Chapter 3

Structure of varieties

Outline:

1. Zariski topology.
2. Irreducible decomposition and dimension.
3. Rational functions.
4. Smooth and singular points.
5. Hilbert functions and dimension
6. Bertini and Bézout Theorems

In Chapter 1 we introduced varieties and ideals and established the algebra-geometry dictionary, and developed basic symbolic algorithms in Chapter 2. We now turn to structural properties of varieties which we will need in subsequent chapters. This begins with the Zariski topology and the notion of genericity, and then the analog of unique factorization for varieties. We introduce rational functions and study maps of projective varieties. After discussing smooth and singular points and tangent spaces, we introduce the notion of dimension. This sets the stage for the fundamental theorems of Bertini-type which deal with the dimension and smoothness of intersections of varieties and their images under maps. This chapter finishes with the Hilbert function and degree of a projective variety and Bézout's Theorem.

3.1 Generic properties of varieties

Many properties in algebraic geometry hold for almost all points of a variety or for almost all objects of a given type. For example, matrices are almost always invertible, univariate polynomials of degree d almost always have d distinct roots, and multivariate polynomials are almost always irreducible. This notion is much stronger than elsewhere in geometry, where 'almost always' may mean the complement of a set of measure zero or the complement of a nowhere dense set. We develop the terminology 'generic' and 'Zariski open' to formalize this notion of almost always in algebraic geometry. Revise this paragraph.

A starting point is the behavior of intersections and unions of affine varieties, which has already had cameo appearences in Chapter 1.

Theorem 3.1.1. *The intersection of any collection of affine varieties is an affine variety. The union of any finite collection of affine varieties is an affine variety.*

Proof. The first statement generalizes Lemma 1.2.11(1). Let $\{I_t \mid t \in T\}$ be a collection of ideals in $\mathbb{K}[x_1, \ldots, x_n]$. Then we have

$$\bigcap_{t\in T} \mathcal{V}(I_t) \;=\; \mathcal{V}\Bigl(\bigcup_{t\in T} I_t\Bigr)\,,$$

as both containments are straightforward. Arguing by induction on the number of varieties shows that it suffices to establish the second statement for the union of two varieties, which is Lemma 1.2.11 (2). □

Theorem 3.1.1 shows that affine varieties have the same properties as the closed sets of a topology on $\mathbb{K}^n$. (See Section A.2 of the Appendix.)

Definition 3.1.2. An affine variety is a *Zariski closed set*. The complement of a Zariski closed set is a *Zariski open set*. The *Zariski topology* on $\mathbb{K}^n$ is the topology whose closed sets are the affine varieties in $\mathbb{K}^n$. The *Zariski closure* of a subset $Z \subset \mathbb{K}^n$ is the smallest variety containing Z, which is $\overline{Z} := \mathcal{V}(\mathcal{I}(Z))$, by Lemma 1.2.4. A subvariety X of $\mathbb{K}^n$ inherits its Zariski topology from $\mathbb{K}^n$, the closed subsets of X are its subvarieties. A subset $Z \subset X$ of a variety X is *Zariski dense* in X if its Zariski closure is X.

Remark 3.1.3. We used these notions implicitly in Chapter 1. You may find it useful to reconsider some of the discussion in light of Zariski open and Zariski closed sets. For example, this terminology allows us to reinterpret the second statement of Theorem 1.3.14 about finite maps: A finite map is *proper*, as it maps closed sets to closed sets. Similarly, Theorem 1.5.6 asserts that maps of projective varieties are proper, as is the projection $\mathbb{P}^n \times \mathbb{K}^m \to \mathbb{K}^m$.

We emphasize that the purpose of this terminology is to aid our discussion of varieties, and not because we will use these notions from topology in an essential way. In a more advanced treatment of algebraic geometry, including sheaves, these topological notions become essential. This Zariski topology behaves quite differently from the usual, or *Euclidean* topology on $\mathbb{R}^n$ or $\mathbb{C}^n$ with which we are familiar, and which is reviewed in Appendix A.2. A topology on a space may be defined by giving a collection of basic open sets which generate the topology. In the Euclidean topology, the basic open sets are (Euclidean) balls. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The *ball* with radius $\epsilon > 0$ centered at $z \in \mathbb{K}^n$ is

$$B(z, \epsilon) \;:=\; \{a \in \mathbb{K}^n \mid \sum |a_i - z_i|^2 \;<\; \epsilon\}\,.$$

In the Zariski topology, the basic open sets are complements of hypersurfaces, called *principal open sets*. Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ and set

$$U_f \;:=\; \{a \in \mathbb{K}^n \mid f(a) \neq 0\}\,. \tag{3.1}$$

In Exercise 4, you are to show that this is an affine variety and to compute its coordinate ring. Such sets occurred in Chapter 1 when discussing local coordinates on projective varieties. In both the Zariski topology and the Euclidean topology the open sets are unions of basic open sets. We give two examples to illustrate the Zariski topology.

Example 3.1.4. The Zariski closed subsets of $\mathbb{K}^1$ consist of the empty set, finite collections of points, and $\mathbb{K}^1$ itself. Thus when $\mathbb{K}$ is infinite the familiar separation property of Hausdorff spaces (any two points are covered by two disjoint open sets) fails spectacularly as any two nonempty open sets meet. ◇

Example 3.1.5. The Zariski topology on a product $X \times Y$ of affine varieties X and Y is in general not the product topology. In the product Zariski topology on $\mathbb{K}^2$, the closed sets are finite unions of sets of the following form: the empty set, points, vertical and horizontal lines $\{a\} \times \mathbb{K}^1$ and $\mathbb{K}^1 \times \{a\}$ for $a \in \mathbb{K}$, and the whole space $\mathbb{K}^2$. On the other hand, $\mathbb{K}^2$ contains a rich collection of other subvarieties (called *plane curves*), such as the cubic plane curves of Section 1.1. ◇

Example 3.1.5 illustrates a general fact: the product Zariski topology on $X \times Y$ is weaker than the Zariski topology on $X \times Y$, when both X and Y are infinite. If $W \subset X$ and $Z \subset Y$ are Zariski closed subsets, then $W \times Z$ is Zariski closed in the product Zariski topology on $X \times Y$, and the same is true for products of Zariski open sets. However, the diagonal $\{(x, x) \mid x \in X\}$ is not closed in the product Zariski topology on $X \times X$, even though it is a subvariety.

We compare the Zariski topology with the Euclidean topology. Recall that a set is nowhere dense in the Euclidean topology if its closure does not contain a ball.

Theorem 3.1.6. *Suppose that $\mathbb{K}$ is one of $\mathbb{R}$ or $\mathbb{C}$. Then*

1. *A Zariski closed set is closed in the Euclidean topology on $\mathbb{K}^n$.*
2. *A Zariski open set is open in the Euclidean topology on $\mathbb{K}^n$.*
3. *A nonempty Euclidean open set is dense in the Zariski topology on $\mathbb{K}^n$.*
4. *$\mathbb{R}^n$ is dense in the Zariski topology on $\mathbb{C}^n$.*
5. *A proper Zariski closed set is nowhere dense in the Euclidean topology on $\mathbb{K}^n$.*
6. *A nonempty Zariski open set is dense in the the Euclidean topology on $\mathbb{K}^n$.*

Proof. For statements 1 and 2, observe that a Zariski closed set $\mathcal{V}(I)$ is the intersection of the hypersurfaces $\mathcal{V}(f)$ for $f \in I$, so it suffices to show this for a hypersurface $\mathcal{V}(f)$. But then Statement 1 (and hence also 2) follows as the polynomial function $f \colon \mathbb{K}^n \to \mathbb{K}$ is continuous in the Euclidean topology, and $\mathcal{V}(f) = f^{-1}(0)$.

Any ball $B(z, \epsilon)$ is dense in the Zariski topology. If a polynomial f vanishes identically on $B(z, \epsilon)$, then all of its partial derivatives do as well. Thus its Taylor series expansion at

z is identically zero. But then f is the zero polynomial. This shows that $\mathcal{I}(B(z,\epsilon)) = \{0\}$, and so $\mathcal{V}(\mathcal{I}(B(z,\epsilon))) = \mathbb{K}^n$, that is, $B(z,\epsilon)$ is dense in the Zariski topology on $\mathbb{K}^n$.

Statement 4 uses the same argument. If a polynomial vanishes on $\mathbb{R}^n$, then all of its partial derivatives vanish and so f must be the zero polynomial. Thus $\mathcal{I}(\mathbb{R}^n) = \{0\}$ and $\mathcal{V}(\mathcal{I}(\mathbb{R}^n)) = \mathbb{C}^n$. In fact, we may replace $\mathbb{R}^n$ by any set containing a Euclidean ball.

For statements 5 and 6, observe that if f is nonconstant, then by 4, the Euclidean closed set $\mathcal{V}(f)$ does not contain a Euclidean ball so $\mathcal{V}(f)$ is nowhere dense. A variety is an intersection of nowhere dense hypersurfaces, so varieties (Zariski closed sets) are nowhere dense. The complement of a nowhere dense set is dense, so nonempty Zariski open sets are dense in $\mathbb{K}^n$. $\square$

Theorem 3.1.6(6) leads to the useful notions of genericity and generic sets and properties. We will use the term "Zariski dense" for dense in the Zariski topology.

Definition 3.1.7. Let X be a variety. A subset $Y \subset X$ is *generic* if it contains a Zariski dense open subset U of X. That is, Y contains a Zariski open set U that is dense in X, $\overline{U} = X$. A property is *generic* if the set of points on which it holds is a generic set. Points of a generic set are called *general* points.

Our notion of which points are general depends on the context, and so care must be exercised in the use of these terms. For example, we may identify $\mathbb{K}^3$ with the set of quadratic polynomials in x via

$$(a,b,c) \longmapsto ax^2 + bx + c\,.$$

Then the general quadratic polynomial does not vanish when $x = 0$. (We just need to avoid quadratics with $c = 0$.) On the other hand, the general quadratic polynomial has two roots, as we need only avoid quadratics with $b^2 - 4ac = 0$. The quadratic $x^2 - 2x + 1$ is general in the first sense, but not in the second, while the quadratic $x^2 + x$ is general in the second sense, but not in the first. Despite this ambiguity due to its reliance on context, general is a very useful concept.

When $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, generic sets are dense in the Euclidean topology, by Theorem 3.1.6(6). Thus generic properties hold almost everywhere, in the standard sense.

Example 3.1.8. A general $n \times n$ matrix is invertible, as invertible matrices form a nonempty principal open subset of $\mathrm{Mat}_{n\times n}(\mathbb{K})$. It is the complement of the variety $\mathcal{V}(\det)$ of singular matrices. The *general linear group* GL_n is the set of all invertible matrices,

$$GL_n \;:=\; \{M \in \mathrm{Mat}_{n\times n} \mid \det(M) \neq 0\} \;=\; U_{\det}\,. \qquad \diamond$$

Example 3.1.9. A general univariate polynomial of degree n has n distinct complex roots. Identify $\mathbb{K}^n$ with the set of univariate polynomials of degree n via

$$(a_1,\dots,a_n) \;\in\; \mathbb{K}^n \;\longmapsto\; x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \;\in\; \mathbb{K}[x]\,. \tag{3.2}$$

The classical discriminant $\mathrm{disc}_n \in \mathbb{K}[a_1,\dots,a_n]$ (see Example 2.1.5) is a polynomial of degree $2n-2$ which vanishes precisely when the polynomial (3.2) has a repeated factor. This identifies the set of polynomials with n distinct complex roots as the set U_{disc}. The discriminant of the quadratic x^2+bx+c is b^2-4c. $\diamond$

Example 3.1.10. A general complex $n \times n$ matrix is semisimple (diagonalizable). We do not show this by providing an algebraic characterization of semisimplicity. Instead we observe that if a matrix $M \in \mathrm{Mat}_{n\times n}$ has n distinct eigenvalues, then it is semisimple. Let $M \in \mathrm{Mat}_{n\times n}$ and consider the (monic) characteristic polynomial of M

$$\chi(x) \ := \ \det(xI_n - M)\,,$$

whose roots are the eigenvalues of M. The coefficients of the characteristic polynomial $\chi(x)$ are polynomials in the entries of M. Evaluating the discriminant at these coefficients gives a polynomial ψ which vanishes when the characteristic polynomial $\chi(x)$ of M has a repeated root.

It follows that the set of matrices with distinct eigenvalues equals the principal open set U_ψ, which is nonempty. Thus the set of semisimple matrices contains an open dense subset of $\mathrm{Mat}_{n\times n}$ and is therefore generic.

When $n=2$, the characteristic polynomial of a generic matrix is

$$\det\left(xI_2 \ - \ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\right) \ = \ x^2 - x(a_{11}+a_{22}) + a_{11}a_{22} - a_{12}a_{21}\,,$$

and so the polynomial ψ is $(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})$. $\diamond$

In each of these examples, we used the following easy fact.

Proposition 3.1.11. *A set $X \subset \mathbb{K}^n$ is generic if and only if there is a nonconstant polynomial that vanishes on its complement, if and only if it contains a principal open set U_f.*

If $X \subset \mathbb{K}^n$ is a variety and $f \in \mathbb{K}[x]$ is a polynomial which is not identically zero on X ($f \notin \mathcal{I}(X)$), then we have the *principal open subset* of X,

$$X_f \ := \ X \smallsetminus \mathcal{V}(f) \ = \ \{x \in X \mid f(x) \neq 0\}\,. \tag{3.3}$$

In Exercise 4, you area sked to show that X_f is an affine variety, and to compute its coordinate ring. Some of this was done in Section 1.4 or 1.5.

Lemma 3.1.12. *Any Zariski open subset U of a variety X is a finite union of principal open subsets.*

Proof. The complement $Y := X \smallsetminus U$ of a Zariski open subset U of X is a Zariski closed subset. The ideal $\mathcal{I}(Y)$ of Y in $\mathbb{K}^n$ contains the ideal $\mathcal{I}(X)$ of X. By the Hilbert Basis Theorem, there are polynomials $f_1,\dots,f_m \in \mathcal{I}(Y)$ such that

$$\mathcal{I}(Y) \ = \ \langle \mathcal{I}(X),\ f_1,\dots,f_m\rangle\,.$$

Then $X_{f_1} \cup \cdots \cup X_{f_m}$ is equal to

$$(X \smallsetminus \mathcal{V}(f_1)) \cup \cdots \cup (X \smallsetminus \mathcal{V}(f_m)) \ = \ X \smallsetminus (\mathcal{V}(f_1) \cap \cdots \cap \mathcal{V}(f_m)) \ = \ X \smallsetminus Y \ = \ U \,. \quad \square$$

In Section 1.4, we introduced the cover of projective space $\mathbb{P}^n = U_0 \cup U_1 \cup \cdots \cup U_n$ by affine charts, where for each $i = 0, \ldots, n$, U_i is the principal open set U_{x_i},

$$U_i \ = \ U_{x_i} \ := \ \{a = [a_0, a_1, \ldots, a_n] \in \mathbb{P}^n \mid a_i \neq 0\} \ \simeq \ \mathbb{K}^n \,.$$

By Lemma 1.4.13, a subset $X \subset \mathbb{P}^n$ is a projective variety if and only if each intersection $X \cap U_i$ is an affine variety, for each $i = 0, \ldots, n$. Thus every projective variety X is covered by affine varieties, $X = X_0 \cup X_1 \cup \cdots \cup X_n$, where $X_i := X \cap U_{x_i}$. We also introduced principal affine subsets U_f and X_f for f a form. We may define the Zariski topology on projective varieties in two equivalent ways: Either extend the definition of Zariski topology to projective space (projective varieties are the closed sets) or use such affine covers of projective varieties to define basic open subsets to generate the topology. Lemma 1.4.13 shows that these are equivalent. A subset $Z \subset X$ of a projective variety X is Zariski closed if and only if it is closed in each principal affine set $X_i = X \cap U_i$ of X. Note that Lemma 3.1.12 holds for projective varieties. Should consolidate this discussion with the previous lemma.

We expand our notion of a variety. A subset $X \subset \mathbb{P}^n$ is a *quasi-projective variety* if it is an open subset of its closure in $\mathbb{P}^n$. That is, if there are projective subvarieties Y, Z of $\mathbb{P}^n$ with $X = Y \smallsetminus Z$. A quasi-projective variety inherits its Zariski topology from that of projective space. A Zariski closed subset of a quasi-projective variety X is its intersection with a projective subvariety Y, and the same for a Zariski open subset of X. A subvariety of a quasi-projective variety X is a Zariski closed subset $Y \subset X$; this will also be a quasi-projective variety. The notion of generic introduced for affine varieties also makes sense for quasi-projective varieties, and it has the same properties. For example, Lemma 3.1.12 holds for quasi-projective varieties. We will henceforth often drop the adjective quasi-projective and simply refer to these as varieties.

We use the existence of a finite affine cover of a variety to define and establish properties of varieties. Any property of a variety that holds under restriction to affine open subsets and may be detected through such restrictions is called *local*. For example, suppose that $\varphi\colon X \to Y$ is a regular map of affine varieties and let $f \in \mathbb{K}[Y]$ be a regular function on Y. Then its pullback $\varphi^*(f)$ is a regular function on X, and we have an induced homomorphism

$$\varphi^* \ : \ \mathbb{K}[Y_f] = \mathbb{K}[Y][\tfrac{1}{f}] \ \longrightarrow \ \mathbb{K}[X_{\varphi^*(f)}] = \mathbb{K}[X][\tfrac{1}{\varphi^*(f)}] \,.$$

Thus maps of affine varieties restrict to principal open subsets. In the exercises, you will show how to glue together maps on a finite affine cover. Being a regular map of affine varieties is a local property.

Let us use this to give a uniform definition of regular map between algebraic varieties. Suppose that $\mathbb{K}$ is algebraically closed, and that X and Y are varieties. A regular map

$\varphi\colon X \to Y$, is a function $\varphi\colon X \to Y$ such that there exists an affine cover $\{V_i \mid i \in I\}$ of Y, and for each $i \in I$ an affine cover $\{U_{i,j} \mid j \in J_i\}$ of $\varphi^{-1}(V_i)$ such that for every i, j, the restriction $\varphi|_{U_{i,j}}\colon U_{i,j} \to V_i$ is a regular map of affine varieties. Observe that this holds for the different types of maps of varieties discussed in Chapter 1. use different indexing sets!

Our definition of finite maps of projective varieties used that the property of a map being finite is local. We illustrate the utility of affine covers by proving a useful property of maps of varieties.

Theorem 3.1.13. *Suppose that $\varphi\colon X \to Y$ is a regular map of varieties with $\overline{\varphi(X)} = Y$. Then $\varphi(X)$ contains a non-empty open subset of Y.*

Since X and Y have finite affine covers $X = \cup U_{i,j}$ and $Y = \cup V_i$ with $\varphi^{-1}(V_i) = \cup_j U_{i,j}$ such that for each i, j the restriction $\varphi|_{U_{i,j}} \to V_i$ us a regular map of affine varieties, there is some i, j such that $\overline{\varphi(U_{i,j})} = V_i$ (the closure being taken in V_i). Thus it suffices to prove the theorem for regular maps of affine varieties. When $\varphi\colon X \to Y$ is a regular map of affine varieties with $\overline{\varphi(X)} = Y$, we have an injection $\varphi^*\colon \mathbb{K}[Y] \hookrightarrow \mathbb{K}[X]$, by Lemma 1.3.11. We regard $K[Y]$ as a subalgebra of $\mathbb{K}[X]$.

Lemma 3.1.14. *There exist $r \geq 0$ and elements $u_1, \ldots, u_r \in \mathbb{K}[X]$ such that*

1. *$u_1, \ldots, u_r$ are algebraically independent over $\mathbb{K}[Y]$.*
2. *Every element of $\mathbb{K}[X]$ is algebraically dependent on $\mathbb{K}[Y]$ and $u_1, \ldots, u_r$.*

After the exercises for this section, the Zariski topology is the default topology; "open" means Zariski open and "closed" means Zariski closed.

Exercises

These are in the wrong order

1. Verify the claim that the collection of affine subvarieties of $\mathbb{K}^n$ form the closed sets in a topology on $\mathbb{K}^n$. (See Section A.2 of the Appendix for definitions.)

2. Prove that a closed set in the Zariski topology on $\mathbb{K}^1$ is either the empty set, a finite collection of points, or $\mathbb{K}^1$ itself.

3. (a) Verify the claim in Example 3.1.5 about the closed sets in the product Zariski topology on $\mathbb{K}^1 \times \mathbb{K}^1$.
   (b) Show that any open set in the product Zariski topology on $\mathbb{K}^1 \times \mathbb{K}^1$ is Zariski open in $\mathbb{K}^2$.
   (c) Find a Zariski open set in $\mathbb{K}^2$ which is not open in the product topology on $\mathbb{K}^1 \times \mathbb{K}^1$.

4. Show that the principal open subset U_f (3.1) is an affine variety by identifying it with $\mathcal{V}(yf-1) \subset \mathbb{K}^{n+1}$. Show that its coordinate ring is $\mathbb{K}[x][\frac{1}{f}]$, the localization of the polynomial ring at f. Deduce that a principal open subset X_f (3.3) of an affine variety is an affine variety.

5. (a) Show that the Zariski topology in $\mathbb{K}^n$ is not Hausdorff if $\mathbb{K}$ is infinite.
   (b) Prove that any nonempty Zariski open subset of $\mathbb{K}^n$ is dense.
   (c) Prove that $\mathbb{K}^n$ is compact in the Zariski topology.

6. Prove that the general triple of points in $\mathbb{R}^2$ are the vertices of a triangle.

7. Suppose that $n \leq m$. Prove that a general $n \times m$ matrix has rank n.

8. Prove that if $W \subset X$ and $Z \subset Y$ are subvarieties of the varieties X and Y, respectively, then $W \times Z$ is closed in the product Zariski topology on $X \times Y$, and that $W \times Z$ is a subvariety of $X \times Y$. Prove that if X is an affine variety, then the diagonal $\{(x,x) \mid x \in X\}$ is a subvariety of $X \times X$.

3.2 Unique factorization for varieties

Make sure to show that a hypersurface has dimension $n-1$.

Every polynomial factors uniquely as a product of irreducible polynomials. A basic structural result about algebraic varieties is an analog of this unique factorization. Any algebraic variety is the finite union of irreducible varieties, and this decomposition is unique.

A polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ is *reducible* if we may factor f nontrivially, that is, if $f = gh$ with neither g nor h a constant polynomial. Otherwise f is *irreducible*. Any polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ may be factored

$$f \;=\; c g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_m^{\alpha_m} \tag{3.4}$$

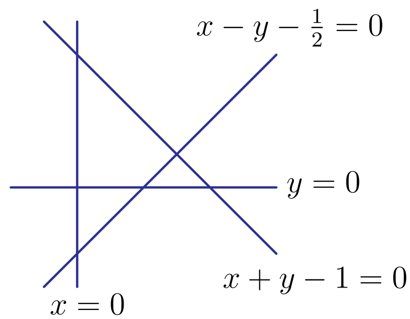
where c is a constant, the exponents α_i are positive integers, each polynomial g_i is irreducible and non-constant, and when $i \neq j$ the polynomials g_i and g_j are not proportional. The factorization (3.4) is essentially unique as any other such factorization is obtained from it by permuting the factors and possibly multiplying each polynomial g_i by a constant. The polynomials g_j are the *irreducible factors* of f.

When $\mathbb{K}$ is algebraically closed, this algebraic property has a consequence for the geometry of hypersurfaces in $\mathbb{K}^n$. Suppose that a polynomial f has a factorization (3.4) into irreducible polynomials. Then the hypersurface $X = \mathcal{V}(f)$ is the union of hypersurfaces $X_i := \mathcal{V}(g_i)$, and this decomposition

$$X \;=\; X_1 \cup X_2 \cup \cdots \cup X_m$$

of X into hypersurfaces X_i defined by irreducible polynomials is unique.

For example, $\mathcal{V}(xy^2(x{+}y{-}1)^3(x{-}y{-}\frac{1}{2}))$ is the union of four lines in $\mathbb{K}^2$.



We will show that this decomposition property is shared by general varieties.

Definition 3.2.1. A variety X is *reducible* if it is the union $X = Y \cup Z$ of proper closed subvarieties $Y, Z \subsetneq X$. Otherwise X is *irreducible*. In particular, if an irreducible variety is written as a union of subvarieties $X = Y \cup Z$, then either $X = Y$ or $X = Z$.

Example 3.2.2. Figure 1.2 in Section 1.2 shows that $\mathcal{V}(xy+z,\ x^2-x+y^2+yz)$ consists of two space curves, each of which is a variety in its own right. Thus it is reducible. To see this, we solve the two equations $xy+z = x^2-x+y^2+yz = 0$. First note that

$$x^2-x+y^2+yz \;-\; y(xy+z) \;=\; x^2-x+y^2-xy^2 \;=\; (x-1)(x-y^2)\,.$$

Thus either $x=1$ or else $x=y^2$. When $x=1$, we have $y+z=0$ and these equations define the line in Figure 1.2. When $x=y^2$, we get $z=-y^3$, and these equations define the cubic curve parameterized by $(t^2, t, -t^3)$.

Figure 3.1 shows another reducible variety. It has six components, one is a surface,

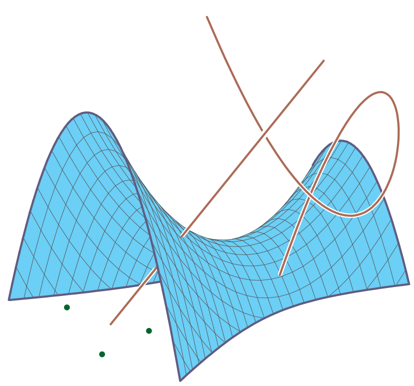


Figure 3.1: A reducible variety

two are space curves, and three are points. $\diamond$

Theorem 3.2.3. *A product $X \times Y$ of irreducible varieties is irreducible.*

Proof. Suppose that $Z_1, Z_2 \subset X \times Y$ are subvarieties with $Z_1 \cup Z_2 = X \times Y$. We assume that $Z_2 \neq X \times Y$ and use this to show that $Z_1 = X \times Y$. For each $x \in X$, identify the subvariety $\{x\} \times Y$ with Y. This irreducible variety is the union of two subvarieties,

$$\{x\} \times Y \;=\; \big((\{x\} \times Y) \cap Z_1\big) \;\cup\; \big((\{x\} \times Y) \cap Z_2\big)\,,$$

and so one of these must equal $\{x\} \times Y$. In particular, we must either have $\{x\} \times Y \subset Z_1$ or else $\{x\} \times Y \subset Z_2$. If we define

$$\begin{aligned} X_1 &= \{x \in X \mid \{x\} \times Y \subset Z_1\}\,, \quad \text{and} \\ X_2 &= \{x \in X \mid \{x\} \times Y \subset Z_2\}\,, \end{aligned}$$

then we have just shown that $X = X_1 \cup X_2$. Since $Z_2 \neq X \times Y$, we have $X_2 \neq X$. We claim that both X_1 and X_2 are subvarieties of X. Then the irreducibility of X implies that $X = X_1$ and thus $X \times Y = Z_1$.

It suffices to show that X_1 is a subvariety of X. For $y \in Y$, set

$$X_y \;:=\; \{x \in X \mid (x, y) \in Z_1\}\,.$$

Since $X_y \times \{y\} = (X \times \{y\}) \cap Z_1$, we see that X_y is a subvariety of X. But we have

$$X_1 \;=\; \bigcap_{y \in Y} X_y\,,$$

which shows that X_1 is a subvariety of X and completes the proof. $\square$

The geometric notion of an irreducible variety corresponds to the algebraic notion of a prime ideal. An ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is *prime* if whenever $fg \in I$ with $f \notin I$, then we have $g \in I$. Equivalently, if whenever $f, g \notin I$ then $fg \notin I$.

Theorem 3.2.4. *An affine variety X is irreducible if and only if its ideal $\mathcal{I}(X)$ is prime.*

Proof. Let X be a variety. First suppose that X is irreducible. Let $f, g \notin \mathcal{I}(X)$. Then neither f nor g vanishes identically on X. Thus $Y := X \cap \mathcal{V}(f)$ and $Z := X \cap \mathcal{V}(z)$ are proper subvarieties of X. Since X is irreducible, $Y \cup Z = X \cap \mathcal{V}(fg)$ is also a proper subvariety of X, and thus $fg \notin \mathcal{I}(X)$.

Suppose now that X is reducible. Then $X = Y \cup Z$ is the union of proper subvarieties Y, Z of X. Since $Y \subsetneq X$ is a subvariety, we have $\mathcal{I}(X) \subsetneq \mathcal{I}(Y)$. Let $f \in \mathcal{I}(Y) - \mathcal{I}(X)$, a polynomial which vanishes on Y but not on X. Similarly, let $g \in \mathcal{I}(Z) - \mathcal{I}(X)$ be a polynomial which vanishes on Z but not on X. Since $X = Y \cup Z$, fg vanishes on X and therefore lies in $\mathcal{I}(X)$. This shows that $\mathcal{I}(X)$ is not prime. $\square$

As a principal ideal $\langle f\rangle$ for $f \in \mathbb{K}[x_1,\ldots,x_n]$ is prime if and only if f is irreducible, Theorem 3.2.4 implies that the unique decomposition of hypersurfaces into unions of hypersurfaces defined by irreducible polynomials is a decomposition of a hypersurface into irreducible hypersurfaces.

We have seen examples of varieties with one, two, four, and six irreducible components. Taking products of distinct irreducible polynomials (or dually unions of distinct hypersurfaces), yields varieties having any finite number of irreducible components. This is all that can occur as Hilbert's Basis Theorem implies that a variety is a union of finitely many irreducible varieties.

Lemma 3.2.5. *Any affine variety is a finite union of irreducible closed subvarieties.*

Proof. An affine variety X either is irreducible or else we have $X = Y \cup Z$, with both Y and Z proper subvarieties of X. We may similarly decompose whichever of Y and Z is reducible, and continue this process, stopping only when all subvarieties obtained are irreducible. *A priori*, this process could continue indefinitely. We show that it must stop after a finite number of steps.

If this process never stops, then X must contain an infinite chain of subvarieties, each properly contained in the previous,

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots .$$

Their ideals form an infinite increasing chain of ideals in $\mathbb{K}[x_1,\ldots,x_n]$,

$$\mathcal{I}(X) \subsetneq \mathcal{I}(X_1) \subsetneq \mathcal{I}(X_2) \subsetneq \cdots .$$

The union I of these ideals is again an ideal. No ideal $\mathcal{I}(X_m)$ is equal to I as the chain of ideals is strict. By the Hilbert Basis Theorem, I is finitely generated, and thus there is some integer m for which $\mathcal{I}(X_m)$ contains these generators. But then $I = \mathcal{I}(X_m)$, a contradiction. $\square$

Lemma 3.2.6. *Let X be a variety with $U \subset X$ a quasiprojective variety that is dense in X. Then X is irreducible if and only if U is irreducible.*

Proof. Assume that U is irreducible and suppose that $X = Y \cup Z$ is the union of two closed subvarieties. Then $U = (U \cap Y) \cup (U \cap Z)$ is the union of two closed subvarieties. As U is irreducible, we may assume that $U = U \cap Y$, but then $X = \overline{U} \subset Y$.

Now assume that X is irreducible and suppose that $U = V \cup W$ is union of two closed subvarieties of U. Then $X = \overline{U} = \overline{V} \cup \overline{W}$ is the union of two closed subvarieties. As X is irreducible, we may assume that $X = \overline{V}$, but then $U = U \cap \overline{V} = V$. $\square$

Corollary 3.2.7. *A variety X is a finite union of irreducible subvarieties.*

Proof. Suppose that $X \subset \mathbb{P}^n$ is a projective variety. Then $X = X_0 \cup X_1 \cup \cdots \cup X_n$ where $X_i = X \cap U_i$. Each affine variety X_i is a finite union of irreducible closed subvarieties

$U_{i,1}, \ldots, U_{i,m_i}$. By Lemma 3.2.6, the closure in $\mathbb{P}^n$ of each $U_{i,j}$ is irreducible. Noting that X equals the union of the closures of the $U_{i,j}$ shows that X is a finite union of irreducible closed subvarieties.

If $X \subset \mathbb{P}^n$ is a quasi-projective variety, then its Zariski closure is a projective variety. Thus $\overline{X}$ may be written as a finite union of irreducible closed subvarieties. The intersection of each of these with X is an irreducible closed subvariety of X, by Lemma 3.2.6. Noting that X is the union of these intersections completes the proof. $\square$

A consequence of the proof of Lemma 3.2.5 and of Corollary 3.2.7 is that any decreasing chain of subvarieties of a given variety must have finite length. When $\mathbb{K}$ is infinite, there are such decreasing chains of arbitrary length. There is however a bound for the length of the longest decreasing chain of irreducible subvarieties.

Definition 3.2.8 (Combinatorial Definition of Dimension)**.** The *dimension* of a variety X is the length of the longest decreasing chain of irreducible subvarieties of X. If

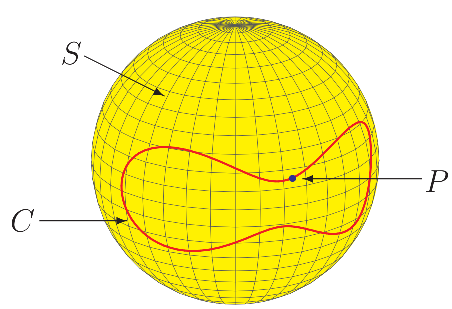
$$X \supset X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_m \supsetneq \emptyset, \tag{3.5}$$

with each X_i irreducible is such a chain of maximal length, then X has dimension m and we write $\dim X = m$. If (3.5) has length $m = \dim X$, the X_i has dimension $m - i$.

The discussion on elimination of bivariate polynomials in Section 2.1, takes care of making this well-founded, at least on $\mathbb{K}^2$.

Since maximal ideals of $\mathbb{C}[x_1, \ldots, x_n]$ have the form $\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $a \in \mathbb{C}^n$, we see that X_m is a point when $\mathbb{K} = \mathbb{C}$. The only problem with this definition is that we cannot yet show that it is well-founded, as we do not yet know that there is a bound on the length of such a chain. In Section 3.5 we shall prove that this definition is correct by relating it to other notions of dimension.

Example 3.2.9. The sphere S has dimension at least two, as we have the chain of irreducible subvarieties $S \supsetneq C \supsetneq P$ as shown below.



It is challenging to show that any maximal chain of irreducible subvarieties of the sphere has length 2 with what we now know. $\diamond$

By Corollary 3.2.7, a variety X may be written as a finite union

$$X \;=\; X_1 \,\cup\, X_2 \,\cup\, \cdots \,\cup\, X_m$$

of irreducible closed subvarieties. We may assume this is irredundant in that if $i \neq j$ then X_i is not a subvariety of X_j. If we did have $i \neq j$ with $X_i \subset X_j$, then we may remove X_i from the decomposition. We prove that this decomposition is unique, which is the main result of this section and a basic structural result about varieties.

Theorem 3.2.10 (Unique Decomposition of Varieties)**.** *A variety X has a unique irredundant decomposition as a finite union of irreducible closed subvarieties*

$$X \;=\; X_1 \,\cup\, X_2 \,\cup\, \cdots \,\cup\, X_m\,.$$

We call these distinguished subvarieties X_i the *irreducible components* of X.

Proof. Suppose that we have another irredundant decomposition into irreducible closed subvarieties,

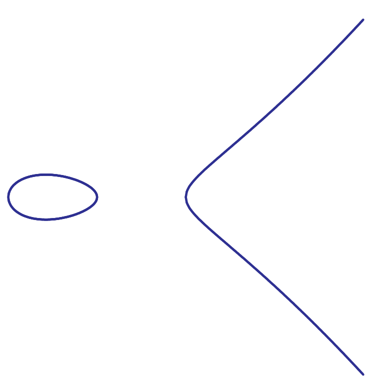
$$X \;=\; Y_1 \,\cup\, Y_2 \,\cup\, \cdots \,\cup\, Y_n\,,$$

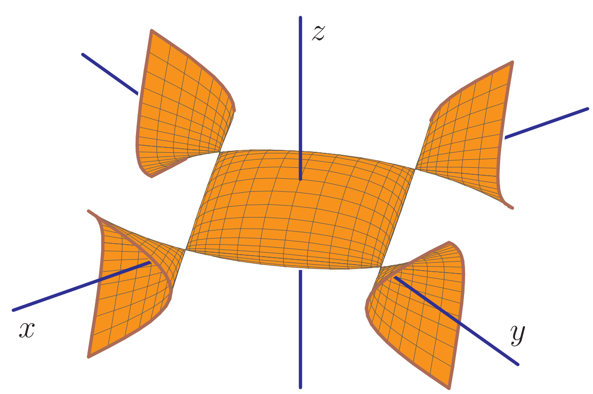
where Y_j is irreducible and closed. Then for each $i = 1, \dots, m$,

$$X_i \;=\; (X_i \cap Y_1) \,\cup\, (X_i \cap Y_2) \,\cup\, \cdots \,\cup\, (X_i \cap Y_n)\,.$$

Since X_i is irreducible, one of these must equal X_i, which means that there is some index j with $X_i \subset Y_j$. Similarly, there is some index k with $Y_j \subset X_k$, so that $X_i \subset X_k$. But then $i = k$, and so $X_i = Y_j$. This implies that $n = m$ and that the second decomposition differs from the first solely by permuting the terms. □

When $\mathbb{K} = \mathbb{C}$, we will show[†] that an irreducible variety is connected in the usual Euclidean topology. We will even show that the smooth points of an irreducible variety are connected. Neither of these facts are true over $\mathbb{R}$. Below, we display the irreducible cubic plane curve $\mathcal{V}(y^2 - x^3 + x)$ in $\mathbb{R}^2$ and the surface $\mathcal{V}((x^2 - y^2)^2 - 2x^2 - 2y^2 - 16z^2 + 1)$ in $\mathbb{R}^3$.





[†]When and where will we show this?

Both are irreducible hypersurfaces. The first has two connected components in the Euclidean topology, while in the second, the five components of smooth points meet at the four singular points. (While intuitive, smooth and singular points will be defined in Section 3.4.)

Theorem 3.2.11. *Suppose that X is irreducible and $f\colon X \to Y$ is a map. Then $\overline{f(X)}$ is an irreducible subvariety of Y.*

Proof. Suppose that the closure Z of $f(X)$ is the union of two subvarieties, $Z = Z_1 \cup Z_2$, with $Z_2 \neq Z$. For each $i = 1, 2$, set $X_i := f^{-1}(Z_i)$, which is closed in X and thus a subvariety. Then $X = X_1 \cup X_2$. Since $Z_2 \neq Z$, we have $X_2 \neq X$. As X is irreducible, this implies that $X = X_1$, and therefore that $Z = Z_1$. □

We close this section with a proof that the resultant polynomial of Section 2.1 is irreducible. This uses facts about irreducibility and dimension that we have not yet established. We should also be able to do the discriminant.

Remark 3.2.12. Suppose that $\mathbb{K}$ is algebraically closed and consider the variety of all triples consisting of a pair of univariate polynomials with a common root, together with a common root,

$$\Sigma \ := \ \{(f, g, a) \in \mathbb{K}_m[x] \times \mathbb{K}_n[x] \times \mathbb{K} \mid f(a) = g(a) = 0\}\,,$$

where $\mathbb{K}_r[x]$ is the $(r{+}1)$-dimensional vector space of polynomials of degree r. This has projections $p\colon \Sigma \to \mathbb{K}_m[x] \times \mathbb{K}_n[x]$ and $\pi\colon \Sigma \to \mathbb{K}$. The image $p(\Sigma)$ is the set of pairs of polynomials having a common root, which is the variety $\mathcal{V}(\mathrm{Res})$ of the resultant polynomial, $\mathrm{Res} \in \mathbb{Z}[f_0, \ldots, f_m, g_0, \ldots, g_n]$, where $f_0, \ldots, g_n$ are the coefficients of f and g,

$$f \ = \ f_0x^m + f_1x^{m-1} + \cdots + f_m \quad \text{and} \quad g \ = \ g_0x^n + g_1x^{n-1} + \cdots + g_n\,.$$

The fiber of π over a point $a \in \mathbb{K}$ consists all pairs of polynomials f, g with $f(a) = g(a) = 0$. Since each equation is linear in the coefficients of the polynomials f and g, this fiber is isomorphic to $\mathbb{K}^m \times \mathbb{K}^n$. Since $\pi\colon \Sigma \to \mathbb{K}$ has irreducible image ($\mathbb{K}$) and irreducible fibers, we see that Σ is irreducible, and has dimension $1 + m + n$.

This implies that $p(\Sigma)$ is irreducible. Furthermore, the fiber $p^{-1}(f, g)$ is the set of common roots of f and g. This is a finite set when $f, g \neq (0, 0)$. Thus $p(\Sigma)$ has dimension $1{+}m{+}n$, and is thus an irreducible hypersurface in $\mathbb{K}_m[x] \times \mathbb{K}_n[x]$. Let F be a polynomial generating the ideal $\mathcal{I}(p(\Sigma))$, which is necessarily irreducible. As $\mathcal{V}(\mathrm{Res}) = p(\Sigma)$, we must have $\mathrm{Res} = F^N$ for some positive integer N. The formula (2.3) shows that $N = 1$ as the resultant polynomial is square-free.

We only need to show that the greatest common divisor of the coefficients of the integer polynomial Res is 1. But this is clear as Res contains the term $f_0^n g_n^m$ with coefficient 1, as we showed in the proof of Lemma 2.1.3. ◇

Exercises

1. Show that the ideal of a hypersurface $\mathcal{V}(f)$ is generated by the *squarefree* part of f, which is the product of the irreducible factors of f, each with exponent 1.

2. Suppose that $\mathbb{K}$ is infinite. For every positive integer n, give a decreasing chain of subvarieties of $\mathbb{K}^1$ of length $n+1$.

3. Suppose that $I_1 \subset I_2 \subset \cdots$ is an increasing chain of ideals in $\mathbb{K}[x_1, \ldots, x_n]$. Show that its union is an ideal of $\mathbb{K}[x_1, \ldots, x_n]$.

4. Prove that the dimension of a point is 0 and the dimension of $\mathbb{K}^1$ is 1.

5. Show that an irreducible affine variety is zero-dimensional if and only if it is a point.

6. Prove that the dimension of an irreducible plane curve is 1 and use this to show that the dimension of $\mathbb{K}^2$ is 2.

7. Write the ideal $\langle x^3 - x, x^2 - y\rangle$ as the intersection of two prime ideals. Describe the corresponding geometry.

8. Show that $f(x, y) = y^2 + x^2(x-1)^2 \in \mathbb{R}[x, y]$ is an irreducible polynomial but that $V(f)$ is reducible.

9. Suppose that f and g are two polynomials on $\mathbb{C}^n$ that are relatively prime. Show that every component of $\mathcal{V}(f, g)$ has dimension $n-2$.

10. Let $f(x, y)$ be a polynomial of total degree n. Show that there is a non-empty Zariski open subset of parameters $(a, b, c, \alpha, \beta, \gamma) \in \mathbb{K}^6$ with $a\beta - \alpha b \neq 0$ such that if A is the affine transformation (2.10), then every monomial $x^i y^j$ with $0 \leq i, j$ and $i + j \leq n$ appears in the polynomial $f(A(x, y))$ with a non-zero coefficient.

11. Use Lemma 2.1.13 to show that $\mathbb{K}^2$ has dimension 2, in the sense of the combinatorial definition of dimension (3.2.8).

12. Use Lemma 2.1.13 and induction on the number of polynomials defining a proper subvariety X of $\mathbb{K}^2$ to show that X consists of finitely many irreducible curves and finitely many isolated points.

3.3 Rational functions and maps

This should be folded into the previous section.

In algebraic geometry, we also use functions and maps between varieties which are not defined at all points of their domains. Working with functions and maps not defined at all points is a special feature of algebraic geometry that sets it apart from other branches of geometry.

Suppose X is any irreducible affine variety. By Theorem 3.2.4, its ideal $\mathcal{I}(X)$ is prime, so its coordinate ring $\mathbb{K}[X]$ has no zero divisors ($0 \neq f, g \in \mathbb{K}[X]$ with $fg = 0$). A ring without zero divisors is called an *integral domain*. In exact analogy with the construction of the rational numbers $\mathbb{Q}$ as quotients of integers $\mathbb{Z}$, we may form the *function field* $\mathbb{K}(X)$ of X as the quotients of regular functions in $\mathbb{K}[X]$. Formally, $\mathbb{K}(X)$ is the collection of all quotients f/g with $f, g \in \mathbb{K}[X]$ and $g \neq 0$, where we identify

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} \iff f_1 g_2 - f_2 g_1 = 0 \text{ in } \mathbb{K}[X]\,.$$

The map $f \mapsto \frac{f}{1}$ embeds $K[X]$ into the function field $\mathbb{K}(X)$.

Example 3.3.1. The function field of affine space $\mathbb{K}^n$ is the collection of quotients of polynomials P/Q with $P, Q \in \mathbb{K}[x_1, \dots, x_n]$. This field $\mathbb{K}(x_1, \dots, x_n)$ is called the *field of rational functions* in the variables $x_1, \dots, x_n$. $\diamond$

Given an irreducible affine variety $X \subset \mathbb{K}^n$, we may also express $\mathbb{K}(X)$ as the collection of quotients f/g of polynomials $f, g \in \mathbb{K}[x_1, \dots, x_n]$ with $g \notin \mathcal{I}(X)$, where we identify

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} \iff f_1 g_2 - f_2 g_1 \in \mathcal{I}(X)\,.$$

Rational functions on an affine variety X do not in general have unique representatives as quotients of polynomials or even as quotients of regular functions.

Example 3.3.2. Let $X := \mathcal{V}(x^2 + y^2 + 2y) \subset \mathbb{K}^2$ be the circle of radius 1 and center at $(0, -1)$. In $\mathbb{K}(X)$ we have

$$-\frac{x}{y} = \frac{y+2}{x}\,.$$ $\diamond$

In Chapter 1, we showed that an affine variety is determined up to embedding in affine space by its coordinate ring, and that there is an equivalence of categories between affine varieties and finitely generated reduced $\mathbb{K}$-algebras. There is not as tight of a correspondence between irreducible varieties and their fields of rational functions. This however enables us to define fields of rational functions for arbitrary irreducible varieties.

Proposition-Definition 3.3.3. *Let X be an irreducible variety and $U, V \subset X$ non-empty affine open subvarieties of X. Then their function fields are equal, $\mathbb{K}(U) = \mathbb{K}(V)$, and we define the function field $\mathbb{K}(X)$ to be this common field.*

Thus the function field of an irreducible variety X depends rather weakly on X as any affine open subset has the same function field.

Proof. Suppose first that $X \subset \mathbb{K}^n$ is is affine. As U is open, there is some $f \in \mathbb{K}[X]$ with $X \smallsetminus U \subset \mathcal{V}(F)$, so that $X_f = X \smallsetminus \mathcal{V}(f) \subset U$. By Exercise 1, $\mathbb{K}[X_f] = \mathbb{K}[X][\frac{1}{f}]$, and $\mathbb{K}[X] \subset \mathbb{K}[X_f]$. But then $\mathbb{K}(X) = \mathbb{K}(X_f)$. As the same holds for U in place of X, we have $\mathbb{K}(X) = \mathbb{K}(U)$.

This does not quite prove the general case, as $U \cap V$ need not be affine. Let $f \in \mathbb{K}[U]$ be such that $U_f \subset U \cap V$, which is an affine subset of both U and V. Then $\mathbb{K}(U_f) = \mathbb{K}(U)$, and the same for V completes the proof. □

A point $x \in X$ is a *regular point* of a rational function $\varphi \in \mathbb{K}(X)$ if φ has a representative f/g with $f, g \in \mathbb{K}[X]$ and $g(x) \neq 0$. From this we see that all points of the principal affine set X_g, which is a neighborhood of x in X, are regular points of φ. Thus the set of regular points of φ is a nonempty open subset of X. This is the *domain of regularity of* φ.

When $x \in X$ is a regular point of a rational function $\varphi \in \mathbb{K}(X)$, we set $\varphi(x) := f(x)/g(x) \in \mathbb{K}$, where φ has representative f/g with $g(x) \neq 0$. The value of $\varphi(x)$ does not depend upon the choice of representative f/g of φ. In this way, φ gives a function from a dense subset of X (its domain of regularity) to $\mathbb{K}$. We write this as

$$\varphi : X \dashrightarrow \mathbb{K}$$

with the dashed arrow indicating that φ is not necessarily defined at all points of X.

The rational function φ of Example 3.3.2 has domain of regularity $X - \{(0,0)\}$. Here $\varphi : X \dashrightarrow \mathbb{K}$ is stereographic projection of the circle onto the line $y = -1$ from the point $(0,0)$.

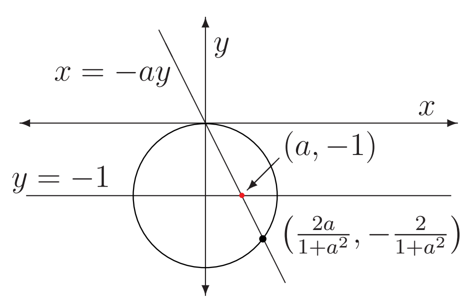


Figure 3.2: Projection of the circle $\mathcal{V}(x^2 + (y-1)^2 - 1)$ from the origin.

Example 3.3.4. Let $X = \mathbb{R}$ and $\varphi = 1/(1+x^2) \in \mathbb{R}(X)$. Then every point of X is a regular point of φ. The existence of rational functions which are regular at every point, but are not elements of the coordinate ring, is a special feature of real algebraic geometry. Observe that φ is not regular at the points $\pm\sqrt{-1} \in \mathbb{C}$. ◇

Theorem 3.3.5. *When $\mathbb{K}$ is algebraically closed, a rational function that is regular at all points of an irreducible affine variety X is a regular function in $\mathbb{K}[X]$.*

Proof. For each point $x \in X$, there are regular functions $f_x, g_x \in \mathbb{K}[X]$ with $\varphi = f_x/g_x$ and $g_x(x) \neq 0$. Let $\mathcal{I}$ be the ideal generated by the denominators g_x of φ for $x \in X$. Then $\mathcal{V}(\mathcal{I}) = \emptyset$, as φ is regular at all points of X.

If we let $g_1, \ldots, g_s$ be generators of $\mathcal{I}$ that are denominators of φ and let $f_1, \ldots, f_s$ be regular functions such that $\varphi = f_i/g_i$ for each i. Then by the Weak Nullstellensatz for X (Theorem 1.3.5(3)), there are regular functions $h_1, \ldots, h_s \in \mathbb{K}[X]$ such that in $\mathbb{K}[X]$,

$$1 \;=\; h_1 g_1 + \cdots + h_s g_s\,.$$

Multiplying this equation by φ, we obtain

$$\varphi \;=\; h_1 f_1 + \cdots + h_s f_s\,,$$

which proves the theorem. □

A list $f_1, \ldots, f_m$ of rational functions gives a *rational map*

$$\begin{aligned} \varphi : X \quad &\dashrightarrow \quad \mathbb{K}^m\,, \\ x \quad &\longmapsto \quad (f_1(x), \ldots, f_m(x))\,. \end{aligned}$$

This rational map φ is only defined on the intersection U of the domains of regularity of each of the f_i. We call U the *domain of* φ and write $\varphi(X)$ for $\varphi(U)$.

Let X be an irreducible affine variety. Since $\mathbb{K}[X] \subset \mathbb{K}(X)$, any regular map is also a rational map. As with regular maps, a rational map $\varphi : X \dashrightarrow \mathbb{K}^m$ given by functions $f_1, \ldots, f_m \in \mathbb{K}(X)$ defines a homomorphism $\varphi^* : \mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}(X)$ by $\varphi^*(g) = g(f_1, \ldots, f_m)$. If Y is an affine subvariety of $\mathbb{K}^m$, then $\varphi(X) \subset Y$ if and only if $\varphi(\mathcal{I}(Y)) = 0$. In particular, the kernel J of the map $\varphi^* : \mathbb{K}[y_1, \ldots, y_m] \to \mathbb{K}(X)$ defines the smallest subvariety $Y = \mathcal{V}(J)$ containing $\varphi(X)$, that is, the Zariski closure of $\varphi(X)$. Since $\mathbb{K}(X)$ is a field, this kernel is a prime ideal, and so Y is irreducible.

When $\varphi : X \dashrightarrow Y$ is a rational map with $\varphi(X)$ dense in Y, then we say that φ is *dominant*. A dominant rational map $\varphi : X \dashrightarrow Y$ induces an embedding $\varphi^* : \mathbb{K}[Y] \hookrightarrow \mathbb{K}(X)$. Since Y is irreducible, this map extends to a map of function fields $\varphi^* : \mathbb{K}(Y) \to \mathbb{K}(X)$. Conversely, given a map $\psi : \mathbb{K}(Y) \to \mathbb{K}(X)$ of function fields, with $Y \subset \mathbb{K}^m$, we obtain a dominant rational map $\varphi : X \dashrightarrow Y$ given by the rational functions $\psi(x_1), \ldots, \psi(x_m) \in \mathbb{K}(X)$ where $x_1, \ldots, x_m$ are the coordinate functions on $Y \subset \mathbb{K}^m$.

Suppose we have two rational maps $\varphi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ with φ dominant. Then $\varphi(X)$ meets the set of regular points of ψ, and so we may compose these maps $\psi \circ \varphi\colon X \dashrightarrow Z$. Two irreducible affine varieties X and Y are *birationally equivalent* if there is a rational map $\varphi : X \dashrightarrow Y$ with a rational inverse $\psi : Y \dashrightarrow X$. By this we mean that the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps on their respective domains. Equivalently, X and Y are birationally equivalent if and only if their function fields are isomorphic, if and only if they have isomorphic open subsets.

For example, the line $\mathbb{K}^1$ and the circle of Figure 3.2 are birationally equivalent. The inverse of stereographic projection from the circle to $\mathbb{K}^1$ is the map from $\mathbb{K}^1$ to the circle given by $a \mapsto (\frac{2a}{1+a^2}, -\frac{2}{1+a^2})$.

Most of the next few paragraphs, up to the example, needs strongly revising, as it now appears in Chapter 1. Let us now consider rational functions and maps of projective varieties. Let $X \subset \mathbb{P}^n$ be a projective variety. Recall that a homogeneous polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$ does not define a function on either $\mathbb{P}^n$ or on X, unless it is a constant, but its vanishing set $\mathcal{V}(f)$ is well defined. However, given two homogeneous polynomials f and g in $\mathbb{K}[x_0, \ldots, x_n]$ which have the same degree, d, the quotient f/g does give a well-defined function, at least on $\mathbb{P}^n - \mathcal{V}(g)$. Indeed, if $[a_0, \ldots, a_n]$ and $[\lambda a_0, \ldots, \lambda a_n]$ are two representatives of the point $a \in \mathbb{P}^n$ and $g(a) \neq 0$, then

$$\frac{f(\lambda a_0, \ldots, \lambda a_n)}{g(\lambda a_0, \ldots, \lambda a_n)} \;=\; \frac{\lambda^d f(a_0, \ldots, a_n)}{\lambda^d g(a_0, \ldots, a_n)} \;=\; \frac{f(a_0, \ldots, a_n)}{g(a_0, \ldots, a_n)}\,.$$

It follows that if $f, g \in \mathbb{K}[X]$ with $g \neq 0$, then the quotient f/g gives a well-defined function on $X - \mathcal{V}(g)$.

Similarly, suppose that $f_0, f_1, \ldots, f_m \in \mathbb{K}[X]$ are elements of the same degree d with at least one f_i non-zero on X. These define a *rational map*

$$\begin{array}{rccl} \varphi : & X & \dashrightarrow & \mathbb{P}^m \\ & x & \longmapsto & [f_0(x), f_1(x), \ldots, f_m(x)]\,. \end{array}$$

Indeed, if $a = [a_0, \ldots, a_n]$ and $\lambda a = [\lambda a_0, \ldots, \lambda a_n]$ are two representatives of a point $x \in X$ where some f_i does not vanish, then

$$[f_0(\lambda a), \ldots, f_m(\lambda a)] \;=\; [\lambda^d f_0(a), \ldots, \lambda^d f_m(a)] \;=\; [f_0(a), \ldots, f_m(a)]\,.$$

This rational map is defined at least on the set $X - \mathcal{V}(f_0, \ldots, f_m)$. A second list $g_0, \ldots, g_m \in \mathbb{K}[X]$ of elements of the same degree (possible different from the degree of the f_i) defines the same rational map if we have

$$\operatorname{rank}\begin{bmatrix} f_0 & f_1 & \cdots & f_m \\ g_0 & g_1 & \cdots & g_m \end{bmatrix} \;=\; 1\,, \qquad \text{i.e., if } \; f_i g_j - f_j g_i \;\in\; \mathcal{I}(X) \;\text{ for } i \neq j\,.$$

The map φ is regular at a point $x \in X$ if there is some system of representatives $f_0, \ldots, f_m$ for the map φ for which $x \notin \mathcal{V}(f_0, \ldots, f_m)$. The set of such points is an open subset of X called the *domain of regularity* of φ. The map φ is *regular* if it is regular at all points of X. The *base locus* of a rational map $\varphi\colon X \dashrightarrow Y$ is the set of points of X at which φ is not regular.

Example 3.3.6. A common example of a rational map is a linear projection. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_m$ be linear forms. These give a rational map φ which is defined at points of $\mathbb{P}^n - E$, where E is the common zero locus of the linear forms $\Lambda_0, \ldots, \Lambda_m$, that is $E = \mathbb{P}(\text{kernel}(M))$, where M is the matrix whose columns are the Λ_i. ⋄

The identification of $\mathbb{P}^1$ with the points on the circle $\mathcal{V}(x^2 + (y-1)^2 - 1) \subset \mathbb{K}^2$ from Example 1.4.3 is an example of a linear projection. Let $X := \mathcal{V}(x^2 + (y-z)^2 - z^2)$ be the plane conic which contains the point $[0,0,1]$. The identification of Example 1.4.3 was the map

$$\mathbb{P}^1 \ \ni \ [a,b] \ \longmapsto \ [2ab, 2a^2, a^2+b^2] \ \in \ X\,.$$

Its inverse is the linear projection $[x,y,z] \mapsto [x,y]$.

Figure 3.3 shows another linear projection. Let C be the cubic space curve with parametrization $[1,t,t^2,2t^3-2t]$ and $\pi\colon \mathbb{P}^3 \dashrightarrow L \simeq \mathbb{P}^1$ the linear projection defined by the last two coordinates, $\pi\colon [x_0,x_1,x_2,x_3] \mapsto [x_2,x_3]$. We have drawn the image $\mathbb{P}^1$ in the picture to illustrate that the inverse image of a point under a linear projection is a linear section of the variety (after removing the base locus). The center of projection is a line,

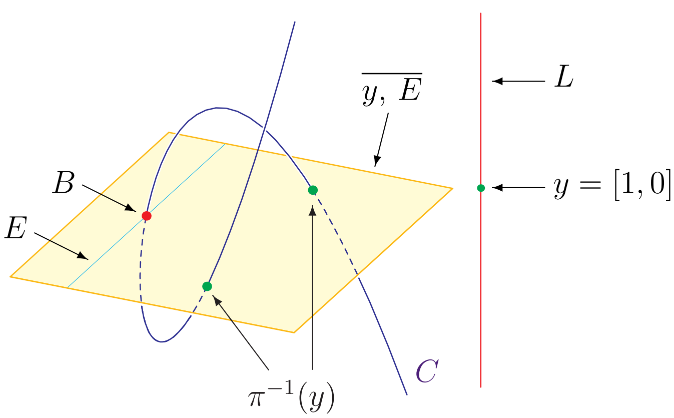


Figure 3.3: A linear projection π with center E.

E, which meets the curve in a point, B.

Exercises

1. Suppose that X is an irreducible affine variety and $0 \neq f \in \mathbb{K}[X]$. Following Exercise 4 of Section 3.1, show that the coordinate ring of the principal open subset X_f (3.3) is $\mathbb{K}[X_f] = \mathbb{K}[X][\frac{1}{f}]$ and that $K[X] \subset \mathbb{K}[X_f]$.

2. Show that irreducible affine varieties X and Y are birationally equivalent if and only if they have isomorphic open sets.

3. We observed that quotients f/g of homogeneous polynomials of the same degree define a function on the principal open set $U_g = \mathbb{P}^n \smallsetminus \mathcal{V}(g)$. The quotient field of the homogeneous coordinate ring of $\mathbb{P}^n$ is graded, and these quotients have degree 0. Show that the degree 0 component of this quotient field is isomorphic to the rational function field $\mathbb{K}(x_1,\ldots,x_n)$.

4. Let $X \subset \mathbb{P}^n$ be a projective variety and suppose that $f, g \in \mathbb{K}[X]$ are homogeneous forms of the same degree with $g \neq 0$. Show that the quotient f/g gives a well-defined function on $X - \mathcal{V}(g)$.

3.4 Smooth and singular points

Algebraic varieties are not manifolds—the very first example of this book ((1.1) in Section 1.1) included the cubic plane curve $\mathcal{V}(y^2 - x^2 - x^3)$. In a neighborhood of the origin, this curve is not a manifold; it has two branches crossing at there. Many other examples likewise have points that do not have a neighborhood in the Euclidean topology which are manifolds, either differentiable or topological. Algebraic varieties have points at which they are differentiable manifolds (smooth points) and others at which they are not manifolds (singular points). We develop some of the basic properties of these smooth and singular points.

Given a polynomial $f \in \mathbb{K}[x]$ and a point $a = (a_1, \ldots, a_n) \in \mathbb{K}^n$, we may write f as a polynomial in new variables $v = (v_1, \ldots, v_n)$, with $v_i := x_i - a_i$ to obtain

$$f \;=\; f(a) \;+\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot v_i \;+\; \cdots\,, \tag{3.6}$$

where the remaining terms have degrees greater than 1 in the variables v. When $\mathbb{K}$ has characteristic zero, this is the usual multivariate Taylor expansion of f at the point a (and the 'derivatives' in (3.6) are derivatives). The coefficient of the monomial v^α in this expansion is the mixed partial derivative of f evaluated at a,

$$\frac{1}{\alpha_1!\alpha_2!\cdots\alpha_n!}\left(\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f\right)(a)\,.$$

In the coordinates v for $\mathbb{K}^n$, the degree one term in the expansion (3.6) is a linear map

$$d_a f \;:\; \mathbb{K}^n \;\longrightarrow\; \mathbb{K}$$

called the *differential* of f at the point a. Note that for any constant $c \in \mathbb{K}$, we have $d_a(c) = 0$ and $d_a(f + c) = d_a f$.

Definition 3.4.1. Let $X \subset \mathbb{K}^n$ be an affine variety with ideal $\mathcal{I}(X)$. The *(Zariski) tangent space* T_aX to X at the point $a \in X$ is the subspace of $\mathbb{K}^n$ annihilated by the collection $\{d_a f \mid f \in \mathcal{I}(X)\}$ of linear maps. Since

$$\begin{aligned} d_a(f+g) &= d_a f + d_a g \\ d_a(fg) &= f(a) d_a g + g(a) d_a f \end{aligned} \tag{3.7}$$

we do not need all the polynomials in $\mathcal{I}(X)$ to define T_aX, but may instead take any finite generating set. ◇

Suppose that $X \subset \mathbb{K}^n$ is an affine variety and $a \in X$. Given a nonzero vector $v \in \mathbb{K}^n$, the map $\ell\colon t \mapsto a + tv$ parameterizes the line through a with direction v. For $f \in \mathcal{I}(X)$, if we expand the composition $f(\ell(t))$ in powers of t, we obtain

$$f(\ell(t)) \;=\; 0 \;+\; t(d_a f \cdot v) \;+\; t^2(\cdots)\,,$$

where we suppress the coefficients of t^2 and of higher powers in t. Here, $d_af \cdot v$ is the usual dot product. When $d_af \cdot v = 0$, the function $f(\ell(t))$ of t vanishes to order at least 2 at $t = 0$. Thus the nonzero vectors in T_aX are the directions of lines through a whose algebraic order of contact with every hypersurface containing X is at least 2. If X is a manifold in $\mathbb{K}^n$ (real or complex as $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), then the Zariski tangent space T_aX is the extrinsic tangent space of X at a. (Extrinsic as it is a linear subspace of $\mathbb{K}^n$.)

Example 3.4.2. Consider the cuspidal cubic $C = \mathcal{V}(f) \subset \mathbb{K}^2$, where $f := y^2 - x^3$. This contains the origin $(0,0)$, and $d_{(0,0)}f$ is the zero linear functional, so that $T_{(0,0)}C = \mathbb{K}^2$, which has dimension two. At every other point $a \in C$, we have $d_af \neq 0$, so that T_aC is one-dimensional. Figure 3.4 shows the cubic, its tangent space at the origin and its

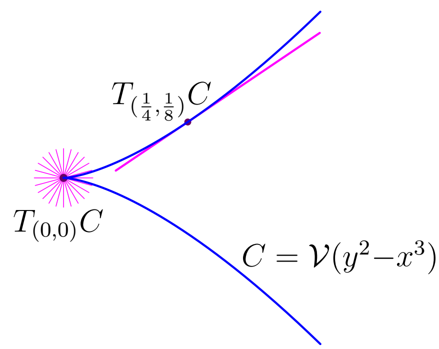


Figure 3.4: Zariski tangent spaces of the cuspidal cubic

tangent space at $(\frac{1}{4}, \frac{1}{8})$. As is customary, we translate the linear subspace T_aC so that its origin is at the point a, to indicate its relation to the variety. $\diamond$

Theorem 3.4.3. *Let X be an affine variety and suppose that $\mathbb{K}$ is algebraically closed. Then the set of points of X whose tangent space has minimal dimension is a nonempty Zariski open subset of X.*

Proof. Let $f_1, \ldots, f_m$ be generators of $\mathcal{I}(X)$. Writing $F = (f_1, \ldots, f_m)$, we let $DF \in \mathrm{Mat}_{m\times n}(\mathbb{K}[x])$ be the matrix whose entry in row i and column j is the partial derivative $\partial f_i/\partial x_j$. For $a \in \mathbb{K}^n$, the components of the vector-valued function

$$\begin{array}{rccc} DF : & \mathbb{K}^n & \longrightarrow & \mathbb{K}^m \\ & v & \longmapsto & DF(a)v \end{array}$$

are the dot products $d_af_1 \cdot v, \ldots, d_af_m \cdot v$ and its kernel is T_aX when $a \in X$.

For each $i = 1, \ldots, \min\{n, m\}$, the *degeneracy locus* $\Delta_i \subset \mathbb{K}^n$ is the variety defined by all $i \times i$ subdeterminants (*minors*) of the matrix DF, and we set $\Delta_{\min\{n,m\}+1} := \mathbb{K}^n$. Since we may expand any $(i+1) \times (i+1)$ minor along a row or column and express it in terms of $i \times i$ minors, these varieties are nested

$$\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_{\min\{n,m\}} \subset \Delta_{\min\{n,m\}+1} = \mathbb{K}^n .$$

By definition, a point $a \in \mathbb{K}^n$ lies in $\Delta_{i+1} \smallsetminus \Delta_i$ if and only if the matrix $DF(a)$ has rank i. In particular, if $a \in \Delta_{i+1} \smallsetminus \Delta_i$, then the kernel of $DF(a)$ has dimension $n - i$.

Let i be the minimal index with $X \subset \Delta_{i+1}$. Then

$$X \setminus (X \cap \Delta_i) \;=\; \{a \in X \mid \dim T_a X = n - i\}$$

is a nonempty open subset of X and $n-i$ is the minimum dimension of a tangent space at a point of X. $\square$

The Zariski tangent space of an affine variety $X \subset \mathbb{K}^n$ is defined extrinsically via a given embedding in affine space. We used this to show that there is a nonempty open subset of X where its tangent space has this minimal dimension. The Zariski tangent space also has an intrinsic definition. For any point $a \in X$ and polynomial $f \in \mathbb{K}[x]$, the differential $d_a f$ is a linear map on $\mathbb{K}^n$ that we may restrict to the Zariski tangent space $T_a X$ of X at a. By the formulas (3.7) and the definition of $T_a X$, this linear map is well-defined for elements $f \in \mathbb{K}[X]$ of the coordinate ring of X. Recall that $\mathfrak{m}_a$ is the maximal ideal of $\mathbb{K}[X]$ consisting of regular functions that vanish at a. Since $d_a f = d_a(f - f(a))$, the formulas (3.7) show that the differential is a linear map from $\mathfrak{m}_a$ to $T_a^* X := \mathrm{Hom}(T_a X, \mathbb{K})$, the space of linear functions on $T_a X$. By the Leibniz formula for d_a (3.7), elements of the square $\mathfrak{m}_a^2$ of $\mathfrak{m}_a$ have zero differential.

Lemma 3.4.4. *For a point $a \in X$, there is a canonical isomorphism $d_a\colon \mathfrak{m}_a/\mathfrak{m}_a^2 \xrightarrow{\sim} T_a^* X$.*

Proof. For a linear form Λ on $\mathbb{K}^n$ and $a \in \mathbb{K}^n$, $d_a(\Lambda - \Lambda(a)) = \Lambda$ on $T_a\mathbb{K}^n = \mathbb{K}^n$. Consequently, if $\ell \in \mathbb{K}[X]$ is the image of $\Lambda - \Lambda(a)$, then $\ell \in \mathfrak{m}_a$ and $d_a \ell$ is the restriction of Λ to $T_a X \subset \mathbb{K}^n$. As every linear form on $T_a X$ is the restriction of a linear form on $\mathbb{K}^n$, we conclude that the map $d_a\colon \mathfrak{m}_a \to T_a^* X$ is surjective.

Suppose that $g \in \mathfrak{m}_a$ and $d_a g$ vanishes on $T_a X$. Let h be a polynomial whose image in $\mathbb{K}[X]$ is g, and let $f_1, \dots, f_m$ be polynomials that generate $\mathcal{I}(X)$. Since $T_a X$ is defined by the vanishing of $d_a f_1, \dots, d_a f_m$, and $d_a h$ vanishes on $T_a X$, there are $\lambda_1, \dots, \lambda_m \in \mathbb{K}$ such that

$$d_a h \;=\; \lambda_1 d_a f_1 \;+\; \lambda_2 d_a f_2 \;+\; \cdots \;+\; \lambda_m d_a f_m\,. \tag{3.8}$$

Set $h_1 := h - (\lambda_1 f_1 + \cdots + \lambda_m f_m)$. If we expand h_1 in the parameters $v_1, \dots, v_n$, where $v_i = x_i - a_i$ (as in (3.6)), then its constant term vanishes (as h and each f_i vanish at a) and its linear terms also vanish, by (3.8). Thus h_1 lies in the ideal $\langle v_1, \dots, v_n\rangle^2$. Since $g \in \mathbb{K}[X]$ is the image of h_1 and $\mathfrak{m}_a \subset \mathbb{K}[X]$ is the image of $\langle v_1, \dots, v_n\rangle$, we conclude that $g \in \mathfrak{m}_a^2$. This completes the proof. $\square$

We may therefore define the Zariski tangent space $T_a X$ independent of any embedding of X to be the vector space $(\mathfrak{m}_a/\mathfrak{m}_a^2)^*$. Suppose that we have a regular map $\varphi\colon X \to Y$ of affine varieties and point $a \in X$. The functorial pullback map $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ sends $\mathfrak{m}_{\varphi(a)}$ to $\mathfrak{m}_a$ as a regular function $g \in \mathbb{K}[Y]$ that vanishes at $\varphi(a)$ has pullback that vanishes at a. This also induces a map $\varphi^*\colon \mathfrak{m}_{\varphi(a)}/\mathfrak{m}_{\varphi(a)}^2 \to \mathfrak{m}_a/\mathfrak{m}_a^2$. Taking linear duals, we obtain a functorial linear map between tangent spaces $d_a\varphi\colon T_a X \to T_{\varphi(a)} Y$.

By Exercise 2, tangent spaces of affine varieties are unchanged in passing to principal affine open subsets. We use this to define Zariski tangent spaces for any variety. Given a variety X and a point $a \in X$, define T_aX to be the Zariski tangent space T_aU for any affine open subset $U \subset X$ containing a.

Suppose that X is irreducible and let m be the minimum dimension of a tangent space of X. By Theorem 3.4.3, the points of X whose tangent space has this minimum dimension form a nonempty open and hence dense subset of X. Call these points of X *smooth* points and write X_{sm} for the nonempty open subset of smooth points. The complement $X \smallsetminus X_{\text{sm}}$ is the set X_{sing} of *singular* points of X. The set of smooth points is dense in X, for otherwise we may write the irreducible variety X as a union $\overline{X_{\text{sm}}} \cup X_{\text{sing}}$ of two proper closed subsets.

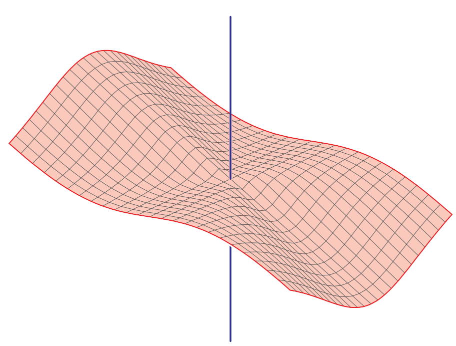
When $\mathbb{K} = \mathbb{C}$, the set of smooth points of X forms a complex manifold of whose dimension at a point $a \in X_{\text{sm}}$ is $\dim_{\mathbb{C}} T_aX$. This is a consequence of standard results about differential manifolds. Similarly, when $\mathbb{K} = \mathbb{R}$, if X has a smooth real point, then the set of smooth points of X is nonempty and forms a manifold whose dimension at a point $a \in X_{\text{sm}}$ is $\dim_{\mathbb{R}} T_aX$. This restriction is necessary, for it is possible that $X_{\text{sm}} = \emptyset$ for a real variety. For example, the real algebraic variety $X = \mathcal{V}(y^2 + x^2)$ has only one real point, the origin, where it is singular as $d_{(0,0)}(y^2 + x^2) = 0$.

Thus when X is smooth and irreducible and $\mathbb{K} = \mathbb{C}$, the dimension of the tangent space to X at a smooth point is equal to its dimension as a manifold. This remains true for any irreducible variety, smooth or not, and for any algebraically closed field. This gives a second definition of dimension which is distinct from the combinatorial definition of Definition 3.2.8.

We have the following facts concerning the locus of smooth and singular points on a real or complex variety. this needs expansion/explanation.

Proposition 3.4.5. *The set of smooth points of an irreducible complex affine subvariety X of dimension d whose complex local dimension in the Euclidean topology is d is dense in the Euclidean topology.*

Example 3.4.6. Irreducible real algebraic varieties need not have this property. The Cartan umbrella $\mathcal{V}(z(x^2 + y^2) - x^3)$



is a connected irreducible surface in $\mathbb{R}^3$ where the local dimension of its smooth points is either 1 (along the z axis or 'handle') or 2 (along the 'canopy' of the umbrella). ⋄

Use this relation of dimension to tangent space to prove some of the theorems about dimension of varieties and then images under maps. This will be a precursor to the Bertini Theorems. These are needed in later sections in Toric varieties and in numerical algebraic geometry. There is also scope to add a couple of pages to this section.

Exercises

1. Using the consequence of Lemma 3.4.4 that the Zariski tangent spaces of an affine variety are intrinsic to give another proof that the cuspidal cubic $\mathcal{V}(y^2 - x^3)$ is not isomorphic to either the parabola $\mathcal{V}(y - x^2)$ or to the line $\mathbb{K}^1$. (This was shown in Example 1.3.1 of Section 1.3 by other means.)

2. Let $X \subset \mathbb{K}^n$ be an affine variety, $a \in X$, and $f \in \mathbb{K}[X]$ a regular function that does not vanish at a. Using the embedding $X_f \hookrightarrow \mathbb{K}^{n+1}$ given by $x \mapsto (x, f(x))$, show that the two tangent spaces T_aX and $T_{(a,f(a))}X_f$ are isomorphic. We recommend using the definition of T_aX as given in Definition 3.4.1.

3. What is the dimension of the Zariski tangent space along the handle of the Cartan umbrella (the locus of points $(0, 0, z)$ for $z \in \mathbb{C}$).

4. Compute some examples of Zariski tangent spaces of differing dimensions.

3.5 Hilbert functions and dimension

The homogeneous coordinate ring $\mathbb{K}[X]$ of a projective variety $X \subset \mathbb{P}^n$ is an invariant of the variety X which determines it up to a linear automorphisms of $\mathbb{P}^n$. Basic numerical invariants, such as the dimension of X, are encoded in the combinatorics of $\mathbb{K}[X]$ and expressed through its Hilbert function. The coordinate ring of an affine variety $Y \subset \mathbb{K}^n$ also has a Hilbert function which encodes its dimension, and it equals the Hilbert function of its projective closure $\overline{Y} \subset \mathbb{P}^n$.

In Section 1.4, we introduced the homogeneous coordinate ring $\mathbb{K}[X] = \mathbb{K}[x_0, \ldots, x_n]/\mathcal{I}(X)$ of a projective variety X. This ring is graded,

$$\mathbb{K}[X] \;=\; \bigoplus_{r=0}^{\infty} \mathbb{K}[X]_r\,,$$

and its degree r component $\mathbb{K}[X]_r$ consists of images of all degree r homogeneous polynomials, that is, it is the quotient $\mathbb{K}[x_0, \ldots, x_n]_r/\mathcal{I}(X)_r$. This is a finite-dimensional $\mathbb{K}$-vector space as $\dim_{\mathbb{K}} \mathbb{K}[x_0, \ldots, x_n]_r = \binom{n+r}{n}$. The most basic numerical invariant of this ring is the *Hilbert function* of X, whose value at $r \in \mathbb{N}$ is the dimension of the r-th graded component of $\mathbb{K}[X]$,

$$\mathrm{HF}_X(r) \;:=\; \dim_{\mathbb{K}}(\mathbb{K}[X]_r)\,.$$

This is also the number of linearly independent degree r homogeneous polynomials on X. We may also define the Hilbert function of a homogeneous ideal $I \subset \mathbb{K}[x_0, \ldots, x_n]$,

$$\mathrm{HF}_I(r) \;:=\; \dim_{\mathbb{K}}\bigl(\mathbb{K}[x_0, \ldots, x_n]_r/I_r\bigr)\,.$$

Note that $\mathrm{HF}_X = \mathrm{HF}_{\mathcal{I}(X)}$.

Example 3.5.1. The space curve C of Figure 3.3 is the image of $\mathbb{P}^1$ under the map

$$\varphi \;:\; \mathbb{P}^1 \ni [s,t] \;\longmapsto\; [s^3, s^2t, st^2, 2t^3{-}2s^2t] \in \mathbb{P}^3\,.$$

If $\mathbb{P}^3$ has coordinates $[w,x,y,z]$, then $C = \mathcal{V}(2y^2 - xz - 2yw,\ 2xy - 2xw - zw,\ x^2 - yw)$. This map has the property that the pullback $\varphi^*(f)$ of a homogeneous form f of degree r is a homogeneous polynomial of degree $3r$ in the variables s,t, and all homogeneous forms of degree $3r$ in s,t occur as pullbacks. Since there are $3r+1$ forms of degree $3r$ in s,t, we see that $\mathrm{HF}_C(r) = 3r+1$. $\diamond$

The Hilbert function of a homogeneous ideal I may be computed using Gröbner bases. First observe that any reduced Gröbner basis of I consists of homogeneous polynomials.

Theorem 3.5.2. *Any reduced Gröbner basis for a homogeneous ideal I consists of homogeneous polynomials.*

Proof. Buchberger's algorithm is friendly to homogeneous polynomials. That is, if f and g are homogeneous, then so is $\mathrm{Spol}(f,g)$. Similarly, the reduction of one homogeneous polynomial by another is a homogeneous polynomial. Since Buchberger's algorithm consists of forming S-polynomials and their reductions, if given homogeneous generators of an ideal, it will compute a reduced Gröbner basis consisting of homogeneous polynomials.

A homogeneous ideal I has a finite generating set B consisting of homogeneous polynomials. Therefore, given a monomial order, Buchberger's algorithm will transform B into a reduced Gröbner basis G consisting of homogeneous polynomials. As reduced Gröbner bases are uniquely determined by the term order, Buchberger's algorithm will transform any generating set into G. □

A consequence of Theorem 3.5.2 is that it is no loss of generality to use graded term orders when computing a Gröbner basis of a homogeneous ideal. Theorem 3.5.2 also implies that the linear isomorphism of Theorem 2.3.3 between $\mathbb{K}[x_0,\dots,x_n]/I$ and $\mathbb{K}[x_0,\dots,x_n]/\mathrm{in}(I)$ respects degree and so the Hilbert functions of I and of $\mathrm{in}(I)$ agree.

Corollary 3.5.3. *Let I be a homogeneous ideal. Then for any term order,* $\mathrm{HF}_I(r) = \mathrm{HF}_{\mathrm{in}(I)}(r)$*, the number of standard monomials of degree r.*

Proof. The image in $\mathbb{K}[x_0,\dots,x_n]/I$ of a standard monomial of degree r lies in the rth graded component. Since the images of standard monomials are linearly independent, we only need to show that they span the degree r graded component of this ring. Let $f \in \mathbb{K}[x_0,\dots,x_n]$ be a homogeneous form of degree r and let G be a reduced Gröbner basis for I. Then the reduction $f \bmod G$ is a linear combination of standard monomials. Each of these will have degree r as G consists of homogeneous polynomials and the division algorithm is homogeneous-friendly. □

Example 3.5.4. In the degree-reverse lexicographic monomial order with $x \succ y \succ z \succ w$, the polynomials

$$\underline{2y^2} - xz - 2yw\,,\ \underline{2xy} - 2xw - zw\,,\ \underline{x^2} - yw\,,$$

form the reduced Gröbner basis for the ideal of the cubic space curve C of Example 3.5.1. The initial terms are underlined, so the initial ideal is the monomial ideal $\langle y^2, xy, x^2\rangle$.

The standard monomials of degree r are exactly the set

$$\{z^a w^b,\ xz^c w^d,\ yz^c w^d \mid a+b=r,\ c+d=r-1\}$$

and so there are exactly $r+1+r+r = 3r+1$ standard monomials of degree r. This agrees with the Hilbert function of C, as computed in Example 3.5.1. ◇

By Corollary 3.5.3 we need only consider monomial ideals when studying Hilbert functions of arbitrary homogeneous ideals. Once again we see how some questions about arbitrary ideals may be reduced to the same questions about monomial ideals, which may be answered using combinatorial arguments.

Because an ideal and its saturation both define the same projective scheme, and because Hilbert functions are difficult to compute, we introduce the Hilbert polynomial.

Definition 3.5.5. Two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ are *stably equivalent* if $f(r) = g(r)$ for r sufficiently large. $\diamond$

We prove the following result at the end of this section.

Proposition-Definition 3.5.6. *The Hilbert function of a homogeneous ideal I is stably equivalent to a polynomial,* HP_I*, called the Hilbert polynomial of* I*.*

The Hilbert polynomial of a projective variety encodes many of its numerical invariants. We explore two such invariants.

Definition 3.5.7. Let $X \subset \mathbb{P}^n$ be a projective variety and suppose that the initial term of its Hilbert polynomial is

$$\mathrm{in}(\mathrm{HP}_X(r)) \;=\; d\frac{r^m}{m!}\,.$$

Then the *dimension of X* is the degree, m, of the Hilbert polynomial and the coefficient d is the *degree of X*. This is the third, and so far, independent definition we have given of dimension. $\diamond$

We computed the Hilbert function of the curve $C \subset \mathbb{P}^3$ of Example 3.5.1 to be $3r+1$. This is also its Hilbert polynomial, and we see that C has dimension 1 and degree 3, which justifies our calling it a cubic space curve.

We may similarly define the dimension and degree of a homogeneous ideal I, using the leading term of its Hilbert polynomial.

Example 3.5.8. In Exercise 3 you are asked to show that if X consists of d distinct points, then the Hilbert polynomial of X is the constant, d. Thus X has dimension 0 and degree d.

Suppose that X is a linear space, $\mathbb{P}(V)$, where $V \subset \mathbb{K}^{n+1}$ has dimension $m{+}1$. We may choose coordinates $x_0, \dots, x_n$ on $\mathbb{P}^n$ so that V is defined by $x_{m+1} = \cdots = x_n = 0$, and so $\mathbb{K}[X] \simeq \mathbb{K}[x_0, \dots, x_m]$. Then $\mathrm{HF}_X(r) = \binom{r+m}{m}$. As a polynomial in r this has initial term $\frac{r^m}{m!}$ and so X has dimension m and degree 1.

Suppose that $I = \langle f \rangle$, where f is a homogeneous polynomial of degree d. Then

$$\bigl(\mathbb{K}[x_0, \dots, x_n]/I\bigr)_r \;=\; \frac{\mathbb{K}[x_0, \dots, x_n]_r}{f \cdot \mathbb{K}[x_0, \dots, x_n]_{r-d}}\,.$$

Since multiplication by f is injective, it follows that $\mathrm{HF}_I(r) = \binom{r+n}{n} - \binom{r-d+n}{n}$, where if $a < n$, then $\binom{a}{n} = 0$. This is a polynomial in r for $r + n \geq d$. By Exercise 4, the leading term of the Hilbert polynomial of I is $d\frac{r^{n-1}}{(n-1)!}$, and so I has dimension $n{-}1$ and degree d. When f is square-free, we have that $I = \mathcal{I}(\mathcal{V}(f))$. Thus the hypersurface defined by f has dimension $n{-}1$ and degree equal to the degree of f. $\diamond$

Remark 3.5.9. Suppose that $Y \subset \mathbb{K}^n$ is an affine variety with coordinate ring $\mathbb{K}[Y]$. This ring is not graded, but it does have an increasing sequence of finite-dimensional subspaces $\mathbb{K}[Y]_{\leq r}$ for $r \in \mathbb{N}$, where $\mathbb{K}[Y]_{\leq r}$ is image in $\mathbb{K}[Y]$ of the linear span of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ of degree at most r. We define the Hilbert function HF_Y of the affine variety Y to be the function whose value at $r \in \mathbb{N}$ is $\dim_{\mathbb{K}} \mathbb{K}[Y]_{\leq r}$. If $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an ideal, then we may define its Hilbert function in the same way.

As in Section 1.4, if $U_0 = \{x \in \mathbb{P}^n \mid x_0 \neq 0\}$, then $U_0 \simeq \mathbb{K}^n$ and we may regard Y as a subvariety of U_0. Let $X \subset \mathbb{P}^n$ be its Zariski closure. In Exercise 5 you will show that for any $r \geq 0$, dehomogenization induces a linear isomorphism $\mathbb{K}[X]_r \xrightarrow{\sim} \mathbb{K}[Y]_{\leq r}$, which implies that $\mathrm{HF}_Y = HF_X$, and thus the equality of affine and projective Hilbert functions. $\diamond$

We need some additional results on inequalities of Hilbert functions and Hilbert polynomials. You are asked to prove the following lemma in Exercise 6

Lemma 3.5.10. *Suppose that $f, g\colon \mathbb{N} \to \mathbb{N}$ are functions that satisfy $f(r) \leq g(r)$ for all $r \in \mathbb{N}$. If, for $r \geq r_0$, f and g are equal to polynomials F and G, respectively, then $\deg F \leq \deg G$. If additionally there is an integer $a \geq 0$ with $g(r-a) \leq f(r)$ for $r \geq a$, then $\deg F = \deg G$ and their leading coefficients are equal.*

Theorem 3.5.11. *Suppose that X is a projective variety of dimension m. Every subvariety of X has dimension at most m, at least one irreducible component of X has dimension m, and the degree of X is the sum of the degrees of its irreducible components of dimension m.*

Proof. Let Y be a subvariety of X. Then the coordinate ring of Y is a quotient of the coordinate ring of X, so $HF_Y(r) \leq HF_X(r)$ for all r. By Lemma 3.5.10, the degree of the Hilbert polynomial of Y is at most the degree of the Hilbert polynomial of X, and thus $\dim Y \leq \dim X$.

Suppose that $X = X_1 \cup \cdots \cup X_k$ is the decomposition of X into irreducible components. Consider the map of graded vector spaces which is induced by restriction

$$\mathbb{K}[X] \longrightarrow \mathbb{K}[X_1] \oplus \mathbb{K}[X_2] \oplus \cdots \oplus \mathbb{K}[X_k]\,.$$

This is injective, which implies the inequality

$$\mathrm{HF}_X(r) \;\leq\; \sum_{i=1}^{r} \mathrm{HF}_{X_i}(r)\,. \tag{3.9}$$

Passing to Hilbert polynomials, Lemma 3.5.10 implies there is a component X_i of X whose Hilbert polynomial has degree at least m, which implies that $\dim X_i = \dim X$.

A consequence of Lemma 3.5.12, which is stated below, is that there is a number $a \geq 0$ such that when $r > a$, we have

$$\sum_{i=1}^{r} \mathrm{HF}_{X_i}(r-a) \;\leq\; \mathrm{HF}_X(r)\,. \tag{3.10}$$

Each Hilbert function is eventually equal to the corresponding Hilbert polynomial, so that the sum in (3.9) is eventually equal to the sum, $\sum_i \mathrm{HP}_{x_i}(r)$ of Hilbert polynomials of the components. By Lemma 3.5.10 and the inequalities (3.9) and (3.10), the polynomial $\sum_i \mathrm{HP}_{x_i}(r)$ has the same degree and leading term as does the Hilbert polynomial of X. But this leading term is the sum of leading terms of the Hilbert polynomials of its components of dimension m, which completes the proof. □

Lemma 3.5.12. *Suppose that $X = X_1 \cup \cdots \cup X_k$ is the decomposition of X into irreducible components. There is a positive integer a such that when $r \geq a$, we have*

$$\sum_{i=1}^{k} \mathrm{HF}_{X_i}(r-a) \;\leq\; \mathrm{HF}_X(r)\,.$$

Proof. For each $i = 1, \ldots, k$, let $f_i \in \mathbb{K}[X]$ be a nonzero element that vanishes on $X \smallsetminus X_i$. As $0 \neq f_i$, it also does not vanish on X_i (for otherwise it would be identically zero on X), so it has nonzero image in the domain $\mathbb{K}[X_i]$. Consider the map $\mu\colon g \mapsto f_i g$ on $\mathbb{K}[X]$. If $g \in \mathcal{I}(X_i)$, then $f_i g$ is identically zero on X, and hence 0 in $\mathbb{K}[X]$, so that $\mathcal{I}(X_i) \subset \ker \mu$, and thus multiplication by f_i gives a well-defined map $\mathbb{K}[X_i] \to \mathbb{K}[X]$.

Consequently, the expression $(g_1, \ldots, g_k) \mapsto f_1 g_1 + \cdots + f_k g_k$ induces a map

$$\varphi \;:\; \mathbb{K}[X_1] \oplus \cdots \oplus \mathbb{K}[X_k] \;\longrightarrow\; \mathbb{K}[X]\,.$$

This is an injection because if $f_1 g_1 + \cdots + f_k g_k = 0$, then $g_i = 0$ for all i. Indeed, the image of $f_1 g_1 + \cdots + f_k g_k$ in $\mathbb{K}[X_i]$, which is $f_i g_i$, is also zero. But this implies that $g_i = 0$ as f_i is a nonzero element of the integral domain $\mathbb{K}[X_i]$.

To complete the proof, let a be any number so that $a \geq \deg(f_i)$ for all i. If for each i we replace f_i by $f_i g_i$ where $0 \neq g_i \in \mathbb{K}[X_i]$ has degree $a - \deg(f_i)$, then we may assume that each f_i has degree a. This the map φ restricts to an injection

$$\varphi \;:\; \mathbb{K}[X_1]_{r-a} \oplus \cdots \oplus \mathbb{K}[X_k]_{r-a} \;\longrightarrow\; \mathbb{K}[X]_r\,.$$

which proves the inequality of the corollary. □

We use combinatorics to prove the following at the end of the section. Do it

Theorem 3.5.13. *When I is a monomial ideal, the degree of HP_I is the dimension of the largest linear subspace contained in $\mathcal{V}(I) \subset \mathbb{P}^n$.*

As $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$, we deduce the following corollary.

Corollary 3.5.14. *If I is a monomial ideal, then the Hilbert polynomials HP_I and $\mathrm{HP}_{\sqrt{I}}$ have the same degree.*

Theorem 3.5.15. *Let X be a subvariety of $\mathbb{P}^n$ and suppose that $f \in \mathbb{K}[X]$ has degree d and is not a zero divisor. Then the ideal $\langle \mathcal{I}(X), f \rangle$ has dimension $\dim(X) - 1$ and degree $d \cdot \deg(X)$.*

If X is irreducible, then every proper subvariety of X has dimension at most $m-1$ and X has a subvariety of dimension $m-1$.

Proof. For $r \geq d$, the degree r component of the quotient ring $\mathbb{K}[x_0, \ldots, x_n]/\langle \mathcal{I}(X), f\rangle$ is

$$\mathbb{K}[X]_r / f \cdot \mathbb{K}[X]_{r-d}\,, \tag{3.11}$$

and so it has dimension $\dim_{\mathbb{K}}(\mathbb{K}[X]_r) - \dim_{\mathbb{K}}(f \cdot \mathbb{K}[X]_{r-d})$.

Suppose that r is large enough so that the Hilbert function of X is equal to its Hilbert polynomial at $r-d$ and all larger integers. Since f is not a zero divisor, multiplication by f is injective. Thus the dimension of the quotient (3.11) is

$$\mathrm{HP}_X(r) - \mathrm{HP}_X(r-d)\,.$$

which is a polynomial of degree $\dim(X)-1$ and leading coefficient $d\cdot\deg(X)/(\dim(X)-1)!$, which is a consequence of Exercise 4.

Suppose now that X is irreducible. Let Y be a proper subvariety of X, and let $0 \neq f \in \mathcal{I}(Y) \subset \mathbb{K}[X]$. Since $\mathbb{K}[X]/\langle f\rangle \twoheadrightarrow \mathbb{K}[X]/\mathcal{I}(Y) = \mathbb{K}[Y]$, we see that the Hilbert polynomial of $\mathbb{K}[Y]$ has degree at most that of $\mathbb{K}[X]/\langle f\rangle$, which is $d-1$.

Suppose that $0 \neq f \in \mathbb{K}[X]$ and let $I = \langle \mathcal{I}(X), f\rangle$, where we write f both for the element $f \in \mathbb{K}(X)$ and for a homogeneous polynomial which restricts to it. If I is radical, then we have just shown that $\mathcal{V}(I) \subset X$ is a subvariety of dimension $d-1$. Otherwise, let $\succ$ be a monomial order, and observe that we have the chain of inclusions

$$\mathrm{in}(I) \;\subset\; \mathrm{in}(\sqrt{I}) \;\subset\; \sqrt{\mathrm{in}(I)}\,. \tag{3.12}$$

Indeed, $I \subset \sqrt{I}$, which implies that $\mathrm{in}(I) \subset \mathrm{in}(\sqrt{I})$. For the other inclusion, let $f \in \sqrt{I}$. Then $f^N \in I$ for some $N \in \mathbb{N}$. But then $\mathrm{in}(f)^N = \mathrm{in}(f^N) \in \mathrm{in}(I)$, and so $\mathrm{in}(f) \in \sqrt{\mathrm{in}(I)}$.

This chain of inclusions (3.12) implies surjections of the corresponding coordinate rings, and therefore the inequalities of Hilbert functions, $\mathrm{HF}_{\mathrm{in}(I)}(r) \geq \mathrm{HF}_{\mathrm{in}(\sqrt{I})}(r) \geq \mathrm{HF}_{\sqrt{\mathrm{in}(I)}}(r)$. This we deduce the corresponding chain of inequalities of degrees of Hilbert polynomials,

$$\deg(\mathrm{HP}_{\mathrm{in}(I)}) \;\geq\; \deg(\mathrm{HP}_{\mathrm{in}(\sqrt{I})}) \;\geq\; \deg(\mathrm{HP}_{\sqrt{\mathrm{in}(I)}})\,.$$

By Corollary 3.5.14, $\deg(\mathrm{HP}_{\mathrm{in}(I)}) = \deg(\mathrm{HP}_{\sqrt{\mathrm{in}(I)}})$, so all three degrees are equal. As $\mathrm{HP}_I = \mathrm{HP}_{\mathrm{in}(I)}$, and the same for $\sqrt{I}$, which is the ideal of $\mathcal{V}(I)$, we conclude that $\mathcal{V}(I)$ is a subvariety of X having dimension $d-1$. $\square$

We may now show that the combinatorial definition (Definition 3.2.8) of dimension agrees with the definition given in terms of Hilbert function.

Corollary 3.5.16 (Combinatorial definition of dimension)**.** *The dimension of a variety X is the length of the longest decreasing chain of irreducible subvarieties of X. If*

$$X \;\supset\; X_0 \;\supsetneq\; X_1 \;\supsetneq\; X_2 \;\supsetneq\; \cdots \;\supsetneq\; X_m \;\supsetneq\; \emptyset\,,$$

is such a chain of maximal length, then X has dimension m.

Proof. Suppose that

$$X \supset X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_m \supsetneq \emptyset$$

is a chain of irreducible subvarieties of a variety X. By Theorem 3.5.11, $\dim(X_{i-1}) > \dim(X_i)$ for $i = 1, \ldots, m$, and so $\dim(X) \geq \dim(X_0) \geq m$.

For the other inequality, we may assume that X_0 is an irreducible component of X with $\dim(X) = \dim(X_0)$. Since X_0 has a subvariety X_1' with dimension $\dim(X_0) - 1$, we may let X_1 be an irreducible component of X_1' with the same dimension. In the same fashion, for each $i = 2, \ldots, \dim(X)$, we may construct an irreducible subvariety X_i of dimension $\dim(X) - i$. This gives a chain of irreducible subvarieties of X of length $\dim(X) + 1$, which proves the combinatorial definition of dimension. □

Similarly, the definition of dimension in terms of tangent spaces (Definition ??????) agrees with the definition given in terms of Hilbert function. For this, I think that we should appeal to differential geometry, such as if $d_x f \neq 0$, and $x \in X$ is a smooth point, then $\mathcal{V}(f)$ is smooth in a neighborhood of x in X.

We now turn to the proof of Hilbert's Theorem 3.5.6, that the Hilbert function of a projective variety or homogeneous ideal is stably equivalent to a polynomial. We prove this for Hilbert functions of a more general class of objects, finitely generated graded modules over a polynomial ring.

A *module* over $\mathbb{K}[x] = \mathbb{K}[x_0, x_1, \ldots, x_n]$ ($\mathbb{K}[x]$*-module*) is a vector space M over $\mathbb{K}$, together with a ring homomorphism $\psi\colon \mathbb{K}[x] \to \operatorname{End}(M)$. Here, $\operatorname{End}(M)$ is the set of linear transformations $M \to M$. This is a $\mathbb{K}$-vector space with a multiplication is induced by composition, and $\operatorname{End}(M)$ is a noncommutative ring. Through ψ, polynomials in $\mathbb{K}[x]$ act as $\mathbb{K}$-linear transformations of the vector space M. We suppress the map ψ; simply writing $f.u$ for $\psi(f)(u)$, the image of $u \in M$ under the linear map $\psi(f)$, for $f \in \mathbb{K}[x]$.

The polynomial ring $\mathbb{K}[x]$ is a $\mathbb{K}[x]$-module, as $\mathbb{K}[x]$ acts linearly on itself by multiplication. An ideal I of $\mathbb{K}[x]$ is a $\mathbb{K}[x]$-module, and ideals are exactly the $\mathbb{K}[x]$-submodules of $\mathbb{K}[x]$. Quotients of modules are also modules, so that a quotient ring $\mathbb{K}[x]/I$ is a $\mathbb{K}[x]$-module. A module M is *finitely generated* if there exist finitely many elements $u_1, \ldots, u_k \in M$ such that every element u of M has an expression

$$u = f_1.u_1 + f_2.u_2 + \cdots + f_k.u_k$$

as a $\mathbb{K}[x]$-linear combination of $u_1, \ldots, u_k$ ($f_1, \ldots, f_k \in \mathbb{K}[x]$).

A module M is *graded* if it has a decomposition

$$M = \bigoplus_{s \in \mathbb{Z}} M_s\,,$$

where for each $s \in \mathbb{Z}$, M_s is a vector subspace of M and the $\mathbb{K}[x]$-action respects this decomposition. That is, for all $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, if $f \in \mathbb{K}[x]_r$ is homogeneous of degree r and $u \in M_s$, then $f.u \in M_{r+s}$.

Lemma 3.5.17. *If M is a finitely generated graded $\mathbb{K}[x]$-module, then each graded component of M is a finite-dimensional vector space.*

Proof. Let $u_1, \dots, u_k$ be generators of M with $u_i \in M_{s_i}$. For $s \in \mathbb{Z}$, every element $u \in M_s$ has an expression

$$u \;=\; f_1.u_1 \;+\; f_2.u_2 \;+\; \cdots \;+\; f_k.u_k$$

as a $\mathbb{K}[x]$-linear combination of $u_1, \dots, u_k$. Here $f_i \in \mathbb{K}[x]_{s-s_i}$. Thus there is a surjection

$$\bigoplus_{i=1}^{k} \mathbb{K}[x]_{s-s_i} \;\longrightarrow\!\!\!\!\rightarrow\; M_s\,.$$

This completes the proof, as the graded components of $\mathbb{K}[x]$ are finite-dimensional. □

The main consequence of Lemma 3.5.17 is that a finitely generated graded module M has a Hilbert function, defined by $HF_M(s) = \dim_{\mathbb{K}} M_s$.

Theorem 3.5.18. *If M is a finitely generated graded $\mathbb{K}[x_0, x_1, \dots, x_n]$-module, then its Hilbert function is stably equivalent to a polynomial of degree $m \leq n$. If as^m is the leading term of this polynomial, then $m!a$ is a nonnegative integer.*

Our proof will use induction on the number of variables, as well as maps of graded modules. A map $\varphi\colon M \to N$ of graded modules is a collection of linear maps

$$\varphi_s \;:\; M_s \;\longrightarrow\; N_s \qquad \text{for } s \in \mathbb{Z}$$

such that for every homogeneous polynomial $f \in \mathbb{K}[x]_r$ and $u \in M_s$ we have

$$\varphi_{r+s}(f.u) \;=\; f.\varphi_s(u) \qquad \text{in } N_{r+s}\,.$$

That is, the map φ is a map of modules that respects the grading (f has degree 0).

A consequence of this definition is that if $f \in \mathbb{K}[x]_r$ and M is a graded module, then multiplication by f is a linear map that sends M_s to M_{r+s}, but it is not *a priori* a map of graded modules. This is remedied by changing the grading in the domain of this multiplication map. Define a new graded module $M(-r)$ by $M(-r)_s := M_{s-r}$. Then multiplication by $f \in \mathbb{K}[x]_r$ is a map of graded modules $M(-r) \to M$.

Proof. When there are no variables, M is a finite-dimensional vector space, and so there is an integer s_0 with $M_s = 0$ for all $s \geq s_0$. In this case, HF_M is stably equivalent to 0, a polynomial of degree -1.

Now suppose that $r \geq 0$ and assume that the theorem holds when there are r variables, that is, for finitely generated $\mathbb{K}[x_0, \dots, x_{r-1}]$-modules. Let M be a finitely generated graded $\mathbb{K}[x_0, \ldots, x_r]$-module, and let $K \subset M$ be the kernel of the linear map induced by multiplication by x_r. This gives an exact sequence of graded vector spaces,

$$0 \longrightarrow K(-1) \longrightarrow M(-1) \xrightarrow{x_r\cdot} M \longrightarrow M/x_r.M \longrightarrow 0\,.$$

For any $s \in \mathbb{Z}$, if we take the dimension of the sth graded components, then the rank-nullity theorem implies that

$$0 \;=\; \dim_{\mathbb{K}} K(-1)_s \;-\; \dim_{\mathbb{K}} M(-1)_s \;+\; \dim_{\mathbb{K}} M_s \;-\; \dim_{\mathbb{K}}(M/x_r.M)_s\,.$$

Using that $M(-1)s = M_{s-1}$, this becomes,

$$\dim_{\mathbb{K}} M_s \;-\; \dim_{\mathbb{K}} M_{s-1} \;=\; \dim_{\mathbb{K}}(M/x_r.M)_s \;-\; \dim_{\mathbb{K}} K(-1)_s\,.$$

Observe that both $K(-1)$ and $M/x_r.M$ are finitely generated modules over $\mathbb{K}[x_0, \ldots, x_{r-1}]$. By our induction hypothesis, the Hilbert function of each is stably equivalent to a polynomial of degree at most $r{-}1$. If m is the degree of the polynomial, then the coefficient a of its leading term as^m has $m!a$ a nonnegative integer. The same is true for their difference, (but the integer could be negative). Let $P(s)$ be this polynomial and s_0 the integer such that for $s \geq s_0$, $P(s) = \dim_{\mathbb{K}}(M/x_r.M)_s - \dim_{\mathbb{K}} K(-1)_s$. Suppose that P has degree m and leading coefficient ad^m.

Then, for $s \geq s_0$, $P(s) = \mathrm{HF}_M(s) - \mathrm{HF}_M(s-1)$, and we have

$$\mathrm{HF}_M(s) \;=\; \mathrm{HF}_M(s_0) \;+\; \sum_{t=s_0}^{s} P(t)\,.$$

By Exercise 4 the right hand side is a polynomial in s of degree $m{+}1$ with leading term $\frac{a}{m+1}s^{m+1}$. This completes the proof. $\square$

Still need to prove the theorem about monomial ideals

Exercises

1. Show that the dimension of the space $\mathbb{K}[x_0, \ldots, x_n]_m$ of homogeneous polynomials of degree m is $\binom{m+n}{n} = \frac{m^n}{n!} +$ lower order terms in m.

2. Let I be a homogeneous ideal. Show that the Hilbert functions HF_I, $HF_{(I\,:\,\mathfrak{m}_0)}$, and $HF_{I_{\geq d}}$ are stably equivalent.

3. Show that if $X \subset \mathbb{P}^n$ consists of d points, then, for r sufficiently large, we have $\mathbb{K}[X]_r \simeq \mathbb{K}^d$, and so $HP_X(r) = d$.

4. Suppose that $f(t)$ is a polynomial of degree d with initial term $a_0 t^m$.

   (a) Show that $f(t) - f(t-1)$ has initial term $m a_0 t^{m-1}$.

   (b) Show that $f(t) - f(t-b)$ has initial term $m b a_0 t^{m-1}$.

   (c) Show that for $t \geq t_0$ the sum
   $$\sum_{s=t_0}^{t} f(s)$$
   is a polynomial of degree $m+1$ in t with initial term $\frac{a_0}{m+1} t^{m+1}$.

5. Let $Y \subset \mathbb{K}^n$ be an affine variety, embedded into projective space $\mathbb{P}^n$ via the identification of U_0 with $\mathbb{K}^n$ as in Remark 3.5.9, and let X be its projective closure. Show that for $r \geq 0$ dehomogenization induces a linear isomorphism of $\mathbb{K}[X]_r$ and $\mathbb{K}[Y]_{\leq r}$.

6. Prove Lemma 3.5.10

7. Compute the Hilbert functions and polynomials the following projective varieties. What are their dimensions and degrees?

   (a) The union of three skew lines in P^3, say $\mathcal{V}(x-w, y-z) \cup \mathcal{V}(x+w, y+z) \cup \mathcal{V}(y-w, x+z)$, whose ideal has reduced Gröbner basis
   $$\langle \underline{x^2} + y^2 - z^2 - w^2,\ \underline{y^2 z} - xz^2 - z^3 + xyw + yzw - zw^2,\ \underline{xyz} - y^2 w - xzw + yw^2,$$
   $$\underline{y^3} - yz^2 - y^2 w + z^2 w,\ \underline{xy^2} - xyw - yzw + zw^2 \rangle$$

   (b) The union of two coplanar lines and a third line not meeting the first two, say the x- and y-axes and the line $x = y = 1$.

   (c) The union of three lines where the first meets the second but not the third and the second meets the third. For example $\mathcal{V}(wy, wz, xz)$.

   (d) The union of three coincident lines, say the x-, y-, and z- axes.
   Draw a picture of these?

8. Show that if $A \subset \mathbb{K}^n$ and $B \subset \mathbb{K}^m$ are subvarieties of degrees a and b, respectively, then $A \times B \subset \mathbb{K}^n \times \mathbb{K}^m$ has degree ab.

3.6 Bertini Theorems

A consequence of a Bertini's Theorem[†] is that if X is a projective variety, then for almost all homogeneous polynomials f of a fixed degree, $\langle \mathcal{I}(X), f\rangle$ is radical and f is not a zero-divisor in $\mathbb{K}[X]$.

Consequently, if Λ is a generic linear form and set $Y := \mathcal{V}(\Lambda) \cap X$, then $\mathcal{I}(Y) = \langle \mathcal{I}(X), \Lambda\rangle$, and so

$$\mathrm{HP}_Y \;=\; \mathrm{HP}_{\langle \mathcal{I}(X),\Lambda\rangle}\,,$$

and so by Theorem 3.5.15, $\deg(Y) = \deg(X)$. If $Y \subset \mathbb{P}^n$ has dimension d, then we say that Y has *codimension* $n-d$.

Corollary 3.6.1 (Geometric meaning of degree)**.** *The degree of a projective variety $X \subset \mathbb{P}^n$ of dimension d is the number of points in an intersection*

$$X \cap L\,,$$

where $L \subset \mathbb{P}^n$ is a generic linear subspaces of codimension d.

For example, the cubic curve of Figure 3.3 has degree 3, and we see in that figure that it meets the plane $z = 0$ in 3 points.

Does this belong anywhere? Suppose that the ideal I generated by the polynomials f_i of (2.16) is not zero-dimensional, and we still want to count the isolated solutions to (2.16). In this case, there are symbolic algorithms that compute a zero-dimensional ideal J with $J \supset I$ having the property that $\mathcal{V}(J)$ consists of all isolated points in $\mathcal{V}(I)$, that is all isolated solutions to (2.16). These algorithms successively compute the ideal of all components of $\mathcal{V}(I)$ of maximal dimension, and then strip them off. One such method would be to compute the primary decomposition of an ideal. Another method, when the non-isolated solutions are known to lie on a variety $\mathcal{V}(J)$, is to saturate I by J to remove the excess intersection.[§]

Exercises

3.7 Notes

Mention about the origin of Zariski topology.

[†] Not formulated here, yet!

[§] Develop this further, either here or somewhere else, and then refer to that place.