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Office Hour 8

Administrative Issues

- Homework 4 is due by October 16 at 11:59pm ET.
- Midterm: Oct 20-Oct23. The withdraw deadline is October 29, 2022 4:00 PM ET, and we plan to have your midterms graded before then. How much before is TBD but we will try our best.
- Practice midterm and solutions are available on course website. I can discuss these more in detail next week before the midterm.
- for HW4, I recommend looking at the gibbs.zip material available in the supplementary resources. That has a good explanation of the Unit 5 material.
- While Aaron's 10/10 office hours were not recorded (my bad!), he put together comprehensive notes regarding HW4 as well as how to computationally find an HPD credible set. Please review these if you are stuck on the implementation aspects of HW4.
- If you need help with latex, try Notability or Mathpix. Also suggest Overleaf, a web-based latex authoring platform with lots of templates and examples.
- Reminder: **No Handwritten Documents are permitted for any submission**

HW4 Guidance

Q1: Metropolis

(a) Bayes estimator (mean of the posterior) and 97% HPD credible set. For the estimator you can just take the mean of your samples, minus the burn-in. For the HPD credible set computation, please refer to Aaron's notes which are available here. We essentially calculate all such intervals that fall within the given range, and we select the shortest such interval and report that as the HPD set.

(b) Two plots: posterior density of θ should just be a histogram

Trace plot: iteration number vs. observed value of the chain at that iteration

(c) Report the acceptance rate of your proposal distribution, based on the simulated values. This value will be in the range $[0, 1]$

Q2: Gibbs

(a) As $\pi\left(\bar{y} \mid \theta, \sigma^2\right) \propto \exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right), \pi(\theta \mid \lambda) \propto \sqrt{\frac{\lambda}{\tau^2}} \exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)$ and $\pi(\lambda) \propto \lambda^{\alpha-1} e^{-\beta\lambda}$ where $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$. The product of the likelihood and the prior is proportional to

$$p(\theta, \lambda, \bar{y}) \propto \exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right) \sqrt{\frac{\lambda}{\tau^2}} \exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right) \lambda^{\alpha-1} \exp(-\beta\lambda)$$

To find the conditional distribution for θ ($\pi(\theta|\lambda, \bar{y})$), we take only the terms involving θ and try to find a distribution for it. Specifically

$\pi(\theta|\lambda, \bar{y}) \propto \exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right) \cdot \exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)$

is proportional to the pdf of the following Normal distribution:

$$\mathcal{N}\left(\frac{\tau^2}{\tau^2 + \lambda\sigma^2/n}\bar{y} + \frac{\lambda\sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\mu, \frac{\tau^2 \cdot \sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\right)$$

One way to show this is by completing the square on θ. The marginal distribution for λ is found via a similar method. Note that in this part, if you find $\pi(\lambda|\theta, \bar{y}) \propto e^{-b\lambda}$ for some constant b, this is enough to conclude that $[\lambda|\theta, \bar{y}]$ is exponentially distributed. When we say $a \propto b$, this means exactly that $a = c \cdot b$ for some constant c.

(b) Implement a Gibbs sampler using the conditional distributions above. Please reference Aaron's computational examples for the specifics.

Note/Appendix: I received a question in last week's OH asking what the following equation from the HW4 Q2 problem statement was about:

$$\frac{\tau}{\pi\left(\tau^2 + (\theta-\tau)^2\right)} \propto \int_0^\infty \sqrt{\frac{\lambda}{2\pi\tau^2}} \exp\left\{-\frac{\lambda}{2\tau^2}(\theta-\mu)^2\right\} \cdot \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\lambda}{2}\right\} d\lambda$$

Explanation: The joint distribution of both λ and θ is obtained by multiplying the pdf's of $\theta|\lambda$ and λ together:

$p(\lambda,\ \theta) \propto \sqrt{\frac{\lambda}{2\pi\tau^2}} \exp\left\{-\frac{\lambda}{2\tau^2}(\theta-\mu)^2\right\} \cdot \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\lambda}{2}\right\}$

To find the marginal distribution of θ, we integrate out λ from the joint likelihood:

$$
\begin{aligned}
p(\theta) &\propto \int_{\Lambda} p(\lambda,\ \theta) d\lambda \\
&= \int_0^\infty \sqrt{\frac{\lambda}{2\pi\tau^2}} \exp\left\{-\frac{\lambda}{2\tau^2}(\theta-\mu)^2\right\} \cdot \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\lambda}{2}\right\} d\lambda \\
&= \frac{1}{\sqrt{2\pi\tau^2}} \int_0^\infty \exp\left\{-\frac{\lambda}{2}\left[\frac{(\theta-\mu)^2}{\tau^2}+1\right]\right\} d\lambda \\
&= \frac{1}{\sqrt{2\pi\tau^2}} \frac{\tau^2}{(\theta-\mu)^2+\tau^2} \cdot (-2) \cdot \left[\exp\left\{-\frac{\lambda}{2}\left[\frac{(\theta-\mu)^2}{\tau^2}+1\right]\right\}\right]_{\lambda=0}^{\lambda=\infty} \\
&= \frac{1}{\sqrt{2\pi\tau^2}} \frac{\tau^2}{(\theta-\mu)^2+\tau^2} \cdot (-2) \cdot [0-1] \\
&\propto \frac{\tau}{\pi[(\theta-\mu)^2+\tau^2]}
\end{aligned}
$$

Thus the marginal distribution of θ follows a Cauchy $\mathcal{C}a(\mu,\tau)$ distribution.