Homework 3

ISyE 6420

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1 Maxwell

. Sample $y_1, \ldots, y_n$, comes from a Maxwell distribution with density

$$f(y \mid \theta) = \sqrt{\frac{2}{\pi}} \theta^{3/2} y^2 e^{-\theta y^2/2}, y \geq 0, \theta > 0.$$

Assume an exponential prior on θ,

$$\pi(\theta) = \lambda e^{-\lambda \theta}, \theta > 0, \lambda > 0.$$

(a) Show that posterior belongs to a Gamma family and depends on data via $\sum_{i=1}^n y_i^2$.

(b) For $\lambda = 1/2$ and $y_1 = 1.4, y_2 = 3.1$, and $y_3 = 2.5$, find the Bayes estimator for θ. How does the Bayes estimator compare to the MLE and prior mean? The MLE for θ is $\hat{\theta} = \frac{3n}{\sum_{i=1}^n y_i^2}$.

(c) Use programming to calculate the 95% equitailed credible set for θ.

(d) Find a prediction for a future single observation. For this, you will need the mean of Maxwell, which is $E[Y] = 2\sqrt{\frac{2}{\pi\theta}}$.

2 Jeremy Mixture

Show that for likelihood $f(x \mid \theta)$ and mixture prior

$$\pi(\theta) = \epsilon \pi_1(\theta) + (1 - \epsilon)\pi_2(\theta), \theta \in \Theta,$$

the posterior is a mixture of

$$\pi(\theta \mid x) = \epsilon' \pi_1(\theta \mid x) + (1 - \epsilon')\, \pi_2(\theta \mid x),$$

where

$$\pi_i(\theta \mid x) = \frac{f(x \mid \theta)\pi_i(\theta)}{m_i(x)}, \quad m_i(x) = \int_\Theta f(x \mid \theta)\pi_i(\theta)d\theta, \quad i = 1, 2, \quad \text{and}$$

$$\epsilon' = \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}.$$

Now we assume $X \mid \theta \sim \mathcal{N}(\theta, 80)$ and the prior for θ is a mixture

$$\theta \sim \pi(\theta) = \frac{2}{3}\mathcal{N}(110, 60) + \frac{1}{3}\mathcal{N}(100, 200).$$

Find the posterior and Bayes estimator for θ if $X = 98$.

3 Mendel's Experiment with Peas

Johann Gregor Mendel (1822-1884) studied the inheritance of seven different features in peas, including height, flower color, seed color, and seed shape. To do so, he first established pea lines with two different forms of a feature, such as tall vs. short height. He grew these lines for generations until they were purebreeds (always produced offspring identical to the parent), then bred them to each other and observed how the traits were inherited.

For the height trait, Mendel's model suggests that 3/4 of the plants grown from a cross between tall and short height strains of pea lines will be of the tall height variety. After breeding 1064 of these plants, 787 resulted as the tall height variety. The reasonable model for the number of tall height results from n experiments is binomial $\mathcal{B}in(n, p)$. Complete a Bayesian model with beta $\mathcal{B}e(15, 5)$ prior on the unknown proportion p.

(a) What are the prior and posterior means?

(b) Find the posterior probability of hypothesis $H_0 : p \leq 3/4$?

(c) Find a 95% equitailed credible set for the true proportion of tall height plants obtained from the given cross.