Numerical Irreducible Decomposition for Multiprojective Varieties

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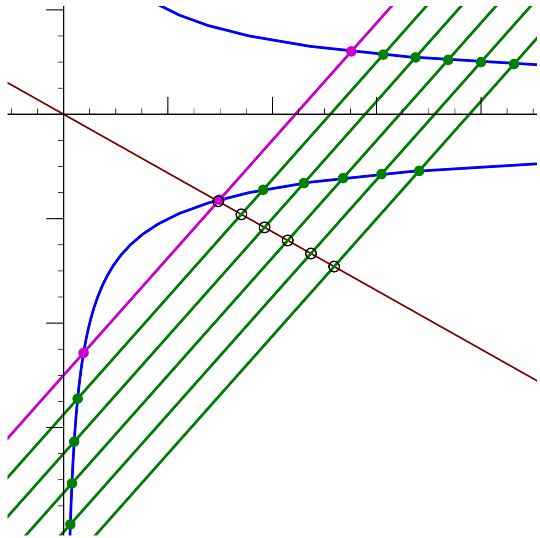

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Work with Anton Leykin and Jose Israel Rodriguez.

Numerical Irreducible Decomposition

$V \subset \mathbb{P}^n$ of dimension m is represented by a witness set $W := V \cap M$, for $M \subset \mathbb{P}^n$ a general linear subspace of codimension m.
$\#W = \deg V$.

If the irreducible decomposition of V is $V^1 \cup \cdots \cup V^s$, then *numerical irreducible decomposition* computes the partition $W = W^1 \sqcup W^2 \sqcup \cdots \sqcup W^s$, where $W^i := V^i \cap M$.

Two steps:

(1) Monodromy. Move the slice M in loops to compute a partition

$$W \;=\; U^1 \sqcup U^2 \sqcup \cdots \sqcup U^t\,,$$

where each U^i lies in one component V^{a_i} of V.

(2) Use the trace test to verify that $U^i = V^{a_i} \cap M$.

Multihomogeneous Witness Sets

A subvariety $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ of dimension m has *multidegrees* $d_{m_1,\ldots,m_r}(V)$ for $0 \leq m_i \leq n_i$ with $m_1 + \cdots + m_r = m$:

$$d_{m_1,\ldots,m_r}(V) \;:=\; \#V \cap (M_1 \times \cdots \times M_r)\,,$$

where $M_i \subset \mathbb{P}^{n_i}$ is a general linear subspace of codimension m_i.

Definition (Hauenstein-Rodriguez) $W_{m_1,\ldots,m_r} := V \cap (M_1 \times \cdots \times M_r)$. These form a *multihomogeneous witness set collection*.

Advantages:

(1) Reflects the structure of V in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.
(2) Smaller than simply embedding V into $\mathbb{P}^{\prod(n_i+1)-1}$ via the Segre embedding, which has an enormous degree.

Hauenstein and Rodriguez: Many algorithms can take advantage of a multihomogeneous witness set collection.
We give some details for numerical irreducible decomposition.

Polymatroid Polytopes

Let $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be irreducible of dimension m. Set

$$S(V) := \{(m_1, \ldots, m_r) \mid d_{m_1,\ldots,m_r}(V) \neq 0\}\,.$$

For $I \subset [n]$, we have the projection

$$\pi_I \colon \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \longrightarrow \prod_{i \in I} \mathbb{P}^{n_i} \ =: \ \mathbb{P}^I\,.$$

By dimension/codimension considerations, $S(V)$ is a subset of

$$\{(m_1, \ldots, m_r) \mid \sum_i m_i = m \text{ and } \sum_{i \in I} m_i \leq \dim \pi_I(V)\}\,.$$

Theorem. (Castillo, Li, and Zhang)

$S(V)$ is this set, $\Pi(V)$, which is a polymatroid polytope.

$\rightsquigarrow$ $\Pi(V)$ *is* the multihomogeneous dimension of V.

It may be determined from a point of V using the local dimension test.

Numerical Irreducible Decomposition

Let $V \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.
This has an (unknown) irreducible decomposition, $V = V^1 \cup \cdots \cup V^s$.

Suppose further that we have all (or part of) a witness set collection $W_{m_1,\ldots,m_r} := V \cap (M_1 \times \cdots \times M_r)$ for V.

Numerical irreducible decomposition computes partitions $\bigsqcup_i W^i_{m_1,\ldots,m_r}$ of $W_{m_1,\ldots,m_r}$, where

$$W^i_{m_1,\ldots,m_r} \ := \ V^i \cap (M_1 \times \cdots \times M_r)\,.$$

Step 0. As we may compute $\Pi(V^i)$ from any point of V^i, we may assume that every component of V has the same support.

Monodromy Break Up

Given $W_{m_1,\ldots,m_r} = V \cap (M_1 \times \cdots \times M_r)$ and an unknown decomposition $V = V^1 \cup \cdots \cup V^s$, numerical irreducible decomposition computes the sets $W^i_{m_1,\ldots,m_r} = V^i \cap (M_1 \times \cdots \times M_r)$.

Monodromy loops (moving the M_i) give a partition whose parts are subsets of the $W^i_{m_1,\ldots,m_r}$. This forms a possibly finer partition than numerical irreducible decomposition.

When $\Pi(V)$ is not a point, membership testing between the partitions for $(m_1, \ldots, m_r)$ and $(m'_1, \ldots, m'_r)$ in $\Pi(V)$ enables further coarsening.

(Developing a reasonable heuristic for gluing adjacent witness sets is on our 'to-do list'.)

Trace Test

(1) Hauenstein and Rodriguez discovered that, despite one's expectations/hope, there is no simple 'multihomogeneous trace test'.

(2) An alternative is to use a witness set collection $W_{m_1,\ldots,m_r}$ for $(m_1,\ldots,m_r) \in \Pi(V)$ to construct a witness set for $Seg(V)$, where $Seg\colon \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{\prod_i (n_i+1)-1}$ is the Segre map. This is not practical because both the ambient dimension and degree are enormous.

(3) Another alternative is to use the witness set collection $W_{m_1,\ldots,m_r}$ to construct a witness set for $V \cap \mathbb{C}^{n_1+\cdots+n_r}$. The ambient dimension remains the same and the degree is the sum of the multidegrees.

We can do better.

Dimension Reduction

When $\dim \Pi(V) < r-1$, components of V are products, $V = W \times U$, where $W \in \mathbb{P}^I$ and $U \subset \mathbb{P}^{[n]\smallsetminus I}$. This reduces to $\dim \Pi(V) = r-1$.

When $\dim \Pi(V) = r-1$, select an $(r-1)$-simplex Δ in $\Pi(V)$ and slice with a product to get $V' := V \cap L^{m'_1} \times \cdots \times L^{m'_r+1}$ such that

(1) V' is a curve, as is each component of V'
(2) $\Pi(V')$ is a simplex.
(3) The witness sets and witness set partitions for V' are those of V restricted to the simplex.

Replace V' by its intersection C with an affine open subset of $L^{m'_1} \times \cdots \times L^{m'_r+1}$. The witness sets of V' can be used to get a witness set for C. This is used for the ordinary trace test in this affine subset.

When $r = 2$

Assume that V is not a product. Given nonzero adjacent multidegrees $d_{l+1,m}$ and $d_{l,m+1}$, $L' \subset \mathbb{P}^a$ and $M' \subset \mathbb{P}^b$ of codimensions l and m containing hyperplanes $L \subset L'$ and $M \subset M'$, then

$$W_{10} := V \cap (L \times M') \text{ and } W_{01} := V \cap (L' \times M)$$

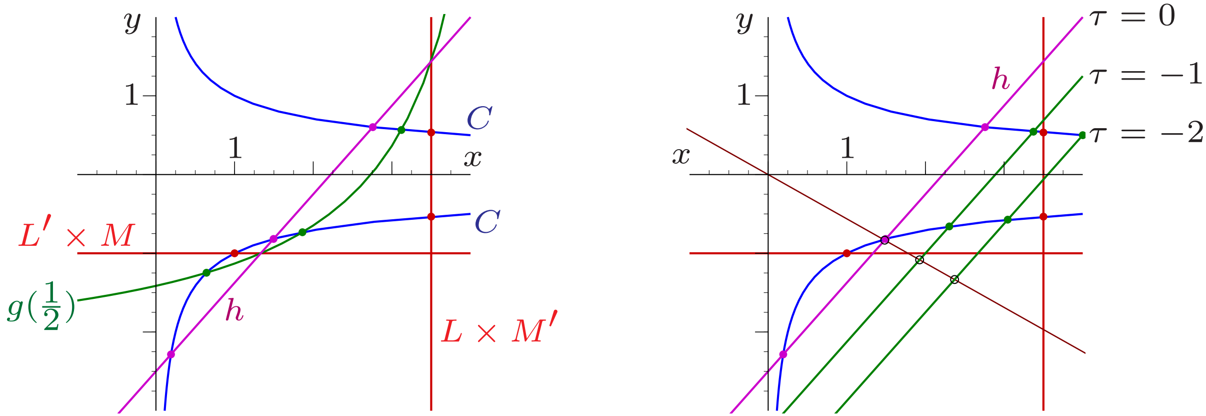
are the corresponding multihomogeneous witness sets.

$C := V \cap (L' \times M')$ is an irreducible curve with multidegrees $d_{10} = d_{l+1,m}$ and $d_{01} = d_{l,m+1}$ having witness sets W_{10} and W_{01}.

Working in an affine patch $\mathbb{C}^p \oplus \mathbb{C}^q$ on $L' \times M'$, C has degree $d_{10} + d_{01}$ and $W_{01} \cup W_{10}$ can be used to get a witness set $W = C \cap H$, which we may use for a trace test in the affine space $\mathbb{C}^n \oplus \mathbb{C}^m$.

Example

Suppose that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is defined locally by $y^2x = 1$.



Left: Linear spaces $x = x_0$ and $y = y_0$, line $H : h = 0$, and the curve $g(\frac{1}{2})$, where $g(t) := (x - x_0)(y - y_0)(1 - t) + th$. These are $g(t)$ at $t = 0,\ \frac{1}{2},\ 1$.

Right: the parallel slices $h = \tau$ are in green, and the averages of witness points ($\frac{1}{3}$ of the trace) lies on the brown line.