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Homework 2

Problem 1

Answer to the problem goes here.

1. Problem 1 part 1 answer here.

$$f_X(x) = \int_x^{+\infty} f(x,y)dy = \int_x^{+\infty} \lambda^2 e^{-\lambda y} dy = \lambda^2 \left(-\frac{1}{\lambda}\right) e^{-\lambda y}|_x^{+\infty} = \lambda e^{-\lambda x}$$

Which is the pdf of exponential distribution (λ)

2. Problem 1 part 2 answer here.

$$f_Y(y) = \int_0^y f(x,y)dx = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 e^{-\lambda y} x|_0^y = \lambda^2 e^{-\lambda y} y = \frac{\lambda^2}{\Gamma(2-1)} y\, e^{-\lambda y}$$

Which is the pdf of gamma distribution $(2, \lambda)$.

3. Problem 1 part 3 answer here

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, y \geq x$$

4. Problem 1 part 4 answer here

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 e^{-\lambda y} y} = \frac{1}{y}$$

Which is pdf of uniform distribution U(0, y).

Problem 2

Answer to the problem goes here.

1. Problem 2 part 1 answer here.

$$L(x|\theta) = f(x_1|\nu,\theta) f(x_2|\nu,\theta) f(x_3|\nu,\theta) = 27\theta^3 (x_1 x_2 x_3)^2 \exp\,(-\theta \sum_{i=3}^{3} x_i^3)$$

$$P(\theta|x) \propto L(x|\theta) f(\theta) = 27\theta^3 (x_1 x_2 x_3)^2 \exp\left(-\theta \sum_{i=3}^{3} x_i^3\right) \frac{5}{2} e^{-\frac{5}{2}\theta} \propto \theta^{4-1} e^{-\theta(\sum_{i=3}^{3} x_i^3 + \frac{5}{2})}$$

Which is the pdf without constant part of gamma distribution $(4, \sum_{i=3}^{3} x_i^3 + \frac{5}{2})$

The posterior distribution follows gamma distribution with parameters $(4, \sum_{i=3}^{3} x_i^3 + \frac{5}{2})$, that is (4,101.5)

2. Problem 2 part 2 answer here:

As gamma mean is $\frac{\alpha}{\beta}$, and variance $\frac{\alpha}{\beta^2}$. In our case,

$$E(\theta|X) = \frac{4}{\sum_{i=3}^{3} x_i^3 + \frac{5}{2}} = \frac{4}{101.5} = 0.03940887$$

$$Var(\theta|X) = \frac{4}{(\sum_{i=3}^{3} x_i^3 + \frac{5}{2})^2} = 0.0003882647$$

Problem 3

Answer to the problem goes here.

3. Problem 3(a) part 1 answer here

$$P(T \geq 5) = \int_5^\infty f(t)dt = \int_5^\infty \lambda e^{-\lambda t} dt = e^{-5\lambda} = e^{-\frac{5}{4}}$$

4. Problem 3(a) part 2 answer here

$$P(T < 10) = \int_0^{10} f(t)dt = \int_0^{10} \lambda e^{-\lambda t} dt = 1 - e^{-10\lambda} = 1 - e^{-\frac{10}{4}}$$

5. Problem 3(a) part 3 answer here

$$P(T \geq 10|T \geq 5) = \frac{P(T \geq 10)}{P(T \geq 5)} = \frac{1 - P(T < 10)}{P(T \geq 5)} = \frac{e^{-\frac{10}{4}}}{e^{-\frac{5}{4}}} = e^{-\frac{5}{4}}$$

6. Problem 3(b) part 1 answer here

$$L(\lambda; t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} f(t_i; \lambda) = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

$$LnL(\lambda) = Ln\left(\lambda^n e^{-\lambda \sum_{i=1}^{n} \lambda_i}\right) = nLn(\lambda) - \lambda \sum_{i=1}^{n} t_i$$

$$\frac{dLnL(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i = 0$$
$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}$$

Therefore,

$$\hat{\lambda} = \frac{3}{\sum_{i=1}^{3} t_i} = \frac{1}{5}$$

7. Problem 3(b) part 2 answer here

$$P(\lambda|x) \propto L(t|\lambda)\pi(\lambda) = \lambda^3 e^{-\lambda \sum_{i=1}^{3} t_i} \frac{1}{\sqrt{\lambda}} = \lambda^{\frac{7}{2}-1} e^{-\lambda \sum_{i=1}^{3} t_i}$$

Which is the pdf without constant part of gamma distribution ($\frac{7}{2}, \sum_{i=1}^{3} t_i$)

The posterior mean is $\frac{7/2}{\sum_{i=1}^{3} t_i} = \frac{7}{30}$, therefore the Bayes estimator is $\frac{7}{30}$.