Administrative Issues

- Homework 3 is due by October 9 at 11:59pm ET.
- Homework 4 is due by October 16 at 11:59pm ET.
- Midterm: Oct 20-Oct23
- If you need help with latex, try Notability or Mathpix. Also suggest Overleaf, a web-based latex authoring platform with lots of templates and examples.
- If you decide to use some programming language for HW3 that is fine. However at the end of the day, you are responsible for reading the documentation associated with that language. Please make sure to enter your probability density functions appropriately!

  Example of this: In this course we say X has Gamma(a, b) distribution if $f(x \mid a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ for $x \geq 0$ but Matlab defines the Gamma (a, b) distribution using $f(x \mid a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$ for $x \geq 0$. Please be careful about these differences and make sure the function does what you think it does! A Gamma$(2, 4)$ variable in our course is equivalent to a Gamma$(2, 1/4)$ in Matlab.
- Reminder: **No Handwritten Documents are permitted for any submission**

HW3 Guidance

Q1: Maxwell

Find the MLE of Maxwell:
We let $\boldsymbol{y} = (y_1, \ldots, y_n)$ and find the likelihood as

$$
\begin{aligned}
L(\theta) = \prod_{i=1}^{n} f\left(x_i \mid \theta\right) &= \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \theta^{3/2} y_i^2 \exp\left\{-\theta y_i^2 / 2\right\} \\
&= \left(\sqrt{\frac{2}{\pi}}\right)^n \theta^{\frac{3}{2} n} (\prod_{i=1}^{n} y_i^2) \exp\left\{-\frac{\theta}{2} \sum_{i=1}^{n} y_i^2\right\} \\
&\propto \theta^{3n/2} \exp\left\{-\frac{\theta}{2} \sum_{i=1}^{n} y_i^2\right\}
\end{aligned}
$$

We have $\log L(\theta) = (3n/2)\log(\theta) - \frac{\theta}{2} \sum_{i=1}^{n} y_i^2$
Thus, $\frac{d}{d\theta} \log L(\theta) = (3n/2\theta) - \frac{1}{2} \sum_{i=1}^{n} y_i^2 = 0$
Rearranging, this gives us $\hat{\theta} = \frac{3n}{\sum_{i=1}^{n} y_i^2}$

(b) Compute 95% equitailed credible set using your favorite software package. Recall that a credible set for θ of size $1-\alpha$ is the interval (a, b) defined as follows:

$$P(a \leq \theta \leq b \mid X) = \int_a^b \pi(\theta \mid X) d\theta = 1 - \alpha$$

An equal-tailed credible set has the additional constraint that

$$P(\theta \geq b \mid X) = \frac{\alpha}{2} \text{ and } P(\theta \leq a \mid X) = \frac{\alpha}{2}$$

Q2: Mixture

(a) Should be relatively straightforward. Don't over-complicate things!

(b) One way to find the posterior distribution given the likelihood $X|\theta \sim N(\theta, \sigma^2)$ where σ^2 is fixed, and prior $\theta \sim N(\theta_0, \sigma_0^2)$, is the following:

$$\pi(\theta \mid x) \propto f(x \mid \theta)\pi(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}}$$

HW4 Guidance

Q1: Simple Metropolis

In this class, the normal distribution is parametrized as follows:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{\sigma^2}\right\}$$

In the BUGS software, this Normal is parametrized in terms of mean and precision ($\frac{1}{\sigma^2}$) instead of mean and variance:

$$f(x|\mu, \frac{1}{\theta}) = \frac{\sqrt{\theta}}{\sqrt{2\pi}} \exp\left\{-\theta(x-\mu)^2\right\}$$

So for example, in BUGS a typical choice for a non-informative prior is `dnorm(0,0.00001)`

Q2: Gibbs

Multiply the likelihood together with the priors. Then to get the conditional distribution, factor out the parameter of interest (e.g. θ). You may have to complete the square to figure out the conditional distribution for θ, but the conditional distribution for λ should be simpler.

Derivation of Beta-Binomial Conjugate Pair

n is the number of binomial observations we are considering in the likelihood. It is up to you to decide if there is a single or multiple observations when completing this homework problem. Derivation of the general case is as follows:

Let $X_i|\theta \sim Bin(m, \theta)$. We have

$$f(x_i \mid \theta) = \binom{m}{x_i} \theta^{x_i}(1-\theta)^{m-x_i}$$

Now we compute the joint likelihood. Note that $\mathbf{X}$ stands for a sample of size n, e.g. $X_1, \ldots, X_n$. We have

$$L = \prod_{i=1}^{n} \binom{m}{x_i} \theta^{x_i}(1-\theta)^{m-x_i} \propto \theta^{\sum_{i=1}^n x_i}(1-\theta)^{nm-\sum_{i=1}^n x_i}$$

The population proportion θ is the parameter of interest. If the prior on θ is Beta $\mathcal{B}e(\alpha, \beta)$ with hyperparameters α and β and density

$$\pi(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1},$$

The posterior is then

$$\begin{aligned}
\pi(\theta|\mathbf{X}) &\propto \theta^{\sum_{i=1}^n x_i}(1-\theta)^{nm-\sum_{i=1}^n x_i} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \theta^{\sum_{i=1}^n x_i+\alpha-1}(1-\theta)^{nm-\sum_{i=1}^n x_i+\beta-1} \\
&\sim \mathcal{B}eta(\sum_{i=1}^{n} x_i + \alpha, nm - \sum_{i=1}^{n} x_i + \beta)
\end{aligned}$$

which matches the form of the posterior in the statbook.