Homework 4

ISyE 6420
Fall 2022

1. Simple Metropolis: Normal Precision – Gamma. Suppose $X = -2$ was observed from the population distributed as $\mathcal{N}\left(0, \frac{1}{\theta}\right)$ and one wishes to estimate the parameter θ. (Here θ is the reciprocal of the variance σ^2 and is called the *precision parameter*). Suppose the analyst believes that the prior on θ is $\mathcal{G}a(1/2, 1)$.

Using Metropolis algorithm, approximate the posterior distribution and the Bayes' estimator of θ. As the proposal distribution, use gamma $\mathcal{G}a(\alpha, \beta)$ with parameters α, β selected to ensure efficacy of the sampling (this may require some experimenting).

(a) Describe the posterior distribution of θ using the Bayes' estimator and the 97% HPD credible set.

(b) Create two plots: one for the posterior density of θ and one trace plot. For the trace plot, the X-axis should be the iteration, and the Y-axis should be the observed value of the chain at that iteration.

(c) Report the acceptance rate of your proposal distribution. That is, what is the probability that the proposal was accepted when you ran the Metropolis algorithm?

2. Normal-Cauchy by Gibbs. Assume that $y_1, y_2, \dots, y_n$ is a sample from $\mathcal{N}(\theta, \sigma^2)$ distribution, and that the prior on θ is Cauchy $\mathcal{C}a(\mu, \tau)$,

$$f(\theta|\mu, \tau) = \frac{1}{\pi} \cdot \frac{\tau}{\tau^2 + (\theta - \mu)^2}.$$

Even though the likelihood for $y_1, \dots, y_n$ simplifies by sufficiency arguments to a likelihood of $\bar{y} \sim \mathcal{N}(\theta, \sigma^2/n)$, a closed form for the posterior is impossible and numerical integration is required.

The approximation of the posterior is possible by Gibbs sampler as well. Cauchy $\mathcal{C}a(\mu, \tau)$ distribution can be represented as a scale-mixture of normals:

$$[\theta] \sim \mathcal{C}a(\mu, \tau) \equiv [\theta|\lambda] \sim \mathcal{N}\left(\mu, \frac{\tau^2}{\lambda}\right), [\lambda] \sim \mathcal{G}a\left(\frac{1}{2}, \frac{1}{2}\right),$$

that is,

$$\frac{\tau}{\pi(\tau^2 + (\theta - \tau)^2)} \propto \int_0^\infty \sqrt{\frac{\lambda}{2\pi\tau^2}} \exp\left\{-\frac{\lambda}{2\tau^2}(\theta - \mu)^2\right\} \cdot \lambda^{\frac{1}{2}-1} \exp\left\{-\frac{\lambda}{2}\right\} d\lambda.$$

The full conditionals can be derived from the product of the densities for the likelihood and priors,

$$[\bar{y}|\theta, \sigma^2] \sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right),$$

$$[\theta|\lambda] \sim \mathcal{N}\left(\mu, \frac{\tau^2}{\lambda}\right),$$

$$[\lambda] \sim \mathcal{G}a\left(\frac{1}{2}, \frac{1}{2}\right).$$

(a) Show that full conditionals are normal and exponential,

$$[\theta|\bar{y}, \lambda] \sim \mathcal{N}\left(\frac{\tau^2}{\tau^2 + \lambda\sigma^2/n}\bar{y} + \frac{\lambda\sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\mu, \frac{\tau^2 \cdot \sigma^2/n}{\tau^2 + \lambda\sigma^2/n}\right),$$

$$[\lambda|\bar{y}, \theta] \sim \mathcal{E}\left(\frac{\tau^2 + (\theta - \mu)^2}{2\tau^2}.\right)$$

(b) Jeremy models the score on his IQ tests as $\mathcal{N}(\theta, \sigma^2)$ with $\sigma^2 = 90$. He places a Cauchy prior on θ: $\mathcal{C}a(110, \sqrt{120})$.

In 10 random IQ tests Jeremy scores $y = [100, 112, 110, 95, 104, 112, 120, 95, 98, 109]$. The average score is 105.5, which is the frequentist estimator of θ. Using the Gibbs sampler described in (a), approximate the posterior mean and variance. Approximate the 94% equi-tailed credible set by sample quantiles.