Algebra Autumn 2023
Frank Sottile
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Twelfth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

## Hand in for the grader Monday 27 November:

61. Let $S$ be a multiplicative subset of an integral domain $R$ with $0 \notin S$. Show that if $R$ is a principal ideal domain, then so is $R[S^{-1}]$.

    Show that if $R$ is a unique factorization domain, then so is $R[S^{-1}]$.

62. Let $R$ be an integral domain, and for each maximal ideal $\mathfrak{m}$ of $R$, show that the localization $R_{\mathfrak{m}}$ is a subring of the quotient field of $R$.

63. Continuing the previous problem, show that the intersection of the rings $R_{\mathfrak{m}}$, as $\mathfrak{m}$ ranges over all maximal ideals of $R$, is $R$ itself.

64. Show that the equation $x^2 + 1 = 0$ has infinitely many solutions in Hamilton's Quaternions, $\mathbb{H}$, which is $\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$, where $ij = k$, $ji = -k$, etc. These are defined in the Example on page 117 of my copy of Hungerford in Section III.1.

65. Let $R$ be a ring and $G$ be an infinite multiplicative cyclic group with generator $\xi$. Prove or disprove: The group ring $R[G]$ is isomorphic to the polynomial ring $R[x]$ in one indeterminate $x$.

66. Show that the polynomial $x + 1$ is a unit in the power series ring $\mathbb{Z}[[x]]$, but not in the polynomial ring $\mathbb{Z}[x]$.

    Show that the polynomial $x^2 + 3x + 2$ is irreducible in $\mathbb{Z}[[x]]$, but not in $\mathbb{Z}[x]$.

67. (a) If $D$ is an integral domain and $c$ is an irreducible element in $D$, show that $D[x]$ is not a principal ideal domain. (Hint: consider the ideal generated by $x$ and $c$.)

    (b) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

    (c) If $\mathbb{F}$ is a field and $n \geq 2$, show that $\mathbb{F}[x_1, \ldots, x_n]$ is not a principal ideal domain. (Hint: show that $x_1$ is irreducible in $\mathbb{F}[x_1, \ldots, x_{n-1}]$.)

68. Let $\mathbb{F}$ be a field. Show that the subring $\mathbb{F}[[x]][x^{-1}]$ of the quotient field of $\mathbb{F}[[x]]$ is a field. This is the field of formal Laurent series in $x$.

69. The *nth cyclotomic polynomial* is

    $$f_n \;:=\; (x^n - 1)/(x - 1) = x^{n-1} + \cdots + 1 \;\in\; \mathbb{Z}[x]\,.$$

    Use Eisenstein's criterion to show that if $p$ is prime, then $f_p(x + 1)$ is irreducible, and deduce that $f_p$ is irreducible.

70. If $c_0, c_1, \ldots, c_n$ are distinct elements of an integral domain $D$, and $d_0, \ldots, d_n$ are elements of $D$, then there is at most one polynomial $f \in D[x]$ of degree $n$ such that $f(c_i) = d_i$ for each $i = 0, \ldots, n$.