Phase limit set of linear spaces and discriminants

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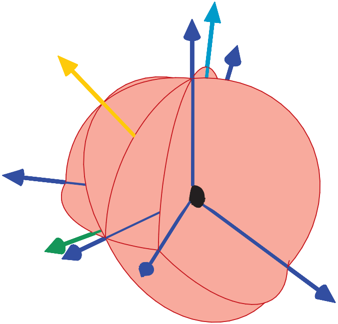

Work with Mounir Nisse

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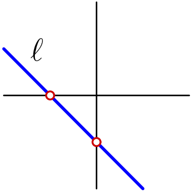
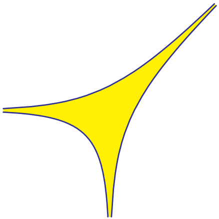
Amoebas and coamoebas

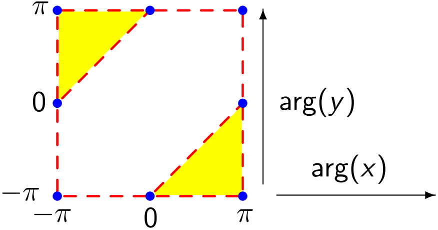
The *amoeba* $\mathcal{A}(X)$ of a very affine variety $X \subset (\mathbb{C}^\times)^n$ is the set of lengths in X and its *coamoeba* $co\mathcal{A}(X)$ is its set of arguments.

Formally, $e^r\theta \mapsto (r, \theta)$ identifies $\mathbb{C}^\times$ with $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = S^1$ is the unit complex numbers. This induces maps $(\mathrm{Log}, \mathrm{Arg})\colon (\mathbb{C}^\times)^n \xrightarrow{\sim} \mathbb{R}^n \times \mathbb{T}^n$. Then $\mathcal{A}(X)$ is the projection of X to $\mathbb{R}^n$ and $co\mathcal{A}(X)$ is its projection to $\mathbb{T}^n$.



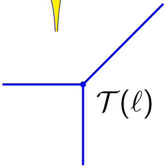
Example: The amoeba and coamoeba of the line $\ell := \mathcal{V}(x+y+1) \subset (\mathbb{C}^\times)^2$ are





Phase limit set

Tropical variety $\mathcal{T}(X)$ is the logarithmic limit set; the cone over limiting directions of $\mathcal{A}(X)$.
It encodes the nonempty initial schemes of X, $\mathcal{T}(X) = \{w \in \mathbb{R}^n \mid \mathrm{in}_w X \neq \emptyset\}$.
It has a (non-unique) fan structure with initial schemes constant on (rel. interiors of) cones.



The phase limit set $\mathcal{P}^\infty(X)$ is the collection of accumulation points of sequences $\{\mathrm{Arg}(x_i) \mid i \in \mathbb{N}\}$ of arguments of unbounded sequences $\{x_i \mid i \in \mathbb{N}\} \subset X$.

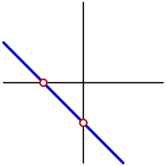
We have $co\mathcal{A}(X) \cup \mathcal{P}^\infty(X) = \overline{co\mathcal{A}(X)}$.

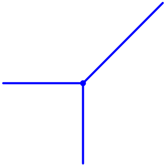
Theorem. (Nisse-S.) *For any fan structure on $\mathcal{T}(X)$,*

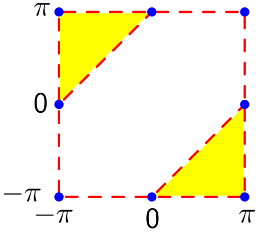
$$\mathcal{P}^\infty(X) \;=\; \bigcup_{\rho \in \mathrm{ray}(\mathcal{T}(X))} \mathrm{Arg}(\mathrm{in}_\rho X)\ .$$

The line

Recall the line $\ell = \mathcal{V}(x+y+1)$, its tropical variety, and coamoeba:



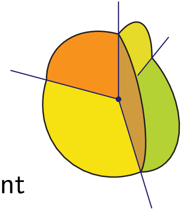




The dashed lines are the phase limit set of ℓ. They are translates of the three subtori in the directions of rays of the tropical variety.

In fact, they are coamoebae of the initial schemes of ℓ.

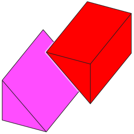
The plane $\Pi := \mathcal{V}(x+y+z+1)$



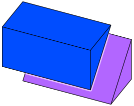
The tropical variety $\mathcal{T}(\Pi)$ of the plane has four rays:

Each ray ρ has a corresponding subtorus $\mathbb{C}^\times_\rho$ which acts freely on the initial scheme $\mathrm{in}_\rho\,\Pi$, with the quotient isomorphic to a line $\mathcal{V}(x+y+1)$.

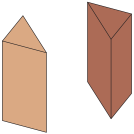
A consequence is that $\mathcal{P}^\infty(\Pi)$ has four components, each a prism over the coamoeba of a line.



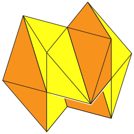
$co\mathcal{A}(\mathrm{in}_{(1,0,0)}(\Pi))$



$co\mathcal{A}(\mathrm{in}_{(0,1,0)}(\Pi))$

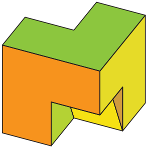


$co\mathcal{A}(\mathrm{in}_{(0,0,1)}(\Pi))$



$co\mathcal{A}(\mathrm{in}_{(-1,-1,-1)}(\Pi))$

Their union is the closure of the coamoeba $co\mathcal{A}(\Pi)$ of the plane, covering a typical point twice.
Note the striking polyhedral structure.



Hyperplane complements

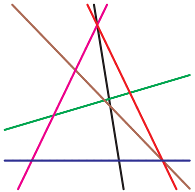
A set $B \subset \mathbb{C}^d$ of linear forms gives a hyperplane arrangement

$$\mathcal{H}_B \ := \ \bigcup\{\mathcal{V}(b) \mid b \in B\} \ \subset \ \mathbb{C}^d\,,$$

and a map $\lambda_B \colon \mathbb{C}^d \to \mathbb{C}^B$ where $\mathbb{C}^d \ni v \mapsto (b(v) \mid b \in B)$.

Intersections of hyperplanes are *flats* of $\mathcal{H}_B$, inducing a matroid structure on the set B.

Example. The column vectors B of $\begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}$ defines a line arrangement in $\mathbb{P}^2$:



The hyperplane complement $\mathcal{H}_B^c := \lambda_B(\mathbb{C}^d) \cap (\mathbb{C}^\times)^B \simeq \mathbb{C}^d \smallsetminus \mathcal{H}_B$ is a very affine variety. We study its coamoeba and phase limit set.

Structure of $\mathcal{P}^\infty(\mathcal{H}^c)$

Using that $\mathcal{P}^\infty(\mathcal{H}^c) =$ accumulation points of arguments,

Theorem. *$\mathcal{P}^\infty(\mathcal{H}^c) \ = \bigcup_L \overline{co\mathcal{A}(\mathcal{H}/L)^c} \times co\mathcal{A}(H|_L)^c$, the union over all flats L of $\mathcal{H}$.*

We refine this. Given a flag $\mathcal{L}\colon L_1 \subset \cdots \subset L_k \subset \mathbb{C}^d$ of flats, set

$$\mathcal{H}(\mathcal{L})^c \ := \ (\mathcal{H}|_{L_1})^c \times \cdots \times ((\mathcal{H}/L_{i-1})|_{L_i})^c \times \cdots \times (\mathcal{H}/L_k)^c \,.$$

Corollary. $\overline{co\mathcal{A}(\mathcal{H}^c)} = \bigcup_{\mathcal{L}\text{ a flag of flats}} co\mathcal{A}(\mathcal{H}(\mathcal{L})^c)$.

Flags of flats $\longleftrightarrow$ cones in $\mathcal{T}(\mathcal{H}^c)$, with $\mathrm{in}_{\mathcal{L}}\,\mathcal{H}^c = \mathcal{H}(\mathcal{L})^c$, recovering the tropical decomposition of the phase limit set. We also relate this to the Bergman fan, which is a different fan structure on $\mathcal{T}(\mathcal{H}^c)$.

(Reduced) Discriminants

When $B \subset \mathbb{Z}^d$, Kapranov showed that the rational map

$$\pi_B \ : \ \mathbb{C}^d \ni z \ \longmapsto \ \prod_{b \in B} b(z)^b \ \in \ \mathbb{P}^{d-1}$$

has image the (reduced) discriminant $D_B \subset \mathbb{P}^{d-1}$.

This monomial map $(x_b \mid b \in B) \mapsto \prod x_b^b$ restricted to the hyperplane complement $\mathcal{H}_B^c \subset (\mathbb{C}^\times)^B$ has been used to study discriminants and their tropicalizations.

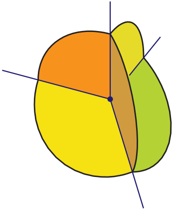
Fact. $co\mathcal{A}(D_B) = \pi_B(co\mathcal{A}(\mathcal{H}_B^c)$, and $\mathcal{P}^\infty(D_B) = \mathcal{P}^\infty(co\mathcal{A}(\mathcal{H}_B^c)$.

Passare and I used this to (re)prove a strong structure theorem when $d = 2$, which motivated this work with Nisse.

Nisse and I have many technical structural results about $\mathcal{P}^\infty(D_B)$.

With the conjecture: $co\mathcal{A}(D_B) \subset \mathcal{P}^\infty(D_B)$, they imply $co\mathcal{A}(D_B)$ has a recursive polyhedral structure, as we saw for the plane.

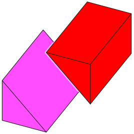
The plane $\Pi := \mathcal{V}(x+y+z+1)$ (reprised)



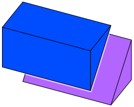
The plane $\Pi := \mathcal{V}(x+y+z+1)$ is a discriminant.
Its tropical variety $\mathcal{T}(\Pi)$ has four rays:

The initial scheme $\mathrm{in}_\rho \Pi$ of a ray has a $\mathbb{C}^\times_\rho$-action with quotient a line $\mathcal{V}(r+s+1)$.

Consequently, $\mathcal{P}^\infty(\Pi)$ has four components, each a prism over the coamoeba of a line.



$co\mathcal{A}(\mathrm{in}_{(1,0,0)}(\Pi))$



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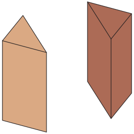

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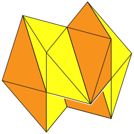

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