Algebra Autumn 2023
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30 October 2023 Tenth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 6 November:

46. Prove that the set N of all nilpotent elements in commutative ring R forms an ideal, and that R/N has no nonzero nilpotent elements.
47. Suppose that R is a division ring. Show that $M_n(R)$ has no proper ideals (so that (0) is a maximal ideal). Show that if $n > 1$ then $M_n(R)$ has zero divisors.
48. Prove that a ring R is a division ring if and only if it has no proper left ideals.
49. Define the binomial coefficient $\binom{n}{k}$ to be $\frac{n!}{k!(n-k)!}$ for integers $0 \leq k \leq n$. Let R be a commutative ring. Prove the binomial theorem:
$$\forall a, b \in R \ \forall n \in \mathbb{N} \qquad (a+b)^n \ = \ \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \ .$$
50. Suppose that R is a commutative ring of characteristic p, a prime number. Prove that the map $a \mapsto a^p$ defined for $a \in R$ is a ring homomorphism. (This is called the *Frobenius homomorphism*.)
51. Prove that the set consisting of zero and all zero divisors in a commutative ring contains at least one prime ideal.