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Contributions to Constructive Polynomial Ideal Theory XXIII: Forgotten Works of Leningrad Mathematician N. M. Gjunter on Polynomial Ideal Theory*

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Abstract

In a 1941 paper (which is condensed here), N.M. Gjunter refers to some of his unknown papers from 1910, 1913 and 1925, which are partially written in context with the 1941 paper. Thus Gjunter had already pursued a constructive theory of polynomial ideals in 1913. Among other results, he proves the inequality attributed to Macaulay/Sperner in 1913 (from 1927 and 1930, respectively). Also discussed are results analogous to more recent work. The author hereby finishes his sequence of articles.

1 Introduction

While working in the library at Humboldt University's I. Mathematical Institute, B. Renschuch accidentally came across N.M. Gjunter's French language paper [12], whose content was used in this series of articles, so that a systematic and precise study could be made. This was achieved with the content of five theses [5,14,15,18,24], which were finished at the same time and coordinated with each other; these works are held in the library of the Pädagogischen Hochschule Karl Liebknecht Potsdam for the disposal of those interested.

The cited work [12] first appeared in 1941 after the death of N.M. Gjunter (thus the mourning border around his name) and mirrors ideas, according to the biography [13], that can be traced back possibly to lectures at Leningrad University and the Herzen Institute, our partner college, respectively, which he gave after his masters dissertation in 1904. But from the start, he evidently had problems from analysis in mind, whose solutions required algebraic means that were, for the most part, imprecisely formulated by Delassus and Riquier [23].

So Gjunter was really a representative of analysis and was classified accordingly in organizational reports. This may be the reason why his work has remained completely unknown to algebraists until now. On the other hand, this also led Gjunter to refer solely to the classical work of Hilbert [17] and apparently not deal with the pertinent work of Hermann [16] and Macaulay [19]. Thus additional difficulties arose in this treatment.

Since Gjunter addresses the Macaulay-Sperner (Macaulay 1927, Sperner 1930) inequality among other things in [12], and cites his own work [7] which appeared in 1913, an interesting question of priority arises, the more so

*[From original paper:] Dedicated to Professor Brehmer on the occasion of his 70th birthday in thankful memory of the valuable suggestions during the production of the book [21].

†[From original paper:] See the theses [5,14,15,18,24] cited in the reference list at the end of this paper; the numbers in square brackets refer to this reference list, unless stated otherwise in the text. The final version of this work and a translation into Russian were completed during G.G. Rasputin's stay in the DDR during the 1985-86 school year.

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since both could have met at the Third International Congress of Mathematicians in 1904 in Heidelberg; in the list of participants, we find on pages 14 and 17, respectively: *Günther, N.M., Privatdozent, St. Petersburg* and *Macaulay, F.S., Professor, London*.

So it was essential to compare the referenced work [12] to the earlier papers [6,7,8,9,10]. Coincidentally, the papers [6,7,8,9] have the same citation number in the reference list in [12]; one paper, designated there with [10], does not exist as an independent publication, but rather as part of paper [8].

This created considerable difficulties in obtaining papers [6,7,8,9], since the corresponding journal issue (better: volume, since every paper is paginated anew) is cited as follows in [12]: *Recueil de l'Institut des Ponts et Chaussèes* [Anthology of the Institute for Bridges and Highways], while the pertinent institute in [13] is designated an institute of traffic system engineering. During his candidacy H. Roloff and a group of librarians in the Leningrad University library succeeded in finding the correct title for the series (see the reference list) and obtaining the corresponding papers.

Thus it was possible to translate the referenced paper [12] and the contexts [6,7,8,9] as faithfully as possible in the five cited theses, and compare them to each other. From the 1925 paper [10], there were interesting things to fish out. Since [12,Chapter VI.C] is of no importance for our theory, it was intentionally omitted in this publication.

Gjunter displays only the theorems in the paper [12]. In order to give the reader an impression of the difficulty of accessing this treatise [12], all essential definitions and theorems are compiled in the two sections that follow. In addition to the Theorems 1 to 15 indicated by Gjunter, more theorems with appended letters are introduced. Definitions are not numbered at all by Gjunter, but are partially highlighted with spaced out print instead. The same is true for the Lemmas and Corollaries. While the terminology has been carried over unchanged, the mathematical notation was partly modernized. In the case of cross references for example, the indicator 136, III, 2 means page 136, chapter 2, paragraph 2.

In the third section, Gjunter's notations are compared to what is common today, while in the four sections that follow, the main results are summarized.

2 Summary of Definitions and Theorems from the Referenced Paper [12] of 1941

Definition 1 (97,foreword). By the *module of forms* with $F_1(x_1,\dots,x_m)$, $\dots$, $F_L(x_1,\dots,x_m)$ as generators, we mean the set of all forms $\Phi = B_1F_1 + \dots + B_LF_L$, B_i, F_j homogeneous polynomials. (Instead of *degree*, the term *dimension* is used and is denoted by n or n_j, respectively; see Definition 21).

Definition 2 (99,I,1). If $x_1,\dots,x_m$ are given, then x_j is called a *predecessor* of x_i precisely when $j < i$. Then

$$x_1^{\alpha_1}\dots x_j^{\alpha_j+1}\dots x_i^{\alpha_i-1}\dots x_m^{\alpha_m}$$

is called a predecessor of

$$x_1^{\alpha_1}\dots x_j^{\alpha_j}\dots x_i^{\alpha_i}\dots x_m^{\alpha_m},$$

just as every predecessor of a predecessor is also.

Definition 3 (99,I,2). The number of monomials of degree n will be denoted by $N(n,m)$; thus $N(n,m) = \binom{n+m+1}{m-1}$.

Definition 4 (99,I,2). Let $\alpha_1 + \dots + \alpha_m = \beta_1 + \dots + \beta_m = n$. The monomials of degree n are called *numbered by ordering* (a) if: $x_1^{\alpha_1}\dots x_m^{\alpha_m}$ has a smaller number than $x_1^{\beta_1}\dots x_m^{\beta_m}$ if and only if the first non-zero difference in $\alpha_1 - \beta_1, \dots, \alpha_m - \beta_m$ is positive.

Definition 5 (100,I,2). Let $\alpha_1 + \dots + \alpha_m = \beta_1 + \dots + \beta_m = n$. The monomials of degree n are called *numbered by ordering* (b) if: $x_1^{\alpha_1}\dots x_m^{\alpha_m}$ has a smaller number than $x_1^{\beta_1}\dots x_m^{\beta_m}$ if and only if the first non-zero difference in $\beta_m - \alpha_m, \dots, \beta_1 - \alpha_1$ is positive.

Definition 6 (100,I,2). An ordering of power products is called *regular* if it is based on comparing differences of exponents of corresponding power products.

Definition 7 (100,I,3). A set (E) of L power products of degree n is called *normalized* if for every power product, it also contains all its predecessors (see Definition 2).

Definition 8 (100,I,3). A set (E) of L power products of degree n is called (a)-*normal* if it consists of the first L power products with respect to ordering (a) (see Definition 4).

Definition 9 (100,I,3). A set (E) of L power products of degree n is called (b)-*normal* if it consists of the first L power products with respect to ordering (b) (see Definition 5).

Definition 10 (101,I,4 resp. 172,V,6). Expressions of the form $F_i = a_1\phi_i^{(1)} + \ldots + a_s\phi_i^{(s)}$ $(i = 1,\ldots,L)$ with indeterminates a_j $(j = 1,\ldots,s)$ and forms $\phi_i^{(j)}$ of degree $n_i^{(j)}$ with $n_i^{(j)} = n_i^{(1)} + n_1^{(j)} - n_1^{(1)}$ are called *generalized forms*.

Definition 11 (101,I,4). Every individual term $a_i x_1^{\alpha_1}\ldots x_m^{\alpha_m}$ of a generalized form is called a *monomial with index*.

Definition 12 (102,I,5). An ordering (a) for monomials with indices is defined:

(1) for monomials with the same $n_i^{(j)}$, by the ordering (a),

(2) $a_1 x_1^{\alpha_1}\ldots x_m^{\alpha_m}$ preceeds $a_2 x_1^{\alpha_1}\ldots x_m^{\alpha_m}$ etc.

(3) For different $n_i^{(j)}$, every monomial is brought into maximal degree by multiplication by x_{m+1}^{δ} and then ordered by (1) and (2), whereby the formal condition of Definition 4 remains satisfied.

Definition 13 (102,I,5). An ordering (b) for monomials with indices is defined:

(1) for monomials with the same $n_i^{(j)}$, by the ordering (b),

(2) $a_1 x_1^{\alpha_1}\ldots x_m^{\alpha_m}$ preceeds $a_2 x_1^{\alpha_1}\ldots x_m^{\alpha_m}$ etc.

(3) For different $n_i^{(j)}$, every monomial is brought into maximal degree by multiplication by x_0^{δ} and then ordered by (1) and (2), whereby the formal condition of Definition 5 remains satisfied.

Definition 14 (102,I,5). The partitioning of $(E) = (K_1)\cup\ldots\cup(K_m)$ into classes given by the ordering (b) with $K_1 = \{x_1^n\}$, $K_2 = \{x_1^{n-1}x_2,\ldots,x_2^n\}$, etc. is called by Gjunter *partitioning in m groups* (see Definition 25).

Definition 15 (102,I,5). By the ordering (bdb) for monomials with indices, we mean partitioning by Definition 14 and then by increasing index within each class.

Definition 16 (103,I,6). Let $(E) = \{F_1,\ldots,F_L\}$ be a set of generalized forms with $\deg\phi_i^{(j)} = n_j$. The set $\{p_1 + \psi_1, \ldots, p_L + \psi_L\}$, the result of transforming by monomials each of smallest number p_i with respect to a fixed ordering that no longer appear in the ψ_i, is called a *well-formed set* of (E); $\{p_1,\ldots,p_L\}$ is called a *representative set*.

Definition 17 (104,I,6). The monomials $p_1,\ldots,p_L$ from Definition 16 appeared in connection with the indices $a_1,\ldots,a_s$. Let Aa_j be the *set of all monomials with index a_j*, where the case $Aa_j = \Phi$ can occur. Then we have $\{p_1,\ldots,p_L\} = Aa_1\cup\ldots\cup Aa_s$. Each set Aa_j is called a *a subset of the representative set*.

Definition 18 (104,I,6). The representative set $\{p_1,\ldots,p_L\}$, as in Definition 16, is called *normalized* if every subset, as in Definition 17, is normalized.

Theorem 1 (104,I,7). *The representative set is normalized by carrying out a general linear transformation.*

Definition 19 (107,I,9). If L linearly independent generalized forms $F_1, \ldots, F_L$ with $\deg \phi_i^{(j)} = n_j$ are multiplied by all $N(k,m)$ monomials of degree k, then the linearly independent subset of the resulting $L \cdot N(k,m)$ generalized forms is called the *derived set of order k* for $\{F_1, \ldots, F_L\}$. For $\deg \phi_i^{(j)} \neq n_j$, we raise the $\phi_i^{(j)}$ to their respective maximal degrees; thus the *derived set of order zero* occurs with $\deg \phi_i^{(j)} = n_j$, from which the higher derivations are formed.

Definition 20 (107,I,9). The *number* of elements in the derived set of order k for $\{F_1, \ldots, F_L\}$ will be denoted by L'_k. In particular, L'_1 is called the *index* of $\{F_1, \ldots, F_L\}$.

Definition 21 (107,I,9). By the *module of forms* with generators $F_1, \ldots, F_L$ ($F_1, \ldots, F_L$ generalized forms with $\deg \phi_i^{(j)} = n_j$), we mean the linearly independent ones among the generalized forms $B_1F_1 + \ldots + B_LF_L$ with forms $B_1, \ldots, B_L$ and $\deg B_1 = \ldots = \deg B_L = k$ ($k = 1, 2, \ldots$) (see Definition 1)

Definition 22 (107/108,I,9 resp. 134,III,1). If $F_1, \ldots, F_L$ are generalized forms with $\deg \phi_i^{(j)} = n_j$ and $\deg B_1 = \ldots = \deg B_L = k$, then every identity of the form $B_1F_1 + \ldots + B_LF_L = 0$ is called a *passivity condition of order k* of $\{F_1, \ldots, F_L\}$.

Definition 23 (108,1,10). The *number* of elements of the derived set of order k for the representative set of $\{F_1, \ldots, F_L\}$ is denoted by L_k.

Theorem 1a (109,I,10). *If L'_k and L_k are stated as in Definitions 20 and 23, respectively, then*

$$L_k \leq L'_k. \tag{28}$$

Theorem 1b (109/110,I,12). *1. The derived set of a normalized set is normalized.*

2. *The derived set of an (a)-normal set is (a)-normal.*
3. *the derived set of a (b)-normal set is not (b)-normal in general. Counterexample: $\{x_1^3, x_2^2x_2, x_1x_2^2\}$ in x_1, x_2, x_3.*

Lemma 1 (111,II,1). *If $(E) = \{p_1, \ldots, p_L\}$ is normalized, then every function ψ that is the left side of a passivity condition of order k is a linear function of products of monomials of degree $k-1$ with functions ϕ that are left sides of passivity conditions of order 1 (see Definition 31).*

Definition 24 (112,II,1). Let $(E) = \{p_1, \ldots, p_L\}$ be a normalized set of monomials of the same degree. In the matrix (x_ip_k), the same power products with larger k are omitted. The remaining passivity conditions of order 1 of $\{p_1 \ldots, p_L\}$ are called *principal conditions*.

Definition 25 (114,II,2). Let $(E) = \{p_1, \ldots, p_L\}$ be a normalized set of L monomials of the same degree, in which a partioning into classes (in Gjunter: a partition into groups) according to Definition 14 is effected: x_1^n belongs to the first class, total $l_1 = 1$, ..., all those monomials which depend only on $x_1, \ldots, x_i$ and in which x_i actually appears belong to the i-th class, total l_i ($i = 1, 2, \ldots, L$). Then $l_1 + \ldots + l_m = L$.

Theorem 1c (114,II,2). *If $l_1, \ldots, l_m$ are the class sizes as in Definition 25 and L_k as in Definition 23, then for $L_k(E^{(k)})$ of a normalized set (E),*

$$L_k = LN(k,m) - \sum_{s=1}^{m} l_s \left[N(k,m) - N(k,m-s+1)\right] = \sum_{s=1}^{m} l_sN(k,m-s+1) \tag{8}$$

If (E_a) is also (a)-normal, then the class sizes will be denoted by $\lambda_1, \ldots, \lambda_m$ (see Definition 28).

Definition 26 (114,II,2). If $(E) = \{p_1, \ldots, p_L\}$ is a normalized set of L monomials of the same degree n with class sizes $l_1, \ldots, l_m$ (as in Definition 25), then the right side of $L_k = \sum_{s=1}^{m} l_sN(k,m-s+1)$ is called the of numerator (E). The numerator is a polynomial of degree $m-1$ in k. Erroneous footnote: numerator = characteristic polynomial; correct: numerator = volume function.

Definition 27 (115,II,3). Let $(E^{(k)})$ be the derived set of order k for the normalized set $(E) = \{p_1, \ldots, p_L\}$. By partitioning into classes as in Definition 25, the quantities $l_1^{(k)}, \ldots, l_m^{(k)}$ are defined. Then $l_1^{(k)} + \ldots + l_m^{(k)} = L_k$.

Theorem 1d (115,II,3). *For the class sizes* $l_1^{(k)}, \ldots, l_s^{(k)}, \ldots l_m^{(k)}$ *of* $(E^{(k)})$ *introduced in Definition 27,*

$$\begin{aligned} l_s^{(k)} &= l_1 N(k-1,s) + l_2 N(k-1,s-1) + \ldots + l_s N(k-1,1) \\ &= \sum_{i=1}^{s} l_i N(k-1,s-i+1). \end{aligned} \tag{9_s}$$

Definition 28 (116/117,II,4). Let $(E_i) = \{p_1, \ldots, p_L\}$ be (a)-normal. The *last* monomial is denoted by $x_1^{\gamma_1} \ldots x_i^{\gamma_i}$, $i \leq m$, $\gamma_1 + \ldots + \gamma_i = n$, $\gamma_i > 0$ and the class sizes will be denoted by $\lambda_1, \ldots, \lambda_m$ (see Theorem 1c).

Theorem 1e (116/117,II,4). *For an (a)-normal set of* L *monomials of degree* n*, the relations* $n = \gamma_1 + \ldots + \gamma_m$*, as well as*

$$L = N(n-\gamma_1-1,m) + \ldots + N(n-\gamma_1-\ldots-\gamma_{m-1}-1,2) + 1 \tag{12}$$

(see Definition 28) and

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= N(n-\gamma_1-2,2)+1, \\ &\vdots \\ \lambda_i &= N(n-\gamma_1-2,i) + \ldots + N(n-\gamma_1-\ldots-\gamma_{i-1}-2,2)+1, \\ &\vdots \\ \lambda_m &= N(n-\gamma_1-2,m) + \ldots + N(n-\gamma_1-\ldots-\gamma_{m-1}-2,2)+1 \end{aligned} \tag{14}$$

exist among $\lambda_1, \ldots, \lambda_m$*,* $\gamma_1, \ldots, \gamma_m$*,* n *and* L

Theorem 2 (118,II,5). *Let* (E) *and* (E_b) *be normalized sets of* L *monomials of the same degree and* (E_b) *be normal as well. Then for derived sets of order* k*,* $L_k(E_b^{(k)}) \geq L_k(E^{(k)})$.

Theorem 3 (119,II,6). *Let* (E) *and* (E_a) *be sets of* L *monomials of the same degree,* (E) *otherwise arbitrary and* (E_a) *(a)-normal. Then for derived sets (of order one),* $L_1(E_a^{(1)}) \leq L_1(E^{(1)})$.

Lemma 2 (126,II,8). *The number of sets*

$$\left\{\gamma_1^{(k)}, \ldots, \gamma_{m-1}^{(k)}\right\} \tag{X}$$

that are formed from each set of $m-1$ *non-negative integers* (X)*, in such a way that at least one difference* $\gamma_1^{(k)} - \gamma_1^{(k+1)}, \ldots, \gamma_{m-1}^{(k)} - \gamma_{m-1}^{(k+1)}$*, namely the first non-zero one, is positive, is bounded.*

Definition 29 (127,II,8). Let $(E) = \{p_1, \ldots, p_L\}$ be a normalized set of monomials of the same degree n and (E_a) the (a)-normal set of L monomials of degree n. Then if $L_k(E^{(k)}) = L_k(E_a^{(k)})$ for all $k = 0, 1, 2, \ldots$, then the normalized set (E) is called *minimal*.

Theorem 4 (127,II,8,9). *For sufficiently large* $k \geq k_{\min}$*, all derived sets* $(E^{(k)})$ *for a normalized set* (E) *are minimal. If* $l_1, \ldots, l_m$ *are defined as in Definition 25, then the auxilliary values* $\Theta_1 := l_2 - 1$*,* $\Theta_s := l_{s+1} - \binom{\Theta_1+s-1}{s} - \binom{\Theta_2+s-2}{s-1} - \ldots - \binom{\Theta_{s-1}+1}{2} - 1$ *(*$s = 2, \ldots, m-1$*) can be computed. For* $k_{\min}$*, the smallest number with*

$$k + \Theta_1 + 1 \geq 0, \ \ldots, \ k + \Theta_{m-1} + 1 \geq 0 \tag{47}$$

can be chosen. Let $x_1^{\gamma_1} \ldots x_m^{\gamma_m}$ *be the last monomial of the (a)-normal comparison set of* L_k *monomials of degree* $n+k$*. Then* $n - \gamma_1 - 1 = \Theta_1, \ldots, n - \gamma_1 - \ldots - \gamma_{m-1} - 1 = \Theta_{m-1}$.

Definition 30 (130,II,9). If (E) is a normalized set of L monomials of the same degree n, then the integers $\gamma_1,\dots,\gamma_{m-1}$ computed in Theorem 4 are called *invariants* of the normalized set (E) (see Definition 28).

Theorem 5 (131,II,10). *Let (E) be a normalized set of L monomials of the same degree n, $(E^{(t)})$ the derived set of order t, and let (l) and $(l^{(t)})$ be the elements of the last class, respectively, as in Definition 25, and further let $(e) := (E)\setminus(l)$, $(e^{(t)}) := (E^{(t)})\setminus(l^{(t)})$. Let K be the number of monomials which depend on x_m and which can be added to (E) without changing (e), and analogously K_t relative to $(E^{(t)})$. Then $K_t = K$, i.e. K_t does not depend on t.*

Addendum to Theorem 5 (133,II,11). *$K \le \gamma_{m-1}$ (see Definition 30).*

Definition 31 (136,III,2). The set of all generalized forms $\{F_1,\dots,F_L\}$ with $\deg\phi_i^{(j)} = n_j$ is called *complete*, if for every k, the passivity conditions of order k are linearly combinable from the passivity conditions of order 1.

Definition 32 (136,III,3). The set of all generalized forms $\{F_1,\dots,F_L\}$ with $\deg\phi_i^{(j)} = n_j$ is called *closed* if for all k, the representative set of the derived set of order k is equal to the derived set of order k for the representative set.

Theorem 6 (137,III,4). *Let $(E) = \{F_1,\dots,F_L\}$ be a set of generalized forms F_i with $\deg\phi_i^{(j)} = n_j$ and representative set $\{p_1,\dots,p_L\}$. If $\{p_1,\dots,p_L\}$ is normalized and $L_1(\{p_1,\dots,p_L\}) = L_1'(E)$ (see Definitions 22,23), then (E) is complete and closed with respect to the chosen ordering (see Definitions 30,31).*

Corollary to Theorem 6 (143,III,5). *If the set $(E) = \{F_1,\dots,F_L\}$ satisfies the conditions of Theorem 6, then the set of left hand sides of their passivity conditions of order 1 (interpreted as generalized forms) is closed.*

Theorem 7 (143,III,6). *If the representative sets of all derived sets for a set (E) are normalized, then there exists a derived set of (E) which satisfies the conditions of Theorem 6.*

Corollary to Theorem 7 (144,III,6). *If the representative sets of all derived sets for a set (E) are normalized, then from a certain derived set on, the representative sets for all derived sets are minimal.*

Definition 33 (146,III,7). If we set $x_1 = \ldots = x_m = 0$ in $(E) = \{F_1,\dots,F_L\}$, then Gjunter speaks of the execution of the transformation (C_i) ($i = 2,\dots,m$).

Theorem 8 (154,IV,2). *Let $F_1,\dots,F_L$ be generalized forms with $\deg\phi_i^{(j)} = n_j$ and $F_j^{(1)} = B_1^{(j)}F_1 + \ldots + B_L^{(j)}F_L$ the passivity conditions of order 1. Let $(E) = \{F_1,\dots,F_L\}$ be complete and l_1 be the class size of the elements dependent only on x_1 of the representative set of the well-formed set of the transformed set (E_1) of (E) after a generalized transformation. Then*

$$\operatorname{rank}\begin{pmatrix} \phi_1^{(1)} & \dots & \phi_1^{(s)} \\ \vdots & & \vdots \\ \phi_L^{(1)} & \dots & \phi_L^{(s)} \end{pmatrix} = l_1 \quad\Rightarrow\quad \operatorname{rank}\begin{pmatrix} B_1^{(1)} & \dots & B_L^{(1)} \\ B_1^{(2)} & \dots & B_L^{(2)} \\ \vdots & & \vdots \end{pmatrix} = L - l_1.$$

Theorem 9 (157,IV,3). *Let $F_1,\dots,F_L$ be generalized forms with $\deg\phi_i^{(j)} = n_j$ and $F_j^{(1)} = B_1^{(j)}F_1 + \ldots + B_L^{(j)}F_L$ the passivity conditions of order one. Let $\{F\} = \{F_1,\dots,F_L\}$ and $\{F_j^{(1)}\}$ be complete. By the substitution*

$$x_1 = a_1^{(1)}y_1,\ \ x_2 = a_2^{(1)}y_1 + y_2,\ \ \dots,\ \ x_m = a_m^{(1)}y_1 + y_m, \tag{16}$$

$\{F_1,\dots,F_L\}$ changes into $\{\overline{F}\} = \{\overline{F}_1,\dots,\overline{F}_L\}$ and $\{F_j^{(1)}\}$ into $\{\overline{F}_j^{(1)}\}$. The passivity conditions of order one of $\{\overline{F}\}$ are split up in such a way that only in $\overline{F}_1^{(1)} = 0,\dots,\overline{F}_{L-l_1}^{(1)} = 0$ do the expressions $y_1\overline{F}_{l_1+1},\dots,y_1\overline{F}_L$ appear, but not in the remaining passivity conditions

$$\Phi_{L-l_1+1}^{(1)} = 0,\dots. \tag{19}$$

If the substitution $b_{hg} := y_1^h a_g$ changes the forms $\{\overline{F}\} = \{\overline{F}_1,\dots,\overline{F}_L\}$ into $\{(\overline{F})\} = \{(\overline{F}_1),\dots,(\overline{F}_L)\}$, then $\{(\overline{F})\}$ is also complete and the passivity conditions of order one of $\{(\overline{F})\}$ are given by (19).

Theorem 10 (159,IV,4). *If $\{F\}, \{\overline{F}\}, \{\overline{F}^{(1)}\}$ are stated as in Theorem 9, then if $\{F\}$ is complete and $\{\overline{F}^{(1)}\}$ is closed under ordering (b), then $\{\overline{F}\}$ is also closed under ordering (b).*

Theorem 10a (161,IV,4). *If a set (E) of generalized forms with $\deg\phi_i^{(j)} = n_j$ is closed under a fixed ordering then it is also closed under the ordering (b).*

Definition 34 (161,IV,5). Let (F) be a set of generalized forms with $\deg\phi_i^{(j)} = n_j$, which is closed and has a normalized representative set. The recursively formed passivity conditions

$$F^{(1)} = \sum_{i=1}^{L} B_i^{(1)} F_i = 0 \tag{31_1}$$

$$\vdots$$

$$F^{(h)} = \sum_{i} B_i^{(h)} F_i^{(h-1)} = 0 \tag{31_h}$$

are called the *passivity conditions of type 1, …, type h.*

Theorem 10b (162,IV,5). *The number of types of different passivity conditions of a closed set in $x_1,\dots,x_m$ is at most $m-1$.*

Theorem 11 (162,IV,6). *If the set (E) is complete and if the left sides of their passivity conditions of all types form complete sets, then the set of all forms $\{\overline{F}\}$ which arise from F via the transformation*

$$x_i = a_i^{(1)} y_1 + \ldots + a_i^{(i)} y_i \quad (i = 1,\dots,m) \tag{3}$$

is closed under the ordering (b).

Theorem 12 (166,V,2). *Hypotheses: Let $(E) = \{f_1,\dots,f_{L+T}\}$ be given with $f_1,\dots,f_L$ linearly independent, but $f_{L+1},\dots,f_{L+T}$ linearly dependent on $f_1,\dots,f_L$. After a general transformation, let $(E^{(k)})$ be closed with linearly independent forms $\Phi_1,\dots,\Phi_{L_k}$ and linear dependent forms $\Phi_{L_k+1},\dots,\Phi_{L_k+T_k}$. The dependency relations lead to*

$$\Omega_j = \sum_{i=1}^{L+T} A_i^{(j)} f_i = 0. \tag{13}$$

Among the $\Phi_1,\dots,\Phi_{L_k}$, there exist $\Lambda_{k+1} = mL_k - L_{k+1}$ passivity conditions $\Phi_1^{(1)} = 0, \dots, \Phi_{\Lambda_{k+1}} = 0$. By multiplying by $y_1,\dots,y_m$, equations (13) lead to the passivity conditions

$$\psi_j = \sum_{i=1}^{L+T} A_i^{(j)} f_i = 0 \tag{16}$$

in which

$$\psi_1,\dots,\psi_{\Delta_{k+1}} \tag{18}$$

are linearly independent with

$$\Delta_{k+1} = N(k+1,m)(L+T) - L_{k+1}.$$

Claim: For $n \geq k+1$, the left side of the passivity conditions of order n form a module with $\psi_1,\dots,\psi_{\Delta_{k+1}}$ as generators, where the ψ_i are interpreted as generalized forms with indices $f_1,\dots,f_{L+T}$.

Lemma 3 (168,V,3). *With the notation of Definition 33 and Theorem 12, we have: If* f_j, ψ_i *are changed into* $\langle f_j \rangle$, $\langle \psi_i \rangle$ *by applying the transformation*

$$y_{i+1} = \ldots = y_m = 0, \tag{C_{i+1}}$$

then we obtain the passivity conditions

$$\psi_j = \sum_{i=1}^{L+T} a_i^{(j)} \langle f_i \rangle = 0 \tag{23}$$

for $\langle f_1 \rangle, \ldots, \langle f_{L+T} \rangle$*, if we carry out the transformation* (C_{i+1}) *in the passivity conditions*

$$\psi_j = \sum_{i=1}^{L+T} A_i^{(j)} f_i = 0. \tag{16}$$

This means that we can choose all linear independent forms from among

$$\langle \psi_j \rangle = \sum_{i=1}^{L+T} \langle A_i^{(j)} \rangle \langle f_i \rangle = 0. \tag{24}$$

Theorem 13 (169,V,4). *With the notation of Theorem 12, the set of all forms*

$$\psi_1, \ldots, \psi_{\Delta_{k+1}} \tag{18}$$

is closed.

Theorem 13a (171,V,5). *For every set of generalized forms (see Definition 10) with* $\deg \phi_i^{(j)} = n_j$*, the number of distinct types of passivity conditions is at most m.*

Theorem 14 (174,V,7). *Hypotheses: Let*

$$f_1, \ldots, f_L$$

be generalized forms (see Definition 10) with $n_i := n_i^{(1)}$, $n_1 = \max\{n_i\}$*. By forming the derived set of order 0 (see Definition 19),*

$$F_{i,h} = \omega_h^{(n_1 - n_i)} f_i \tag{42$'$}$$

arises with $\deg \phi_{i,h} = n_j$*, to which the hypotheses of Theorem 12 are applied with* $(E) = \{f_1, \ldots, f_{L+T}\} = \{F_{i,h}\}$ *and*

$$\psi_1, \ldots, \psi_{\Delta_{k+1}} \tag{48}$$

$$\Omega_1, \ldots, \Omega_{\Delta_{k+1}} \tag{48$'$}$$

$$\blacksquare_1, \ldots, \blacksquare_\Lambda \tag{48*}$$

Claim: Then for $n \geq k$*, the left sides of the passivity conditions of order n of the set of generalized forms (39) form a module with the set of generalized forms* ($48'$) *as generators.*

Theorem 15 (175,V,8). *Under the hypotheses of Theorem 14, the set of generalized forms* (48^*) *is closed under ordering (b).*

Theorem 15a (180,V,9). *Under the hypotheses of Theorem 14, the number of types of distinct types of passivity conditions is at most* $m-1$ *for the set* (48^*) *and at most m for the set (39).*

3 Modern Version of Definitions and Theorems

For lack of space, we will content ourselves with noting the important changes in terminology.

In today's usage, Gjunter's ordering (a) in Definitions 4, 8, 12 correspond to *graded lexicographic ordering*; the ordering (b) in Definitions 5, 9, 13 to *pure lexicographic order* and also to *inverse lexicographic order* in French literature.

For the term *normalized* in Definition 7, we propose the term *quasilexicographic*.

In Definitions 8, 9, (a)-normal and (b)-normal would be replaced by *graded lexicographic segment* with the ideal $\mathfrak{b}_\pi^*$ and *pure lexicographic segment* with the ideal $\mathfrak{c}_\pi^*$, respectively.

The *well-formed set* in Definition 16 is a *Gröbner basis* (precisely: a V-basis) under the the hypotheses indicated. We would like to retain the terms *representative* and *representative set*; in the English literature (see [2]) we find the terms *leading terms* and *leading monomials*, respectively.

In Definitions 19, 20 and 23 we substitute: For the forms $F_1,\ldots,F_L$ of the same degree g with $\mathfrak{a}=(F_1,\ldots,F_L)=(p_1+\psi_1,\ldots,p_L+\psi_L)$, the *derived set of order k* is the vector space $\mathfrak{M}(g+k;\mathfrak{a})$ with $L_h'=V(g+k;\mathfrak{a})$ and $L_k=V(g+k;p_1,\ldots,p_L)=V(g+k;\mathfrak{a}\pi)$.

The inequalities of Theorems 2 and 3 can be described by the inequalities $V(t;\mathfrak{c}_\pi^*)\geq V(t;\mathfrak{a}\pi)$ and $V(g+1;\mathfrak{a})\geq V(g+1;\mathfrak{b}_\pi^*)$, respectively, and the minimality in Definition 29 is described by $V(g+1;\mathfrak{a}\pi)=V(g+1;\mathfrak{b}_\pi^*)$.

In today's usage, *Completeness* in Definition 31 means that the syzygy module has a basis of only linear syzygies. *Closure* in Definition 32 is characterized by $V(t;\mathfrak{a})=V(t;\mathfrak{a}_\pi)$. Theorems 6, 7 and the Corollary to Theorem 7 can then be appropriately formulated.

Theorems 10b, 13a, 15a comprise Hilbert's Theorem on the termination of the syzygy chain, i.e. the validity of the inequality $L(\mathfrak{a})\leq n+1$.

In addition to these transliterations, a more precise rendering of terminology (e.g. by quantifications) given in the second chapter will still prove to be necessary, see [14].

4 Results of Collation, Main Results

For this, can be summarily evaluated:

1. For the interesting theorems 1b, C,2,9,...,15 in [12], for which the pure lexicographic order (ordering (b)) is used, no context exists in earlier work. All other theorems (in partucular 1,3,...,8) appeared in work prior to 1925, the majority in 1913.
2. [12,Theorem 3] shows that Gjunter formulated the Macaulay-Sperner inequality 14 years before Macaulay and proved it in an entirely different way, which we study in the next chapter.
3. For the number of operations, Gjunter indicates algorithmic bounds up until [12,Theorem 4] and degree bounds thereafter, which are extremely high.
4. Because of the necessary carry over to transformed ideals (prior linear substitution of variables with indeterminate coefficients), no practical hand calculation is possible.
5. Gjunter's proof of the termination of syzygy chains, thus the inequality $L(\mathfrak{a})\leq n+1$, is not uninteresting, since he shows that the number of variables (thus $x_0,\ldots,x_n$; for him: $x_1,\ldots,x_m$) is an upper bound for $L(\mathfrak{a})$. This leads to a successive "freezing" of variables; so the last variable can no longer be replaced.
6. Through Gjunter's Theorem 6 in [12] (in the context of [9]), relations to new areas of research (Dubreil 1949, Mora & Möller 1984) and to Gröbner Basis theory (Buchberger since 1970) ensue. We will go into this in Chapters 6 and 7.
7. Consequently, N.M. Gjunter, together with F.S. Macaulay, can be regarded historically as founders of a constructive polynomial ideal theory.

5 Gjunter's Method of Proving the Macaulay-Sperner Inequality

Let $F_1, \ldots, F_L$ be forms of the same degree g and $\mathfrak{a} := (F_1, \ldots, F_L) = (p_1 + \psi_1, \ldots, p_L + \psi_L)$.

If $U \ldots$ denotes forms, power products, ideals, etc. after applying a general transformation

$$y_i = u_{i1}x_1 + \ldots + u_{im}x_m \qquad (i = 1, \ldots, m) \tag{1}$$

then let

$$U(p_1, \ldots, p_L) = (Up_1, \ldots, Up_L) = (q_1 + \psi_1, \ldots, q_L + \psi_L).$$

Moreover, if $\mathfrak{b}^*_\pi := (\pi_1, \ldots, \pi_L)$ is a lexicographic segment ideal, then for the derived sets (in Gjunter's terminology), the fundamental numeric relations are

$$L'_1(\mathfrak{a}) = L'_1(U\mathfrak{a}), \tag{2}$$

$$L'_1(p_1 + \psi_1, \ldots, p_L + \psi_L) \geq L_1(p_1, \ldots, p_L) \quad \text{i.e.} \tag{3}$$

(2) Invariance of the number of linearly independent forms under general transformations

(3) from the linear independence of forms multiplied by $x_1, \ldots, x_m$, equality of the first power products follows, but not conversely; thus the symbol $>$ can occur (Theorem 1a). From this follows

$$\begin{aligned} L'_1(p_1 + \psi_1, \ldots, p_L + \psi_L) \geq L_1(p_1, \ldots, p_L) &= L'_1(Up_1, \ldots, Up_L) \\ &\geq L_1(q_1, \ldots, q_L). \end{aligned}$$

Now if $q_1, \ldots, q_L$ were the first L power products in graded lexicographic order, then we would have $q_i = \pi_i$ and $(q_1, \ldots, q_L) = (\pi_1, \ldots, \pi_L)$, whereby $L_1(q_1, \ldots, q_L) = L_1(\pi_1, \ldots, \pi_L)$ and hence $L'_1(\mathfrak{a}) \geq L_1(\pi_1, \ldots, \pi_L)$, thus the Macaulay-Sperner inequality (Theorem 3) would be proved trivially.

In light of the indeterminacy of the coefficients u_{ik} in (1), we can still hope for the validity of $q_i = \pi_i$. Delassus took this for granted, but Gjunter sharply refuted this in [6,12]. In [6] he gives

$$(x_1^2, x_1x_2, x_2^2) \subset K[x_1, x_2, x_3] \tag{4}$$

as a counterexample. In particular, if we substitute (1) into the three basisi power products in (4) and perform Gaussian elimination (GE) with respect to the graded lexicographic order, thus in the sequence $y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^2$, then while eliminating y_1y_2, we also throw away y_1y_3 unexpectedly (in spite of the indeterminacy of u_{ik}), yielding $(q_1, q_2, q_3) = (y_1^2, y_1y_2, y_2^2)$. This was computed detail in [18].

However, since (4) represents a pure lexicographic segment, one could hope that this effect does not occur in the pure lexicographic order. As shown in [18], the example (x_1^2, x_1x_2, x_1x_3) yields the representative set $\{y_1^2, y_1y_2, y_1y_3\}$ after (1) and (GE) are applied with respect to the sequence $y_1^2, y_1y_2, y_2^2, y_1y_3, y_2y_3, y_3^2$; eliminating y_1y_2 throws away y_2^2 as well.

Nevertheless, Gjunter was able to show in Theorem 1, that $\{q_1, \ldots, q_L\}$ is a quasilexicographic set (he calls this a *normalized* set), which must then be used in [12,Definition 7]; the definitions of normalization given in [7,8,9,10] are not insightful and not equivalent to Definition 7, see [24].

Thus the proof of the Macaulay-Sperner inequality is reduced to

$$L_1(q_1, \ldots, q_L) \geq L_1(\pi_1, \ldots, \pi_L), \tag{5}$$

where $\{q_1, \ldots, q_L\}$ is quasilexicographic.

As evident from the compilation in the second chapter, many properties of graded lexicographic segments (for Gjunter: (a)-normal) carry over to quasilexicographic sets (for Gjunter: normalized sets). Thus Gjunter's method of proof consists of successively converting a quasilexicographic set of power products of the same degree into a graded lexicographic segment by means of a certain *exchange principle*, in the course of which only (2) and (3) appear (eventually several times) and which finally leads to (5), which we already said proves Theorem 3.

In view of the property just cited, if the set $\{q_1, \ldots, q_L\}$ is quasilexicographic, but not graded lexicographic, then there are power products missing. We consider the last gap of this type. Gjunter's exchange principle consists of exchanging a certain number of power products occurring after the last gap with the same number of power products after q_L, and applying (1) to the exchanged set. In any case, the representative set of the resulting transformed set contains not only every element before the last gap, but also the first element of the last gap, which was missing until now.

After several applications of this principle, all gaps are successively closed. By differentiating many cases, whose need is first illuminated by complicated examples in [5] (which Gjunter misses entirely), it is shown in [7,12] that this exchange method is always feasible.

Applying the principle to (4) leads (see [5,18]) to replacing x_2^2 with x_2x_3:

$$\{x_1^2, x_1x_2, x_2^2\} \to \{x_1^2, x_1x_2, x_2x_3\} \to \{y_1^2, y_1y_2, y_1y_3\}.$$

We can regard Gjunter's proof ideas as simpler than those of Sperner [25], but this does not mean that it is shorter.

6 The Proof of Dubreil's Inequality

If the Macaulay-Sperner inequality holds for the symbol $>$, then with the appearance of the equal sign, we must still add $L_1' - L_1$ power products to the graded-lexicographic segment and then form the derived set. Macaulay [19] and Sperner [25] prove that by iterating this process, the equal sign must appear. This was also already proved by Gjunter [12,Theorem 4]. The degree t_2, from where this occurs on, was defined by Dubreil [4] as the *second regularity index*. The *first regularity index*, he calls the degree t_1, above which the Hilbert function changes into the characteristic polynomial. It follows from the degrees τ_{xi} of the first elements of the 1-st,$\ldots,(k-1)$-th syzygy modules and τ_{ki} of the elements of the first row of the k-th (last) syzygy module, that

$$\begin{aligned} t_1 &= \max\{\tau_{11} + \ldots + \tau_{k-1,1} + \tau_{ki} - n\}, \quad \text{see [21, p. 256, w.l.o.g.]} \\ t_1 &= \tau_{11} + \ldots + \tau_{k-1,1} + \tau_{k,s_k} - n. \end{aligned} \tag{6}$$

Now Dubreil [4] attempted to prove the inequality

$$t_1 \leq t_2 \tag{7}$$

(its importance is put into perspective by Gröbner bases theory). Now t_2 is the first regularity index of a derived set, so $t_2 = \tau_{11}' + \ldots + \tau_{k'-1,1}' + \tau_{k',s_{k'}}' - n$ and we would be finished if we could show that $\tau_{x1}' \geq \tau_{x1}$ and $k' \geq k$. But as shown in [14], both hold. By forming derived sets, trivial components arise; therefore, by a theorem of Gröbner (see [21,Theorem 39, p. 246]), $k' = n+1$, verifying $k' \geq k$. Gjunter's Theorem 13 says that for all syzygy modules, the corresponding passivity conditions for the derived sets are linear combinations of multiplied passivity conditions of the final set, whereby $\tau_{x1}' \geq \tau_{x1}$ follows.

7 Connection to Gröbner Bases

As established in connection with Definition 5 in Part XXII of this article series, Gjunter's treatment leads to Gröbner bases for the case of equal degrees. By Theorem 7, Gjunter's *canonical form* is the V-basis (see also XXII, Example 1) raised to maximal degree. The same is true for Gröbner bases of syzygy modules (see XXII, Chapter 4). In [15], it was also shown that Buchberger's Axioms (G1) and (G2) in [2] correspond to Gjunter's two conditions for well-formed sets.

8 Gjunter's 1930 Conference Report

Here we deal with the Russian paper [11], which first became available to us and could be evaluated by us only after completion of the theses [5,14,15,18,24]. Gjunter refers to this Russian paper in [12]. We quote from the end of the foreword in [11]:

> ”Therefore I decided on this lecture because some results from my old papers were unfortunately printed in specialized publications which are scarcely available and unknown to mathematicians, and thus are almost forgotten. By the way, they are still not covered today in any other papers. Moreover, the content of paragraphs 7 and 8 of this paper have never been published anywhere until now and currently exists only in the manuscript of a larger work being prepared for publication.”

This passage should admittedly be modified in view of the papers [16,19,25] available by 1930. Except for this, Gjunter was aware of the meaning of his studies, through which he, along with Macaulay, are credited as founders of a constructive polynomial ideal theory.

However, [11] is in no way merely a summary of [12], but rather represents a selection of themes of [12], in fact in a different arrangement, which we briefly contrast as a reading aid below:

Arrangement of chapters in [12] from 1941 (every chapter is subdivided into paragraphs without subheadings).

1. Foreword
2. Basic Terminology
3. On the Set of Normalized Sets
4. On the Set of Generalized Forms
5. On Passivity Conditions
6. On Passivity Conditions in the Module of Algebraic Forms
7. Applications

Paragraph divisions in [11] from 1930 (no further subdivisions)

1. Introduction
2. On the Set of Monomials
3. On Generalized Forms
4. On Normalized Sets
5. On the Set of Generalized Forms
6. On Passivity Conditions of Closed Sets
7. The Case of Arbitrary Sets of the Same Degree
8. The Case of Sets of Different Degrees
9. Some Applications

Contents from §... in [11]	1	2	3	4	5	6	7	8	9
are contained in Chap... in [12]	0	1	3	2	3 after Thm. 7	4	5	5 after §6	6

With this publication, we hope very much to have called attention to the algebraic works of N.M. Gjunter listed here and to have inspired further analysis.

9 Afterword to this Series of Articles by B. Renschuch

The goal (via my academic teachers Hermann Ludwig Schmid, Wolfgang Gröbner and Ott-Heinrich Keller) of this series of articles was to indicate practically feasible methods to allow computation of examples. Thus I had in mind computation ”by hand”, as carried out by the 106 total diploma and state exam candidates. However, the need for inputting them into modern computing devices soon became clear. Perhaps it will also be possible to treat open problems of effective computation from fundamental ideals and primary decomposition. On the question of determining bases of minimal length, H. Bresinsky and B. Renschuch will yet publish something in *Beiträge zur Algebra und Geometrie* **24**. On the question of forming set theoretic complete intersections by H-ideals, Mrs. Ute Meinhold will share something with us for monomial ideals. Otherwise, I hope that younger mathematicians will find this series of articles from the ever expanding international literature, which (using possibly other types of methods) will lead to a solution of these problems. Perhaps the examples given in this series of articles will be helpful; they were computed by the 106 candidates, who at this time are warmly thanked. To this hope, I am also encouraged by the fact that polynomial ideal theory is applicable to problems in numerical mathematics, for which I acknowledge Professors Hans Kaiser (Potsdam) and Georgij G. Rasputin (Archangelsk). The reader can learn the details from one of Rasputin and Renschuch's coauthored papers in *Beiträge zur Algebra und Geometrie* **23**, which (like the conclusion of this paper) originated during G.G. Rasputin's stay at the Pädagogisches Hochschule Karl Liebknecht Potsdam in the 1985-86 school year. For this successful collaboration, Georgij G. Rasputin is sincerely thanked.

Further thanks go to my friend Professor Wolfgang Vogel (Martin Luther University, Halle) who coordinated these articles, both here and abroad, with the work of members and cooperation partners of the *Algebraic Geometry* research group.

With that, I would like to end this series of articles, say goodbye and thank all my readers for valuable advice. Special thanks in this regard goes to Henrik Bresinsky (Orono, ME) for so many written and oral (during his five month stay in the DDR) suggestions. In conclusion, I thank Professor Wilfried Gerstmeyer, district school board member of the Frankfurt (Oder) district, for the special challenge of aspirants and diploma candidates from his district. For the technical side, I thank finally those responsible for the Scientific Journal, Professor Fritz-Joachim Schütte and Mrs. Ursula Kramm, as well as Miss Carola Sperlich and Mr. Fritz Reinicke (Berlin).

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