Foundations of Mathematics — Math 300H Section 970

YOUR NAME

Sixth Homework: Due 18 October 2022

Use English when possible. Answers should not just be symbols.

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1. Please give formal definitions of the following mathematical terms that we have used in our course. These should be, in Frank's terms, proper mathematical definitions.

   (a) A real number x is *rational*.

   (b) A real number x is *irrational*.

   (c) The *Cartesian product* $A \times B$ of two sets A and B.

   (d) For a real number x, the *absolute value* $|x|$ of x.

   (e) The set *theoretic difference*, $A - B$ of sets A and B.

   (f) That $a|b$, for integers a and b.

   (g) That $a \equiv b \mod m$, for integers m, a, and b.

2. Please give a formal definition of the empty set, $\emptyset$, and of the subset relation, that is, for sets A and B, give a definition of what it means to say that $A \subseteq B$.

   Use these in a (short) proof that "For every sey A, $\emptyset \subseteq A$"

3. Let P, Q, and R be statements. Give useful negations of the following statements.

   (a) $P \Rightarrow Q$.

   (b) $P \Rightarrow (Q \vee R)$.

   (c) $P \Rightarrow (Q \wedge R)$.

   (d) $(P \vee Q) \Rightarrow R$.

   (e) $(P \wedge Q) \Rightarrow R$.

   (f) $(P \Rightarrow Q) \Rightarrow R$

   (g) $P \Rightarrow (Q \Rightarrow R)$

4. Recall that a real number x is *positive* if $x > 0$. Consider the statement P: "The sum of two real numbers is positive".

   (a) Write P as a statement of the form: "some quantifier. . . , if . . . , then . . . ."

   (b) Write the contrapositive of this statement in this form.

   (c) Write the converse of this statement in this form.

   (d) Write $\sim P$ in this form.

   (e) Prove whichever of P or $\sim P$ is true.

5. Let a, b, and c be integers.

   (a) Write a crisp and correct proof of the statement that "If $a|b$ and $b|c$, then $a|c$".

   (Also, rewrite the statement using quantifiers.)

   (b) Do the same for the statement "If $a|b$ and $a|c$, then $a|bc$".