# Bayes U4L13 Notes

Greg Schreiter

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## 1 Bayesian Prediction

Recall $m(x) = \int f(x|\theta)\pi(\theta)d\theta$ is the marginal distribution, which is sometimes called the **prior predictive distribution**.

$$f(x_{n+1}|x_1,\dots,x_n) = \int f(x_{n+1}|\theta)\pi(\theta|x_1,\dots,x_n)d\theta$$

The above is called the **posterior predictive distribution**.

$$\hat{X}_{n+1} = \int x_{n+1} \times f(x_{n+1}|x_1,\dots,x_n)dx_{n+1} = \mathbb{E}(X_{n+1}|X_1,\dots,X_n)$$

The above is called the **predictive mean** (prediction for $X_{n+1}$ ).

$$\int \left(x_{n+1} - \hat{X}_{n+1}\right)^2 f(x_{n+1}|x_1,\dots,x_n)dx_{n+1}$$

The above is called the **predictive variance**.

## 2 Example 1

Observations from Exponential distribution with Gamma prior on $\lambda$

$x_1,\dots,x_n \sim Exp(\lambda),\ \ f(x_i) = \lambda e^{-\lambda x_i},\ \ \pi(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta\lambda},\ \lambda \geq 0$

**Likelihood**:

$$\begin{aligned} L(\lambda|x_1,\dots,x_n) &= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \\ &= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \end{aligned}$$

**Posterior**:

$$\begin{aligned} L(\lambda|x_1,\dots,x_n)\pi(\lambda) &= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda} \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1}\lambda^n e^{-\lambda \sum_{i=1}^n x_i} e^{-\beta\lambda} \\ &= C\lambda^{\alpha+n-1} e^{-\lambda(\sum_{i=1}^n x_i + \beta)} \\ &\propto Gamma(\alpha+n, \sum_{i=1}^{n} x_i + \beta) \end{aligned}$$

The pdf of the posterior is then

$$\pi(\lambda|x_1,\dots,x_n) = \frac{(\sum_{i=1}^n x_i + \beta)^{\alpha+n}}{\Gamma(\alpha+n)} \lambda^{\alpha+n-1} e^{-\lambda(\sum_{i=1}^n x_i+\beta)} \text{ for } \lambda > 0$$

We also need the help of the Gamma function (not distribution) to help solve the next problem:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \quad \text{for } z > 0.\ \text{We also have the result that } \Gamma(z+1) = z\Gamma(z)$$

**Posterior predictive distribution**:

$$\begin{aligned}
f(x_{n+1}|x_1,\ldots,x_n) &= \int_0^\infty \lambda e^{-\lambda x_{n+1}}\pi(\lambda|x_1,\ldots,x_n)d\lambda \\
&= \int_0^\infty \lambda e^{-\lambda x_{n+1}}\frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{\Gamma(\alpha+n)}\lambda^{\alpha+n-1}e^{-\lambda(\sum_{i=1}^n x_i+\beta)}d\lambda \\
&= \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{\Gamma(\alpha+n)}\int_0^\infty \lambda^{\alpha+n}e^{-\lambda(x_{n+1}+\sum_{i=1}^n x_i+\beta)}d\lambda \\
&\text{Substituting } u = \lambda(x_{n+1}+\sum_{i=1}^n x_i+\beta),\ du = (x_{n+1}+\sum_{i=1}^n x_i+\beta)d\lambda \\
&= \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{\Gamma(\alpha+n)}\int_0^\infty \left(\frac{u}{x_{n+1}+\sum_{i=1}^n x_i+\beta}\right)^{\alpha+n}e^{-u}\frac{du}{(x_{n+1}+\sum_{i=1}^n x_i+\beta)} \\
&= \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{\Gamma(\alpha+n)(x_{n+1}+\sum_{i=1}^n x_i+\beta)^{\alpha+n+1}}\int_0^\infty u^{\alpha+n}e^{-u}du \\
&= \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}\Gamma(\alpha+n+1)}{\Gamma(\alpha+n)(x_{n+1}+\sum_{i=1}^n x_i+\beta)^{\alpha+n+1}} \\
&= \frac{(\sum_{i=1}^n x_i+\beta)^{\alpha+n}(\alpha+n)\Gamma(\alpha+n)}{\Gamma(\alpha+n)(x_{n+1}+\sum_{i=1}^n x_i+\beta)^{\alpha+n+1}} \\
&= \frac{(\alpha+n)(\sum_{i=1}^n x_i+\beta)^{\alpha+n}}{(x_{n+1}+\sum_{i=1}^n x_i+\beta)^{\alpha+n+1}}
\end{aligned}$$

Thus $x_{n+1}+\sum_{i=1}^n x_i+\beta$ has a Pareto distribution with parameters $\sum_{i=1}^n x_i+\beta$ and $\alpha+n$.

# 3 Example 2

If $X \sim Pa(c,\alpha)$, then $f(x) = \frac{\alpha}{c}(\frac{c}{x})^{\alpha+1},\ x \geq c$. We have

$$E[X] = \frac{\alpha c}{\alpha-1},\ \alpha > 1$$

and

$$Var(X) = \frac{\alpha c^2}{(\alpha-1)^2(\alpha-2)},\quad \alpha > 2$$

We have $x_{n+1}+\sum_{i=1}^n x_i+\beta \sim Pa(\sum_{i=1}^n x_i+\beta, \alpha+n)$
Then

$$\begin{aligned}
E\hat{X}_{n+1} &= EX_{n+1} \\
&= \frac{(\sum_{i=1}^n x_i+\beta)(\alpha+n)}{\alpha+n-1} - \sum_{i=1}^n x_i - \beta \\
&= \frac{\sum_{i=1}^n x_i+\beta}{\alpha+n-1}
\end{aligned}$$

**Exercise for reader**: show

$$\hat{\sigma}^2_{x_{n+1}} = \frac{(\sum_{i=1}^n x_i + \beta)^2(n+\alpha)}{(\alpha+n-1)^2(\alpha+n-2)}$$

For example if $x_1 = 2.1$, $x_2 = 5.5$, $x_3 = 6.4$, $x_4 = 8.7$, $x_5 = 4.9$, $x_6 = 5.1$, $x_7 = 2.3$ and $\lambda \sim Ga(2,1)$, then

$\hat{X}_8 = \frac{9}{2}$, $\hat{\sigma}^2_{x_8} = 26.0357$

This is easier if only $\hat{X}_{n+1}$ is wanted:

$$\hat{X}_{n+1} = \int_\theta \mu(\theta)\pi(\theta|x_1,\ldots,x_n)d\theta$$

where $\mu(\theta) = \int xf(x|\theta)d\theta$ is the mean of X.