Algebra Autumn 2023
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23 October 2023 **Ninth Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 30 October:

39. The wreath product $S_m \wr S_n$ of symmetric groups is the semidirect product $(S_m)^n \rtimes_\varphi S_n$ where φ is the action of S_n on $(S_m)^n$ permuting the factors of $(S_m)^n$.

    (a) For $(\pi_1, \ldots, \pi_n\,,\,\omega) \in S_m \wr S_n$ ($\pi_i \in S_m$ and $\omega \in S_n$) define the map from $[m] \times [n]$ to itself by

    $$(\pi_1, \ldots, \pi_n\,,\,\omega).(i,j) \;:=\; (\pi_{\omega(j)}(i)\,,\,\omega(j))\,.$$

    (Here, $[m] := \{1, \ldots, m\}$ and the same for $[n]$. Show that this defines an action of $S_m \wr S_n$ on $[m] \times [n]$.

    (b) Using this action or any other methods show that $S_2 \wr S_2 \simeq D_4$, the dihedral group with 8 elements.

    (c) This action realizes $S_2 \wr S_3$ as a sugroup of S_6. What are the cycle types of permutations in $S_2 \wr S_3$? For each cycle type, how many elements of $S_2 \wr S_3$ have that cycle type?

40. Prove that the converse to Lagrange's Theorem holds for nilpotent groups. That is, if G is a finite nilpotent group and n divides the order of G, then G has a subgroup of order n.

    Hint: first prove it for p-groups.

41. Prove (without using the Feit-Thompson Theorem) that the following two statements are equivalent:

    (a) Every group of odd order is solvable.

    (b) The only simple groups of odd order are the abelian groups of prime order.

42. Let $H = \mathbb{Z}_3$ and $K = \mathbb{Z}_4$, and consider the homomorphism $\varphi\colon K \to \mathrm{Aut}(\mathbb{Z}_3)$ which sends the generator of $\mathbb{Z}_4$ to multiplication by -1. Show that $H \rtimes_\varphi K$ is a nonabelian group of order 12 that is not isomorphic to either A_4 or D_6 (the dihedral group of symmetries of the regular hexagon).

43. Use semidirect products to classify all groups of order 28 up to isomorphism. (There are four isomorphism classes.)

44. Consider the ring $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$ of endomorphisms of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$. Prove that $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$ is noncommutative.

45. Let R be a ring such that for every $a \in R$, there is a unique $b \in R$ such that $aba = a$. Prove that:

    (a) R has no zero divisors.

    (b) $bab = b$, where a, b are as above.

    (c) R is a division ring.