Algebra Autumn 2013
Frank Sottile
31 October 2013

Ninth Homework

Hand in to Frank Tuesday 5 November:

47. Let R be a ring, I an ideal of R, and $n \geq 1$ an integer. Define $M_n(I)$ to be the set of $n \times n$ matrices with entries in I.

    (1) Prove that for any ideal I of R, $M_n(I)$ is an ideal of $M_n(R)$.

    (2) Prove that any ideal of $M_n(R)$ has the form $M_n(I)$ for I an ideal of R.

Hand in to Frank Thursday 7 November:

48. Let R be a ring that contains the complex numbers in its center (every element of $\mathbb{C}$ commutes with every element of R). Suppose that $a, b \in R$ are elements such that $qab = ba$ for some non-zero $q \in \mathbb{C}^\times$, which we assume is not a root of unity for simplicity. Prove the *q-binomial theorem*: For all positive integers n, we have

$$(a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}\,, \qquad \text{where} \qquad \binom{n}{k}_q \;:=\; \frac{(n)_q!}{(k)_q!(n-k)_q!}\,,$$

and $(j)_q! := (j)_q(j-1)_q \cdots (3)_q(2)_q(1)_q$ with $(a)_q = 1 + q + \cdots + q^{a-1}$ and $(0)_q! = 1$. *Hint: you may want to first prove a recursion involving* $\binom{n}{k}_q$.

Hand in for the grader Thursday 7 November:

49. Prove that a ring R is a division ring if and only if it has no proper left ideals.

50. Determine all prime and maximal ideals in the ring $\mathbb{Z}_m$ of integers modulo a positive integer m.

51. Let a, b be elements of a ring R. Prove that $1 - ab$ is invertible in R if and only if $1 - ba$ is invertible in R.

52. Suppose that R is a division ring. Show that $M_n(R)$ has no proper ideals (so that (0) is a maximal ideal). Show that if $n > 1$ then $M_n(R)$ has zero divisors.

53. Prove that if $I_1 \subset I_2 \subset \cdots$ is a chain of ideals in a ring R then $\bigcup_{i \geq 1} I_i$ is an ideal of R.

    Let $A = (a_1, \ldots, a_n)$ be a nonzero finitely generated ideal of a ring R. Prove there exists an ideal I of R that is maximal with respect to the property that $A \not\subset I$.

54. Let R be a commutative ring and $I \subset R$ and ideal. Its *radical* is

$$\sqrt{I} \;:=\; \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}\,.$$

Prove that $\sqrt{I}$ is an ideal of R containing I and that $\sqrt{I}/I$ is the nilradical of R/I.