Foundations of Mathematics Math 300 Sections 902, 905

YOUR NAME

Third Homework: Due 14 September 2020

1. Rewrite the following English sentences (which are mathematical statements) as sentences involving quantifiers.
   (a) A trangle has three sides.
   (b) The square of a real number is nonnegative.
   (c) Some Aggies are not Human.
   (d) An integer is necessarily prime or composite.
   (e) Some even numbers are divisible by two and are divisible by seven.
   (f) The sum of two even integers is an odd integer.
   (g) Irrational numbers are real.
2. Negate each of the quantified statements from Question 1, again as English sentences.
3. Recall the following property of the integers:
   "If n is an integer, then there is an integer m with the property that $n + m = 0$."
   (a) Write this as a statement involving quantifiers.
   (b) Give a useful negation of this statement.
   (c) What is this property called?
4. Negate each of the following statements (which are important definitions in mathematics). Assume that the symbols f, K, a, and l are defined.
   (a) For every $x \in K$, if $x \neq 0$, then there is a $y \in K$ such that $xy = 1$.
   (b) For every real number $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in \mathbb{R}$ with $x \neq a$ and $|x - a| < \delta$, then $|f(x) - l| < \epsilon$.
   (c) For every real number $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \mathbb{R}$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
5. Is the following statement a tautology?
   $$(\forall x \in U)(P(x)) \longrightarrow (\exists x \in U)(P(x)).$$
   Why or why not? Justify your assertions.

6. Prove the following statement:

   For integers a, b, and c, if $a|b$ and $a|c$, then $a|(b+c)$.

   (a) Construct a "know-show" table for a proof of this statement. You may find it useful to recycle LaTeX code from HW1.

   (b) Write your proof in paragraph form.

7. Prove or find counterexamples to following statements. Write negations of the false statements in English.

   (a) For all integers a, we have $\sqrt{a^2} = a$.

   (b) For all integers a, b, c with $a \neq 0$, if $a|(bc)$ then $a|b$ or $a|c$.

   (c) For all integers a, b with $a \neq 0$, if $a|b$, then $a^2|b^2$.

   (d) For all real numbers x, y we have $\sqrt{x^2 + y^2} > 2xy$.

   (e) For all integers a, b, and c with $a \neq 0$, if a divides $(b-1)$ and a divides $(c-1)$, then a divides $(bc-1)$.

   (f) For all integers a, b, and c with $a \neq 0$, if a divides both $b-c$ and $b+c$, then a divides b.

8. Let n be a positive integer and consider the statement we explored about congruence modulo n:

   For any integers a, b, c, d if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $(a+c) \equiv (b+d) \mod n$.

   (a) Construct a "know-show" table for a proof of this statement. You may find it useful to recycle LaTeX code from HW1.

   (b) Write your proof in paragraph form.

9. Repeat the previous question, but replace addition by multiplication.

10. Prove or find counterexamples to following statements.

    (a) If a is an integer with $a \equiv 2 \mod 6$, then $a^2 \equiv 4 \mod 6$.

    (b) If a is an integer with $a^2 \equiv 4 \mod 6$, then $a \equiv 2 \mod 6$.

11. Consider Statement (e) in Problem 7.

    (a) Rewrite this as a statment involving congruences.

    (b) Formulate a useful generalization of you statement in part (a) of this problem.