1. Let $f = (x^2-2)(x^2-3)(x^2-5) \in \mathbb{Q}[x]$.
   Prove that every subfield E of the splitting field F of f over $\mathbb{Q}$ is Galois over $\mathbb{Q}$.

   The splitting field of f is $F := \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$; this has a basis over $\mathbb{Q}$ consisting of

   $$\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}, \sqrt{30}\}\ ,$$

   so that $[F\colon \mathbb{Q}] = 8$. As $\mathbb{Q}$ has characteristic zero (also f is separable), this is a Galois extension. Its Galois group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (written additively), or better $\{\pm 1\}^3$. The action of a triple $\varepsilon := (\varepsilon_2, \varepsilon_3, \varepsilon_5) \in \{\pm 1\}^3$ on the field generators is $\varepsilon(\sqrt{2}, \sqrt{3}, \sqrt{5}) = (\varepsilon_2\sqrt{2}, \varepsilon_3\sqrt{3}, \varepsilon_5\sqrt{5})$.

   Alternative: F is a splitting field, hence Galois over $\mathbb{Q}$. Visibly, it contains seven quadratic extensions of $\mathbb{Q}$ as subfields, so its Galois group G must have seven normal subgroups of index 2. The only possibility is $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   This Galois group is abelian, so each of its 16 subgroups is normal. Thus, by the Galois correspondence, (intermediate fields E correspond to subgroups, with intermediate Galois extensions corresponding to normal subgroups), every intermediate field is Galois over $\mathbb{Q}$.

2. Suppose that K is a finite field with characteristic p.
   Show that every element of K has a unique pth root in K.

   An rth root of an element $a \in K$ is an element x in some field extension such that $x^r = a$, or rather a root of the polynomial $x^r - a$. Thus we need to show that for $a \in K$, there is an $x \in K$ with $x^p = a$, and that x is the unique solution to this equation.

   Recall that as K has characteristic p, the map $K \ni b \mapsto b^p$ is a field homomorphism (it is clearly multiplicative, and it turns out to also be additive, due to the characteristic). Field homomorphisms are injective ($\{0\}$ is the only ideal) as K is finite. As K is a finite set, injectivity implies surjectivity, so thsat this pth power map is an automorphism, called the *Frobenius automorphism*. (I expect that you will just begin with the Frobenius isomorphism.)

   Since the Frobenius map $x \mapsto x^p$ is surjective and injective as a map on K, every element $a \in K$ has a pth root (surjectivity) and this pth root is unique (injectivity).

3. Let $n > 0$ be an integer. What is the radical of the zero ideal in the ring $\mathbb{Z}/n\mathbb{Z}$?
   (Recall that for I an ideal of a commutative ring R, its radical is
   $\sqrt{I} := \{r \in R \mid \exists m > 0 \text{ with } r^m \in I\}$, and there is a second characterization.)

   We use the correspondence between ideals of $\mathbb{Z}/n\mathbb{Z}$ and ideals of $\mathbb{Z}$ that contain $n\mathbb{Z}$, which preserves primality. Since the radical of an ideal I is the intersection of the prime ideals that contain I, the nilradical of $\mathbb{Z}/n\mathbb{Z}$ (radical of the zero ideal) is the image in $\mathbb{Z}/n\mathbb{Z}$ of the radical of $n\mathbb{Z}$.

   The prime ideals of $\mathbb{Z}$ that contain the ideal $n\mathbb{Z}$ are exactly the ideals $p\mathbb{Z}$ for p a prime divisor of n. Recall that, for $a, b \in \mathbb{Z}$, the intersection $a\mathbb{Z} \cap b\mathbb{Z} = \gcd\{a, b\}\mathbb{Z}$ (this is the definition of greatest common divisor). Thus the intersection of all prime ideals of $\mathbb{Z}$ that contain $n\mathbb{Z}$ is the ideal $\eta\mathbb{Z}$, where η be the product of the prime numbers that divide n. (I'll call this the *squarefree* part of n.)

   Thus the desired nilradical is $\eta\mathbb{Z}/n\mathbb{Z}$, where η is the squarefree part of n.