Permutation invariant functions

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April 2025

Let $f : 2^X \to \mathbb{R}, S \mapsto f(S)$ be a real-valued set function where $X = \{x_1, \ldots, x_n\}$ is a set. It was proven in [2] that f can be written canonically as

$$f(S) = g\left(\sum_{x \in S} \phi(x)\right) \tag{1}$$

for functions ϕ and g.

Proof: First, to prove the sufficient condition, we check that the right-hand-side of Eq. 1 is invariant to any permutation σ of the elements of X: $f(\sigma(S)) = g\left(\sum_{x\in\sigma(S)} \phi(x)\right) = g\left(\sum_{x\in S} \phi(x)\right)$ because of the commutativity property of the addition. To prove necessity, let $c : X \to \mathbb{N}, x \mapsto c(x)$ be a count function such that $c(x) \neq c(x) \Leftrightarrow x \neq x'$. Let $\phi(x)$ be any positive function such that $\phi(x) \neq \phi(x')$ for any $x, x' \in X$. For example, we may choose $\phi(x) = \exp(c(x))$. Then for any two distinct subsets S and S' of 2^X, we have $\sum_{x\in\S} \phi(x) \neq \sum_{x\in\S'} \phi(x)$ since the difference is

$$\sum_{x\in S\Delta S'} \phi(x) > 0,$$

where $S\Delta S' = (S\backslash S') \cup (S'\backslash S)$ denotes the symmetric difference (non-empty since subsets are distinct).

Thus $\{\phi(S) : S \in 2^X\}$ is a collection of $|2^X| = 2^n$ distinct points in $\mathbb{R}$. We may then choose g to be the Lagrange polynomial interpolating those 2^n points $\{(\sum_{x\in\S} \phi(x), f(S)) : S \in 2^X\}$.

For example, the Heron formula for the area of a triangle is invariant to the triangle vertex permutation, and can thus be written using the canonical form of Eq. 1. See [1].

□

References

[1] Connor Hainje and David W Hogg. A formula for the area of a triangle: Useless, but explicitly in deep sets form. *arXiv preprint arXiv:2503.22786*, 2025.

[2] Manzil Zaheer, Satwik Kottur, Siamak Ravanbakhsh, Barnabas Poczos, Russ R Salakhutdinov, and Alexander J Smola. Deep sets. *Advances in neural information processing systems*, 30, 2017.