Graphs Whose Spectral Band Functions are Perfect Morse Functions

Minisymposium on Spectral Theory and Applications of Schrödinger Operators

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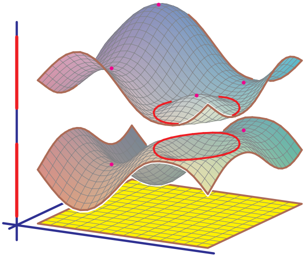
Work with Matthew Faust of Michigan State.



Not Morse functions

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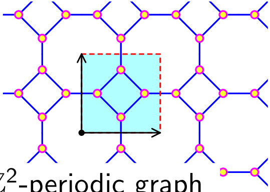
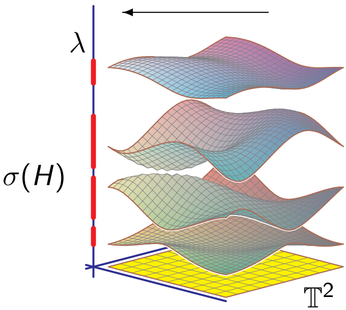
Periodic Graph Operators

A $\mathbb{Z}^d$-periodic graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with periodic functions $V\colon \mathcal{V} \to \mathbb{R}$ and $E\colon \mathcal{E} \to \mathbb{R}$ (a *labeling*)
$\rightsquigarrow$ a discrete periodic operator $H = V + \Delta$ on $\ell_2(\mathcal{V})$.
Δ is a weighted graph Laplacian.



a $\mathbb{Z}^2$-periodic graph



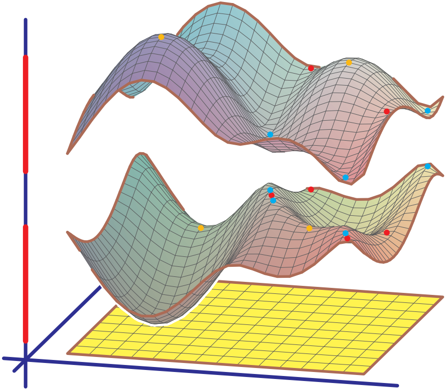
As H is self-adjoint, its spectrum $\sigma(H)$ is a closed subset of $\mathbb{R}$.

Viewed through the lens of unitary characters $\mathbb{T}^d$ of $\mathbb{Z}^d$, the spectrum is the image of $|\mathcal{V}/\mathbb{Z}^d|$ *spectral band functions* $\lambda_i\colon \mathbb{T}^d \to \mathbb{R}$.

This also gives a *Floquet matrix* $H(z)$ of Laurent polynomials such that $\det(H(z) - \lambda_i(z)) = 0$.

Spectral Edges Conjecture

Kuchment: Physicists typically assume that all extrema of spectral band functions are nondegenerate critical points, which implies many important physical properties.
The *spectral edges conjecture* for a periodic graph Γ posits that this occurs for a general labeling of Γ.



Easy fact: If $z \in \mathbb{T}^d$ satisfies that $z^2 = 1$ (z is a *corner point*), then it is a critical point of every spectral band function.

By the Morse inequalities, if every critical point of a spectral band function is nondegenerate, then it has at least 2^d critical points.

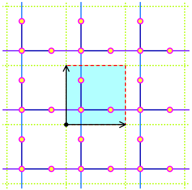
(This is because 2^d is the total Betti number of $\mathbb{T}^d$.)

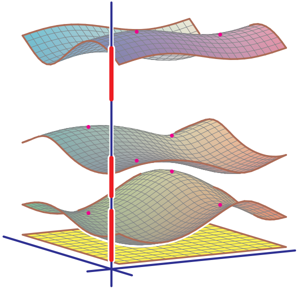
Perfect Morse Functions

A spectral band function has at least 2^d critical points.

When it has exactly 2^d critical points, all nondegenerate, then the spectral band function is a *perfect Morse function*.

The Lieb lattice is a graph whose spectral band functions can be perfect Morse functions:



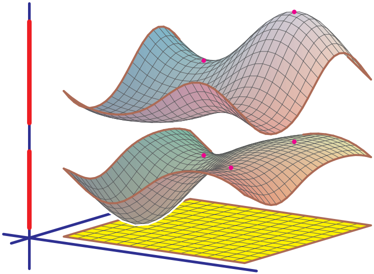


We seek conditions which imply that all spectral band functions are perfect Morse functions.

This very strong condition implies the spectral edges conjecture.

Minimally Supported Periodic Graphs

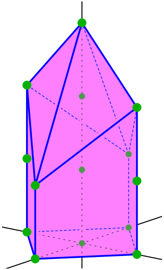
The general Floquet matrix $H(z)$ of a connected periodic graph Γ has a monomial $z^{\pm\alpha} \Longleftrightarrow \Gamma$ has an edge between vertices v and $\alpha+v$.



Such edges span $\mathbb{Z}^d$.

The Floquet matrix $H(z)$ is *minimally supported* if the only monomials of z in $\det H(z)$ are z^0, z_i, z_i^{-1}, for $i \in [d]$.

A graph is *minimally supported* if its general Floquet matrix is minimally supported.



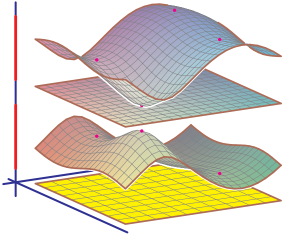
This is a property of the Newton polytope of $\det(H(z) - \lambda)$.

Faust-Robinson-S.: *If Γ is minimally supported, each spectral band has at most 2^d isolated critical points.*

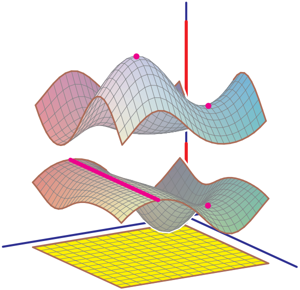
Flat Band Trichotomy

A labeled graph Γ has a *flat band* if $D(z,\lambda) := \det(H(z)-\lambda)$ has a factor $\lambda-\lambda_0$.

Γ has a *flat sub-band* if, after substituting ± 1 for z_i in $D(z,\lambda)$ for some *but not all* coordinates, there is a factor $\lambda-\lambda_0$. This implies a $\mathbb{T}^r$ of critical points for some $r < d$.

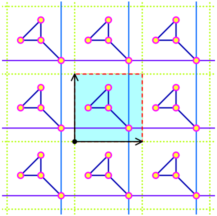


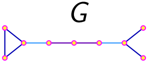
Trichotomy (Faust-S.): *If a labeled $\mathbb{Z}^d$ periodic graph Γ is minimally supported, then either*
(1) Γ has a flat band.
(2) Γ has a flat sub-band.
(3) Every spectral band function is a perfect Morse function.

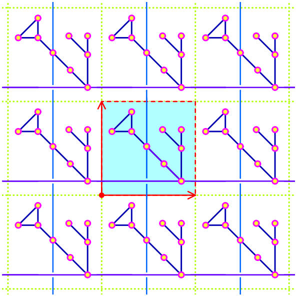


These can be wild. html

Isthmus connected graphs







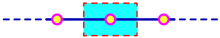
A periodic graph Γ is *isthmus-connected* if it has a fundametal domain G with an induced path such that each adjacent fundamental domain is connected through a unique vertex in the interior of the path.

Isthmus (Faust-S.): *Let Γ be an isthmus connected graph. For any choice of non-zero edge parameters, there is an explcitly described dense open subset U of potentials with the property:*

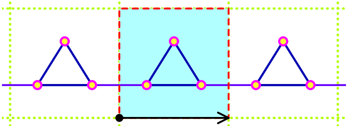
For any labeling of Γ with these edge weights and potential $V \in U$, every spectral band function is a perfect Morse function.

Parallel Extensions

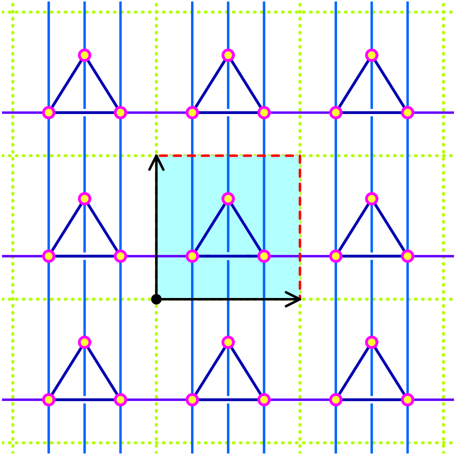
Let $\mathbb{Z}_*$ be the 1-dimensional square lattice.



The parallel extension of a $\mathbb{Z}^d$-periodic labeled graph Γ is the $\mathbb{Z}^{d+1}$-periodic graph $\mathbb{Z}_* \times \Gamma$.



minimally connected



Parallel Extension (Faust-S.):
If every spectral band function of a labeled periodic graph Γ is perfect Morse function, then every spectral band function of its parallel extension is a perfect Morse function.