Algebra II Winter 2021 — 19 April

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Thirteenth Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 26 April.

1. Prove Fermat's little theorem. For $a \in \mathbb{Z}$ and p a prime, $a^p \equiv a \bmod p$, using the structure of $\mathbb{Z}/p\mathbb{Z}$. (Hint: show that $a^{p-1} \equiv 1 \bmod p$.)

2. Suppose that F is a field of (prime) characteristic $p > 0$. Show that for $a, b \in F$, $(a+b)^p = a^p + b^p$. (If you use binomial coefficients, some care is needed in going from characteristic 0 to characteristic p).

   Deduce that the map $\phi\colon F \to F$ defined by $a \mapsto a^p$ is a field homomorphism.

3. Show that every element of a finite field may be written as the sum of two squares.

4. Show that the algebraic closure of a finite field F is Galois over F.

5. Show that the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is $|\mathbb{C}|$. What is the cardinal number of the Galois group of the extension $\mathbb{C}/\mathbb{Q}$?

6. Let I be a nonzero ideal of a principal ideal domain R. Show that R/I is both Noetherian and Artinian.

7. Prove that an Artinian integral domain is a field.