Webs and Maximally Inflected Curves?

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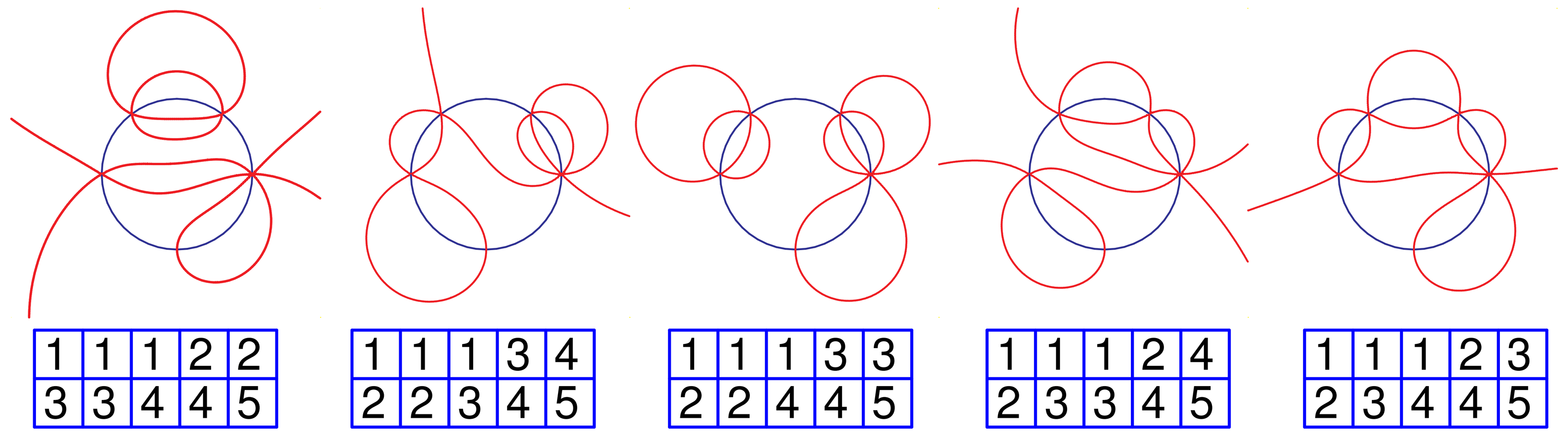
Maximally Inflected Curves

A real rational curve $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^n$ of degree $d+n$ is *maximally inflected* if its $N = (n+1)d$ ramification points (generalized flexes) lie in $\mathbb{RP}^1$.

Purbhoo: For given ramification, these are in bijection with tableaux of shape $(n+1) \times d$ (a consequence of the Shapiro Conjecture).

Purbhoo: The geometry of these curves encode most tableau combinatorics.

$n = 1$: The map $\varphi \mapsto \varphi^{-1}(\mathbb{RP}^1) \cap \mathcal{H}$ ($\mathcal{H}$ is upper-half plane of $\mathbb{CP}^1$) is a bijection between curves and nets preserving tableaux, and this persists when ramification points collide:



$\varphi^{-1}(\mathbb{RP}^1)$ and tableaux of ramified maximally inflected sextics $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$

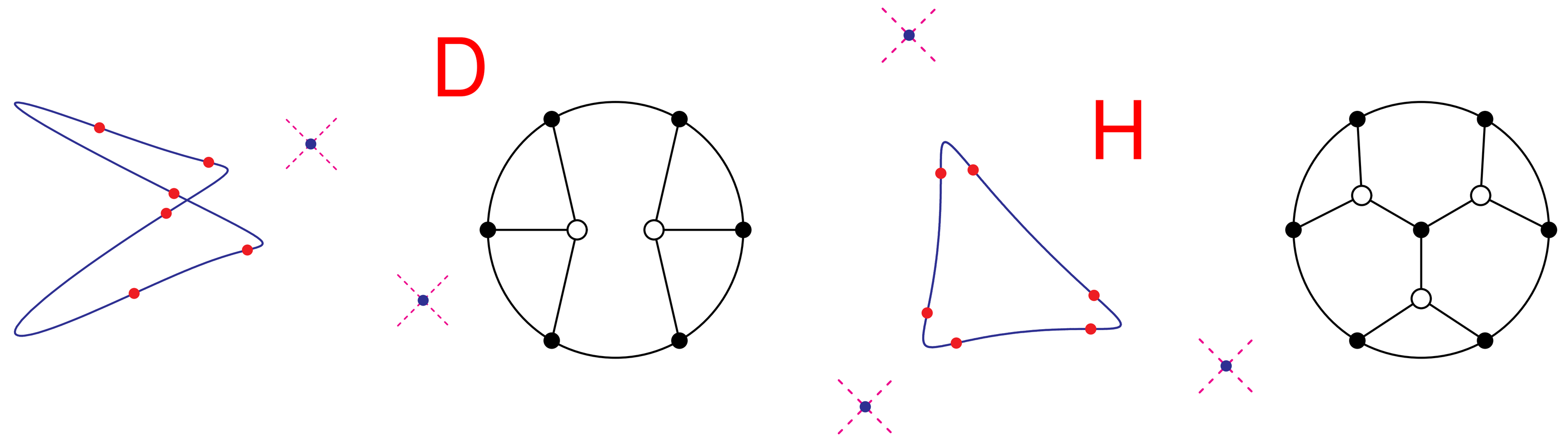
Plane curves

A rational plane curve of degree $d+2$ has $\frac{1}{2}d(d+1)$ double points. For real curves, these include nodes $\times$ and *solitary points* $\times$.

The *Welschinger invariant* of a curve φ is the parity of the number of its solitary points.

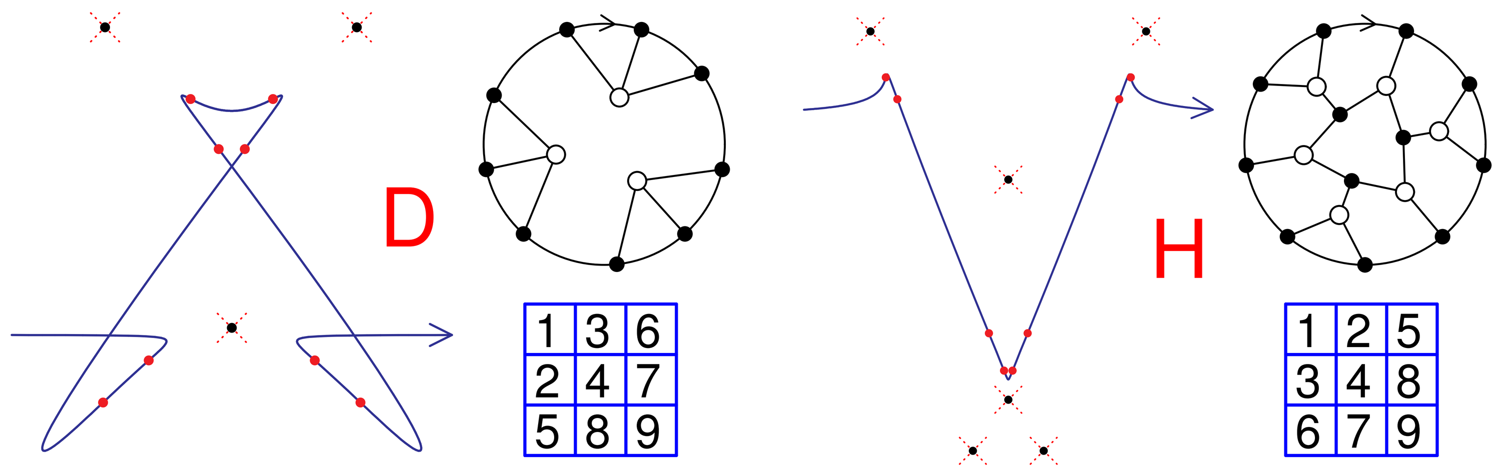
We have a non-rigorous method associating a tableau to a curve.

Maximally inflected quartics and their webs:

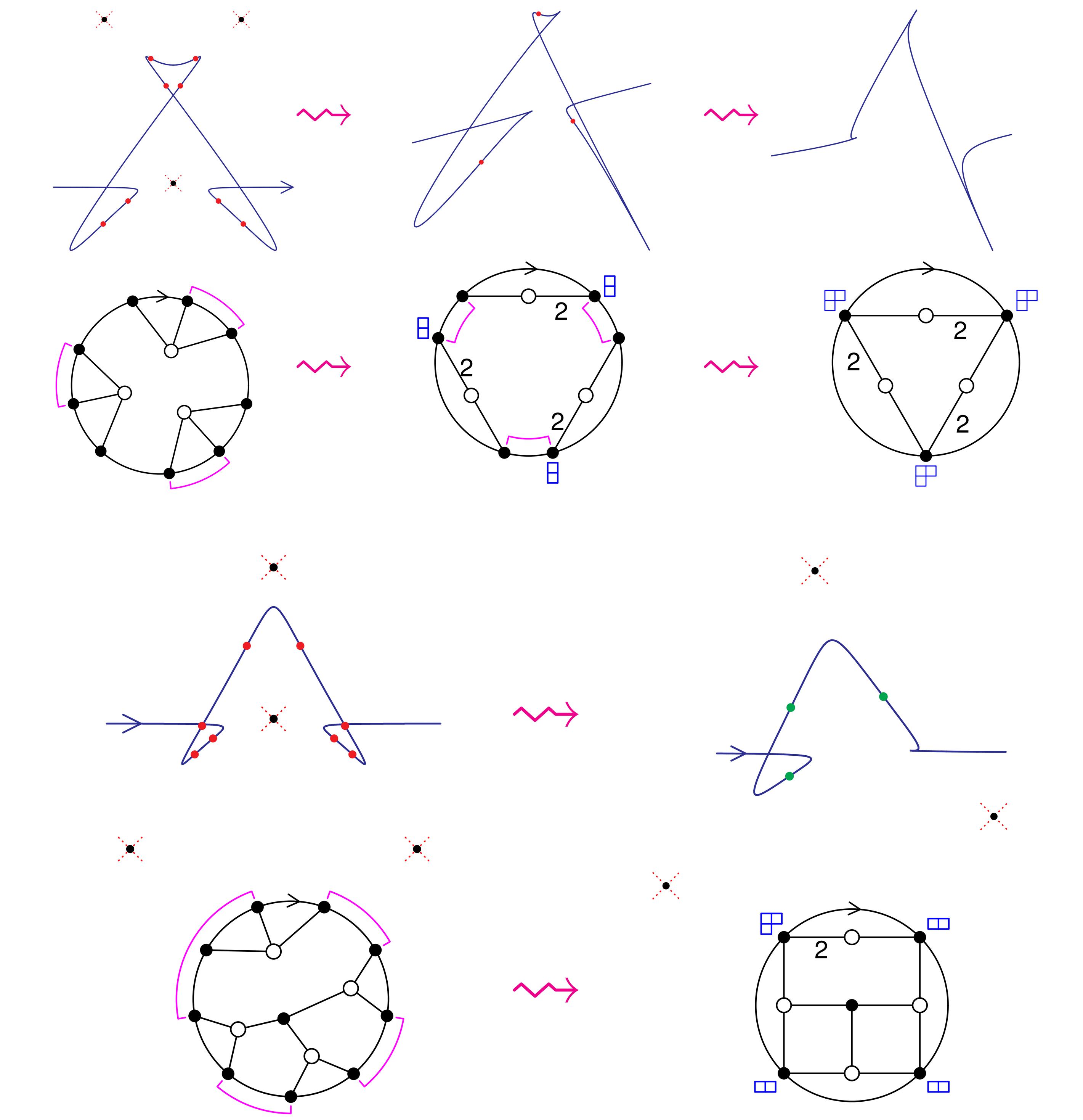


Conjecture. The number of solitary points equals the number of regions $(-d-1)$.

This holds for quintics with simple flexes,



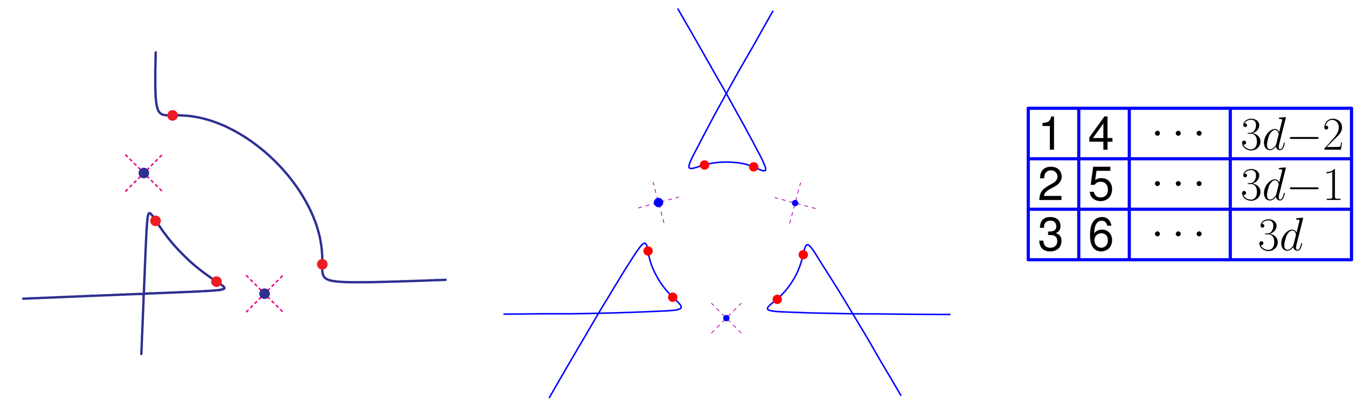
and when ramification points collide:



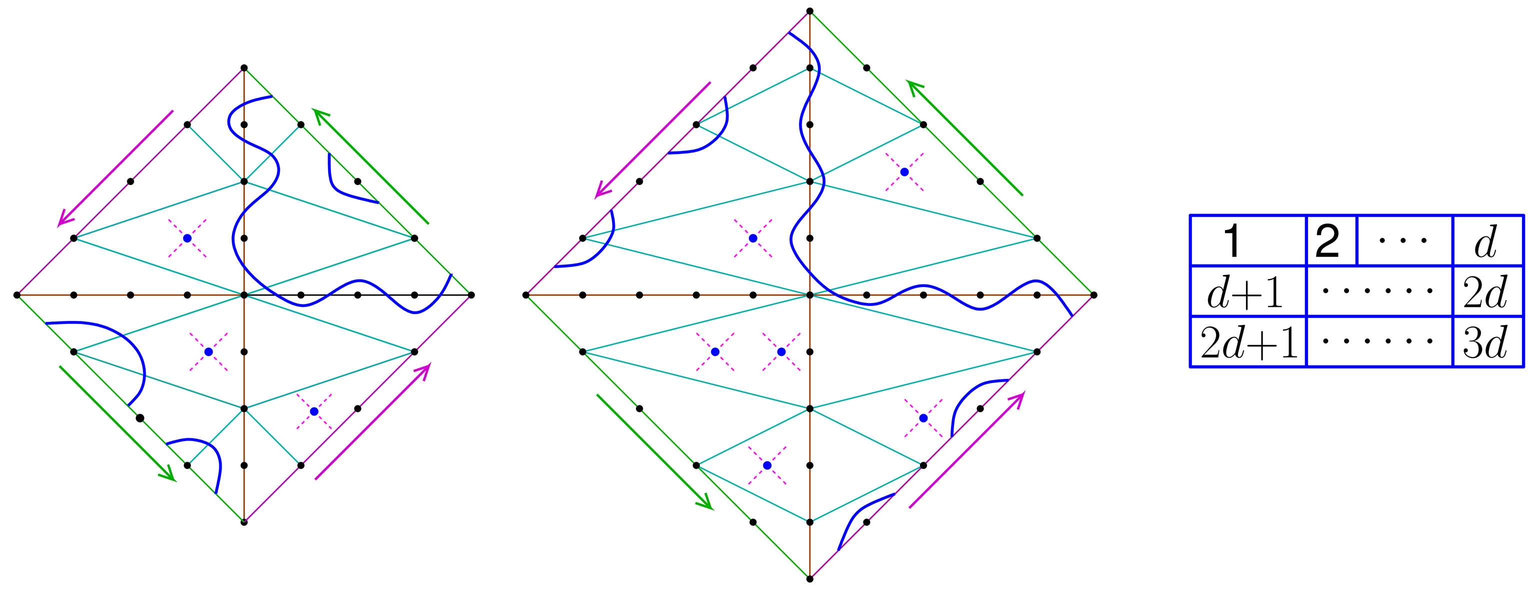
Constructions

[Kh-S] Construct maximally inflected plane curves of degrees $d+2$ generalizing D and H.

Deforming d lines tangent to a conic:



Patchworking rational Harnack curves:



For both constructions, the conjecture on solitary points and regions of the web holds.

Colliding ramification on curves gives a well-defined notion of degeneration and ramification of webs.

Bibliography

[E-H] D. Eisenbud and J. Harris. *Divisors on general curves and cuspidal rational curves*, Invent. Math. **74**, (1983), 371–418.

[Kh-S] V. Kharlamov and F. Sottile. *Maximally Inflected Real Rational Curves*, Moscow Math. J., **3** (2003), 947–987.

[MTV] E. Mukhin, V. Tarasov, and A. Varchenko. *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, Ann. Math., **170** (2009) 863–881.

[P] K. Purbhoo. *Jeu de taquin and a monodromy problem for Wronskians of polynomials*, Adv. Math., **224** (2010), 827–862.