Algebra Autumn 2013
Frank Sottile
7 November 2013

Tenth Homework

Hand in to Frank Tuesday 12 November:

55. Let $R \subset \mathbb{C}$ be set whose elements are

$$R \;:=\; \{a + b\left(\tfrac{1+\sqrt{-19}}{2}\right) \;\mid\; a, b \in \mathbb{Z}\}\,.$$

Show that R is a subring of $\mathbb{C}$ that is a principal ideal domain but not a Euclidean domain.

Hand in for the grader Thursday 14 November:

56. For $\alpha = a + b\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}]$, (the Gaußian integers) set $N(\alpha) := a^2 + b^2$. Determine the units in $\mathbb{Z}[\sqrt{-1}]$. Show that if $N(\alpha)$ is prime, then α is irreducible. Show that the same conclusion holds if $N(\alpha) = p^2$, where p is a prime in $\mathbb{Z}$ that is congruent to 3 modulo 4.

57. Show that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain. (Hint: Show that it is a Euclidean domain.)

58. Using that $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain, show that every prime p that is congruent to 1 modulo 4 is the sum of two squares.

    Hint: use the cyclicity of the group of units of $\mathbb{Z}_p^*$ (an earlier homework problem) to show that there is a number $n \in \mathbb{Z}$ with $n^2 \cong -1 \pmod{p}$. Then use this to show that p is reducible in $\mathbb{Z}[\sqrt{-1}]$.

59. Let S be a multiplicative subset of an integral domain R with $0 \notin S$. Show that if R is a principal ideal domain, then so is $R[S^{-1}]$.

60. Let $p \in \mathbb{Z}$ be a prime number, so that (p) is a prime ideal. What can be said about the relation between the quotient ring $\mathbb{Z}_p$ and the localisation $\mathbb{Z}_{(p)}$?