Homework 2

ISyE 6420
Fall 2022

1. 2-D Density Tasks. If

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y,\ \lambda > 0 \\ 0, & \text{else} \end{cases}$$

Show that:
(a) marginal distribution $f_X(x)$ is exponential $\mathcal{E}(\lambda)$.
(b) marginal distribution $f_Y(y)$ is Gamma $\mathcal{G}a(2, \lambda)$.
(c) conditional distribution $f(y|x)$ is shifted exponential, $f(y|x) = \lambda e^{-\lambda(y-x)},\ y \ge x$.
(d) conditional distribution $f(x|y)$ is uniform $\mathcal{U}(0, y)$.

2. Weibull Lifetimes. A lifetime X (in years) of a particular device is modeled by a Weibull distribution

$$f(x|\nu, \theta) = \nu\theta x^{\nu-1} \exp\left\{-\theta x^{\nu}\right\},\ x \ge 0,$$

with shape parameter $\nu = 3$ and unknown rate parameter θ. The lifetimes of $X_1 = 3, X_2 = 4$, and $X_3 = 2$ are observed. Assume that an expert familiar with this type of devices suggested an exponential prior on θ with rate parameter $\lambda = \frac{5}{2}$.

(a) For the prior suggested by the expert, find the posterior distribution of θ.

(b) What are the posterior mean and variance? No need to integrate if you recognize to which family of distributions the posterior belongs.

3. Silver-Coated Nylon Fiber. Silver-coated nylon fiber is used in hospitals for its anti-static electricity properties, as well as for antibacterial and antimycotic effects. In the production of silver-coated nylon fibers, the extrusion process is interrupted from time to time by blockages occurring in the extrusion dyes. The time in hours between blockages, T, has an exponential $\mathcal{E}(\lambda)$ distribution, where λ is the rate parameter.

(a) Suppose $\lambda = 1/4$, find the probabilities that
(i) a run continues for at least 5 hours.
(ii) a run lasts less than 10 hours.
(iii) a run continues for at least 10 hours, given that it has lasted 5 hours.

(b) Now suppose that the rate parameter λ is unknown, but there are three measurements of interblockage times, $T_1 = 3, T_2 = 5$, and $T_3 = 7$.

(i) How would a classical statistician estimate λ ?

(ii) What is the Bayes estimator of λ if the prior is $\pi(\lambda) = \frac{1}{\sqrt{\lambda}}$, $\lambda > 0$.

Hint. In (ii) of (b), the prior is not a proper distribution, but the posterior is. Identify the posterior from the product of the likelihood from (i) and the prior, no need to integrate.