NUMERICAL REAL ALGEBRAIC GEOMETRY

Homotopy Continuation: Exercises

(1) Suppose that we use the straight-line homotopy

$$H(x;t) \;=\; t(x^2+1) \,+\, (1-t)(x^2-1) \qquad \text{for} \quad t\in[0,1]$$

to compute $\pm\sqrt{-1}$ numerically. Sketch the path taken by the roots of $H(x;t)$ for $t\in[0,1]$. Discuss the suitability of the predictor-corrector path tracking algorithm for this homotopy.

(2) For an arbitrary complex number $\gamma\in\mathbb{C}$, consider the homotopy

$$H(x;t) \;=\; t(x^2+1) \,+\, \gamma(1-t)(x^2-1) \qquad \text{for} \quad t\in[0,1]\,.$$

Sketch the paths taken by the roots of $H(x;t)$ for $t\in[0,1]$ for different values of γ. (For example, $\gamma\in\{e^{i\frac{3\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{6}}\}$.)

(3) Interpreting the coefficents of vectors $\mathbf{v}$ and $\mathbf{w}$ in a convex combination $\tau\mathbf{v}+(1-\tau)\mathbf{w}$ as points in $\mathbb{CP}^1$, plot curves $t\mathbf{v}+\gamma(1-t)\mathbf{w}$ for $t\in[0,1]$ and different values of γ, say those in the previous exercise together with $\gamma=1$, in $\mathbb{CP}^1$. That is, plot $t/(t(1-\gamma)+\gamma)$ for $t\in[0,1]$.

(4) Download and install PHCPack and Bertini onto a computer, and test them on a suite of polynomial systems in three variables of your choosing. Compare their performance. Push your computer, and look for the limits of computability.

Bonus: Install Singular or Macaulay2 or CoCoA, and try to compute the dimension and degrees of the varieties defined by the equations from the first part of this exercise.

Fewnomial bounds: Exercises

(1) Compute the number of non-zero complex solutions to the system

$$10x^{106} + 11y^{53} - 11y = 10y^{106} + 11x^{53} - 11x = 0\,.$$

How many of them are real?

(2) Give a system involving the monomials 1, x, y, xyz^8, z, z^2, z^3 that has 9 or more real solutions. Can you find a system with more real solutions?

(3) Exhibit a system of two polynomials in the variables x, y involving a total of 4 different monomials that has three positive real solutions.

(4) Exhibit a system of three polynomials in the variables x, y, z involving a total of 5 different monomials that has four positive real solutions.

(5) Find the Gale dual system to each system of functions below. For each, also compute the number of complex solutions (non-zero, and off the hyperplanes), the number of real solutions, and the number of positive solutions.

(a)
$$\begin{cases} 1 + 3x^3y - 6x^2y^2 + 5y^3 &= 0 \\ 4 + 5x^3y - yx^2y^2 - 9y^3 &= 0 \end{cases}$$

(b)
$$\begin{cases} 2 - 3xy^3 + 4x^2z + x^2y^2 - 6y^3 + 11xy^2z^5 &= 0 \\ 6 + 2xy^3 - x^2z + 2x^2y^2 + 7y^3 - 6xy^2z^5 &= 0 \\ -1 - xy^3 + 3x^2z - 3x^2y^2 + 2y^3 + 3xy^2z^5 &= 0 \end{cases}$$

(c)
$$\begin{cases} x^{-1}(x+y-1)y^{-1}(x+2y-6)^2(y+2z-6) &= 1 \\ x(x+y-1)^{-3}y^2(x+2y-6)(y+2z-6)^{-2} &= 1 \end{cases}$$

(d)
$$\begin{cases} x^2(x+y-1)^3y(x+2y-6)^2(y+2z-6) &= 1 \\ x^{-4}(x+y-1)^3y(x+2y-6)^{-2}(y+2z-6)^{-3} &= 1 \end{cases}$$

(e)
$$\begin{cases} x(x+y-1)^7y(x+2y-6)^5(y+2z-6)^3 &= 1 \\ x^{-6}(x+y-1)^{-4}y^2(x+2y-6)^{-3}(y+2z-6)^{-6} &= 1 \end{cases}$$

(6) Challenge project: Construct a system of two polynomials in two variables, having a total of 5 monomials that has six or more real positive solutions.