Foundations of Mathematics   Math 300 Sections 902, 905

YOUR NAME

Sixth Homework: Due 12 October 2020

Definition: The *Fibonacci sequence* $\{f_n \mid n \geq 1\}$ is defined by $f_1 = f_2 = 1$ and for $n \geq 2$, $f_{n+1} = f_n + f_{n-1}$.

1. [20] Please do parts (a) and (b) of Problem 17 for Section 4.3 on page 198 in the .pdf of the Sundstrom book.

2. [10] Write a proof in paragraph form of the inequality $3^n > 1 + 2^n$ for $n \geq 2$ using mathematical induction.

3. [10] Find a number M and which of $\{<, =, >\}$ such that $\forall n \geq M,\ 2^n\ (<, =, >)\ n!$ is true, and then write a proof in paragraph form of this assertion using mathematical induction.

4. [10] For which natural numbers n do there exist nonnegative integers x and y such that $n = 4x + 7y$? Justify your conclusion.

5. [10] Consider the sequence $\{a_n \mid n \in \mathbb{N}\}$ defined by $a_1 = 1$, $a_2 = 3$ and for each $n \in \mathbb{N}$, $a_{n+2} = 3a_{n+1} - 2a_n$. Calculate the first eight elements of this sequence.

   Conjecture a formula for a_n and prove it using induction.

6. [10] Consider the sequence $\{a_n \mid n \in \mathbb{N}\}$ defined by $a_1 = a_2 = 1$ and for each $n \in \mathbb{N}$, $a_{n+2} = \frac{1}{2}\left(a_{n+1} + \frac{2}{a_n}\right)$. Calculate the first six elements of this sequence.

   Prove, for all $n \in \mathbb{N}$, that $1 \leq a_n \leq 2$.

7. [10] Compute the first 15 terms of the Fibonacci sequence. Note that the recursion $f_{n+1} = f_n + f_{n-1}$ may be rewritten $f_{n-1} = f_{n+1} - f_n$. Use this to extend the Fibonacci sequence to *negative* integers and compute the values of f_n for $-10 \leq n \leq 0$. Conjecture a formula for f_{-n} for $n \in \mathbb{N}$ and prove it by induction.

8. [10] Prove that for every $n \in \mathbb{N}$, f_{5n} is a multiple of 5.

9. [10] Look up the term *Pythagrean triple* (it is in our book). Investigate the following

   Conjecture. *For each natural number n, the numbers $f_n f_{n+3}$, $2f_{n+1}f_{n+2}$, and $(f_{n+1}^2 + f_{n+2}^2)$ form a Pythagorean triple.*

   If true, provide a proof, and if false, a counterexample.

10. [20] Extra credit for the ambitious and bored. For $k > 2$, find (and prove) a formula for f_{n+k} in terms of f_n and f_{n+1}. Use it to give a proof by induction that for all $n, k \in \mathbb{N}$ the Fibonacci number f_{nk} is a multiple of f_n.

    There will be a special assignment for this on Gradescope; hand it in separately. Your proof needs to be correct for full credit.