0^+} \frac{P(x)}{x} = k,$$

*i.e.*, $P'_+(0)$ exists and equals $k$.

2.1. *Corollary.* It is easily seen that if the above sufficient condition is satisfied then

$$P'_+(0) = \lim_{r\to\infty} \frac{P(x_r)}{x_r}.$$

3. THEOREM. *If $P(x)$ is a monotone non decreasing and continuous function of $x$ in any interval $(0, \alpha)$ with $P(0) = 0$, then the necessary and sufficient condition for the existence of $P'_+(0)$ is that*

$$\lim_{r\to\infty} \frac{P(x_r)}{Q(x_r)} \text{ exists,} \tag{5}$$

*where $\{x_r\}$ is a sequence of positive values of $x$ having the limit zero, provided*

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 1; \tag{6}$$

*and $Q(x)$ is any differentiable function of $x$ with $Q(0) = 0$ and $Q'_+(0) \neq 0$.*

From the existence of $Q'_+(0)$ it follows that

$$\lim_{r\to\infty} \frac{Q(x_r)}{x_r}$$

must exist. Suppose, therefore,

$$\lim_{r\to\infty} \frac{Q(x_r)}{x_r} = k, \tag{7}$$

where $k \neq 0$, and $\{x_r\}$ is any sequence of positive values of $x$, in which $x_r \to 0$ as $r \to \infty$. Now,

$$\lim_{r\to\infty} \frac{P(x_r)}{Q(x_r)} = \lim_{r\to\infty} \left[ \frac{x_r}{Q(x_r)} \cdot \frac{P(x_r)}{x_r} \right] = \frac{1}{k} \lim_{r\to\infty} \frac{P(x_r)}{x_r}, \tag{8}$$

because from (7), $k \neq 0$.

Hence if $P'_+(0)$ exists, so that $\lim_{r\to\infty} [P(x_r)/x_r]$ exists, then $\lim_{r\to\infty} [P(x_r)/Q(x_r)]$ must exist. Also if $\lim_{r\to\infty} [P(x_r)/Q(x_r)]$ exists, then $\lim_{r\to\infty} [P(x_r)/x_r]$ must exist, which by (6) of the hypothesis and §2 guarantees the existence of $P'_+(0)$. Hence the proposition.

3.1. *Corollary.* From (8) it is easily seen that if the conditions of §3 are satisfied so that $P'_+(0)$ must exist, then

$$P'_+(0) = Q'_+(0) \cdot \lim_{r\to\infty} \frac{P(x_r)}{Q(x_r)}.$$

3.2. It should be noted that the conditions given in §2 and §3 are not independent; but that the condition in §3 is a consequence of that in §2.

*Sufficient Conditions*

We establish now three sufficient conditions for the differentiability of $P(x)$ at $x = 0$. We postpone for a future investigation the question whether these conditions are also necessary, and whether there exist other necessary conditions for the differentiability of $P(x)$ at $x = 0$.*

4. THEOREM. *If $P(x)$ is a monotone non-decreasing and continuous function of $x$ in any interval $(0, \alpha)$ with $P(0) = 0$, then $P'_+(0)$ exists provided*

$$\operatorname*{Lim}_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r - x_{r+1}}$$

*exists, where $\{x_r\}$ is a sequence of positive values of $x$ having the limit zero such that*

$$\operatorname*{Lim}_{r\to\infty} \frac{x_{r+1}}{x_r} = 1.$$

Let

$$\frac{P(x_r)-P(x_{r+1})}{x_r - x_{r+1}} = k + \eta_r,$$

where $k$ is some definite number, and $\eta_r \to 0$ with $1/r$. Hence

$$P(x_r)-P(x_{r+1}) = (k+\eta_r)(x_r - x_{r+1}).$$

Similarly

$$P(x_{r+m})-P(x_{r+m+1}) = (k+\eta_{r+m})(x_{r+m} - x_{r+m+1}),$$

for $m = 1, 2, 3, \ldots$ . Therefore,

$$P(x_r)-P(0) = kx_r + \sum_{m=0}^{\infty} \eta_{r+m}(x_{r+m} - x_{r+m+1}).$$

But it is obvious that

$$0 \lesssim \sum_{m=0}^{\infty} \eta_{r+m}(x_{r+m} - x_{r+m+1}) < \bar{\eta}_r x_r,$$

where $\bar{\eta}_r$ is the greatest value of $|\eta_n|$, $n \geq r$. Hence

$$\operatorname*{Lim}_{r\to\infty} \frac{P(x_r)}{x_r} = k, \tag{9}$$

because by hypothesis $\bar{\eta}_r \to 0$ as $r \to \infty$.

But (9), by §2, ensures the existence of $P'_+(0)$. Hence the proposition.

4.1. *Corollary* It is seen from (9) and §2.1 that when the conditions of §4 are satisfied, then

$$P'_+(0) = \operatorname*{Lim}_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r - x_{r+1}}.$$

* It may be remarked here that the question whether the differentiability at $x = 0$ of $P(x)$ as a step function depends upon the existence of the metric density at $x = 0$ of the set of values of $x$ corresponding to the lines of invariability of $P(x)$ has already been answered by me (Shukla, 1948).

5. THEOREM. *If $P(x)$ is a monotone non-decreasing and continuous function of $x$ in any interval $(0, \alpha)$ with $P(0) = 0$, then $P'_+(0)$ exists and equals zero, provided*

$$\lim_{r \to \infty} \frac{P(x_r) - P(x_{r+1})}{x_r - x_{r+1}} = 0,$$

*where $\{x_r\}$ is a sequence of positive values of $x$ having the limit zero, such that*

$$\lim_{r \to \infty} \frac{x_{r+1}}{x_r} \neq 0.$$

Let us consider any $x > 0$. There must then be a value of $r$, say $n$, such that

$$x_{n+1} \leq x \leq x_n.$$

Also, because of the monotone character of $P(x)$, we have, for any $x$ lying in $(x_{n+1}, x_n)$,

$$0 < \frac{P(x)}{x} < \frac{P(x_n)}{x_{n+1}}. \tag{10}$$

But by hypothesis,

$$\frac{P(x_n) - P(x_{n+1})}{x_n - x_{n+1}} = \eta_n,$$

where $\eta_n \to 0$ as $n \to \infty$. Hence

$$P(x_n) - P(x_{n+1}) = \eta_n(x_n - x_{n+1}).$$

Similarly

$$P(x_{n+m}) - P(x_{n+m+1}) = \eta_{n+m}(x_{n+m} - x_{n+m+1}),$$

for $m = 1, 2, 3, \ldots$. Therefore,

$$P(x_n) - P(0) = \sum_{m=0}^{\infty} \eta_{n+m}(x_{n+m} - x_{n+m+1}).$$

But

$$0 < \sum_{m=0}^{\infty} \eta_{n+m}(x_{n+m} - x_{n+m+1}) < \bar{\eta}_n x_n,$$

where $\bar{\eta}_n$ is the greatest* value of $\eta_r$, $r \geq n$. Hence

$$0 < \frac{P(x_n)}{x_{n+1}} < \bar{\eta}_n \frac{x_n}{x_{n+1}}. \tag{11}$$

$$i.e. \quad \lim_{n \to \infty} \frac{P(x_n)}{x_{n+1}} = 0,$$

because by hypothesis $\lim\limits_{n \to \infty} [x_n/x_{n+1}]$ is finite, and $\bar{\eta}_n \to 0$ as $n \to \infty$. Therefore, from (10), $\lim\limits_{x \to 0^+} [P(x)/x]$ exists and is equal to zero. That is, $P'_+(0)$ exists and equals zero.

* It may be noted that $\eta_n \geq 0$ for all values of $n$ from and after some fixed one; because

$$P(x_n) \geq P(x_{n+1}),$$

and

$$x_n > x_{n+1}.$$

6. THEOREM. *If $P(x)$ is a monotone non-decreasing and continuous function of $x$ in any interval $(0, \alpha)$ with $P(0) = 0$, then $P'_+(0)$ exists and equals zero, provided*

$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_{r+1}} = 0, \tag{12}$$

*where $\{x_r\}$ is any sequence of positive values of $x$ having the limit zero, such that*

$$\lim_{r\to\infty} \frac{x_{r+1}}{x_r} = 0; \tag{13}$$

*and provided*

$$\frac{P(x_r)-P(x_{r+1})}{x_r-x_{r+1}},$$

*which due to* (12) *and* (13) *has necessarily to tend towards zero for $r \to \infty$, does so monotonically.*

From
$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_{r+1}} = 0,$$
it follows that
$$\lim_{r\to\infty} \frac{P(x_r)-P(x_{r+1})}{x_r-x_{r+1}} = 0,$$

because by hypothesis $(x_r-x_{r+1}) \succ x_{r+1}$.

Now, from (10) and (11) of §5, it can be easily proved that here also for $x_{n+1} \leq x \leq x_n$, we have

$$0 \leq \frac{P(x)}{x} \leq \frac{P(x_n)}{x_{n+1}} < \frac{P(x_n)-P(x_{n+1})}{x_n-x_{n+1}} \times \frac{x_n}{x_{n+1}},$$

because due to the monotone character of $\eta_n$, the $\bar{\eta}_n$ of (11) can be replaced here by

$$\eta_n = \frac{P(x_n)-P(x_{n+1})}{x_n-x_{n+1}}.$$

But
$$\lim_{n\to\infty} \left[\frac{P(x_n)-P(x_{n+1})}{x_n-x_{n+1}} \times \frac{x_n}{x_{n+1}}\right] = \lim_{n\to\infty} \frac{P(x_n)-P(x_{n+1})}{x_{n+1}\{1-(x_{n+1}/x_n)\}} = 0.$$

because by hypothesis,

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0 = \lim_{n\to\infty} \frac{P(x_n)-P(x_{n+1})}{x_{n+1}}.$$

Therefore,
$$\lim_{x\to 0^+} \frac{P(x)}{x}$$

exists and is equal to zero. That is, $P'_+(0)$ exists and equals zero.

Hence the proposition.

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# ON THE EQUATION $2^x - 3^y = 2^X + 3^Y$

By

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§1. Aaron Herschefeld has proved that the equation

$$2^x - 3^y = C$$

has got at most one solution when $C$ is large. But it is known that, when $2^x \geq 3^y$,

$$\lim_{y \to \infty} (2^x - 3^y) = \infty.$$

These two results together mean that the equation

$$2^X - 3^Y = 2^x - 3^y \tag{1}$$

has got only a finite number of solutions.

The proof of this result is based on Siegel's theorem on rational approximation to algebraic irrationals. The very nature of the proof of Siegel's theorem is against giving any idea about the number of solutions for the corresponding inequality, except that it is finite.* As such, we do not get any idea about the number of solutions of (1). It is highly probable that the only solutions of (1) are:—

$$-1 = 2 - 3 = 2^3 - 3^2,$$

$$5 = 2^3 - 3 = 2^5 - 3^3,$$

and

$$13 = 2^4 - 3 = 2^8 - 3^5.$$

I wonder whether the theory of Pell's equation will not enable one to prove this conjecture.

The object of this paper is to find all the solutions of

$$2^x - 3^y = 3^Y - 2^X, \tag{2}$$

$$2^x - 3^y = 2^X + 3^Y, \tag{3}$$

and

$$3^y - 2^x = 2^X + 3^Y. \tag{4}$$

These three equations are completely solved by elementary methods. For our proof, some of the solutions of

$$2^x - 3^y = C \tag{5}$$

are required. From this to consider all cases when $|C| \leq 101$ is not a lengthy process. I give the proof of this also. But this was proved by Herschefeld also. Since my present proof of this result is different from my original proof which I got in 1925, it is not unlikely that my proofs are different from that of Herschefeld.

* If it is possible to give an independent (not relative) upper bounds for the highest solution for the inequality in Siegel's theorem, we can draw many interesting conclusions.

I take this opportunity to put in print a conjecture which I gave during the conference of the Indian Mathematical Society held at Aligarh.

Arrange all the powers of integers like squares, cubes etc. in increasing order as follows:

$$1,\ 4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 32,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 125,\ 128,\ \ldots\ .$$

Let $a_n$ be the $n$th member of this series so that $a_1 = 1$, $a_2 = 4$, $a_3 = 8$, $a_4 = 9$, etc. Then

$$\textit{Conjecture}:\text{—}\quad \underline{\lim}(a_n - a_{n-1}) = \infty.$$

§2. *Lemma (1)*: If $3^m \mid (2^x-1)$, then $2.3^{m-1} \mid x$;

if $3^m \mid (2^x+1)$, then $3^{m-1} \mid x$ and $x$ is odd;

if $2^n \mid (3^y-1)$, then $2^{n-2} \mid y$;

and $3^y+1$ is a multiple of 4 but not of 8 when $y$ is odd, and $3^y+1$ is a multiple of 2 but not of 4 when $y$ is even. It is easy to verify all these.

*Lemma (2)*: With respect to the modulus 24, when $r \geq 1$,

$$3^{2r-1} \equiv 3,\quad 3^{2r} \equiv 9,\quad 2^{2r+1} \equiv 8 \quad\text{and}\quad 2^{2r+2} \equiv 16.$$

*Lemma (3)*: If $2^x - 3^y = a$, $x \geq 3$, $y \geq 1$ and $0 < a \leq 100$, then the possible values for $a$ are 5, 7, 13, 23, 29, 31, 37, 47, 53, 55, 61, 71, 77, 79, 85, 95.

*Lemma (4)*: If $3^y - 2^x = a$, $x \geq 3$, $y \geq 1$ and $0 < a \leq 100$, then the possible values for $a$ are 1, 11, 17, 19, 25, 35, 41, 43, 49, 59, 65, 67, 73, 83, 89, 91, 97.

Lemma (2) is obvious. Lemmas (3) and (4) follow from lemma (2).

*Lemma (5)*: When $x > n$, $y > m$, if either $n \geq 3$ or $m \geq 3$, then

$$2^x - 3^y \neq 3^n - 2^m.$$

If impossible, let $2^x - 3^y = 3^n - 2^m$, then

$$2^m-(2^{x-m}+1) = 3^n(3^{y-n}+1).$$

Therefore, from lemma (1),

$$m \leq 2. \tag{6}$$

Let $n \geq 3$. Then $9 \mid (x-m)$ so that $19 \mid (2^{x-m}+1)$. So $19 \mid (3^{y-n}+1)$. Therefore $9 \mid (y-n)$. Consequently $7 \mid (3^{y-n}+1)$. So $7 \mid (2^{x-m}+1)$. But 7 does not divide $2^a+1$ for any $a$. So

$$n \leq 2. \tag{7}$$

From (6), (7), the lemma follows.

*Lemma (6)*: When $x > n$, $y > m$, if either $n \geq 3$ or $m \geq 3$, then

$$2^x - 3^y \neq 2^m + 3^n.$$

If $2^x - 3^y = 2^m + 3^n$, then

$$2^m(2^{x-m}-1) = 3^n(3^{y-n}+1).$$

From lemma (1),

$$m \leq 2. \tag{8}$$

Let $n \geq 3$. Then $18 \mid (x-m)$, so that 19 and 73 divide $2^{x-m}-1$ and consequently $3^{y-n}+1$ also. But $19 \mid 3^{y-n}+1$ when $y-n$ is odd. Since $3^6+1=730$, $73 \mid 3^{y-n}+1$ when $y-n$ is even. These two are contradictory. Hence

$$n \leq 2. \tag{9}$$

From (8), (9), the lemma follows.

*Lemma* (7): If (a) $n \geq 3$, or (b) $m \geqq 5$, then

$$3^y-2^x \neq 2^m+3^n,$$

provided $x>n$, $y>m$.

If $3^y-2^x = 2^m+3^n$, then

$$2^m(2^{x-m}+1) = 3^n(3^{y-n}-1).$$

Proceeding as in lemma (5),

$$n \leq 2. \tag{10}$$

Let $m \geq 5$. Then $8 \mid (y-n)$, so that $41 \mid (3^{y-n}-1)$ and consequenty $2^{x-m}+1$ also. Hence $10 \mid (x-m)$. Therefore $25 \mid (2^{x-m}+1)$ and $25 \mid (3^{y-n}-1)$. So $20 \mid (y-n)$ so that 11 divides $3^{y-n}-1$ and consequently $2^{x-m}+1$. Then $x-m$ is odd. This is a contradiction. So

$$m \leq 4. \tag{11}$$

From (10), (11), the lemma follows.

*Lemma* (8): When $x>t$ and $y>3$, $2^x-3^y \neq 2^t-3^3$.

If $2^x-3^y = 2^t-3^3$, then

$$2^t(2^{x-t}-1) = 3^3(3^{y-3}-1).$$

Then $18 \mid (x-t)$, so that 19 divides $2^{x-t}-1$ and consequently $3^{y-3}-1$ also. So $18 \mid (y-3)$. But $757 \mid (3^9-1)$ so that 757 divides $3^{y-3}-1$ and consequently $2^{x-t}-1$. But $2^{252}-1$ is not a multiple of 757. Hence $x-t$ is a multiple of 27. Therefore, since $x-t$ is even, $81 \mid (2^{x-t}-1)$, which is impossible. Hence the lemma.

*Lemma* (9): When $x>r$, $y>s$ and $6 \leq r \leq 9$, $2^x-3^y \neq 2^r-3^s$.

If $2^x-3^y = 2^r-3^s$, then

$$2^r(2^{x-r}-1) = 3^s(3^{y-s}-1).$$

Since $r \geq 6$, $16 \mid (y-s)$. But $193 \mid (3^{16}-1)$ so that 193 is a divisor of $3^{y-s}-1$ and so of $2^{x-r}-1$. Hence $x-r$ is a multiple of 96. But 257 is a divisor of $2^{96}-1$, so of $2^{x-r}-1$ and so of $3^{y-s}-1$. Since 3 is a primitive root of 257, $256 \mid (y-s)$. So $2^{10}$ is a factor of $3^{y-s}-1$. Since $r \leq 9$, this is impossible. Hence the lemma.

*Lemma* (10): If $y \geq 4$, then $3^y-2^x \neq 1$.

If possible, let $3^y-2^x = 1$. Since $y \geq 4$, $x \geq 2$. Then $y$ is even so that $y = 2r$. Then

$$2^x = 3^{2r}-1 = (3^r-1)(3^r+1).$$

Both $3^r-1$ and $3^r+1$ cannot be multiples of 4. So when $r \geq 2$, either $3^r-1$ or $3^r+1$ is divisible by an odd prime, which should not be. So we get the lemma.

*Lemma* (11): When $x>r$, $2^x-3^y \neq 2^r \pm 63m$, where $m \neq 0$.

If $2^x - 3^y = 2^r \pm 63m$, then

$$2^r(2^{x-r}-1) = 3^2(3^{y-2} \pm 7m),$$

so $6 \mid (x-r)$. Hence $7 \mid (2^{x-r}-1)$ and so $7 \mid (3^{y-2} \pm 7m)$, which is impossible. Hence the lemma.

*Lemma (12)*. When $x \geq 2$, the only solutions of $3^y - 2^x = z^2$ are $3^2 - 2^3 = 1^2$ and $3^4 - 2^5 = 7^2$.

Let $3^y - 2^x = z^2$. Then $z$ is odd and so $y = 2r$. Therefore

$$2^x = 3^{2r} - z^2 = (3^r - z)(3^r + z),$$

so that $2 = 3^r - z$ and $2^{x-1} = 3^r + z$. Adding and dividing by 2, $2^{x-2}+1 = 3^r$. But by (11), the only solutions of $2^X + 1 = 3^Y$ are $2+1 = 3$ and $2^3 + 1 = 3^2$. Hence the lemma.

I think that $3^y - 2 = z^2$ has only one solution.

*Lemma (13)*: $2^x - 3^y \neq 53$.

Let $2^x - 3^y = 53$; $50 + 3 = 53$. So $2^x - 50 = 3(3^{y-1}+1)$. So $y-1$ is even. When $y - 1 = 4m+2$, $3^{y-1}+1$ is a multiple of 5. Hence $y-1 = 4m$.

Again $2^5 + 21 = 53$. So

$$2^5(2^{x-5}-1) = 3(3^{y-1}+7).$$

$3^{y-1}+7$ is a multiple of $2^5$ only when $y - 1 = 8m+6$. Hence the lemma follows.

*Lemma (14)*: $2^x - 3^y = 7$ has only one solution.

If $2^x - 3^y = 7$, from lemma (1), $x = 2r$, $y = 2s$. So

$$7 = 2^{2r} - 3^{2s} = (2^r - 3^s)(2^r + 3^s).$$

Hence $1 = 2^r - 3^s$ and $2^r + 3^s = 7$. So $r = 2$, $s = 1$. The lemma is proved.

§8. Table A gives all the solutions of $a = 2^x - 3^y$ when $1 \leq a \leq 100$. Table B gives all the solutions of $a = 3^y - 2^x$ when $1 \leq a \leq 100$. The numbers within brackets give the lemmas by which we get that further solutions are impossible.

TABLE A for $2^x - 3^y$

| | | | |
|---|---|---|---|
| $1 = 2 - 3^0 = 2^2 - 3$ | | $5 = 2^5 - 3^3 = 2^3 - 3$ | (8) |
| $7 = 2^4 - 3^2$ | (14) | $13 = 2^8 - 3^5 = 2^4 - 3$ | (9) |
| $23 = 3^3 - 2^2 = 2^5 - 3^2$ | (5) | $29 = 3^3 + 2 = 2^5 - 3$ | (6) |
| $31 = 3^3 + 2^2 = 2^5 - 3^0$ | (6) | $37 = 2^6 - 3^3$ | (9) |
| $47 = 2^7 - 3^4$ | (9) | $53 =$ no. | (13) |
| $55 = 2^6 - 3^2$ | (9) | $61 = 2^6 - 3$ | (9) |
| $71 = 2^3 + 63$ | (11) | $77 = 3^4 - 2^2$ | (5) |
| $79 = 3^4 - 2$ | (5) | $85 = 3^4 + 2^2$ | (6) |
| $95 = 2^5 + 63$ | (11) | | |

TABLE B for $3^y - 2^x$

| | | | |
|---|---|---|---|
| $1 = 3-2 = 3^2-2^3$ | (10) | $5 = 3^2-2^2$ | |
| $7 = 3^2-2$ | | $11 = 3^3-2^4$ | (8) |
| $17 = 3^4-2^6$ | (9) | $19 = 3^3-2^3$ | (8) |
| $23 = 3^3-2^2$ | | $25 = 3^3-2$ | (8) |
| $35 = 3^3+2^3$ | (7) | $41 = 2^5+3^2$ | (7) |
| $43 = 3^3+2^4$ | (7) | $49 = 3^4-2^5$ | (12) |
| $59 = 2^5+3^3$ | (7) | $65 = 2^6+3^0 = 3^4-2^4$ | (7) |
| $67 = 2^6+3$ | (7) | $73 = 2^6+3^2 = 3^4-2^3$ | (7) |
| $77 = 3^4-2^2$ | | $79 = 3^4-2$ | |
| $83 = 3^4+2$ | (7) | $89 = 3^4+2^3$ | (7) |
| $91 = 2^6+3^3$ | (7) | $97 = 3^4+2^4$ | (7) |

*N.B.* In the above table, the expression of $a$ in the form $2^m+3^n$ shows the corresponding lemma is applicable.

**Theorem I**: When $1 \leq a \leq 100$, tables A and B give all the solutions of $a = 2^x-3^y$ and $a = 3^y-2^x$.

The theorem follows from the tables and lemmas (3) and (4).

§4. **Theorem II**: 1, 5, 7, 23 are the only positive numbers which can be simultaneously expressed by the forms $2^x-3^y$ and $3^Y-2^X$; and their solutions are:—

$$1 = 3-2 = 3^2-2^3 = 2^2-3; \qquad 5 = 3^2-2^2 = 2^3-3 = 2^5-3^3;$$

$$7 = 3^2-2 = 2^4-3^2; \qquad 23 = 2^5-3^2 = 3^3-2^2.$$

From lemma (5), if $x > n$, $y > m$ and $2^x-3^y = 3^n-2^m$, then $n \leq 2$ and $m \leq 2$. So the possible values for $a$ are $3-2 = 1$, $3^2-2 = 7$, and $3^2-2^2 = 5$, when $x$ and $y$ are restricted as above. Hence from theorem I, we get

$$1 = 3-2 = 3^2-2^3 = 2^2-3; \quad 5 = 3^2-2^2 = 2^3-3 = 2^5-3^3 \quad \text{and} \quad 7 = 3^2-2 = 2^4-3^2.$$

But from $3^2-2^2 = 2^5-3^3$ we get the additional number $23 = 2^5-3^2 = 3^3-2^2$. Other interchanges do not give any new number. Hence the theorem.

**Theorem III**: 5, 7, 13, 29 247 are the only numbers which can be expressed in both the forms $2^x-3^y$ and $2^X+3^Y$; and the solutions are:

$$5 = 2+3 = 2^3-3 = 2^5-3^3; \quad 7 = 2^2+3 = 2^4-3^2; \quad 29 = 2+3^3 = 2^5-3;$$

$$13 = 2^2+3^2 = 2^4-3 = 2^8-3^5; \quad 247 = 2^2+3^5 = 2^8-3^2.$$

From lemma (6), if $x > n$, $y > m$ and $2^x-3^y = 3^n+2^m$, then $n \leq 2$ and $m \leq 2$. So if $a = 2^x-3^y = 2^m+3^n$ with $x > m$, $y > n$, then the possible values for $a$ are

$$2+3 = 5, \quad 2^2+3 = 7, \quad 2+3^2 = 11 \quad \text{and} \quad 2^2+3^2 = 13.$$

Hence from theorem I, we get the solutions for 5, 7, 11 and 13. By interchanging the members of above solutions, we get 29 and 247.

**Theorem IV**: The numbers which can be represented both by $3^y-2^x$ and $2^X+3^Y$ are 5, 7, 11, 17, 19, 25, 65 and 73. Their representations are:—

$$5 = 2+3 = 3^2-2^2; \quad 7 = 3^2-2 = 2^2+3; \quad 19 = 3^3-2^3 = 2^4+3;$$

$$17 = 3^4-2^6 = 2^3+3^2 = 2^4+3^0; \quad 11 = 3^3-2^4 = 2^3+3 = 2+3^2;$$

$$25 = 3^3-2 = 2^4+3^2; \quad 65 = 3^4-2^4 = 2^6+3^0; \quad 73 = 3^4-2^3 = 2^6+3^2.$$

From lemma (7), if $x > m$, $y > n$ and $3^y-2^x = 2^m+3^n$, then $n \leq 2$ and $m \leq 4$. Hence if $a = 3^y-2^x = 2^m+3^n$ with $y > n$, $x > m$, then the possible values of $a$ are:—

$$2+3 = 5, \quad 2+3^2 = 11, \quad 2^2+3 = 7, \quad 2^2+3^2 = 13, \quad 2^3+3 = 11,$$
$$2^3+3^2 = 17, \quad 2^4+3 = 19, \quad 2^4+3^2 = 25.$$

From theorem I, we get the solutions for the above values of $a$. 65 and 73 are got from the above solutions by transfering the members to opposite sides.

DEPARTMENT OF PURE MATHEMATICS,
CALCUTTA UNIVERSITY

# ON THE INEQUALITY SATISFIED BY THE DERIVATIVE OF ORDER $n$ OF A FUNCTION

By

H. Sircar

(Received October 24, 1944)

The following commends itself on a perusal of a paper by J. Soula (1932) with a similar title. The inequality depends upon the orthogonal system chosen.

1. Let $f(x)$ be a function admitting continuous derivatives upto the order $n$ in the closed interval $(a, b)$, $a < b$, which may be finite or infinite with $a = 0$ and $b = \infty$, and $\lim_{x=\infty} f(x) = f(\infty)$.

Let $(\phi_n(x))$ be a normalised orthogonal system of functions in $(a, b)$ so that

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0, \quad m \neq n\ ; \quad m, n = 1, 2, \ldots,$$

and

$$\int_a^b \phi_n^2(x)dx = 1, \quad n = 1, 2, \ldots .$$

Let us suppose that $\phi_n(x)$ can be expressed in the form

$$\phi_n(x) = \psi(x)\,\frac{d^n}{dx^n}\,F_n(x), \tag{A}$$

where $F_n(x)$ and $\psi(x)$ satisfy the following conditions ;

(i) $F_n^{(p)}(x) = 0$, $p = 0, 1, 2, \ldots, n-1$, $F_n^{(p)}(x)$ denoting the $p^{th}$ derivative of $F_n(x)$,

(ii) $F_n(x)$ does not change sign in $(a, b)$,

(iii) $f(x)/\psi(x)$ is continuous in $(a, b)$.

Let us consider the sequence $(\gamma_n)$ of constants defined by

$$\gamma_n = \int_a^b F_n(x) f^{(n)}(x)\,dx\ ;$$

when $(0, \infty)$ is the interval, the integral is assumed to be convergent.

By integration by parts and by the property $(i)$

$$\gamma_n = (-)^n \int_a^b f(x)\,\frac{d^n}{dx^n}\,F_n(x)dx = (-)^n \int_a^b \frac{f(x)}{\psi(x)}\,\phi_n(x)dx = (-)^n\,C_n, \tag{B}$$

where

$$C_n = \int_a^b \frac{f(x)}{\psi(x)}\,\phi_n(x)\,dx.$$

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It is well-known that

$$\sum_{i=1}^{k} C_i^2 \leq \int_a^b \frac{f^2(x)}{\psi^2(x)}\,dx,$$

($k$ is an arbitrary positive integer), when $(a, b)$ is a finite interval; but even if $(0, \infty)$ be the interval, the law can be established by an exactly similar reasoning.

In particular,

$$C_i^2 \leq \int_a^b \frac{f^2(x)}{\psi^2(x)}\,dx, \quad i = 1, 2, \ldots .$$

Therefore from (B)

$$\gamma_n^2 = C_n^2 \leq \int_a^b \frac{f^2(x)}{\psi^2(x)}\,dx. \tag{C}$$

But the continuity of $f^{(n)}(x)$ in $(a, b)$ and the property (*ii*) of $F_n(x)$ give

$$\gamma_n = \int_a^b F_n(x) f^{(n)}(x)\,dx = f^{(n)}(\xi) \int_a^b F_n(x)dx, \tag{D}$$

$\xi$ lying between $a$ and $b$; the result is true when $(a, b)$ denotes a finite interval or the infinite interval $(0, \infty)$, the integral being assumed convergent.

Hence from (C) and (D)

$$| f^{(n)}(\xi) | \left| \int_a^b F_n(x)dx \right| \leq \sqrt{\left[ \int_a^b \frac{f^2(x)}{\psi^2(x)}\,dx \right]}.$$

Therefore, if $\beta_n$ denote the lower bound of $| f^{(n)}(x) |$ in $(a, b)$, since

$$\int_a^b F_n(x)\,dx \neq 0,$$

we have

$$\beta_n \leq \sqrt{\left[ \int_a^b \frac{f^2(x)}{\psi^2(x)}\,dx \right]} \bigg/ \left| \int_a^b F_n(x)\,dx \right|. \tag{I}$$

2. We can form a similar estimate for any other finite interval $(x_0, x_0+2c)$, $c \geq 0$, from a knowledge of (I) for a finite interval, by a linear transformation of the form

$$x = Az + B, \quad A = \frac{2c}{b-a}, \quad B = x_0 - \frac{2ac}{b-a}.$$

Since

$$dx = A dz \quad \text{and} \quad \frac{d^n}{dz^n} = A^n \frac{d^n}{dx^n},$$

we have

$$\alpha_n = \text{lower bound of } | \phi^{(n)}(x) | \text{ in } (x_0, x_0+2c)$$

$$\leq \frac{1}{A^n} \cdot \sqrt{\left[ \frac{1}{A} \int_{x_0}^{x_0+2c} \frac{f^2(x)}{\psi^2(x)}\,dx \right]} \bigg/ \left| \int_a^b F_n(x)dx \right|, \quad \left( A = \frac{2c}{b-a} \right), \tag{II}$$

where $\phi(z)/\psi(z) = f(x)/\psi(x)$.

The substitution $x = Az + x_0$, $A > 0$, but arbitrary, transforms the $z$-interval $(0, \infty)$ to the $x$-interval $(x_0, x_0+2c)$ and the formula (II) holds for the interval $(x_0, x_0+2c)$ with $c = \infty$.

3. We consider some applications with well-known normalised orthogonal systems.

(*i*) With

$$\phi_n(x) = \left[\frac{2^{2n-1}}{\pi}\right]^{1/2} \frac{T_n(x)}{(1-x^2)^{1/4}},$$

where

$$T_n(x) = \text{the } n^{th} \text{ Tschebyscheffian polynomial}$$

$$= \frac{(-)^n 2.n!}{(2n)!}(1-x^2)^{1/2}\frac{d^n}{dx^n}[(1-x^2)^{n-\frac{1}{2}}], \quad n \geq 1.$$

for which

$$\psi(x) = (1-x^2)^{1/4}, \quad F_n(x) = \frac{(-)^n n!2^{n+\frac{1}{2}}}{(2n)!\sqrt{\pi}}(1-x^2)^{n-\frac{1}{2}},$$

and

$$\int_a^b F_n(x)\,dx = \frac{(-)^n n!2^{n+\frac{1}{2}}}{(2n)!\sqrt{\pi}}\int_{-1}^1 (1-x^2)^{n-\frac{1}{2}}dx = \frac{(-)^n 2^{n+\frac{1}{2}}}{(2n)!}\Gamma(n+\tfrac{1}{2}), \quad (a=-1, \quad b=1),$$

$$\beta_n \leq \frac{(2n)!}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}\sqrt{\left[\int_{-1}^1 \frac{f^2(x)}{\sqrt{(1-x^2)}}\,dx\right]},$$

$$\alpha_n = \frac{(2n)!}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}\cdot\frac{1}{c^n}\sqrt{\left[\frac{1}{c}\int_{x_0}^{x_0+2c}\frac{f^2(x)}{\sqrt{(1-x^2)}}\,dx\right]}.$$

(*ii*) With $\phi_n(x) = \sqrt{[(2n+1)/2]}P_n(x)$, where $P_n(x)$ is the $n^{th}$ Legendre's polynomial, for which

$$\psi(x) = 1, \quad F_n(x) = \sqrt{\left[\frac{2n+1}{2}\right]}\frac{1}{2^n n!}(x^2-1)^n,$$

and $a=-1$, $b=1$,

$$\int_a^b F_n(x)\,dx = \sqrt{\left[\frac{2n+1}{2}\right]}\frac{1}{2^n n!}\int_{-1}^1 (x^2-1)^n\,dx = \sqrt{\left[\frac{2}{2n+1}\right]}\frac{2^n n!}{(2n)!}(-)^n,$$

we obtain J. Soula's results.

(*iii*) With

$$\phi_n(x) = \frac{e^{-x/2}L_n(x)}{n!},$$

where

$$L_n(x) = \text{the } n^{th} \text{ Laguerre polynomial} = e^x\frac{d^n}{dx^n}(x^n e^{-x}),$$

for which

$$\psi(x) = e^{x/2}, \quad F_n(x) = \frac{1}{n!}x^n e^{-x},$$

and $a=0$, $b=\infty$ and

$$\int_a^b F_n(x)dx = \text{the convergent integral } \frac{1}{n!}\int_0^\infty x^n e^{-x}\,dx = 1,$$

$$\beta_n \leq \sqrt{\left[\int_0^\infty e^{-x}f^2(x)dx\right]}.$$

DEPARTMENT OF MATHEMATICS,
DACCA UNIVERSITY.

*Reference*

Soula, J., (1932), *Mathematica*, 6, 86.

# ON THE APPLICATION OF THE CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION TO CERTAIN QUATERNARY FORM

By

D. P. Banerjee

*(Communicated by the Secretary—Received March 30, 1945)*

Ramanujan (1927), Gupta (1943), myself (1942) proved certain congruence properties of Ramanujan's function $T(n)$. Here I have considered new and interesting applications of the congruence properties to the possibility of solutions of certain quaternary equations. In Sec. I, I shall prove certain new congruence properties and in Sec. II, apply some congruence properties to some quarternary forms.

## Sec. I

Since (Ramanujan, 1927)

$$T(5) \equiv 0 \pmod 5)$$

and

$$T(p^\lambda) = T(p)T(p^{\lambda-1}) - p^{11}T(p^{\lambda-2}), \quad \lambda \geq 2,$$

where $p$ is prime number and the Ramanujan's function $T(n)$ is defined by the equation

$$\sum_1^\infty T(n)x^n = x\{(1-x)(1-x^2)\ldots\}^{24}.$$

Then

$$T(5^n) \equiv 0 \pmod{5^n}. \tag{1}$$

We know (Gupta, 1943)

$$T(4m+3) \equiv 0 \pmod 4,$$

then

$$T(4m+3)n \equiv 0 \pmod 4, \tag{2}$$

if $(4m+3, n) = 1$ and $3n$ is not of the form $4m+3$.

Also (Gupta, 1943, p. 21)

$$T(8m+7) \equiv 0 \pmod 8,$$

then

$$T(8m+7)n \equiv 0 \pmod 8, \tag{3}$$

if $(8m+7, n) = 1$ and $7n$ is not of the form $8m+7$.

Since

$$(1-x)^{11} = 1-x^{11}+11J,$$

where $J$ is any integral power series in $x$,

$$\sum_1^\infty T(n)x^n = [1+\sum_1^\infty(-1)^n(x^{11n(3n-1)/2}+x^{11n(3n+1)/2})][1+\sum_1^\infty(-1)^r(x^{r(3r-1)/2}+x^{r(3r+1)/2})]+11J.$$

Equating the coefficients of $x^8$, $x^{19}$, $x^{29}$, we have

$$T(8) \equiv 0 \pmod{11}, \quad T(19) \equiv 0 \pmod{11} \quad \text{and} \quad T(29) \equiv 0 \pmod{11}.$$

Hence

$$\left.\begin{array}{ll} T(8m) \equiv 0 \pmod{11}, & \text{where } (8, m) = 1 \\ T(19m) \equiv 0 \pmod{11}, & \text{where } (19, m) = 1 \\ T(29m) \equiv 0 \pmod{11}, & \text{where } (29, m) = 1. \end{array}\right\} \tag{4}$$

SEC. II

Here I shall consider the following quaternary equations,

(A) $4n = a_1^2 + a_2^2 + a_3^2 + a_4^2$, where $n$, $a_1$, $a_2$, $a_3$, $a_4$ are positive odd integers,

(B) $8n = \sum_1^8 a_r^2$, where $n$, $a_r$'s are positive odd integers,

(C) $2n = a_1^2 + a_2^2$, where $a_1$, $a_2$, $n$ are positive odd integers,

(D) $4n = a_1^2 + a_2^2 + a_3^2 + a_4^2 + du^2$, where $n$, $a_1$, $a_2$, $a_3$, $a_4$, $u$ are positive odd integers.

Let $\mu_1(n)$, $\mu_2(n)$, $\mu_3(n)$ and $\mu_4(n)$ denote the total number of solutions of the equations A, B, C, D respectively. To avoid ambiguity "solution" will be used where the order $a_1, a_2, a_3 \ldots$ are relevant and representation when it is not so.

Now

$$\sum_1^\infty T(n)x^n = x\{(1-x^8)(1-x^{16})\ldots\}^3 + 2J = \sum_1^\infty x^{(2\mu+1)^2} + 2J.$$

Hence

$$T(n) = \text{odd}, \quad \text{if } n = (2\mu+1)^2. \tag{5}$$

Again

$$\sum_1^\infty T(n)x^n = x[(1-x^2)(1-x^4)\ldots]^{12} + 2J = x[\sum_0^\infty x^{\mu(\mu+1)}]^4 + 2J = \sum_1^\infty \mu_1(n)x^n + 2J.$$

Hence

$$\mu_1(n) \equiv T(n) \pmod 2,$$

or

$$\begin{aligned} \mu_1(n) &= \text{odd}, \quad \text{if } n = (2\mu+1)^2, \\ &= 0 \text{ or even}, \quad \text{if } n \neq (2\mu+1)^2. \end{aligned} \tag{6}$$

Since

$$\sum_1^\infty T(n)x^n = n[\sum_1^\infty (-1)^n(2\mu+1)x^{\mu(\mu+1)/2}]^8 + 2J = \sum_1^\infty \mu_2(n)x^n + 2J.$$

Hence as before

$$\begin{aligned} \mu_2(n) &= \text{odd}, \quad \text{if } n = (2\mu+1)^2, \\ &= 0 \text{ or even}, \quad \text{if } n \neq (2\mu+1)^2. \end{aligned} \tag{7}$$

We know (Gupta, 1943, p. 19)

$$\sum_1^\infty T(n)x^n = \sum_1^\infty \mu_3(n)x^n + 4J,$$

$$\mu_3(n) \equiv T(n) \pmod 4.$$

Hence

$$\begin{aligned} \mu_3(n) &= \text{odd}, \quad \text{if } n = (2\mu+1)^2, \\ &= 0 \text{ or even}, \quad \text{if } n \neq (2\mu+1)^2. \end{aligned} \tag{8}$$

To complete the discussion we consider the equation (D)

$$\mu_4(n) = 0, \quad \text{if } \mu_1\left(n - \frac{d}{4}u^2\right) = 0,$$

*i.e.*, if

$$n - \frac{d}{4}u^2 \equiv 7 \pmod 8.$$

Since $\mu_1(8m+7) = 0$. If $d \equiv 0 \pmod 8$,

$$\mu_4(n) = \mu_1(8m+7) = 0. \tag{9}$$

If $d = 4d_1$, then

$$n - d_1u^2 = 8m+7,$$

if

$$n = d_1(2\mu+1)^2+7.$$

Hence

$$\mu_4[7+d_1(2\mu+1)^2] = 0.. \tag{10}$$

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——— (1942), *Proc. Nat. Acad, Sc., India*, 12, Part 2.
Gupta, H C., (1943), *Proc. Benares Math. Soc.*, 5, 17.
Ramanujan, (1927), *Collected Papers*, 230.

# CORRECTION TO MY PAPER "BERTRAND'S POSTULATE"*

By

S. S. Pillai

(*Received February 19, 1945*)

I am sorry that there is a confusion of notation and thank Mr. R. C. Bose for drawing my attention to it. The confusion may be avoided as follows:—

In line 12 page 97, omit $t$, replace $n!$ by $n$. At the end of next line, insert $t = M(n)$. In the line beginning with (*iv*), on page 98, replace $M(n)$ by $M(N)$.

If preferred, lemma (3) may be omitted and lemma (4) may be proved as follows:—

Since

$$M(n!) = \sum_i [n/p^i],$$

$$\log N = \sum_{p \leq 2n} \log p \,\{\sum_i ([2n/p^i] - 2[n/p^i])\} = S_1 + S_2 + S_3 + S_4,$$

where the summations in $S_1$, $S_2$, $S_3$, $S_4$ are for primes in (*i*), (*ii*), (*iii*), (*iv*) respectively.

$$[2n/a] - 2[n/a] \leq 1,$$

So

$$S_1 = \theta(2n) - \theta(n);$$

when $n \geq 2$,

$$S_3 = \theta(2n/3) - \theta(\sqrt{2n}) \leq \theta(2n/3) - \pi(\sqrt{2n}) \log 2;$$

and

$$S_4 \leq \sum_{p \leq \sqrt{(2n)}} \log p \sum_{p^i \leq 2n} 1 \leq \sum \log p \times \frac{\log 2n}{\log p} = \pi(\sqrt{2n}) \log 2n.$$

In $S_2$, $1 \leq n/p < 3/2$ and $2 \leq 2n/p < 3$, so that

$$[n/p] = 1, \quad [2n/p] = 2.$$

Hence $S_2 = 0$. From the above results for $S_1$, $S_2$, $S_3$, $S_4$, the lemma is immediate.

* Pillai, S. S., (1944), *Bull. Cal. Math. Soc.*, 36, 97.

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# CALCUTTA MATHEMATICAL SOCIETY

## *Report of the Council for the year 1944 to the Annual General Meeting of the Calcutta Mathematical Society*

The Council of the Calcutta Mathematical Society have the pleasure to submit the following report on the general concerns of the Society for the year 1944, as required by the provisions of Rule 25.

*The Council.*—The Council of the Society for the year 1944 consisting of the Officers and Members elected at the last Annual General Meeting together with the Editorial Secretary, was constituted as follows:—

*President*

Prof. N. R. Sen

*Vice-Presidents*

Prof. C. V. Hanumanta Rao, Prof. D. N. Sen, Mr. S. Gupta, Prof. F. W. Levi, Dr. C. N. Srinivasiengar.

*Treasurer*

Mr. S. C. Ghosh

*Secretary*

Dr. S. K. Chakrabarty

*Editorial Secretary*

Dr. S. Ghosh (upto March, 1944),
Mr. S. Gupta (from April, 1944).

*Members of the Council*

Prof. M. R. Siddiqi, Dr. H. Bagchi, Mr. R. C. Bose, Mr. A. C. Choudhury, Mr. B. M. Sen, Prof. M. N. Saha, Dr. S. S. Pillai, Dr. S. R. Sen Gupta, Prof. V. V. Narlikar, Dr. R. N. Sen, Mr. N. N. Ghosh.

The Council of the Society held 6 meetings during the year. At a meeting held on the 16th March, 1944, the Council at the request of the University decided to take the charge of publishing the Bulletin which was being done by the Calcutta University. At the request of the Council Dr. R. N. Sen took the publishers declaration on behalf of the Calcutta Mathemetical Society. The Council also decided to award the Krishnakumari

Ganesh Prasad Prize and Medal for 1946 to the author of the best theses on "The Development of the Concept of Number in Hindu Mathematics before 1600 A.D." and an announcement has already been made to that effect.

There were altogether 6 meetings of the Society in which 24 papers were read.

*Membership.*—The Council record with regret the death of the following 2 members of the Society during the year under review:

Mr. Mohitmohan Ghosh and Mr. R. V. Shastry

During the year under review 8 new members were elected.

*Publication*:—Five numbers of the Bulletin of the Calcutta Mathematical Society were published during the year (*viz.* Nos. 3 and 4 of Vol. 35 and Nos. 1, 2 and 3 of Vol. 36) so that at the close of the year the Bulletin was in arrear by one number only, and it is a pleasure to announce that this last number is also practically ready and we hope to publish it within the next fortnight. In that case for the first time within the last few years we shall have a good margin of time before the next number will be officially due. Such a state of affair could not be a possibility, particularly in these difficult times due to the conditions of war, except for the efficient handling of the situation by the Editorial Secretary Mr. S. Gupta. The Council take this opportunity to convey its grateful thanks to Mr. Gupta for his very valuable services to the Society.

In a circular letter on the subject of economy of paper the Government of India suggested to curtail the publications of the Society to a very great extent. The Council considered the situation carefully and decided to take steps in order to diminish the number of pages to be published in a year in the Bulletin, as far as possible without seriously disturbing the cause of the Society. At the request of the Council of the Society the Government of India have allowed a liberal concession to the Society in this matter, and for this act of kindness the Society is very much indebted to it.

The Rockfeller Foundation Grant has very kindly alloted to the Society a grant of Rs. 250 for the year 1944, for the improvement of the Bulletin of the Society. The Council has gladly accepted this grant and have already taken steps for the improvement of the Bulletin. The future issues of the Bulletin will show to what extent the desired improvement has been made possible. The Council takes this opportunity to convey its thanks to the Rockfeller Foundation for this award.

The Calcutta University Press which is printing the Bulletin free of charge since its first publication have continued to grant the same privileges even at this difficult times. Without the very sympathetic and active co-operation of the University Press the regular publication of Bulletin would certainly be not a possibility. The Council takes this opportunity to convey its thanks to the officers and members of the staff concerned for this act of co-operation.

*Exchanges*:—During the year under review the Society entered into one new exchange relation so that the total number of Societies, Libraries, etc. to whom the Society sends its Bulletin for favour of exchange stood at 113 at the close of the year. Due to war-time difficulties it has not been possible to send Bulletins to several institutions and to receive their's in exchange. It has, however, been arranged to revive the exchange relationship as soon as the external circumstances permit.

*Finance* :—The financial position of the Society can be seen from the audited statement of account of the Society for the year under review as will presently be submitted by the auditors. The surplus in the general fund amounting to Rs. 461-12-0 revealed by the auditors' report is only apparent. A sum of Rs. 250 received from the Rockfeller Foundation Grant and also a sum of Rs 100 which is to be paid to Messrs. G. E. Stechert & Co., towards our subscription for the Mathematical Reviews aud Physical Reviews for 1944 are to be set apart from it. The actual expense in the item of Books and Journals is Rs. 125-8-0, which includes the cost of journals subscribed by the Society. This has not been included in the audit report as it has not actually been paid due to obvious difficulties.

# CALCUTTA MATHEMATICAL SOCIETY

## Receipts and Disbursements Accounts of the Calcutta Mathematical Society for the Year Ending 31st December, 1944

*Receipts*

| | | | | Rs. | As. | P. |
|---|---|---|---|---|---|---|
| 1. Opening Balance :— | | | | | | |
| (a) Cash with Secretary | ... | | 13 13 3 | | | |
| (b) Balance at Bank | | | | | | |
| (i) Imperial Bank of India | | 875 2 6 | | | | |
| (ii) Do. (K.K.G.P. Fund) | | 77 5 3 | | | | |
| (iii) Bengal Central Bank | | 412 11 0 | | | | |
| (iv) P.O. Savings Bank | | 551 11 0 | 1,916 13 9 | | | |
| (c) G. P. Notes (General Fund) (Face value Rs. 6,000) | ... | | 5,663 11 6 | | | |
| (d) G. P. Notes (K.K.G.P. Fund) (Face value Rs. 2,000) | ... | | 1,937 7 9 | 9,581 | 14 | 3 |
| 2. Subscriptions | ... | | 803 10 0 | | | |
| 3. Admission Fee | ... | | 50 0 0 | | | |
| 4. Sale Proceeds | ... | | 210 14 0 | | | |
| 5. Grant from Rockfeller Foundations | ... | | 250 0 0 | 1,314 | 8 | 0 |
| 6. Interest A/c. :— | | | | | | |
| (a) P.O. Savings Bank | ... | | 9 10 3 | | | |
| (b) G. P. Notes (General Fund) | ... | | 210 0 0 | | | |
| (c) Do. (K. K. G. P. Fund) | ... | | 70 0 0 | 289 | 10 | 3 |
| Total | ... | | | 11,136 | 0 | 6 |

*Disbursements*

| | | | | Rs. | As. | P. |
|---|---|---|---|---|---|---|
| 1. Establishment | ... | | 517 10 0 | | | |
| 2. Meetings | ... | | 63 2 6 | | | |
| 3. Books and Journals (including charges for book binding) | ... | | 25 8 3 | | | |
| 4. Bulletins :— | | | | | | |
| (a) Papers and Blocks | ... | 282 0 9 | | | | |
| (b) Postage | ... | 58 14 9 | 340 15 6 | | | |
| 5. Printing and Stationery | ... | | 40 12 0 | | | |
| 6. Postage | ... | | 49 0 6 | | | |
| 7. Bank Charges | ... | | 8 14 0 | | | |
| 8. Miscellaneous (including conveyance charges) | | | 26 7 6 | 1,072 | 6 | 3 |
| *Closing Balance :—* | | | | | | |
| (a) Cash with Secretary | ... | | 5 5 0 | | | |
| (b) Balance at Bank | | | | | | |
| (i) Imperial Bank of India | ... | 1,088 2 6 | | | | |
| (ii) Do. (K. K. G. P. Fund) | ... | 146 13 3 | | | | |
| (iii) Bengal Central Bank | ... | 653 0 0 | | | | |
| (iv) Do. (Suspense A/c.) | ... | 12 13 0 | | | | |
| (v) P. O. Savings Bank | ... | 561 5 3 | 2,457 2 0 | | | |
| (c) G. P. Notes (General Fund) (Face value Rs. 6,000) | ... | | 5,663 11 6 | | | |
| (d) G. P. Notes (K. K. G. P. Fund) (Face value Rs. 2,000) | ... | | 1,937 7 9 | 10,063 | 10 | 3 |
| Total | ... | | | 11,136 | 0 | 6 |

We beg to report that we have examined the accounts of the Calcutta Mathematical Society for the year ending 31st December, 1944. The accounts have been properly kept and we note with pleasure that the expenses have been kept as low as possible under the present circumstances.

B. SEN
B. C. CHATTERJEE
*Auditors*

# ON A NEW REDUCTION THEOREM OF MATRICES

By

N. N. Ghosh

*(Received May 22, 1945)*

Let $M$ denote the $m \times n$ matrix

$$M = \sum_{p,q=1}^{m,n} \mu_{pq}\, e_{pq} \tag{1}$$

with real or complex elements, where the $e$'s are matrix units. Associated with the above there are $n$ fundamental column-matrices

$$(M)^i = \sum_{p=1}^{m} \mu_{pi}\, e_{p1}, \quad (i = 1, 2, \ldots, n), \tag{2}$$

and $m$ fundamental row-matrices

$$(M)_a = \sum_{q=1}^{n} \mu_{aq}\, e_{1q}. \quad (a = 1, 2, \ldots, m). \tag{3}$$

The object of the present paper is to develop the general method of expressing $M$, in an infinite variety of ways, as the sum of a number of products of the type $C^h R_h$, where the matrices $C^h$ and $R_h$ belong respectively to the systems linearly dependent on (2) and (3). This method of resolution proceeds by successive steps in accordance with the choice of an arbitrary sequence of matrices *auxiliary* to $M$. A class of identities involving the product of a determinant and any one of its minors expressed as an aggregate of products of pairs of minors is intimately associated with the process. Typical instances of such identities supplemented by a proof* are given by Muir (1930).

With this reduction theorem, moreover, the problem of factorization of $M$ is linked up. For, let $r$ be the rank of $M$, we then obtain the reduction formula actually in the form

$$M = \sum_{h=1}^{r} C^h R_h, \tag{4}$$

where

$$C^h = \sum_{p=1}^{m} \gamma_{ph}\, e_{p1}, \qquad R_h = \sum_{q=1}^{n} \rho_{hq}\, e_{1q}.$$

Hence the $(p, q)$th element of $M$ in (4) can be expressed as

$$\sum_{h=1}^{r} \gamma_{ph}\, \rho_{hq},$$

which is identical with the $(p, q)$th element in the product $M = \Gamma P$, where

$$\Gamma = \sum_{p,h=1}^{m,r} \gamma_{ph}\, e_{ph}, \qquad P = \sum_{h,q=1}^{r,n} \rho_{hq}\, e_{hq}.$$

The reduction of a matrix in the form (4) has also some physical significance attached to it. I have already utilized it in the analysis of stress and strain in an Euclidean hyperspace, where the general stress and strain matrices are real symmetric square matrices (Ghosh, 1942) and also in the resolution of generalized couples and infinitesimal rotations in connection with an $n$-dimensional rigid motion, where the representative square matrices are real and skew-symmetric (Ghosh, 1943).

* The determinant (13) defined below supplies an alternative proof which is simpler.

1. Let us denote the general minor determinant of $M$ of order $s$ with the principal diagonal elements $\mu_{ai}, \mu_{bj}, \ldots, \mu_{dl}$, formed according to the scheme

$$\begin{pmatrix} ij & \ldots. & l \\ & \mu & \\ ab & \ldots. & d \end{pmatrix}, \tag{5}$$

by the symbol $\mu_{aibj..dl}$. We define then a column-matrix of the $s$th class, linearly dependent on (2), by

$$(M)^{i\ldots kl}_{a\ .c} = \sum_{p=1}^{m} \mu_{ai\ldots ckpl}\, e_{p1}, \tag{6}$$

and a row-matrix of the $s$th class, linearly dependent on (3), by

$$(M)^{i\ .\ k}_{a.\ .cd} = \sum_{q=1}^{n} \mu_{ai\ldots ckdq}\, e_{1q}. \tag{7}$$

The indices $l$ and $d$ occurring respectively in (6) and (7) will be called the *effective* indices. It must be noticed that by means of (5) we can form only $s$ distinct matrices of the $s$th class having distinct effective indices belonging to the type (6) or (7). We further define the matrix of the $s$th class *auxiliary* to $M$ by

$$M^{i.\ l}_{a\ldots d} = \sum_{p,q=1}^{m,n} \frac{\mu_{aibj..dlpq}}{\mu_{aibj\ .dl}}\, e_{pq}. \tag{8}$$

In view of the above definition the class-number of the matrix $M$ is 0.

2. The general formula showing the linear dependence of the column-matrix (6) to those of lower classes can be written down by noting the symmetry in the following typical relations:

$$\left.\begin{aligned} (M)^{ijfg}_{abx} &= \mu_{aibjxf}(M)^{g} - \mu_{aibjxg}(M)^{f} - \mu_{aibgxf}(M)^{j} - \mu_{agbjxf}(M)^{i}, \\ \mu_{ai}(M)^{ijfg}_{abx} &= \mu_{aibjxf}(M)^{ig}_{a} - \mu_{aibjxg}(M)^{if}_{a} - \mu_{aibgxf}(M)^{ij}_{a}, \\ \mu_{aibj}(M)^{ijfg}_{abx} &= \mu_{aibjxf}(M)^{ijg}_{ab} - \mu_{aibjxg}(M)^{ijf}_{ab}. \end{aligned}\right\} \tag{9}$$

Corresponding to the row-matrices represented by (7), the typical relations are as follows:

$$\left.\begin{aligned} (M)^{ijf}_{abxy} &= \mu_{aibjxf}(M)_{y} - \mu_{aibjyf}(M)_{x} - \mu_{aiyjxf}(M)_{b} - \mu_{yibjxf}(M)_{a}, \\ \mu_{ai}(M)^{ijf}_{abxy} &= \mu_{aibjxf}(M)^{i}_{ay} - \mu_{aibjyf}(M)^{i}_{ax} - \mu_{aiyjxf}(M)^{i}_{ab}, \\ \mu_{aibj}(M)^{ijf}_{abxy} &= \mu_{aibjxf}(M)^{ij}_{aby} - \mu_{aibjyf}(M)^{ij}_{abx}. \end{aligned}\right\} \tag{10}$$

Particularly important for the purpose of this paper is the set of *mixed* relations involving the matrices (6), (7) and (8) of the following type:

$$\left.\begin{aligned} \mu_{aibjxf}(M - M^{ijf}_{abx}) &= (M)^{f}(M)^{ij}_{abx} + (M)^{j}(M)^{if}_{axb} + (M)^{i}(M)^{jf}_{bxa}, \\ \mu_{aibjxf}\cdot\mu_{ai}(M^{i}_{a} - M^{ijf}_{abx}) &= (M)^{if}_{a}(M)^{ij}_{abx} + (M)^{ij}_{a}(M)^{if}_{axb}, \\ \mu_{aibjxf}\cdot\mu_{aibj}(M^{ij}_{ab} - M^{ijf}_{abx}) &= (M)^{ijf}_{ab}(M)^{ij}_{abx}. \end{aligned}\right\} \tag{11}$$

There is an alternative way of expressing (11) obtained by an exchange of the effective indices between the pair forming each of the product-terms in the right-hand side. Thus, for the first two, the right-hand sides are equivalent to

$$\left.\begin{aligned}&(M)^{ij}_{ab}(M)_x+(M)^{ij}_{ax}(M)_b+(M)^{ji}_{bx}(M)_a,\\&(M)^{ij}_{ab}(M)^i_{ax}+(M)^{ij}_{ax}(M)^i_{ab},\end{aligned}\right\}\tag{12}$$

the third remaining unchanged.

It should be noted in (11) or (12) that the difference of a pair of $M$'s belonging to any two classes is expressible as the sum of a certain number of products of the type $C^hR_h$, this number being always equal to the difference of the two class-numbers.

3. The identity necessary to establish general relations of the types (9) and (11) is derived by the two-fold expansion of the determinant of order $2s+1-t$ represented according to the following scheme:

$$\left|\begin{array}{c:c}D_{s-t,\,s-t} & D_{s-t,\,s+1}\\ \hdashline D_{s+1,\,s-t} & D_{s+1,\,s+1}\end{array}\right|,\tag{13}$$

where

$D_{s+1,\,s+1}$ stands for $\mu_{a_1i_1a_2i_2\ldots a_si_spq}$,

$D_{s-t,\,s-t}$ stands for $\mu_{a_1i_1\ldots a_{s-t}i_{s-t}}$, a minor of $D_{s+1,\,s+1}$,

$D_{s-t,\,s+1}$ consists of the first $s-t$ rows of $D_{s+1,\,s+1}$ and $D_{s+1,\,s-t}$ consists of $s$ rows of zeros except the last containing the elements $\mu_{pi_1}, \mu_{pi_2}, \ldots, \mu_{pi_{s-t}}$. It may be noted that the three particular determinants leading to the relations (11) are obtained from (13) when we put, in succession, (*i*) $s=3$, $t=3$; (*ii*) $s=3$, $t=2$; (*iii*) $s=3$. $t=1$. The relations (9) will also follow from these determinants if we replace the suffix $q$ in each by $g$.

To establish general relations of the types (10) and (12) the procedure is similar, but the determinant defined above requires modification; so that, in the symbolic representation (13), $D_{s-t,\,s+1}$ now consists of $s$ columns of zeros except the last containing the elements $\mu_{a_1q}, \mu_{a_2q}, \ldots, \mu_{a_{s-t}q}$ and $D_{s+1,\,s-t}$ consists of the first $s-t$ columns of $D_{s+1,\,s+1}$, the other $D$'s remaining unchanged.

4. For a reduction of $M$ in the form (4), let us consider a pair of similar sequences $(ab\ldots cd\ldots)$ and $(ij\ldots kl\ldots)$, containing not less than $r$ distinct integers arranged in a definite order, the former being chosen out of the natural numbers $1, 2, \ldots, m$ and the latter out of $1, 2, \ldots, n$. We form then the sequence of auxiliary matrices $M^i_a, M^{ij}_{ab}, \ldots, M^{ij\ldots kl}_{ab\ldots cd}, \ldots\ldots$ with class-numbers differing by unity, till we arrive at the last containing $r-1$ pairs of indices, since $M$ is of rank $r$. Referring to the last member in the set of formulae (11) we can write, in succession.

$$\left.\begin{aligned}&M-M^i_a=\frac{(M)^iM_a}{\mu_{ai}},\qquad M^i_a-M^{ij}_{ab}=\frac{(M)^{ij}_a(M)^i_{ab}}{\mu_{ai}\,\mu_{aibj}},\ \ldots\\&M^{ij\ldots k}_{ab\ldots c}-M^{ij\ldots kl}_{ab\ldots cd}=\frac{(M)^{ij\ldots kl}_{ab\ldots c}(M)^{ij\ldots k}_{ab\ldots cd}}{\mu_{aibj\ldots ck}\,\mu_{aibj\ldots ckdl}},\end{aligned}\right\}\tag{14}$$

and so on, which are, in fact, the successive terms of the required expression (4). If desired, we may omit any number of the auxiliary matrices in the above sequence and consider only the remaining ones in the order of ascending class-numbers. This exclusion is a necessity if some of the auxiliary matrices become infinite. The general set of formulae (11) or (12), will then also supply the necessary modification required in the series of equations (14), where the class-numbers differ by more than unity. In any case, it is obvious that the number of terms in (4) is equal to the rank of $M$.

A particular case of the above reduction has been dealt with in a previous paper (Ghosh, 1944), where $M$ is a square matrix of the Hermitian type.

5. We conclude this paper by finding out a set of relations which may be regarded as inverse to (9) and (10). Let us take the first member of (10) along with the following three allied equations:

$$(M)^{jfg}_{bxya} = \mu_{bjxfyg}(M)_a - \mu_{bjxfag}(M)_y - \mu_{bjafyg}(M)_x - \mu_{ajxfyg}(M)_b,$$

$$(M)^{ifg}_{axyb} = \mu_{aixfyg}(M)_b - \mu_{aixfbg}(M)_y - \mu_{aibfyg}(M)_x - \mu_{bixfyg}(M)_a,$$

$$(M)^{ijg}_{abyx} = \mu_{aibjyg}(M)_x - \mu_{aibjxg}(M)_y - \mu_{aixjyg}(M)_b - \mu_{xibjyg}(M)_a$$

Solving these four equations, we obtain

$$\left.\begin{aligned} \mu_{aibjxfyg}(M)_a &= \mu_{ai}(M)^{jfg}_{bxya} + \mu_{aj}(M)^{ifg}_{axyb} + \mu_{af}(M)^{ijg}_{abyx} + \mu_{ag}(M)^{ijf}_{abxy}, \\ \mu_{aibjxfyg}(M)_b &= \mu_{bi}(M)^{jfg}_{bxya} + \mu_{bj}(M)^{ifg}_{axyb} + \mu_{bf}(M)^{ijg}_{abyx} + \mu_{bg}(M)^{ijf}_{abxy}, \end{aligned}\right\} \tag{15}$$

and two others for $(M)_x$ and $(M)_y$.

To express $(M)^i_{ab}$ in the form (15) we make use of the identity

$$(M)^i_{ab} = \mu_{ai}(M)_b - \mu_{bi}(M)_a,$$

and get the equation

$$\mu_{aibjxfyg}(M)^i_{ab} = \mu_{aibj}(M)^{ifg}_{axyb} + \mu_{aibg}(M)^{ijf}_{abxy} + \mu_{aibf}(M)^{ijg}_{abyx}. \tag{16}$$

Proceeding a step further we derive the equation

$$\mu_{aibjxfyg}(M)^{ij}_{abx} = \mu_{aibjxf}(M)^{ijg}_{abyx} + \mu_{aibjxg}(M)^{ijf}_{abxy}. \tag{17}$$

An inspection of the formulae (15), (16) and (17) suggests the general law which enables one to write down similar equations in other cases. Corresponding to the set (9), we are led to the following inverse set of equations:

$$\left.\begin{aligned} \mu_{aibjxfyg}(M)^i &= \mu_{ai}(M)^{ijfg}_{bxy} + \mu_{bi}(M)^{ijfg}_{axy} + \mu_{xi}(M)^{ijgf}_{aby} + \mu_{yi}(M)^{ijfg}_{abx}, \\ \mu_{aibjxfyg}(M)^{ij}_a &= \mu_{aibj}(M)^{ijfg}_{axy} + \mu_{aixj}(M)^{ijgf}_{aby} + \mu_{aiyj}(M)^{ijfg}_{abx}, \\ \mu_{aibjxfyg}(M)^{ijf}_{ab} &= \mu_{aibjxf}(M)^{ijgf}_{aby} + \mu_{aibjyf}(M)^{ijfg}_{abx}. \end{aligned}\right\} \tag{18}$$

*References*


Ghosh, N. N., (1942), *Bull. Cal. Math. Soc.* **34**, 150.
Ghosh, N. N., (1943), *Bull. Cal. Math. Soc.*, **35**, 119.
Ghosh, N. N., (1944), *Bull. Cal. Math. Soc.*, **36**, 89.
Muir, T., (1930), *A treatise on the theory of determinants*, 182.

# PROBLEMS OF THIN PLATES WITH CIRCULAR HOLES

By

Bibhutibhusan Sen

(Received May 7, 1945)

*Introduction*

Boundary value problems of circular disks with forces acting inside the disks have been discussed by the author in two previous papers (Sen, 1944). The object of this paper is to solve a different type of problems, namely, that of finding the stresses in an infinite plate, when either a force in the plane of the plate or a couple with its axis normal to the plate, acts at a point *outside* a circular hole, not far away from its centre. The distribution of stresses in an infinite plate due to forces applied to the internal boundary of a circular hole was obtained by Bickley (1928). The method used in this paper, which is different from that of Bickley, is as follows.

We assume that the plate is in a state of generalized plane stress and that its middle surface is given by the plane $z=0$. Taking the origin at the centre of the circular hole of radius $a$, we first write down the average stresses $\breve{xx}_1$, $\breve{xy}_1$, $\breve{yy}_1$, due to the force or the couple as the case may be, acting at a point (outside the hole) on the ***hypothesis*** that the plate is infinite in extent and that there is no hole.

Putting

$$\left.\begin{aligned} r.\breve{rx}_1 &= x.\breve{xx}_1+y.\breve{xy}_1, \\ r.\breve{ry}_1 &= x.\breve{xy}_1+y.\breve{yy}_1, \end{aligned}\right\} \tag{A}$$

where $r^2 = x^2+y^2$, we can calculate the values of $r.\breve{rx}_1$ and $r.\breve{ry}_1$ on the edge $r=a$. To annul these stresses on the boundary of the hole we must superimpose a stress system $r.\breve{rx}_2$, $r.\breve{ry}_2$ given by

$$\left.\begin{aligned} r.\breve{rx}_2 &= x.\breve{xx}_2+y.\breve{xy}_2, \\ r.\breve{ry}_2 &= x.\breve{xy}_2+y.\breve{yy}_2, \end{aligned}\right\} \tag{B}$$

where $\breve{xx}_2$, $\breve{xy}_2$ and $\breve{yy}_2$ are average stresses possible in a thin elastic plate such that

$$\left.\begin{aligned} [\breve{rx}_2]_{r=a} &= -[\breve{rx}_1]_{r=a}, \\ [\breve{ry}_2]_{r=a} &= -[\breve{ry}_1]_{r=a}. \end{aligned}\right\} \tag{C}$$

Expressions for $r.\breve{rx}_2$ and $r.\breve{ry}_2$ in terms of the boundary values have been given in the previous papers and also in a different form elsewhere (Sen, 1938). These results are

$$\left.\begin{aligned} r.\breve{rx}_2 &= \mathbf{Re}\left[\frac{r^2-a^2}{4}.\frac{f(z)-zf'(z)}{z}+aL(z)\right], \\ r.\breve{ry}_2 &= \mathbf{Re}\left[i\frac{r^2-a^2}{4}.\frac{f(z)-zf'(z)}{z}+aM(z)\right]. \end{aligned}\right\} \tag{D}$$

where $i = \sqrt{(-1)}$, $z = x+iy$, **Re** denotes real part and $L(z)$, $M(z)$ are functions of $z$ such that

$$\left.\begin{aligned}[\overset{\smile}{rx}_2]_{r=a} &= [\mathbf{Re}\ L(z)]_{r=a},\\ [\overset{\smile}{ry}_2]_{r=a} &= [\mathbf{Re}\ M(z)]_{r=a}.\end{aligned}\right\} \tag{E}$$

Moreover, we have

$$\left.\begin{aligned}\overset{\smile}{xx}_2+\overset{\smile}{yy}_2 &= \mathbf{Re}\ f(z),\\ f(z) &= 2az^{-1}[L(z)+iM(z)]\end{aligned}\right\} \tag{F}$$

and

except when $f(z) \propto z^{-1}$ [*cf.* Art. 3(b) (Sen, 1938)].

In these problems $r.\overset{\smile}{rx}_1$ and $r.\overset{\smile}{ry}_1$ will be already known. Hence from (C) the values of $r.\overset{\smile}{rx}_2$ and $r.\overset{\smile}{ry}_2$ on the edge can be found. Then we can obtain the functions $L(z)$ and $M(z)$ which vanish at infinity and satisfy the relations (E). The values of $L(z)$ and $M(z)$ being known, $f(z)$ can be found from (F), and hence from (D), the values of $r.\overset{\smile}{rx}_2$ and $r.\overset{\smile}{ry}_2$ can be completely determined. The resultant expressions $r.\overset{\smile}{rx}$ and $r.\overset{\smile}{ry}$ at any point are then given by

$$\left.\begin{aligned}r.\overset{\smile}{rx} &= r.\overset{\smile}{rx}_1+r.\overset{\smile}{rx}_2,\\ r.\overset{\smile}{ry} &= r.\overset{\smile}{ry}_1+r.\overset{\smile}{ry}_2.\end{aligned}\right\} \tag{G}$$

### 1. *Solution for a force acting at a point near a circular hole*

Let an isolated force $T$ act at the point $A(c, 0)$ $(c > a)$ in the direction of the $x$-axis, that is, in the direction of the line joining the centre $O$ to the point $A$. If there had been no hole in this infinite plate, we would have the corresponding average stresses (Coker and Filon, 1931, p 327), given by

$$\overset{\smile}{xx}_1 = \frac{T}{4\pi}\left[-\frac{(3+\sigma)\,(x-c)}{r_1^{\,2}}+\frac{2(1+\sigma)\,(x-c)y^2}{r_1^{\,4}}\right], \tag{1.1a}$$

$$\overset{\smile}{xy}_1 = -\frac{T}{4\pi}\left[\frac{(3+\sigma)y}{r_1^{\,2}}-\frac{2(1+\sigma)y^3}{r_1^{\,4}}\right], \tag{1.1b}$$

$$\overset{\smile}{yy}_1 = \frac{T}{4\pi}\left[\frac{(1-\sigma)\,(x-c)}{r_1^{\,2}}-\frac{2(1+\sigma)y^2(x-c)}{r_1^{\,4}}\right], \tag{1.1c}$$

where $\sigma$ is Poisson's ratio and $r_1^{\,2} = (x-c)^2+y^2$. Hence from (A) we have

$$r.\overset{\smile}{rx}_1 = \frac{T}{4\pi}\left[\frac{(3+\sigma)\,(x-r^2/c)c}{r_1^{\,2}}+\frac{(1+\sigma)y^2}{r_1^{\,2}}+\frac{(1+\sigma)y^2(r^2-c^2)}{r_1^{\,4}}\right], \tag{1.2a}$$

$$r.\overset{\smile}{ry}_1 = \frac{T}{4\pi}\left[-\frac{(1-\sigma)cy}{r_1^{\,2}}-\frac{2(1+\sigma)xy}{r_1^{\,2}}+\frac{2(1+\sigma)cy^3}{r_1^{\,4}}\right]. \tag{1.2b}$$

At a point $P(x, y)$ on the circular edge we have

$$[r.\overset{\smile}{rx}_1]_{r=a} = \frac{T}{4\pi}\left[\frac{(3+\sigma)(x-a^2/c)c}{AP^2}+\frac{(1+\sigma)y^2}{AP^2}+\frac{(1+\sigma)y^2(a^2-c^2)}{AP^4}\right], \tag{1.3a}$$

$$[r.\overset{\smile}{ry}_1]_{r=a} = \frac{T}{4\pi}\left[-\frac{(1-\sigma)cy}{AP^2}-\frac{2(1+\sigma)xy}{AP^2}+\frac{2(1+\sigma)cy^3}{AP^4}\right]. \tag{1.3b}$$

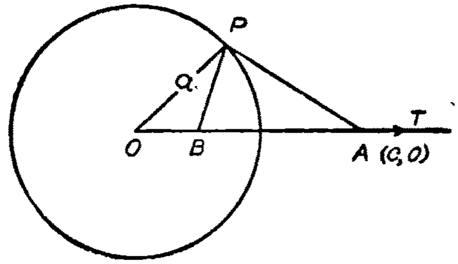


Fig.1

If $B$ be the inverse point of $A$ with respect to the circle we know that $c.(AP)^{-1} = a.(BP)^{-1}$. Hence expressing $AP$ in terms of $BP$ we find from (C) that expressions $r.\breve{rx}_2$ and $r.\breve{ry}_2$ to be determined must be such that

$$[r.\breve{rx}_2]_{r=a} = -\frac{T}{4\pi}\left[\frac{(3+\sigma)a^2(x-a^2/c)}{c.BP^2} + \frac{(1+\sigma)a^2y^2}{c^2.BP^2} - \frac{(1+\sigma)a^4(c^2-a^2)y^2}{c^4.BP^4}\right], \tag{1.4a}$$

$$[r.\breve{ry}_2]_{r=a} = -\frac{T}{4\pi}\left[-\frac{(1-\sigma)a^2y}{c.BP^2} - \frac{2(1+\sigma)a^2xy}{c^2.BP^2} + \frac{2(1+\sigma)a^4y^3}{c^3.BP^4}\right]. \tag{1.4b}$$

In terms of $z$ we have the above results as

$$[r.\breve{rx}_2]_{r=a} = -\frac{T}{4\pi}\,\mathrm{Re}\left[\frac{(3+\sigma)a^2}{z-a^2/c} + \frac{(1+\sigma)}{2c}.\frac{z^2-a^2}{z-a^2/c} - \frac{(1+\sigma)a^2(z^2-a^2)}{2c^2(z-a^2/c)^2} - \frac{(1+\sigma)}{2c}.\frac{(z^2-a^2)}{z}\right]_{r=a}, \tag{1.5a}$$

$$[r.\breve{ry}_2]_{r=a} = -\frac{T}{4\pi}\,\mathrm{Re}\left[-\frac{(1-\sigma)ia^2}{c(z-a^2/c)} - \frac{(1+\sigma)iz^2}{z-a^2/c} + \frac{(1+\sigma)iz(z^2-a^2)}{2c(z-a^2/c)^2} + \frac{(1+\sigma)}{2c}.\frac{i(z^2+a^2)}{z}\right]_{r=a}. \tag{1.5b}$$

The last term in each bracket on the right hand sides of the above expressions which gives zero values on $r=a$, has been added to secure the vanishing of stresses $\breve{xx}_2$, $\breve{xy}_2$, $\breve{yy}_2$ at an infinite distance. We find from (E) that we can put

$$aL(z) = -\frac{T}{4\pi}\left[\frac{2a^2}{c(z-a^2/c)} + \frac{(1+\sigma)}{c}.\frac{z^2}{z-a^2/c} - \frac{(1+\sigma)}{2c}.\frac{z(z^2-a^2)}{(z-a^2/c)^2} - \frac{(1+\sigma)}{2c}.\frac{(z^2-a^2)}{z}\right], \tag{1.6a}$$

$$aM(z) = -\frac{T}{4\pi}\left[-\frac{(1-\sigma)ia^2}{c(z-a^2/c)} - \frac{(1+\sigma)iz^2}{z-a^2/c} + \frac{(1+\sigma)iz(z^2-a^2)}{2c(z-a^2/c)^2} + \frac{(1+\sigma)}{2c}.\frac{i(z^2+a^2)}{z}\right], \tag{1.6b}$$

which give

$$f(z) = 2a\,\frac{L(z)+iM(z)}{z}$$

$$= -\frac{T}{2\pi}\left[(3-\sigma)\left\{\frac{1}{z-a^2/c} - \frac{1}{z}\right\} + \frac{2(1+\sigma)z}{c(z-a^2/c)} - \frac{(1+\sigma)}{c}.\frac{(z^2-a^2)}{(z-a^2/c)^2} - \frac{(1+\sigma)}{c}\right]. \tag{1.7}$$

Substituting the above values of $L(z)$, $M(z)$ and $f(z)$ in (D) we can obtain the values of $r.\breve{rx}_2$ and $r.\breve{ry}_2$ at any point.

Since $\widetilde{rr} = 0$ on the edge $r = a$, the hoop stress at the point $P(a, \theta)$ is

$$[\widetilde{\theta\theta}]_{r=a} = [\widetilde{rr}+\widetilde{\theta\theta}]_{r=a} = [\widetilde{xx}_1+\widetilde{yy}_1+\mathbf{Re}\, f(z)]_{r=a}$$

$$= -\frac{T}{2\pi}\left[\frac{(1+\sigma)(x-c)}{AP^2}+\frac{(3-\sigma)(x-a^2/c)}{BP^2}-\frac{(3-\sigma)x}{a^2}+\frac{(1+\sigma)}{c}\right.$$

$$\left.+\frac{2(1+\sigma)a^2(x-a^2/c)}{c^2.BP^2}-\frac{2(1+\sigma)a^2(c^2-a^2)y^2}{c^3.BP^4}\right]. \qquad (1.8)$$

In the particular case when $c = 2a$ and $\sigma = 0.25$, we have

$$[\widetilde{\theta\theta}]_{r=a} = \frac{T}{2\pi}.I_1(\theta), \qquad (1.9)$$

where $I_1(\theta)$ is given by the relation

$$4(5-4\sin\theta)^2 I_1(\theta) = 22\cos 3\theta - 76\cos 2\theta + 32\cos\theta + 15{\cdot}25. \qquad (1.10)$$

The graph of the function $I_1(\theta)$ is given in Fig. 2.

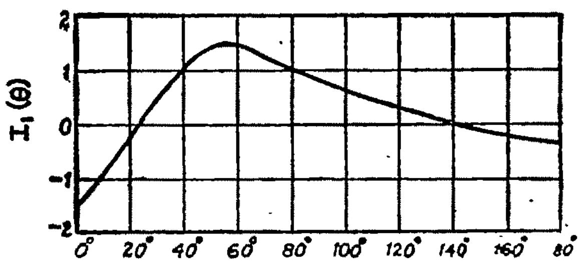


Fig. 2

When the force acts at the point $(a, 0)$, $c = a$ and hence

$$AP^2 = BP^2 = 2a^2(1-\cos\theta).$$

In this case

$$[\widetilde{\theta\theta}]_{r=a} = \frac{T}{2\pi a}[2+(3-\sigma)\cos\theta]. \qquad (1.11)$$

This result tallies with that deduced from Art. 9 of Bickley's paper.

## 2. *Solution for a couple*

Let a couple of moment $Q$ with its axis perpendicular to the plate act at the point $A(c, 0)$. Had there been no hole, the average stresses in the infinite plate due to this couple would have been given by (Coker and Filon, 1931, p. 362)

$$\widetilde{xx}_1 = \frac{Q}{\pi}.\frac{(x-c)y}{r_1^4}, \qquad (2.1a)$$

$$\widetilde{xy}_1 = \frac{Q}{2\pi}.\frac{y^2-(x-c)^2}{r_1^4}, \qquad (2.1b)$$

$$\breve{yy}_1 = -\frac{Q}{\pi}\cdot\frac{(x-c)y}{r_1^4}, \tag{2.1c}$$

where as before

$$r_1^2 = (x-c)^2+y^2.$$

Hence from (A) we have

$$\left.\begin{aligned} r.\breve{rx}_1 &= \frac{Qy(r^2-c^2)}{2\pi r_1^4}, \\ r.\breve{ry}_1 &= \frac{Q}{2\pi r_1^4}[2cr^2-x(r^2+c^2)]. \end{aligned}\right\} \tag{2.2}$$

$B$ being the inverse point of $A$ with respect to the circle, we have at any point $P(x, y)$ on the edge $r = a$

$$[r.\breve{rx}_1]_{r=a} = \frac{Qy(a^2-c^2)}{2\pi.AP^4} = \frac{Qy}{2\pi}\cdot\frac{a^4(a^2-c^2)}{c^4BP^4}, \tag{2.3a}$$

$$[r.\breve{ry}_1]_{r=a} = \frac{Q}{2\pi.AP^4}[2ca^2-x(a^2+c^2)] = \frac{Qa^4}{2\pi c^4BP^4}[2ca^2-x(a^2+c^2)]. \tag{2.3b}$$

From (C) we then have

$$[r.\breve{rx}_2]_{r=a} = \frac{Qy(c^2-a^2)a^4}{2\pi c^4BP^4} = \frac{Qa^2}{2\pi c^2}\left[\mathbf{Re}\left(\frac{iz}{(z-a^2/c)^2}\right)\right]_{r=a}, \tag{2.4a}$$

$$[r.\breve{ry}_2]_{r=a} = \frac{Qa^4}{2\pi c^4BP^4}[x(a^2+c^2)-2ca^2] = \frac{Qa^2}{2\pi c^2}\left[\mathbf{Re}\left(\frac{z}{(z-a^2/c)^2}\right)\right]_{r=a}. \tag{2.4b}$$

On reference to (E) we find that we can write $aL(z)$ and $aM(z)$ in the forms

$$\left.\begin{aligned} aL(z) &= \frac{Qa^2}{2\pi c^2}\cdot\frac{iz}{(z-a^2/c)^2}, \\ aM(z) &= \frac{Qa^2}{2\pi c^2}\cdot\frac{z}{(z-a^2/c)^2}, \end{aligned}\right\} \tag{2.5}$$

so that

$$f(z) = 2a\,\frac{L(z)+iM(z)}{z} = \frac{2Qa^2}{\pi c^2}\cdot\frac{i}{(z-a^2/c)^2}. \tag{2.6}$$

Substituting the above values of $L(z)$, $M(z)$ and $f(z)$ in (D) we get the values of $r.\breve{rx}_2$ and $r.\breve{ry}_2$. The hoop stress $\breve{\theta\theta}$ at any point $P(a, \theta)$ on the circular hole where $\breve{rr} = 0$, is given by

$$\begin{aligned} [\breve{\theta\theta}]_{r=a} &= [\breve{rr}+\breve{\theta\theta}]_{r=a} = [\breve{xx}_1+\breve{yy}_1+\mathbf{Re}\,f(z)]_{r=a} = [\mathbf{Re}\,f(z)]_{r=a} \\ &= \frac{2Qa^2}{\pi c^2.BP^4}\,2a\sin\theta(a\cos\theta-a^2/c) = \frac{4Q}{\pi}\cdot\frac{c\sin\theta(c\cos\theta-a)}{(c^2+a^2-2ac\cos\theta)^2} \end{aligned} \tag{2.7}$$

Taking the particular case in which $c = 2a$, we have

$$[\breve{\theta\theta}]_{r=a} = \frac{32Q}{\pi c^2}\cdot I_2(\theta), \tag{2.8}$$

where

$$I_2(\theta) = \frac{\sin\theta\,(2\cos\theta - 1)}{(5-4\cos\theta)^2}. \tag{2.9}$$

The graph of the function $I_2(\theta)$ is given in Fig. 3.

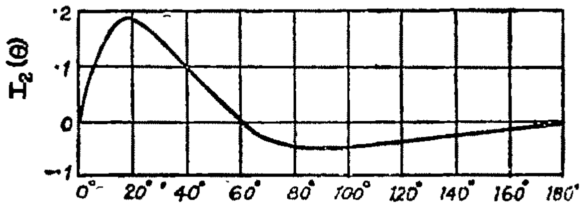

Fig. 3

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# MATRIX TREATMENT OF A RIGID BODY MOTION IN COMPLEX SPACE

By

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In two previous papers (Ghosh, 1940, 1943), some characteristic properties of a 'rigid' body moving in an Euclidean $n$-space were obtained by a matrix method. The object of the present paper is to extend those investigations into a complex domain of $n$ dimensions. Representing a complex vector by means of a Hermitian matrix of a certain type, the development proceeds in close analogy to that in real space. Thus a complex vector has not only a real magnitude but also the cosine of the angle between a pair of such vectors is real. Further, the kinetic energy, the magnitude of angular velocity, the moment of a couple and the work done by a force or a couple are all defined by real quantities. Moreover, under this general scheme, a note-worthy feature is that the symmetric and the skew-symmetric matrices, representing different physical entities in real space, are both merged into the Hermitian matrices, which, as a matter of fact, play a fundamental role in the present analysis.

Let the rigid body be initially at rest in a certain vector space at time $t=0$. Radiating from a common origin let vectors $H$, $K$, ... specify different particles in the body. Denoting the co-ordinates of a typical vector $H$ by $(h_1, h_2, \ldots, h_n)$ let us represent it by means of the Hermitian matrix

$$H = \sum_{p=2}^{n+1} h_{p-1}e_{p1} + \sum_{p=2}^{n+1} \bar{h}_{p-1}e_{1p}, \tag{1}$$

where the $e$'s denote matrix units and the bar over an element denotes its conjugate complex. Associated with (1) we have the Hermitian matrix

$$iH = \sum_{p=2}^{n+1} ih_{p-1}e_{p1} - \sum_{p=2}^{n+1} i\bar{h}_{p-1}e_{1p}, \tag{2}$$

representing the vector with co-ordinates $(ih_1, ih_2, \ldots, ih_n)$. We shall call it the vector *adjoint* to $H$. It is to be noticed that $H$ *multiplied* by $i$ does not represent a vector. If $h$'s be split up into real and pure imaginary parts according to the relations $h_{p-1} = a_{p-1} + ib_{p-1}$, the matrix (1) takes the form

$$H = \sum_{p=2}^{n+1} a_{p-1}(e_{p1} + e_{1p}) + \sum_{p=2}^{n+1} b_{p-1}i(e_{p1} - e_{1p}), \tag{3}$$

which involves $2n$ elementary Hermitian matrices

$$\epsilon_p = e_{p1} + e_{1p}, \quad i\epsilon_p = i(e_{p1} - e_{1p}), \quad (p = 2, 3, \ldots, n+1), \tag{4}$$

serving as *co-ordinate vectors* of the complex space.

Again, introducing the diagonal matrix

$$U_1 = \sum_{p=2}^{n+} e_{pp}, \tag{5}$$

the column and the row parts of $H$ in (1) may be expressed as $U_1H$ and $HU_1$ respectively. Referring to (2) we have then

$$U_1(iH) = i.U_1H, \quad (iH)U_1 = -i.HU_1. \tag{6}$$

If $\alpha$ be a complex number then $\alpha H$ is a vector in the plane of $H$ and $iH$ and we have the general relations

$$U_1(\alpha H) = \alpha.U_1H, \quad (\alpha H)U_1 = \bar{\alpha}.HU_1. \tag{7}$$

The square of the length of the vector $H$ is then given by the matrix $HU_1H$ of type $e_{11}$. In fact, all vectors $\alpha H$ have the same length, provided $\alpha\bar{\alpha} = 1$. As for the angle between two vectors $H$ and $K$ we shall take the real quantity

$$\frac{1}{2}\frac{HU_1K + KU_1H}{\sqrt{(HU_1H)}\sqrt{(KU_1K)}} \tag{8}$$

to define its cosine, denoted by $\cos(H, K)$. In accordance with this definition the cosine of the angle between a pair of adjoint vectors $H$ and $iH$ vanishes, so that they are mutually orthogonal. Thus the $2n$ coordinate vectors (4) are all orthogonal to one another and they are of the same length equal to unity. When (8) vanishes by virtue of $HU_1K = 0$, the orthogonality between $H$ and $K$ also involves that between $H$ and $iK$ and the orthogonality is said to be *complete*.

Forming the product of a pair of matrices $H$, $K$ we get

$$HK = \begin{Bmatrix} H \\ K \end{Bmatrix} e_{11} + U_1HKU_1, \tag{9}$$

where

$$\begin{Bmatrix} H \\ K \end{Bmatrix} \text{ represents } \sum_{p=2}^{n+1} \bar{h}_{p-1}k_{p-1}. \tag{10}$$

The product of three matrices $H$, $K$, $L$ may be expressed as

$$HKL = \begin{Bmatrix} H \\ K \end{Bmatrix} LU_1 + \begin{Bmatrix} K \\ L \end{Bmatrix} U_1H. \tag{11}$$

Generally the product of an even number of matrices is of type (9) and for an odd number the product is of type (11). It follows from (11) that $H$ satisfies the characteristic equation

$$H^3 = \begin{Bmatrix} H \\ H \end{Bmatrix} H. \tag{12}$$

Let us consider now the elementary Hermitian matrices $iU_1(KH - HK)U_1$ and $U_1(KH + HK)U_1$, belonging to the general type

$$\Omega = \sum_{p,q=2}^{n+1} \omega_{p-1,q-1}\, e_{pq}, \quad \omega_{pq} = \bar{\omega}_{qp}. \tag{13}$$

Putting the first in the form $U_1(iK)HU_1 + U_1H(iK)U_1$, we represent it by the bracket notation $[H, K]$, then the second is represented by $-[H, iK]$.

We note the following properties of these symbols:

$$[H, K] = [iH, iK] = -[K, H] = -[iK, iH], \tag{14}$$

$$[H, iK] = -[iH, K] = [K, iH] = -[iK, H], \tag{15}$$

$$[(m_1+in_1)H+(a_1+ib_1)K,\ (m_2+in_2)H+(a_2+ib_2)K] = (m_1n_2-n_1m_2)[H, iH]+(a_1b_2-b_1a_2)$$

$$\times [K, iK]+(a_2m_1-a_1m_2+b_2n_1-b_1n_2)[H, K]+(b_2m_1-b_1m_2-a_2n_1+a_1n_2)[H, iK], \tag{16}$$

$$[H, K]^2 = -\frac{1}{2}\begin{Bmatrix}H\\H\end{Bmatrix}[K, iK]-\frac{1}{2}\begin{Bmatrix}K\\K\end{Bmatrix}[H, iH]+\left[\begin{Bmatrix}K\\H\end{Bmatrix}H, iK\right], \tag{17}$$

where the small letters denote real numbers.

The Hermitian matrix $-\frac{1}{2}[H, iH]$, that is, $U_1H^2U_1$ will be called a *simple* Hermitian matrix of type (13). It has been proved elsewhere (Ghosh, 1944), that the general Hermitian matrix (13) is always expressible as a sum of a number of such simple Hermitian matrices. To express $[H, iK]$, for instance, in the desired form, we may use the identity

$$2lmU_1(HK+KH)U_1 = U_1(lH+mK)^2U_1-U_1(lH-mK)^2U_1, \tag{18}$$

where $l$, $m$ are arbitrary real numbers.

2. We are now in a position to define a rigid body motion. Let the coordinate axes (4) be fixed in space and for simplicity, suppose the motion takes place about a fixed point at the origin. The mutual distances of points in the body and the angles between any pair of lines will remain unaltered if at time $t$ any representative vector $H$ is transformed into $X$ in accordance with an equation

$$X = \Lambda H+H\Lambda^\dagger, \tag{19}$$

where $\Lambda$ is a unitary matrix of the type

$$\Lambda = \sum_{p,q=2}^{n+1} \lambda_{p-1,q-1}\, e_{pq}, \tag{20}$$

depending on the time such that $\Lambda = U_1$ when $t=0$, and $\Lambda^\dagger$ is the transposed of $\Lambda$ with complex conjugate elements, so that $\Lambda\Lambda^\dagger = U_1$. Differentiating (19) with respect to the time, the velocity at time $t$ is given by

$$\dot{X} = \dot{\Lambda}H+H\dot{\Lambda}^\dagger,$$

or

$$\dot{X} = \dot{\Lambda}\Lambda^\dagger X+X\Lambda\dot{\Lambda}^\dagger. \tag{21}$$

Now the relation $\Lambda\Lambda^\dagger = U_1$ gives

$$\dot{\Lambda}\Lambda^\dagger+\Lambda\dot{\Lambda}^\dagger = 0,$$

and consequently

$$-i\dot{\Lambda}\Lambda^\dagger = i\Lambda\dot{\Lambda}^\dagger$$

is a Hermitian matrix of type (13) which we denote by $\Phi$. Rewriting (21) in the form

$$\dot{X} = i\Phi.X-X.i\Phi \tag{22}$$

the acceleration at time $t$ is expressed as

$$\ddot{X} = (i\dot{\Phi}-\Phi^2)X-X(i\dot{\Phi}-\Phi^2). \tag{23}$$

From (22) it follows that

$$\dot{X}U_1X+XU_1\dot{X} = 0,$$

that is, $X$ and $\dot{X}$ are orthogonal, hence (22) represents an instantaneous 'infinitesimal unitary rotation' (Weyl, 1928) about the origin. The square of the magnitude of the *angular velocity* involved in it will be defined by the real quantity

$$\frac{1}{2}\sum_{p,q=1}^{n+1}\phi_{pq}\bar{\phi}_{pq}, \tag{24}$$

which is half the *trace* of the Hermitian $\Phi^2$.

If $X$ be an eigen-vector of $\Phi$, (22) reduces to the form

$$\dot{X} = a.iX, \tag{25}$$

where $a$ is a real number. Corresponding to its adjoint we have similarly

$$i\dot{X} = -aX. \tag{26}$$

Thus every vector in the plane of $X$ and $iX$, where $X$ is an eigen-vector of $\Phi$, undergoes rotation in the *same plane* along the direction of its adjoint vector.

When $\Phi$ is a simple Hermitian of the type $U_1L^2U_1$, the corresponding rotation will be called *simple*. The equation (22) then reduces to

$$\dot{X} = i\begin{Bmatrix}L\\X\end{Bmatrix}U_1L-i\begin{Bmatrix}X\\L\end{Bmatrix}LU_1 = i\begin{Bmatrix}L\\X\end{Bmatrix}L. \tag{27}$$

Hence the instantaneous simple rotation takes place in the plane of $L$ and $iL$, the particle at any point $X$ moving along the vector $i\begin{Bmatrix}L\\X\end{Bmatrix}L$, which is orthogonal to $X$.

If the eigen-vectors corresponding to the eigen-values of $\Phi$ be all known then the general rotation (22) may be decomposed into a certain number of simple rotations, occurring in a set of completely orthogonal planes of the type, defined in (27). For a decomposition in an oblique set of planes, on the other hand, we may use the formula (Ghosh, 1944)

$$\Phi = \sum_{r=1}^{s} p_rU_1L_r^2U_1, \tag{28}$$

where $p$'s are real numbers and $s$ denotes the rank of $\Phi$. It may be noted that the set of vectors $L_r$ may be obtained in an infinite variety of ways, the vectors, however, all belonging to a linear subspace uniquely associated with $\Phi$.

3. Let $dm$ be the mass of the particle at $H$, the rigid body being at rest, then $\Sigma dmH = mH'$, where $m$ is the total mass of the body and the vector $H'$ corresponds to the centre of mass. At time $t$, let $H'$ be transformed into $X'$ in accordance with (19), we have then

$$X-X' = \Lambda(H-H')+(\bar{H}-H')\Lambda^{\dagger}, \tag{29}$$

whence $\Sigma dmX = mX'$, also $\Sigma dm\dot{X} = m\dot{X}'$, $\Sigma dm\ddot{X} = m\ddot{X}'$.

The *kinetic energy* at time $t$ will be defined by

$$\tfrac{1}{2}\Sigma dm\dot{X}U_1\dot{X} = \tfrac{1}{2}m\dot{X}'U_1\dot{X}' + \tfrac{1}{2}\Sigma dm(\dot{X}-\dot{X}')U_1(\dot{X}-\dot{X}'),$$

which by means of (22) may be written as

$$\tfrac{1}{2}\Sigma dm\dot{X}U_1\dot{X} = \tfrac{1}{2}m\dot{X}'U_1\dot{X}' + \tfrac{1}{2}\Sigma dm(X-X')\Phi^2(X-X'). \tag{30}$$

The Hermitian matrix $\Sigma dmU_1H^2U_1$ of type (13) will be called the *matrix of inertia*. Its eigen-vectors will determine the principal axes of inertia. Analogous to real space, we have the following relations:

$$\Sigma dmU_1H^2U_1 = mU_1H'^2U_1 + \Sigma dmU_1(H-H')^2U_1, \tag{31}$$

$$\Sigma dmU_1(X-X')^2U_1 = \Lambda\{\Sigma dmU_1(H-H')^2U_1\}\Lambda^\dagger. \tag{32}$$

To define the *moment of momentum* at time $t$ about the fixed origin, we take the Hermitian matrix

$$\Sigma dmU_1(iX.\dot{X}+\dot{X}.iX)U_1 = i\Sigma dmU_1(X\dot{X}-\dot{X}X)U_1.$$

Introducing the vector $X'$ and making use of (22), this is expressible in the form

$$i.mU_1(X'\dot{X}'-\dot{X}'X')U_1 + \Phi\Sigma dmU_1(X-X')^2U_1 + \{\Sigma dmU_1(X-X')^2U_1\}\Phi, \tag{33}$$

involving the matrix of inertia. It may be noted that in the kinetic energy expression (30) the elements of the matrix of inertia are also involved.

4. We pass on next to the representation of a force acting at a point in a rigid body. Analogous to the representation in a real space, we shall take the Hermitian matrix

$$P + iU_1(HP-PH)U_1 \tag{34}$$

to represent a force $P$ acting at the point $H$. Using bracket symbol we may express the above in the form

$$P + [P, H]. \tag{35}$$

A force $P$ at the point $iH$ is then represented by

$$P + [P, iH]. \tag{36}$$

Since $[P, P] = 0$, the point of application of the force $P$ in the above may by transferred anywhere along its *line of action*. Combining a force $-P$ at the origin with (35), we notice that $[P, H]$ represents a couple in the plane formed by $P$ and $H$. If $x$ be determined by means of the equation

$$PU_1(H-xP)+(H-xP)U_1P = 0,$$

then $[P, H]$ reduces to a form $[P, H_0]$ in which $P$ and $H_0$ are orthogonal and $H_0$ may be called the *arm* of the couple. Referring to (16), we have the identity

$$[m_1P+a_1H,\ m_2P+a_2H] = (a_2m_1-a_1m_2)[P, H], \tag{37}$$

so that couples lying in the same plane are multiples of one another.

Further, we have

$$[(m_1+in_1)P,\ (m_1+in_1)H] = (m_1^2+n_1^2)[P, H], \tag{38}$$

which shows the equivalence of a set of couples.

A system of couples acting on a rigid body can be combined to form a general Hermitian matrix of type (13). The square of the moment of the generalized couple $\Omega$ will be defined by the real quantity

$$\frac{1}{2}\sum_{p,q=2}^{n} \omega_{pq}\,\bar{\omega}_{pq}, \tag{39}$$

which is half the *trace* of the Hermitian $\Omega^2$. Corresponding to the couple $[P, H]$ the expression (39) reduces to

$$\begin{Bmatrix} P \\ P \end{Bmatrix}\begin{Bmatrix} H \\ H \end{Bmatrix} - \frac{1}{2}\begin{Bmatrix} P \\ H \end{Bmatrix}^2 - \frac{1}{2}\begin{Bmatrix} H \\ P \end{Bmatrix}^2. \tag{40}$$

A couple $[P, iP]$ will be called *simple*. It represents a force $P$ acting at $iP$ and a force $-P$ acting at the origin. From (38) it follows that $[\alpha P, i\alpha P]$ is equivalent to $[P, iP]$ provided $\alpha\bar{\alpha} = 1$. The square of the moment of the couple is given by the real positive quantity

$$2\begin{Bmatrix} P \\ P \end{Bmatrix}^2. \tag{41}$$

The three sets of couples $[\epsilon_p, i\epsilon_p]$, $[\epsilon_p, i\epsilon_q]$ and $[\epsilon_p, \epsilon_q]$ are called the *coordinate couples*. They represent the elementary Hermitians $-2e_{pp}$, $-(e_{qp}+e_{pq})$, $i(e_{qp}-e_{pq})$ respectively. In terms of these coordinate couples a generalized couple $\Omega$ is of the form

$$\Omega = -\frac{1}{2}\sum_{p=2}^{n+1}\omega_{p-1,\,p-1}[\epsilon_p, i\epsilon_p] - \sum_{p,q=2}^{n+1} a_{p-1,\,q-1}[\epsilon_p, i\epsilon_q] - \sum_{p,q=2}^{n+1} b_{p-1,\,q-1}[\epsilon_p, \epsilon_q], \quad p < q, \tag{42}$$

where $\omega_{pq}$ has been replaced by $a_{pq}+ib_{pq}$.

As remarked in (28) a generalized couple $\Omega$ of rank $s$ can always be expressed in the form

$$\Omega = \sum_{r=1}^{s} l_r[M_r, iM_r], \tag{43}$$

where $l$'s are real numbers and the set of $s$ vectors $M_r$ are, in general, oblique to one another. We shall now obtain an expansion theorem for an arbitrary vector in terms of the $M$'s. Suppose we have determined a set of reciprocal vectors $M_r^*$ (Ghosh, 1944), such that

$$M_p^* U_1 M_q = M_p U_1 M_q^* = \delta_{pq}\, e_{11},$$

where $\delta_{pq}$ denotes the *Kronecker delta*. Let us assume that an arbitrary vector $N$ is expressible in the form

$$N = \sum_{r=1}^{s} (a_r + ib_r)M_r, \tag{44}$$

then we have

$$NU_1M_p^* + M_p^*U_1N = 2a_p\, e_{11},$$

$$i(NU_1M_p^* - M_p^*U_1N) = 2b_p\, e_{11},$$

whence

$$a_p + ib_p = \begin{Bmatrix} M_p^* \\ N \end{Bmatrix}.$$

Thus the required expansion of $N$ is given by

$$N = \sum_{r=1}^{s} \begin{Bmatrix} M_r^* \\ N \end{Bmatrix} M_r. \tag{45}$$

It may be noted that interchanging $M_r$ and $M_r^*$ in the above we get an alternative expansion.

A system of forces acting at different points of a rigid body is represented by the Hermitian matrix $P+\Omega$, where $P$ denotes the resultant force and $\Omega$ the resultant couple. By means of (43) and (45) we can express it in the form

$$\sum_{r=1}^{s} \alpha_r M_r + \sum_{r=1}^{s} l_r [M_r, iM_r],$$

where $\alpha_r$ is a complex number.

Transforming the above, by means of (38), into the form

$$\sum_{r=1}^{s} \alpha_r M_r + \sum_{r=1}^{s} \frac{l_r}{\alpha_r \bar{\alpha}_r} [\alpha_r M_r, i\alpha_r M_r], \tag{46}$$

it follows that the system of forces is reducible only to a set of $s$ forces acting at definite points.

5. Let us now introduce the generalized notion of work in complex space. Suppose the point of application $X$ of a force $P$ be displaced to $X+dX$, then as a measure of work we shall take the real matrix

$$\tfrac{1}{2}(P.U_1.dX + dX.U_1.P) \tag{47}$$

of type $e_{11}$. Referring to (8) this is equivalent to

$$\surd(P.U_1.P)\surd(dX.U_1\,dX)\cos(P,\, dX). \tag{48}$$

If $dX$ be along the vector $iP$, the work evidently vanishes. Writing the vectors $P$ and $dX$ in the form (3)

$$P = \sum_{j=2}^{n+1} (p'_{j-1}e_j + p''_{j-1}ie_j),$$

$$dX = \sum_{j=2}^{n+1} (dx'_{j-1}e_j + dx''_{j-1}ie_j),$$

the expression (47) becomes

$$e_{11} \sum_{j=2}^{n+1} (p'_{j-1}dx'_{j-1} + p''_{j-1}dx''_{j-1}). \tag{49}$$

If $dX$ be replaced by $idX$, the expression for work is of the form

$$e_{11} \sum_{j=2}^{n+1} (-p'_{j-1}dx''_{j-1} + p''_{j-1}dx'_{j-1}). \tag{50}$$

We now define the force $P$ as a function of $X$ by means of the $2n$ equations

$$\begin{aligned} p'_{j-1} &= p'_{j-1}(x'_1x'_2\ldots x'_nx''_1\ldots x''_n), \\ p''_{j-1} &= p''_{j-1}(x'_1x'_2\ldots x'_nx''_1\ldots x''_n), \end{aligned} \tag{51}$$

then (49) or (50) may be integrated between given limits to calculate the work done for some finite displacement in a field of force.

Proceeding to the case of a rigid body, let the force $P$ acting at the point $X$ undergo infinitesimal displacement, consisting of a translation and a rotation, defined by

$$\delta X = \delta A + i\delta\Phi.X - X.i\delta\Phi. \tag{52}$$

Then the work done by the force is given by

$$\tfrac{1}{2}(PU_1\delta A + \delta AU_1P + P.i\delta\Phi.X - X.i\delta\Phi.P). \tag{53}$$

Combining the above with the work done by a force $-P$ acting at $Z$, we get the work done by the couple $[P, X-Z]$, represented in the form

$$\tfrac{1}{2}i\{P\delta\Phi(X-Z) - (X-Z)\delta\Phi P\}. \tag{54}$$

By evaluating the matrix products involved in the above, it follows that the work done by any couple $\Gamma$ may be expressed in the symmetrical form

$$e_{11}\frac{1}{2}\sum_{j,k=1}^{n} \delta\phi_{jk}\gamma_{kj}, \tag{55}$$

where $\delta\phi_{jk}$ and $\gamma_{kj}$ are a pair of mutually conjugate elements of $\delta\Phi$ and $\Gamma$ respectively. It may be noted that the work done by the simple couple $[P, iP]$ is given by $-P\delta\Phi P$, which is expressible in the form (55). Hence the work done by the generalized couple $\Omega$ is given by the matrix

$$e_{11}\frac{1}{2}\sum_{j,k=1}^{n} \delta\phi_{jk}\,\omega_{kj}. \tag{56}$$

6. Analogy with the real space is further maintained if we adopt the usual dynamical principles and form the general equations of motion of a rigid body in complex space. Let the force on the element of mass $dm$ at the point $X$ at time $t$ be represented by

$$Pdm + [Pdm, X],$$

then, by applying D'Alembert's principle, the equations of motion are

$$\Sigma Pdm + \Sigma[Pdm, X] = \Sigma dm\ddot{X} + \Sigma[dm\ddot{X}, X].$$

Introducing the matrix $X'$ which corresponds to the centre of mass of the rigid body at time $t$, the above is expressible in the form

$$\Sigma Pdm + \Sigma[Pdm, X-X'] = m\ddot{X}' + \frac{d}{dt}\Sigma[dm(\dot{X}-\dot{X}'), X-X']. \tag{57}$$

For a suitable choice of axes further simplification will be possible. The transformation formulae for a set of moving axes follow in a similar manner as in real space, where the orthogonal matrix is to be replaced by a unitary matrix.

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# ON AN INTEGRAL EQUATION ASSOCIATED WITH THE EQUATION FOR HYDROGEN ATOM

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## I

The Schrödinger functions '$\phi$' characterising the stationary states of hydrogen atom are now very familiar things in analysis as also the differential equation which they satisfy, namely,

$$\nabla^2\phi + \frac{8\pi^2 m}{h^2}\left(E + \frac{e^2}{r}\right)\phi = 0. \tag{1,1}$$

The associated functions $M$, defined by the relation,

$$\phi = \int M \exp.\, 2\pi\iota(lx + my + nz)\, dl\, dm\, dn, \tag{1,2}$$

can be utilised for defining the probability in momentum space if the momentum variables are introduced by the relations $p_x = hl$, $p_y = hm$, $p_z = hn$.

When inversion is possible, (1,2) implies also

$$M(l, m, n) = \int \phi \exp. -2\pi\iota(lx + my + nz)\, dx\, dy\, dz \tag{1,3}$$

and solutions of (1,1) can be utilised to calculate $M$'s. Elsasser (1933) has followed this method and arrived at fairly complicated formula. Another alternative would be to set up an appropriate equation for them, and investigate its solutions. This is an integral equation, whose complete solution is presented here. The analysis presents several interesting features, and leads to expressions of $M$'s, which can be immediately utilised to study their properties or to apply them to physical problems.

## II

If we use the semi-convergent integral

$$\frac{1}{R} = \frac{1}{\pi}\int \frac{dl\, dm\, dn}{l^2 + m^2 + n^2} \exp.\, 2\pi\iota(lx + my + nz), \quad R^2 = x^2 + y^2 + z^2, \tag{2,1}$$

then combination with (1,2) leads to the following sex-tuple integral for $\phi/R$, after a change of variable and order of integration:

$$\frac{\phi}{R} = \frac{1}{\pi}\int \frac{M(\lambda - l,\ \mu - m,\ \nu - n)}{l^2 + m^2 + n^2} \exp.\, 2\pi\iota(\lambda x + \mu y + \nu z)\, dl\, dm\, dn\ \ d\lambda\, d\mu\, d\nu.$$

$M$, and

$$\frac{1}{\pi}\int \frac{M(\lambda - l,\ \mu - m,\ \nu - n)\, dl\, dm\, dn}{l^2 + m^2 + n^2}$$

can thus be regarded as Fourier transforms of $\phi$ and $\phi/R$.

Using then in the Schrödinger equation (1,1) we deduce the following integral equation:

$$(l^2+m^2+n^2-k^2)M = \lambda\int \frac{M(l-x,\ m-y,\ n-z)}{x^2+y^2+z^2}dx\,dy\,dz, \tag{A}$$

where

$$k^2 = 2mE/h^2, \text{ and } \lambda = 2me^2/\pi h^2.$$

$M$ is here assumed to be finite and single-valued throughout the domain of integration, as also

$$\int |M|^2\,dx\,dy\,dz = 1.$$

(A) is the characteristic integral equation for $M$-functions of hydrogen; for $k^2 < 0$, it leads to the discrete spectrum, while $k^2 > 0$: yields the continuous spectrum.

It is easy to transform (A) to the Fredholm type, when $k^2 < 0 = -a^2$. We first transform the origin, *i,e.*, put $l-x = x'$, etc, in the right side. (A) becomes

$$(l^2+m^2+n^2-k^2)M = -\int \frac{\lambda M(x'y'z')dx'dy'dz'}{[(l-x')^2+(m-y')^2+(n-z')^2]}; \tag{2,32}$$

when $k^2 = -a^2$, we make a similarity transformation

$$l = l'a, \quad x' = ax'', \quad \text{etc.},$$

and put

$$M(l, m, n) = M'(l', m', n')$$

whereby

$$dx'\,dy'\,dz' = a^3\,dx''\,dy''\,dz''$$

and

$$(l^2+m^2+n^2+a^2) = a^2(l'^2+m'^2+n'^2+1). \text{ etc.}$$

The relation becomes

$$(l^2+m^2+n^2+1)M = -\frac{\lambda}{a}\int \frac{M'(x'y'z')dx'dy'dz'}{(l-x')^2+(m-y')^2+(n-z')^2}.$$

If further

$$\sqrt{(1+l^2+m^2+n^2)}M = \phi,$$

then

$$\phi(A) = \kappa\int \frac{\phi(P)dv_P}{[(l-x)^2+(m-y)^2+(n-z)^2]\sqrt{(1+l^2+m^2+n^2)}\sqrt{(1+x^2+y^2+z^2)}}, \tag{2,33}$$

where

$$\kappa = \frac{e^2}{\pi h}\left(\frac{2m}{-E}\right)^{\frac{1}{2}}.$$

(2,33) is thus seen of the standard form

$$\phi(A) = -\lambda\int \phi(P)K(A, P)dv_P,$$

where the kernel is symmetrical in $A$ and $P$. Such a transformation, will however make the kernel imaginary within the domain, if $k^2 > 0$; to have an uniform treatment to cover both the cases, we will not utilise the transformation mentioned above.

III

We require the following simple result in our subsequent calculations. If $r_1$ and $r_2$ are the distances of a point $P(x, y, z)$ from two fixed points $A(a, b, c)$ and $F(f, g, h)$, the integral

$$I = \int \frac{dx\, dy\, dz}{r_1^2 r_2^2}$$

extended over the whole domain easily transforms to

$$\frac{2\pi}{c}\int_1^\infty \int_{-1}^1 \frac{dX dY}{X^2 - Y^2}$$

in bipolar co-ordinates with $AF = 2c$,

$$dx\, dy\, dz = c^3(\cosh^2\xi - \cos^2\eta)\sinh\xi \sin\eta\, d\xi\, d\eta\, d\phi,$$

and

$$X = \cosh\xi \quad \text{and} \quad Y = \cos\eta,$$

whence by easy integration,

$$I = \frac{\pi^3}{2c} = \frac{\pi^3}{AF}. \tag{3,1}$$

Again, if

$$A(a, b, c) = \lambda\int \frac{M(a-x,\, b-y,\, c-z)dx\, dy\, dz}{x^2+y^2+z^2} = -\lambda\int \frac{M(x'y'z')\, dx'\, dy'\, dz'}{(a-x')^2 + (b-y')^2 + (c-z')^2} \tag{3,2}$$

we have, by multiplying the equation with

$$\frac{da\, db\, dc}{(f-a)^2 + (g-b)^2 + (h-c)^2}$$

and integrating over the whole domain,

$$\int \frac{A(a\, b\, c)\, da\, db\, dc}{[(f-a)^2 + (g-b)^2 + (h-c)^2]} = -\lambda\int\int \frac{M(x'y'z')\, dx'\, dy'\, dz'\, da\, db\, dc}{[(a-x')^2 + \ldots]\ [(f-a)^2 + \ldots]}.$$

Changing the order of integration and utilising (3,1) we have

$$\int \frac{M(x'y'z')dx'dy'dz'}{[(f-x')^2 + (g-y')^2 + (h-z')^2]^{1/2}} = -\frac{1}{\pi^3\lambda}\int \frac{A(abc)\, da\, db\, dc}{[(f-a)^2 + (g-b)^2 + (h-c)^2]}$$

$$= \frac{1}{\pi^3\lambda}\int \frac{A(f-a', \ldots)da'db'dc'}{a'^2 + b'^2 + c'^2};$$

hence, operating with $\nabla^2_{(fgh)}$ on both sides, as

$$\nabla^2_{(fgh)}\int \frac{M(x'y'z')dx'dy'dz'}{[(f-x')^2 + \ldots]^{1/2}} = -4\pi M(f, g, h)$$

from potential theory, and

$$\nabla^2_{(fgh)}\{A(f-a,\ g-b,\ h-c)\} = (\nabla^2 A)(f-a,\ g-b,\ h-c),$$

we have finally

$$M(f, g, h) = -\frac{1}{4\pi^4\lambda}\int \frac{(\nabla^2 A)(f-a, g-b, h-c)\, da\, db\, dc}{a^2 + b^2 + c^2}$$

$$= \frac{1}{4\pi^4\lambda}\int \frac{(\nabla^2 A')\, dx'dy'dz'}{[(f-x')^2 + (g-y')^2 + (h-z')^2]} \tag{3,3}$$

as a solution of (3, 2).

This important result furnishing a solution of integral equation (3,2) with obvious restrictions about the nature of the function $A$, enables us to tackle our present problem.

IV

According to a well-known theorem due to Hobson, if an operator $S\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ is constructed by substituting $\frac{d}{dx}$, ..., etc. for $x, y, z$ in a solid harmonic $S_n$ of degree $n$ (a positive integer),

$$S_n\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)F(r^2) = 2^n S_n(x, y, z)\left(\frac{d}{dr^2}\right)^n F. \tag{4,1}$$

Let us assume with regard to the integral equation (A), that

$$M(x, y, z) = S_n(x, y, z)f(r)$$

or

$$M(x, y, z) = S_n\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\mathsf{A}(r^2) \tag{4,2}$$

in view of (4,1); also $(r^2-k^2)M(x, y, z)$ can be written as

$$S_n\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\mathsf{B}(r^2) \tag{4,3}$$

with the same operator, as the same surface-harmonic will occur in both. Also as

$$S_n\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\mathsf{A} = 2^n S_n(x, y, z)\left(\frac{d}{dr^2}\right)^n\mathsf{A} = M,$$

and

$$S_n\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\mathsf{B} = 2^n S_n(x, y, z)\left(\frac{d}{dr^2}\right)^n\mathsf{B} = (r^2-k^2)M,$$

we have

$$\left(\frac{d}{dr^2}\right)^n\mathsf{B} = (r^2-k^2)\left(\frac{d}{dr^2}\right)^n\mathsf{A} = \left(\frac{d}{dr^2}\right)^{n-1}[\mathsf{A}(r^2-k^2)-n\mathsf{A}]$$

implying

$$\frac{d\mathsf{B}}{dr} = \frac{d}{dr}[\mathsf{A}(r^2-k^2)]-2n\mathsf{A}r. \tag{4,4}$$

We have also from (A)

$$S_n\left[\frac{d}{da}, \ldots\right]\mathsf{B} = \lambda\int\frac{S_n\left[\frac{d}{d(a-x)}, \ldots\right]\mathsf{A}\,dxdydz}{x^2+y^2+z^2}$$

$$= \lambda S_n\left(\frac{d}{da}, \ldots\right)\int\frac{\mathsf{A}[(a-x)^2+\ldots]\,dx\,dy\,dz}{x^2+y^2+z^2}.$$

Hence removing the operator $S_n$

$$\mathsf{B} = \lambda\int\frac{\mathsf{A}[(a-x)^2+\ldots]\,dx\,dy\,dz}{x^2+y^2+z^2}, \tag{4,51}$$

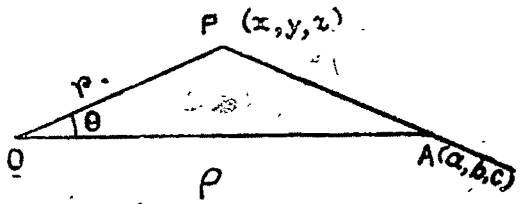
and hence on account of (3,3)

$$\mathsf{A} = -\frac{1}{4\pi^4\lambda}\int \frac{(\nabla^2\mathsf{B})(a-x, b-y, c-z)dx\,dy\,dz}{x^2+y^2+z^2}. \tag{4,52}$$

We now perform the integration; assuming

$$(a-x)^2+(b-y)^2+(c-z)^2 = f^2 = r^2+\rho^2-2r\rho\cos\theta$$

according to the accompanying figure




we have, as $fdf = r\rho\sin\theta$,

$$\rho\mathsf{B}(\rho^2) = 2\pi\lambda\int_0^\infty \frac{dr}{r}\int_{f^2=(\rho-r)^2}^{f^2=(\rho+r)^2} \tfrac{1}{2}\mathsf{A}(f^2)\,df^2$$

$$= 2\pi\lambda\int_0^\infty \frac{dx}{x}\,[G_1[\rho+x]-G_1(\rho-x)],$$

where

$$\frac{1}{x}\frac{dG}{dx} = \mathsf{A}(x^2)$$

and $G_1$, an even function of $x$. Similarly from (4,52)

$$\rho\mathsf{A} = -\frac{1}{2\pi^3\lambda}\int_0^\infty \frac{dx}{x}\,[L(\rho+x)-L(\rho-x)],$$

where

$$\frac{dL}{xdx} = \nabla^2\mathsf{B}(x^2);$$

hence integrating and putting

$$\frac{dQ}{dx} = L, \quad \text{or} \quad \frac{1}{x}\frac{d^2Q}{dx^2} = \nabla^2\mathsf{B}$$

we have

$$G_1(\rho) = -\frac{1}{2\pi^3\lambda}\int\frac{dx}{x}\,[Q(\rho+x)-Q(\rho-x)],$$

where

$$\frac{1}{x}\frac{d^2Q}{dx^2} = \nabla^2\mathsf{B} = \frac{1}{x}\frac{d^2}{dx^2}(x\mathsf{B}),$$

$$\text{or} \quad Q = x\mathsf{B},$$

and therefore an odd function of $x$.

Putting finally

$$\frac{x\mathsf{B}(x^2)}{2\pi^2\lambda} = G_2 = \frac{Q}{2\pi^2\lambda},$$

we have the following relations between $G_1$ and $G_2$ in the skew-reciprocal form:

$$G_2(\rho) = \frac{1}{\pi}\int_0^\infty \frac{G_1(\rho+x)-G_1(\rho-x)}{x}\,dx, \tag{4.53}$$

$$G_1(\rho) = -\frac{1}{\pi}\int_0^\infty \frac{G_2(\rho+x)-G_2(\rho-x)}{x}\,dx. \tag{4,54}$$

If the functions $G_1$ and $G_2$ are introduced in (4,4) we have

$$\frac{d}{dr}\left[\frac{G_2}{r}\right] = \frac{1}{2\pi^2\lambda}\left[\frac{d}{dr}\left(\frac{r^2-k^2}{r}\cdot\frac{dG_1}{dr}\right)-2n\frac{dG_1}{dr}\right],$$

or

$$G_2 = \frac{1}{2\pi^2\lambda}\left[(x^2-k^2)\frac{dG_1}{dx}-2nxG_1\right],$$

or

$$\frac{G_2}{(x^2-k^2)^{n+1}} = \frac{1}{2\pi^2\lambda}\frac{d}{dx}\left[\frac{G_1}{(x^2-k^2)^n}\right]. \tag{4,55}$$

The skew-reciprocal relations (4,53) and (4,54) then at once suggest that

$$\frac{G_1}{(x^2-k^2)^{n+1}} = -\frac{1}{2\pi^2\lambda}\frac{d}{dx}\left[\frac{G_2}{(x^2-k^2)^n}\right] \tag{4,56}$$

is true at the same time.

(4,55) and (4,56), lead at once by integration to

$$G_1 = \mathfrak{A}(x^2-k^2)^n\cos\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{x-k}{x+k}\right|\right),$$

$$G_2 = -\mathfrak{A}(x^2-k^2)^n\sin\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{x-k}{x+k}\right|\right), \tag{4,57}$$

when $k^2 \gtreqless 0$; when $k^2 = -a^2$, on the other hand,

$$G_1 = \mathfrak{B}(x^2+a^2)^n\cos\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{x}{a}\right),$$

$$G_2 = -\mathfrak{B}(x^2+a^2)^n\sin\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{x}{a}\right). \tag{4,58}$$

The two equivalent forms of $M$ are

$$M(x,y,z) = \mathfrak{A}S_n(x,y,z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(r^2-k^2)^n\cos\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{r-k}{r+k}\right|\right)\right]$$

or

$$= -2\pi^2\lambda\frac{\mathfrak{A}S_n(x,y,z)}{r^2-k^2}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left[\frac{(r^2-k^2)^n}{r}\sin\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{r-k}{r+k}\right|\right)\right] \tag{4,591}$$

for the case: $k^2 > 0$; while, if $k^2 = -a^2$,

$$M(x, y, z) = \mathfrak{B}\, S_n(x, y, z) \left( \frac{1}{r} \frac{d}{dr} \right)^{n+1} \left[ (r^2 + a^2)^n \cos\left( \frac{2\pi^2\lambda}{a} \tan^{-1} \frac{r}{a} \right) \right]$$

or

$$= -2\pi^2\lambda\, \mathfrak{B}\, \frac{S_n(x, y, z)}{r^2 + a^2} \left( \frac{1}{r} \frac{d}{dr} \right)^{n} \left[ \frac{(r^2 + a^2)^n}{r} \sin\left( \frac{2\pi^2\lambda}{a} \tan^{-1} \frac{r}{a} \right) \right]. \qquad (4,592)$$

## V

As $M$ is to be single-valued, we see, in the case of $k^2 < 0 = -a^2$, that a restriction comes in regarding the choice of constants of the problem.

$\tan^{-1}(r/a)$ is a multiple-valued function of the form $\theta + N\pi$; if therefore $\cos\left( \frac{2\pi^2\lambda}{a} \tan^{-1} \frac{r}{a} \right)$ and $\sin\left( \frac{2\pi^2\lambda}{a} \tan^{-1} \frac{r}{a} \right)$ are single-valued, it follows that $\pi^2\lambda/a$ must be an integer $= N$. This gives discrete energy-values:

$$k^2 = \frac{2mE}{h^2} = -\frac{\pi^4\lambda^2}{N^2},$$

or

$$E = -\frac{2\pi^2 e^4 m}{h^2 N^2}$$

agreeing with the well-known result. Also as

$$2 \tan^{-1} \frac{r}{a} = \cos^{-1} \frac{a^2 - r^2}{a^2 + r^2},$$

$$\cos\left( N \cos^{-1} \frac{a^2 - r^2}{a^2 + r^2} \right) = \frac{f_N(r^2)}{(a^2 + r^2)^N}$$

we see that $M = 0$ unless $N \geq n + 1$.

Again starting from the series

$$\frac{1}{1 - 2h \cos\theta + h^2} = \sum_1 h^{N-1} \frac{\sin N\theta}{\sin\theta}, \quad \frac{1 - h\cos\theta}{1 - 2h\cos\theta + h^2} = \sum_0 h^N \cos N\theta, \qquad (5,1)$$

and writing

$$\cos\theta = \frac{a^2 - r^2}{a^2 + r^2} \quad \text{and} \quad \sin\theta = \frac{2ar}{a^2 + r^2}$$

we have

$$\frac{2a(a^2 + r^2)}{\{(1-h)^2 a^2 + (1+h)^2 r^2\}} = \Sigma\, h^{N-1} \frac{(a^2 + r^2)}{r} \sin N\theta, \qquad (5,2)$$

$$\frac{a^2(1-h) + r^2(1+h)}{\{(1-h)^2 a^2 + (1+h)^2 r^2\}} = \Sigma\, h^N \cos N\theta. \qquad (5,3)$$

Multiplying (5,2) with $(a^2+r^2)^{n-1}$ and differentiating $n$ times with regard to $r$, we have

$$\frac{1}{[(1-h)^2a^2+(1+h)^2r^2]^{n+1}}$$

$$=\frac{(-1)^n}{n!\,2^{2n+1}\,a^{2n+1}}\sum h^{N-n-1}\left(\frac{d}{dr^2}\right)^n\left[\frac{(a^2+r^2)^n}{r}\sin\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right)\right]; \quad (5,4)$$

from (5,3) we deduce similarly

$$\frac{1-h^2}{[(1-h)^2a^2+(1+h)^2r^2]^{n+2}}$$

$$=\frac{(-1)^{n+1}}{2^{2n+1}a^{2n+2}}\cdot\frac{1}{(n+1)!}\sum h^{N-n-1}\left(\frac{d}{dr^2}\right)^{n+1}\left[(a^2+r^2)^n\cos\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right)\right]. \quad (5,5)$$

Remembering the Gegenbauer expansion

$$\frac{1}{(\alpha^2-2\alpha h\cos\theta+h^2)^\nu}=\frac{1}{\alpha^{2\nu}}+\sum_\mu\frac{(2h)^\mu}{\alpha^{2\nu+\mu}}F^\nu_\mu(\cos\theta)$$

we can easily deduce, if $\nu=n+1$, $\alpha=1$ and

$$\cos\theta=\frac{a^2-r^2}{a^2+r^2},$$

the following results:

$$\frac{1}{[a^2(1-h)^2+r^2(1+h)^2]^{n+1}}=\frac{1}{(a^2+r^2)^{n+1}}\left[1+\sum_1(2h)^\lambda F^{n+1}_\lambda\left(\frac{a^2-r^2}{a^2+r^2}\right)\right], \quad (5,6)$$

$$\frac{1-h^2}{[a^2(1-h)^2+r^2(1+h)^2]^{n+2}}=\frac{1}{(a^2+r^2)^{n+2}}\left[1+\sum_\lambda\frac{(n+1+\lambda)}{n+1}(2h)^\lambda F^{n+1}_\lambda\left(\frac{a^2-r^2}{a^2+r^2}\right)\right]. \quad (5,7)$$

Comparing (5,4) and (5,5) with (5,6) and (5,7) we have

$$\frac{1}{(a^2+r^2)}\left(\frac{d}{dr^2}\right)^n\left[\frac{(a^2+r^2)^n}{r}\sin N\theta\right]=\frac{(-1)^n2^{N+n}a^{2n+1}n!}{(a^2+r^2)^{n+2}}F^{n+1}_{N-n-1}\left(\frac{a^2-r^2}{a^2+r^2}\right),$$

and

$$\left(\frac{d}{dr^2}\right)^{n+1}\left[(a^2+r^2)^n\cos N\theta\right]=\frac{(-1)^{n+1}2^{N+n}a^{2n+2}N.n!}{(a^2+r^2)^{n+2}}F^{n+1}_{N-n-1}\left(\frac{a^2-r^2}{a^2+r^2}\right).$$

Hence

$$M(x,y,z)=\mathfrak{B}\,S_n(x,y,z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(r^2+a^2)^n\cos\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right)\right]$$

or

$$=-2Na.\frac{\mathfrak{B}S_n(x,y,z)}{a^2+r^2}\left(\frac{d}{rdr}\right)^n\left[\frac{(a^2+r^2)^n}{r}\sin\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right)\right]$$

$$=C\,S_n(x,y,z)\,F^{n+1}_{N-n-1}\left(\frac{a^2-r^2}{a^2+r^2}\right).$$

## VI

When $k^2 < 0$, the problem can be transformed to a homogeneous integral equation of the Fredholm type, with a symmetrical kernel

$$\phi(A) = \lambda \int \phi(P) K(A, P)\, dv_P.$$

The results that we have obtained above enable us also to say that

$$\phi = C\sqrt{1+x^2+y^2+z^2}\, S_n(x, y, z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(1+x^2+y^2+z^2)^n \cos\left(N\cos^{-1}\frac{1-r^2}{1+r^2}\right)\right]$$

$$= D\frac{S_n(x, y, z)}{(1+x^2+y^2+z^2)^{n+3/2}} F_\lambda^{n+1}\left(\frac{1-r^2}{1+r^2}\right) \tag{6,1}$$

are eigen-functions corresponding to the eigen-value $-(n+1+\lambda)/\pi^2$ for the kernel

$$K(A, P) = \frac{1}{(x-x')^2+(y-y')^2+(z-z')^2\sqrt{(1+x^2+y^2+z^2)}\sqrt{(1+x'^2+y'^2+z'^2)}}.$$

Writing

$$M(x, y, z) = \frac{\mathfrak{C}\, S_n(x, y, z)}{(a^2+r^2)^{n+2}} F_\lambda^{n+1}\left(\frac{a^2-r^2}{a^2+r^2}\right) = \mathfrak{C}\, Y_n(\theta, \phi)\frac{r^n}{(a^2+r^2)^{n+2}} F_\lambda^{n+1}\left(\frac{a^2-r^2}{a^2+r^2}\right),$$

also

$$\int M^2 dx dy dz = 1 = \mathfrak{C}^2 \int Y_n^2(\theta, \phi) d\omega \int \frac{r^{2n+2}}{(a^2+r^2)^{2n+4}}\left[F_\lambda^{n+1}\left(\frac{a^2-r^2}{a^2+r^2}\right)\right]^2 dr.$$

To determine the radial integral, we use (5,7) and write

$$I = \int \frac{(1-h^2)(1-t^2) r^{2n+2} dr}{[a^2(1-h)^2+r^2(1+h)^2]^{n+2}[a^2(1-t)^2+r^2(1+t)^2]^{n+2}}$$

$$= \sum_\lambda \sum_\sigma \frac{(n+1+\lambda)(n+1+\sigma)}{(n+1)^2}(2h)^\lambda (2t)^\sigma \int\int \frac{F_\lambda^{n+1} F_\sigma^{n+1} r^{2n+2}}{(a^2+r^2)^{2n+4}} dr,$$

which reduces to

$$I = \frac{(-1)^{n+1}}{(n+1)!}\cdot\frac{(1-h^2)(1-t^2)}{(1+h)^{2n+4}(1+t)^{2n+4}}\cdot\left\{\frac{d}{d(p+q)}\right\}^{n+1}\int_0^\infty \frac{dr}{[p+r^2][q+r^2]},$$

where

$$p = \frac{a^2(1-h)^2}{(1+h)^2}, \qquad q = \frac{a^2(1-t)^2}{(1+t)^2},$$

when by easy calculation

$$I = \frac{\pi}{2^{4n+5} a^{2n+5}}\cdot\frac{(2n+1)!}{n!\,(n+1)!}\,\frac{(1+h)(1+t)}{(1-ht)^{2n+3}}. \tag{6.2}$$

Equating coefficient of $(ht)^\lambda$

$$\int \frac{r^{2n+2}}{(a^2+r^2)^{2n+4}}\left(F_\lambda^{n+1}\right)^2 dr = \frac{\pi(2n+1+\lambda)!}{2^{4n+2\lambda+5} a^{2n+5}(n+1+\lambda)\,\lambda!\,\{n!\}^2};$$

hence

$$\overline{M}\text{ (normalised)} = 2^{2n+\lambda+3}\, a^{n+3}\left(\frac{N}{2\pi a}\right)^{\frac{1}{2}} n!\left\{\frac{\lambda!}{(N+n)!}\right\}^{\frac{1}{2}} \frac{F_{\lambda}^{n+1}\left(\dfrac{a^2-r^2}{a^2+r^2}\right)}{(a^2+r^2)^{n+2}}\, \overline{Y}_n(\theta,\phi), \qquad (6{,}3)$$

where $\overline{Y}_n(\theta, \phi)$ is taken to be a normalised spherical harmonic.

We can normalise the eigen-functions of the homogeneous integral equation, for which similarly the value of

$$\int \frac{r^{2n+2}}{(1+r^2)^{2n+3}}\left(F_{\lambda}^{n+1}\right)^2 dr$$

will be necessary; for it, we utilise again (5,7) and write

$$I = \int_0^{\infty} \frac{(1-h^2)r^{2n+2}dr}{[(1-h)^2+(1+h)^2r^2]^{n+2}[(1-t)^2+(1+t)^2r^2]^{n+1}}$$

$$= \sum_{\lambda}\sum_{\sigma} (2h)^{\lambda}(2t)^{\sigma}\frac{(n+1+\lambda)}{(n+1)}\int \frac{F_{\lambda}^{n+1}F_{\sigma}^{n+1}r^{2n+2}dr}{(1+r^2)^{2n+3}},$$

after easy integration

$$I = \frac{\pi}{2^{4n+4}}\cdot\frac{(2n+1)!}{n!\,(n+1)!}\cdot\frac{1}{(1-ht)^{2n+2}}. \qquad (6{,}4)$$

Equating coefficient we have

$$\int \frac{\{F_{\lambda}^{n+1}\}^2\, r^{2n+2}\,dr}{(1+r^2)^{2n+3}} = \frac{\pi}{2^{4n+2\lambda+4}(n!)^2(n+1+\lambda)}\cdot\frac{(2n+1+\lambda)!}{\lambda!}.$$

The normalised eigen-functions of the homogeneous equation is

$$\psi = \frac{2^{2n+\lambda+2}\, n!\,(n+1+\lambda)^{\frac{1}{2}}}{\sqrt{\pi}}\cdot\left(\frac{\lambda!}{(2n+1+\lambda)!}\right)^{\frac{1}{2}}\frac{Y_n(\theta,\phi)r^n}{(1+r^2)^{n+3/2}}F_{\lambda}^{n+1}\left(\frac{1-r^2}{1+r^2}\right). \qquad (6{,}5)$$

The completeness of the eigen-functions series will mean the equality

$$K(x, y) = \sum_n \frac{\pi^2\psi_r^n(x)\psi_r^n(y)}{(n+1+\lambda)}$$

(multiple summations as there are repeated roots) leading to the relation

$$\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2][1+r^2]^{1/2}[1+r'^2]^{1/2}}$$

$$= \sum_1 \frac{\pi^2}{N}\sum_{n=0}^{n=N}\frac{2^{2N+2n+2}}{\pi}(n!)^2 N\left[\frac{(N-n-1)!}{(N+n)!}\right]\frac{r^n}{(1+r^2)^{3/2}}\cdot\frac{r'^n}{(1+r'^2)^{3/2}}F_{\lambda}^{n+1}F_{\lambda}'^{n+1}$$

$$\times\frac{(2n+1)}{4\pi}\left[\sum_1^n 2P_n^m(\mu)P_n^m(\mu')\frac{(n-m)!}{(n+m)!}\cos m(\phi-\phi')+P_n(\mu)P_n(\mu')\right],$$

and leading to an expansion of

$$\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2} = \sum_N 2^{2N}\sum_0^{N-1} 2^{2n}F_{N-n-1}^{n+1}\left(\frac{1-r^2}{1+r^2}\right)F_{N-n-1}^{n+1}\left(\frac{1-r'^2}{1+r'^2}\right)\frac{r^n r'^n}{(1+r^2)^{n+1}(1+r'^2)^{n+1}}$$

$$\times\left[\frac{(N-n-1)!}{(N+n)!}\right](n!)^2\left[P_n(\mu)P_n(\mu')+2\sum_1^n\frac{(n-m)!}{(n+m)!}P_n^m(\mu)P_n^m(\mu')\cos m(\phi-\phi')\right], \qquad (6{,}7)$$

by a change of axis making $(x' y' z')$ on the $z$ axis

$$\frac{1}{r^2+r'^2-2rr'\mu} = \sum_N 2^{2N} \sum_n 2^{2n} F^{n+1}_{N-n-1}\left(\frac{1-r^2}{1+r^2}\right).F^{n+1}_{N-n-1}\left(\frac{1-r'^2}{1+r'^2}\right)\frac{r^n r'^n}{(1+r^2)^{n+1}(1+r'^2)^{n+1}}$$

$$\times (n\,!)^2\left[\frac{(N-n-1)\,!}{(N+n)\,!}\right](2n+1)\,P_n(\mu) \tag{6,8}$$

from the well-known addition theorem of $P_n(\mu)$.

To verify the correctness of the result we note that

$$\frac{2}{2[r^2+r'^2-2rr'\mu]} = \frac{2}{(1+r^2)(r'^2+1)\left[1-\frac{1-r^2}{1+r^2}.\frac{1-r'^2}{1+r'^2}-\frac{4rr'}{(1+r^2)(1+r'^2)}\cos\phi\right]};$$

hence writing

$$\cos\theta = \frac{1-r^2}{1+r^2}, \qquad \cos\theta' = \frac{1-r'^2}{1+r'^2},$$

we have

$$\frac{1}{r^2+r'^2-2rr'\mu} = \frac{2}{(1+r^2)(1+r'^2)[1-\cos\theta\cos\theta'-\sin\theta\sin\theta'\cos\phi]}$$

$$= \frac{4}{(1+r^2)(1+r'^2)}\sum_1^N \frac{\sin N\chi}{\sin\chi}, \tag{6,81}$$

if $\pi/2 > \chi > 0$ from (5,1), and $\cos\chi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi$; hence

$$\frac{\sin N\chi}{\sin\chi} = \sum_{n=0}^{N-1} 2^{2(N-1)}(2n+1)(n\,!)^2\left[\frac{(N-n-1)\,!}{(N+n)\,!}\right]$$

$$\times \sin^n\theta\sin^n\theta' F^{n+1}_{N-n-1}(\cos\theta).F^{n+1}_{N-n-1}(\cos\theta').P_n(\cos\phi), \tag{6,9}$$

a result which is a particular case of a general class of identity deduced by Gegenbauer (Whittaker, 1927).

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# FINITE STRIAN IN AELOTROPIC ELASTIC BODIES—I

By

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*(Communicated by Prof. N. R. Sen)*

## 1 *Introduction*

Aelotropic elastic bodies have been the subject of a number of recent papers by Taylor (1939), Green (1939, 1942)) and others.* The treatment in all of them is based on the small strain theory and a number of results have been obtained for aelotropic plates.

The pressing needs of the present war have produced many aelotropic materials which can be deformed to such an extent without exceeding the elastic limits that the small strain theory cannot be applied to them. For such materials it is proposed to extend the theory of finite strain, which has been developed for isotropic bodies (Seth, 1935, 1939A, 1939B, 1941; Shepperd and Seth, 1936), to some types of crystalline aelotropic bodies.

Even in the case of finite strain the strain invariants are relatively small. For example, if $I_1, I_2, I_3$ are the first three invariants, it is seen (Murnaghan, 1937) that in the case of uniform hydrostatic pressure $I_1$ varies between $-0.030$ and $-0.225$, $I_2$ between $0.001$ and $0.0169$ and $I_3$ between $-0.000001$ and $-0.00042$. Moreover, the inclusion of the third degree terms in the potential energy function makes the equation involved very intricate even in the case of isotropic bodies, and only very simple cases like uniform tension and uniform hydrostatic pressure can be successfully dealt with. Hence, at the expense of some exactness, we shall assume the stress-strain relation to be linear. This assumption implies that, though the strain components are not very large, which is generally the case if we remain within the limits of elasticity, the deflections produced in the body by external forces are so large that in the expression for the strain components the squares and products of the deflections and their derivatives cannot be neglected. The results thus obtained are not devoid of interest, even though they may not have the accuracy obtained by including higher degree terms in the stress-strain relations.

In what follows the co-ordinates used refer to the actual position of a point of the material in the strained condition. The components of strain are given by equations of the form (Seth, 1935)

$$s_x = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right], \tag{1.1}$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right]. \tag{1.2}$$

* In the papers by Taylor and Green references to the works of Mitchell, Hokubo, Sen, Huber and others are given.

In matrix form the stress-strain relation can be written as

$$(\widehat{xx}, \widehat{yy}, \widehat{zz}, \widehat{yz}, \widehat{zx}, \widehat{xy}) = \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{vmatrix} (s_x, s_y, s_z, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}) \tag{2}$$

where $c_{rs} = c_{sr}$ $(r, s = 1, 2, \ldots, 6)$.

## 2. *Cylinder under uniform tension*

The ends of a cylinder are subjected to a uniform stress $T$, the sides being free from any applied force. The condition of the problem are met by assuming a state of strain given by

$$\left.\begin{aligned} u &= x(1-p)+a_1x+b_1z, \\ v &= y(1-q)+a_2z+b_2x, \\ w &= z(1-r)+a_3x+b_3y, \end{aligned}\right\} \tag{3}$$

$p$, $q$, $r$, $a$'s and $b$'s being all constants. The $z$-axis may be assumed to be parallel to the generators of the cylinder.

For a rigid body displacement the strain components $s_x$, $\sigma_{yz}$, etc. all vanish. The displacement in such a case represents a rotation about an axis (Love, p. 69-70), and this can be expressed in terms of three unknown constants. In the expression for $u$, $v$, $w$ given by (3) we need therefore have no more than six unknown constants. Accordingly we put $b_1$, $b_2$, $b_3$ all equal to zero.

The strain components are given by

$$s_x = \tfrac{1}{2}(1-p^2-a_3^2), \quad s_y = \tfrac{1}{2}(1-q^2-a_1^2), \quad s_z = \tfrac{1}{2}(1-r^2-a_2^2). \tag{4.1}$$

$$\sigma_{yz} = a_2q, \quad \sigma_{zx} = a_3r, \quad \sigma_{xy} = a_1p. \tag{4.2}$$

The stress-strain relations give

$$C_{13}T_1 = 1-p^2-a_3^2, \quad C_{23}T_1 = 1-q^2-a_1^2, \quad C_{33}T_1 = 1-r^2-a_2^2, \tag{4.3}$$

$$C_{34}T_1 = 2a_2q, \quad C_{35}T_1 = 2a_3r, \quad C_{36}T_1 = 2a_1p, \tag{4.4}$$

where $T_1 = 2T/\Pi$, $\Pi$ being the determinant given in (2), and the capital $C$'s are the minors of the small $c$'s in (2).

Putting

$$t_1 = 1 - C_{13}T_1 = 1 + 2\sigma_{31}T/E_{33},$$

$$t_2 = 1 - C_{23}T_1 = 1 + 2\sigma_{32}T/E_{33},$$

$$t_3 = 1 - C_{33}T_1 = 1 - 2T/E_{33},$$

$$t_4 = C_{34}T_1, \quad t_5 = C_{35}T_1, \quad t_6 = C_{36}T_1,$$

and, eliminating all the unknown constants in (4.3) and (4.4), excepting $p$, we get

$$p^4(t_4^2 - 4t_2t_3) - p^2(t_1t_4^2 + t_2t_5^2 - t_3t_6^2 - 4t_1t_2t_3) + \tfrac{1}{4}t_5^2t_6^2 - t_1t_3t_6^2 = 0, \tag{5}$$

which gives two values for $p^2$, which being known, (4.3) and (4.4) give the values of the remaining five constants.

It appears, therefore, that in an aelotropic cylinder an uniform tension $T$ applied to its plane ends can produce shearing stress in it. Such is not the case when the cylinder is isotropic.

In the particular case when the cylinder possesses three orthogonal planes of symmetry all $c_{rs}$'s excepting $c_{44}$, $c_{55}$, $c_{66}$ and those for which $r, s = 1, 2, 3$, vanish. In such a case no shear is produced by a uniform tension. One root of (5) is now zero, and the other is given by

$$p^2 = t_1 = 1 + \frac{2\sigma_{31}T}{E_{33}}. \tag{5.1}$$

The corresponding values of $q$ and $r$ are

$$q^2 = t_2 = 1 + \frac{2\sigma_{32}T}{E_{33}}, \qquad r^2 = t_3 = 1 - \frac{2T}{E_{33}}, \tag{5.2}$$

$\sigma$'s and $E$'s denoting the Poisson's ratios and Young's modulii respectively.

As is to be expected the displacements in the $x$ and $y$ directions are not the same as in the isotropic case. In fact we find

$$\frac{1-p^2}{1-q^2} = \frac{\sigma_{31}}{\sigma_{32}} = \frac{c_{12}c_{33} - c_{22}c_{13}}{c_{12}c_{13} - c_{11}c_{23}}.$$

For Topaz, a rhombic crystal, we know (Love, pp. 163-64) that

$$c_{11}=2870, \quad c_{22}=3000, \quad c_{33}=3560, \quad c_{23}=900, \quad c_{31}=1280,$$

$$c_{12}=860, \quad c_{44}=1100, \quad c_{55}=1330, \quad c_{66}=1350,$$

and we find $\sigma_{31}/\sigma_{32} = 2.1$.

For Barytes we have

$$c_{11}=800, \quad c_{22}=907, \quad c_{33}=1074, \quad c_{23}=275, \quad c_{31}=273, \quad c_{12}=468,$$

$$c_{44}=293, \quad c_{55}=122, \quad c_{66}=283,$$

and the corresponding value of $\sigma_{31}/\sigma_{32}$ is 1.3.

In the general case (5) gives two values of $p^2$. We can show that both these values are positive.

From (5.1) and (5.2) we see that $t_1$, $t_2$, $t_3$ are all positive. If we put $t_5 = t_6 = 0$, we get

$$a_3 = a_1 = 0, \quad p^2 = t_1, \quad q^2 = t_2,$$

and

$$4t_2 r^2 = 4t_2 t_3 - t_4^2.$$

Hence

$$4t_2 t_3 > t_4^2,$$

and similarly

$$4t_1 t_3 > t_5^2.$$

From (5) we now see that the product of the two values of $p^2$ is positive. If therefore one value is positive, the other must also be positive.

Let the two values be represented by $p_1^2$ and $p_2^2$ and let $p_1^2 > p_2^2$. We know (Love, p. 163) that some aelotropic materials like Pyrites expand slightly in a lateral direction when extended in the direction of a principal axis. In such cases $p_2$ should be used. In general it will be found that $p_1$ is to be taken.

It should be mentioned that the small strain theory for aelotropic bodies does not and cannot give these two results simultaneously. From physical considerations we at once conclude that $p_1$ is greater than one and $p_2$ is less than one.

### 3. *Case of hydrostatic pressure*

Let the uniform hydrostatic pressure be $P$, so that

$$\widehat{xx} = \widehat{yy} = \widehat{zz} = -P, \quad \widehat{yz} = \widehat{zx} = \widehat{xy} = 0.$$

For the displacements we can assume the same forms as given in (3), $b_1$, $b_2$, $b_3$ being all zero. We have now.

$$\Pi s_x = -P \sum_1^3 C_{1r}, \quad \Pi s_y = -P \sum_1^3 C_{2r}, \quad \Pi s_z = -P \sum_1^3 C_{3r},$$

$$\Pi \sigma_{yz} = -P \sum_1^3 C_{4r}, \quad \Pi \sigma_{zx} = -P \sum_1^3 C_{5r}, \quad \Pi \sigma_{xy} = -P \sum_1^3 C_{6r}.$$

Puttting $T_1 = -2P/\Pi$ and

$$t_r\,(r = 1, 2, 3) = 1 - T_1 \sum_{s=1}^{3} C_{rs},$$

$$t_r\,(r = 4, 5, 6) = \sum_{s=1}^{3} C_{rs},$$

we get the same equation as given in (5) to determine $p$. This again will have two roots, one greater than the other.

When the material has three orthogonal planes of symmetry the shearing stress effect is absent, one value of $p^2$ is zero and the other is given by

$$p^2 = 1 + \frac{2P}{\Pi} \sum_{r=1}^{3} C_{1r},$$

with similar expression for $q^2$ and $r^2$.

In the case of aelotropic materials subjected to hydrostatic pressure the following points may be noticed:

(1) The constants in $u$, $v$, $w$ are not the same as in the isotropic case.

(2) Shearing stress may be produced in the material.

### 4. *Rectangular plate bent by terminal couples*

We suppose that an initially plane rectangular plate is bent into the form of a circular cylinder with two edges as generators. Two faces of the plates get bent into right cylindrical surfaces of inner radius $a$ and outer radius $b$ and the other two into axial terminating planes given by $\theta = \pm\alpha$. The $z$-axis is taken as the axis of the cylinder.

As in the isotropic case we assume the displacements to be given by

$$u = x - f(r), \quad v = y - A\theta, \quad w = \alpha z, \tag{6}$$

$A$ and $\alpha$ being constants. $f(r)$ is a function of $r = (x^2+y^2)^{\frac{1}{2}}$ only. These displacements give the strain components as

$$s_x = \frac{1}{2}\left(1 - \frac{f'^2 x^2}{r^2} - \frac{A^2 y^2}{r^4}\right), \quad s_y = \frac{1}{2}\left(1 - \frac{f'^2 y^2}{r^2} - \frac{A^2 x^2}{r^4}\right), \quad s_z = \alpha - \tfrac{1}{2}\alpha^2, \tag{7.1}$$

$$\sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad \sigma_{xy} = -\left(\frac{f'^2 xy}{r^2} - \frac{A^2 xy}{r^4}\right). \tag{7.2}$$

We assume that the aelotropy is of the hexagonal or rhombohedral type. A particular case of it is when the plate is transversely isotropic and the axis of symmetry is the axis of the cylinder. The stress-strain relations are given by

$$\left.\begin{aligned}
\widehat{xx} &= c_{11}s_x + (c_{11} - 2c_{66})s_y + c_{13}s_z,\\
\widehat{yy} &= (c_{11} - 2c_{66})s_x + c_{11}s_y + c_{13}s_z,\\
\widehat{zz} &= c_{13}s_x + c_{13}s_y + c_{33}s_z,\\
\widehat{yz} &= 0, \quad \widehat{zx} = 0,\\
\widehat{xy} &= c_{66}\sigma_{xy}.
\end{aligned}\right\} \tag{8}$$

Combined with the values of $s_x$, $\sigma_{yz}$, etc. we get from these relations the stresses in polar co-ordinates as

$$\widehat{rr} = \frac{1}{2}c_{11}\left(2 - f'^2 - \frac{A^2}{r^2}\right) - c_{66}\left(1 - \frac{A^2}{r^2}\right) + c_{13}(\alpha - \tfrac{1}{2}\alpha^2), \tag{9.1}$$

$$\widehat{r\theta} = 0, \tag{9.2}$$

$$\widehat{\theta\theta} = \frac{1}{2}c_{11}\left(2 - f'^2 - \frac{A^2}{r^2}\right) - c_{66}(1 - f'^2) + c_{13}(\alpha - \tfrac{1}{2}\alpha^2), \tag{9.3}$$

$$\widehat{zz} = \frac{1}{2}c_{13}\left(2 - f'^2 - \frac{A^2}{r^2}\right) + c_{33}(\alpha - \tfrac{1}{2}\alpha^2), \tag{9.4}$$

$$\widehat{rz} = 0, \quad \widehat{\theta z} = 0. \tag{9.5}$$

All body-stress equations are satisfied excepting

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0,$$

which gives

$$\frac{d}{dr}(f'^2)+\frac{2c_{66}}{c_{11}}\frac{f'^2}{r}+\frac{2(c_{66}-c_{11})}{c_{11}}\frac{A^2}{r^3}=0,$$

from which we have

$$f'^2=\frac{l}{r^{c_0}}-\frac{A^2}{r^2}, \tag{10}$$

where $l$ is a constant and $c_0 = 2c_{66}/c_{11}$.

To satisfy the boundary conditions over $r=a$ and $r=b$ we must have $\widehat{rr}=0$ over both of them. Thus we get

$$\tfrac{1}{2}c_{11}\left(2-\frac{l}{a^{c_0}}\right)-c_{66}\left(1-\frac{A^2}{a^2}\right)+c_{13}(\alpha-\tfrac{1}{2}\alpha^2)=0, \tag{11.1}$$

$$\tfrac{1}{2}c_{11}\left(2-\frac{l}{b^{c_0}}\right)-c_{66}\left(1-\frac{A^2}{b^2}\right)+c_{13}(\alpha-\tfrac{1}{2}\alpha^2)=0. \tag{11.2}$$

Over the plane ends we have

$$\int_a^b r.\widehat{zz}\,dr=0,\quad \frac{1}{a}\int_a^b r^2.\widehat{zz}\,dr=-M_2, \tag{12}$$

where $M_2$ is the couple in the axial plane per unit arc between $\theta$ and $\theta+d\theta$.

The first condition gives $\alpha=0$ or $\alpha=2$. The latter value corresponds to the case of the cylinder turned upside down. The second condition gives

$$M_2=-\frac{c_{13}}{a}\left[\frac{1}{3}(b^3-a^3)-\frac{1}{2}\frac{2-c_0}{3-c_0}(b^2-a^2).\frac{b^{3-c_0}-a^{3-c_0}}{b^{2-c_0}-a^{2-c_0}}\right]. \tag{13}$$

On the straight edges $\theta=\pm\alpha$ we should have

$$\int_a^b \widehat{\theta\theta}\,dr=0,\qquad \int_a^b r.\widehat{\theta\theta}\,dr=M_1.$$

The first is identically satisfied on account of (11). $M_1$, the moment per unit length, is given by

$$M_1=c_{66}\left[\tfrac{1}{2}(b^2-a^2)-\frac{(2-c_0)(\log b-\log a)\,(ab)^{2-c_0}\,(b^{c_0}-a^{c_0})}{c_0(b^{2-c_0}-a^{2-c_0})}\right]. \tag{14}$$

For an isotropic thin plate $M_1$, $M_2$ reduce to the values given by Love (1927, p. 554).

The accompanying table gives the values of $10^2M_1/a^2c_{66}$ and $10^2M_1/a^2c_{11}$ for varying values of $c_0$ and $t=b/a$. As is to be expected the couple to be applied to the straight edges increases with the thickness of the plate.

From (13) it is obvious that as $c_0\to 0$, $M_2\to 0$. Keeping $c_{66}$ as constant we see that $M_1$ remains finite. It should therefore be possible to bend such a plate into a cylindrical shape by applying couples only in the straight edges. In such a case $c_{11}$ is much greater than $c_{66}$. Some of the aelotropic specimens referred to in the Introduction belong to this class.

| $t$ | $c_0=0$ | | $c_0=\frac{1}{2}$ | | $c_0=1$ | | $c_0=\frac{3}{2}$ | | $c_0=2$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\frac{10^2 M_1}{c_{66}a^2}$ | $\frac{10^2 M_1}{c_{11}a^2}$ | $\frac{10^2 M_1}{c_{66}a^2}$ | $\frac{10^2 M_1}{c_{11}a^2}$ | $\frac{10^2 M_1}{c_{66}a^2}$ | $\frac{10^2 M_1}{c_{11}a^2}$ | $\frac{10^2 M_1}{c_{66}a^2}$ | $\frac{10^2 M_1}{c_{11}a^2}$ | $\frac{10^2 M_1}{c_{66}a^2}$ | $\frac{10^2 M_1}{c_{11}a^2}$ |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1.1 | 0.5 | 0 | 0.1 | 0.025 | 0.01 | 0.005 | 0.008 | 0.005 | 0 | 0 |
| 1.2 | 1.5 | 0 | 0.3 | 0.075 | 0.12 | 0.06 | 0.06 | 0.045 | 0 | 0 |
| 1.3 | 3.1 | 0 | 0 6 | 0.15 | 0.40 | 0.20 | 0·20 | 0.15 | 0 | 0 |
| 1.4 | 5.2 | 0 | 1.4 | 0 35 | 0.89 | 0.45 | 0.45 | 0.33 | 0 | 0 |
| 1 5 | 7 7 | 0 | 2.5 | 0 63 | 1.68 | 0.84 | 0.85 | 0.63 | 0 | 0 |

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# ON A CERTAIN CLASS OF MULTIPLE INTEGRALS

By

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The problem of evaluation and reduction of a class of multiple integrals arising out of certain recent work in mathematical statistics by the author and other persons (Roy, 1939, 1945; Fisher, 1939) offers some features of pure mathematical interest. This problem was solved by the author sometime ago and it is proposed in this paper to detach the solution from its statistical aspects. The principle underlying the solution is that the evaluation of the $p$-fold multiple integral is ultimately made to depend upon that of a $(p-1)$-fold integral and so on and on until we reach one-fold integrals which can be immediately evaluated by the incomplete $B$-function tables.

The integrand is denoted by

$$F\{m_p\,;\, n_p\,;\, x_p : m_{p-1}\,;\, n_{p-1}\,;\, x_{p-1} : \ldots\ldots\, m_2\,;\, n_2\,;\, x_2 : m_1\,;\, n_1\,;\, x_1\} \tag{1}$$

which stands for the determinant

$$\begin{vmatrix} \dfrac{x_p^{m_p}}{(1+x_p)^{n_p}} & \dfrac{x_p^{m_{p-1}}}{(1+x_p)^{n_{p-1}}} & \ldots\ldots & \dfrac{x_p^{m_2}}{(1+x_p)^{n_2}} & \dfrac{x_p^{m_1}}{(1+x_p)^{n_1}} \\ \dfrac{x_{p-1}^{m_p}}{(1+x_{p-1})^{n_p}} & \dfrac{x_{p-1}^{m_{p-1}}}{(1+x_{p-1})^{n_{p-1}}} & \ldots\ldots & \dfrac{x_{p-1}^{m_2}}{(1+x_{p-1})^{n_2}} & \dfrac{x_{p-1}^{m_1}}{(1+x_{p-1})^{n_1}} \\ \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ \dfrac{x_2^{m_p}}{(1+x_2)^{n_p}} & \dfrac{x_2^{m_{p-1}}}{(1+x_2)^{n_{p-1}}} & \ldots & \dfrac{x_2^{m_2}}{(1+x_2)^{n_2}} & \dfrac{x_2^{m_1}}{(1+x_2)^{n_1}} \\ \dfrac{x_1^{m_p}}{(1+x_1)^{n}} & \dfrac{x_1^{m_{p-1}}}{(1+x_1)^{n_{p-1}}} & \ldots\ldots & \dfrac{x_1^{m_2}}{(1+x_1)^{n_2}} & \dfrac{x_1^{m_1}}{(1+x_1)^{n_1}} \end{vmatrix}$$

The variables $x_1, x_2, \ldots\ldots, x_p$ each goes from 0 to $\infty$, and the $2p$ parameters $(m_1, \ldots, m_p, n_1, \ldots, n_p)$ are subject to the following restrictions

$$n_p > n_{p-1} > n_{p-2} > \ldots\ldots > n_2 > n_1 > m_p > m_{p-1} > \ldots\; . > m_2 > m_1 > 1,$$

and they differ by integers (with the added restriction that $m_i - m_j = n_i - n_j$, where $i > j$) but might be otherwise non-integral. The class of integrals with which we are concerned fall into the following broad categories:

$$(i)\quad \int_0^X dx_p \int_0^{x_p} dx_{p-1} \ldots \int_0^{x_2} dx_1 F\{m_p\,;\, n_p;\, x_p : m_{p-1}\,;\, n_{p-1}\,;\, x_{p-1} : \ldots\, m_1\,;\, n_1\,;\, x_1\}, \tag{1.1}$$

$$(ii)\quad \int_{x_{p-1}}^{\infty} dx_p \int_{x_{p-2}}^{x_p} dx_{p-1} \ldots \int_{x_1}^{\infty} dx_2 \int_X^{\infty} dx_1 F\{m_p\,;\, n_p\,;\, x_p : m_{p-1}\,;\, n_{p-1}\,;\, x_{p-1} : \ldots\, m_1\,;\, n_1\,;\, x_1\}, \tag{1.2}$$

$$(iii) \quad \int_{x_{p-1}}^{\infty} dx_p \int_{x_{p-2}}^{\infty} dx_{p-1} \ldots \int_{x_s}^{\infty} dx_{s+1} \int_{X}^{\infty} dx_s \int_0^{x_s} dx_{s-1} \ldots \int_0^{x_2} dx_1 F\{m_p;\, n_p;\, x_p :\ \ldots\ m_1;\, n_1;\, x_1\}. \quad (1.3)$$

The order of integration is from $x_1 \to x_2 \to x_3 \to \ldots x_p$ in $(i)$, $x_p \to x_{p-1} \to x_{p-2} \to \ldots x_1$ in $(ii)$, and in $(iii)$ it is from $x_p \to x_{p-1} \to \ldots x_s$ on one side and $x_1 \to x_2 \to \ldots x_s$ on the other side.

Integrals (1.1), (1.2) and (1.3) which are all really functions of $X$, can be conveniently denoted respectively by the following pseudo-determinantal expressions:

$$F\left\{X;\ \left\| \begin{array}{cccc} m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \end{array} \right\| \right\}, \qquad (1.4)$$

$$F\left\{\left\| \begin{array}{cccc} m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \end{array} \right\|;\ X \right\}, \qquad (1.5)$$

and

$$F\left\{ \begin{array}{ll} & \begin{array}{cccc} m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \\ \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \end{array} \\ X; & \begin{array}{cccc} (m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1) \end{array} \\ & \begin{array}{cccc} \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ m_p, n_p & m_{p-1}, n_{p-1} & \ldots\ldots\ldots\ldots & m_1, n_1 \end{array} \end{array} \right\}. \qquad (1.6)$$

In this paper it is proposed to sketch in outline the evaluation (or reduction) of (1.4). The method is such that a student of mathematics will be immediately able to use it to reduce (1.5) and (1.6) which need not thus be tackled seperately.

Looking at (1), (1.4)-(1.6) one can easily see that each of (1.4)-(1.6) would be zero if any two columns of the corresponding pseudo-determinant were the same, that is, if $m_i = m_j$ and $n_i = n_j$, $(i, j = 1, 2, \ldots, p$, and $i \neq j)$. Furthermore if say (1) and (1.4) are taken together it would be evident that the incomplete multiple integral (1.4) can be written as

$$\Sigma \pm F(X;\ m'_p,\ n'_p;\ m'_{p-1},\ n'_{p-1};\ \ldots;\ m'_1,\ n'_1). \qquad (1.7)$$

Here $F(X;\ m'_p,\ n'_p;\ m'_{p-1},\ n'_{p-1};\ \ldots;\ m'_1,\ n'_1)$ stands for the multiple integral

$$\int_0^X \frac{x_p^{m'_p}\, dx_p}{(1+x_p)^{n'_p}} \cdot \int_0^{x_p} \frac{x_{p-1}^{m'_{p-1}}\, dx_{p-1}}{(1+x_{p-1})^{n'_{p-1}}} \ \ldots\ldots \int_0^{x_2} \frac{x_1^{m'_1}\, dx_1}{(1+x_1)^{n'_1}}, \qquad (1.8)$$

and where $(m'_p,\ n'_p;\ m'_{p-1},\ n'_{p-1};\ \ldots\ldots;\ m'_1,\ n'_1)$ is any permutation of the $p$-pairs of quantities $(m_p,\ n_p;\ m_{p-1},\ n_{p-1};\ \ldots\ldots;\ m_1,\ n_1)$, the positive or the negative sign in (1,7) being taken according as it is an even or odd permutation (just as in the case of opening out a determinant). It is obvious that associated with (1.5) and (1.6) we should have expressions like

$$\left.\begin{array}{l} F(m'_p,\, n'_p\,;\; m'_{p-1},\, n'_{p-1}\,;\; \ldots\ldots;\; m'_1,\, n'_1,\, X), \\ \text{and} \\ F(m'_p,\, n'_p\,;\, m'_{p-1},\, n'_{p-1}\,;\ldots\ldots;\; m'_s,\, n'_s\,;\; X\,;\; m'_{s-1},\, n'_{s-1}\,;\ldots;\; m'_1,\, n'_1), \end{array}\right\} \quad (1.9)$$

with obvious interpretation similar to that of (1.8). But we need not consider this seperately. It will do if we proceed with (1.4) directly.

The reduction of (1.4) depends upon the following simple mathematical auxiliaries:

$$(a)\quad \int_0^X \frac{x^m dx}{(1+x)^n} Q(x) = -\frac{1}{n-1}\left[\frac{x^m}{x^{n-1}}\,Q(x)\right]_0^X + \frac{1}{n-1}\int_0^X \frac{x^m dx}{(1+x)^{n-1}}\frac{\partial Q}{\partial x} + \frac{m}{n-1}\int_0^X \frac{x^{m-1}dx}{(1+x)^{n-1}}\,Q$$

$$= -\frac{1}{n-1}\,\frac{X^m}{(1+X)^{n-1}} + \frac{1}{n-1}\int_0^X \frac{x^m dx}{(1+x)^{n-1}}\frac{\partial Q}{\partial x} + \frac{m}{n-1}\int_0^X \frac{x^{m-1}dx}{(1+x)^{n-1}}\,Q. \quad (2)$$

Here $Q(x)$ is any function which conforms to the usual conditions of integrability and is zero for $x = 0$. The result is easily derived by integration by parts.

(b) Considering $F(X\,;\; m_p,\, n_p\,;\; m_{p-1},\, n_{p-1}\,;\; \ldots\ldots;\; m_1,\, n_1)$, if we permute the pairs of parameters over the different places we can easily convince ourselves that

$$\Sigma F(X\,;\; m'_p,\, n'_p\,;\; m'_{p-1},\, n'_{p-1},\; \ldots;\; m'_1,\, n'_1) = \prod_{s=1}^{p} F(X\,;\; m_s,\, n_s), \quad (2.1)$$

where

$$F(X\,;\; m_s,\, n_s) = \int_0^X \frac{x^{m_s}dx}{(1+x)^{n_s}}.$$

The mechanism by which the relation (2.1) is obtained can be easily visualised if we consider the case of $p = 2$. Here

$$F(X\,;\; m_2,\, n_2\,;\; m_1,\, n_1) + F(X\,;\; m_1,\, n_1\,;\; m_2,\, n_2)$$

$$= \int_0^X \frac{x_2^{m_2}dx_2}{(1+x_2)^{n_2}} \cdot \int_0^{x_2} \frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}} + \int_0^X \frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}} \cdot \int_0^{x_2} \frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}$$

$$= \int_0^X \frac{x^{m_2}dx}{(1+x)^{n_2}} \cdot \int_0^X \frac{x^{m_1}\,dx}{(1+x)^{n_1}} = F(X\,;\; m_2,\, n_2).F(X\,;\; m_1,\, n_1).$$

With the auxiliaries (a) and (b) the reduction of (1.4) can now be proceeded with. It has been observed that the reduction of (1.4) depends upon the reduction of terms like those in (1.7). Now suppose that in any $F(X\,;\; m'_p,\, n'_p\,;\; \ldots;\; m'_1,\, n'_1)$ the largest of the pairs is $m'_s$ and $n'_s$ equal respectively to $m_p$ and $n_p$. Then using (a) we should have

$$F(X\,;\; m'_p,\, n'_p\,;\; m'_{p-1},\, n'_{p-1}\,;\; \ldots;\; m'_1,\, n'_1)$$

$$= -\frac{1}{n_p-1}\,F(X\,;\, m'_p,\, n'_p\,;\; \ldots;\; m'_{s+1}+m_p,\; n'_{s+1}+n_p-1\,;\; m'_{s-1},\, n'_{s-1}\,;\; \ldots;\; m'_1,\, n'_1)$$

$$+\frac{1}{n_p-1}\,F(X\,;\; m'_p,\, n'_p\,;\; \ldots;\; m'_{s+1},\, n'_{s+1}\,;\; m'_{s-1}+m_p,\; n'_{s-1}+n_p-1\,;\; \ldots;\; m'_1,\, n'_1)$$

$$+\frac{m_p}{n_p-1}\,F(X\,;\, m'_p,\, n'_p\,;\, m'_{p-1},\, n'_{p-1}\,;\ldots;\, m'_{s+1},\, n'_{s+1}\,;\; m_p-1,\; n_p-1\,;\; m'_{s-1},\, n'_{s-1}\,;\; \ldots;\; m'_1,\, n'_1). \quad (3)$$

The above relation will hold for $s = p-1,\;\; p-2,\, \ldots,\, 2.$

For $s = p$ we should have a slightly different result:

$$F(X;\ m_p, n_p;\ m'_{p-1}, n'_{p-1};\ \dots;\ m'_1, n'_1)$$
$$= -\frac{1}{n_p-1} F_0(X;\ m_p, n_p-1).F(X;\ m'_{p-1}, n'_{p-1};\ m'_{p-2}, n'_{p-2};\ \dots;\ m'_1, n'_1)$$
$$+\frac{1}{n_p-1} F(X;\ m_p+m'_{p-1}, n_p+n'_p-1;\ m'_{p-2}, n'_{p-2};\ \dots;\ m'_1, n'_1)$$
$$+\frac{m_p}{n_p-1} F(X;\ m_p-1, n_p-1;\ m'_{p-1}, n'_{p-1};\ \dots;\ m'_1, n'_1), \tag{3.1}$$

where $F_0(X;\ m_p, n_p-1)$ stands for $X^{m_p}/(1+X)^{n_p-1}$.

For $s = 1$ we should have

$$F(X;\ m'_p, n'_p;\ m'_{p-1}, n'_{p-1};\ \dots;\ m_p, n_p)$$
$$= -\frac{1}{n_p-1} F(X;\ m'_p, n'_p;\ \dots;\ m'_2+m_p, n'_2+n_p-1)$$
$$+\frac{m_p}{n_p-1} F(X;\ m'_p, n'_p;\ \dots;\ m'_k, n'_k;\ \dots;\ m'_2, n'_2;\ m_p-1, n_p-1). \tag{3.2}$$

The left hand side of (3), (3.1) and (3.2) are all $p$-fold incomplete integrals, the first two terms on the right hand side of (3), the second term and the second factor of the first term on the right hand side of (3.1) are all $(p-1)$-fold incomplete integrals. The last terms on the right hand side of (3)-(3.2) are again $p$-fold incomplete integrals but with the largest pair $(m_p, n_p)$ reduced by 1 each to $(m_p-1, n_p-1)$. As a more convenient notation one can replace

(i) $F(X;\ m'_p, n'_p;\ \dots;\ m'_{s+1}+m_p, n'_{s+1}+n_p-1;\ m'_{s-1}, n'_{s-1};\ \dots;\ m'_1, n'_1)$

by

$$F(X;\ m'_p, n'_p;\ \dots;\ m'_{s+1}, n'_{s+1};\ \overleftarrow{m_p}, \overleftarrow{n_p-1};\ m'_{s-1}, n'_{s-1};\ \dots;\ m'_1, n'_1), \tag{3.3}$$

where $\overleftarrow{m_p}$ is to be summed with '$m$' on the left and $\overleftarrow{n_p-1}$ with '$n$' on the left.

(ii) $F_0(X;\ m_p, n_p-1).F(X;\ m'_{p-1}, n'_{p-1};\ \dots;\ m'_1, n'_1)$

by

$$F(X;\ \overleftarrow{m_p}, \overleftarrow{n_p-1};\ m'_{p-1}, n'_{p-1};\ \dots;\ m'_1, n'_1). \tag{3.4}$$

(iii) $F(X;\ m'_p, n'_p;\ \dots;\ m'_{s+1}, n'_{s+1};\ m'_{s-1}+m_p, n'_{s-1}+n_p-1;\ m'_{s-2}, n'_{s-2};\ \dots;\ m'_1, n'_1)$

by

$$F(X;\ m'_p, n'_p;\ \dots;\ m'_{s+1}, n'_{s+1};\ \overrightarrow{m_p}, \overrightarrow{n_p-1};\ m'_{s-1}, n'_{s-1};\ \dots;\ m'_1, n'_1), \tag{3.5}$$

where $\overrightarrow{m_p}$ is to be summed with '$m$' to the right and $\overrightarrow{n_p-1}$ with '$n$' to the right.

Using the results (3)-(3.2) and the notation just introduced we see that

$$F\left\{X;\ \left\|\begin{matrix} m_p, n_p & m_{p-1}, n_{p-1} & \dots\dots & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & \dots\dots & m_1, n_1 \\ \dots & \dots & \dots & \dots \\ m_p, n_p & m_{p-1}, n_{p-1} & \dots\dots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$
= -\frac{1}{n_p-1} F \left\{ X; \left\| \begin{array}{cccc}
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1\ n_1
\end{array} \right\| \right\}
$$

$$
+ \frac{1}{n_p-1} F \left\{ X; \left\| \begin{array}{cccc}
\overrightarrow{m_p}, \overrightarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\overrightarrow{m_p}, \overrightarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
\square & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1
\end{array} \right\| \right\}
$$

$$
+ \frac{m_p}{n_p-1} F \left\{ X; \left\| \begin{array}{cccc}
m_p-1, n_p-1 & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
m_p-1, n_p-1 & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
m_p-1, n_p-1 & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1
\end{array} \right\| \right\} \qquad (3.6)
$$

where in the second term on the right hand side of (3.6), $\square$ in the pseudo-determinant is a dummy to denote that the corresponding term in the formal expansion is not to be considered at all. It is introduced to express the pseudo-determinant in a complete form.

(3.6) can now be simplified thus:

$$
F \left\{ X; \left\| \begin{array}{cccc}
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1 \\
\cdots, \cdots & \cdots, \cdots & \cdots\cdots & \cdots, \cdots \\
\overleftarrow{m_p}, \overleftarrow{n_p-1} & m_{p-1}, n_{p-1} & \cdots\cdots & m_1, n_1
\end{array} \right\| \right\}
$$

$$
= F_0(X;\ m_p, n_p).F \left\{ X; \left\| \begin{array}{cccc}
m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_1, n_1 \\
\cdots, \cdots & \cdots, \cdots & \cdots\cdots & \cdots, \cdots \\
m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_1, n_1
\end{array} \right\| \right\}
$$

$$
-F \left\{ X; \left\| \begin{array}{ccccccc}
m_{p-1}+m_p, n_{p-1}+n_p-1 & & m_{p-2}+m_p, n_{p-2}+n_p-1 & & \cdots & m_1+m_p, n_1+n_p-1 & \\
m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1 \\
\cdots, & \cdots & \cdots, & \cdots & \cdots & \cdots, & \cdots \\
m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1
\end{array} \right\| \right\}
$$

$$+F\left\{X;\left\|\begin{matrix} m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1 \\ m_{p-1}+m_p, & n_{p-1}+n_p-1 & m_{p-2}+m_p, & n_{p-2}+n_p-1 & \cdots & m_1+m_p, & n_1+n_p-1 \\ \cdots\cdots, & \cdots\cdots & \cdots\cdots, & \cdots\cdots & \cdots & \cdots, & \cdots \\ m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1 \end{matrix}\right\|\right\}$$

$$-\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\ +\ \cdots\cdots\cdots\cdots\cdots$$

$$\pm F\left\{X;\left\|\begin{matrix} m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1 \\ m_{p-1}, & n_{p-1} & m_{p-2}, & n_{p-2} & \cdots & m_1, & n_1 \\ \cdots\cdots, & \cdots\cdots & \cdots\cdots, & \cdots\cdots & \cdots & \cdots, & \cdots \\ m_{p-1}+m_p, & n_{p-1}+n_p-1 & m_{p-2}+m_p, & n_{p-2}+n_p-1 & \cdots & m_1+m_p, & n_1+n_p-1 \end{matrix}\right\|\right\}$$

(the last term being $+ve$ or $-ve$ according as $p$ is odd or even)

$$= F_0(X;\ m_p,\ n_p).F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots\cdots & m_1, n_1 \\ \cdots\cdots, \cdots\cdots & \cdots\cdots; \cdots\cdots & \cdots\cdots\cdots & \cdots, \cdots \\ m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$-F(X;\ m_{p-1}+m_p,\ n_{p-1}+n_p-1).F\left\{X;\left\|\begin{matrix} m_{p-2}, n_{p-2} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \\ m_{p-2}, n_{p-2} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ m_{p-2}, n_{p-2} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$+F(X;\ m_{p-2}+m_p,\ n_{p-1}+n_p-1).F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \\ m_{p-1}, n_{p-1} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ m_{p-1}, n_{p-1} & m_{p-3}, n_{p-3} & \cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$-\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\ +\ \cdots\cdots\cdots\cdots\cdots$$

$$\pm F(X;\ m_1+m_p,\ n_{p-1}+n_p-1).F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_2, n_2 \\ m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_2, n_2 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_2, n_2 \end{matrix}\right\|\right\}$$

$$= F_0(X;\ m_p,\ n_p).F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_1, n_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$+\sum_{s=p-1}^{1}(-1)^{p-s-1}F(X\,;m_s+m_p,\,n_s+n_p-1)$$

$$\times F\left\{X;\left\|\begin{matrix} m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\\ m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\} \quad (3.7)$$

Similarly

$$F\left\{X;\left\|\begin{matrix} \overrightarrow{m_p},\overrightarrow{n_p-1} & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\\ \overrightarrow{m_p},\overrightarrow{n_p-1} & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ \overrightarrow{m_p},\overrightarrow{n_p-1} & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\}$$

$$=\sum_{s=p-1}^{1}(-1)^{p-s-1}F(X;\;m_s+m_p,\;n_s+n_p-1)$$

$$\times F\left\{X;\left\|\begin{matrix} m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\\ m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\} \quad (3.8)$$

Combining (3.7) and (3.8) we have now

$$F\left\{X;\left\|\begin{matrix} m_p,n_p & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_p,n_p & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\}$$

$$=-\frac{1}{n_p-1}F_0(X;\,m_p,n_p-1).F\left\{X;\left\|\begin{matrix} m_{p-1},n_{p-1} & m_{p-2},n_{p-2} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_{p-1},n_{p-1} & m_{p-2},n_{p-2} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\}$$

$$+\frac{2}{n_p-1}\sum_{s=p-1}^{1}(-1)^{p-s-1}F(X;\;m_s+m_p\;\;n_s+n_p-1)$$

$$\times F\left\{X;\left\|\begin{matrix} m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_{p-1},n_{p-1} & \cdots\cdots & m_{s+1},n_{s+1} & m_{s-1},n_{s-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\}$$

$$+\frac{m_p}{n_p-1}.F\left\{X;\left\|\begin{matrix} m_p-1,n_p-1 & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\\ m_p-1,n_p-1 & m_{p-1},n_{p-1} & \cdots\cdots & m_1,n_1\end{matrix}\right\|\right\} \quad (4)$$

Proceeding along the reccursion chain suggested by (4) we should have

$$F\left\{X;\left\|\begin{matrix} m_p, n_p & m_{p-1}, n_{p-1} & \cdots\cdots\cdots & m_1, n_1 \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots\cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\}$$

$$= -F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots\cdots\cdots & m_1, n_1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ m_{p-1}, n_{p-1} & m_{p-2}, n_{p-2} & \cdots\cdots\cdots\cdots & m_1, n_1 \end{matrix}\right\|\right\} Q$$

$$+\sum_{s=p-1}^{1}(-1)^{p-s-1} F\left\{X;\left\|\begin{matrix} m_{p-1}, n_{p-1} & \cdots & m_{s+1}, n_{s+1} & m_{s-1}, n_{s-1} & \cdots & m_1, n_1 \\ \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots \\ m_{p-1}, n_{p-1} & \cdots & m_{s+1}, n_{s+1} & m_{s-1}, n_{s-1} & \cdots & m_1, n_1 \end{matrix}\right\|\right\} R \tag{4.1}$$

where

$$Q = \sum_{r=1}^{m_p-m_{p-1}} \{{}_{m_p}P_{r-1}/{}_{n_p-2+r}P_r\} F_0(X;\ m_p-r+1,\ n_p-r), \tag{4.2}$$

and

$$R = \sum_{r=1}^{m_p-m_{p-1}} \{{}_{m_p}P_{r-1}/{}_{n_p-2+r}P_r\} F(X;\ m_s+m_p-r+1,\ n_s+n_p-r) \tag{4.3}$$

${}_nP_r$ being the pemutation of $n$ things taken $r$ at a time.

The reduction of the $p$-fold pseudo-determinant is thus thrown back on that of $(p-1)$ and $(p-2)$-fold integrals and these again back on $(p-2)$ and $(p-3)$-fold and so on, until we get to one fold integrals, that is, to functions of the type $F(X;\ m,\ n)$ which are immediately evaluated from the incomplete $B$-function tables.

The method indicated is such that in (1.4) we could have reduced (by successive integration by parts) any column to the one immediately following it and not merely the first column to the second. As soon as one column becomes equal to the consecutive one the pseudo-determinant, of course, becomes zero, which is really the secret of the reduction outlined. As pointed out earlier this method can be used (without any difficulty) by the student of mathematics to get exactly similar reductions for (1.5) and (1.6). Such reduction has actually been made by the author for statistical purposes but is hardly worth-while giving in full for the mathematical readers.

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# ON SOME INTEGRALS INVOLVING BESSEL FUNCTIONS

By

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The object of the present note is to obtain certain integrals involving Bessel Functions. The method adopted is that of the Operational Calculus. It may be noted here that changes in the order of integration of the infinite integrals that are effected can be easily justified on account of the absolute convergence of the double integrals.

1. A given function $h(t)$ is related symbolically to another function $f(p)$, given by the Laplace transform

$$f(p) = p\int_0^\infty e^{-pt}\,h(t)\,dt, \tag{1}$$

where it is assumed that the integral converges (Mellin, 1902).

The relation between $f(p)$ and $h(t)$ may be denoted by

$$f(p) \doteqdot h(t).$$

When $f(p)$ is known, $h(t)$ may be found back by means of the Bromwich-Wagner Theorem (Bromwich, 1916)

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{f(p)}{p}\,e^{pt}\,dp. \tag{2}$$

2. We have (Watson, 1922, p. 425)

$$\int_0^\infty \frac{J_0(az)dz}{z^2+k^2} = \frac{\pi}{2k}[I_0(ak)-L_0(ak)], \quad R(k)>0, \quad a>0.$$

Writing $k = \sqrt{p}$ and multiplying both sides by $p$, we get

$$\int_0^\infty \frac{pJ_0(az)dz}{z^2+p} = \frac{\pi}{2}\sqrt{p}.[I_0(a\sqrt{p})-L_0(a\sqrt{p})]. \tag{3}$$

Since we have

$$\frac{p}{p+z^2} \doteqdot \exp(-z^2t),$$

by changing the order of integration of the double integral

$$\int_0^\infty J_0(az)dz\int_0^\infty \exp\{-(z^2+p)t\}dt,$$

we get

$$\int_0^\infty \frac{pJ_0(az)dz}{p+z^2} \doteqdot \int_0^\infty J_0(az)\exp(-z^2t)\,dz.$$

Noting that

$$\int_0^\infty \exp(-z^2t).J_0(az)dz = \frac{1}{2}\sqrt{\frac{\pi}{t}}\exp(-a^2/8t).I_0(a^2/8t),$$

and

$$\sqrt{(p)}.L_0(a\sqrt{p}) \doteqdot 0,$$

we get, on interpretation of (3),

$$\sqrt{(p)}.I_0(a\sqrt{p}) \doteqdot \frac{1}{\sqrt{(\pi t)}}\exp(-a^2/8t).I_0(a^2/8t). \tag{4}$$

Again we have (Bose, 1944)

$$\int_0^\infty \frac{J_{2n+1}(az)\,dz}{z(z^2+k^2)} = \frac{(-1)^n\pi}{2k^2}[I_{2n+1}(ka)-L_{2n+1}(ka)].$$

Putting $k = \sqrt{p}$ and then on interpretation of the result as before, we get after a bit of reduction

$$I_{2n+1}(a\sqrt{p}) \doteqdot \frac{(-1)^n}{2(2n+1)}\frac{a}{\sqrt{(\pi t)}}\exp(-a^2/8t).[I_n(a^2/8t)+I_{n+1}(a^2/8t)], \tag{5}$$

since it can be easily seen that

$$L_{2n+1}(\sqrt{p}) \doteqdot 0.$$

We have proved before (Bose, 1944)

$$\int_0^1 P_n(1-2y^2)yk[I_0(ky)-L_0(ky)]dy = (-1)^n[I_{2n+1}(k)-L_{2n+1}(k)].$$

Let us put $k = \sqrt{p}$ and proceed exactly as before. On interpretation and writing $a$ for $1/8t$ and $y$ for $y^2$, we get

$$\int^1 P_n(1-2y)e^{-ay}I_0(ay)dy = \frac{1}{2n+1}e^{-a}[I_n(a)+I_{n+1}(a)]. \tag{6}$$

Let us next consider the relation (Watson, 1922, p. 338)

$$I_{2n+1}(\sqrt{p})-L_{2n+1}(\sqrt{p}) = \frac{1}{2^{2n}\Gamma(\frac{1}{2})\Gamma(2n+\frac{3}{2})}\int_0^{\pi/2} p^{n+\frac{1}{2}}\exp(-\sqrt{p}.\cos\theta).\sin^{4n+2}\theta\,d\theta. \tag{7}$$

Goldstein (1931) has proved that

$$p^{\mu/2}\exp(-\sqrt{p}.r) \doteqdot \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{(2t)^{\frac{1}{2}\mu}}\exp(-r^2/4t).D_{\mu-1}\left(\frac{r}{\sqrt{(2t)}}\right),$$

whence we have

$$p^{n+1/2}\exp(-\sqrt{p}.\cos\theta) \doteqdot \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{(2t)^{n+1/2}}\exp(-\cos^2\theta/8t).D_{2n}\left(\frac{\cos\theta}{\sqrt{(2t)}}\right).$$

On interpretation of both sides of (7), we get, after some simplification and writing $x = 1/\{2\sqrt{(2t)}\}$;

$$\int_0^{\pi/2}\exp(-x^2\cos^2\theta).D_{2n}(2x\cos\theta)\sin^{4n+2}\theta\,d\theta = \frac{(-1)^n\sqrt{\pi}.\Gamma(2n+\frac{3}{2})}{2(2n+1)x^{2n}} \times\exp(-x^2).[I_n(x^2)+I_{n+1}(x^2)]. \tag{8}$$

Making use of the relation

$$2nI_n(z) = z[I_{n-1}(z)-I_{n+1}(z)],$$

we get from (5)

$$\sqrt{p}.[I_{2n}(\sqrt{p})-I_{2n+2}(\sqrt{p})] \doteqdot \frac{(-1)^n}{\sqrt{(\pi t)}}\exp(-1/8t).[I_{n+1}(1/8t)+I_n(1/8t)].$$

Giving to $n$ the values 0, 1, ..., $n-1$ in succession and adding and making use of (4), we get

$$\sqrt{p}.I_{2n}(\sqrt{p}) \doteqdot \frac{(-1)^n}{\sqrt{(\pi t)}} \exp(-1/8t).I_n(1/8t). \tag{9}$$

Employing (9) and the relation

$$\frac{p}{(p+a)^{m+1}} \doteqdot \frac{e^{-at}t^m}{\Gamma(m+1)}, \; R(m+1) \geq 0,$$

in Goldstein's result (1931)

$$\int_0^\infty \frac{\phi(t)g(t)}{t}\,dt = \int_0^\infty \frac{\psi(t)f(t)}{t}\,dt,$$

where $\qquad \phi(p) \doteqdot f(t), \; \psi(p) \doteqdot g(t),$

and then making use of the result

$$\int_0^\infty J_n(at)t^{n+1}\exp(-pt^2)\,dt = \frac{a^n}{(2p)^{n+1}}\exp(-a^2/4p), \; (R(n) \geq -1), \text{ when } a = i,$$

we get, after considerable simplification,

$$\int_0^\infty \frac{e^{-y}I_n(y)y^n}{(a+y)^{n+3/2}}dy = \frac{(-1)^n\sqrt{\pi}.2^n.a^{n-1/2}}{\Gamma(n+\frac{3}{2})}e^{2a}, \; (a>0). \tag{10}$$

We now write $b^2/a^2$ for $a$ and $y^2$ for $y$ in (10) and multiply both sides by $b^{2n+3}.\exp(-\frac{1}{2}b^2)$. Integrating with respect to $b$ between the limits zero and infinity and changing the order of integration and making use of the result (Varma, 1937)

$$\int_0^\infty \frac{x^{2n}\exp(-\frac{1}{2}x^2)}{(x^2+a^2)^n}\,dx = 2^{n-1/2}\Gamma(n+\tfrac{1}{2})\exp(\tfrac{1}{4}a^2).D_{-2n}(a), \; (n > -\tfrac{1}{2}),$$

we get after some reductions

$$\int_0^\infty \exp\{(a^2-1)y^2\}.I_n(y^2)\,D_{-2n-3}(2ay)y^{2n+1}dy = \frac{(-1)^n\Gamma(2n+\frac{3}{2})a}{2\sqrt{2}.\Gamma(2n+3)(a^2-1)^{2n+3/2}}, \tag{11}$$

($n \geq -\frac{1}{2}$; $a$ is real and $|a| > 1$).

We have the relation (Watson, 1922, p. 338)

$$\int_0^{\pi/2} \{I_0(x\sin\theta) - L_0(x\sin\theta)\}\sin\theta\,d\theta = \frac{1-e^{-x}}{x}, \tag{11'}$$

whence writing $\sqrt{p}$ for $x$, we have

$$\int_0^{\pi/2} \sqrt{p}.\{I_0(\sqrt{p}.\sin\theta) - L_0(\sqrt{p}.\sin\theta)\}\sin\theta\,d\theta = 1-\exp(-\sqrt{p}). \tag{12}$$

On interpretation and making use of the result

$$\exp(-\sqrt{p}) \doteqdot \left(\frac{2}{\pi}\right)^{\frac{1}{2}}.\exp(-1/4t).D_{-1}\left(\frac{1}{\sqrt{(2t)}}\right)$$

and writing $1/8t = a^2$, we get

$$\int_0^{\pi/2} \exp(-a^2\sin^2\theta).I_0(a^2\sin^2\theta)\sin\theta\,d\theta = \frac{\sqrt{\pi}}{2\sqrt{2}.a}\{1-\sqrt{(2/\pi)}.\exp(-2a^2).D_{-1}(2a)\}. \tag{13}$$

Again multiplying both sides of (11′) by $x$ and writing $ax$ for $x$, we get

$$\int_0^{\pi/2} ax\{I_0(ax \sin\theta) - L_0(ax\sin\theta)\}\sin\theta\, d\theta = 1 - e^{-ax}.$$

Multiplying both sides by $P_n(1-2x^2)$ and integrating with respect to $x$ between the limits zero and one, and making use of the results (Cooke, 1924)

$$\int_0^1 P_n(1-2x^2)x^{2m+1}dx = \frac{(-1)^n\{\Gamma(m+1)\}^2}{2\Gamma(m-n+1)\Gamma(m+n+2)},\ (m > -1),$$

$$\int_0^1 P_n(1-2x^2)e^{-ax}\,dx = I_{n+\frac{1}{2}}(\tfrac{1}{2}a)K_{n+\frac{1}{2}}(\tfrac{1}{2}a),$$

we get, on writing $2a$ for $a$,

$$\int_0^{\pi/2}\{I_{2n+1}(2a\sin\theta) - L_{2n+1}(2a\sin\theta)\}d\theta = (-1)^n\left[\frac{1}{2n+1} - I_{n+\frac{1}{2}}(a)K_{n+\frac{1}{2}}(a)\right]. \quad (14)$$

Writing $a = \sqrt{p}$, we deduce for the above

$$\int_0^{\pi/2}\left[\frac{4n\sqrt{p}.I_{2n}(2\sqrt{p}.\sin\theta)}{\sin\theta} - p\{L_{2n+1}(2\sqrt{p}.\sin\theta) - L_{2n-1}(2\sqrt{p}.\sin\theta)\}\right]d\theta$$

$$= (-1)^n\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right)p + (-1)^n p\{I_{n+\frac{1}{2}}(\sqrt{p})K_{n+\frac{1}{2}}(\sqrt{p}) + I_{n-\frac{1}{2}}(\sqrt{p})K_{n-\frac{1}{2}}(\sqrt{p})\},\ (n>0). \quad (15)$$

With the help of the result

$$\int_0^\infty J^2{}_{n+\frac{1}{2}}(az)z\,\exp(-z^2t)dz = a^{n+1/2}\exp(-a/2t).I_{n+\frac{1}{2}}(a/2t),$$

it can be easily shown that

$$pI_{n+\frac{1}{2}}(\sqrt{p})K_{n+\frac{1}{2}}(\sqrt{p}) \doteqdot \frac{1}{t}\exp(-1/2t).I_{n+\frac{1}{2}}(1/2t).$$

On interpretation of (15) and writing $a = 1/2t$, we get

$$\int_0^{\pi/2} I_n(a\sin^2\theta)\exp(-a\sin^2\theta).\frac{d\theta}{\sin\theta} = \frac{1}{2}\sqrt{\left(\frac{\pi a}{2n^2}\right)}e^{-a}\{I_{n+\frac{1}{2}}(a) + I_{n-\frac{1}{2}}(a)\},\ (n>0) \quad (16)$$

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# INFORMATION AND THE ACCURACY ATTAINABLE IN THE ESTIMATION OF STATISTICAL PARAMETERS

By

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*Introduction*

The earliest method of estimation of statistical parameters is the method of least squares due to Markoff. A set of observations whose expectations are linear functions of a number of unknown parameters being given, the problem which Markoff posed for solution is to find out a linear function of observations whose expectation is an assigned linear function of the unknown parameters and whose variance is a minimum. There is no assumption about the distribution of the observations except that each has a finite variance.

A significant advance in the theory of estimation is due to Fisher (1921) who introduced the concepts of *consistency*, *efficiency* and *sufficiency* of estimating functions and advocated the use of the maximum likelihood method. The principle accepts as the estimate of an unknown parameter $\theta$, in a probability function $\phi(\theta)$ of an assigned type, that function $t(x_1, \ldots, x_n)$ of the sampled observations which makes the probability density a maximum. The validity of this principle arises from the fact that out of a large class of unbiassed estimating functions following the normal distribution the function given by maximising the probability density has the least variance. Even when the distribution of $t$ is not normal the property of minimum variance tends to hold as the size of the sample is increased.

Taking the analogue of Markoff's set up Aitken (1941) proceeded to find a function $t(x_1, \ldots, x_n)$ such that

$$\int t\phi(\theta)\pi dx_i = \theta$$

and

$$\int (t-\theta)^2 \phi(\theta)\pi\, dx_i \quad \text{is minimum.}$$

Estimation by this method was possible only for a class of distribution functions which admit sufficient statistics. Some simple conditions under which the maximum likelihood provides an estimate accurately possessing the minimum variance, even though the sample is finite and the distribution of the estimating function is not normal, have emerged.

The object of the paper is to derive certain inequality relations connecting the elements of the *Information Matrix* as defined by Fisher (1921) and the variances and covariances of the estimating functions. A class of distribution functions which admit estimation of parameters with the minimum possible variance has been discussed.

The concept of distance between populations of a given type has been developed starting from a quadratic differential metric defining the element of length.

*Estimation by minimising variance*

Let the probability density $\phi(x_1, \ldots, x_n; \theta)$ for a sample of $n$ observations contain a parameter $\theta$ which is to be estimated by a function $t = f(x_1, \ldots, x_n)$ of the observations. This estimate may be considered to be the best, if with respect to any other function $t'$, independent of $\theta$, the probabilities satisfy the inequality

$$P(\theta-\lambda_1 < t < \theta+\lambda_2) \nless P(\theta-\lambda_1 < t' < \theta+\lambda_2) \tag{2.1}$$

for all positive $\lambda_1$ and $\lambda_2$ in an interval $(0, \lambda)$. The choice of the interval may be fixed by other considerations depending on the frequency and magnitude of the departure of $t$ from $\theta$. If we replace the condition (2.1) by a less stringent one that (2.1) should be satisfied for all $\lambda$ we get as a necessary condition that

$$E(t-\theta)^2 \ngtr E(t'-\theta)^2, \tag{2.2}$$

where $E$ stands for the mathematical expectation. We may further assume the property of unbiassedness of the estimating functions *viz.*, $E(t)=\theta$, in which case the function $t$ has to be determined subject to the conditions $E(t)=\theta$ and $E(t-\theta)^2$ is minimum.

As no simple solution exists satisfying the postulate (2.1) the inevitable arbitrariness of these postulates of unbiassedness and minimum variance needs no emphasis. The only justification for selecting an estimate with minimum variance from a class of unbiassed estimates is that a necessary condition for (2.1) with the further requirement that $E(t)=\theta$ is ensured. The condition of unbiassedness is particularly defective in that many biassed estimates with smaller variances lose their claims as estimating functions when compared with unbiassed estimates with greater variances. There are, however, numerous examples where a slightly biassed estimate is preferred to an unbiassed estimate with a greater variance. Untill a unified solution of the problem of estimation is set forth we have to subject the estimating functions to a critical examination as to its bias, variance and the frequency of a given amount of departure of the estimating function from the parameter before utilising it.

*Single parameter and the efficiency attainable*

Let $\phi(x_1, \ldots, x_n)$ be the probability density of the observations $x_1, x_2, \ldots, x_n$, and $t(x_1, \ldots, x_n)$ be an unbiassed estimate of $\theta$. Then

$$\int \cdots \int t\phi\pi\, dx_i = \theta. \tag{3.1}$$

Differentiating with respect to $\theta$ under the integral sign, we get

$$\int \cdots \int t \frac{d\phi}{d\theta} \pi\, dx_i = 1 \tag{3.2}$$

if the integral exists, which shows that the covariance of $t$ and $\dfrac{1}{\phi}\dfrac{d\phi}{d\theta}$ is unity. Since the square of the convariance of two variates is not greater than the product of the variances of the variates we get using $V$ and $C$ for variance and covariance

$$V(t)V\left(\frac{1}{\phi}\frac{d\phi}{d\theta}\right) \nless \left\{C\left(t, \frac{1}{\phi}\frac{d\phi}{d\theta}\right)\right\}^2 \tag{3.3}$$

which gives that

$$V(t) \nless 1/I$$

where

$$I = V\left(\frac{1}{\phi}\frac{d\phi}{d\theta}\right) = E\left\{-\frac{d^2 \log \phi}{d\theta^2}\right\} \tag{3.4}$$

is the intrinsic accuracy defined by Fisher (1921). This shows that *the variance of any unbiassed estimate of $\theta$ is greater than the inverse of $I$ which is defined independently of any method of estimation.* The assumption of the normality of the distribution function of the estimate is not necessary.

If instead of $\theta$ we are estimating $f(\theta)$, a function of $\theta$, then

$$V(t) \nless \{f'(\theta)\}^2/I. \tag{3.5}$$

If there exists a sufficient statistic $T$ for $\theta$ then the necessary and sufficient condition is that $\phi(x;\ \theta)$ the probability density of the sample observations satisfies the equality

$$\phi(x;\ \theta) = \Phi(T,\ \theta)\psi(x_1,\ \ldots,\ x_n), \tag{3.6}$$

where $\psi$ does not involve $\theta$ and $\Phi(T, \theta)$ is the probability density of $T$. If $t$ is an unbiassed estimate of $\theta$ then

$$\theta = \int t\phi\pi dx_i = \int f(T)\Phi(T,\ \theta)dT \tag{3.7}$$

which shows that there exists a function $f(T)$ of $T$, independent of $\theta$ and is an unbiassed estimate of $\theta$. Also

$$\int (t-\theta)^2\phi\pi dx_i = \int [t-f(T)]^2\phi\pi dx_i + \int [f(T)-\theta]^2\Phi(T,\ \theta)dT \geq \int [f(T)-\theta]^2\Phi(T,\ \theta)dT, \tag{3.8}$$

which shows that

$$V[f(T)] \ngtr V(t) \tag{3.9}$$

and hence we get the result that *if a sufficient statistic and an unbiassed estimate exist for $\theta$, then the best unbiassed estimate of $\theta$ is an explicit function of the sufficient statistic.* It usually happens that instead of $\theta$, a certain function of $\theta$ can be estimated by this method for a function of $\theta$ may admit an unbiassed estimate.

It also follows that if $T$ is a sufficient statistic for $\theta$ and $E(T) = f(\theta)$, then there exists no other statistic whose expectation is $f(\theta)$ with the property that its variance is smaller than that of $T$.

It has been shown by Koopman (1936) that under certain conditions, the distribution function $\phi(x, \theta)$ admitting a sufficient statistic can be expressed as

$$\phi(x,\ \theta) = \exp\ (\Theta_1 X_1 + \Theta_2 + X_2), \tag{3.10}$$

where $X_1$ and $X_2$ are functions of $x_1, x_2, \ldots, x_n$ only and $\Theta_1$ and $\Theta_2$ are functions of $\theta$ only. Making use of the relation

$$\int \exp(\Theta_1 X_1 + \Theta_2 + X_2)\,\pi dx_i = 1, \tag{3.11}$$

we get

$$E(X_1) = -\frac{d\Theta_2}{d\Theta_1} \text{ and } V(X_1) = -\frac{d^2\Theta_2}{d\Theta_1^{\,2}}. \tag{3.12}$$

If we choose $-\dfrac{d\Theta_2}{d\Theta_1}$ as the parameter to be estimated we get the minimum variance attainable is by (3.5)

$$\left\{\frac{d}{d\theta}\,\frac{d\Theta_2}{d\Theta_1}\right\}^2 \Big/ \left\{\frac{d^2\Theta_2}{d\Theta_1^{\,2}}\cdot\frac{d\Theta_1}{d\theta}\right\} = -\frac{d^2\Theta_2}{d\Theta_1^{\,2}} = V(X_1). \tag{3.13}$$

Hence $X_1$ is the best unbiassed estimate of $-\dfrac{d\Theta_2}{d\Theta_1}$. Thus for *the distributions of the type* (3.10), *there exists a function of the observations which has the maximum precision as an estimate of a function of* $\theta$.

*Case of several parameters*

Let $\theta_1, \theta_2, \ldots, \theta_q$ be $q$ unknown parameters occuring in the probability density $\phi(x_1, \ldots, x_n;\ \theta_1, \theta_2, \ldots, \theta_q)$ and $t_1, t_2, \ldots, t_q$ be $q$ functions independent of $\theta_1, \theta_2, \ldots, \theta_q$ such that

$$\int \cdots \int t_i\, \phi \pi\, dx_j = \theta_i. \tag{4.1}$$

Differentiating under the integral sign with respect to $\theta_i$ and $\theta_j$, we get, if the following integrals exist,

$$\int \cdots \int t_i \frac{\partial \phi}{\partial \theta_i} \pi\, dx_k = 1, \tag{4.2}$$

and

$$\int \cdots \int t_i \frac{\partial \phi}{\partial \theta_j} \pi\, dx_k = 0. \tag{4.4}$$

Defining

$$E\left[-\frac{\partial^2 \log \phi}{\partial \theta_i \partial \theta_j}\right] = I_{ij}, \tag{4.4}$$

and

$$E(t_i - \theta_i)(t_j - \theta_j) = V_{ij}, \tag{4.5}$$

we get the result that the matrix of the determinant

$$\begin{vmatrix} V_{ii} & 0 & \ldots & 1 & \ldots & 0 \\ 0 & I_{11} & \ldots & I_{1i} & \ldots & I_{1q} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & I_{i1} & \ldots & I_{ii} & \ldots & I_{iq} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & I_{q1} & \ldots & I_{q2} & \ldots & I_{qq} \end{vmatrix} \tag{4.6}$$

being the dispersion matrix of the stochastic variates $t_i$ and $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}$ $(j = 1, 2, \ldots, q)$ is positive definite or semi-definite. If we assume that there is no linear relationship of the type

$$\Sigma\, \lambda_j \frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j} = 0 \tag{4.7}$$

among the variables $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}$ $(i = 1, 2, \ldots, q)$ then the matrix $\| I_{ij} \|$, which is known as the information matrix due to $\theta_1, \theta_2, \ldots, \theta_q$, is positive definite in which case there exists a matrix $\| I^{ij} \|$ inverse to $\| I_{ij} \|$. From (4.6) we derive that

$$V_{ii} - I^{ii} \geq 0 \tag{4.8}$$

which shows *that minimum variance attainable for the estimating function of $\theta_i$ when $\theta_1, \theta_2, \ldots, \theta_q$ are not known is $I^{ii}$, the element in the i-th row and the i-th column of the matrix $\| I^{ij} \|$ inverse to the information matrix $\| I_{ij} \|$.*

The equality is attained when

$$t_i - \theta_i = \Sigma\, \mu_j \frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}. \tag{4.9}$$

We can obtain a generalisation of (4.8) by considering the dispersion matrix of $t_1, t_2, \ldots, t_i$ and $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_r}$ $(r = 1, 2, \ldots, q)$

$$\begin{vmatrix} V_{11} & \cdots & V_{1i} & 1 & 0 & \cdots & 0 & \cdots & 0 \\ V_{21} & \cdots & V_{2i} & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ V_{i1} & \cdots & V_{ii} & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 1 & \cdots & 0 & I_{11} & I_{12} & \cdots & I_{1i} & \cdots & I_{1q} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & I_{q1} & I_{q2} & \cdots & I_{qi} & \cdots & I_{qq} \end{vmatrix} \tag{4.10}$$

This being positive definite or semi-definite we get the result that the determinant

$$| V_{rs} - I^{rs} | \geq 0, \; (r, s = 1, 2, \ldots, i) \tag{4.11}$$

for $i = 1, 2, \ldots, q$. The above inequality is evidently independent of the order of the elements so that, in particular, we get that the determinant

$$\begin{vmatrix} V_{ii} - I^{ii}, & V_{ij} - I^{ij} \\ V_{ji} - I^{ji}, & V_{jj} - I^{jj} \end{vmatrix} \geq 0, \tag{4.12}$$

which gives the result that if $V_{ii} = I^{ii}$, so that maximum precision is attainable for the estimation of $\theta_i$, then $V_{ij} = I^{ij}$ for $(j = 1, 2, \ldots, q)$.

In the case of the normal distribution

$$\phi(x\,;\, m, \sigma) = \text{const.}\ \exp - \tfrac{1}{2}\{\Sigma(x_i - m)^2/\sigma^2\}, \tag{4.13}$$

we have

$$I_{mm} = n/\sigma^2, \ I_{m\sigma} = 0, \ I_{\sigma\sigma} = 2n/\sigma^2. \tag{4.14}$$

Since the mean of observations $(x_1+x_2+\ldots+x_n)/n$ is the best unbiassed estimate of the parameter $m$ and the maximum precision is attainable *viz.*, $V_{mm} = I^{mm}$, it follows that any unbiassed estimate of the parameter $\sigma$ is uncorrelated with the mean of observations for $V_{m\sigma} = I^{m\sigma} = 0$. *Thus in the case of the univariate normal distribution any function of the observations whose expectation is a function of $\sigma$ and independent of $m$ is uncorrelated with the mean of the observations.* This can be extended to the case of multivariate normal populations where any unbiassed estimates of the variances and covariances are uncorrelated with the means of the observations for the several variates.

If there exists no functional relationships among the estimating functions $t_1, t_2, \ldots, t_q$ then $\| V^{ij} \|$ the inverse of the matrix $\| V_{ij} \|$ exists in which case we get that the determinant

$$| V^{rs} - I_{rs} |, \ (r, s = 1, 2, \ldots, i) \tag{4.15}$$

is greater than or equal to zero for $i = 1, 2, \ldots, q$, which is analogous to (4.11).

If a sufficient set of statistics $T_1, T_2, \ldots, T_q$ exist for $\theta_1, \theta_2, \ldots, \theta_q$ then we can show as in the case of a single parameter that the best estimating functions of the parameters or functions of parameters are explicit functions of the sufficient set of statistics.

Koopman (1936) has shown that under some conditions the distribution function $\phi\,(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_q)$ admitting a set of statistics $T_1, T_2, \ldots, T_q$ sufficient for $\theta_1, \theta_2, \ldots, \theta_q$ can be expressed in the form

$$\phi = \exp(\Theta_1 X_1 + \Theta_2 X_2 + \ldots + \Theta_q X_q + \Theta + X) \tag{4.16}$$

where $X$'s are independent of $\theta$'s and $\Theta$'s are independent of $x$'s. Making use of the relation

$$\int \phi\, dv = 1, \tag{4.17}$$

we get

$$\left.\begin{aligned} E(X_i) &= -\frac{\partial \Theta}{\partial \Theta_i}, \\ V(X_i) &= -\frac{\partial^2 \Theta}{\partial \Theta_i^2}, \\ \operatorname{cov}(X_i X_j) &= -\frac{\partial^2 \Theta}{\partial \Theta_i \partial \Theta_j}. \end{aligned}\right\} \tag{4.18}$$

This being the maximum precision available we get that for this class of distribution laws there exist functions of observations which are the best possible estimates of functions of parameters.

### *Loss of Information*

If $t_1, t_2, \ldots, t_q$, the estimates of $\theta_1, \theta_2, \ldots, \theta_q$, have the joint distribution $\Phi(t_1, t_2, \ldots,$

$t_q$; $\theta_1, \theta_2, \ldots, \theta_q$) then the information matrix on $\theta_1, \theta_2, \ldots, \theta_q$ due to $t_1, t_2, \ldots, t_q$ is $\| F_{ij} \|$ where

$$F_{ij} = E\left\{-\frac{\partial^2 \log \Phi}{\partial\theta_i \partial\theta_j}\right\}. \tag{5.1}$$

The equality

$$I_{ij} = (I_{ij} - F_{ij}) + F_{ij} \tag{5.2}$$

effects a partition of the covariance between $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}$ and $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}$ as within and between the regions formed by the intersection of the surfaces for constant values of $t_1, t_2, \ldots, t_q$. Hence we get that the matrices

$$\| I_{ij} - F_{ij} \| \text{ and } \| F_{ij} \| \tag{5.3}$$

which may be defined as the dispersion matrices of the quantities $\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}$ $(i = 1, 2, \ldots, q)$ within and between the meshes formed by the surfaces of constant values of $t_1, t_2, \ldots, t_q$, is positive definite or semidefinite. This may be considered as a generalisation of Fisher's inequality $I_{ii} \geq F_{ii}$ in the case of a single parameter.

If $I_{ii} = F_{ii}$, then it follows that $I_{ij} = F_{ij}$ for all $j$ for otherwise the determinant

$$\begin{vmatrix} I_{ii}-F_{ii} & I_{ij}-F_{ij} \\ I_{ij}-F_{ij} & I_{jj}-F_{jj} \end{vmatrix} < 0. \tag{5.4}$$

If in the determinant

$$| I_{ij} - F_{ij} |, \ (i, j = 1, 2, \ldots, q), \tag{5.5}$$

the zero rows and columns are omitted, the resulting determinant will be positive and less than the determinant obtained by omitting the corresponding rows and columns in $| I_{ij} |$. If we represent the resulting determinants by dashes, we may define the loss of information in using the statistics $t_1, t_2, \ldots, t_q$ as

$$| I_{ij} - F_{ij} |' / | I_{ij} |'. \tag{5.6}$$

If $\Phi$ is the joint distribution of $t_1, t_2, \ldots, t_q$ the estimates of $\theta_1, \theta_2, \ldots, \theta_q$ with the dispersion matrix $\| V_{ij} \|$ then we have the relations analogous to (4.11) and (4.15) connecting the elements of $\| V_{ij} \|$ and $\| F_{ij} \|$ defined above. Proceeding as before we get that the determinants

$$| V_{rs} - F^{rs} | \text{ and } | F_{rs} - V^{rs} |, \ (r, s = 1, 2, \ldots, i), \tag{5.7}$$

are greater than or equal to zero for all $i = 1, 2, \ldots, q$.

*The population space*

Let the distribution of a certain number of characters in a population be characterised by the probability differential

$$\phi(x, \theta_1, \ldots, \theta_q)dv. \tag{6.1}$$

The quantities $\theta_1, \theta_2, \ldots, \theta_q$ are called population parameters. Given the functional form in $x$'s as in (6.1) which determines the type of the distribution function, we can generate different populations by varying $\theta_1, \theta_2, \ldots, \theta_q$. If these quantities are represented in a space of $q$ dimensions, then a population may be identified by a point in this space which may be defined as the population space (P.S).

Let $\theta_1, \theta_2, \ldots, \theta_q$ and $\theta_1+d\theta_1, \theta_2+d\theta_2, \ldots, \theta_q+d\theta_q$ be two contiguous points in (P.S). At any assigned value of the characters of the populations corresponding to these contiguous points, the probability densities differ by

$$d\phi(\theta_1, \theta_2, \ldots, \theta_q) \tag{6.2}$$

retaining only first order differentials. It is a matter of importance to consider the relative discrepancy $d\phi/\phi$ rather than the actual discrepancy. The distribution of this quantity over the $x$'s summarises the consequences of replacing $\theta_1, \theta_2, \ldots, \theta_q$ by $\theta_1+d\theta_1, \ldots, \theta_q+d\theta_q$. The variance of this distribution or the expectation of the square of this relative discrepancy comes out as the positive definite quadratic differential form

$$ds^2 = \Sigma\Sigma\, g_{ij} d\theta_i d\theta_j, \tag{6.3}$$

where

$$g_{ij} = E\left(\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_i}\right)\left(\frac{1}{\phi}\frac{\partial\phi}{\partial\theta_j}\right). \tag{6.4}$$

Since the quadratic form is invariant for transformations in (P.S) it follows that $g_{ij}$ form the components of a covariant tensor of the second order and is also symmetric for $g_{ij} = g_{ji}$ by definition. This quadratic differetial form with its *fundamental tensor* as the elements of the *Information matrix* may be used as a suitable measure of divergence between two populations defined by two contiguous points. The properties of (P.S) may be studied with this as the *quadratic differential metric* defining the element of length. The space based on such a metric is called the Riemanian space and the geometry associated with this is the Riemanian geometry with its definitions of distances and angles.

*The distance between two populations*

If two populations are represented by two points $A$ and $B$ in (P.S) then we can find the distance between $A$ and $B$ by integrating along a geodesic using the element of length

$$ds^2 = \Sigma\Sigma\, g_{ij} d\theta_i d\theta_j. \tag{7.1}$$

If the equations to the geodesic are

$$\theta_i = f_i(t), \tag{7.2}$$

where $t$ is a parameter, then the functions $f_i$ are derivable from the set of differential equations

$$\sum_1^q{}_j\, g_{jk}\frac{d^2\theta_j}{dt^2} + \sum_1^q{}_{jl}[jl.\, k]\frac{d\theta_j}{dt}\frac{d\theta_l}{dt} = 0, \tag{7.3}$$

where $[jl, k]$ is the Christoffel symbol defined by

$$[jl, k] = \tfrac{1}{2}\left[\frac{\partial g_{jk}}{\partial\theta_l} + \frac{\partial g_{lk}}{\partial\theta_j} + \frac{\partial g_{jl}}{\partial\theta_k}\right]. \tag{7.4}$$

The estimation of distance, however, presents some difficulty. If the two samples from two populations are large then the best estimate of distance can be found by substituting the maximum likelihood estimates of the parameters in the above expression for distance. In the case of small samples we can get the fiducial limits only in a limited number of cases.

We apply the metric (7.1) to find the distance between two normal populations defined by $(m_1, \sigma_1)$ and $(m_2, \sigma_2)$ the distribution being of the type

$$\phi(x, m, \sigma) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp. -\tfrac{1}{2} \frac{(x-m)^2}{\sigma^2}. \tag{7.5}$$

The quantities $g_{ij}$ defined above have the values

$$g_{11} = 1/\sigma^2, \quad g_{12} = 0, \quad g_{22} = 2/\sigma^2, \tag{7.6}$$

so that the element of length is obtained from

$$ds^2 = \frac{(dm)^2}{\sigma^2} + \frac{2}{\sigma^2}(d\sigma)^2. \tag{7.7}$$

If $m_1 \neq m_2$ and $\sigma_1 \neq \sigma_2$ then the distance comes out as

$$D_{AB} = \sqrt{2} \log \frac{\tan \theta_1/2}{\tan \theta_2/2} \tag{7.8}$$

where

$$\theta_i = \sin^{-1}\sigma_i/\beta \text{ and } \beta = \sigma_1^2 + [(m_1 - m_2)^2 - 2(\sigma_2^2 - \sigma_1^2)]/8(m_1 - m_2)^2. \tag{7.9}$$

If $m_1 = m_2$ and $\sigma_1 \neq \sigma_2$

$$D_{AB} = \sqrt{2} \log (\sigma_2/\sigma_1). \tag{7.10}$$

If $m_1 \neq m_2$ and $\sigma_1 = \sigma_2$

$$D_{AB} = \frac{m_1 - m_2}{\sigma}. \tag{7.11}$$

*Distance in tests of significance and classification*

The necessity for the introduction of a suitable measure of distance between two populations arises when the position of a population with respect to an assigned set of characteristics of a given population or with respect to a number of populations has to be studied. The first problem leads to tests of significance and the second to the problem of classification. Thus if the assigned values of parameters which define some characteristics in a population are $\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_q$ represented by the point $O$, and the true values are $\theta_1, \theta_2, \ldots, \theta_q$ represented by the point $A$, then we can define the divergence from the assigned sets of parameters by $D_{AO}$, the distance defined before in the (P.S). The testing of the hypothesis

$$\bar{\theta}_i = \theta_i, \quad (i = 1, 2, \ldots q), \tag{8.1}$$

may be made equivalent to the test for the significance of the estimated distance $D_{AO}$ on the large sample assumption If $D_{AO} = \psi(\theta_1, \ldots; \theta_q; \bar{\theta}_1, \ldots, \bar{\theta}_q)$ and the maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_q$ are $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_q$, then the estimate of $D_{AO}$ is given by

$$\hat{D}_{AO} = \psi(\hat{\theta}_1, \ldots, \hat{\theta}_q; \bar{\theta}_1, \ldots, \theta_q). \tag{8.2}$$

The covariances between the maximum likelihood estimates being given by the elements of the information matrix, we can calculate the large sample approximation to the variance of the estimate of $D_{AO}$ by the following formula

$$V(\hat{D}_{AO}) = \Sigma\Sigma \frac{\partial\psi}{\partial\theta_i} \frac{\partial\psi}{\partial\theta_j} \operatorname{cov}(\hat{\theta}_i \hat{\theta}_j) \tag{8.3}$$

We can substitute the maximum likelihood estimates of $\theta_1, \theta_2, \ldots, \theta_q$ in the expression for variance. The statistic

$$w = \frac{\hat{D}_{AO}}{[V(\hat{D}_{AO})]^{\frac{1}{2}}} \tag{8.4}$$

can be used as a normal variate with zero mean and unit variance to test the hypothesis (8.1).

If the hypothesis is that two populations have the same set of parameters then the statistic

$$w = \frac{\hat{D}_{AB}}{[V(\hat{D}_{AB})]^{\frac{1}{2}}}, \tag{8.5}$$

where $\hat{D}_{AB}$ is the estimate of the distance between two populations defined by two points $A$ and $B$ in (P.S) can be used as (8.4). The expression for variance has to be calculated by the usual large sample assumption.

If the sample is small the appropriate test will be to find out a suitable region in the sample space which affords the greatest average power over the surfaces in the (P.S) defined by constant values of distances. The appropriate methods for this purpose are under consideration and will be dealt with in a future communication.

The estimated distances can also be used in the problem of classification, It usually becomes necessary to know whether a certain population is closer to one of a number of given populations when it is known that populations are all different from one another. In this case the distances among the populations taken two by two settle the question. We take that population whose distance from a given population is significantly the least as the one closest to the given population.

This general concept of distance between two statistical populations (as different from tests of significance) was first developed by Prof. P.C. Mahalanobis. The generalised distance defined by him (Mahalanobis, 1936) has become a powerful tool in biological and anthropological research. A perfectly general measure of divergence has been developed by Bhattacharya (1942) who defines the distance between populations as the angular distance between two points representing the populations on a unit sphere. If $\pi_1, \pi_2, \ldots, \pi_k$ are the proportions in a population consisting of $k$ classes then the population can be represented by a point with coordinates $\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_k}$ on a unit sphere in a space of $k$ dimensions. If two populations have the proportions $\pi_1, \pi_2, \ldots, \pi_k$ and

$\pi_1', \pi_2', \ldots, \pi_k'$ the points representing them have the co-ordinates $\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_k}$ and $\sqrt{\pi_1'}, \sqrt{\pi_2'}, \ldots, \sqrt{\pi_k'}$. The distance between them is given by

$$\cos^{-1}\{\sqrt{(\pi_1\pi_1')}+\sqrt{(\pi_2\pi_2')}+\ldots+\sqrt{(\pi_k\pi_k')}\}. \tag{8.6}$$

If the populations are continuous with probability densities $\phi(x)$ and $\psi(x)$ the distance is given by

$$\cos^{-1}\int\sqrt{\{\phi(x)\psi(x)\}}\,dx. \tag{8.7}$$

The representation of a population as a point on a unit sphere as given by Bhattacharya (1942) throws the quadratic differential metric (7.1) in an interesting light. By changing $\theta_1, \theta_2, \ldots, \theta_q$ the parameters occuring in the probability density, the points representing the corresponding populations describe a surface on the unit sphere. It is easy to verify that the element of length $ds$ connecting two points corresponding to $\theta_1, \theta_2, \ldots, \theta_q$ and $\theta_1+d\theta_1, \ldots, \theta_q+d\theta_q$ on this is given by

$$ds^2 = \Sigma(d\phi)^2/\phi = \Sigma\Sigma\, g_{ij}d\theta_i d\theta_j, \tag{8.8}$$

where $g_{ij}$ are the same as the elements of the quadratic differential metric defined in (7.1).

Further aspects of the problems of distance will be dealt with in an extensive paper to be published shortly.

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# ON THE RELATION BETWEEN CERTAIN TYPES OF TACTICAL CONFIGURATIONS

By

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If $v$ elements are arranged in $b$ sets of $k$ elements each such that each element occurs in just $r$ sets and any pair of elements occurs in just $\lambda$ of the sets, then the arrangement is a $(v, b, r, k, \lambda)$ configuration (well known in statistics as a balanced incomplete block design). More than one arrangement or solution for the same configuration may exist and in that case two solutions which can be identified by setting up a correspondence between the elements and sets of one with those of the other are called isomorphic.

2. When $v=b$ and consequently $r=k$, the arrangement will be called a symmetrical configuration $(v, v, r, r, \lambda)$. By suppressing an assigned set and all the treatments contained in it from a symmetrical configuration, we obtain a $(v-k,\ v-1,\ r,\ r-\lambda, \lambda)$ configuration called a 'residual' configuration. Again if from the symmetrical configuration one assigned set is suppressed and in the remaining sets only the elements of the assigned set are retained, we obtain a $(k,\ v-1,\ r-1,\ \lambda,\ \lambda-1)$ configuration called the 'derived' configuration. The processes of obtaining a 'residual' and a 'derived' configuration are known as 'residuation' and 'derivation' respectively (Ayyangar, 1943). Sometimes we are given the residual and the derivative and we have to build up the parent symmetrical configuration by suitable combination of the sets of the former two configurations with the addition of a new set containing all the elements of the derivative —this process will be called 'integration'.

3. Unlike derivation and residuation, integration is not always possible. The arrangement given for the (16, 24, 9, 6, 3) configuration by K. N. Bhattacharya (1943) and any arrangement for the (9, 24, 8, 3, 2) configuration cannot be integrated to yield a solution for the (25, 25, 9, 9, 3) symmetrical configuration. Moreover, from a single solution of a symmetrical configuration, we may obtain more than one non-isomorphic residuals or derivatives. Again we can integrate one residual and one derived configuration to yield more than one non-isomorphic solution for the parent symmetrical configuration (Nandi, 1943 and 1944).

For all symmetrical configurations with $\lambda = 2$, there is a unique derivative $\{k, k(k-1)/2, k-1, 2, 1\}$ consisting of all possible pairs of $k$ elements. In this case given any residual configuration, integration if possible is unique in the sense that we cannot obtain more than one non-isomorphic solution for the symmetrical configuration.

To see this let $k = 2n+1$ and denote the elements of the derivative by $a_1, a_2, \ldots, a_{2n+1}$. Suppose integration is possible. Then the following $n$ sets

$$(a_1,\ a_2);\ (a_3,\ a_4);\ (a_5,\ a_6);\ \ldots\ldots;\ (a_{2n-1},\ a_{2n})$$

are adjoined uniquely to suitably chosen sets of the residual configurations. Next $2n$ sets

$$(a_1, a_{2n+1});\ (a_2, a_{2n+1});\ \ldots\ldots;\ (a_{2n}, a_{2n+1})$$

are also adjoined uniquely. The remaining sets are finally adjoined also in a unique manner.

When $k$ is even, the argument is similar.

4. We will now consider, in particular, the integration of the two configurations (15, 21, 7, 5, 2) and (7, 21, 6, 2, 1) yielding the symmetrical one (22, 22, 7, 7, 2). The solution for the symmetrical is known to be non-existent (Hussain, 1944). Hence if it is shown that in this case integration is possible except in a few cases where the residual is easily shown to be non-existent, then the residual also can have no solution.

To see this denote the fifteen elements of the residual configuration under study by the natural numbers 1, 2, 3, ..., 15. Let one set of the configuration contain the five elements 1, 2, 3, 4, 5. Then out of the twenty remaining sets it is easy to see (Nandi, 1944) that ten sets will contain a pair of the above five elements and ten other sets will contain one element each, every one of 1, 2, 3, 4, 5 being repeated twice. Thus we get the pattern:

(1, 2, 3, 4, 5); (1, 2); (1, 3); (1, 4); (1, 5); (2, 3); (2, 4); (2, 5); (3, 4);
(3, 5); (4, 5); (1); (1); (2); (2); (3); (3); (4); (4); (5); (5);

each bracket represents a set. Let us name the sets in the following order $A_0$, $A_1$, $A_2$, ... $A_{10}$, $B_1$, $B_2$, ..., $B_{10}$. The set $A_0$ will be called the initial set, the sets $A_1$, $A_2$, ..., $A_{10}$ each of which has two elements in common with the initial set will be called the sets of the first kind and the remaining ten sets $B_1$, $B_2$, ..., $B_{10}$ sets of the second kind.

Now the following property of the residual configuration exists: There are just three sets of the first kind which have one element in common with each other and two elements in common with any assigned set of the first kind.

If we consider four sets $A_1$, $A_2$, $A_3$, $A_4$, then one pair of these must have two elements in commen, otherwise the number of elements in the configuration will exceed fifteen. By a proper renaming of the elements it is possible to make $A_1$, $A_2$ have two elements common. Then completing the set $A_1$ with new elements 6, 7, 8, we consider the allocation of the new elements in the other sets, two such ways of allocating the same element which can be made identical by a renaming of the elements being isomorphic will be regarded as the same.

There are two such ways of placing 6 such that it occurs in both $A_1$ and $A_2$ giving us the patterns:

[α]: (1, 2, 3, 4, 5); (1, 2, 6, 7, 8); (1, 3, 6); (1, 4); (1, 5); (2, 3); (2, 4, 6); (2, 5);
(3, 4); (3, 5); (4, 5); (1); (1); (2); (2); (3, 6); (3); (4, 6); (4), (5, 6); (5, 6).

[β]: (1, 2, 3, 4, 5); (1, 2, 6, 7, 8), (1, 3, 6); (1, 4); (1, 5); (2, 3); (2, 4); (2, 5);
(3, 4); (3, 5); (4, 5, 6); (1); (1); (2, 6); (2); (3, 6); (3); (4, 6); (4); (5, 6); (5).

In both of the above cases enumeration of the possibilities of placing 7 and 8 reveals that there are just three sets of the first kind having two common elements with $A_1$. A step further with enumeration and it is found that except in a few cases where the configuration does not have any solution, the above three sets have one element in common with one another. Since $A_1$ is any assigned set of the first kind, the above property generally holds.

The derived configuration (7, 21, 6, 2, 1) consists of all possible pairs of 7 elements denoted by 16, 17, ..., 22.

To the initial set let us adjoin the pair (16, 17) and to the set $A_1$ the pair (18, 19). Since we can find three sets of the first kind which have two elements in common with $A_1$, suppose $A_2'$, $A_3'$, $A_4'$ are the chosen sets and we adjoin to these three pairs formed out of 20, 21, 22, say, in the order (20, 21); (21, 22); (20, 22). Again considering the set $A_2'$, we must have the set $A_1$ and two other sets $A_5'$, $A_6'$ which have two elements in common with $A_2'$ and we adjoin pairs (18, 22) and (19, 22) consistent with the previous ones in some order to $A_5'$ and $A_6'$ respectively—which fails in a few cases which are enumerated and found not to yield any solution. Repeating the process we find that integration is possible except in a few cases where no solution is obtained.

Hence the residual configuration (15, 21, 7, 5, 2) has got no solution.

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# SHOWER PRODUCTION BY MESONS IN COSMIC RADIATION

BY

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The rapid development of the phenomena connected with cosmic radiation, both from the theoretical and experimental side makes it now possible to study the nature and properties of the fundamental particles with a higher degree of precision than has been so far possible. The nature of the two groups of particles *viz.*, the electron group and the proton group are more or less well known. The difficulty arises, however, in respect of the meson group, which can be considered as the latest addition to the group of elementary particles. Since its first postulation by Yukawa, purely from the theoretical standpoint, various attempts have been made to study its nature and properties. Yukawa's theory, however, requires that this particle, which at the present time, is called meson, should have a rest mass of approximately 180 times the electron mass. Unfortunately it has not yet been possible to know definitely from observations what is the exact mass of the meson. Nor has it yet been established that all mesons have identical rest mass. Another difficulty about the nature of the mesons relates to its spin and satistics. Even from the theoretical considerations it has not as yet been possible to say what would be the spin of the meson, or even whether the meson has a unique spin. Various modifications of the meson theory have been made in order to explain the observed phenomena. On the other hand results of observations connected with the shower generations by mesons and also the origin of meson lead to diverging conclusions, and several authors are inclined to believe that cosmic ray mesons are different from the Yukawa particles. Such a possibility cannot, however, be discarded and we shall have to depend on the results of future observations and theory to decide over the issue. Hamilton, Heitler and Peng (1943) have tried to show that cosmic ray mesons and those postulated in the theory of nuclear forces are identical particles. Following Møller and Rosenfeld they have assumed that both the vector and the pseudoscalar mesons exist in the cosmic radiation. Moreover it has been assumed that only pseudoscalar mesons are found at sea level and that the vector mesons have a much smaller life time which consequently decays practically at the point where they are created, and the decay products are responsible for the observed absorption curves recorded by Millikan and others in the high atmosphere.

A critical study of the different properties of the meson, mentioned above, particularly the nature of its spin and statistics can be made by comparing the results of observation connected with the shower generation by mesons, with those predicted from the theory. It is now believed that the majority of the soft component of the cosmic rays available in the low atmosphere are produced by mesons although a portion of these soft particles is also associated with Auger showers (*A*-showers). From the theoretical standpoint we know that electrons and positrons can be produced by mesons according to the following processes.

(1) The meson may produce a very fast secondary electron by direct collision which will subsequently produce a shower by cascade multiplications. This will be denoted in the present paper as $B$-shower. (2) The meson may radiate a high energy quantum which subsequently produces the shower. This will be denoted as $C$-shower. (3) The meson decays and the decay products, which contain electrons, multiply according to the cascade process. This will be denoted as $D$-shower. Bhabha (1938) and later on Bhabha, Carmichael and Chou (1939) worked out theoretically the nature of the $B$-showers. It is evident that these results depend critically on the results of the cascade theory of showers, and hence the inaccuracies in the results of the cascade theory naturally affected the results obtained by the authors mentioned above. A better treatment was not possible without improving the results of the cascade theory, a serious defect in which was the practically total neglect of the effect of ionisation loss in the cascade process, as developed by the previous authors. Bhabha and the present author (Bhabha and Chakarbarty, 1942 and 1943) have developed the cascade theory quite accurately both from the mathematical and physical standpoint. These results can now be used for an accurate estimate of the shower intensity. The cross-sections for the different processes mentioned above depends critically on the spin of the meson and it is possible that a proper study of the shower phenomena will give definite indications regarding the nature of the meson spin. All the different observations of the showers produced by mesons can be roughly divided into two classes, *viz.*, (*i*) observations made in a Wilson chamber which usually give the average number of shower particles, (*ii*) observations made with Geiger-Muller counters or ionization chambers, which usually give the probability of getting exactly a given number of particles or that of getting more than a given number of particles. Observations made by Lovell (1939), Seren (1942), Hazen (1943) etc., falls in class (*i*) whereas those made by Schein and Gill (1939) falls in class (*ii*). Christy and Kusaka (1941) worked out theoretically the probabilities for the production of large bursts by mesons with a view to the determination of the meson spin. But their calculations are defective for various reasons, and I have shown in another paper* (Chakrabarty, 1942) that their conclusions cannot be maintained. There I have only considered the probability for large bursts, and consequently the effect of the process (1) mentioned above, *viz.*, the knock-on process, was completely ignored, since in the region under consideration in I, the knock-on process produces an insignificant contribution as compared to the radiation process. This has also been shown there roughly from a general consideration. It is the purpose of the present paper to study small showers, and consequently a region in which the effect of the knock-on process predominates. In the last section the shower generation by the decay process has also been discussed. So that the present paper and the previous one (Chakrabarty, 1942) give a more or less complete theory of the shower generations by mesons, through all the different processes hitherto known in quantum mechanics. In addition to the showers of soft particles there is, however, a possibility of the formation of meson showers, and

* To be denoted henceforth as I.

this has been actually observed by various authors. But its probability will be very small as compared to the electron showers produced by mesons, and so will be completely ignored in the present paper. Some authors suggest also the possibility of the occurence of double knock-on showers or even triple knock on showers. Such an assumption will, however, make the interpretation of the results of observation difficult. We shall not take into consideration in the present paper the possibility of such processes.

A meson in its passage through matter will produce secondary electrons or photons which will then multiply according to the cascade theory, and these shower particles will come out associated with the parent meson. The differential effective cross-section $Q(W, E_0)dE_0$, for the production by a meson of energy $W$ of a secondary electron or a quantum having energies between $E_0$ and $E_0+dE_0$, which ultimately produces the $B$-showers or $C$-showers, taking different spins and magnetic moments have been given by various authors (*cf.* Chakrabarty, 1942). Since we are concerned with highly energetic mesons, we can take $W \gg Mc^2$, where $M$ is the mass of the meson. With this approximation we have, for the knock-on process,

(*i*) Spin 0, magnetic moment 0,

$$Q_c(W, E_0)dE_0 = 2\pi r_0^2 mc^2 Z\left[1-\frac{E_0}{E_{0m}}\right]\frac{dE_0}{E_0^2}, \tag{1a}$$

(*ii*) Spin $\frac{1}{2}$, magnetic moment $\dfrac{e\hbar}{2Mc}$,

$$Q_c(W, E_0)dE_0 = 2\pi r_0^2 mc^2 Z\left[1-\frac{E_0}{E_{0m}}+\frac{1}{2}\frac{E_0^2}{W^2}\right]\frac{dE_0}{E_0^2}, \tag{1b}$$

(*iii*) Spin 1, magnetic moment $\dfrac{e\hbar}{2Mc}$,

$$Q_c(W, E_0)dE_0 = 2\pi r_0^2 mc^2 Z\left[1-\frac{E_0}{E_{0m}}+\frac{1}{3}\frac{E_0^2}{W^2}+\frac{1}{3}\frac{mE_0}{M^2c^2}\left\{1-\frac{E_0}{E_{0m}}+\frac{1}{2}\frac{E_0^2}{W^2}\right\}\right]\frac{dE_0}{E_0^2}, \tag{1c}$$

where $r_0=e^2/mc^2$ and $E_{0m}$ is the maximum energy which can be communicated to an electron in a free collision, and is given by

$$E_{0m} = W[1+M^2c^2/(2mW)]^{-1}. \tag{2}$$

Similarly the expression for $Q(W, E_0)dE_0$ for the radiation process are given by the following equations,

(*i*) Spin 0, magnetic moment 0,

$$Q_r(W, E_0)dE_0 = \frac{16}{3}\frac{Z^2r_0^2}{137}(m/M)^2\left(\frac{W-E_0}{E_0}\right)\left[\log\frac{2W(W-E_0)}{Mc^2E_0}-\frac{1}{2}\right]\frac{dE_0}{W}, \tag{3a}$$

when the screening of the atomic nuclei is neglected, and

$$Q_r(W, E_0)dE_0 = \frac{16}{3}\cdot\frac{Z^2r_0^2}{137}(m/M)^2\log\left(137.\frac{M}{m}Z^{-\frac{1}{3}}\right)\left(\frac{W-E_0}{E_0}\right)\frac{dE_0}{W} \tag{3b}$$

when the screening is complete.

The expressions for case of spin $\frac{1}{2}$ and 1 are given in I. Since we shall not use them explicitly in the present paper, we do not include them here. It may be mentioned here that the emission of photons by mesons takes place at much smaller distances from the centre of nucleus than the emission of photons by electrons. Consequently, in the theory of the radiation processes of mesons the screening of the nuclear field by the outer electrons can be neglected to a greater extent than in the corresponding theory of electrons. The results of the analysis given in I, show this behaviour. For convenience in calculations we shall, however, take into consideration the form given by (3*b*) instead of (3*a*). The results of the present analysis will show that for the region under consideration in the present paper the radiation process produces a small contribution as compared to that of the knock-on process. A consideration of (3*b*) in place of (3*a*) will, however, give a slightly higher probability than is actually the case, but this will certainly give a correct estimate of the order of its contribution in the different cases considered in the present paper. It may also be noted that the influence of the spin on the collision probability of mesons manifest itself only for very close collisions. The theoretical predictions depend essentially on the hypothesis that the electro-magnetic field of meson can be described in the ordinary way even at distances smaller than $10^{-13}$ cms., from the centre of the meson itself. So far this hypothesis lacks any experimental support, although some theoretical justification for it can be found in Oppenheimer's arguments (Oppenheimer, Serber and Snyder, 1940; Oppenheimer, 1941). At any rate the validity of the formulae expressing the probabilities of large energy transfers from mesons to electrons cannot yet be considered as established.

Let us first calculate the average number of electrons and positrons which accompany the passage of a meson of energy $W$ and which are due either to the knock-on process and the ensuing cascade ($B$-showers), or to the radiation process and the ensuing cascade ($C$-showers). If $n(W, t)$ be the average number of soft particles produced by a meson of energy $W$ in traversing a thickness $t$ (in characteristic units) of material, then we have

$$n(W, t) = l\sigma \int_0^{\epsilon} Q(W, E_0)\, dE_0 \int_0^t \overline{N}(E_0, t')\, dt' \tag{4}$$

where $\overline{N}(E_0, t')$, is the average number of electrons and positrons in a cascade shower produced by a particle or quantum of energy $E_0$, in traversing a thickness $t'$ of the material. $\epsilon$ will be taken as $E_{0m}$ and $W$, for the $B$-showers and $C$-showers respectively. $\sigma$ represents the number of atoms per cubic centimeter of the substance.

From the results of the cascade theory (Bhabha and Chakrabarty, 1942), we have

$$\overline{N}(E_0, t') \approx \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{E_0}{\beta g_0(s)}\right)^{s-1} \cdot \frac{1}{s-1} \cdot \frac{D-\lambda_s}{\mu_s-\lambda_s} \cdot \exp(-\lambda_s t')\, ds \tag{5}$$

where $\lambda_s$, $\mu_s$, $g_0(s)$ etc., are all functions of $s$ and have been defined previously. So that using (5) in (4) we have after some simplifications,

$$n(W, t) = l\sigma . \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\{g_0(s)\}^{-s+1}}{(s-1)} . \frac{D-\lambda}{\mu-\lambda} . \left\{\frac{1-\exp(-\lambda_s t)}{\lambda_s}\right\} \phi(W, s)\, ds, \tag{6}$$

where

$$\phi(W, s) = \int_0^{c} Q(W, E_0)(E_0/\beta)^{s-1}\, dE_0, \tag{7}$$

where $c$ is any real number greater than 2.

The values of $Q(W, E_0)$ are given by equations (1) or (3) according as the $B$ or the $C$-showers are taken into considerations. Using equations (1), (3), (6), (7), we have after some simplifications, the following expressions for the average number of soft particles, produced by a meson of energy $W$, in traversing a thickness $t$ of the material, depending on the spin of the meson, *viz.*,

$$n_c^0(W, t) = A\{L_1(t) - L_2(t)\}, \tag{8a}$$

$$n_c^{\frac{1}{2}}(W, t) = n_c^0(W, t) + \tfrac{1}{2}A(E_{0m}/W)^2 L_3(t), \tag{8b}$$

$$n_c^1(W, t) = n_c^0(W, t) + \tfrac{1}{3}A\Big[(E_{0m}/W)^2 L_3(t) + \frac{m}{M}\cdot\frac{E_{0m}}{Mc^2}\Big\{L_2(t) - L_3(t) + \tfrac{1}{2}(E_{0m}/W)^2 L_4(t)\Big\}\Big], \tag{8c}$$

where $n_c^0$, $n$ $n_c^1$ represent the average numbers in $B$-showers, for a meson of spin 0, $\frac{1}{2}$ and 1 respectively, and

$$L_1(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-2)y_m\}.\psi(s, t)\frac{1}{s-2}\, ds, \tag{9a}$$

$$L_2(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-2)y_m\}.\psi(s, t)\frac{1}{s-1}\, ds, \tag{9b}$$

$$L_3(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-2)y_m\}.\psi(s, t)\frac{1}{s}\, ds, \tag{9c}$$

$$L_4(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-2)y_m\}.\psi(s, t)\frac{1}{s+1}\, ds, \tag{9d}$$

and

$$\psi(s, t) = \{g_0(s)\}^{-s+1}\cdot\frac{1}{s-1}\cdot\frac{D-\lambda_s}{\mu_s-\lambda_s}\cdot\frac{1}{\lambda_s}(1-\exp\lambda_s t), \tag{9e}$$

where

$$y_m = \log(E_{0m}/\beta) \text{ and } A = 2\pi r_0^2 l\sigma Z(mc^2/\beta). \tag{10}$$

Proceeding in a similar way for the $C$-showers, we have in the case of spin 0 and complete screening,

$$n_r^0(W, t) = A'\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\{(s-1)u_0\}.\psi(s, t).\frac{1}{s(s-1)}\, ds, \tag{11}$$

where

$$u_0 = \log(W/\beta) \text{ and } A' = \frac{16}{3}\cdot\frac{Z^2 r_0^2}{137}(m/M)^2\, l\sigma \log\left(137\frac{M}{m}Z^{-\frac{1}{3}}\right). \tag{12}$$

It can now be easily seen that for a given value of $W$ in all the different cases $n(W, t)$ gradually increases and after a sufficiently large value of $t$ it attains its asymptotic value obtained by making $t\to\infty$, in the above equations. This maximum value of $n(W, t)$ is defined as the average number of soft particles in equilibrium with the meson. The layer of a material of thickness equal to or greater than the value of $t$ for which $n(W, t)$ practically attains its maximum value has been defined previously as an "infinitely thick" layer (Bhabha, 1938). The integrals occurring in (9) and (11) can be easily calculated by the saddle-point method and we can get the values of $n(W, t)$ for different values of $W$ and $t$ and for different spins of the meson. For the purpose of comparing with the results of observation and also with the previous theoretical estimates made by Bhabha, the values of $n(W, t)$ when $t \rightarrow \infty$, have been worked out and the results have been given in Table I. The values of $n(W, t)$ for other values of $t$, can be similarly calculated. These calculations are no doubt straight forward but tedious. The calculations have been made for the case of Pb, but any other material can be similarly taken into consideration. As in I we have taken $M = 177m$ and for Pb, $A = 5{\cdot}133 \times 10^{-2}$; $A' = 9{\cdot}850 \times 10^{-6}$.

TABLE I

The average number of soft particles accompanying a meson, below an infinitely thick layer of Pb.

| $u_0$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|
| $y_m$ | 2.829 | 4.650 | 6.282 | 7.673 | 8.867 | 9.949 | 11.00 |
| $n_c^0$ | .02431 | .07263 | .1199 | .1568 | .1804 | .2156 | .2505 |
| $n_c^{\frac{1}{2}}$ | .02440 | .07815 | .1216 | .1605 | 1860 | .2221 | .2577 |
| $n_c^1$ | .02440 | .07820 | .1222 | .1643 | 2018 | .2725 | .4100 |
| $n_r^0$ | $3.801 \times 10^{-4}$ | $1.059 \times 10^{-3}$ | $3.049 \times 10^{-3}$ | $7.342 \times 10^{-3}$ | $2.187 \times 10^{-2}$ | $5.831 \times 10^{-2}$ | .1581 |
| $n_c^{\frac{1}{2}}$(B) | | | .190 | | | | .34 |

The last row in the table gives the values obtained by Bhabha, assuming that the meson has a spin of half a unit, and has a mass of about 100 times that of the electron. Table I shows that for low energy mesons and even up to an energy of $10^{10}$ e.v. the average number is practically independent of the meson spin. If therefore a test is to be made as to the nature of the meson spin, by a measurement of the average number, one has to exclude all mesons having energies less than, say, $10^{11}$ e.v. when the material used is Pb. But in this region the contribution of the $C$-showers are comparable to that of the $B$-showers, and as the energy of the meson increases the contribution of the $C$-showers outweighs that of the $B$-showers. Hazen (1943) has shown that the average number of soft particles in equilibrium with a meson under a layer of Pb is about $7{\cdot}5 \pm 0{\cdot}5$ per cent. This result when compared with those given in Table I suggest that the mean energy of the meson at the level where the observations were made is of the order of $\beta e^6 (\sim 2{\cdot}5 \times 19^9$ e.v.), or slightly higher, as in fact is otherwise known. Seren (1942), however, has obtained a slightly higher value, *viz.*, about 10 per cent. Similar experiments in which only those mesons, having energies greater than a definite value are allowed to pass through the Wilson chamber, can give results, which when compared with those given in Table 1, will give a good estimate of the energy spectra of the mesons. As there exists a certain amount of ambigüity about the nature of the energy spectra of the meson, we do not integrate the average number $n(W, t)$ over the spectra of the incident meson. This however, can be easily done whenever required.

Let us next consider the probability of getting *more* than $N$ particles either in a $B$-shower or a $C$-shower, produced by a meson of given energy, in traversing an infinitely thick layer of the material. Following Bhabha and Heitler (1937) we assume that the fluctuation obeys the Poisson law. The calculations of Scott and Uhlenbeck (1942) show that in the actual cosmic ray problem, the fluctuations are much smaller than that deduced from the Furry model and that the Poisson distribution approaches reality. It has been shown in I that the probability $P(N>, W)$ of a meson of energy $W$, emerging accompanied by a shower containing more than $N$ particles from an infinitely thick layer of a substance is given by

$$P(N>, W) = l\sigma \int_0^{\varepsilon} Q(W, E_0) J_{N>}(y_0) dE_0, \tag{13}$$

where

$$J_{N>}(y_0) = 8{\cdot}21. \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) N_m^{N-s+1} t_m^{1/2}}{\Gamma(N+1)(N-s+1)^{3/2}} ds, \tag{14}$$

in which $N_m$ represents the maximum number of particles produced at a depth $t_m$ by an electron of energy $\beta . \exp y_0$ according to the cascade process. $c$ is any real number greater than zero but less than $(N+1)$. The values of $J_{N>}(y_0)$ for different values of $y_0$ and $N$ can be obtained by evaluating the integral in (14) by the saddle-point method. Some such values of $J_N\ (y_0)$ together with the values of $s_0$ which determine the saddle-point is given in Table II. It appears that for a given $y_0$ when $N \gg N_m$, the saddle-point lies in the neighbourhood of $N_m$ and $J_{N>}(y_0)$ tends to zero, but when $N \ll N_m$, $s_0$ lies in the neighbourhood of $(N+1)$ and but is always less than $(N+1)$ and $J_N\ (y_0)$ in that case is finite and for a fixed $N$, its value gradually increases with $y_0$.

The values of $J_{N>}(y_0)$ for two extreme cases *viz.*, when $N \gg N_m$ and again when $N \ll N_m$ can be obtained analytically from (14).*

We have

$$J_{N>}(y_0) = 8{\cdot}21\, t_m^{\frac{1}{2}} . \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \phi(s) ds$$

$$\approx 8{\cdot}21\, t_m^{\frac{1}{2}} \exp \phi(s_0) / \{2\pi\phi''(s_0)\}^{\frac{1}{2}}, \tag{15}$$

where $\quad \phi(s) = \log \Gamma(s) + (N' - s + 1) \log N_m - \log \Gamma(N+1) - \frac{3}{2} \log (N - s + 1). \quad (16)$

Consequently when $N \gg N_m$ we have $s_0 \approx N_m$ so that $\phi''(s_0) \approx N_m^{-1}$ provided $N_m \gg 1$, so that from (15) we get,

$$J_{N>}(y_0) \approx 8{\cdot}21 t_m^{\frac{1}{2}} \frac{\Gamma(N_m) N_m{}^{N-N_m+1}}{\Gamma(N+1)(N-N_m+1)^{3/2}} . \frac{1}{\sqrt{(2\pi/N_m)}} .$$

$$\approx 8{\cdot}21\, t_m^{\frac{1}{2}} (eN_m/N)^{N+1} . \exp(-N_m - 1)/N. \tag{17}$$

* A similar calculation is also possible for $J_N(y_0)$ defined in I. Bhabha pointed out this to me in a private communication.

TABLE II

The values of $J_{N>}(y_0)$ for different values of $N$ and $y_0$

| $N$ | $y_0$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | $z_0$ | 1·586 | 1·996 | 2·275 | 2·470 | 2·590 | 2·669 | 2·724 | |
| | $J_{N>}(y_0)$ | 1·473 | 4·111 | 7·806 | 11·67 | 15·52 | 19·35 | 23·12 | |
| 3 | $z_0$ | 2·000 | 2·623 | 3·112 | 3·388 | 3·544 | 3·640 | 3·704 | 3·750 |
| | $J_{N>}(y_0)$ | ·8207 | 3·181 | 7·094 | 10·96 | 14·85 | 18·68 | 22·50 | 26·31 |
| 5 | $z_0$ | 2·570 | 3·755 | 4·756 | 5·235 | 5·465 | 5·593 | 5·674 | 5·728 |
| | $J_{N>}(y_0)$ | ·1784 | 2·121 | 5·953 | 9·828 | 13·81 | 17·69 | 21·48 | 25·34 |
| 10 | $z_0$ | 3·160 | 5·370 | 8·435 | 9·330 | 10·29 | 10·50 | 10·62 | 10·69 |
| | $J_{N>}(y_0)$ | $6·238\times10^{-4}$ | ·1091 | 4·280 | 8·150 | 11·99 | 15·92 | 19·83 | 23·74 |
| 50 | $z_0$ | | | | 31·35 | 48·22 | 49·94 | 50·36 | 50·54 |
| | $J_{N>}(y_0)$ | | | | $4·633\times10^{-3}$ | 5·593 | 11·94 | 15·82 | 19·62 |

using Stirling's approximation for the Γ-function. So that in this case $J_{N>}(y_0)$ tends to zero

On the other hand when $N_m \gg N$, we have easily form (16) $z_0 = N+1-\delta$, where $\delta = 1·5 \log (N_m/N+1)$ and $\phi''(z_0) \approx 1·5\delta^{-2}$

$$J_{N>}(y_0) \approx 3·21\, t_m^{\frac{1}{2}} \frac{\Gamma(N+1-\delta)N_m^{\delta}}{\Gamma(N+1)\,\delta^{3/2}} \cdot \frac{1}{\sqrt{(3\pi\delta^2)}}$$

$$= \frac{3·21}{\sqrt{(3\pi)}} \cdot t_m^{\frac{1}{2}} \cdot N_m^{\delta} \left\{ \frac{2}{3} \log \frac{N_m}{N+1} \right\}^{\frac{1}{2}}.$$

But
$$N_m^{\delta} = \exp\left[1·5\left\{1+\frac{\log N+1}{\log N_m - \log N+1}\right\}\right] \approx e^{1·5}$$

and $\log N_m \approx y_0$ and $t_m \approx y_0$ when $N_m \gg N$ and hence $y_0 \gg 1$. We thus have

$$J_{N>}(y_0) \approx 1·07 . y_0^{\frac{1}{2}} e^{1·5} (2 \log N_m/\pi)^{\frac{1}{2}}$$

$$\approx 1·07 (2\pi)^{-\frac{1}{2}} y_0 e^{1·5} = 3·83 y_0. \tag{18}$$

When the values of $J_{N>}(y_0)$ given in Table II are plotted against $y_0$ it appears that except in the immediate neighbourhood of $y_N$ *i.e.*, the value of $y_0$ for which $N_m = N$, the curve is very well represented by a straight line the slope of which is given by the asmyptotic value $J_{N>}(y_0)$. We can therefore with a high order of accuracy, take

$$J_{N>}(y_0) = 3·83 y_0 - a(N) \tag{19}$$

where $a(N)$ is independent of $y_0$ but depends on $N$. When $y_0 < y_N$, we can, according to (17) take $J_{N>}(y_0) \approx 0$. For different values of $N$, the values of $y_N$ can be easily calculated. The values of $a(N)$ also can be easily calculated from the results given in Table II. In Table III the values of $a(N)$ thus obtained for some given $N$ have been given.

TABLE III

Values of $y_N$ and $a(N)$ for different values of $N$

| $N$ | 2 | 3 | 5 | 10 | 50 |
|---|---|---|---|---|---|
| $a(N)$ | 11·29 | 11·96 | 13·00 | 14·80 | 18·65 |
| $y_N$ | (2·948) | (3·123) | 3·674 | 4·572 | 6·447 |

We can therefore take $J_{N>}(y_0)$ occurring in (13) its value as given by (19) provided the lower limit of the $y_0$-integration be taken as $y_N$ or $a_N/3{\cdot}83$, whichever is larger. These values have also been given in Table III. The terms within brackets represent the values of $a(N)/3{\cdot}83$ and not of $y_N$.

Substituting the different values of $Q(W, E_0)dE_0$ as given by (1) and (3) and that of $J_{N>}(y_0)$ as given by (19) in (13) we have after simplifications the following values for the probabilities $P(N_>, W)$ depending on the meson spin and also on the nature of the shower. For $B$-showers we have,

$$P^0_c(N_>, W) = A[\{3{\cdot}83(y_N+1)-a(N)\}\exp(-y_N)-\{(3{\cdot}83-a(N))(y_m+1) \\ +1{\cdot}915(y_m^2-y_N^2)+a(N)y_N\}\exp(-y_m)], \quad (20a)$$

$$P^{\frac{1}{2}}_c(N_>, W) = P^0_c(N_>, W)+A.\tfrac{1}{2}\exp(-2u_0)[\{3{\cdot}83(y_m-1)-a(N)\}\exp y_m \\ -\{3{\cdot}83(y_N-1)-a(N)\}\exp y_N], \quad (20b)$$

$$P^1_c(N_>, W) = P^0_c(N_>, W)+\tfrac{2}{3}\{P^{\frac{1}{2}}_c(N_>, W)-P^0_c(N_>, W)\} \\ +A.\tfrac{1}{3}(m/M)(\beta/Mc^2)[\{1{\cdot}915(y_m^2-y_N^2)-a(N)(y_m-y_N)-3{\cdot}83(y_m-1)+a(N)\} \\ +\exp(y_N-y_m)\{3{\cdot}83(y_N-1)-a(N)\} \\ +\tfrac{1}{4}\exp(-2u_0)\{(1{\cdot}915.\overline{2y_m-1}-a(N))\exp(2y_m)-(1{\cdot}915.\overline{2y_N-1}-a(N))\exp(2y_N)\}], \quad (20c)$$

where $P^0_c$, $P^{\frac{1}{2}}_c$ and $P^1_c$ represent the values of $P(N_>, W)$ for the $B$-showers and for meson spins 0, $\frac{1}{2}$ and 1 units respectively.

For the case of $C$-showers and for spin 0 we have,

$$P^0_\gamma(N_>, W) = A'[\{1{\cdot}915(u_0+y_N)-a(N)\}(u_0-y_N)-\{3{\cdot}83(u_0-1)-a(N)\} \\ +\{3{\cdot}83(y_N-1)-a(N)\}\exp(y_N-u_0)]. \quad (21)$$

The values $P(N_>, W)$ given by the above equations have been calculated for some different values of $N$ *viz.*, 2, 10, and 50 and also for some different values of $u_0$ and are given in Table IV. The dependence of $P(N_>, W)$ on the nature of the material comes through $A$, $A'$ and $y_m$. We have taken the case of Pb for our calculations but similar calculations can be made also in the case of any other material. The figures in the Table show that for $N=2$, the dependence of $P(N_>, W)$ on the meson spin is very small, but this increases for a given $N$ as $u_0$ increases and also increases for a given $u_0$ as $N$

TABLE IV

The values of $P(N_>, W) \times 10^3$ in Pb for different values of $N$ and $u_0$

| $N$ \ $u_0$ | | 6·0 | 7·0 | 8·0 | 9·0 | 10·0 | 11·0 | 12·0 |
|---|---|---|---|---|---|---|---|---|
| 2 | $P_c^0(N_>, W)$ | 2·510 | 6·668 | 8·762 | 9·640 | 10·01 | 10·17 | 10·25 |
| | $P_c^{\frac{1}{2}}(N_>, W)$ | 2·510 | 6·770 | 8·849 | 9·691 | 10·04 | 10·18 | 10·25 |
| | $P_c^1(N_>, W)$ | 2·566 | 6·832 | 9·049 | 10·07 | 10·60 | 10·97 | 11·28 |
| | $P_r^0(N_>, W)$ | ·0965 | ·1939 | ·3283 | ·5002 | ·7097 | ·9569 | 1·241 |
| 10 | $P_c^0$ | | 1·234 | 2·394 | 2·973 | 3·239 | 3·368 | 3·426 |
| | $P_c^{\frac{1}{2}}$ | | 1·298 | 2·460 | 3·015 | 3·261 | 3·378 | 3·430 |
| | $P_c^1$ | | 1·317 | 2·576 | 3·270 | 3·911 | 3·984 | 4·233 |
| | $P_r^0$ | | ·0945 | ·1945 | ·3319 | ·5069 | ·7195 | ·9699 |
| 50 | $P_c^0$ | | | ·2053 | ·4763 | ·6344 | ·7196 | ·7617 |
| | $P_c^{\frac{1}{2}}$ | | | ·2438 | ·5076 | ·6519 | ·7279 | ·7653 |
| | $P_c^1$ | | | ·2772 | ·6365 | ·9085 | 1·142 | 1·351 |
| | $P_r^0$ | | | ·0622 | ·1585 | ·2944 | ·4678 | ·6809 |

increases. The incident energy spectra of mesons has not yet been quantitatively ascertained, but it is more or less established that it roughly varies as some negative power of $W$. Consequently if $P(N_>, W)$ be integrated over the incident meson spectra, the final result will depend almost entirely on the values of $P(N_>, W)$ corresponding to the smaller values of $u_0$. This shows that if $N$ be small then the spin effect will not be appreciable, unless the low energy mesons, say up to $10^{10}$ e.v., be sorted out from the incident mesons. The effect of the $C$-showers is practically negligible in comparison with those of the $B$-showers, when $N$ is small, but the relative contribution gradually increases as $N$ increases. That this will occur was shown in I from general considerations. The dependence of $P(N_>, W)$ on the nature of the production process is also evident from Table IV. It shows that for any given value of $N$ the value of $P_r/P_c$ increases as $u_0$ increases and again for a given $u_0$, this increases as $N$ increases, a feature which is also otherwise evident. It thus appears that for small showers the effect of the $B$-showers predominates whereas for large burst the $C$-showers give the entire contribution to $P(N_>, W)$. The only source of uncertainty in the above calculations is the nature of the fluctuations. Unless the actual form for the fluctuation differs considerably from a Poisson's distribution, the results given in Tables II and IV can be considered as fairly accurate. The approximation made by (19) is very good for smaller values of $N$. But as $N$ increases the difference between the exact value and approximate value increases. For the calculations of large bursts where the $C$-showers predominate this approximation is not necessary and we have shown in (I) how the exact expression for $J_{N_>}(y_0)$ given by (14) can be taken into the calculations.

Let us now consider the formation of $D$-showers. It has been observed from the measurements of the variation of cosmic ray intensity with altitude that the number of cosmic ray mesons is more strongly reduced by a layer of air than by a dense absorber equivalent to the air layer with regard to the ionization loss. Such anomalous absorption is interpreted on the hypothesis that mesons are unstable and has a short life period. The nature of the decay products will, however, depend on the spin of the meson. Consequently if the meson decays into an electron, this electron, while moving further, will produce cascades and we shall get the $D$-showers. Hamilton, Heitler and Peng (1943) postulates that the primary protons have an extremely short range, the majority of the mesons which are produced by protons are thus produced in a thin layer of the atmosphere. The vector mesons decay almost immediately and only the pseudoscalar mesons travel through the atmosphere. The vector mesons thus ultimately produce the $D$-showers in the high atmosphere, and are responsible for the observed absorption curves of Millikan, Neher and Pickering (1942). A study of the $D$-showers may therefore determine the nature of the primary cosmic rays. If the $D$-showers can explain the above curves, then it may be possible to discard the hypothesis that the primary cosmic rays contain also electrons. In that case the only charged component in the primary will be protons, and this protons will through different processes produce the $A$, $B$, $C$ and $D$-showers. Experimental evidences are now more in favour of a single primary charged component, *viz.*, protons, but the results cannot as yet claim to be sufficiently accurate in order to make a definite conclusion on the matter.

The probability that a meson of momentum $p$ decays in the layer between $t_1$ and $t_1+dt_1$ is given by $b/pt_1$, where $b=Mlt_0/\tau_0$ and $\tau_0$ is the life period of the meson at rest and $t_0$ is the sea level depth in radiation units below the top of the atmosphere. In another paper* I have shown that if $\bar{n}_D(W, t')$ be the average number of soft particles produced in a $D$-shower by a meson of energy $W$ at a depth $t'$ below the meson producing layer then

$$\bar{n}_D(W, t') \approx \left(\frac{W+a\alpha}{W}\right)\{1-(\alpha/t')^2\}$$

$$\times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(W/a)^{s-1}\left\{\frac{a}{\beta g_0(s)}\right\}^{s-1}\frac{1}{s(s-1)}\cdot\frac{D-\lambda_s}{\mu_s-\lambda_s}\exp(-\lambda_s t')ds \qquad (23)$$

where $a \approx \beta = 1{\cdot}03\times10^8$ e.v., and $\alpha$ is the depth below the top of the meson producing layer in the atmosphere. The values of $\bar{n}_D$ have been calculated for some different values of $W$ and $t'$ and also for two different values of $\alpha$ *viz.*, $\alpha=1$ and $\alpha=2$.

The results obtained there show that the rate of rise in the ionization before the maximum is reached is more rapid and at the same time after the maximum the rate of decrease in ionization is slower, if it be a $D$-shower than if it be an ordinary cascade shower produced by a soft particle having the same energy as that of the meson producing the $D$-showers. Both these features possibly give better fit with the results of observation. The observational data at present available are not very accurate

* To be shortly published.

(*cf.* Chakrabarty, 1943) and as such are quite insufficient for making any definite conclusion on these points.

It thus appears that a proper study of the showers produced both in the atmosphere and also in heavier elements will give definite indications about the nature and properties of meson, and also the nature of the primary cosmic rays. It will then also be possible to say definitely, how the mesons available in cosmic radiation are related to the mesons postulated in the nuclear theories.

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# ON THE CONSTRUCTION OF AFFINE DIFFERENCE SETS

By

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*Introduction.* James Singer (1938) by using the Finite Projective Geometry $PG(2, p^n)$, proved the following theorem of the theory of numbers: *Given an integer $s \geqq 2$ of the form $p^n$ ($p$ being a prime) we can find $s+1$ integers*

$$d_0, d_1, d_2, \ldots, d_s \tag{1.1}$$

*such that among the $s(s+1)$ differences*

$$d_i - d_j \quad (i, j = 0, 1, 2, \ldots, s;\ i \neq j) \tag{1.2}$$

*reduced modulo $s^2+s+1$, the integers $1, 2, 3, \ldots, s^2+s$ occur exactly once.*

One of the authors R. C. Bose (1942), by using the Finite Affine Geometry $EG(2, p^n)$ proved the following analouge of Singer's theorem:

*Given an integer $s \geqq 2$ of the form $p^n$($p$ being a prime) we can find $s$ integers*

$$d_1, d_2, \ldots, d_s \tag{1.3}$$

*such that among the $s(s-1)$ differences*

$$d_i - d_j \quad (i, j = 1, 2, \ldots, s;\ i \neq j) \tag{1.4}$$

*reduced modulo $s^2-1$, all the positive inegers less than $s^2-1$ and not divisible by $q = s+1$, occur exactly once.*

The set (1.1) possessing the property envisaged in Singer's theorem may be called a *Projective Difference Set*. In contrast the set (1.3) possessing the property envisaged in Bose's analouge of Singer's theorem, may be called an Affine Difference Set. The object of this paper is to obtain a quick method of constructing an *Affine Difference Set*, for the special case when $s$ is a prime.

2. **Lemma** I. *If $x^2+ax+b=0$, is the irreducible equation satisfied by the primitive element $x$ of $GF(p^2)$, $a$ and $b$ belonging to $GF(p)$, then $x^{p+1}=b$, and $b$ is a primitive root of $GF(p)$.*

*Proof.* Since $x \to x^p$ is an automorphism of $GF(p^2)$, we must have

$$x^{p^2} + a^p x^p + b^p = 0. \tag{2.1}$$

But $a^p = a$, $b^p = b$. Hence

$$x^{p^2} + x^p + b = 0 \tag{2.2}$$

which shows that the other root of (2.1) is $x^p$. Hence

$$x.x^p = b,$$

or

$$x^{p+1} = b. \tag{2.3}$$

Now all the non-null elements of $GF(p)$ are given by

$$x^{q(p+1)}, \quad q = 0, 1, 2, \ldots, p-1, \tag{2.4}$$

since

$$\{x^{q(p+1)}\}^{(p-1)} = (x^q)^{p^2-1} = 1. \tag{2.5}$$

Hence $b^0, b^1, b^2, \ldots, b^{p-2}$ are all the non-null elements of $GF(p)$ which shows that $b$ is a primitive element of $GF(p)$.

COROLLARY (1). *$x^t$ is an element of $GF(p)$ if and only if*

$$t \equiv 0 \pmod{p+1}. \tag{2.6}$$

3. **Theorem** I. *If $x^2+ax+b=0$ is the irreducible equation satisfied by a primitive element $x$ of $GF(p^2)$, $a$ and $b$ belonging to $GF(p)$, and if $f_1 = 0$ and $f_2, f_3, \ldots, f_p$ are defined by*

$$f_{m+1} = \frac{b}{a-f_m} \pmod{p}, \ m = 1, 2, \ldots, p-1, \tag{3.1}$$

*then $f_1, f_2, \ldots, f_p$ are all incongruent* (mod $p$).

Every element of $GF(p^2)$ is uniquely expressible in the form $hx+k$ where $h$ and $k$ belong to $GF(p)$. Let

$$x^u = hx+k, \quad h \neq 0, \ k \neq 0. \tag{3.2}$$

Then

$$x^{u+1} = hx^2+kx = h(-ax-b)+kx = (k-ah)x-bh = h^*x+k^* \tag{3.3}$$

where

$$h^* = k-ah, \quad k^* = -bh. \tag{3.4}$$

If therefore we have

$$x^m = h_m x+k_m, \quad m = 1, 2, \ldots, p \tag{3.5}$$

then $h_1 = 1$, $k_1 = 0$, and

$$\left.\begin{aligned} h_{m+1} &= k_m - ah_m, \\ k_{m+1} &= -bh_m. \end{aligned}\right\} \tag{3.6}$$

Now $h_m \neq 0$ for $m = 1, 2, \ldots, p$, since the vanishing of $h_m$ would mean that $x^m$ is an element of $GF(p)$, contrary to the fact (*cf.* Corollary (1)) that $x, x^2, \ldots, x^p$ do not belong to $GF(p)$.

Let us set

$$f_m = k_m/h_m \tag{3.7}$$

which gives $f_1=0$. Clearly $f_m$ belongs to $GF(p)$. We shall show that $f_1, f_2, \ldots, f_p$ are all different.

If possible let $f_i = f_j$, $1 \leqq i < j \leqq p$. Then

$$\frac{k_i}{h_i} = \frac{k_j}{h_j}. \tag{3.8}$$

Let $h_j = ch_i$, where $c \neq 0$ is an element of $GF(p)$. Then $k_j = ck_i$. Now

$$\begin{gathered} x^j = h_j x + k_j, \quad x^i = h_i x + k_i, \\ \therefore \ x^{j-i} = c \end{gathered} \tag{3.9}$$

which is absurd since $1 \leq j-i \leq p$. Now from (3.6) and (3.7),

$$f_{m+1} = \frac{b}{a-f_m}$$

which proves the theorem.

COROLLARY (2). *If $GF(p)$ is extended by the adjunction of the symbol $\infty$, then the triply transitive group $G$ of degree $p+1$ and order $p(p^2-1)$ given by the linear fractional transformations*

$$x^* = \frac{\alpha x+\beta}{\gamma x+\delta} \pmod p$$

*contains a cyclic subgroup of degree and order $p+1$, generated by*

$$x^* = \frac{b}{a-x}$$

*where $x^2+ax+b=0$ is the irreducible eqution satisfied by the primitive element $x$ of $GF(p^2)$.*

*Proof*:

$$x^p = h_p x + k_p,$$

$$\therefore \quad b = x^{p+1} = h_p x^2 + k_p x$$

$$= h_p(-ax-b)+k_p x$$

$$= (k_p - ah_p)x - bh_p,$$

$$\therefore \quad h_p = -1, \quad k_p = -a,$$

$$\therefore \quad f_p = \frac{k_p}{h_p} = a,$$

$$\therefore \quad f_{p+1} = \frac{b}{a-f_p} = \infty,$$

$$f_{p+2} = \frac{b}{a-f_{p+1}} = 0 = f_1.$$

Thus $f_1, f_2, \ldots, f_p, f_{p+1}$ are all the different symbols of $GF(p)$ extended by the adjunction of $\infty$, and are cyclically permuted into one another by the transformation

$$x^* = \frac{b}{a-x}.$$

4. **Theorem** II. *If $x$ is a primitive element of $GF(p^2)$, and if we set*

$$x^{c_m} = x+m \tag{4.1}$$

*where $m$ takes all the values $1, 2, \ldots, p = 0$ of $GF(p)$, then*

$$c_1, c_2, \ldots, c_m \tag{4.2}$$

*is an Affine Difference Set.*

*Proof*: Firstly no difference

$$c_i - c_j, \ (1 \leq i < j \leq p), \tag{4.3}$$

is divisible by $p+1$. If possible let the opposite be the case, Then

$$\frac{x+i}{x+j} = x^{c_i - c_j} = k \text{ (an element of } GF(p)). \tag{4.4}$$

This would make $x$ an element of $GF(p)$, which is absurd.

We shall next show that

$$c_i - c_j = c_u - c_v \pmod{p^2-1}, \quad (1 \leqq i < j \leqq p,\ 1 \leqq u < v \leqq p), \tag{4.5}$$

is impossible unless $i = u$, $j = v$. For otherwise

$$(x+i)(x+v) = (x+j)(x+u). \tag{4.6}$$

This equation must be an identity, for otherwise $x$ would belong to $GF(p)$. Hence $i$, $v$ must be some permutation of $j$, $u$. Since $i \neq j$, we have $i = u$, $j = v$.

This shows that the differences $c_i - c_j \pmod{p^2-1}$, are all different, where the suffixes $i$, $j$ can take all unequal values between 1 and $p$. These differences thus give $p(p-1)$ numbers incongruent (mod $p^2-1$) and not divisible by $p+1$. But there can only be $p(p-1)$ different numbers incongruent (mod $p^2-1$) and not divisible by $p+1$. Thus every number not divisible by $p+1$, can be represented exactly once as a difference of two $c$'s (mod $p^2-1$). This shows that $c_1, c_2, \ldots, c_m$ is an Affine Difference Set.

5. We shall now prove the following rule for the generation of an Affine Difference Set.

**Theorem** III. *An Affine Difierence Set is generated by taking $d_1=1$ and*

$$d_{m-1} = 1 + d_m - (p+1) \text{ ind } (f_m - a), \tag{5.1}$$

*where ind (N) is the number t, uniquely defined (mod $p-1$), by*

$$b^t \equiv N \pmod p \tag{5.2}$$

*and $x$, $a$, $b$, $f_1, f_2, \ldots, f_p$ are as in theorem* I.

Let us set

$$x^{d_m} = x + f_m, \quad m = 1, 2, \ldots, p, \quad 0 \leqq d_m < p^2-1, \tag{5.3}$$

From Theorem I, $f_1, f_2, \ldots, f_p$ are all the non-null elements of $GF(p)$. Hence from theorem II, the numbers $d_1, d_2, \ldots, d_p$ given by (5.3) must form an Affine Difference Set. We have to show that they can be generated by the recurrence formula (5.1).

Since $f_1 = 0$, we have $d_1 = 1$. Now

$$\begin{aligned} \therefore \quad x^{1+d_m} &= x^2 + f_m x = (-ax - b) + f_m x \\ &= (f_m - a)\left\{x + \frac{b}{a - f_m}\right\} = (f_m - a)(x + f_{m+1}) \\ &= b^{\text{ind } (f_m - a)}(x + f_{m+1}) = x^{(p+1) \text{ ind } (f_m - a)}(x + f_{m+1}), \end{aligned}$$

$$\therefore \quad 1 + d_m = (p+1) \text{ ind } (f_m - a) + d_{m+1}, \tag{5.4}$$

or

$$d_{m+1} = 1 + d_m - (p+1) \text{ ind } (f_m - a), \tag{5.5}$$

which proves our theorem.

6. **Example.** *Let $p=17$, and let us generate an Affine Difference Set $d_1, d_2, \ldots, d_{17}$ such that any number not divisible by 18, can be represented just once as $d_i-d_j$ (mod 288).*

The first thing is to find an irreducible equation $x^2+ax+b=0$ satisfied by a primitive root of $GF(p^2)$, *i.e*, a minimum function for $GF(p^2)$. This problem has already been considered in an earlier paper (Bose, Chowla and Rao, 1944). It was shown that $x^2+ax+10$ is a minimum function for $GF(17^2)$ if $a=1, 3, 4, 13, 14$ or $16$. Let us choose the value 16 for $a$. Thus

$$b = 10, \quad a = 16 = -1 \qquad (\text{mod } 17), \tag{6.1}$$

$$f_{m+1} = \frac{b}{a-f_m} = \frac{7}{1+f_m} \quad (\text{mod } 17), \tag{6.2}$$

$$\begin{aligned} d_{m+1} &= 1+d_m-(p+1) \text{ ind } (f_m-a) \quad (\text{mod } 288) \\ &= 1+d_m-18 \, ind \, (1+f_m) \qquad ,, \end{aligned} \tag{6.3}$$

The initial values $f_1=0$ and $d_1=1$ have also to be remembered. The calculation of the difference set may now be arranged as follows:

| $m$ | $f_m$ (mod 17) | $1+f_m$ (mod 17) | ind $(1+f_m)$ (mod 16) | $1-18$ ind $(1+f_m)$ (mod 288) | $d_m$ (mod 288) |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 16 | 1 | 1 |
| 2 | 7 | 8 | 14 | 37 | 2 |
| 3 | 3 | 4 | 4 | 217 | 39 |
| 4 | 6 | 7 | 9 | 127 | 256 |
| 5 | 1 | 2 | 10 | 109 | 95 |
| 6 | 12 | 13 | 12 | 73 | 204 |
| 7 | 11 | 12 | 15 | 19 | 277 |
| 8 | 2 | 3 | 11 | 91 | 8 |
| 9 | 8 | 9 | 6 | 181 | 99 |
| 10 | 14 | 15 | 2 | 253 | 280 |
| 11 | 5 | 6 | 5 | 199 | 245 |
| 12 | 4 | 5 | 7 | 163 | 156 |
| 13 | 15 | 16 | 8 | 145 | 31 |
| 14 | 10 | 11 | 13 | 55 | 176 |
| 15 | 13 | 14 | 3 | 235 | 231 |
| 16 | 9 | 10 | 1 | 271 | 178 |
| 17 | 16 | 0 | — | — | 161 |

Thus the required Affine Difference Set is

$$1,\ 2,\ 8,\ 31,\ 39,\ 95,\ 99,\ 156,\ 161,\ 176,\ 178,\ 204,\ 231,\ 245,\ 256,\ 277,\ 280.$$

If we form the 272 mutual differences of these numbers and reduce them (mod 288), we shall get every number less than 288 and not divisible by 18, just once.

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# ON A THEOREM IN THE THEORY OF PARTITION

By

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Ramanujan (Hardy, 1938, p. 294) first considered the congruence properties of the function of partition. In this note I shall consider the new congruence property of the functions of partition not known before and give for reference the table containing the values of $Q(n)$ for $n$ upto 70.

The generating function (Hardy, p. 278) of the function of partition $p(n)$ is

$$\sum_0^\infty p(n)x^n = \frac{1}{(1-x)(1-x^2)\ldots\ldots}$$

$$= 1+\frac{x}{(1-x)^2}+\frac{x^4}{(1-x)^2(1-x^2)^2}+\frac{x^9}{(1-x)^2(1-x^2)(1-x^3)^2}+\ldots\ldots$$

$$=1+\frac{x}{1-x^2}+\frac{x^4}{(1-x^2)(1-x^4)}+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}+\ \ldots\ +2J$$

where $J$ is an integral power series in $x$.

Table I

| $n$ | $Q(n)$ | $n$ | $Q(n)$ | $n$ | $Q(n)$ | $n$ | $Q(n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 18 | 5 | 35 | 139 | 53 | 1731 |
| 2 | 0 | 19 | 6 | 36 | 33 | 54 | 133 |
| 3 | 1 | 20 | 7 | 37 | 185 | 55 | 2244 |
| 4 | 1 | 21 | 8 | 38 | 37 | 56 | 157 |
| 5 | 1 | 22 | 8 | 39 | 261 | 57 | 2630 |
| 6 | 1 | 23 | 19 | 40 | 46 | 58 | 178 |
| 7 | 1 | 24 | 11 | 41 | 335 | 59 | 3422 |
| 8 | 2 | 25 | 22 | 42 | 52 | 60 | 209 |
| 9 | 2 | 26 | 12 | 43 | 463 | 61 | 4135 |
| 10 | 2 | 27 | 34 | 44 | 63 | 62 | 236 |
| 11 | 2 | 28 | 16 | 45 | 600 | 63 | 5141 |
| 12 | 3 | 29 | 47 | 46 | 72 | 64 | 276 |
| 13 | 3 | 30 | 18 | 47 | 806 | 65 | 6136 |
| 14 | 3 | 31 | 70 | 48 | 87 | 66 | 312 |
| 15 | 4 | 32 | 23 | 49 | 1033 | 67 | 7541 |
| 16 | 5 | 33 | 95 | 50 | 98 | 68 | 361 |
| 17 | 5 | 34 | 26 | 51 | 1359 | 69 | 8845 |
| | | | | 52 | 117 | 70 | 408 |

Then (Hardy, p. 275)

$$\sum_0^\infty p(n)x^n = (1+x)(1+x^3)(1+x^5)\ldots\ +2J = \sum_0^\infty Q(n)x^n+2J,$$

where $Q(n)$ is the function of partition of $n$ into parts which are both odd and unequal. Hence

$$p(n) \equiv Q(n) \pmod 2.$$

Therefore $p(n)$ is even when $Q(n) = 0$, or even and odd when $Q(n) =$ odd. Again

$$(1-x^2)(1-x^4)\ldots\ldots \sum_0^\infty Q(n)(-x)^n = (1-x)(1-x^2)\ldots\ldots$$

Hence

$$(-1)^n[Q(n)-Q(n-2)-Q(n-4)+\ldots] = (-1)^m \text{ or } 0$$

according as $n$ is or is not of the form $\frac{1}{2}m(3m\pm 1)$.

Using these formulae we have the Table I.

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# SYMMETRICAL INCOMPLETE BLOCK DESIGNS WITH $\lambda = 2$, $k = 8$ OR 9

By

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## §1. *Introduction*

In an incomplete block design $v$ varieties, each replicated $r$ times, are placed in blocks each containing $k(<v)$ different varieties in such a manner that every pair of varieties occurs together in $\lambda$ blocks.

The following conditions among the integers $b$, $v$, $r$, $k$, are obviously necessary :—

$$bk = vr, \tag{1}$$

and

$$r(k-1) = \lambda(v-1). \tag{2}$$

Fisher (1940) has shown that we must have also

$$b \geq v; \quad r \geq k. \tag{3}$$

Moreover, since $v < k$, (2) gives

$$r > \lambda. \tag{4}$$

In the so-called symmetrical incomplete block designs,

$$b = v, \quad r = k. \tag{5}$$

In the special case of symmetrical incomplete Block designs with $\lambda = 2$,

$$v = b = 1 + \frac{k(k-1)}{2}. \tag{6}$$

The designs for the cases $k = 5$, 6 and 7 have already been studied in full in two papers due to appear in Sankhya Vol. 7. It was found that

1. The design with $k = 5$ has only one independent* solution;
2. The design with $k = 6$ has just three independent solutions;
3. The design with $k = 7$ has no solutions.

The purpose of the present paper is to study the designs for $k = 8$ or 9. It has been shown that

4. The design with $k = 8$ has no solutions;
5. The design with $k = 9$ has at least two independent solutions other than the well known (Bose, 1939; Fisher and Yates, 1938 and 1942) standard solution depending on an initial block formed of the 9 different integers obtained from the successive powers of $2^4$ reduced (mod 37), *viz.*, the biquadratic residues of 37.

* Two solutions may be said to be "dependent" when one can be derived from the other by re-naming the varieties. If this is not possible, they may be said to be "independent" or non-isomorphic.

In the method used, the designs were sought to be expressed in a standard form. The varieties occurring in the first block were named serially from 1 to $k$. In accordance with a general property of Symmetrical Incomplete Block designs, that there must be just $\lambda$ varieties common to any two blocks, all the blocks other than the first must contain some one of the $\binom{k}{2}$ pairs of varieties formed out of the $k$ varieties of the first block. As each pair must occur just twice ($\because \lambda = 2$), each of the other blocks will contain a different pair of the varieties $1, 2, \ldots, k$. Two varieties of the other blocks are thus already known and they are placed in successive blocks in the order

$$1, 2;\ 1, 3;\ \ldots;\ 1, k;\ 2, 3;\ 2, 4;\ \ldots;\ 2, k;\ \ldots;\ k-2, k-1;\ k-2, k;\ k-1, k.$$

It is convenient to refer to such a first block as the initial block (or $i$-block), and the varieties contained in it as the initial varieties (or $i$-varieties). The other blocks may be called non-initial blocks (or $j$-blocks) and the other varieties, non-initial varieties (or $j$-varieties).

It is evident that the $j$-blocks can be uniquely specified by naming the initial variety-pairs contained in them. The location of a $j$-variety in the $r(=k)$ blocks in which it occurs can therefore be expressed by naming the corresponding initial variety-pairs which, as have seen, are all different. Again any given $j$-variety must occur twice with each of the initial varieties. Hence in the $k$ pairs of these initial varieties, each of the $k$ numbers must occur twice. We can designate the $k$ pairs concisely by means of a chain of $k$ numbers consisting of one or more cycles as follows:

Arrange the pairs such that the last element of a pair is the same as the first element of the next pair. As soon as the last element of the pair is the same as the first element of the starting pair a cycle is closed. If all the pairs have not been already used in the cycle, start again with a pair not included in the first cycle, and complete a new cycle. In this way the whole set of the $k$ pairs corresponding to the given $j$ can be expressed by a chain of $k$ elements consisting of one or more cycles. We observe that these cycles must contain 3 or more elements. For in the process of formation of cycles out of a given number of pairs as outlined above, only one element cannot be left out; nor can one pair be left out, for this would mean that the varieties of this pair have not occurred more than once, which cannot be true.

It therefore follows that for $k = 5$, there is only one type of chain, *viz.*, $(abcde)$; the possible types

for $k = 6$ are $(abc)$ $(def)$ and $(abcdef)$;
for $k = 7$ are $(abc)$ $(defg)$ and $(abcdefg)$;
for $k = 8$ are $(abc)$ $(defgh)$, $(abcd)$ $(efgh)$ and $(abcdefgh)$;
for $k = 9$ are $(abc)$ $(def)(ghi)$, $(abc)$ $(defghi)$, $(abcd)$ $(efghi)$ and $(abcdefghi)$.

A cycle can be begun at any element and can be read either forward or backward. One cycle of $m$ elements can be put in $2m$ different forms. So the number of different cycles with $m$ elements $= \frac{1}{2}(m-1)!$. Thus, for $n = 3, 4, 5, 6$ there are respectively 1, 3, 12 and 60 different cycles. A cycle of 3 elements is unchanged by any permutation and may be called a simple cycle. The non-initial varieties, $j$, are named conveniently according to the following scheme:

The variety common to the blocks (1, 2) and (1, 3) is called $k+1$,
,, ,, (1, 2) and (1, 4) ,, $k+2$,
,, ,, (1, 2) and (1, $k$) ,, $2k-2$,

This completes the block (1, 2), and places one non-initial variety in each of the blocks (1, 3) to (1, $k$).

Then the variety common to the blocks (1, 3) and (1, 4) is called $2k-1$,
,, ,, ,, (1, 3) and (1, 5) ,, $2k$,
,, ,, ,, (1, 3) and (1, $k$) ,, $3k-5$.

This completes the block (1, 3) and places 2 non-initial varieties in each of the blocks (1, 4), ..., (1, $k$).

In this way all the varieties, $j$, are placed in the blocks (1, 2), (1, 3), ..., (1, $k$). It will be seen that this is only a convenient system of naming the varieties, and does not involve any loss of generality. Any possible design with $\lambda = 2$ admits of this standard placing of the first $k$ blocks. To make this more clear, the standard placing for $k = 6$ is shown below:

| Block Nos. | Varieties. | | | | | |
|---|---|---|---|---|---|---|
| 1: | 1 | 2 | 3 | 4 | 5 | 6 |
| 2: | 1 | 2 | 7 | 8 | 9 | 10 |
| 3: | 1 | 3 | 7 | 11 | 12 | 13 |
| 4: | 1 | 4 | 8 | 11 | 14 | 15 |
| 5: | 1 | 5 | 9 | 12 | 14 | 16 |
| 6: | 1 | 6 | 10 | 13 | 15 | 16 |
| 7: | 2 | 3 | | | | |
| 8: | 2 | 4 | | | | |
| 9: | 2 | 5 | | | | |
| 10: | 2 | 6 | | | | |
| 11: | 3 | 4 | | | | |
| 12: | 3 | 5 | | | | |
| 13: | 3 | 6 | | | | |
| 14: | 4 | 5 | | | | |
| 15: | 4 | 6 | | | | |
| 16: | 5 | 6 | | | | |

In this system the first $k$ blocks of any two solutions of a design must be identical. It is only in the manner of placing the varieties, $j$, in the next $(b-k)$ blocks not containing 1 that two solutions of a design (both put in the standard form) may differ. It thus appears that if two standardised solutions are not identical, block by block, it must be due to differences in the system of chains corresponding to their respective non-initial varieties.

As for the construction of the chains, it is clear that the chain for $(k+1)$ must contain the block-pair (1, 2) and (1, 3), *i.e.*, the appropriate cycle, if begun with 12 must end with 3. This consideration helps in quickly constructing the possible chains. As an easy example, in the case of $k=5$ the chains for the varieties 6 to 11 must be of the forms shown below:

| Variety. | Chains. | Variety. | Chains. |
|---|---|---|---|
| 6: | (12453); (12543) | 9: | (13254); (13524) |
| 7: | (12354); (12534) | 10: | (13245); (13425) |
| 8: | (12345); (12435) | 11: | (14235); (14325) |

Now, we must take a "consistent" set of chains. A set of chains may-be called consistent if between any two of them—

(*i*) three consecutive numbers of one do not coincide with three consecutive numbers of the other (read forward or backward),

and (*ii*) just two pairs of consecutive numbers of one coincide with two pairs of consecutive numbers of the other (read forward or backward). It may be added that the numbers at the beginning and end of a cycle are regarded as consecutive.

The violation of (*i*) indicates that between two blocks 3 varieties are common. Thus if the chains for $j_1$ and $j_2$ are $(abcdefgh)$ and $(abcd'e'f'g'h')$; the blocks $(a, b)$ and $(b, c)$ both contain $j_1$ and $j_2$, so that the three varieties $b$, $j_1$ and $j_2$ are common to these blocks, which is contrary to the general property of S.I.D. ref. to above. Therefore the condition (*i*) is necessary. The violation of (*ii*) obviously means that either more or less than 2 blocks contain the corresponding pair of varieties which is contrary to the condition $\lambda=2$.

A set of chains satisfying the conditions (*i*) and (*ii*) fully satisfy the conditions of the design, and therefore lead to a solution. For instance, if we adopt the chain (12435) as given above for the case $k=5$, the complete set of consistent chains will be:

6: (12453); 7: (12534); 8: (12345); 9: (13254); 10: (13425) and 11: (14235)

as can be easily verified. The remaining chains

6: (12543); 7: (12354); 8: (12435); 9: (13524); 10: (13245) and 11: (14325)

are also seen to be a consistent set. The solutions corresponding to these sets are written out in full as easy illustrations of building up complete solutions.

| Soln. for set 1. | | | | | Soln. for set 2. | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 6 | 7 | 8 | 1 | 2 | 6 | 7 | 8 |
| 1 | 3 | 6 | 9 | 10 | 1 | 3 | 6 | 9 | 10 |
| 1 | 4 | 7 | 9 | 11 | 1 | 4 | 7 | 9 | 11 |
| 1 | 5 | 8 | 10 | 11 | 1 | 5 | 8 | 10 | 11 |
| 2 | 3 | 8 | 9 | 11 | 2 | 3 | 7 | 10 | 11 |
| 2 | 4 | 6 | 10 | 11 | 2 | 4 | 8 | 9 | 10 |
| 2 | 5 | 7 | 9 | 10 | 2 | 5 | 6 | 9 | 11 |
| 3 | 4 | 7 | 8 | 10 | 3 | 4 | 6 | 8 | 11 |
| 3 | 5 | 6 | 7 | 11 | 3 | 5 | 7 | 8 | 9 |
| 4 | 5 | 6 | 8 | 9 | 4 | 5 | 6 | 7 | 10 |

We notice that the transposition (4, 5) makes the cycles for 6 identical for the two sets. But this transposition applied to the set 1 convert the cycle of 7 into that of 8 as in set 2.

| | | | | |
|---|---|---|---|---|
| ,, | 8 | ,, | 7 | ,, |
| ,, | 9 | ,, | 10 | ,, |
| ,, | 10 | ,, | 9 | ,, |
| ,, | 11 | ,, | 11 | ,, |

On applying the transposition (4, 5) and the consequential transpositions (7, 8), (9, 10) to the first solution we arrive at the second solution. This proves the isomorphism of the design with $k=5$. We notice that a set of consistent chains remains consistent after any permutation; the $j$-varieties receive consequential changes to correspond with the altered chains in accordance with the scheme of standard placing; and the blocks are re-arranged among themselves. It is easy to see that no new solution is obtained by any permutation of the chains of a given solution.

### §2. *Impossibility of the Design with $k=8$.*

This design requires 29 varieties of which 21 are non-initial, requiring 21 chains. In this case the chain-types are (123) (45678); (1234) (5678) and (12345678). We will call these $A$, $B$ and $C$ respectively. The type (12345) (678) is the same as $A$; we will, however, designate it by $A'$. We will first test the possibility of a solution using chains of type $B$ only. Take the chain 10: (1234) (5678) as the standard. The complete list of chains for the $j$-varieties 9, 11, 12, 13 and 14 consistent with this is shown below:—

| Serial No. of chains. | Variety 9 : | Variety 11 : | Variety 12 : | Variety 13 : | Variety 14 : |
|---|---|---|---|---|---|
| 1 | (1253) (4687) | (1245) (3687) | (1246) (3578) | (1247) (3568) | (1248) (3567) |
| 2 | (1253) (4768) | (1245) (3768) | (1246) (3758) | (1247) (3568) | (1248) (3576) |
| 3 | (1263) (4578) | (1265) (3748) | (1256) (3748) | (1257) (3468) | (1258) (3647) |
| 4 | (1263) (4758) | (1275) (3468) | (1276) (3548) | (1257) (3486) | (1268) (3457) |
| 5 | (1273) (1275) | (1275) (3486) | (1286) (3457) | (1267) (3548) | (1268) (3475) |
| 6 | (1273) (4586) | (1285) (3647) | (1286) (3475) | (1287) (3546) | (1278) (3546) |
| 7 | (1283) (4657) | | | | |
| 8 | (1283) (4576) | | | | |
| etc., | | | | | |

The second cycle of standard chain is unchanged by the permutations:

(5678); (5876); (57) (68); (58) (67); (57); (56) (78) and (68).

Looking down the column of chains for the variety 9, we see that these transformations change the first chain into the others. So it is enough to consider the standard chain of 10 in conjunction with the first chain of 9. Trying out the combination as shown below we see that we can at most find 5 consistent chains out of the 6 needed. Therefore, there can be no solution of the design consisting of chains of type $B$ only.

| | | | | | |
|---|---|---|---|---|---|
| 10: | (1234) (5678) | | | | |
| 9: | (1253) (4768) | | | | |
| 11: | (1245) (3768); | (1265) (3748); | (1275) (3486) | | |
| 12: | ? | (1286) (3457); | (1246) (3758); | (1256) (3748); | (1246) (3758) |
| 13: | ? | ? | (1267) (3548) | ? | (1287) (3546) |
| 14: | ? | ? | ? | ? | ? |

We now take the standard chain of type $A$, *viz.*, 9: (123) (45678) and seek a solution containing chains of other types without restriction. It appears that there are 51 chains consistent with this standard belonging to each of the varieties 10, 11, 12, ..., 17. However, by using either singly or in combination, the permutatives

$$(12);\ (45)\ (68);\ (46)\ (78);\ (47)\ (56);\ (48)\ (57)\ \text{and}\ (67)\ (85)$$

which leave the standard chain unchanged, it was found possible to reduce the number of effective chains. Thus in conjunction with the standard chain for variety 9, the number of "independent" chains for variety 10 was reduced from 51 to 15. The number of chains of type $A$ was reduced from 5 to 3, of type $B$ from 4 to 1, of type $A'$ from 6 to 2 and of type $C$ from 36 to 9. It is, therefore, enough to consider these 15 pairs of chains corresponding to the varieties 9 and 10, and look for consistent chains for the other $j$-varieties. It would be too elaborate to give here the complete set of consistent chains for each of these pairs, and the successive process of elimination, to arrive at a consistent set of chains for as many $j$-varieties as possible. The result of such processes is summarised below:

It is possible not only to obtain a consistent set of chains for all the $j$-varieties in the second block, but in many cases to obtain chains for the next $j$-variety 15 in the 3rd block. In one case it is even possible to obtain a chain for another $j$-variety, 16, in the third block. But it is not possible to get any chain for the $j$-variety 17. This proves the impossibility of the design containing a chain of type $A$.

It also follows that no solution can be obtained containing a chain of type $A'$. For by a suitable substitution of $i$-varieties, a chain of type $A'$ can be transformed to one of type $A$, and a whole set of consistent chains would be found containing a chain of type $A$, which we have proved by exhaustion to be impossible.

Actually, however, in order to avoid or minimise the chance of omission of a consistent chain further on, all the 51 chains for the variety 9 were used in the search for a solution. It therefore remains to examine whether a solution is possible containing chains of types $B$ and $C$ only. As we have already seen that a solution containing chains of type $B$ only, is impossible, there must be at least one chain of type $C$. Take as a standard of this type, the chain 14: (12345678). By applying the set of permutations that leave this chain unchanged, it is found on examination of the chains belonging to the $j$-variety 9 that 35 chains of type $C$ are reduced to 29, and 5 chains of type $B$ are reduced to 4. We are thus left with 33 chains for the variety 9 which, in combination with the standard chain for 14, may if at all, lead to a solution. On examination of all the chains, however, it is seen that by successive formation, only two sets of chains can be obtained going upto the $j$-variety 16, but none for the $j$-variety 17. Hence there is no solution containing a chain of type $C$.

This completes the proof of the impossibility of the Design with $k = 8$.

### §3. *A new solutions of the Design with $k = 9$*

This design requires 37 varieties of which 28 are non-initial requiring therefore 28 chains. The possible chain-types are

$$(123)\ (456)\ (789);\ (123)\ (456789);\ (1234)\ (56789);\ \text{and}\ (123456789)$$

which we may designate by $A, B, C, D$, respectively. The types (123456) (789) and (123456) (6789) which are included in the above will be designated as $B'$ and $C'$ for distinction.

In the classical solution the initial block is 1, 7, 9, 10, 12, 16, 26, 33, 34. The $(m+1)$th block is obtained from this adding $m$ to each of these numbers and reducing (mod 37). On examining this solution it appears that there are 19 chains of type $D$, 4 of type $C$ and 5 of type $C'$; there are none of types $A$, $B$ and $B'$. The method of exhaustion, as outlined before, may perhaps be used but the patience and labour involved will certainly be very great. The possibility of constructing a design containing chains of type $A$ only has been tested and resulted in the discovery of a new solution.

Let us take the standard chain as (123)(456)(789) corresponding to the variety 10. Chains consistent with this and belonging to $j$-varieties 11 to 16 are listed below:

11 : (124)(357)(689); (124)(358)(679); (124)(359)(678); (124)(367)(589); (124)(368)(579); (124)(369)(578)

12 : (125)(347)(689); (125)(348)(679); (125)(349)(678); (125)(367)(489); (125)(368)(479); (125)(369)(478)

13 : (126)(347)(589); (126)(348)(579); (126)(349)(578); (126)(357)(489); (126)(358)(479); (126)(359)(478)

14 : (127)(348)(569); (127)(349)(568); (127)(358)(469); (127)(359)(468); (127)(368)(459); (127)(369)(458)

15 : (128)(347)(569); (128)(349)(567); (128)(357)(469); (128)(359)(467); (128)(367)(459); (128)(369)(457)

16 : (129)(347)(568); (129)(348)(567); (129)(357)(468); (129)(358)(467); (129)(367)(458); (129)(368)(457)

We notice that the first chain for 11 can be transformed into the other chains for the same variety by transpositions which keep the standard chain unaltered. It is enough, therefore, to take one chain, say the first one, for 11. So we consider the pair of chains

10: (123) (456) (789)
11: (124) (357) (689)

We notice that this pair is unchanged by the transposition (89)

With this pair, the consistent chains for 12 are

(125)(348)(679); (125)(349)(678); (125)(368)(479) and (125)(369)(478).

But the first of these is transposed to the 2nd and the 3rd to the 4th by the transposition (89). Therefore it is enough to consider only the first and the 3rd chains for 12. In this way the working is carried to the chain for 16, and it is found that no set of consistent chains upto this can be found with one of the variations at 12. The only consistent set obtained is the following:—

10: (123)(456)(789)
11: (124)(357)(689)
12: (125)(368)(479)
13: (126)(349)(578)
14: (127)(348)(569)
15: (128)(359)(467)
16: (129)(367)(458)

These complete the placing of the 7 varieties in block No. 2. We have now to place 6 more varieties in block No. 3, *i.e.*, the block (1, 3).

The chains for variety 17 consistent with the standard chain are the following

17: (134)(257)(689); (134)(258)(679); (134)(259)(678);
(134)(267)(589); (134)(268)(579); (134)(269)(578).

We notice that the first and the last of these contain triads already used, the 4th and the 5th are inconsistent with the group already obtained. So we are left with the 2nd and the 3rd. In this way, examining 6 chains for the varieties 17 to 22, we get only one the set given below which is consistent with the previous sets and consistent among themselves :—

17: (134)(259)(678)
18: (135)(267)(489)
19: (136)(248)(579)
20: (137)(258)(469)
21: (138)(269)(457)
22: (139)(247)(568)

This completes the placing of the varieties in block No. 3.

After this, there is an unique way of obtaining the remaining chains. These are listed below :—

| | | |
|---|---|---|
| 23: (145)(278)(369) | 28: (156)(289)(347) | 33: (168)(237)(459) |
| 24: (146)(279)(358) | 29: (157)(239)(468) | 34: (169)(235)(478) |
| 25: (147)(236)(589) | 30: (158)(234)(679) | 35: (178)(249)(356) |
| 26: (148)(256)(379) | 31: (159)(246)(378) | 36: (179)(268)(345) |
| 27: (149)(238)(567) | 32: (167)(245)(389) | 37: (189)(257)(346) |

This is a consistent set of 28 chains providing a solution for the design with $k = 9$. It needs hardly to be pointed out that all the 48 triads have been used and no triad has occurred more than once.

### §4. *Conjugate Designs and another new solution for the Design $k = 9$*

A symmetric incomplete block design being given and the blocks numbered in any manner, we can write down corresponding to every variety a set of $r$ numbers which represent the block Nos. containing that particular variety. For the $v$ varieties there will be $v$ sets. These $v$ sets, each containing $r$ numbers, may be called a design conjugate to the given design. We will suppose, as in the case of $\lambda = 2$ that the numbering of the blocks has been made according to an initial block in the standard manner *i.e.*, the successive blocks are arranged in such a manner that the $k$ digits representing the variety may form an increasing set of number. In these designs $b = v$; $r = k$; each block-pair contains $\lambda$ varieties in common. A conjugate design has, therefore, the same parameters as the original design from which it is derived. For non-symmetric designs $b > v$; therefore the conjugate design is impossible as it would violate Fisher's condition $b \geq v$. We can prove that conjugates of a given design corresponding to different initial blocks are isomorphic.

It is interesting to note that the conjugate of a design is not necessarily the same solution of the design though the conjugate of a conjugate is the identical design. This is brought out by the example of the solution of the design $k=9$, with all the chains formed of 3 simple cycles, as given in this paper. The chains of the conjugate solution are recorded below.

| | | | |
|---|---|---|---|
| 10 : | (123)(475869) | 24 : | (146)(237598) |
| 11 : | (124)(367859) | 25 : | (147)(256938) |
| 12 : | (125)(348967) | 26 : | (148)(275369) |
| 13 : | (126)(359748) | 27 : | (149)(253786) |
| 14 : | (127)(354689) | 28 : | (156,(239487) |
| 15 : | (128)(365497) | 29 : | (157)(249836) |
| 16 : | (129)(346578) | 30 : | (158)(264739) |
| 17 : | (134)(265897) | 31 : | (159)(274386) |
| 18 : | (135)(246978) | 32 : | (167)(285349) |
| 19 : | (136)(254789) | 33 : | (168)(243975) |
| 20 : | (137)(248659) | 34 : | (169)(245837) |
| 21 : | (138)(259467) | 35 : | (178)(234596) |
| 22 : | (139)(267548) | 36 : | (179)(236485) |
| 23 : | (145)(238679) | 37 : | (189)(235674) |

It appears that all these chains are formed of one simple cycle and another cycle of 6 elements. The two solutions are, therefore *non-isomorphic*.

The classical solution of the design with $k=9$, is self-conjugate, *i.e.*, the conjugate of this solution is isomorphic with it.

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# A DISCUSSION ON THE EXACTNESS OF THE LORENTZ-DIRAC CLASSICAL EQUATIONS

By

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## 1. *Introduction*

The classical relativistic equations of motion of a radiating electron in an electromagnetic field were derived by Dirac (1938) by regarding the electron as a point-charge and applying the principles of conservation of energy and momentum. These equations take into account appropriately the effect of radiation damping on the motion of the electron. If $(x_0, x_1, x_2, x_3)$ denote the time and space co-ordinates of the electron in a Lorentz frame of reference in flat space-time, and $v_\mu$ is the velocity four-vector given by $v_\mu = (\dot{x}_0, \dot{x}_1, \dot{x}_2, \dot{x}_3)$, $\mu = 0, 1, 2, 3$, where dots denote differentiation with respect to the proper time $s$, then the Dirac equations of motion of the electron are

$$\left.\begin{aligned} m\dot{v}_\mu - \tfrac{2}{3}e^2(\ddot{v}_\mu + \dot{v}^2 v_\mu) &= ev_\sigma F^\sigma_\mu, \\ v^2 = \dot{x}_0^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2 &= 1 \end{aligned}\right\}, \tag{1}$$

where $F^{\mu\nu}$ are the usual field quantities describing the external electromagnetic field, and the scalar product notation

$$a_\mu b^\mu = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$$

is used. The units are chosen so that the velocity of light is unity. In terms of the electric and magnetic field vectors $\mathbf{E}$ and $\mathbf{H}$, these equations are

$$\left.\begin{aligned} m\dot{\mathbf{v}} - \tfrac{2}{3}e^2\{\ddot{\mathbf{v}} + (\ddot{x}_0^2 - \dot{\mathbf{v}}^2)\mathbf{v}\} &= e(\dot{x}_0\mathbf{E} + \mathbf{v}\times\mathbf{H}), \\ \dot{x}_0^2 - \mathbf{v}^2 &= 1 \end{aligned}\right\} \tag{2}$$

where $\mathbf{v}$ denotes the three-dimensional spatial part of the velocity four-vector $v_\mu$. When the velocity of the electron is small compared to the velocity of light, then the equations take the non-relativistic form

$$m\dot{\mathbf{v}} - \tfrac{2}{3}e^2\ddot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{H}), \tag{3}$$

where dots now denote differentiation with respect to the time $x_0$.

The equations (3) are the same as those obtained by Lorentz, who used the spherical model of the electron and calculated the force of retardation of the electromagnetic field inside the electron, obtaining for this damping force a series in ascending powers of the radius $r$ of the electron, thus

$$\mathbf{R} = \tfrac{2}{3}e^2\ddot{\mathbf{v}} + (\ )r_0 + (\ )r_0^2 + \ \dots\ , \tag{4}$$

If $r_0$ is considered small then $\mathbf{R}$ is approximately $\frac{2}{3}e^2\ddot{\mathbf{v}}$, and hence the equations of motion as derived by this method are the same as the equations (3).

Although both the above methods lead to the same equations of motion, the derivation by Lorentz makes them necessarily approximate, whereas the method of Dirac gives room to hope that these equations may be exact, within the limits of the classical theory. The suggestion that these equations appear to be exact was made by Dirac at the time he derived them; and all subsequent authors who have worked with these equations or who have extended the method of derivation of the equations of motion to spinning particles in electromagnetic and meson fields (Bhabha, 1939, 1941), have gone on the hypothesis that the equations are exact.

The purpose of this paper is to point out certain evidence discounting this suggestion of exactness.

## 2. *Free Electron*

The simplest application of the equations of motion is to investigate the motion of a free electron. This problem has been already considered by Dirac (*loc. cit.*, p. 156). In a suitable Lorentz frame in which the motion is rectilinear, the velocity $v$ is given by the equation

$$a\dot{v}-\ddot{v}+v\dot{v}^2/(1+v^2) = 0, \tag{5}$$

where $a = 3m/2e^2$. The equation (5) has solution of the form

$$v = \sinh\,(Ae^{as}+B), \tag{6}$$

where $A$ and $B$ are arbitrary constants. The general motion, with $A$ non-zero, is such that the electron steadily increases its velocity, while rapidly losing energy by radiation. Ultimately the velocity tends to the velocity of light. Such a self-accelerating motion has never been observed. Dirac's comments on this solution are of interest here and are quoted below:—

"One would be inclined to say that there is a mistake in sign in our equations and that we ought to have $e^{-as}$ insead of $e^{as}$ in (6). With this alteration we should have a theory in which, if an electron is disturbed in any way and then left alone, it would rapidly settle down into a state of constant velocity with emission of radiation while it is settling down. This would be a reasonable behaviour for an electron according to our present-day physical ideas. However it is not possible to tamper with the signs in our theory to obtain this result without getting equations of motion which would make the electron in the hydrogen atom spiral outwards, instead of spiralling inwards and ultimately falling into the nucleus, as it should in the classical theory. We are therefore forced to keep the signs in (6) as they are."

Thus, Dirac did not make any alterations in the signs, but retained the equations of motion (1), getting over the difficulty that is referred to above by adopting the following device. He postulated that not every solution of the equations of motion need necessarily correspond to a motion that is observable in Nature. The equations of motion have two types of solutions—physical solutions and non-physical solutions. The solution (6) with $A$ non-zero should be regarded as a non-physical solution. The physical solution has $A$ zero and corresponds to a motion of uniform velocity, which is the motion observed in Nature. In this way it was found possible to resolve the difficulty and to preserve the assumption that the equations are exact. There remained one defect in the theory, and that was the absence of an adequate criterion to distinguish between the physical and non-physical solutions. Recently, Eliezer and Mailvaganam (1945) have suggested that the physical solution should satisfy, apart from the equations of motion, an additional physical principle which has been called the principle of field-balance—that is, after a sufficient lapse of time the emitted and absorbed fields should tend to balance each other. If this result proves to be of general validity then we will have a satisfactory way of discriminating between the physical and non-physical solutions.

### 3. *Electron Disturbed by a Pulse*

A further difficulty arises when we apply the equations of motion to consider the behaviour of an electron which is initially at rest and which is disturbed by a pulse of electromagnetic radiation. If the duration of the pulse is supposed to be infinitesimally small and the electric field is given by

$$E_x = k\delta(t-y), \quad E_y = E_z = 0,$$

then the equations of motion give

$$a\dot{v}-\ddot{v} = \kappa\delta(t), \tag{7}$$

where $\kappa = 3k/2e$, and dots denote differentiation with respect to $t$. The solution that satisfies the boundary conditions at $t = 0$ and at $t = -\infty$ is seen to be of the form,

$$\left.\begin{aligned} v &= Ce^{at}, & t<0, \\ &= \kappa/a+(C-\kappa/a)e^{at}, & t>0, \end{aligned}\right\} \tag{8}$$

where $C$ is an arbitrary constant.

If the electron is initially at rest till $t = 0$, then $C = 0$. This gives

$$v = (\kappa/a)(1-e^{at}), \quad t>0,$$

which is a non-physical solution. It appears that the problem does not have a physically allowable solution.

Dirac again got over the difficulty by suggesting the following solution. Choose $C$ so that the final motion is one of uniform velocity, that is, take $C = \kappa/a$. Then

$$v = (\kappa/a)\, e^{at}, \quad t<0,$$

which implies that the electron is gradually building up an acceleration of such amount that the pulse just counter-balances it when the pulse reaches the electron at time $t = 0$. The difference between such a motion and one of uniform velocity is too small to be physically observable. This solution appears to contradict elementary ideas of causality; but a natural interpretation can be given if we suppose that the electron behaves as though it has a finite size of order $a^{-1}$.

### 4. *Electron in a Field of Potential* $\mu|x|$

The solution given by Dirac for the above problem looks rather artificial, but being the only solution of the equations of motion which does not differ appreciably from observable results, it has to be assumed to be the appropriate one, if the equations of motion are assumed to be exact. But when we consider the motion of an electron which moves in the field of potential $\mu|x|$, where $(x, y, z)$ are cartesian co-ordinates, then the above method does not help and all the solutions of the equations of motion appear to be of non-physical nature. This potential corresponds to the field due to a thin infinite plate in the $yz$-plane.

For simplicity we consider rectilinear motion normal to the plate. Then the velocity $v$ is given by

$$\left.\begin{aligned} a\dot{v}-\ddot{v}+v\dot{v}^2/(1+v^2) &= \alpha(1+v^2)^{\frac{1}{2}}, \quad \text{for } x<0 \\ a\dot{v}-\ddot{v}+v\dot{v}^2/(1+v^2) &= -\alpha(1+v^2)^{\frac{1}{2}}, \quad \text{for } x>0 \end{aligned}\right\}, \tag{9}$$

where $\alpha = 3\mu/2e$. We take the case when $\alpha$ is positive. We solve the equations by transforming to $\phi$ where $v = \sinh\phi$. The equations then become

$$a\dot{\phi}-\ddot{\phi} = \pm\alpha, \tag{10}$$

and the solution has the form

$$\phi = Ae^{as}+B\pm\alpha s/a. \tag{11}$$

Suppose that initially the electron is moving towards a point $O$ of the plate from the left, and reaches $O$ when $s = 0$ with velocity $v_0$. Till $s = 0$, the velocity will be of the form

$$v = \sinh\,(A_0e^{as}+B_0+\alpha s), \quad s<0; \tag{12}$$

as $s$ increases from $s = 0$, the electron will begin to move into the region $x>0$, and therefore the velocity will have the form

$$v = \sinh\,(A_1e^{as}+B_1-\alpha s), \quad s>0, \tag{13}$$

as long as $x>0$. The boundary conditions at $s = 0$ require that the velocity and acceleration should be continuous at $s = 0$, and therefore

$$\left.\begin{aligned} A_0+B_0 &= A_1+B_1, \\ aA_0+\alpha &= aA_1-\alpha. \end{aligned}\right\} \tag{14}$$

If $A$ is non-zero the motion after $s = 0$ is non-physical. Hence, following the notation of the previous example, we take $A_1 = 0$, thus obtaining a solution which corresponds to a physical motion after $s = 0$. Then

$$A_0 = -2\alpha/a, \quad B_1 = B_0-2\alpha/a = \sinh^{-1}v_0.$$

Hence for $s>0$, $x>0$, the solution is of the form

$$v = \sinh\,(\sinh^{-1}v_0-\alpha s). \tag{15}$$

Therefore

$$x = \alpha^{-1}\{\cosh\,(\sinh^{-1}v_0)-\cosh\,(\sinh^{-1}v_0-\alpha s)\}. \tag{16}$$

From (15) and (16) we see that as $s$ increases from $s = 0$, $v$ gradually decreases and vanishes, and then becomes negative. That is, the electron moves towards the right with decreasing velocity, comes to rest, and then commences to move backwards towards $O$ with gradually increasing speed. It reaches $O$ again at $s = 2\alpha^{-1}\sinh^{-1}v_0 = s_1$, say, and then goes over into the region of space $x < 0$. The motion in this region will be given by a solution of the form

$$v = \sinh\,(A_2 e^{\alpha s} + B_2 + \alpha s), \quad s > s_1. \tag{17}$$

The conditions of continuity at $x = 0$ require that

$$\left.\begin{aligned} A_2 + B_2 + \alpha s_1 &= B_0 - 2\alpha/a - \alpha s_1 \\ aA_2 + \alpha &= -\alpha. \end{aligned}\right\} \tag{18}$$

Hence

$$A_2 = -2\alpha/a, \quad B_2 = B_0 - 2\alpha s_1 = 2\alpha/a - 8\sinh^{-1}v_0. \tag{19}$$

We see that $A_2$ is non-vanishing. The solution after $s = s_1$ is such that the electron continues to move away from $O$ towards the left with ever increasing velocity This motion is clearly non-physical.

It appears that we cannot adjust the value of the arbitrary constants in such a way that the motion is physical after each instant at which the electron passes the point $x = 0$. The equation of motion do not have a physical solution.

## 5. *Discussion*

All the evidence of the above work points to the suggestion that the Lorentz-Dirac equations of motion are not exact. Dirac did not follow up his suggestion that a change in sign of some of the terms in his equations may be necessary, because he had thought that any alteration in sign would lead to a set of equations of motion according to which the electron in the hydrogen atom would spiral outwards, instead of spiralling inwards and falling into the nucleus.

The present author (Eliezer, 1943) has applied the Lorentz-Dirac equations to investigate the motion of an electron in the hydrogen atom, and arrived at the unexpected result that these equations do not allow the electron to spiral inwards and fall into the nucleus. This result now provides added reason to suppose that some modification of the theory is necessary.

The author has examined the possibility of modifying the Maxwell-Dirac theory, while retaining the idea of deriving equations of motion from the conservation laws. There are two modified forms of the theory which are of interest. One of these is seen to lead to equations of motion which differ from the corresponding Lorentz-Dirac equations only in the signs of certain terms. The result of the change of sign is that a free electron does not accelerate itself, nor does the electron in the hydrogen atom spiral outwards and escape to infinity. In the other form of the theory we postulate that in the classical theory the radiation field emitted by an electron, if such a field is emitted at all, has no retarding or accelerating effect on the motion of the electron. This assumption is a reasonable one, when we consider the electron as a point-charge. Starting with this assumption equations of motion are derived from the conservation laws. A detailed account of these two forms of the modified theory will be published shortly.

*Summary*

The paper gives a discussion of the applications of the Lorentz-Dirac equations of motion to investigate the motion of a free electron, of an electron disturbed by a pulse, and of an electron in the field of a charged thin infinite plate. The self-accelerating motions in the first case, the artificial nature of the only allowable physical solution in the second case, and the absence of a physically allowable solution in the third case—all point to the suggestion that the equations of motion are not exact. A preliminary account of suitable modifications of the theory is given.

The author wishes to express his gratitude to Professor P. A. M. Dirac under whose supervision this work was commenced.

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# ON THE PETERSEN-MORLEY THEOREM

By

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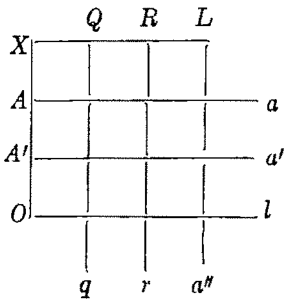
1. The theorem was established by Morley (1898) algebraically. Two geometrical proofs were given by Lyons and Firth (1934); the method used is that of approximations through special cases, and is not what is usual in this subject A verification by symbols was given by Baker (1936).

I give a direct geometrical proof based on known properties of the plane Desargues configuration and the idea of involutions.

2. We take the theorem in projective form. Given a plane $\Pi$ and therein a Desargues configuration of two triangles $ABC$ and $A'B'C'$ in perspective, with $O$ for centre and $LMN$ for axis: Through $A$, $B$, $C$ are drawn three arbitrary skew lines $a$, $b$, $c$ in space; through $A'$, $B'$, $C'$ are drawn the transversals $a'$, $b'$, $c'$ to pairs of these: for clearness $a'$ is the transversal from $A'$ to $b$ and $c$. Let $a''$ be the transversal from $L$ to $a$ and $a'$; let $b''$ be the transversal from $M$ to $b$ and $b'$; finally let $c''$ be the transversal from $N$ to $c$ and $c'$. Then the theorem states that there passes through $O$ a single transversal to all three lines $a''$, $b''$, $c''$.

3. In the Desargues figure on the plane $\Pi$ let $P$, $Q$. $R$ be the points of intersection of $BC'$ and $B'C$, of $CA'$ and $C'A$, of $AB'$ and $A'B$, respectively. Because the triangles $AB'C'$ and $A'BC$ are in perspective with $O$ for centre, it follows that $LQR$ is a line namely the axis. So also $MRP$ and $NPQ$ are lines.




4. Denote by $p$, $q$, $r$ the lines of intersection of the planes $bc'$ and $b'c$, of $ca'$ and $c'a$, of $ab'$ and $a'b$. Then the lines $p$, $q$, $r$ pass through the points $P$, $Q$, $R$ respectively because the lines $a$, $a'$, ..., have been drawn through the points $A$, $A'$, ... .

5. The pair of lines $a$, $a'$ and the pair of lines $q$, $r$ make a skew quadrilateral with four corners. The line $OAA'$ is a secant to the former pair. The line $LQR$ is a secant to the latter pair, for as seen in § 4 the point $Q$ lies on $q$ and the point $R$ on $r$. But these two secants, both lying in the plane $\Pi$, intersect say in $X$.

The five points $X$, $Q$, $R$, $A$, $A'$ and the four corners mentioned fix a quadric having $a$, $a'$, $q$, $r$ as generators and also the lines $XQR$ and $XAA'$; but these last two lines are the same as $LQR$ and $OAA'$ which are thus two generators through $L$ and $O$ respectively. The other generator through $L$ is $a''$ and thus intersects the other generator through $O$ *viz.*, the transversal from $O$ to $q$, $r$. This in effect is what German writers speak of as the $g_4$-$s_4$ theorem on the quadric.

6. Therefore denoting the secants from $O$ to $q$, $r$ to $r$, $p$ and to $p$, $q$ by $l$, $m$, $n$ respectively we find that $a''$ meets $l$, and $b''$ meets $m$, and $c''$ meets $n$.

Thus the planes $(O, a'')$, $(O, b'')$, $(O, c'')$ are respectively the *same* as the planes $(L, l)$, $(M, m)$, $(N, n)$.

7. Again in the plane $\Pi$ let the projections from $O$ of $P$, $Q$, $R$ on to the line $LMN$ be called $P_0$, $Q_0$, $R_0$. We have seen in §3 that $LQR$ is a line; it follows on considering the quadrangle $OPQR$ that the pairs of points $P_0L$, $Q_0M$, $R_0N$ are in involution.

8. Also from the definition in §6 of the lines $m$ and $n$ it follows that the plane of $O$, $m$, $n$ contains the line $p$; and therefore also the point $P$ in view of §4; and therefore also the point $P_0$ by the definition of $P_0$ in §7.

In other words, the planes $(m, n)$, $(n, l)$, $(l, m)$ are met by the line $LMN$ in the points $P_0$, $Q_0$, $R_0$ respectively.

9. Here we have three lines through $O$ namely $l$, $m$, $n$ and three collinear points $L$, $M$, $N$ associated with them in order; the planes $(m, n)$, $(n, l)$, $(l, m)$ are met by the line $LMN$ in the three points $P_0$, $Q_0$, $R_0$ respectively; and finally the pairs of points $P_0L$, $Q_0M$, $R_0N$ are in involution, as seen in §7.

We deduce at once that the planes $(L, l)$, $(M, m)$, $(N, n)$ are coaxial, for the involution property just mentioned is easily recognised to be the necessary and sufficient condition for the desired coaxiality. What we need for our present purpose is the sufficiency. That the condition stated is sufficient follows on considering the three plane-pairs $(m, n)$, $(l, L)$ and $(n, l)$, $(m, M)$ and $(l, m)$, $(n, N)$ and observing that all three plane-pairs pass through the three lines $l$, $m$, $n$. Thus the three planes $(l, L)$ and $(m, M)$ and $(n, N)$ are coaxial.

These three planes are however, as shown in §6, just the planes $(O, a'')$ and $(O, b'')$ and $(O, c'')$ which are thus shown to have a common line. Therefore the lines $a''$, $b''$, $c''$ permit a transversal through $O$. That completes the proof of the theorem.

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# NOTE ON TRANSVERSALS WHICH MEET CONSECUTIVE GENERATORS OF A RULED SURFACE AT A CONSTANT ANGLE

By

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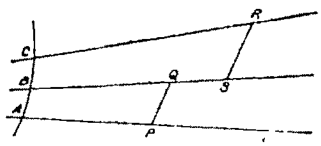
1. Line of striction of the generators of a ruled surface is the locus of points where the common perpendiculars to consecutive generators meet them. In this note, transversals of consecutive generators meeting them at a constant angle are considered. They are found to meet the generators at the central points. Such transversals generate a ruled surface the generators of which have the same line of striction. A new measure called the "Skewness of distribution" of the generators of the ruled surface is introduced and certain properties of this quantity are discussed. It is found that it is connected with first curvature of the surface.

2. Let $ABC$ be the directrix, $AP$, $BQ$, $CR$ three consecutive generators of the ruled surface. Let $PQ$, $SR$ be transversals which meet pairs of consecutive generators at the same angle $\alpha$. We shall call such transversals as the $\alpha$-transversals.




Let the unit vector along $PQ$ with the usual notation* be denoted by

$$\delta = d \cos\alpha + \frac{d'}{a} \cos\beta + \frac{d \times d'}{a} \cos\gamma$$

here

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Now because

$$\frac{\delta . \{d + d'\delta s + \ldots\}}{| d + d'\delta s + \ldots |} = \cos\alpha$$

we get

$$\cos\alpha + a \cos\beta . \delta s + \ldots\ldots = \cos\alpha\,(1 + a^2\delta s^2 + \ldots)^{\frac{1}{2}}$$

whence

$$\cos\beta = 0 \quad i.e., \ \beta = \frac{\pi}{2} \text{ and } \gamma = \frac{\pi}{2} - \alpha.$$

Hence

$$\delta = d \cos\alpha + \frac{d \times d'}{a} \sin\alpha.$$

Now $P$ and $S$ are consecutive points on the curve which is the locus of the feet of the $\alpha$-tranversals and therefore $PS$ in the limit is the tangent to the curve.

$$\overline{PS} = \overline{PQ} + \overline{QS},$$

* The notation followed is that of Weatherburn's Differential Geometry vol. 1.

or

$$\overline{PS} = \xi\delta + \eta(d + d'\delta s + \ldots)$$

where $\xi$ and $\eta$ are infinitesimals.

$$\therefore \quad \overline{PS}.d' = \{\xi\delta + \eta(d + d'\delta s + \ldots)\}.d' = \text{an infinitesimal of the second order.}$$

Hence if $T$ denote the unit tangent to the locus of $P$, $T.d' = 0$ and therefore $P$ is the central point of the generator. Hence we have the theorem that " *Transversals which meet consecutive generators of a ruled surface at a constant angle, meet the generators at the central points and hence the locus of the feet of such transversals is the line of striction of the generators.*"

3. Taking the line of striction as the Directrix, the vector position of $Q$ is $t\delta s + \eta(d + d'\delta s + \ldots)$. $Q$ also lies on $PQ$ whose equation is $r = \xi\delta$.

$$\therefore \quad t\delta s + \eta(d + d'\delta s + \ldots) = \xi\{d \cos\alpha + \frac{d \times d'}{a} \sin\alpha\}.$$

Now because

$$t = d \cos\theta + \frac{d \times d'}{a} \sin\theta \,*$$

where $\theta$ is the angle at which the line of striction meets the generator, we have solving the equations and neglecting infinitesimals of second and higher orders

$$\eta = \delta s \frac{\sin(\theta - \alpha)}{\sin\alpha}. \tag{3.1}$$

Hence "the $\alpha$-transversal meets the consecutive generator at a distance

$$\delta s \frac{\sin(\theta - \alpha)}{\sin\alpha}$$

from the central point of the consecutive generator." In particular, the common perpendicular meets the consecutive generator at a distance $-\delta s \cos\theta$ from the central point.

4. These $\alpha$-transversals generate a ruled surface which has clearly the same curve as the line of striction. In order to find the parameter of distribution of the $\alpha$-transversals on the ruled surface generated by them we first put $[d, d', d''] = a^3\gamma$ where dashes denote differentiation with respect to the arc of the line of striction.

Now

$$\delta = d \cos\alpha + \frac{d \times d'}{a} \sin\alpha,$$

$$\delta' = d' \cos\alpha + \frac{d \times d''}{a} \sin\alpha - \frac{d \times d'}{a^2} a' \sin\alpha.$$

But

$$d'' = -a^2 d + \frac{a'}{a} d' + a\gamma(d \times d') \tag{4.1}$$

$$\therefore \quad \delta' = d' \cos\alpha + \frac{\sin\alpha}{a} d \times \{-a^2 d + \frac{a'}{a} d' + a\gamma(d \times d')\} - \frac{d \times d'}{a^2} a' \sin\alpha = d'(\cos\alpha - \gamma \sin\alpha).$$

Hence

$$A = |\,\delta'\,| = a\,|\cos\alpha - \gamma \sin\alpha\,|.$$

* Here $\beta$, the parameter of distribution is equal to $-\sin\theta/a$.

Now because the $\alpha$-transversals meet the line of striction at an angle $\theta-\alpha$, *the parameter of distribution $B$ of the $\alpha$-transversals is given by*

$$B=-\frac{\sin(\theta-\alpha)}{a\,|\cos\alpha-\gamma\sin\alpha|}. \tag{4.2}$$

In particular, *the parameter of the distribution of the common perpendiculars is* $\cos\theta/a\gamma$.

5. *The quantity $\gamma$ introduced above is independent of the Directrix chosen.* Let $d$, $d_1$, $d_2$, be three consecutive generators so that

$$d_1=d+d'\delta s+d''\frac{\delta s^2}{2!}+\ldots,$$

$$d_2=d+d'(\delta s+\delta s_1)+d''\frac{(\delta s+\delta s_1)^2}{2!}+\ldots.$$

The volume of the parallelopiped formed by unit vectors along its three generators is given by

$$V=\frac{[d,\,d_1,\,d_2]}{|d_1|\,|d_2|}=\frac{1}{2}\frac{[d,\,d',\,d'']\delta s\delta s_1(\delta s+\delta s_1)+\ldots}{|d_1|\,|d_2|}$$

$$=\tfrac{1}{2}[d,\,d',\,d'']\delta s\delta s_1(\delta s+\delta s_1)+\text{higher order infinitesimals.}$$

The shortest distances between pairs of these three generators are

$$\frac{D\delta s}{a},\quad \left\{\frac{D}{a}+\frac{d}{ds}\left(\frac{D}{a}\right)\delta s+\ldots\ldots\right\}\delta s_1 \text{ and } \frac{D}{a}(\delta s+\delta s_1).$$

Hence if $p$ denote the product of these three shortest distances we have

$$\underset{\delta s\to 0,\delta s_1\to 0}{Lt}\ \frac{V}{p}=\frac{a^3[d,\,d',\,d'']}{D^3}=\frac{a^6\gamma}{a^6\beta^3}=\frac{\gamma}{\beta^3} \tag{5.1}$$

where $\beta$ is the parameter of distribution. Now $V$, $p$, $\beta$ are independent of the Directrix chosen and therefore $\gamma$ is also so. *This quantity $\gamma$ may be called the "skewness of distribution" of the generators and its vanishing means that the generators are all parallel to the same plane* as in the case of a paraboloid.

6. The skewness of distribution of the $\alpha$-transversals is given by

$$\Gamma=[\delta,\,\delta',\,\delta'']/A^3$$

where

$$A=a\,|\cos\alpha-\gamma\sin\alpha|.$$

Now

$$\delta=d\cos\alpha+\frac{d\times d'}{a}\sin\alpha$$

$$\delta'=d'(\cos\alpha-\gamma\sin\alpha)$$

$$\delta''=d''(\cos\alpha-\gamma\sin\alpha)-d'\gamma'\sin\alpha.$$

$$\therefore\quad [\delta,\,\delta',\,\delta'']=a^6(\cos\alpha-\gamma\sin\alpha)^2(\gamma\cos\alpha+\sin\alpha).$$

Hence

$$\Gamma=\frac{\gamma\cos\alpha+\sin\alpha}{|\cos\alpha-\gamma\sin\alpha|}. \tag{6.1}$$

Putting $\alpha+\frac{\pi}{2}$ for $\alpha$ we get $\Gamma'$, the skewness of distribution of the $\left(\frac{\pi}{2}+\alpha\right)$-transversals.

$$\Gamma' = \frac{\cos\alpha-\gamma\sin\alpha}{|\gamma\cos\alpha+\sin\alpha|}. \tag{6.2}$$

Hence $\Gamma\Gamma'=\pm 1$. Hence we get the theorem that "*The skewness of distribution of the mutually perpendicular transversals of the consecutive generators of a ruled surface are (numerically) reciprocals to one another.*"

For a developable surface, the skewness $=[t,\ t',\ t'']/|t'|^2$, where $t$ denotes the unit tangent to the edge of regression. Hence *the skewness of distribution of the generators of a developable surface is $\rho/\sigma$, where $\rho$ and $\sigma$ are the radii of curvature and torsion of its edge of regression.*

It can be easily verified that the skewness of distribution for the two systems of generators of a hyperboloid of one sheet at a point of the principal elliptic section are equal and opposite.

7. The quantity $\gamma$ introduced above occurs in the curvature properties of the surface. Taking the line of striction as the Directrix we have

$$\begin{aligned} H^3J &= (t'+ud'').\{(d\times t)+u(d\times d')\}-2\cos\theta[t,\ d',\ d] \\ &= u^2[d,\ d',\ d'']+u\{[d,\ t,\ d'']+[d,\ d',\ t']\}+2\cos\theta[d,\ d',\ t]. \end{aligned}$$

where $J$ denotes the first curvature of the surface. Now

$$t = d\cos\theta+\frac{d\times d'}{a}\sin\theta$$

Therefore

$$\begin{aligned} t' &= d'\cos\theta+\frac{d\times d''}{a}\sin\theta-d\sin\theta\frac{d\theta}{ds}+\frac{d\times d'}{a}\cos\theta\frac{d\theta}{ds}-\frac{d\times d'}{a^2}a'\sin\theta \\ &= -d\sin\theta\frac{d\theta}{ds}+d'(\cos\theta-\gamma\sin\theta)+\frac{d\times d'}{a}\cos\theta\frac{d\theta}{ds}. \end{aligned}$$

Substituting this value for $t'$ and the value of $d''$ in §4 we get

$$H^3J = a^3\gamma u^2-a^2u\frac{d\beta}{ds}+a\sin\theta(\cos\theta+\gamma\sin\theta). \tag{7.1}$$

In particular, $J_0$ the first curvature at the the central point in given by

$$J_0 = \frac{a}{\sin\theta}(\gamma+\cot\theta) = -\frac{\gamma+\cot\theta}{\beta}.$$

From the value of $t'$ we get if $k$ denotes the curvature of the line of striction

$$k^2 = \left(\frac{d\theta}{ds}\right)^2+a^2(\cos\theta-\gamma\sin\theta)^2.$$

Now because $d\theta/ds$ is the geodesic curvature of the line of striction, we get $k_n$ the normal curvature of the surface along the line of striction is given by

$$k_n = a(\cos\theta-\gamma\sin\theta) = -\frac{\sin^2\theta}{\beta}(\cot\theta-\gamma). \tag{7.2}$$

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# VIBRATION OF AIR-COLUMNS IN CLOSED PIPES

By

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(*Communicated by Mr. R. N. Mohanty—Received November 10, 1945*)

In a recent paper Mohanty and Patnaik (1944) have deduced formulae giving the normal modes of vibration of air columns in a pipe open at both the ends and fitted symmetrically with thin movable frictionless pistons each of mass $M$ and each controlled by a spring of strength $m^2$. In this note the normal modes of vibration are deduced in a pipe, either closed at both the ends or closed at one end and open at the other, the pipe being fitted symmetrically with $n$ pistons of the type referred to above.

It is assumed that the air in the pipe is at atmospheric pressure throughout. The notations used here are the same as in the paper cited above, *viz.*, the origin is at a closed end, $\xi_r$ $(r = 1, 2, \ldots, n-1)$ represents the displacement at a distance $x$ at the time $t$ of the air between the $r$th and the $(r+1)$th pistons, and $\xi_0$, $\xi_n$ represent the displacements respectively of the air between the origin and the first piston, and the last piston and the other end of the pipe.

Suppose $a$ is the distance between adjacent pistons, $c$, the velocity of the vibrations in air and $\gamma_r e^{ipt}$, the oscillations of the $r$th piston, where $p$ is the angular frequency of one of the normal modes. We have also the solution

$$\xi_r = \{A_r \cos (px/c) + B_r \sin (px/c)\} e^{ipt}, \quad (r = 0, 1, \ldots, n). \tag{1}$$

Taking first the case of the pipe closed at both the ends, the boundary conditions are

$$\xi_0 = 0, \text{ when } x = 0; \quad \xi_0 = \gamma_1 e^{ipt}, \text{ when } x = a;$$

$$\xi_r = \gamma_r e^{ipt}, \text{ when } x = ra; \quad \xi_r = \gamma_{r+1} e^{ipt}, \text{ when } x = (r+1)a, \quad (r = 1, 2, \ldots, n-1);$$

and

$$\xi_n = \gamma_n e^{ipt}, \text{ when } x = na; \quad \xi_n = 0, \text{ when } x = (n+1)a = l.$$

We therefore get the following equations:

$$\xi_0 = \frac{\gamma_1 \sin (px/c)}{\sin (pa/c)} e^{ipt}, \tag{2}$$

$$\xi_r = \frac{\gamma_{r+1} \sin \{p(x-ra)/c\} - \gamma_r \sin \{p(x-\overline{r+1}\, a)/c\}}{\sin (pa/c)} e^{ipt}, \quad (r = 1, 2, \ldots, n-1), \tag{3}$$

and

$$\xi_n = -\frac{\gamma_n \sin\{p(x-\overline{n+1}\, a)/c\}}{\sin (pa/c)} e^{ipt}. \tag{4}$$

Since, the equation of motion of the $r$th piston is given by

$$M \frac{\partial^2}{\partial t^2} (\gamma_r e^{ipt}) + Mm^2 (\gamma_r e^{ipt}) = -c^2 \rho S \Big[ \Big( \frac{\partial \xi_{r-1}}{\partial x} \Big) - \Big( \frac{\partial \xi_r}{\partial x} \Big) \Big]_{x=ra} \quad (r = 1, 2, \ldots, n),$$

where $\rho$ is the density of air in the pipe and $S$ the sectional area of the pipe, we get, substituting from equations (2), (3) and (4), symplifying, and putting

$$X = \frac{M}{c\rho S}\,\frac{m^2-p^2}{p}\,\sin\frac{pa}{c}+2\cos\frac{pa}{c},$$

we get

$$\gamma_1 X-\gamma_2+0\;+\;\ldots\ldots\ldots\ldots\;=0, \tag{5}$$

$$\ldots\ldots\;+0-\gamma_{r-1}+\gamma_r X-\gamma_{r+1}+0\;\ldots\ldots\;=0,\quad (r=2,3,\ldots,n-1) \tag{6}$$

$$\ldots\ldots\ldots\ldots\;+0-\gamma_{n-1}+\gamma_n X=0. \tag{7}$$

The frequencies of the normal modes are therefore given by the determinantal equation

$$T_n \equiv \begin{vmatrix} X & -1 & 0 & . & . & . & . & . & . & . \\ -1 & X & -1 & 0 & . & . & . & . & . & . \\ 0 & -1 & X & -1 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & -1 & X & -1 & 0 \\ . & . & . & . & . & . & 0 & -1 & X & -1 \\ . & . & . & . & . & . & . & 0 & -1 & X \end{vmatrix} = 0. \tag{8}$$

Using the result

$$T_n = \frac{\sin\,(n+1)\phi}{\sin\phi},$$

where $\cos\phi = X/2$, we get

$$\frac{\sin\,(n+1)\phi}{\sin\phi} = 0,$$

or

$$\phi = \frac{q\pi}{n+1}$$

where $q$ is an integer other than an exact multiple of $(n+1)$. Thus the frequencies of the normal modes are given by the relation

$$\frac{X}{2} = \cos\frac{q\pi}{n+1},$$

or

$$\frac{M}{2c\rho S}\,\frac{m^2-p^2}{p} = \cot\frac{pa}{c}-\cos\frac{q\pi}{n+1}\,\operatorname{cosec}\frac{pa}{c} = \cot\frac{pa}{c}-\alpha\,\operatorname{cosec}\frac{pa}{c}, \tag{9}$$

where $\alpha$ stands for $\cos\dfrac{q\pi}{n+1}$ and has, therefore, only $n$ different values all lying between $-1$ and $+1$. If $n$ be odd, one value of $\alpha$ is zero $\{$when $q=(n+1)/2\}$.

If the pipe contains a gas other than air and the pressure be different from atmospheric pressure, equation (9) still holds good, $\rho$ being the density of the gas and $c$ the velocity of the vibrations in the gas.

If the pipe be closed at one end only and open at the other, the boundary conditions for $\xi_n$ are $\xi_n = \gamma_n e^{ipt}$, when $x = na$ and $\frac{\partial \xi_n}{\partial x} = 0$, when $x = (n+1)a = l$. Equations (2) and (3) remain unaltered but equation (4) is changed to

$$\xi_n = \frac{\gamma_n \cos \{p(x-\overline{n+1}a)/c\}}{\cos (pa/c)} e^{ipt}. \tag{10}$$

The frequencies of the normal modes are given in this case by the determinantal equation

$$\Delta_n \equiv \begin{vmatrix} X & -1 & 0 & . & . & . & . & . & . & . & . \\ -1 & X & -1 & 0 & . & . & . & . & . & . & . \\ 0 & -1 & X & -1 & 0 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 & -1 & X & -1 & 0 \\ . & . & . & . & . & . & . & 0 & -1 & X & -1 \\ . & . & . & . & . & . & . & . & 0 & -1 & X-\sec\theta \end{vmatrix} = 0, \tag{11}$$

where $\theta$ stands for $pa/c$.

$$\Delta_n = T_n - T_{n-1} \sec \theta = \frac{\sin (n+1)\phi}{\sin \phi} - \frac{\sin n\phi}{\sin \phi} \sec \theta.$$

Hence, equation (11) is equivalent to the equation

$$\cot^2 \frac{\theta}{2} = \cot \frac{\phi}{2} \Big/ \cot \frac{(2n+1)\phi}{2} . \tag{12}$$

In case of the pipe open at both the ends the following pair of equations is obtained (see the paper referred to).

$$\cot^2 \frac{\theta}{2} = \cot \frac{\phi}{2} \Big/ \cot \frac{n\phi}{2},$$

and

$$\cot^2 \frac{\theta}{2} = -\cot \frac{\phi}{2} \cot \frac{n\phi}{2}.$$

It will be seen that in all the three cases the frequencies of the normal modes are given in the form of determinantal equations, *viz.*, $D_n = 0$, $\Delta_n = 0$ and $T_n = 0$. The determinants $D_n$, $\Delta_n$ and $T_n$ are connected by the relations

$$D_n = (X - \sec \theta)\Delta_{n-1} - \Delta_{n-2},$$

$$\Delta_n = T_n - T_{n-1} \sec \theta, \text{ and } T_n = \frac{\sin (n+1)\phi}{\sin \phi}.$$

The simplest case is that of the pipe closed at both the ends. Equation (9) obtained in this case can be easily solved graphically, the values of $p$ thus determined giving the angular frequencies of the normal modes.

In conclusion, I express my thankfulness to Mr. R. N. Mohanty for his interest in this work.

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*Reference*

Mohanty, R. N., and Patnaik, B. C., (1941), *Bull. Cal Math Soc.*, **36**, 79-82.

# NOTE ON THE LARGE-SCALE MOTION IN VISCOUS STARS*

By

N. R. Sen and N. L. Ghosh

(Received December 17, 1945)

## Introduction

Not much is known about the motion of a mass of viscous fluid under its own gravitational, and external forces. A recent attempt has been made by Gunnar Randers (1941) to study the large-scale motions in a stellar body of this type. The motion, as may be expected, is extremely complicated, and at this stage only a recording of the very general features of the motion, as has been done by Randers may be considered useful.

We limit ourselves exclusively to the case of axial symmetry. This case has been studied somewhat in detail by Randers. The principal results he has obtained are that in the case of symmetry about the equatorial plane in addition to the axial symmetry (*i*) there is in the meridianal quadrant no point or line (excepting boundary and the axis) of extremum circulation in the fluid; (*ii*) on the external boundary the gradient of the angular velocity $\omega$, is entirely along it, *i.e.*, on the boundary $\partial\omega/\partial n$ vanishes; (*iii*) if the angular velocity $\omega$ is monotone along both the polar and equatorial radius it must increase inwards; (*iv*) the stream-lines in a meridian plane are a set of closed curves encircling a point near the equatorial corner of the quadrant, so that there are two closed vortex-lines in the fluid one in each of the hemispheres. The other important conclusions relate to the order of the veloicty components in the fluid and certain other consequences when the meridianal velocity is small.

The results recorded in this note all relate to Randers' analysis from which it is shown that some further important conclusions may be drawn. In what follows we have preserved Randers' notation, except for using $\omega$ for $W$ as the symbol for angular velocity, and $r$ for $R$ as that for axial distance.

## 2. The hydrodynamical equations and the boundary conditions

We briefly give below the relevant equations obtained by Randers. From the equation of hydrodynamical motion

$$\rho\frac{d\mathbf{V}}{dt} = -\rho\nabla\Phi-\nabla p+\mu\{\tfrac{1}{3}\nabla\,\mathrm{div}\,\mathbf{V}+\nabla^2\mathbf{V}\}, \tag{1}$$

where the external force is $-\nabla\Phi$, the $\varphi$-equation in the case of steady motion is obtained as

$$\rho\left[\mathbf{V}\nabla v+\frac{uv}{r}\right] = \mu\left[\nabla^2 v-\frac{v}{r^2}\right] \tag{2}$$

where $u$ and $v$ stand for the $r$ and $\varphi$ components of $\mathbf{V}$ respectively.

* This note is the result of several discussions by the authors on the article *Large-scale motion in stars* by Gunnar Randers in the *Astrophysical Journal* (1941), 94, 109.

This combined with the equation of continuity

$$\operatorname{div} \rho \mathbf{V} = 0 \tag{3}$$

gives

$$\operatorname{div}\{r^2(\rho \mathbf{V}\omega - \mu \nabla \omega)\} = 0 \tag{4}$$

where we have replaced $v$ by $r\omega$.

Putting

$$r^2(\rho \mathbf{V}\omega - \mu \nabla \omega) = \operatorname{rot} \mathbf{b} \tag{5}$$

and

$$\rho \mathbf{V} = \operatorname{rot} \mathbf{a} \tag{6}$$

we obtain

$$r^2\omega \operatorname{rot} \mathbf{a} - \mu r^2 \nabla \omega = \operatorname{rot} \mathbf{b}. \tag{7}$$

Now putting

$$r\mathbf{a}_\phi = A, \quad r\mathbf{b}_\phi = B,$$

the $z$- and $r$-components of equation (7) can be written as

$$\omega r \frac{\partial A}{\partial r} - \mu r^2 \frac{\partial \omega}{\partial z} = \frac{1}{r}\frac{\partial B}{\partial r}, \tag{8a}$$

$$\omega r \frac{\partial A}{\partial z} + \mu r^2 \frac{\partial \omega}{\partial r} = \frac{1}{r}\frac{\partial B}{\partial z}. \tag{8b}$$

From equations (8) Randers deduces the following

$$\omega r^2 dA - dB = \mu r^3 \frac{\partial \omega}{\partial n} . ds \tag{9}$$

where $ds$ and $dn$ are the elements of any arc and the corresponding normal respectively, $dn$ being directed to the left of $ds$.

Equations (8) and (9) are important in Randers' analysis. Our discussion below, excepting Section 3, is also based on them. Randers' boundary conditions reduce to $A = 0$, $B = 0$, on the equatorial and polar radii as well as on the boundary of the meridian section. In addition $\partial \omega/\partial n$ vanishes on the curved boundary. This corresponds to the vanishing of the azimuthal stress component on the boundary. There are two other boundary conditions corresponding to the vanishing of the two other stress components on the boundary. These latter involve the density, the meridianal velocity components, and their first derivatives.

The condition $\partial \omega/\partial n = 0$, on the curved boundary, we shall refer to below as Randers' boundary condition. In the following discussion we shall consider the motion to be steady and make use of equations (8) and (9). Further, on account of the symmetry assumed we shall take $\partial \omega/\partial r = 0$, on the polar axis, and $\partial \omega/\partial z = 0$, on the equatorial radius of the meridian section, in addition to Randers' boundary condition.

## 3. On rotating viscous fluid

We consider the possibility of a mass of viscous fluid rotating steadily about the polar axis with differential rotation. There will be surfaces of equal rotation within the fluid, and this rotation will change from one such surface to another so that we may consider the whole mass of fluid to be made up of thin shells, each rotating as a rigid body and gliding over the adjoining shells, resisted by viscous forces.

If we put all velocity components except $v$ equal to zero, we obtain,

$$0 = \mu\left[\nabla^2 v - \frac{v}{r^2}\right], \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

For a viscous fluid $\mu \neq 0$, hence we have putting $v = r\omega$

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{3}{r}\frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial z^2} = 0. \tag{10}$$

From this we conclude that the existence of viscosity imposes a restriction on $\omega$ given by equation (10) in case of steady motion of differential rotation about an axis. The equations of motions now all become independent of $\mu$. The other two equations are the same as for frictionless fluids. The consideration of viscosity only imposes a restriction on $\omega$ given by (10), and a further restriction, $\partial\omega/\partial n = 0$, on the external boundary, the other two boundary conditions being now identfcally satisfied. If there be differential rotations of frictionless fluid which also obey these additional restrictions, these motions will exist even when the fluid has any viscosity. It is quite possible the only solution is $\omega =$ *constant*. If, however, such differential rotation be possible, we find further important restrictions regarding the rotation. These are put by our boundary conditions and given by the following two theorems.

**Theorem 1.** *When a viscous mass of gravitating fluid has steady rotation about an axis which is an axis of symmetry, and the surfaces of equal rotation have no conical point on the axis, then the angular velocity cannot steadily increase or decrease from the axis parallel to the equatorial radius.*

Let $OAB$ be a quadrant of the meridian section, the coordinates $r$ and $z$ being measured along $OA$ and $OB$ respectively. The angular velocity $\omega$ at any point of the quadrant may be regarded as a function of $r$ and $z$, which satisfies the differential equation (10). We multiply this equation by $drdz$, and integrate over the quadrant. Applying Gauss's integral we have

$$\int \frac{\partial \omega}{\partial n}\, ds + 3\int \frac{1}{r}\frac{\partial \omega}{\partial r}.drdz = 0 \tag{11}$$

where the first integral is taken over the three boundaries. On account of the symmetry assumed $\partial\omega/\partial z = 0$ on $OA$, and on $OB$ $\partial\omega/\partial r = 0$ when the level surfaces are smooth on the polar axis, and further by Randers' condition $\partial\omega/\partial n = 0$ on $AB$. Hence we must have

$$\int \frac{1}{r}\frac{\partial \omega}{\partial r}\, drdz = 0. \tag{12}$$

This is an impossibility if $\partial\omega/\partial r$ is positive or negative everywhere, that is, if $\omega$ either always increases or decreases everywhere parallel to the equatorial radius.

It may be noted that equation (12) is consistent with $\partial\omega/\partial r = 0$, that is when the surface of equal rotation are planes parallel to the equator; but this is inconsistent with the surface condition $\partial\omega/\partial n = 0$, except when the boundary is a cylinder. But $\omega =$ *constant* is an exception, so that uniform rotation of a viscous fluid is surely a possibility.

Some of the excluded cases implied by Theorem 1 are of great interest. To establish them completely, we first prove that in order that steady differential rotation may be

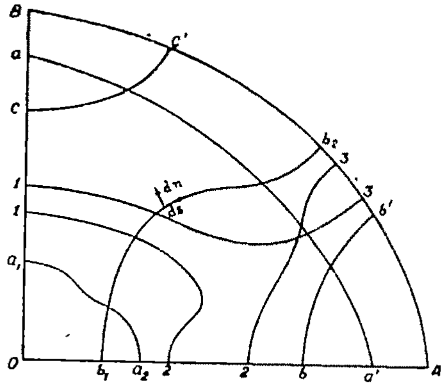


FIG. 1

possible, $\omega$ should not only not change monotonically in the direction of $r$ but $\omega$ cannot also be an extremum on a level line inside the quadrant, except when such lines have 'folds' of a particular type.

**Theorem 2.** *Under conditions of Theorem 1, if the $\omega$ level lines on the meridianal quadrant be smooth curves between any two boundaries and have no folds (hump or trough) over the equatorial axis, then excepting the axial boundary and the equatorial plane there can be no surface in the fluid on which the angular velocity may be an extremum.*

The proof given below will apply equally to level lines of the type $aa'$, $bb'$, $cc'$, 12, and 23, as on such lines there is no extremum of $z$, and so no hump or trough over $OA$. For instance, suppose 12 is a curve on which $\omega$ is an extremum, and between 12 and $AB$ there is no other curve on which $\omega$ has this property. Then integrating equation (10) multiplied by $drdz$ over the area $12AB$, and arguing as in Theorem 1, we obtain equation (12), since $\partial\omega/\partial n$ will vanish over the entire boundary. But, for the assumption made, $\partial\omega/\partial r$ will be of the same sign in case of curves of the type 12 in this area, and hence equation (12) cannot be satisfied. Hence for $\omega$ level lines of the type 12 there cannot be a line of extremum $\omega$ inside the quadrant. Similarly, for $\omega$ level lines of the type 23 there cannot be a line of extremum $\omega$ within the quadrant. We note the above argument fails in the case of $\omega$ lines having humps, or troughs with respect to the equatorial radius, in which case $\partial\omega/\partial r$ changes sign within the area considered. This happens in the case, for example, of lines of the type 13.

Combining Theorems 1 and 2, we may conclude that no steady differential rotation of the above symmetrical type of a viscous fluid is possible in which the $\omega$ level lines have no folds over the equatorial axis. As special case we note that steady differential rotation of a viscous fluid with smooth ovals as $\omega$ level surfaces is not possible. Similarly, the surfaces produced by the rotation of $bb'$, or $cc'$, about the axis of symmetry, are also excluded as $\omega$ level surfaces.

The point which is of importance in the above theorems is that they involve the monotonic behaviour of $\omega$ in *one direction* only, *namely, perpendicular to the axis of* *symmetry*. Further, the theorems are independent of pressure density relation in the fluid. It is known that differential rotation is not possible in a fluid in which the pressure is a function of density alone. A proof of this is also involved in the work of G. Randers. This proof is dependent on the boundary conditions. It is, however, of interest to note that if the pressure is a function of the density alone, then *independent of the boundary conditions* there can be no steady rotation of a viscous fluid except when the whole fluid rotates as a rigid body with uniform angular velocity. Equation (10) is only the $\varphi$-component of the vector equation (1). In the case when $u = 0$, $w$ (component parallel to $z$-axis) $= 0$, the $r$- and $z$-components of this equation are the same as those for a non-viscous fluid with differential rotation about the $z$-axis. If we now assume $p$ to be a function of $\rho$ only, then by Poincaré's theorem $\omega$ should be a function of $r^2$ only. Equation (10) now reduces to

$$\frac{d^2\omega}{dr^2}+\frac{3}{r}\frac{d\omega}{dr}=0$$

of which the general solution is

$$\omega=\frac{A}{r^2}+B.$$

If $\omega$ is to be bounded on the axis $A = 0$ which also ensures zero circulation on the axis. Hence the only allowable value of $\omega$ is a constant.

Considering the results of Theorems 1 and 2, it seems very probable that stable rotating configurations of viscous fluid will be possible only when $\omega$ is constant all throughout the mass.

## 4. Some properties of general motion

In the general case of motion with axial symmetry we have to satisfy the differential equations (8a) and (8b). We assume symmetry about the equatorial plane The boundary conditions become $\partial\omega/\partial n = 0$, on the quadrantal boundary, on which further $A = 0$ and $B = 0$.

Apart from the differential equation the boundary conditions alone introduce some restrictions on the values of angular rotation $\omega$. On the quadrant $OAB$ of the meridian section we note that in the case of symmetry about the equatorial plane the gradiant of $\omega$ vanishes at each of the three corners $O$, $A$ and $B$. From this we conclude that either all the points $O$, $A$ and $B$ are the points of maximum or minimum $\omega$, or, one, two, or all the three boundaries (polar and equatorial radii, and the curved boundary) are lines of maximum or minimum $\omega$. In the first case there will be at least three different families of surfaces forming level surfaces, and in some cases there will be maximum or minimum of $\omega$ on the boundary. Leaving out these cases which may arise as the result of the action of very complicated factors within stellar bodies, we shall examine only those conditions under which $\omega$ may always increase or decrease inwards parallel to the polar, or equatorial radius.

We can have all the three boundaries of the quadrant as lines of maximum, or minimum $\omega$; to this we add the two cases when the curved boundary and only one of the two straight axes are lines of maximum, or minimum $\omega$. But these possibilities are all excluded by the general property (i) proved by Randers, according to which the $\Omega$ lines, that is the lines of equal circulation should all start at right angles from the equatorial radius and end in the curved boundary. It is not necessary to postulate any particular variation of $\omega$ in these cases. The case of the two straight boundaries being lines of maximum or minimum $\omega$ is incompatible with the assumtion that $\omega$ is monotone parallel to the polar or equatorial axis; besides in this case there will be a point of extremum $\omega$ over the curved boundary.

We can also have only one of the three boundaries as a line of maximum or minimum $\omega$. Then curves of the type 11, 22, 33, may be possible types of $\omega$ level lines, and in space $\omega$ level surfaces will be those generated by the rotation of these lines about the axis of symmetry.

The curves of the type $bb'$, and $cc'$ as level lines are ruled out by Randers' theorem (*iii*), namely, that if the angular velocity be monotone in both $r$ and $z$ directions, it must increase inwards. But Randers' theorem is proved under the assumption that $\omega$ is monotone in both the $r$ and $z$ directions. In the theorems of this section we only assume that a $\omega$ level line goes from one boundary to another, and has no "fold" over the axis of symmetry, so that $\omega$ need not necessarily be monotone parallel to the equatorial radius. The absence of "fold" over the axis of symmetry is indicated by the property that along a $\omega$ level line the co-ordinate $r$ never attains a maximum, or minimum value. Under this restricting condition further properties of the $\omega$ level lines are given by the following two theorems.

**Theorem 8.** *Under conditions of symmetry assumed in Section 4, the angular velocity $\omega$ cannot attain a maximum or minimum value within the fluid on a $\omega$ level line which joins any two of the boundaries of a meridianal section and has no fold over the axis of symmetry.*

We take the dynamical equation (9) and integrate this equation over a level line on which $\omega$ is an extremum, say over $b_1b_2$ (Fig. 1). The directions of $ds$ and $dn$ have been indicated in the figure, $dn$ lying to the left of the positive direction of $ds$. On all the boundaries both $A$ and $B$ are zero; $A = constant$ are stream lines, and Randers has proved that $A$ increases from the boundary inside the fluid, so that $A$ is always positive. If $\omega$ is an extremum over $b_1b_2$, we have

$$\omega \int r^2 dA = 0,$$

or, by partial integration

$$\omega \int A d(r^2) = 0 \tag{13}$$

the integral being taken over the level line $b_1b_2$. Unless $\omega$ vanishes, since $A$ is positive this equation cannot be satisfied when there is no extremum of $r^2$ on the $\omega$ level line, in other words this level line has no fold over the axis of symmetry. But $\omega$ cannot vanish on a level line inside the fluid, since then it would be a $\Omega$ level line on which $\Omega = 0$. But the only $\Omega$ line on which $\Omega = 0$ is the axis of symmetry, as has been proved by Randers. Hence the theorem is proved for $\omega$ level lines of type $b_1b_2$. Similar proof would apply to level lines of the type $a_1a_2$ and 13.

**Theorem 4.** *Under conditions of symmetry assumed in Section 4, if the $\omega$ level lines in the fluid be of the type such as $a_1a_2$, and 13 which have no folds over the axis of symmetry, then the $\omega$ values will increase inwards (i.e., from a corner A, or B towards the interior) on these lines; if the $\omega$ level lines be of the type such as $b_1b_2$, with no folds over the axis of symmetry, then the $\omega$ values will decrease inwards on these lines.*

The proof follows as above from integration of equation (9) over the $\omega$ line. For instance, for integration over the $b_1b_2$ level line we have as before

$$\omega \int Ad(r^2) + \mu \int r^3 \frac{\partial\omega}{\partial n}\,ds = 0. \tag{14}$$

For integration from $b_1$ to $b_2$, the first integral is positive, as by our assumption there is no maximum of $r^2$ on the level line; further if the $\omega$ values increase inwards towards the axis on the level line, $\partial\omega/\partial n$ is also positive, and the above equation cannot be satisfied. Since by Theorem 3, $\omega$ should vary monotonically on the successive $\omega$ lines, it must decrease inward on the level lines of type $b_1b_2$.

For level lines of the type $a_1a_2$, and 13, equation (14) will still be true. Similar arguments will show that on these level lines $\omega$ must increase inwards.

We again stress the point that Theorems 3 and 4 give us new information in the way that for their validity it is only assumed that $r$ does not attain extremum value on the $\omega$ lines, which is less stringent than the condition that $\omega$ is monotone in both $r$ and $z$ directions.

When the $\omega$ line is of the type 13 the equatorial plane rotates as a rigid plane and has the maximum rotation. For the type $aa'$, or $a_1a_2$ the boundary surface is a $\omega$ level surface. Since $\partial\omega/\partial n$ vanishes on the boundary, there is in these cases a thin surface layer in which every particle has the same $\omega$ value. The arguments given in Section 6 will show that this layer will not, however, rotate as a rigid body with an angular velocity $\omega$.

## 5. A PROPERTY OF ANGULAR ROTATION

It has been shown by Randers that if the angular velocity $\omega$ be monotone parallel to both the equatorial and polar radii, then it will increase inwards. This combined with our Theorem 4 shows that $\omega$ level lines cannot be of the type $bb'$ on which $\omega$ must increase outwards.

The above theorem of Randers enunciates a property of the first derivatives of $\omega$, namely of $\partial\omega/\partial r$ and $\partial\omega/\partial z$, derived from the dynamical equations. Similarly, the dynamical equations determine another property of the second derivatives. A typical stream line is $PQR$ (Fig. 2). There will in the meridianal plane be a current such as in the direction from $P$ to $Q$ which we shall call *current below the surface*. Similarly there will be a current such as $QR$ which we shall call *current in the equatorial region*. Then we have

**Theorem 5.** *Under conditions of Section 4, the angular velocity $\omega$ cannot decrease in both the $r$ and $z$ directions slower than $1/r^3$, anywhere in the current below the surface.*

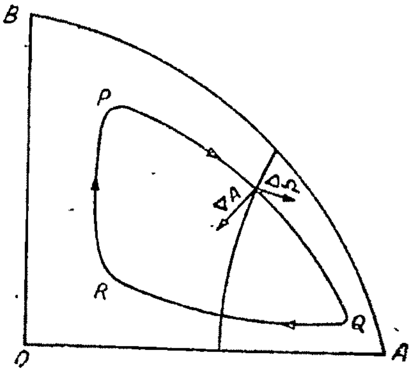


FIG 2.

From (8a) and (8b) by eliminating $B$ we obtain

$$[\nabla A \times \nabla \Omega] = \mu \operatorname{div} (r^3 \nabla \omega), \tag{15}$$

$$\operatorname{div} \equiv \frac{\partial}{\partial r} + \frac{\partial}{\partial z},$$

where $\Omega = r^2\omega$, $\nabla$ represents the operator gradient, and the bracket on the left represents the vector product. From Fig. 2 it is apparent that $\nabla\Omega$ is to the left of $\nabla A$ in the current below the surface. Hence $\operatorname{div}(r^3\nabla\omega)$ *must be positive*. This means

$$\frac{\partial}{\partial r}(r^3 \partial\omega/\partial r) + \frac{\partial}{\partial z}(r^3 \partial\omega/\partial z)$$

must be positive. From this on the first hand we conclude that $r^3\partial\omega/\partial r$ and $r^3\partial\omega/\partial z$ cannot both decrease outwards; in other words (since both the derivatives of $\omega$ are assumed to be negative) at any point in the current below the surface $\omega$ cannot decrease in *both* $r$ and $z$ directions slower than $1/r^3$.

We further note that if in the current below the surface $\omega$ increases inwards parallel to the equatorial axis, then $\partial^2\omega/\partial r^2$ and $\partial^2\omega/\partial z^2$ cannot both be negative at any point.

Analogous properties of the second derivatives of $\omega$ can also be established for the current in the equatorial region.

## 6. On the stream lines in symmetrical motion

From the picture of the meridianal stream lines in case of symmetry about an axis and the equator we can have an idea of how the particles of fluid will move in the general case. The motion, of course, is equivalent to an anxial rotation with meridianal current. The stream lines $A$ = *constant* on the meridianal quadrant are the projections of the paths of the particles on this plane by circles parallel to the equator. If we rotate a closed curve $A$ = *const.* about the axis we obtain a tubular surface. A particle once on this surface will lie entirely on this surface at all subsequent times. Since the meridianal section of this tube is the path $A$ = *const.*, and on this curve there is no velocity component in the direction of the normal in this plane, there will be no velocity component perpendicular to the surface of the tube at any time. A particle will move on one of these tubes from the poleside top towards the bottom of this tube in a curve lying on that side of its surface which faces the boundary of the stellar body (the circulation on the meridianal plane in the upper hemisphere is clockwise), then pass round the bottom and move up the tubular surface on a curve lying on that side of this tube which faces the axis. Even in the case of steady motion it is not necessary that the path of the particle on the tube should come back on itself by the time a rotation in azimuth is completed. It may go up and down the tubular surface several times (or not even once) by the time a rotation round the axis is completed, and proceed to describe an exactly similar path displaced with respect to the previous one on the surface. The paths may indeed never be closed at all. There will be such tubes one within another on which the fluid particles will move. The particles on the surface will circulate on the surface and the equatorial plane, and return to the surface through the polar axis. What has been said at the end of Section 4 will now be clear. The fluid particles, so long as they lie within the surface layer spoken of there, will all have the same $\omega$, but they will wander into the region of equatorial plane and the axis, when they will have different angular velocities.

It may be noticed there is according to this picture enough scope for the mixing of the material of the deep interior with that of external region. For symmetrical motions of the type considered there will then be due to convection no lack of stirring of the stellar material. Stagnation in the process of energy generation cannot be expected to be general.

## 7. Solutions of equations in two cases

Though the equations of motion and the boundary conditions for the steady motion of a viscous fluid in a stellar body with axial and equatorial symmetry obtained in §2 do not appear to be complicated, the construction of a simple case in which complete integration is possible, paying due attention to the requirements at the boundary, does not appear to be at all easy. Such an analytical solution would be valuable in studying the simpler characteristics of some typical motions. It has not indeed been possible to find a simple distribution in $\omega$ satisfying all the requisite conditions and giving the typical stream lines described above. Nevertheless, complete solution of the equations giving the stream lines has been discussed in two cases which are of interest from other points of view.* A general point of interest which easily comes out from the analysis below is that the *pattern of meridianal stream lines is independent of the co-efficient of viscosity $\mu$, when this coefficient is constant throughout the configuration.*

Case 1. Jeans (1921) has discussed the case for which the law of rotation is

$$\omega = f(\theta)/R^2 \tag{16}$$

* The complete boundary conditions have not been attended to. The surface conditions $p_{rs} = p_{\epsilon s} = 0$ have been ignored. The boundaries, strictly speaking, are to be kept in form by surface forces.

$R$, $\theta$ being the polar coordinates of a point in the quadrant, in some detail and has concluded that such rotation is quite plausible in a stellar body excepting in the internal core. But no boundary condition has been taken account of in that discussion. The boundary which will satisfy Randers' boundary condition for this $\omega$ is not simple. We have, however, taken a circular boundary $R = a$ for our integration. We can make the boundary a stream line though Randers' boundary condition will not be satisfied there. The flow is expected to be different from that described in Section 4 also on account of the fact that the origin is a singularity.

Eliminate $B$ from equations (8). We obtain a partial differential equation in $A$, which can be solved by the subsidiary equations

$$\frac{dr}{\partial(\omega r^2)/\partial z} = \frac{dz}{-\partial(\omega r^2)/\partial r} = \frac{d(A/\mu)}{\partial(r^3\partial\omega/\partial z)/\partial z + \partial(r^3\partial\omega/\partial r)/\partial r}.$$

∴ The combination of $\mu$ with $A$ shows that the pattern of meridianal stream lines is independent of $\mu$, and is determined by the distribution of $\omega$ only. But since the velocity components $u$ and $w$ will depend on $\mu$, the pattern of stream lines in space on the tubular surface will in general depend on $\mu$. It will be convenient to introduce polar co-ordinates $R$, $\theta$ and write these equations as

$$\frac{RdR}{\partial(\omega R^2\cos^2\theta)/\partial\theta} = \frac{Rd\theta}{-\partial(\omega R^2\cos^2\theta)/\partial R} = \frac{d(A/\mu)}{\partial(r^3\partial\omega/\partial z)/\partial z + \partial(r^3\partial\omega/\partial r)/\partial r} \tag{17}$$

putting

$$r = R\cos\theta, \quad z = R\sin\theta.$$

The third denominator can be calculated for $\omega$ given by (16). It takes the form $(1/R)Y(\theta)$, where

$$Y(\theta) = \cos^2\theta(\cos\theta . f'' - 3\sin\theta . f' - 2\cos\theta . f). \tag{18}$$

and

$$f' = \partial f/\partial R, \quad f'' = (\partial^2 f/\partial R^2),$$

An integral of (17) is obviously

$$\omega R^2\cos^2\theta = \text{constant}, \tag{19}$$

which for the present value of $\omega$ takes the form $\theta = const.$ The general integral is then furnished by

$$Y(\theta)dR = (1/\mu)\,\partial(f\cos^2\theta)/\partial\theta . dA,$$

$\theta$, for the time being being regarded as a constant. This furnishes the integral

$$A = \frac{\mu\cos^2\theta(\cos\theta . f'' - 3\sin\theta . f' - 2\cos\theta . f)}{\partial(f\cos^2\theta)/\partial\theta}\,r + F(\theta)$$

$F(\theta)$ being an arbitrary function of $\theta$. We can now choose $F(\theta)$ such that for $R = a$, $A = 0$, independently of $\theta$.

This gives

$$A/\mu = (a-R).\frac{\cos^2\theta(2.\cos\theta . f + 3\sin\theta . f' - \cos\theta . f'')}{\partial(f\cos^2\theta)/\partial\theta}. \tag{20}$$

We shall now further choose $f(\theta)$ such that the two axes also become stream-lines.

We take the simple case

$$\omega = \frac{B \sin\theta}{R^2}.$$

The stream lines will be given by $A =$ const, where

$$A/\mu = 4(a-R) \sin\theta \cos^2\theta/(3 \cos^2\theta - 2) \tag{21}$$

so that

$$R = a-(A/\mu)(3 \cos^2\theta - 2)/4 \sin\theta \cos^2\theta. \tag{21a}$$

The stream line patterns are independent of the constant $B$ which was proved generally by G. Randers.

The axes are the stream lines $A = 0$. Also on $R = a$, $A$ vanishes for all values of $\theta$, except $\theta_0$, where $\cos\theta_0 = \sqrt{(2/3)}$. On the line $\theta = \theta_0$, $A = \infty$. The stream-lines are shown in figure 3, and join the origin to the point $P(\theta = \theta_0, R = a)$. The line $A = \infty$ divides the quadrant into two parts. On the lower part we have stream lines corresponding to all positive values of $A$, and on the upper part those with negative values of $A$. By calculating the velocities it may be shown that the flow is from the point $P$ to the centre $O$, along different paths on two sides of $OP$, on which the velocities are finite. Along $PO$ the motion is from $P$ to $O$, with infinite velocity. This flow is entirely of a different pattern from what has been described in Section 4, and resembles that between a source ($P$), and a sink ($O$). It has no resemblance whatsoever with the pattern of closed stream lines we are looking for. It may be noted that as the distribution of $\omega$ does not satisfy Randers' boundary condition, the boundary is to be kept in form by external constraint.

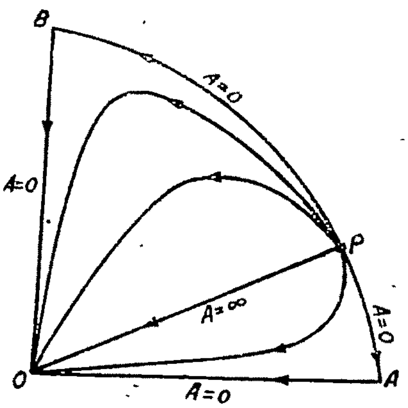


FIG 3.

The case $\omega = B/R^2$, can be easily deduced from this. The stream lines are

$$A/\mu = (R-a)\cos^2\theta/\sin\theta \tag{22}$$

and have the same pattern as in Fig. 3. The point $P$ now comes on the equator so that only negative values of $A$ are now allowed.

Case 2. The surface of the sun in the lower latitudes has a distribution of angular velocity of the type (Unsöld, 1938)

$$\omega = a - b \sin^2\theta. \tag{23}$$

If we assume this to be the distribution of $\omega$ within a stellar body, particles on circular cones with apex at the centre will have the same angular rotation. If the boundary of the meridianal section be a circle then the condition $\partial\omega/\partial n = 0$ will be satisfied on this boundary.

As before (19) is an integral of the equations (17). There is another integral which gives the general integral

$$A/\mu = \tfrac{1}{2}bR\cos\theta\,(a-b\sin^2\theta)^{\frac{1}{2}}\int^{\theta}\frac{\cos^4 t-4\sin^4 t-3\sin^2 t\cos^2 t}{(a-b\sin^2 t)^{3/2}}\,dt+F(\omega R^2\cos^2\theta).$$

To get a completely integrable case we put $a = b$ Then we have the general integral

$$A/\mu = R\cos^2\theta\left[\tfrac{5}{2}\log\frac{1+\sin\theta}{1-\sin\theta}-2\log(\sec\theta+\tan\theta)-2\tan\theta\sec\theta\right]+F(R\cos^2\theta).$$

We now choose the arbitrary function $F$ such that $A = 0$, on the boundary $R = a$ (assumed circular) for all values of $\theta$. This gives

$$F = -R\cos^2\theta\Big[\tfrac{5}{2}\log\frac{1+\sqrt{(1-R/a.\cos^2\theta)}}{1-\sqrt{(1-R/a.\cos^2\theta)}}-2\log\{\sqrt{(a/R)}.\sec\theta+\sqrt{(a/R.\sec^2\theta-1)}\}$$
$$-2\sqrt{(a/R)}.\sec\theta.\sqrt{(a/R.\sec^2\theta-1)}\Big].$$

Hence the stream lines ($A$ = const) will be given by

$$A/\mu = R\cos^2\theta\Big[\tfrac{3}{2}\log\frac{(1+\sin\theta)\{1-\sqrt{(1-R/a.\cos^2\theta)}\}}{(1-\sin\theta)\{1+\sqrt{(1-R/a.\cos^2\theta)}\}}$$
$$-2\tan\theta\sec\theta+2\sec\theta\sqrt{\{a/R.(a/R.\sec^2\theta-1)\}}\Big].$$

It can be shown that $\theta = 0$, and $\theta = \pi/2$, *i.e.*, the equatorial and polar radii are not stream lines. The stream line pattern is very peculiar. Generally, the lines are curves concave to the centre going from the polar to the equatorial axis, bulging out towards the middle and moving nearer the centre before they meet the equatorial axis. The stream line $A = 0$ is the circular boundary, and a small curve between the point on the equator (end of equatorial radius) and a neighbouring point on the equatorial radius (this part of the line is to be completed by its image in the lower quadrant). Within the loop of this small curve the stream lines are given by negative values of $A$; in the rest of the quadrant the stream lines are given by positive values of $A$. At the centre $A/\mu = 2$. The stream lines cut only the polar axis normally, but not the equatorial radius.* This pattern also can hardly correspond to actual motion in stellar bodies. These two cases only point to the difficulty of constructing a case of possible motion true to the necessary mathematical conditions.

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* The authors desire to thank Mr. G. Bandyopadhyay for drawing the pattern of these stream lines.

# ON PARALLELISM IN RIEMANNIAN SPACE—II

By

R. N. Sen

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§1. This paper is a continuation of a previous one (Sen, 1944) in which, with the introduction of an orthogonal ennuple ${}_{\tau}h^i$ ($\tau$, $i = 1, 2, \ldots, n$) at each point of a Riemannian space whose metric is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j,$$

where

$$g_{ij} = \sum_{\tau} {}^{\tau}h_i\, {}^{\tau}h_j, \quad {}^{\tau}h_i = g_{ij}\, {}_{\tau}h^j,$$

the parallel displacement of a vector $V^l$ defined by

$$dV^l + \nabla^l_{ij} V^i dx^j = 0 \tag{1.1}$$

where

$$\nabla^l_{ij} = \tfrac{1}{2}\, {}_{\tau}h^l \left( \frac{\partial\, {}^{\tau}h_i}{\partial x^j} + \frac{\partial\, {}^{\tau}h_j}{\partial x^i} \right)$$

was considered, and its connection with Levi-Civita parallelism for the expression of some fundamental invariants was undertaken.

With reference to the letters used as indices, small Latin italics letters are used for coordinate indices while capital and small Roman letters are used for indices referring to the ennuple. The signs of summation with respect to indices when they occur once above and once below are, as usual, left out, but other $\Sigma$'s are retained.

Using the notation ( ) with a subscript to denote covariant derivative with respect to (1.1),

$$({}^{\tau}h_i)_j + ({}^{\tau}h_j)_i = 0 \tag{1.2}$$

whence

$$(g_{ij})_k + (g_{jk})_i + (g_{ki})_j = 0, \tag{1.3}$$

$$({}^{\tau}h_i)_{jk} + ({}^{\tau}h_j)_{ki} + ({}^{\tau}h_k)_{ij} = 0. \tag{1.4}$$

If $R^t_{jkl}$ denotes the curvature tensor formed with respect to (1.1),

$$({}^{\tau}h_j)_{kl} - ({}^{\tau}h_j)_{lk} = ({}^{\tau}h_l)_{kj} = {}^{\tau}h_t R^t_{jkl}. \tag{1.5}$$

Also, if we put $R_{ijkl} = g_{ti} R^t_{jkl}$, then

$$(g_{ij})_{kl} + (g_{ik})_{lj} + (g_{il})_{jk} = R_{jlki} + R_{lkji} + R_{kjli}. \tag{1.6}$$

This is seen by multiplying (1.5) by ${}^{\tau}h_i$, summing for $\tau$, and arranging.

Now, if $K^t_{jkl}$ is the curvature tensor formed with respect to the Levi-Civita parallelism, and $K_{ijkl} = g_{ti} K^t_{jkl}$, then, as is well-known,

$$K_{ijkl} + K_{ijlk} = 0, \quad K_{ijkl} + K_{jikl} = 0, \quad K_{ijkl} = K_{klij}, \quad K_{ijkl} + K_{iklj} + K_{iljk} = 0. \tag{1.7}$$

These properties are not, of course, all independent.

Obviously, $R_{ijkl}$ possesses only the first and the last of the four properties (1.7) by virtue of the very nature of construction of a curvature tensor and of the symmetric property of $\nabla^l_{ij}$ in $i$ and $j$.

The tensor $R^t_{jkl}$ can be expressed as

$$R^t_{jkl} = (\Lambda^t_{jk})_l - (\Lambda^t_{jl})_k + \Lambda^t_{sl}\Lambda^s_{jk} - \Lambda^t_{sk}\Lambda^s_{jl} \tag{1.8}$$

where

$$\Lambda^t_{ij} = \tfrac{1}{2}\,{}_T h^t\left(\frac{\partial\, {}^T h_i}{\partial x^j} - \frac{\partial\, {}^T h_j}{\partial x^i}\right) = {}_T h^t({}^T h_i)_j.$$

Also, putting

$$g^{kl}(g^{ij})_l = (g^{ij})^k,$$

it can be seen from (1.3) that

$$(g^{ij})^k + (g^{jk})^i + (g^{ki})^j = 0. \tag{1.9}$$

§2. In the paper referred to it was shown that

$$K_{ijkl} - R_{ijkl} = g^{st}\{(g_{jk})_s(g_{il})_t - (g_{jl})_s(g_{ik})_t\} + (g_{jk})_{il} - (g_{jl})_{ik}, \tag{2.1}$$

Put

$$P_{ijkl} = g^{st}\{(g_{jk})_s(g_{il})_t - (g_{jl})_s(g_{ik})_t\}, \tag{2.2}$$

$$A_{ijkl} = (g_{jk})_{il} - (g_{jl})_{ik}. \tag{2.3}$$

It is seen that $P_{ijkl}$ possesses all the properties (1.7) while $A_{ijkl}$ possesses only the first and the last of these properties.

Put

$$\bar{A}_{ijkl} = A_{ijkl} + A_{jilk} + A_{klij} + A_{lkji}. \tag{2.4}$$

Since

$$(g_{jk})_{il} = \sum_T\{{}^T h_j({}^T h_k)_{il} + {}^T h_k({}^T h_j)_{il} + ({}^T h_j)_i({}^T h_k)_l + ({}^T h_j)_l({}^T h_k)_i\},$$

it follows in consequence of (1.2), (1.4) and (1.5) that

$$\bar{A}_{ijkl} = 3\{R_{ijkl} + R_{jilk} + R_{klij} + R_{lkji}\} + 4\sum_T\{2({}^T h_j)_i({}^T h_k)_l + ({}^T h_j)_l({}^T h_k)_i + ({}^T h_j)_k({}^T h_i)_l\}.$$

Put

$$\bar{R}_{ijkl} = R_{ijkl} + R_{jilk} + R_{klij} + R_{lkji}, \tag{2.5}$$

$$Q_{ijkl} = \sum_T\{2({}^T h_j)_i({}^T h_k)_l + ({}^T h_j)_l({}^T h_k)_i + ({}^T h_j)_k({}^T h_i)_l\}. \tag{2.6}$$

Finally, substitute $jilk$, $klij$ and $lkji$ for $ijkl$ in turn in (2.1) and add the four expressions so obtained. Then

$$K_{ijkl} - \bar{R}_{ijkl} = P_{ijkl} + Q_{ijkl}, \tag{2.7}$$

where each of the tensors $\bar{R}_{ijkl}$, $P_{ijkl}$, $Q_{ijkl}$ possesses all the properties (1.7).

The relation (2.7) shows how the Riemann-Christofell tensor is expressed in terms of the parallelism (1.1) by tensors which have the same properties with respect to the indices as belong to the former.

The contracted curvature tensor and other tensors and invariants arising in that connection can be expressed as follows:

Let

$$K_{jk} = g^{il}K_{ijkl}, \quad \bar{R}_{jk} = g^{il}\bar{R}_{ijkl}, \quad P_{jk} = g^{il}P_{ijkl},$$

and ${}_1R_{jk}$ and ${}_1R'_{jk}$ be the symmetric and skew-symmetric parts of $g^{il}R_{ijkl}$. That is,

$${}_1R_{jk} = \tfrac{1}{2}g^{il}(R_{ijkl} + R_{ikjl}), \quad {}_1R'_{jk} = \tfrac{1}{2}g^{il}(R_{ijkl} - R_{ikjl}).$$

Similarly let ${}_2R_{jk}$ and ${}_2R'_{jk}$ be the symmetric and skew-symmetric parts of $g^{il}R_{jilk}$.

For the sake of convenience, consider also ${}_1A_{jk}$, ${}_1A'_{jk}$ and ${}_2A_{jk}$, ${}_2A'_{jk}$ formed from $g^{il}A_{ijkl}$ and $g^{il}A_{jilk}$ in the same manner.

Further let

$$K = g^{jk}K_{jk}, \quad P = g^{jk}P_{jk} \quad {}_1R = g^{jk}{}_1R_{jk} \quad {}_2R = g^{jk}{}_2R_{jk}.$$

From (2.5)

$$\bar{R}_{jk} = 2({}_1R_{jk} + {}_2R_{jk}).$$

From (2.1)

$$K_{jk} - {}_1R_{jk} = P_{jk} + {}_1A_{jk}, \tag{2.8}$$

$${}_1R'_{jk} = -{}_1A'_{jk}, \quad {}_2R'_{jk} = -{}_2A'_{jk}. \tag{2.9}$$

From (2.3)

$${}_1A_{jk} = g^{il}[(g_{jk})_{il} - \tfrac{1}{2}\{(g_{jl})_{ik} + (g_{kl})_{ij}\}].$$

Therefore, by (1.3) and (1.6)

$${}_1A_{jk} = \tfrac{3}{2}g^{il}(g_{jk})_{il},$$

$$-\tfrac{1}{2}g^{il}[\{(g_{jl})_{ik} + (g_{ji})_{kl} + (g_{jk})_{li}\} + \{(g_{kl})_{ij} + (g_{ki})_{jl} + (g_{kj})_{li}\}] = \tfrac{3}{2}g^{il}(g_{jk})_{il}$$

$$-\tfrac{1}{2}g^{il}[\{R_{lkij} + R_{kilj} + R_{ilkj}\} + \{R_{ljil} + R_{jilk} + R_{iljk}\}] = \tfrac{3}{2}g^{il}(g_{jk})_{il} + {}_1R_{jk} + {}_2R_{jk}.$$

Therefore, from (2.8)

$$K_{jk} = 2\,{}_1R_{jk} + {}_2R_{jk} + P_{jk} + \tfrac{3}{2}g^{il}(g_{jk})_{il}, \tag{2.10}$$

$$\therefore \quad K = 2\,{}_1R + {}_2R + P + \tfrac{3}{2}g^{jk}g^{il}(g_{jk})_{il}. \tag{2.11}$$

Again, from (2.3)

$${}_1A'_{jk} = \tfrac{1}{2}g^{il}\{(g_{kl})_{ij} - (g_{jl})_{ik}\}$$

and

$$g^{il}(g_{kl})_{ij} = g^{il}(g_{ki})_{lj} = \tfrac{1}{2}g^{il}\{(g_{kl})_{ij} + (g_{ki})_{lj}\} = -\tfrac{1}{2}g^{il}(g_{il})_{kj}.$$

Therefore from (2.9)

$${}_1R'_{jk} = \tfrac{1}{4}g^{il}\{(g_{il})_{kj} - (g_{il})_{jk}\}. \tag{2.12}$$

Also, it is easily seen (Eisenhart, 1927) that ${}_1R'_{jk}$ is the curl of a vector:

$${}_1R'_{jk} = \frac{1}{2}\left(\frac{\partial T_j}{\partial x^k} - \frac{\partial T_k}{\partial x^j}\right), \tag{2.12'}$$

where (Sen, 1944)

$$T^t_{ij} = \nabla^t_{ij} - \left\{ {t \atop ij} \right\} = g^{st}(g_{ij})_s, \quad T_j = T^t_{tj}, \tag{2.13}$$

Equality of these two values of ${}_1R'_{jk}$ can be easily established.

Lastly,

$${}_2A'_{jk} = \tfrac{1}{2}g^{il}[\{(g_{ik})_{jl} - (g_{ij})_{kl}\} + \{(g_{il})_{kj} - (g_{il})_{jk}\}].$$

Therefore, from (2.9) and (2.12)

$${}_2R'_{jk} + 2\,{}_1R'_{jk} = \tfrac{1}{2}g^{il}\{(g_{ij})_{kl} - (g_{ik})_{jl}\}. \tag{2.14}$$

§3. We shall now give a divergence formula for a skew-symmetric contravariant tensor of the second rank. For this purpose we introduce the tensor $B_{ijkl}$ defined by

$$B_{ijkl} = g^{st}\{(g_{sk})_i(g_{jl})_t - (g_{sl})_i(g_{jk})_t\}. \tag{3.1}$$

It may be seen that like $A_{ijkl}$, the tensor $B_{ijkl}$ possesses only the first and the last of the properties (1.7), and that

$$\bar{B}_{ijkl} = B_{ijkl} + B_{jikl} + B_{klij} + B_{lkji} = 2P_{ijkl}, \tag{3.2}$$

where $P_{ijkl}$ is defined by (2.2).

Now, let $v^i$ be a contravariant vector, and $\xi^{ij}$ a skew-symmetric contravariant tensor.

If $\{\ \}$ with a subscript denotes covariant derivative with respect to Levi-Civita parallelism, then, in consequence of (2.13) and the skew-symmetric nature of $\xi^{ij}$

$$(v^i)_i = \{v^i\}_i + v^k g^{ij}(g_{ik})_j, \tag{3.3}$$

$$(\xi^{ij})_j = \{\xi^{ij}\}_j + \xi^{ik} g^{jl}(g_{jk})_l. \tag{3.4}$$

Supposing for the sake of convenience $(\xi^{ij})_j = \eta^i$, we have from (3.3)

$$(\eta^i)_i = \{\eta^i\}_i + \eta^k g^{ij}(g_{ik})_j.$$

Therefore from (3.4)

$$(\xi^{ij})_{ji} = \{\{\xi^{ij}\}_j + \xi^{ik} g^{jl}(g_{jk})_l\}_i + [\{\xi^{kj}\}_j + \xi^{kl} g^{jl}(g_{ij})_l] g^{st}(g_{sk})_t$$

$$= \{\xi^{ij}\}_{ji} + \{\xi^{ik}\}_i g^{jl}(g_{jk})_l + \xi^{ik}\{g^{jl}\}_i(g_{jk})_l + \xi^{ik} g^{jl}\{(g_{jk})_l\}_i + \{\xi^{kj}\}_j g^{st}(g_{sk})_t + \xi^{ki} g^{jl} g^{st}(g_{ij})_l(g_{sk})_t.$$

For simplification of the above we notice the following:

$$\{\xi^{ij}\}_{ji} = 0, \quad \text{(Levi-Civita, 1929)},$$

$$\{\xi^{ik}\}_i g^{jl}(g_{jk})_l + \{\xi^{kj}\}_j g^{st}(g_{sk})_t = 0, \qquad \because \quad \xi^{ij} + \xi^{ji} = 0;$$

$$\xi^{ik}\{g^{jl}\}_i(g_{jk})_l = 0, \qquad \because \quad \{g^{jl}\}_i = 0;$$

$$\xi^{ki} g^{jl} g^{st}(g_{ij})_l(g_{sk})_t = 0,$$

for

$$g^{jl}(g_{ij})_l = g^{jl}(g_{il})_j = \tfrac{1}{2} g^{jl}\{(g_{ij})_l + (g_{il})_j\} = -\tfrac{1}{2} g^{jl}(g_{jl})_i.$$

$$\therefore \quad \xi^{ki} g^{jl} g^{st}(g_{ij})_l(g_{sk})_t = \tfrac{1}{4}\xi^{ki} g^{jl} g^{st}(g_{jl})_i(g_{st})_k = 0, \qquad \because \quad \xi^{ki} + \xi^{ik} = 0.$$

Lastly,

$$\{(g_{jk})_l\}_i = (g_{jk})_{li} + g^{st}\{(g_{sk})_l(g_{ji})_t + (g_{sj})_l(g_{ki})_t + (g_{jk})_s(g_{li})_t\}.$$

And since $\xi^{ik}$ is skew-symmetric

$$\xi^{ik} g^{jl} g^{st}(g_{jk})_s(g_{li})_t = 0, \quad \xi^{ik} g^{jl} g^{st}(g_{sj})_l(g_{ki})_t = 0.$$

Therefore finally (interchanging $i$ and $l$) we have

$$(\xi^{ij})_{ji} = \xi^{lk} g^{ij}\{(g_{jk})_{il} + g^{st}(g_{sk})_i(g_{jl})_t\}. \tag{3.5}$$

This formula may be written in a more convenient form:

From (3.5) we have

$$(\xi^{ij})_{ji} = \xi^{kl} g^{ij}\{(g_{jl})_{ik} + g^{st}(g_{sl})_i(g_{jk})_t\} = -\xi^{lk} g^{ij}\{(g_{jl})_{ik} + g^{st}(g_{sl})_i(g_{jk})_t\}.$$

Adding this to (3.5),

$$(\xi^{ij})_{ji} = \tfrac{1}{2}\xi^{lk} g^{ij}[\{(g_{jk})_{il} - (g_{jl})_{ik}\} + g^{st}\{(g_{sk})_i(g_{jl})_t - (g_{sl})_i(g_{jk})_t\}]$$

or,

$$(\xi^{ij})_{ji} = \tfrac{1}{2}\xi^{lk} g^{ij}\{A_{ijkl} + B_{ijkl}\}, \tag{3.6}$$

where $A_{ijkl}$ and $B_{ijkl}$ are defined by (2.3) and (3.1) respectively.

This is the required divergence formula in terms of the parallelism (1.1).

It thus appears that besides $R_{ijkl}$, the tensors $A_{ijkl}$, $B_{ijkl}$, $P_{ijkl}$, $Q_{ijkl}$ are some of the important tensors in Riemannian geometry from the point of view of the parallelism (1.1).

§4. The relation (2.7) and the tensors associated with it can be expressed in terms of the components with respect to four members I, J, K, L of the ennuple and geometrical interpretations given.

The coefficients of rotation with respect to the Levi-Civita parallelism and the parallelism (1.1) are the quantities $\gamma_{IJK}$, $\beta_{IJK}$ respectively defined by* (Thomas, 1934)

$$\gamma_{IJK} = \{{}^{I}h_{i}\}_{j}\,{}_{J}h^{i}\,{}_{K}h^{j}, \qquad \beta_{IJK} = ({}^{I}h_{i})_{j}\,{}_{J}h^{i}\,{}_{K}h^{j}, \tag{4.1}$$

whence

$$\gamma_{IJK} + \gamma_{JIK} = 0, \qquad \beta_{IJK} + \beta_{IKJ} = 0.$$

They are connected by the relation (Sen, 1944)

$$\gamma_{IJK} = \beta_{IJK} + \beta_{JKI} + \beta_{KJI}. \tag{4.2}$$

Corresponding to the well-known four-index symbol of Ricci

$$\gamma_{IJKL} = K_{ijkl}\,{}_{I}h^{i}\,{}_{J}h^{j}\,{}_{K}h^{k}\,{}_{L}h^{l}$$

which is expressed in terms of the coefficients of rotations $\gamma$'s, we form the symbol

$$\begin{aligned}\beta_{IJKL} &= R_{ijkl}\,{}_{I}h^{i}\,{}_{J}h^{j}\,{}_{K}h^{k}\,{}_{L}h^{l}\\ &= \frac{d\beta_{IJK}}{ds_{L}} - \frac{d\beta_{IJL}}{ds_{K}} + \sum_{T}\{2\beta_{IJT}\beta_{TKL} + \beta_{IKT}\beta_{TJL} + \beta_{ILT}\beta_{TJK}\}\\ &= \frac{d\beta_{ILK}}{ds_{J}} - \sum_{T}(\beta_{ILT}\beta_{TJK} + \beta_{IKT}\beta_{TLJ}),\end{aligned} \tag{4.3}$$

and put

$$\bar\beta_{IJKL} = \beta_{IJKL} + \beta_{JILK} + \beta_{KLIJ} + \beta_{LKJI} \tag{4.4}$$

It is then seen from (4.2) or (2.7) that

$$\begin{aligned}\gamma_{IJKL} = \bar\beta_{IJKL} &+ \sum_{T}\{(\beta_{ILT} + \beta_{LIT})(\beta_{JKT} + \beta_{KJT}) - (\beta_{IKT} + \beta_{KIT})(\beta_{JLT} + \beta_{LJT})\}\\ &+ \sum_{T}\{2\beta_{TJI}\beta_{TKL} + \beta_{TIL}\beta_{TJK} + \beta_{TKI}\beta_{TJL}\}.\end{aligned} \tag{4.5}$$

The Riemannian curvature in the orientation determined by the two lines I and J of the ennuple follows immediately from (4.5) as

$$\gamma_{IJIJ} = \bar\beta_{IJIJ} + \sum_{T}\{(\beta_{IJT} + \beta_{JIT})^{2} - 4\beta_{IIT}\beta_{JJT} - 3(\beta_{TIJ})^{2}\}. \tag{4.6}$$

The scalar curvature is given by $-\sum_{IJ}\gamma_{IJIJ}$, and is the same as that given by $K$ in (2.11).

The expressions in the second and third terms in the right-hand side of (4.5) are respectively the $n$-uplet tensors

$$P_{ijkl}\,{}_{I}h^{i}\,{}_{J}h^{j}\,{}_{K}h^{k}\,{}_{L}h^{l} \quad\text{and}\quad Q_{ijkl}\,{}_{I}h^{i}\,{}_{J}h^{j}\,{}_{K}h^{k}\,{}_{L}h^{l},$$

where $P_{ijkl}$ and $Q_{ijkl}$ are defined by (2.2) and (2.6).

* Thomas has made use of the fundamental vectors ${}_{T}h^{i}$ to construct various sets of scalar invariants, including these coefficients, and establish relations and identities among them by the use of normal coordinates in what he calls affine space of distant parallelism, and not specially in the metric or Riemannian space. See pp. 48, 104, 115, 129, 139.

Also, for $A_{ijkl}$ and $B_{ijkl}$ defined by (2.3) and (3.1), we have

$$A_{ijkl}\,{}_{\mathrm{I}}h^{i}\,{}_{\mathrm{J}}h^{j}\,{}_{\mathrm{K}}h^{k}\,{}_{\mathrm{L}}h^{l} = \bar{\beta}_{\mathrm{IJKL}} - \beta_{\mathrm{IJKL}} + \sum_{\mathrm{T}}\{2\beta_{\mathrm{TJI}}\beta_{\mathrm{TKL}} + \beta_{\mathrm{TIL}}\beta_{\mathrm{TJK}} + \beta_{\mathrm{IKI}}\beta_{\mathrm{TJL}}\},$$

$$B_{ijkl}\,{}_{\mathrm{I}}h^{i}\,{}_{\mathrm{J}}h^{j}\,{}_{\mathrm{K}}h^{k}\,{}_{\mathrm{L}}h^{l} = \sum_{\mathrm{T}}\{(\beta_{\mathrm{JLT}} + \beta_{\mathrm{LJT}})(\beta_{\mathrm{TKI}} + \beta_{\mathrm{KTI}}) - (\beta_{\mathrm{JKT}} + \beta_{\mathrm{KJT}})(\beta_{\mathrm{TLI}} + \beta_{\mathrm{LTI}})\}.$$

Let these four $n$-uplet tensors be denoted by $p_{\mathrm{IJKL}}$, $q_{\mathrm{IJKL}}$, $a_{\mathrm{IJKL}}$ and $b_{\mathrm{IJKL}}$. They are expressed in terms of the coefficients of rotation which have a geometrical meaning, namely that $\beta_{\mathrm{IJK}}$ is the rate of change of the scalar product of the vectors ${}_{\mathrm{I}}h$ and ${}_{\mathrm{J}}h$ for a displacement in the direction of ${}_{\mathrm{K}}h$ in which ${}_{\mathrm{I}}h$ is moved by local and ${}_{\mathrm{J}}h$ by parallel displacement (1.1). We may also look at these $n$-uplet tensors in the following way:

Let

$$f_{\mathrm{PQ}} = g_{ij}\,{}_{\mathrm{P}}h^{i}\,{}_{\mathrm{Q}}h^{j}.$$

Evidently $f_{\mathrm{PQ}}$ is the scalar product of ${}_{\mathrm{P}}h$ and ${}_{\mathrm{Q}}h$ and is equal to $\delta^{\mathrm{P}}_{\mathrm{Q}}$. It is known that $f_{\mathrm{PQ}}$ does not undergo any change when ${}_{\mathrm{P}}h$ and ${}_{\mathrm{Q}}h$ are moved along a curve by Levi-Civita parallelism.

Let $f_{\mathrm{PQ,R}}$ denote the rate of change of $f_{\mathrm{PQ}}$ when ${}_{\mathrm{P}}h$ and ${}_{\mathrm{Q}}h$ are given parallel transport (1.1) along ${}_{\mathrm{R}}h$. Then

$$f_{\mathrm{PQ,R}} = (g_{ij})_{k}\,{}_{\mathrm{P}}h^{i}\,{}_{\mathrm{Q}}h^{j}\,{}_{\mathrm{R}}h^{k}.$$

Also, let $f_{\mathrm{PQ,R,S}}$ denote the rate of change of $f_{\mathrm{PQ,R}}$ when ${}_{\mathrm{P}}h$, ${}_{\mathrm{Q}}h$ and ${}_{\mathrm{R}}h$ are given parallel transport (1.1) along ${}_{\mathrm{S}}h$. Then

$$f_{\mathrm{PQ,R,S}} = (g_{ij})_{kl}\,{}_{\mathrm{P}}h^{i}\,{}_{\mathrm{Q}}h^{j}\,{}_{\mathrm{R}}h^{k}\,{}_{\mathrm{S}}h^{l}.$$

It may then be seen that

$$p_{\mathrm{IJKL}} = \sum_{\mathrm{T}}(f_{\mathrm{IL,T}}f_{\mathrm{JK,T}} - f_{\mathrm{IK,T}}f_{\mathrm{JL,T}}), \tag{4.7}$$

$$b_{\mathrm{IJKL}} = \sum_{\mathrm{T}}(f_{\mathrm{JL,T}}f_{\mathrm{TK,I}} - f_{\mathrm{JK,T}}f_{\mathrm{TL,I}}), \tag{4.8}$$

$$a_{\mathrm{IJKL}} = f_{\mathrm{JK,I,L}} - f_{\mathrm{JL,I,K}}, \tag{4.9}$$

$$q_{\mathrm{IJKL}} = a_{\mathrm{IJKL}} - (\bar{\beta}_{\mathrm{IJKL}} - \beta_{\mathrm{IJKL}}). \tag{4.10}$$

Evidently these $n$-uplet tensors possess the same properties with regard to the indices as are possessed by the corresponding tensors expressed in terms of the general coordinates, and they are related in the same manner.

Finally, let $v^{i}$ be a contravariant vector. Then

$$v_{\mathrm{K}} = v^{i}\,{}^{\mathrm{K}}h_{i}.$$

By covariant differentiation with respect to (1.1)

$$\frac{dv_{\mathrm{K}}}{ds_{\mathrm{L}}} = [(v^{i})_{k}\,{}^{\mathrm{K}}h_{i} + v^{i}({}^{\mathrm{K}}h_{i})_{k}]\,{}_{\mathrm{L}}h^{k},$$

where

$$\frac{d}{ds_{\mathrm{L}}} = \sum_{k}{}_{\mathrm{L}}h^{k}\,\frac{\partial}{\partial x^{k}},$$

$$\therefore\quad \sum_{\mathrm{K}}\frac{dv_{\mathrm{K}}}{ds_{\mathrm{K}}} = (v^{i})_{i} + \sum_{\mathrm{JK}} v_{\mathrm{J}}\,\beta_{\mathrm{KJK}},$$

Similarly, for Levi-Civita parallelism,

$$\sum_k \frac{dv_k}{ds_k} = \{v^i\}_i + \sum_{jk} v_j \gamma_{kjk}.$$

If all the congruences of the ennuple, which are autoparallels with respect to (1.1), are also geodesics (Levi-Civita, 1927), then by (4.2)

$$\gamma_{kjk} = \beta_{kjk} = 0, \quad j, k = 1, 2, \ldots, n$$

and therefore

$$\{v^i\}_i = (v^i)_i.$$

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