Foundations of Mathematics — Math 300 Sections 902, 905

YOUR NAME

Tenth Homework: — Due 19 November 2020

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1. [14] Do all parts of Problem 16 in the Exercises for Section 7.2 in the Sundstrom book.

2. [10] Let $A = \{0, 1\}$. Determine all the relations R on A.

   Which of these are equivalence relations?

3. [8] Let A be a set. As a function is also a relation, what can be said if a function $f\colon A \to A$ is also an equivalence relation? (E.g. is this possible, and if so, what does this say about f?)

4. [10] Consider the relation $\sim$ on the power set $\mathcal{P}(U)$ of some set U, where for $A, B \subseteq U$, we have $A \sim B$ if and only if there is a bijection $f\colon A \to B$. Prove that this is a equivalence relation. Determine the equivalence class $[\emptyset]$.

5. [8] Consider the relation $\sim$ on the power set $\mathcal{P}(U)$ of some set U, where for $A, B \subseteq U$, we have $A \sim B$ if and only if $A \cap B = \emptyset$. Is this an equivalence relation? If not, is it reflexive, symmetric, or transitive? Justify your conclusions.

6. [12] Consider the relation q on $\mathbb{Z} \times \mathbb{Z}$ where, for integers a, b, c, d, we have $(a, b) \sim (c, d)$ if $ad = bc$. Show that this is not an equivalence relation. What if we restrict to $\mathbb{Z} \times \mathbb{N}$?

7. [8] Determine all the congruence classes (equivalence classes) for the relation on the integers $\mathbb{Z}$ of congruence modulo 5.

8. [8] The relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ if $3a + 4b \equiv 0 \mod 7$ is an equivalence relation (you can check this, but it is not necessary). Determine all distinct equivalence classes for this equivalence relation.

9. [8] Compute the addition and multiplication tables for $\mathbb{Z}_5$.

10. [14] For $n \in \mathbb{N}$, let $s(n)$ denote the sum of the digits of n, expressed in base 10. That is, if we write $n = a_k \ldots a_1 a_0$ in base 10 so that

$$n \;=\; \left(a_k \cdot 10^k\right) + \left(a_{k-1} \cdot 10^{k-1}\right) + \cdots + (a_1 \cdot 10) + a_0\,.$$

   (a) Use mathematical induction to prove that for all $n \in \mathbb{N}$, $10^n \equiv 1 \mod 9$. Thus $[10^n]_9 = [1]_9$.

   (b) Use this to prove that $[n]_9 = [s(n)]_9$ and deduce that $9|n$ if and only if $9|s(n)$.

   (c) Show that for $a, b \in \mathbb{Z}$, we have $[a \cdot b]_9 = [s(a) \cdot s(b)]_9$. This is the idea behind *casting out nines*.