Foundations of Mathematics Math 300H Section 970

YOUR NAME

Fourth Homework: 20 September 2022

Use English when possible. Answers should not just be symbols.

Definition: An integer a is *even* if there is an integer k such that $n = 2k$. An integer a is *odd* if there is an integer k such that $n = 2k+1$.

Definition: Let a and b be integers. We say that a *divides* b and write $a|b$ if there is an integer c such that $ac = b$.

Definition: Let a, b, and m be integers. We say that a and b are *congruent modulo* m if m divides their difference, $a - b$. That is, when $m|(a - b)$. We write $a \equiv b \mod m$ when this occurs.

Definition: Let A and B be sets. We say that $A = B$ (A equals B) of they have the same elements/members. That is, if $\forall x(x \in A \Leftrightarrow x \in B)$.

Definition: We say that $A \subseteq B$ if for every x, if $x \in A$, then $x \in B$.

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1. Write the converse and contrapositive of the following conditional statements:
   - (a) If it rains, then the grass is wet.
   - (b) $\alpha^2 = 25$ if $\alpha = 5$.
   - (c) The integer a is odd only if $3a$ is odd.
   - (d) “Inattentive when bored”.
   - (e) “Quiet is necessary for sleep”.
   - (f) “Pepperoni is necessary for Pizza”.

2. Consider the following “proof” that if m and n are even, then $m+n$ is even:

   We know that $n = 2t$ and $m = 2t$, so $m+n = 2t + 2t = 4t$. Therefore $m+n$ is even.
   - (a) Criticize (discuss its shortcomings).
   - (b) Write a correct proof of this statement in paragraph form.

3. Which of the following statement are true and which are false? Justify your conclusions. (E.g. give a proof or a counterexample.)
   - (a) If a, b, and c are integers, then $ab + ac$ is an even integer.
   - (b) If a, b, and c are integers with both b and c odd integers, then $ab+ac$ is an even integer.

For these next two, sketch it first (perhaps in a table form), and then write it in paragraph form.

4. Let $x \in \mathbb{Z}$. Prove that if 2^{2x} is an odd integer, then 2^{-2x} is an odd integer.

5. Recall that for integers m, n we sat that m divides n, written $m|n$, if there is an integer p such that $n = mp$. Prove the following statement:

   For integers a, b, and c, if $a|b$ and $a|c$, then $a|(b + c)$.

6. Write a proof in paragraph form of the statement: For all integers a and b, if $a|b$ and $b|a$, then either $a = b$ or $a = -b$.

7. Write a proof in paragraph form of the statement: For all integers a, b, m if $a \equiv b \mod m$, then $a^2 \equiv b^2 \mod m$.

8. Write a proof in paragraph form of the statement: For all integers a, b, m, n if $a \equiv b \mod m$ and $n|m$, then $a \equiv b \mod n$.

9. Write a proof in paragraph form of the statement: Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.