Extra calculus problems

1. Let $\mathbf{a} := \langle 3, 4, 12\rangle$ and $\mathbf{b} := \langle -2, 7, -3\rangle$. Compute $\|\mathbf{a}\|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$, as well as the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, and the area of the parallelogram spanned by the two vectors $\mathbf{a}$ and $\mathbf{b}$.

2. Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$, where
$$\mathbf{a} \;=\; \langle -1, -2, 2\rangle \qquad \mathbf{b} \;=\; \langle 3, 3, 4\rangle \;.$$

3. Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$, where
$$\mathbf{a} \;=\; 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \qquad \mathbf{b} \;=\; \mathbf{i} + 6\mathbf{j} - 2\mathbf{k} \;.$$

4. (a) Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$, where
$$\mathbf{a} \;=\; \langle 12, 3, 4\rangle \qquad \mathbf{b} \;=\; \langle -1, 5, 2\rangle \;.$$
(b) Compute the cross product $\mathbf{a}\times\mathbf{b}$.

5. If a vector $\mathbf{v}$ is orthogonal to vectors $\mathbf{w}$ and $\mathbf{u}$, is it true that $\mathbf{v} = \mathbf{w}\times\mathbf{u}$?

6. Find a unit vector that is orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$.

7. Find a vector orthogonal to the plane spanned by the points
$$(1, 0, 0)\,, \qquad (0, 2, 0)\,, \quad \text{and} \quad (0, 0, 3)\,.$$
Find the area of the triangle with these vertices.

8. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Prove that
$$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) \;=\; 2(\mathbf{a}\times\mathbf{b}) \;.$$

9. Find the volume of the parallelepiped determined by the vectors $2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\mathbf{i}-\mathbf{j}$, and $2\mathbf{i}+3\mathbf{k}$.

10. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where
$$P \;=\; (0, 1, 2) \qquad Q \;=\; (2, 4, 5) \qquad R \;=\; (-1, 0, 1) \qquad S \;=\; (6, -1, 4) \;.$$

11. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where
$$P \;=\; (1, 1, 1) \qquad Q \;=\; (2, 1, 7) \qquad R \;=\; (3, 4, -7) \qquad S \;=\; (9, -4, 7) \;.$$

12. Find the volume of the parallelepiped spanned by the three vectors

$$\langle 1, 4, -7\rangle \qquad \langle 2, -1, 4\rangle \qquad \langle 0, 1, 2\rangle \ .$$

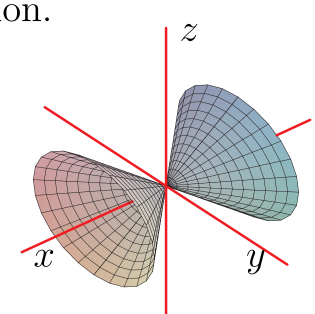
13. Find the equation of the plane that contains the point $(-1, -3, 2)$ and is perpendicular to the line

$$x \ = \ 3 + 4t\,, \qquad y \ = \ 1 - 3t\,, \qquad z \ = \ 2 - t\,.$$

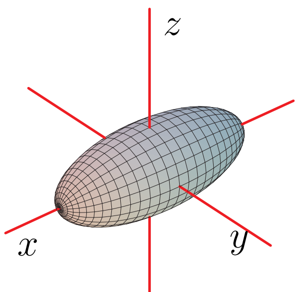
14. Identify the picture of the quadric defined by each equation.

All axes have the same orientation.

(a)	$2x^2 + 8z^2 = y^2 + 2$	(e)	$z = x^2 + y^2$
(b)	$4x^2 + 25y^2 + 25z^2 = 25$	(f)	$z = -xy$
(c)	$x^2 = 1 + y^2 + z^2$	(g)	$x^2 = y^2 + 2z^2$
(d)	$3x^2 + 4y^2 + 4z^2 + 4xy - 4xz = 4$		



I



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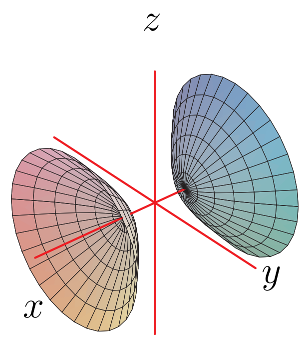

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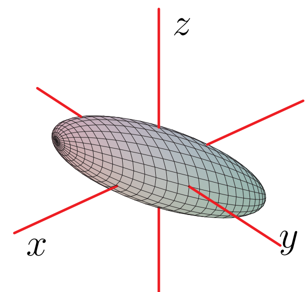

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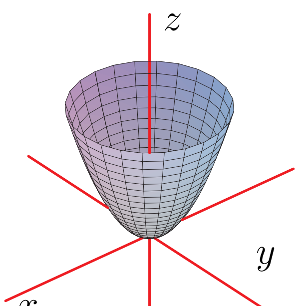

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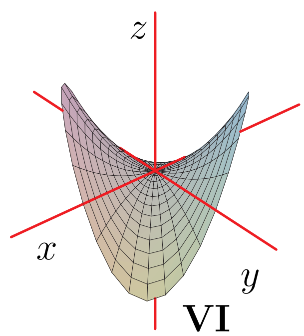

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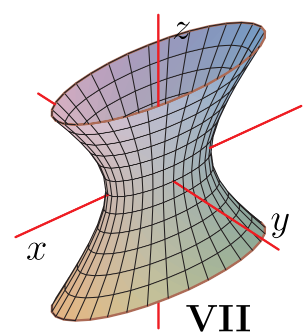


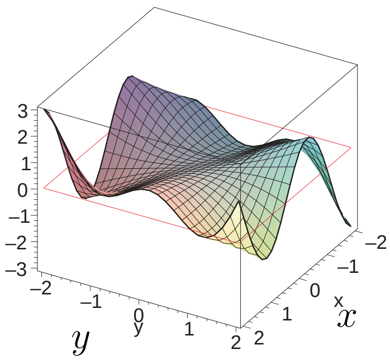
VII

15. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\Omega$, $I = 0.08A$, $dV/dt = -0.01V/s$, and $dR/dt = 0.03\Omega/s$.

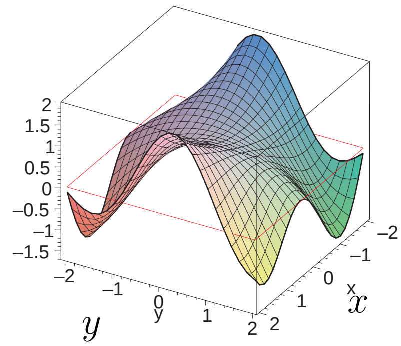
16. Find the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is implicitly defined as a function of x and y by the equation

$$xe^y + yz + ze^x \ = \ 0\,.$$

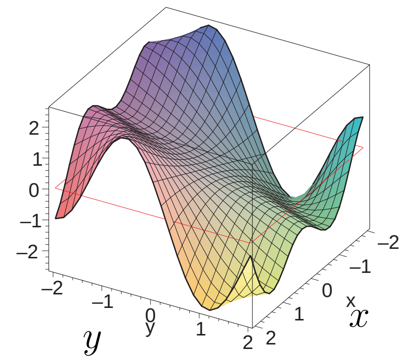
17. Identify which of the following three pictures is the graph of a function $f = f(x, y)$, and which are graphs of its partial derivatives f_x and f_y.



(a)

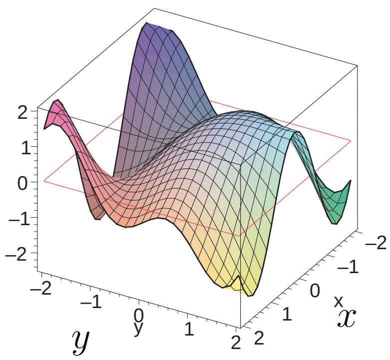


(b)

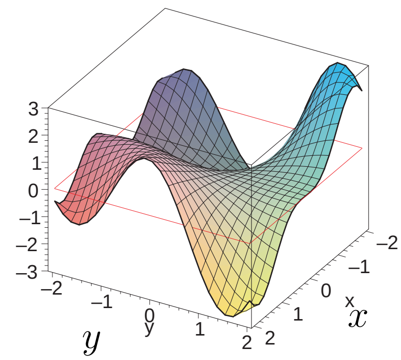


(c)

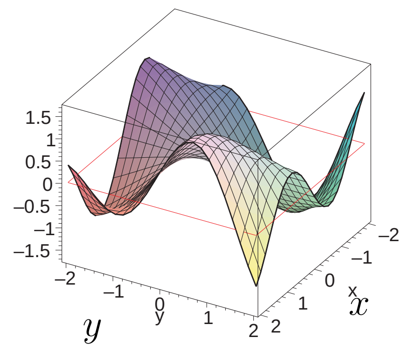
18. Identify which of the following three pictures is the graph of a function $f = f(x, y)$, and which are graphs of its partial derivatives f_x and f_y.



(a)



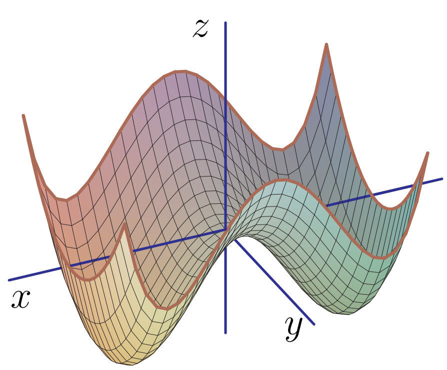
(b)

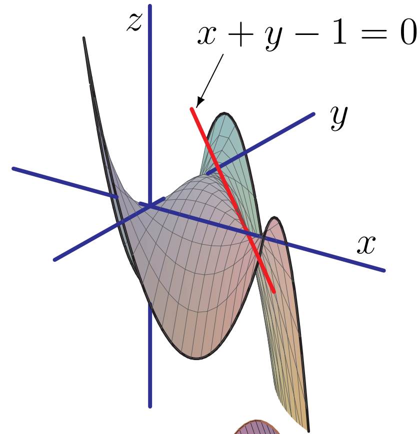


(c)

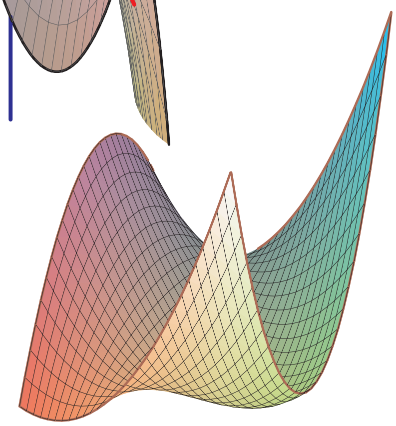
19. Estimate the length of the vector using differentials $\langle 3.02, 1.97, 5.99 \rangle$.

20. Find the critical points of the function $f(x, y) = x^4 - 2x^2 + y^2$ and use the second derivative test to classify each (local maximum, local minimum, and saddle point).





21. Find all critical points of the function $f(x, y) = xy - x^2y - xy^2$ and use the second derivative test to classify each (local maximum, local minimum, and saddle point). (You do not get credit if you only appeal to the picture for your answer.)

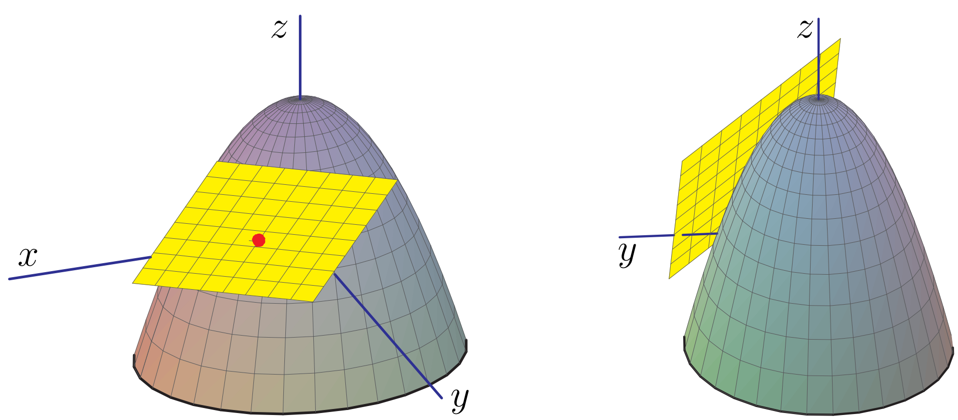


22. Find and classify all local extreme values and saddle points of the function $f(x, y) = x^2 + y^2 + x^2y + 1$.

23. Find an equation of the plane tangent to the surface

$$6z \;=\; 26 - 2x^2 - 3y^2$$

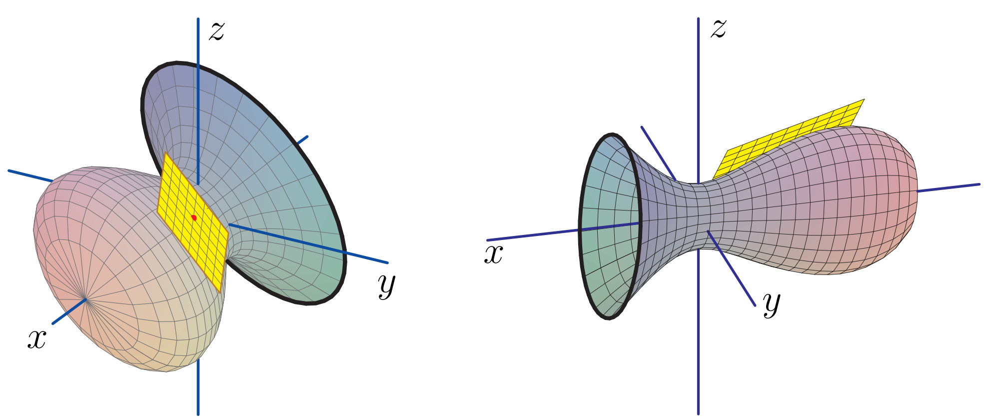
at the point $(2, 2, 1)$.



24. Find an equation of the plane tangent to the surface

$$x^3 - 3x^2 + y^2 + yz + z^2 \;=\; 1$$

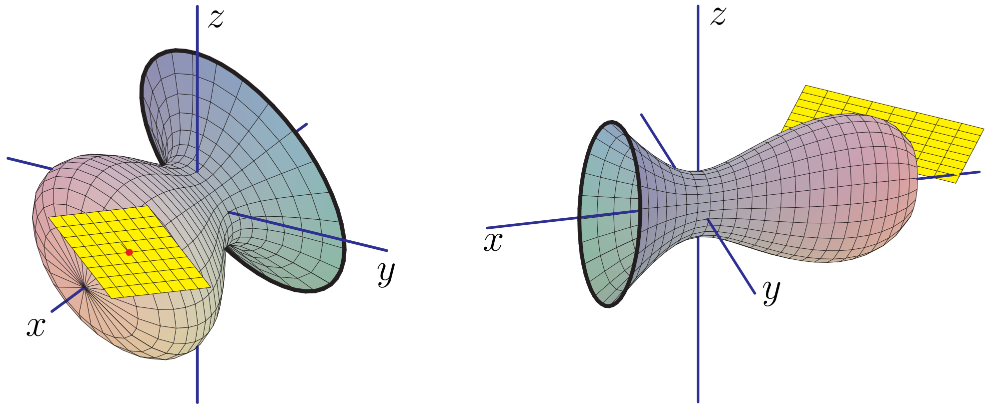
at the point $(1, 1, 1)$.



25. Find an equation of the plane tangent to the surface

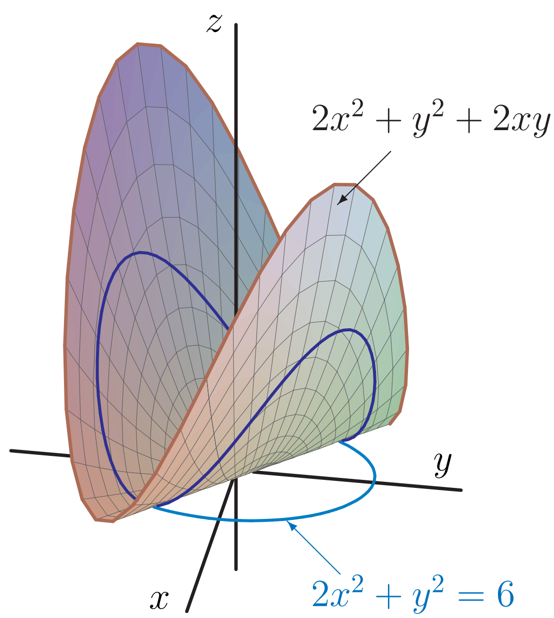
$$x^3 - 3x^2 + y^2 + yz + z^2 \ = \ 1$$

at the point $(1+\sqrt{3}, 1, 1)$.

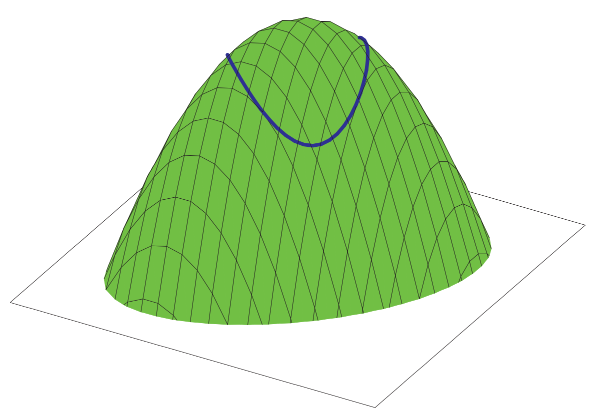


26. Find the extreme values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 10$ on the circle $x^2 + y^2 = 16$, using Lagrange's method of multipliers.

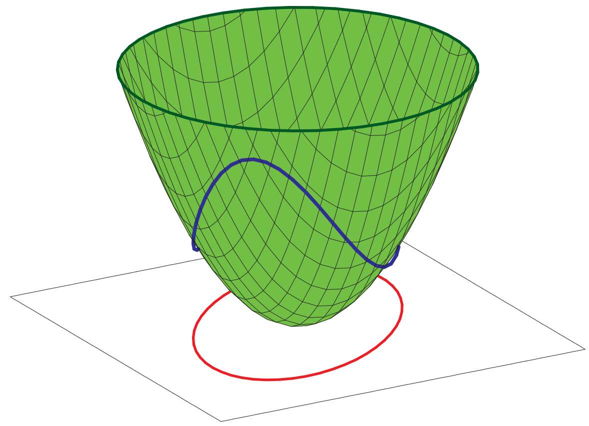
27. Use Lagrange multipliers to find the maximum and minimum values that the function $f(x, y) = x^2 + y^2 + 2xy$ takes on the ellipse $2x^2 + y^2 = 6$.



28. Use Lagrange multipliers to find the extreme values of the function $f(x, y) := 4 + xy - x^2 - y^2$ on the circle $x^2 + y^2 = 1$, that is, the highest and lowest points of the displayed curve. The surface is the graph of f.

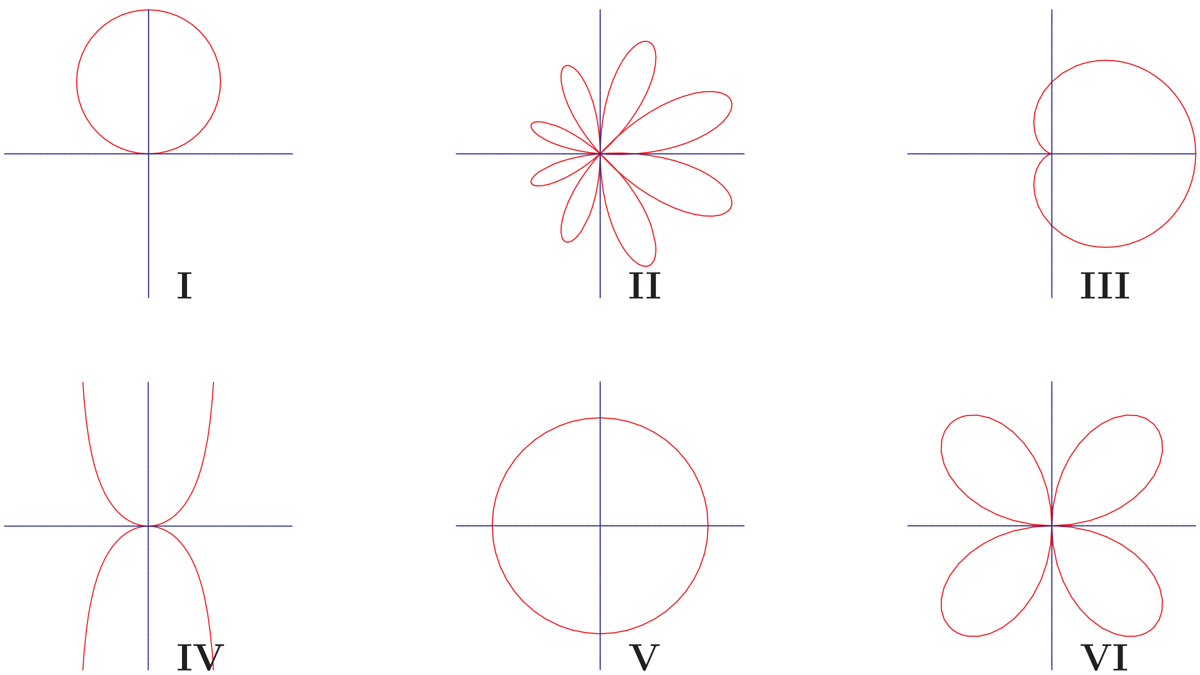


29. Use Lagrange multipliers to find the extreme values of the function $f(x,y) := \frac{1}{2}(x^2+y^2)$ on the ellipse $x^2+y^2-xy=4$, that is, the highest and lowest points of the displayed curve. The surface is the graph of f.

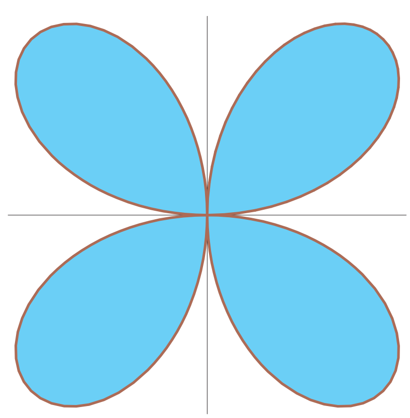


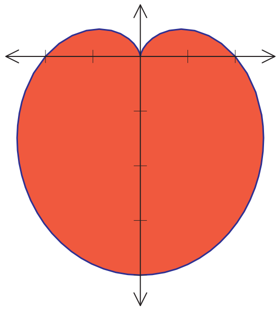
30. Identify the polar curve with its polar equation.

(a) $r=\cos(\theta)-1$	(b) $r=3/2$	(c) $r=2\sin(2\theta)$
(d) $r=\cos(\theta)+3\sin(4\theta)$	(e) $r=\tan(\theta)$	(f) $r=2\sin(\theta)$

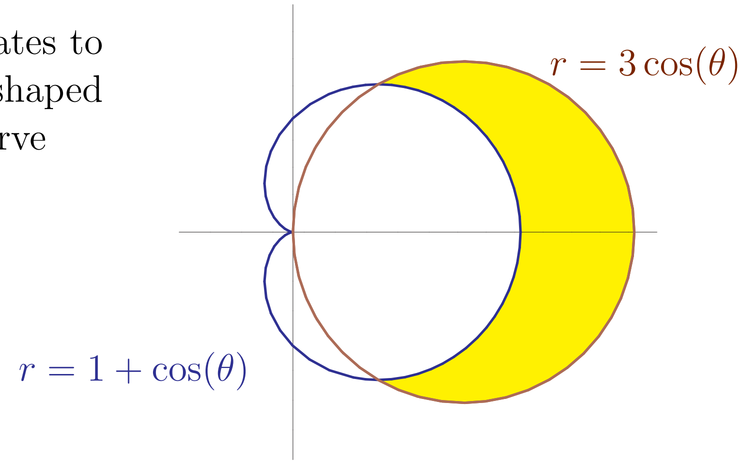


31. Use integration in polar coordinates to find the area enclosed by all four loops in the four-leaved rose $r=2\sin(2\theta)$.

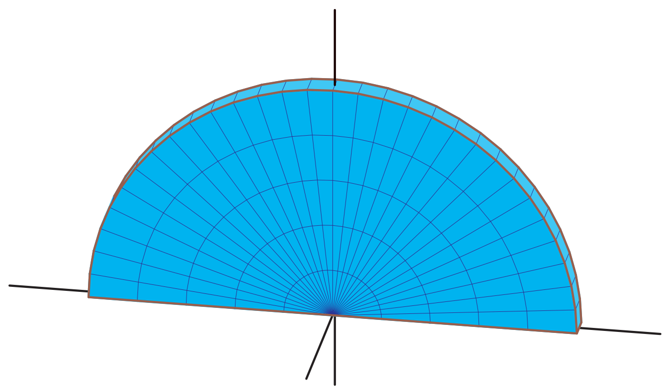


32.  Find the area enclosed by the cardioid $r = 2 - 2\sin(\theta)$.

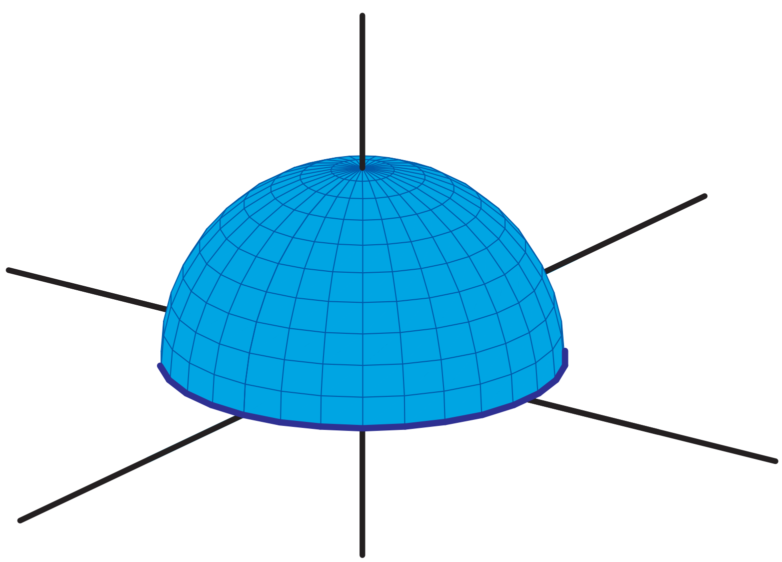
33. Use integration in polar coordinates to find the area of the shaded lune-shaped region which lies between the curve $r = 3\cos(\theta)$ and $r = 1 + \cos(\theta)$.



34. Suppose that a metal plate of uniform mass density k is cut in the shape of a half disc of radius a, and is lying in the (y, z)-plane as shown. Determine its centre of mass.



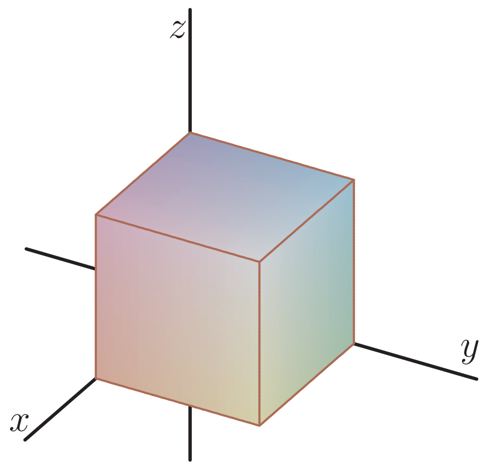
35. Find the center of mass of a solid hemisphere H of radius 2 if its mass density is proportional to the distance from the center of its base. (Hint: Exploit the symmetry of the problem to reduce this to two integrals.)



36. Set up, but do not solve, an integral to find the volume of the solid that lies between the two paraboloids $z = 2x^2 + y^2 - 1$ and $z = 8 - x^2 - 2y^2$ and above the circle $x^2 + y^2 \le 3$.

    Do this both in polar (r, θ) coordinates and in rectilinear (x, y) coordinates. This should be a double integral.

37. Find the center of mass of the unit cube $0 \le x, y, z \le 1$ in the positive orthant if the mass density is $\rho = xyz$. Hint: You only need to evaluate 2 integrals, if you exploit the symmetry of the problem.



38. Evaluate the integral by changing to spherical coordinates

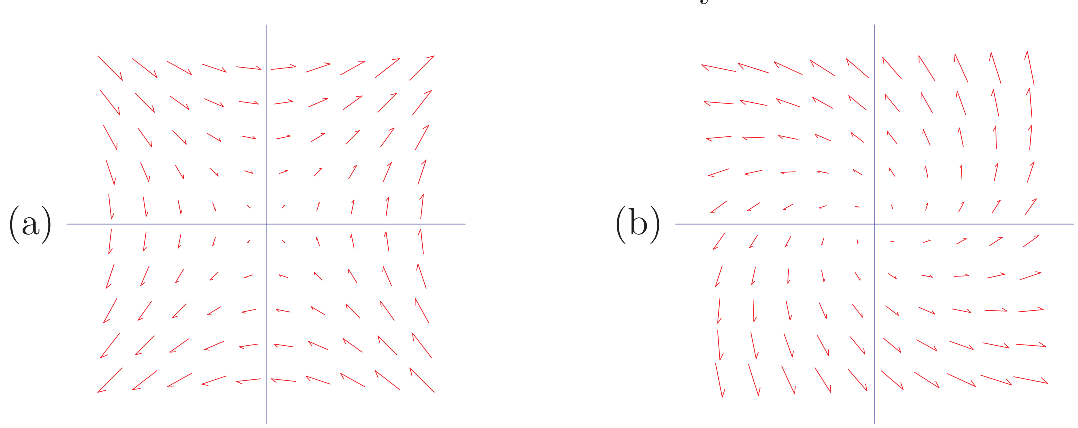
$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2 + y^2 + z^2}\, dz dy dx$$

39. Set up integrals to compute the volume of a sphere of radius $k > 0$ in each of the THREE coordinate systems: Rectilinear (x, y, z), Cylindrical (r, θ, z), and Spherical (ρ, ψ, θ). Evaluate two of them.

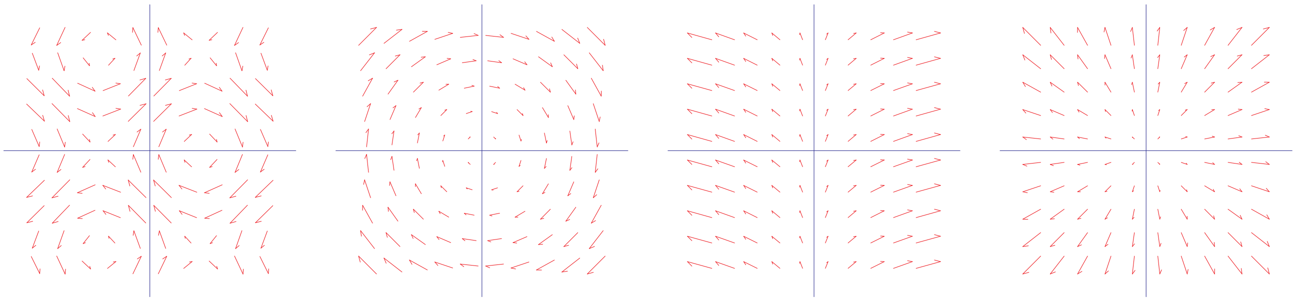
40. Evaluate the triple integral by converting it into cylindrical coordinates.

$$\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz dx dy$$

41. Give two properties or consequences of a vector field $\mathbf{F}$ being conservative. Which of the vector fields below is **not** conservative? Why?



42. Pick out the 2 of the four vector fields that are conservative



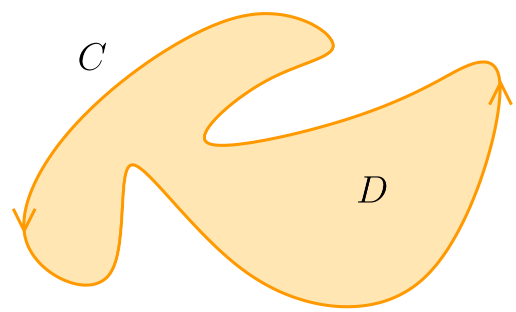
(a) (b) (c) (d)

43. Show that the line integral $\int_C 2x \sin y \, dx + (x^2 \cos y - 3y^2)dy$ is independent of path and evaluate the integral where C is any path from $(-1, 0)$ to $(5, 1)$.

44. Show that the line integral $\int_C (y - \frac{1}{x^2})dx + (x - \frac{1}{y^2})dy$ is independent of path and evaluate this integral where C is any path from $(-1, 1)$ to $(4, 2)$.

45. Show that the line integral $\int_C (2y^2 - 12x^3y^3)dx + (4xy - 9x^4y^2)dy$ is independent of path and evaluate the integral where C is any path from $(-1, 0)$ to $(5, 1)$.

46. Suppose that C is a positively oriented, piecewise smooth, simple closed curve in the plane that bounds a region D, and suppose that P and Q are functions with continuous partial derivatives on an open region that containd D. Give the conclusions of Green's Theorem.



Use Green's Theorem to evaluate the integral

$$\int_C (y^2 - \arctan(x))dx + (3x + \sin(y))dy$$

where C is the positively oriented boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.

47. State Green's Theorem. Assume that the coordinates of the vector field $\mathbf{F}$ have continuous partial derivatives where they are defined.

48. Compute the curl $(\nabla\times)$ and divergence $(\nabla\cdot)$ of the following vector fields.

(a) $\mathbf{F} = (x+3y-5z)\mathbf{i} + (z-3y)\mathbf{j} + (5x+6y-z)\mathbf{k}$

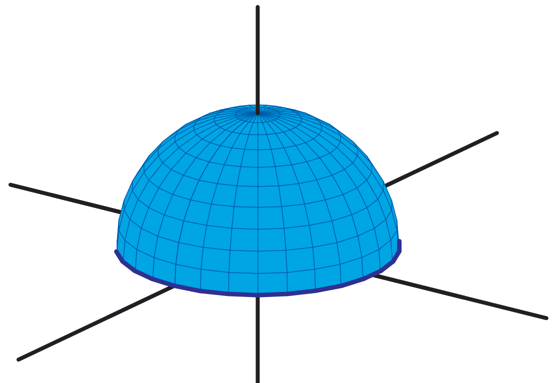
(b) $\mathbf{F} = xe^y\mathbf{i} + \sqrt{z}\sin(x-y)\mathbf{j} + y\ln z\mathbf{k}$.

49. State Green's Theorem, Stokes's Theorem, and Gauß's Theorem†. Assume that the coordinates of the vector field $\mathbf{F}$ have continuous partial derivatives where they are defined. Define all objects.

50. Evaluate the surface integral

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

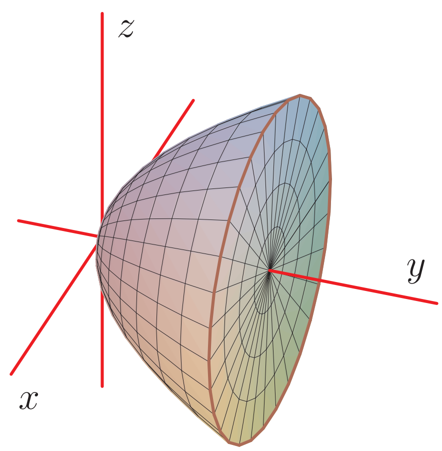
where $\mathbf{F} = zy\mathbf{i} - zx\mathbf{j} + (x^2+y^2)\mathbf{k}$ and S is the upper hemisphere of the sphere of radius 1 with upward orientation.



51. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (x^2+y^2)\mathbf{k}$. Use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of that part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant. Suppose that C is oriented counter-clockwise when viewed from above.

52. Let $\mathbf{F} = (2xz + \ln(1+y^2z^2))\mathbf{i} + \frac{e^z}{1+x^2}\mathbf{j} - z^2\mathbf{k}$. Use Gauß's Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is the boundary of the ellipsoid $x^2 + 4y^2 + 3z^2 = 12$.

53. Evaluate the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{j} - z\mathbf{k}$ and S is the surface consisting of the paraboloid $y = x^2 + z^2$ for $0 \le y \le 1$ and the disc $y = 1$ and $x^2 + z^2 \le 1$.



† a.k.a. the divergence theorem