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Midterm

Problem 1

Answer to the problem goes here.

1. Problem 1 part 1 answer here.

As the mean rate of defection in plant A is 0.02. $\frac{\alpha_0}{\alpha_0+\beta_0} = 0.02, \alpha_0 = 0.02 * 100 = 2, \beta_0 = 98$

I postulate beta prior for the defective rate for plant A to be $Beta(\alpha_0 = 2, \beta_0 = 98)$

As the mean rate of defection in plant B is 0.1. $\frac{\alpha_0}{\alpha_0+\beta_0} = 0.1, \alpha_0 = 0.1 * 100 = 10, \beta_0 = 90$

I postulate beta prior for the defective rate for plant A to be $Beta(\alpha_1 = 10, \beta_1 = 90)$

2. Problem 1 part 2 answer here.

Denote $P(D|A) = 0.02, P(D|B) = 0.1$

During inspection, $P(A) = 0.6, P(B) = 0.4$

$P(D) = P(D|A)P(A) + P(D|B)P(B) = 0.02 * 0.6 + 0.4 * 0.1 = 0.052$

Denote number of defections during inspection follow $Bin(n, p)$

$\pi(p|x) \propto f(x|p)\pi(p) \propto p^{x+\alpha-1}(1-p)^{n-x+\beta-1}$

$p|X \sim Beta(x+\alpha, n-x+\beta)$

In plant A:
$p|X \sim Beta(x+\alpha_0, n-x+\beta_0), plugging\ in\ n = 200, x = 15 \Rightarrow p|X \sim Beta(17{,}283)$

In plant B:
$p|X \sim Beta(x+\alpha_1, n-x+\beta_1), plugging\ in\ n = 200, x = 15 \Rightarrow p|X \sim Beta(25{,}275)$

3. Problem 1 part 3 :

   In plant A: [0.03346604, 0.08544675]

   In plant B: [0.05483958, 0.11707831]

4. Problem1 part 4:

   In plant A: [0.03175030, 0.08313973]

   In plant B: [0.05307317, 0.11483440]

Problem 2

$$p(\theta_2|y) \propto \int \quad f(\theta_1)f(\theta_2)f(y|\theta_1,\theta_2)d\theta_1$$

$$= f(\theta_2)\int_{-\infty}^{+\infty} f(y|\theta_1,\theta_2)f(\theta_1)d\theta_1$$

$$= f(\theta_2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(1-(\theta_1+\theta_2))^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\theta_1)^2}{2}\right)d\theta_1$$

$$= f(\theta_2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(1-(\theta_1+\theta_2))^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\theta_1)^2}{2}\right)d\theta_1$$

$$= f(\theta_2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(1-2\theta_1-2\theta_2+2\theta_1^2+2\theta_1\theta_2+\theta_2^2}{2}\right)\frac{1}{\sqrt{2\pi}}d\theta_1$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(1-2\theta_2+\theta_2^2)}{2}\right)f(\theta_2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(-2\theta_1+2\theta_1^2+2\theta_1\theta_2)}{2}\right)d\theta_1 =$$

$$= \frac{1}{2\pi}\exp\left(-\frac{(1-2\theta_2+\theta_2^2)}{2}\right)f(\theta_2)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi * \frac{1}{2}}}\exp\left(-\frac{(\theta_1-\frac{(1-\theta_2)}{2})^2}{2*\frac{1}{2}}\right)\exp\left(\frac{(1-\theta_2)^2}{4}\right)\sqrt{2\pi*\frac{1}{2}}d\theta_1$$

$$= f(\theta_2)\frac{1}{2\pi}\exp\left(-\frac{(1-2\theta_2+\theta_2^2)}{2}\right)\exp\left(\frac{(1-\theta_2)^2}{4}\right)\sqrt{2\pi*\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{(1-\theta_2)^2}{4}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\theta_2)^2}{2}\right) \propto \exp\left(-\frac{(\theta_2-\frac{1}{3})^2}{4/3}\right)$$

Which is the density of $N(\frac{1}{3},\frac{2}{3})$

Similarly, $p(\theta_1|y) \propto \int \quad f(\theta_1)f(\theta_2)f(y|\theta_1,\theta_2)d\theta_2$

Using the similar integration approach as above, the marginal distribution of θ_1 is $N(\frac{1}{3},\frac{2}{3})$

Problem 3

$$f(y,\theta,\tau^2) \propto \left(\prod_{i=1}^{n} f(y_i\,|\theta)\right)\pi(\theta|\tau^2)\pi(\tau^2)$$

$$= \left(\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y_i-\theta)^2}{2}}\right)\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{\theta^2}{2\tau^2}}\frac{1}{\Gamma(1)}(\tau^2)^{-2}e^{-\frac{1}{\tau^2}}$$

$$\propto e^{-\frac{\sum_{i=1}^{n}(y_i-\theta)^2}{2}}(\tau^2)^{-\frac{1}{2}}e^{-\frac{\theta^2}{2\tau^2}}(\tau^2)^{-2}e^{-\frac{1}{\tau^2}}$$

Thus,

$$\pi(\theta|y,\tau^2) \propto e^{-\frac{\sum_{i=1}^{n}(y_i-\theta)^2}{2}} e^{-\frac{\theta^2}{2\tau^2}} = e^{-\frac{\tau^2\sum_{i=1}^{n}y_i^2-2\theta\tau^2\sum_{i=1}^{n}y_i+n\theta^2\tau^2}{2\tau^2}} e^{-\frac{\theta^2}{2\tau^2}} \propto e^{-\left(\frac{\theta-\frac{\tau^2\sum_{i=1}^{n}y_i}{n\tau^2+1}}{2\tau^2/n\tau^2+1}\right)^2}$$

Which is the kernel of Normal $N(\frac{\tau^2\sum_{i=1}^{n}y_i}{n\tau^2+1}, \frac{\tau^2}{n\tau^2+1})$

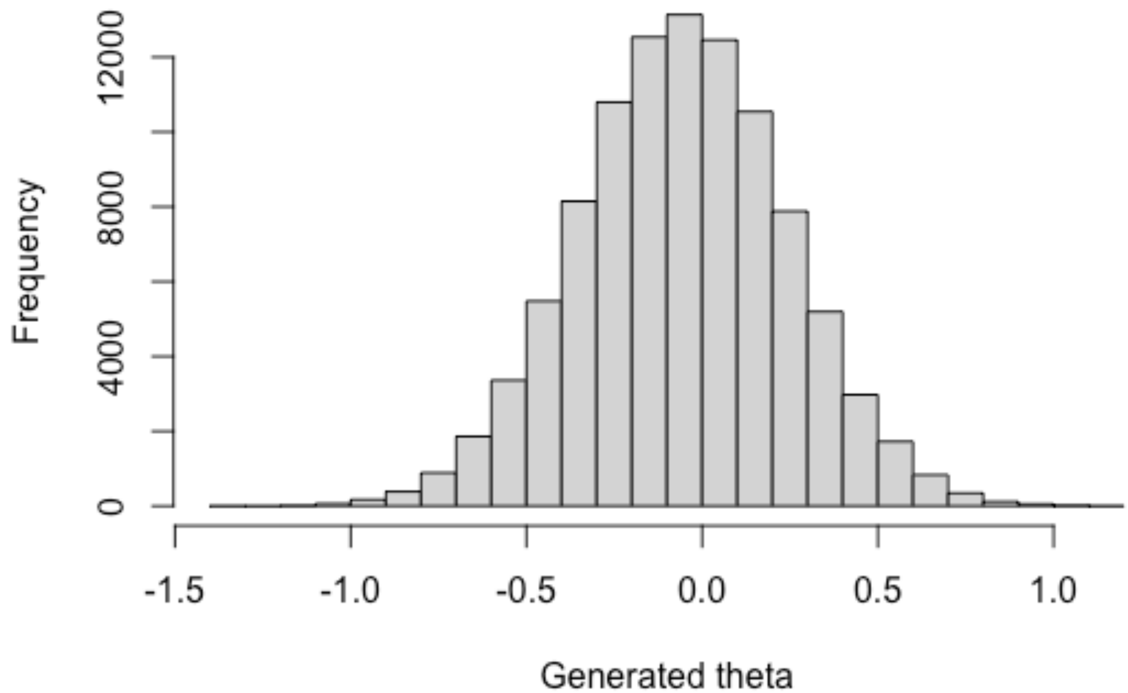
As $\sum_{i=1}^{n} y_i = -0.63, n = 10$, the conditional distribution follows Normal $N(\frac{-0.63\tau^2}{10\tau^2+1}, \frac{\tau^2}{10\tau^2+1})$

$$\pi(\tau^2|\theta,y) \propto (\tau^2)^{-\frac{1}{2}} e^{-\frac{\theta^2}{2\tau^2}} (\tau^2)^{-2} e^{-\frac{1}{\tau^2}} = (\tau^2)^{-\frac{5}{2}} e^{-\frac{\theta^2+2}{2\tau^2}} = (\tau^2)^{-\frac{3}{2}-1} e^{-\frac{\theta^2+2}{2}/\tau^2}$$

Which is the kernel of inverse gamma $(\frac{3}{2}, \frac{\theta^2+2}{2})$

By implementing the Gibbs sampling (generate 100,000 samples and use 1,000 samples as burn-in), I have the results below.

Posterior density plot of theta:



Posterior mean: -0.05653624

95% equi-tailed credible interval of θ: [-0.6406627, 0.5286508]