Algebra Autumn 2023
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18 September 2023

Fifth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Read Hungerford, Section I.7 on Categories.

Hand in to Frank Monday 25 September: (Have this on a separate sheet of paper.)

1. True or False, with justification. Given a collection of groups $\{H_\alpha \mid \alpha \in I\}$ then the Cartesian product $\prod\{H_\alpha \mid \alpha \in I\}$ is generated by its collection of subgroups $\iota_\alpha(H_\alpha)$ for $\alpha \in I$, where, for $h \in H_\alpha$, the element $\iota_\alpha(h)$ takes value h at α, and is the identity at $\beta \in I \smallsetminus \{\alpha\}$.

Hand in for the grader Monday 25 September: (Have this separate from #1.)

17. Show that the symmetric group S_n is generated by
    (a) The transpositions $(1,2),\ (1,3),\ \ldots\ ,\ (1,n)$.
    (b) The transpositions $(1,2),\ (2,3),\ \ldots\ ,\ (n-1,n)$.
    (c) The transposition $(1,2)$ and the n-cycle $(1,2,\ldots,n)$.
18. Show that the group defined by generators a, b and relations $a^2 = e$, $b^3 = e$ is infinite and nonabelian.
19. If $f\colon G \to H$ and $g\colon K \to L$ are homomorphisms of groups, then there is a unique homomorphism $h\colon G * K \to H * L$ between their free products such that $h|_G = f$ and $h|_K = g$.
20. Give an example to show that the direct product (in Hungerford, weak direct product) is not a coproduct in the category of all groups. It suffices to consider the case of two factors. That is, find a group G and groups H, K that have homomorphisms $f_H\colon H \to G$ and $f_K\colon K \to G$ for which there is no homomorphism $f\colon H \times K \to G$ such that $f|_H = f_H$ and $f|_K = f_K$.
21. Following this last question up, show that the diret product is a coproduct in the category of abelian groups. That is, suppose $\{H_\alpha \mid \alpha \in I\}$ is a family of abelian groups indexed by a set I, and G is an abelian group such that there are homomorphisms $f_\alpha\colon H_\alpha \to G$ for $\alpha \in I$. Prove there is a unique map $f\colon \bigoplus\{H_\alpha \mid \alpha \in I\} \to G$ such that for each $\alpha \in I$ we have $f_\alpha = f \circ \iota_\alpha$, where $\iota_\alpha\colon H_\alpha \hookrightarrow \bigoplus\{H_\alpha \mid \alpha \in I\}$ is the canonical injection.

    Deduce that this property determines the direct product $\bigoplus\{H_\alpha \mid \alpha \in I\}$ of abelian groups up to unique automorphism.