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Homework 4

ISyE 6420
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Problem 1

Answer to the problem goes here.

1. Problem 1 part 1 answer here.

   The prior $\pi(\theta) \propto \theta^{\alpha_0 - 1} e^{-\beta_0 \theta}$, the likelihood $f(x|\theta) = \frac{1}{\sqrt{2\pi}} \theta^{\frac{1}{2}} e^{-\frac{1}{2}\theta x^2}$

   Posterior: $\pi(\theta|x) \propto \pi(\theta) f(x|\theta) \propto \theta^{\alpha_0 - \frac{1}{2}} e^{-\theta(\frac{1}{2}x^2 + \beta_0)}$

   As $\alpha_0 = \frac{1}{2}, and\ \beta_0 = 1, x = -2$

   $\pi(\theta|x) \propto e^{-\theta\left(\frac{1}{2}x^2 + 1\right)} = e^{-3\theta}$, which is the density of Gamma(1,3)

   Then the posterior mean is 1/3.

$$\int_C \pi(\theta|x) d\theta \geq 97\%$$

   R code: qgamma(0.03,shape = 1,scale = (1/3),lower.tail = F)

   The HPD credible set is [0,1.17]

   $\gamma = \frac{\pi(\theta')q(\theta|\theta\prime)}{\pi(\theta)q(\theta\prime|\theta)} = \frac{e^{-\theta\prime(\frac{1}{2}x^2+1)} q(\theta|\theta\prime)}{e^{-\theta(\frac{1}{2}x^2+1)} q(\theta\prime|\theta)} = \frac{e^{-3\theta\prime} q(\theta|\theta\prime)}{e^{-3\theta} q(\theta\prime|\theta)}$

   $\rho(\theta_n, \theta') = 1 ^\wedge \frac{e^{-3\theta'} q(\theta_n|\theta')}{e^{-3\theta_n} q(\theta'|\theta_n)}$

   $q(\theta|\theta')$ and $q(\theta'|\theta)$ can be determined by Gamma (α, β). The metropolis hasting can be used as the following steps.

   Step1: Start with arbitrary x_0 from the support of target π.

   Step2: At stage n, generate proposal from Gamma (α, β) for the chosen α, β

   Step3: $\theta_{n+1} = \theta'$ with probability $\rho(\theta_n, \theta')$,

   And $\theta_{n+1} = \theta_n$ with probability $1 - \rho(\theta_n, \theta')$

   (generate U~U(0,1) and accept proposal if $U \leq \rho(\theta_n, \theta')$)

   Step4: increase n and go to step 2.

   I select Gamma (1,3) as proposal (independent metropolis), then

   $\rho(\theta, \theta') = 1 ^\wedge \frac{e^{-3\theta\prime} e^{-3\theta_n}}{e^{-3\theta_n} e^{-3\theta\prime}}$

   Which is equal to 1

The Bayesian estimator is 0.3381.

R code:

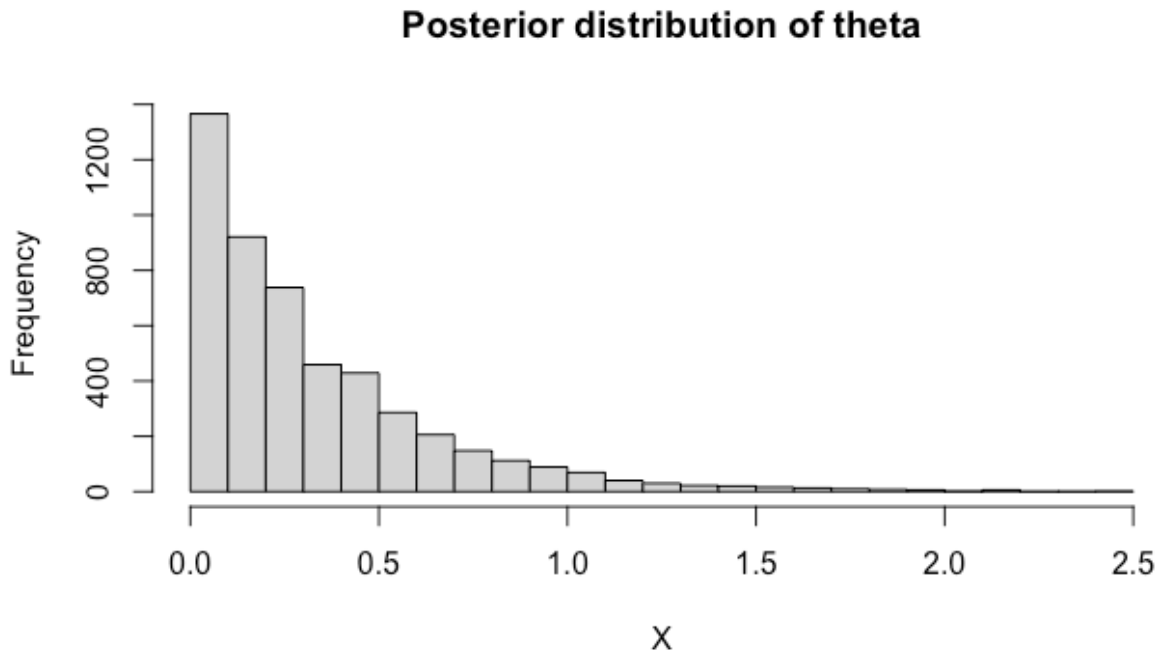
```
N=5000
X<-rep(0,N)
Acp = 0
for (i in 2:N) {
  proposal <- rgamma(1,shape = 1,scale = (1/3))
  U <- runif(1)
  if(U<=1){
    X[i]=proposal
    Acp = Acp+1
  }else{
     X[i] = X[i-1]}
}
mean(X)
```

2. Problem 1 part 2 answer here.

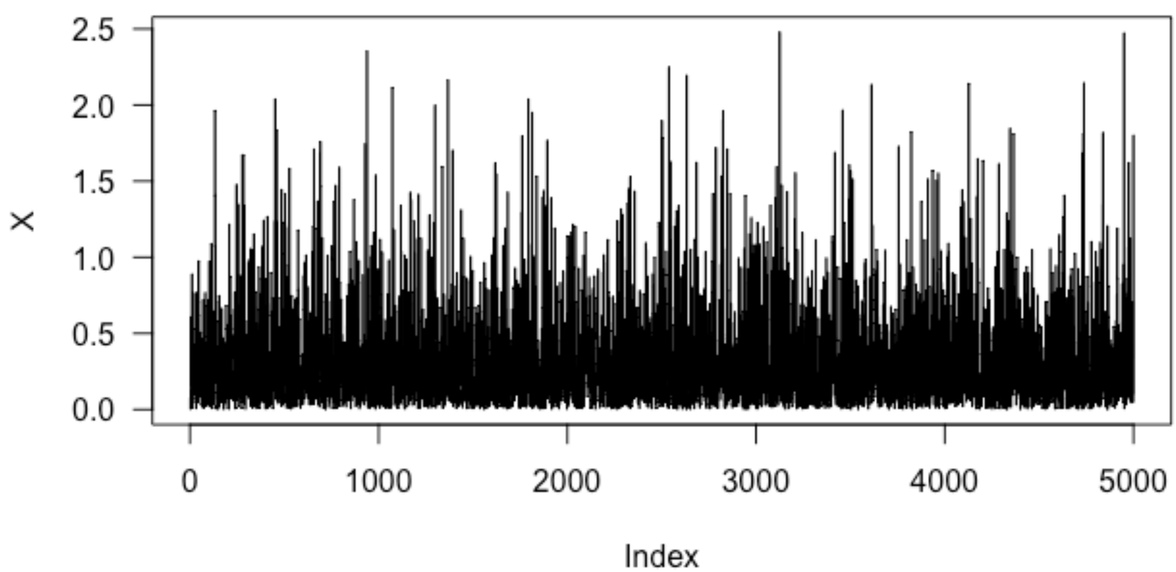
R code:

```
hist(X,main = "Posterior distribution of theta",breaks = 30)
plot(X,type = "s",las=1)
```

Density plot:


Posterior distribution of theta
Frequency
0
400
800
1200
0.0
0.5
1.0
1.5
2.0
2.5
X

Trace plot:


X
0.0
0.5
1.0
1.5
2.0
2.5
0
1000
2000
3000
4000
5000
Index

3. Problem 1 part 3 answer here

   R code:

   Acp/(N-1)

   The acceptance rate is 1.

Problem 2

Answer to the problem goes here.

1. Problem 2 part 1 answer here.

   The product of the likelihood and the prior is proportional to

$$\exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right)\sqrt{\frac{\lambda}{\tau^2}}\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)\lambda^{\alpha-1}\exp(-\beta\lambda)\,,\alpha=\beta=1/2$$

$$\text{Then } \pi(\theta|\bar{y},\lambda)\propto\exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)=$$

$$\exp\left(-\frac{\theta^2-2\theta\frac{\bar{y}n\tau^2+\lambda\sigma^2\mu}{n\tau^2+\lambda\sigma^2}+(\frac{n\tau^2\bar{y}^2+\lambda\sigma^2\mu^2}{n\tau^2+\lambda\sigma^2})^2}{\frac{2\sigma^2\tau^2}{n\tau^2+\lambda\sigma^2}}\right)=\exp\left(-\frac{(\theta-(\frac{\tau^2\bar{y}}{\tau^2+\frac{\lambda\sigma^2}{n}}+\frac{\frac{\lambda\sigma^2}{n}\mu}{\tau^2+\frac{\lambda\sigma^2}{n}}))^2}{2\frac{\frac{\tau^2\sigma^2}{n}}{\tau^2+\frac{\lambda\sigma^2}{n}}}\right)$$

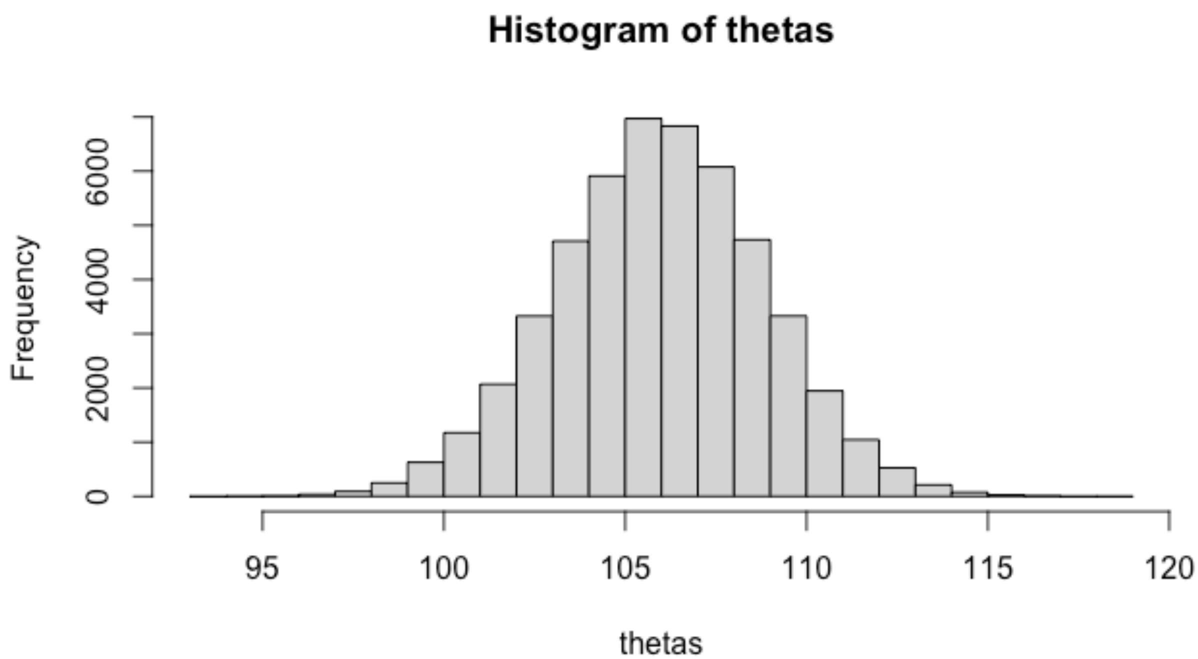
   Which is the density of $N(\frac{\tau^2}{\tau^2+\frac{\lambda\sigma^2}{n}}\bar{y}+\frac{\frac{\lambda\sigma^2}{n}}{\tau^2+\frac{\lambda\sigma^2}{n}}\mu,\frac{\frac{\tau^2\sigma^2}{n}}{\tau^2+\frac{\lambda\sigma^2}{n}})$

$$\pi(\lambda|\bar{y},\theta)\propto\sqrt{\lambda}\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)\lambda^{\alpha-1}\exp(-\beta\lambda)\,,\alpha=\beta=1/2$$

$$\pi(\lambda|\bar{y},\theta)\propto\exp\left(-\frac{(\theta-\mu)^2+\tau^2}{2\tau^2}\lambda\right)$$

   Which is the density of Exp $(\frac{(\theta-\mu)^2+\tau^2}{2\tau^2})$

2. Problem 2 part 2 answer here:



By burnin first 1000 observations, we have posterior samples.

Posterior mean: 105.9539

Posterior variance: 8.222889

The 94% credible set is [100.5183, 111.296]

R code:

```
thetas = rep(0,50000)
lambdas = rep(0,50000)
lambda = 1
theta = 110
for (i in 1:50000){
  mean_theta = (105.5*120)/(120+lambda*90/10)+(lambda*90*110)/(10*120+lambda*90)
  var_theta = (120*90)/(10*120+lambda*90)
  lamba_mean = (120+(theta-110)^2)/(2*120)

  newtheta = rnorm(1,mean=mean_theta,sd=sqrt(var_theta))
```

```
  newlambda = rexp(1,rate = lamba_mean)

  thetas[i] = newtheta
  lambdas[i] = newlambda

  theta = newtheta
  lambda = newlambda

}

mean(thetas[1000:50000])
var(thetas[1000:50000])
summary(thetas[1000:50000])
quantile(thetas[1000:50000],0.03)
quantile(thetas[1000:50000],0.97)
hist(thetas,breaks = 30)
```