Problem 1

Answer to the problem goes here.

1. Problem 1 part 1 answer here.

$$f(\theta|y) \propto f(y|\theta)f(\theta) = \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \theta^{3/2} y_i^2 e^{-\frac{\theta y_i^2}{2}} \lambda e^{-\lambda\theta} \propto \theta^{\frac{3n}{2}} e^{-\lambda\theta - \frac{\theta}{2}\sum_{i=1}^{n} y_i^2}$$
$$= \theta^{\frac{3n}{2}+1-1} e^{-\theta(\lambda + \frac{1}{2}\sum_{i=1}^{n} y_i^2)}$$

Which is the pdf of Gamma distribution ($\frac{3n}{2} + 1, \lambda + \frac{1}{2}\sum_{i=1}^{n} y_i^2$)

2. Problem 1 part 2 answer here.

The posterior mean is $\frac{\frac{3n}{2}+1}{\lambda + \frac{1}{2}\sum_{i=1}^{n} y_i^2}$. By plugging the number, $\frac{\frac{3*3}{2}+1}{0.5+\frac{1}{2}(1.4^2+3.1^2+2.5^2)} = 0.5844846$

The MLE is $\frac{3n}{\sum_{i=1}^{n} y_i^2} = \frac{9}{1.4^2+3.1^2+2.5^2} = 0.5050505$

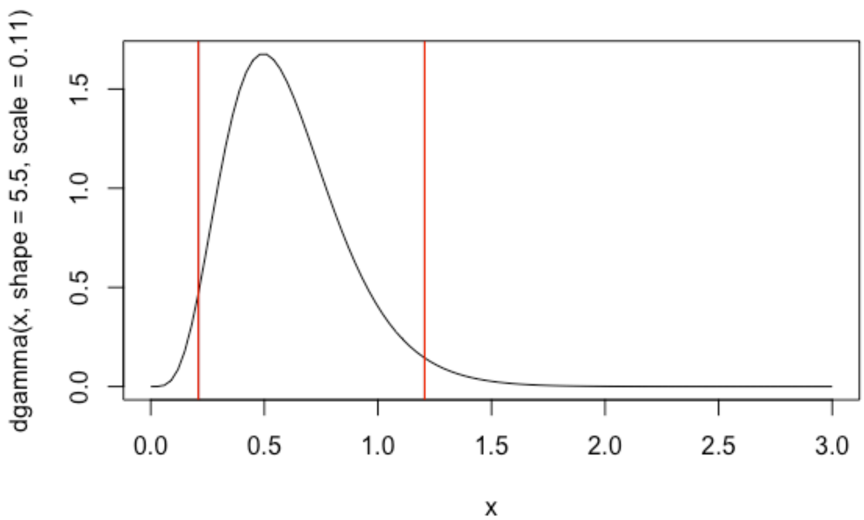
The prior mean is $\frac{1}{\lambda} = 2$

3. Problem 1 part 3 answer here

R code:

```
plot(0, 0, xlim = c(0, 3), ylim = c(0, 1), type = "n")
curve(dgamma(x, shape = 5.5, scale = 0.11), from = 0, to = 3)
q1<-qgamma(0.025,shape = 5.5, scale = 0.11)
q2<-qgamma(0.975,shape = 5.5, scale = 0.11)
abline(v=c(q1,q2),col="red")
```

The 95% equitailed credible set is [0.2098662, 1.205603]



4. Problem 1 part 4 answer here

   Gamma distribution $(\frac{3n}{2}+1, \lambda+\frac{1}{2}\sum_{i=1}^{n} y_i^2)$, posterior gamma (5.5, 9.41)

   As $E(Y) = 2\sqrt{\frac{2}{\pi\theta}}$

$$\hat{y}_{n+1} = \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta = \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{9.41^{5.5}}{\Gamma(5.5)}\theta^{5.5-1}e^{-9.41\theta}d\theta$$

$$= (\frac{9.41^{5.5}}{\Gamma(5.5)})\frac{\Gamma(5)}{9.41^5}2\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{9.41^5}{\Gamma(5)}\theta^{5-1}e^{-9.41\theta}d\theta = 2.244499$$

Problem 2

Answer to the problem goes here.

1. Problem 2 part 1 answer here.

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{m(x)} = \frac{f(x|\theta)[\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)]}{\int_\Theta f(x|\theta)[\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)]d\theta}$$

$$= \frac{\epsilon f(x|\theta)\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)f(x|\theta)}{\epsilon\int_\Theta f(x|\theta)\pi_1(\theta)d\theta + (1-\epsilon)\int_\Theta f(x|\theta)\,\pi_2(\theta)d\theta}$$

$$= \frac{\epsilon\pi_1(\theta|x)m_1(\theta) + (1-\epsilon)\pi_2(\theta|x)m_2(x)}{\epsilon m_1(\theta) + (1-\epsilon)m_2(x)}$$

$$= \epsilon'\pi_1(\theta|x) + (1-\epsilon')\pi_2(\theta|x)$$

2. Problem 2 part 2 answer here:

$$\pi(\theta|x) \propto f(x|\theta)\pi(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}}$$

$$= \frac{1}{2\pi\sigma\sigma_0} e^{-\frac{\theta^2 - \frac{2(\sigma^2\theta_0+\sigma_0^2 x)}{\sigma_0^2+\sigma^2}\theta + \frac{\sigma^2\theta_0^2+\sigma_0^2x^2}{\sigma_0^2+\sigma^2}}{\frac{2\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}}}$$

Which could be represented by $N(\frac{\sigma^2\theta_0+\sigma_0^2 x}{\sigma_0^2+\sigma^2}, \frac{\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2})$

Therefore, $\pi_1(\theta) \sim N(\frac{60*98+80*110}{80+60}, \frac{60*80}{80+60})$, i.e. N (104.9, 34.3)

$\pi_2(\theta) \sim N(\frac{200*98+80*100}{80+200}, \frac{200*80}{80+200})$, i.e. N (98.6, 57.1)

$$m(x) = \int_{-\infty}^{\infty} f(x|\theta)\pi(\theta)d\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}} d\theta$$

$$= \frac{\sqrt{2\pi\frac{\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}}}{2\pi\sigma\sigma_0} e^{\frac{(\frac{\sigma^2\theta_0+\sigma_0^2x}{\sigma_0^2+\sigma^2})^2 - \frac{\sigma^2\theta_0^2+\sigma_0^2x^2}{\sigma_0^2+\sigma^2}}{2\frac{\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}}}$$

As $\pi_1(\theta) \sim N(110,60)$, $m_1(98) = 0.0202$

As $\pi_2(\theta) \sim N(100,200)$, $m_2(98) = 0.0237$

$$\epsilon' = \frac{\frac{2}{3}m_1(98)}{\frac{2}{3}m_1(98) + \frac{1}{3}m_2(98)} = 0.63$$

Then the posterior is $0.63 * \mathrm{N}\,(104.9, 34.3) + 0.37 * \mathrm{N}\,(98.6, 57.1)$

The bayes estimator is expectation of this mixture distribution 0.63*104.9+0.37*98.6=102.6

Problem 3

Answer to the problem goes here.

3. Problem 3(a) part 1 answer here
   Beta prior mean: $\frac{\alpha}{\alpha+\beta} = \frac{15}{20} = 0.75$

Posterior distribution: Beta (15+787, 5+1064-787), i.e. Beta(802, 282)

The posterior mean is 802/(802+282)=0.7398524

4. Problem 3(a) part 2 answer here:
   R code: pbeta(0.75,802,282)

$$P\left(p \leq {}^{3}/_{4}\right) = 0.7754435$$

5. Problem 3(a) part 3 answer here:

   R code:

qbeta(0.025,802,282): 0.7133363

qbeta(0.975,799,281): 0.7655302