Algebra II Winter 2021 — 23 March

Frank Sottile — Tenth Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 29 March.

1. Let F/K be a field extension with intermediate fields F_1, F_2 that are finite extensions of K.
   (a) Write F_1F_2 for the subfield of F generated by F_1 and F_2. Prove that $[F_1F_2\colon K] \leq [F_1\colon K]\cdot[F_2\colon K]$ (Hint: Let $\alpha_1,\dots\alpha_n$ be a basis for F_1 over K and write $F_1 = K(\alpha_1,\dots\alpha_n)$, and the same for F_2.)
   (b) If the two indices $[F_1\colon K]$ and $[F_2\colon K]$ are relatively prime, show that we obtain an equality in part (a).
2. Let F/K be a field extension. If $u \in F$ is algebraic over K of odd degree, then so is u^2, and $K(u) = K(u^2)$.
3. (a) Let $F := \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Determine $[F\colon \mathbb{Q}]$ and give a basis of F over $\mathbb{Q}$.
   (b) Do the same for $F := \mathbb{Q}(\sqrt{-1}, \sqrt{3}, \omega)$, where $\omega \neq 1$ is a cube root of 1.
4. Let K be a field and $x_1,\dots,x_n$ indeterminates. Let u be an element of the function field $K(x_1,\dots,x_n)$. Show that either $u \in K$ or u is transcendental over K.
5. Let $u = x^4/(x^2+1)$ be an element of the function field $\mathbb{C}(x)$. Show that $\mathbb{C}(x)$ is a simple algebraic extension of $\mathbb{C}(u)$, and determine the degree of this extension.
6. Let F/K be a field extension, with intermediate fields F_1, F_2 that are Galois extensions of K.
   (a) Show that $F_1 \cap F_2$ is a Galois extension of K.
   (b) Show that F_1F_2 is a Galois extension of K.