Workshop on Real Algebraic Geometry in Geometric Modeling

CONTENTS

1. Summary

Geometric modeling and algebraic geometry both study curves and surfaces generated by polynomials, but for very different purposes. In algebraic geometry these objects are studied for their theoretical interest, while geometric modeling uses them to build computer models for industrial design and manufacture. Interactions between these fields have been increasing in recent years. This comes in part from the sophisticated techniques developed in geometric modeling to manipulate curves and surfaces and in part from the growth of computational algebraic geometry, which places an emphasis on concrete problems and algorithms. While the objects of geometric modeling are real (as opposed to complex) curves and surfaces, there has been relatively little interaction between geometric modeling and real algebraic geometry. The purpose of this proposal is to strengthen these ties between geometric modeling and algebraic geometry, particularly with real algebraic geometry.

In the Winter/Spring of 2004, the Mathematical Sciences Research Institute (MSRI) in Berkeley California is hosting a semester-long program entitled "Topological Aspects of Real Algebraic Varieties", with a concentration on applications of real algebraic geometry in April. The organizers for this program are Selman Akbulut, Grisha Mikhalkin, Victoria Powers, Boris Shapiro, Frank Sottile (the Principal Investigator), and Oleg Viro. Members of the program who will be in residence include several computer scientists and mathematicians who have been among the leaders in the interactions between algebraic geometry and geometric modeling. In addition, there are many members of the program (including the author of this proposal) who have closely related interests and a desire to participate in such cross-cultural scientific interaction. The MSRI will hold a small workshop on the theme of real and computational algebraic geometry in geometric modeling over the weekend of April 3-4 as part of this larger program. This proposal asks for funding to invite people to this workshop from outside the MSRI program, with some to stay for a week after the workshop, providing a longer period of sustained interactions.

While some key scientific contributions to this interface between geometric modeling and algebraic geometry have been made by US-based scientists, there currently is more activity in Europe. In part, this is because of a successful NSF-funded workshop in Vilnius, Lithuania in August 2002 `<http://www.mif.vu.lt/cs2/cagl/aggm/>` which was perhaps the first to focus on the interactions between these two fields, and helped to highlight the potential for future interactions. The proceedings of that conference, which are being published by the American Mathematical Society, are a collection of surveys intended to provide a foundation for further work at the interface of geometric modeling and algebraic geometry. Since then, there has been another workshop (COMPASS) `<http://www.ag.jku.at/compass/>` in Austria along similar lines. The MSRI workshop will bring together researchers from both fields and will be important for the evolution of these scientific links in US.

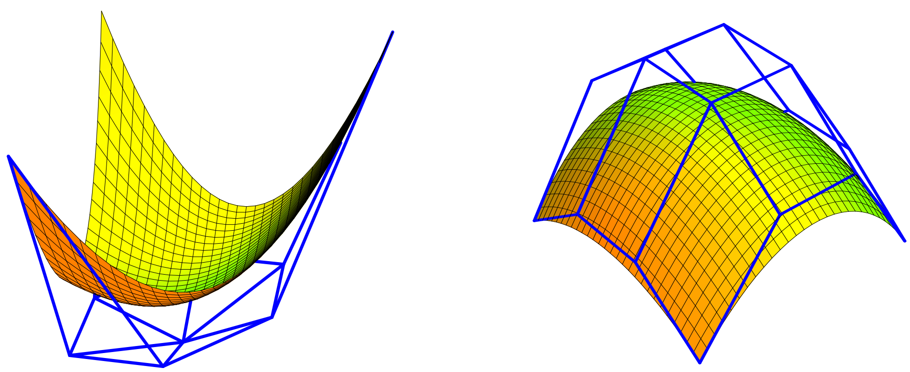
2. Scientific Description

Geometric modeling uses curves and surfaces to build computer models of geometric objects. These representations are cheaper to construct, easier to manipulate, and simpler to analyze than the physical models they are fast replacing in manufacturing. The basic unit in such a computer model is a surface patch. As its name implies, this is a piece of the surface of an object typically represented parametrically

$$D \ni (u,v) \;\longmapsto\; (x(u,v),\, y(u,v),\, z(u,v))\,, \tag{1}$$

over some set $D \subset \mathbb{R}^2$ by easy-to-compute rational functions x, y, z.

Traditionally, these patches are triangular Bézier patches and tensor product Bézier patches, which are parametrized by triangles and rectangles, respectively. We show these in Figure 1. These Bézier patches are well-understood, easy to manipulate, and



Cubic Bézier Patch   Bicubic Tensor Product Patch

Figure 1. Beźier patches

have algorithms for fast generation such as de Castlejau's algorithm and blossoming. A key to their utility is that they are described by minimal data. For example, the vertices in each 'cage' in the pictures above are called the control points, as they determine the patch. Moving the control points affects the patch in a well-understood way.

Associated to a parametric surface (1) is its implicit equation $F(x, y, z)$, which satisfies

$$F(x(u,v),\, y(u,v),\, z(u,v)) \;=\; 0$$

for all u, v (not just $(u, v) \in D$). The implicit equation has many uses. For example, it is needed to determine the intersection of two surface patches. An early interaction between geometric modeling and algebraic geometry was by Tom Sederberg [12], who applied the classical Dixon's resultant to find the implicit equation for such a rational surface. Since then, other classical resultant formulas have found uses in geometric modeling.

Currently, many people in geometric modeling borrow tools from classical algebraic geometry. For example, Bajaj and his collaborators developed GANITH [1], a suite of software tools for geometric modeling based upon algebraic geometry. Hoffman's book [9] on solid modeling advocates using Gröbner bases for calculations, such as finding intersections of surfaces and locating singularities. Winkler and his collaborators developed CASA [15], a computer algebra package for constructive algebraic geometry that implements the classical analysis of singularities in order to robustly render real algebraic curves including surface/surface intersections. These developments have been facilitated in no small part by the books of Cox, Little,and O'Shea [3, 4], which introduced many ideas and techniques from algebraic geometry to a wider audience.

Geometric modeling has also led to new insights in algebraic geometry. The technique of moving planes and/or quadrics of geometric modeling led to new insights on syzygies [6, 7], and some of the recent work on sparse resultants was initiated by the geometric modeling community. We will invite Amit Khetan, an NSF postdoctoral fellow, to the workshop and to spend some additional time at MSRI. He recently found a new class of optimal resultant formulas using deep results of David Eisenbud, the MSRI director, and is collaborating with Carlos D'Andrea of Berkeley to extend the method of moving planes and quadrics.

Many questions including locating singularities and determining intersections require the solution of polynomial equations, either numerically or symbolically. One robust method for this involves the afore-mentioned resultants, others use Gröbner bases or even sophisticated numerical techniques. One expected participant, Ming Zhang, is now working at M.D. Anderson on computer aided drug design. This discipline involves solving large systems of polynomial equations of low degree, and he has been using techniques both from algebraic geometry and from geometric modeling in his work. We hope that his participation will highlight some algebraic geometric challenges in his field.

These might be called classical interactions as they deal with objects from classical algebraic geometry. The Vilnius workshop tried to deepen the connections between the subjects by focusing on more modern developments in algebraic geometry, including toric varieties. The MSRI workshop intends to further this trend by emphasizing real algebraic geometry, in addition to toric varieties and computational algebraic geometry.

A shortcoming of the Bézier patches we encountered previously is that they only come in two shapes, namely triangular or rectangular. This limits their usefulness in some special situations, such as blending, when there is a polygonal be filled with more than four sides. Figure 2 (due to Karčiauskas) show such a blending using 5- and 6-sided patches. Greater design flexibility is afforded by Krasauskas's toric surface patches [11], which may have any (convex) polygonal shape. Warren [14] defined a hexagonal patch as a cubic Bézier triangle with base points at three vertices—blowing them up to three new sides. Krasauskas's theory of toric surface patches realizes this as a toric surface patch and gives a method to subdivide it into six rectangular patches. Figure 3 shows two views of a hexagonal patch. (The grid in these pictures reflects Krasauskas's subdivision.) These pictures help illustrate the beginnings of a very appealing dictionary between concepts in toric geometry and in geometric modeling. The vertices of the control polyhedra of these pictures are the control points, and their

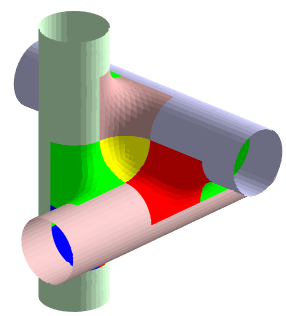


FIGURE 2. Blending cylinders

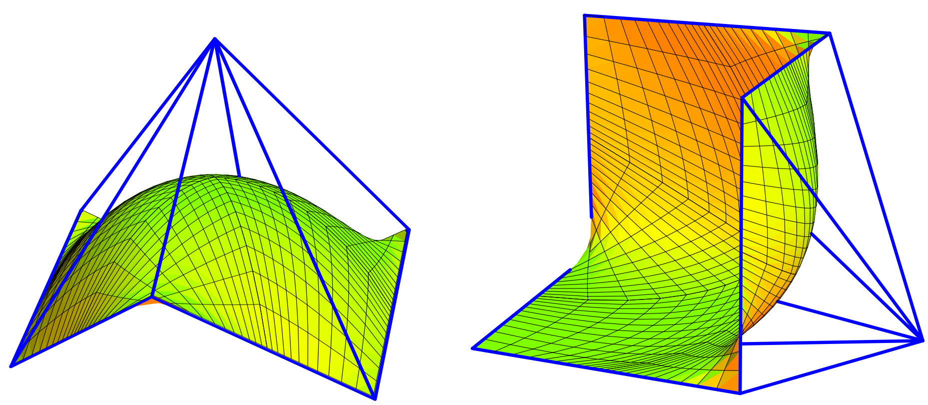


FIGURE 3. Hexagonal Toric Patch

edges are the line segments between the control points we have drawn. These are related to regular polyhedral subdivisions of the domain polygon D in exactly the same way that a patch is related to a projective toric variety. Namely, both the regular polyhedral subdivision and the toric variety lie in a large dimensional space, and the choice of control points defines a projection to 3-dimensional space. The image of the toric variety under this projection is the toric surface patch and the image of the polydehral subdivision is the control polyhedron. A regular subdivision encodes the limiting behavior of a toric variety under a toric deformation. Toric deformations are a fundamental technique in real algebraic geometry which underlie many constructions and are the basis for the best numerical algorithms we have for solving polynomial systems.

Toric surface patches are not yet known to have all of the desirable properties of Bézier patches (which are special cases), and more work is needed to determine if these esoteric objects can be made to be practical. The presence of experts on real toric varieties such as Sturmfels and Viro (who are in residence all Spring at MSRI) as well as Krasauskas (who is a senior visitor for April) at the workshop could help begin to clear up these issues.

As noted earlier, toric surface patches are manifestly objects from real algebraic geometry. This has already led to some significant advances. Notably Delaunay [8] studied non-standard real structures which give toric patches having the same flexibility and control as do ordinary patches, but with very interesting geometries: Some are cylindrical, which may lead to the novel modeling of shapes such as handles by relatively few patches or even a single patch. This is an example of the design flexibility that toric patches may allow.

There are obvious theoretical issues with the use of toric surface patches, such as controlling the boundary curves and smooth gluing of adjacent patches. Optimal parametrizations are another. Cox, Krasauskas, and Mustaţă [5] showed that the classical Delzant construction of a toric variety as a quotient of affine space leads to the universal rational parametrization of toric surfaces. This can be used to construct Bézier patches and more general toric patches of optimal degree on a fixed real toric surface (including those with non-standard real structure) having given boundary rational curves. Parametrizations of Bézier patches have many nice properties, including linear precision, which also provides numerical stability of some algorithms. It is not clear and in fact unlikely that many toric patches have rational parametrizations possessing linear precision. An alternative approach is to use the inverse image of the moment map. (The moment map of a toric variety comes from symplectic geometry, which is featured in a different program at the MSRI this Spring.) The inverse of the moment map is an algebraic (not necessarily rational) map that does give a parametrization possessing linear precision [13]. While this vector-valued algebraic function is in general multi-valued, it has a unique value with positive coordinates, and this may be effectively numerically computed using iterative proportional scaling. We hope to further explore these issues of parametrizations at the MSRI workshop.

Other algebraic surfaces, such as classical Del Pezzo surfaces may also be practical for geometric modeling. There is increasing evidence that these may have properties suited for geometric modeling. For example, Karčiauskas's pentagonal patch [10] is a (non-toric) Del Pezzo surface, and Warren's hexagonal patch [14] is toric Del Pezzo surface. Kharlamov, who is a senior researcher in the MSRI program, is the world's expert on real algebraic surfaces, and was a participant at the Vilnius workshop. Schicho, whom we intend to invite is not only a leader in symbolic computation and actively involved in geometric modeling, but has been studying real Del Pezzo surfaces with an eye for these applications. (This was a topic discussed at the COMPASS workshop in Austria)

There are also promising possibilities for parametrizations of these other rational surfaces. The universal rational parametrizations of Cox, Krasauskas, and Mustaţă of toric patches uses the Cox coordinate ring of the associated toric variety. Del Pezzo surfaces also have manageable Cox rings [2], (most algebraic varieties do not) and it is possible that this will lead to a similarly useful set of parametrizations for these surfaces. Since some of these surfaces can be made to degenerate to toric surfaces, there is the possibility that toric surfaces will emerge as the correct control structures for these exotic surface patches.

There are other topics that this workshop will likely address, as its goal is to further the collaboration between these two subjects. Some senior participants from the

Computer Aided Geometric Design community will be asked to address exactly this question of further mathematical challenges. We have already identified three, which are beyond the expertise of the Principal Investigator to discuss here:

- Parameterization of Level sets and Algebraic surface patches (A-patches).
- Differential properties of level sets (principal curvatures)
- Vector field topology (critical points, dual stream surfaces)

References

[1] C. Bajaj and A. Royyappa, The GANITH algebraic geometry toolkit, *Proceedings of the first annual conference on the implementation of computer algebra systems*, Lecture Notes in Computer Science 429, Springer-Verlag, pp. 268-269, 1990.

[2] V. Batyrev, O. Popov. The Cox ring of a Del Pezzo surface. `math.AG/0309111`.

[3] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, Algorithms*, Springer-Verlag, 1992.

[4] ———, *Using Algebraic Geometry*, Springer-Verlag, GTM 185, 1998.

[5] D. Cox, R. Krasauskas, and M. Mustaţă, Universal Rational Parametrizations and Toric Varieties, *Topics in Algebraic Geometry and Geometric Modeling*, Contemporary Mathematics, bf 334, AMS, 2003.

[6] D. Cox, T. Sederberg, and F. Chen, The moving line ideal basis of planar rational curves, *Computer Aided Geometric Design*, vol 15, (1998), pp. 803–827.

[7] D. Cox, M. Zhang, and R. Goldman, On the validity of implicitization by moving quadrics for rational surfaces with no base points, *Journal of Symbolic Computation*, vol. 29, (2000), pp. 419–440.

[8] Claire Delaunay, *Real structures on smooth compact toric surfaces*, *Topics in Algebraic Geometry and Geometric Modeling*, Contemporary Mathematics, bf 334, AMS, 2003.

[9] C. Hoffman, *Geometric and Solid Modeling*, Morgan Kaufman, San Mateo, California, 1989.

[10] K. Karčiauskas, Rational m-sided surface patches, in: R. Cripps (ed.), *The Mathematics of Surfaces VIII, Information Geometers*, 355–368, 1998.

[11] R. Krasauskas, Toric surface patches, *Adv. Comput. Math.* **17** (2002), no. 1-2, 89–133.

[12] T. Sederberg, *Implicit and Parametric Curves and Surfaces for Computer Aided Geometric Design*, Ph.D. Thesis, Department of Mechanical Engineering, Purdue University, 1983.

[13] F. Sottile Toric ideals, real toric varieties, and the moment map, in *Topics in Algebraic Geometry and Geometric Modeling*, Contemporary Mathematics, bf 334, AMS, 2003.

[14] J. Warren, Creating multisided rational Bézier surfaces using base points. *ACM Transactions on Graphics*, 11, 127–236, 1992.

[15] F. Winkler, R. Hemmecke, and E. Hillgarter, The CASA System, *Handbook of Computer Algebra: Foundations, Applications, Systems*, Springer-Verlag. 2000.

3. Personnel

This workshop will be open to any who wish to attend. We expect several people from the MSRI program, as well as some others from the mathematical community in the Bay Area. Some expected participants, their position and affiliation in April 2004, and research interests are given below. These people have all expressed interest in this topic and in this workshop. In particular, Krasauskas, Piene, and Gonzalez-Vega will attend. The role of toric geometry in geometric modeling is due to Krasauskas, and both Piene and Gonzalez-Vega are leaders among European algebraic geometers with an interest in geometric modeling and Computer-Aided Geometric Design (CAGD).

C. D'Andrea, Postdoc at Berkeley. *Algebraic Geometry and Resultants*
F. Ardilla, Postdoc at MSRI. *Computational Geometry*
L. Gonzalez-Vega, Professor at MSRI. *Real Algebraic Geometry and CAGD*
S. Hosţen, Professor at San Fransisco State University. *Toric Varieties*
S. Kharlamov, Professor at MSRI. *Real Algebraic Geometry*
R. Krasauskas, Professor at MSRI. *Geometric Modeling*
R. Piene, Professor at MSRI. *Algebraic Geometry and CAGD*
M. Rojas, Professor at MSRI. *Real Algebraic Geometry*
F. Sottile, Professor at MSRI. *Real Algebraic Geometry*
B. Sturmfels, Professor at MSRI. *Real Algebraic Geometry, Toric Varieties*
T. Theobald, Postdoc at MSRI. *Real Algebraic Geometry, Computational Geometry.*
O. Viro, Professor, MSRI. *Real Algebraic Geometry*

There are also a number of others whom we hope to invite. These people have been contacted and have agreed in principle to attend. The purpose of this grant is to enable their attendance. We give their affiliation and research interests.

C. Bajaj, Professor at U. Texas. *CAGD and visualization*
T. Beck, Graduate Student at Linz. *Computational Geometry and CAGD*
R. Goldman, Professor at Rice U. *Geometric Modeling*
B. Hassett, Professor at Rice U. *Algebraic Geometry*
K. Karčiauskas, Professor at Vilnius University. *Real Algebriac Geometry and Geometric Modeling*
A. Khetan, Postdoc at U. Massachusetts, *Resultants and Computational Algebraic Geometry*
M. Lucian, Research Scientist at Boeing Corporation. *Geometric Modeling*
M. Peternell, Professor at Vienna. *Computational Geometry and CAGD*
J. Peters, Professor at Florida. *Geometric Modeling*
J. Schicho, Professor at Linz. *Computational Algebraic Geometry and CAGD*
I. Soprounov, Postdoc at U. Massachusetts. *Resultants and Toric Geometry*
M. Zhang, Research Scientist M.D. Anderson Corporation. *Geometric Modeling, Drug Design*

We particularly want to have some people make extended visits to prolong the period of intensive contact between geometric modeling and real algebraic geometry. This makes the most sense for participants from Europe, as crossing 9 time zones for a weekend is excessive. We intend to invite Beck, Karčiauskas, Khetan, Peternell, and Schicho for extended visits.

A feature of the interactions between geometric modeling and algebraic geometry is the mathematical sophistication that is increasingly necessary to address problems arising from practical considerations gleaned through years of experience. A consequence of this is that we have identified only one young participant from the computer science community as a potential participant (Beck from Linz), but several younger mathematicians. This is perhaps inevitable as students in computer science do not (and cannot be expected to) have the mathematical background necessary to begin working on these problems requiring knowledge of algebraic geometry. It seems necessary to recruit mathematicians to work on certain theoretical issues arising in this area of computer science.

This is exactly what we are doing. Besides the current list of invited participants, we expect some Berkeley graduate students to attend, and we will try to identify some other young scientists to invite. The senior scientists will lend their experience to this endeavor and will gain a greater appreciation of the potentially useful interactions between geometric modeling and algebraic geometry. This will be useful as it is they who will be directing the work of others.

A CV and Bibliography for the Principal Investigator are at the end.

4. Scientific Program

The scientific program of this workshop will be planned by Sottile in consultation with Krasauskas and Gonzalez-Vega, who have a better knowledge of geometric modeling than Sottile. The program will include some longer presentations on mathematical challenges in geometric modeling, as well as shorter presentations of current research. It will also have a session devoted to an open discussion of future directions for these interactions. The presence of Miriam Lucian, a research scientist at Boeing Corporation, will help keep these discussions focused on potential applications. Other time will be set aside for less formal interactions, such as extended coffee breaks and planned meals.

We expect these discussions to continue in the following week as the people we plan to invite for a week (Beck, Karčiauskas, Khetan, Peternell, and Schicho) interact with the members of the MSRI community.

5. Outcomes

The success of this workshop will be measured by its long-term affect on these two communities and by the future contacts and collaborations between researchers in this area between algebraic geometry and geometric modeling. Thus its concrete outcomes will not be immediately apparent or easily quantifiable. It will take sustained interactions to bring these two communities closer. Nevertheless, we do expect that some research and collaborations will result directly from this meeting. The participants have been carefully selected for their willingness to engage in cross cultural scientific exchanges.

We do not plan to publish a proceedings, although this may change if there is sufficient interest among the participants. The proceedings of the Vilnius conference already providen a fairly up-to-date collection of surveys that can form the foundation for further scientific exchanges. There may not yet have been enough work in between these areas since then to warrant a proceedings. Also, while conference proceedings are valued in the Computer Science Community, they are not highly regarded within Mathematics. In fact, it is considered inadvisable for an untenured mathematician to publish their research in a conference proceedings.

In lieu of a proceedings, we will create a web page for the meeting. This would include material from the talks, such as detailed outlines or slides, and also a summary of the open discussion. This would be maintained at the MSRI WWW site `www.msri.org`, and include electronic versions of preprints arising from the meeting.

6. Facilities and Equipment Available.

The workshop will take place in the MSRI building, located on a hill overlooking the Berkeley campus of the University of California. We will use the MSRI lecture hall, which is sufficient for our purposes. Workshop participants have full access to computers and printers in two computing laboratories and the building is supplied with a wireless network. There is ample open space at MSRI with blackboards and an environment that is congenial for informal discussions. The participants who stay at the MSRI for a longer period would have additional access to resources, including office space.

For the weekend, there is regular bus service to the Lawrence Hall of Science just below the MSRI parking lot. Some participants will drive their cars to the Institute. Lunch is available nearby either at the Lawrence Hall of Science or in Tilden Park (a 10 minute walk from the institute) This arrangement has worked well at other MSRI weekend events.

7. Budget

The MSRI will administer this grant and allow the workshop to use its facilities. In addition, it is supporting this workshop by funding the visits of the (approximately 10) participants who are in residence at MSRI in this period. The source of this funding is largely through their continuing agreement with the Division of Mathematical Sciences in the NSF. This proposal only asks for funds to invite outside participants and thereby leverage the concentration of MSRI visitors in April with interests in interactions between geometric modeling and real algebraic geometry.

The table below lists the projected costs of the workshop in terms of supporting participants. The travel is covered up to predetermined maximums and the per diem ($90/day) to cover hotel and food. For the weekend meeting, participants will stay two days, but three nights. Taking travel time into account, we will offer 3 days at $90 per day. Participants, Beck, Karčiauskas, Khetan, Peternell, and Schicho will be invited for an additional week, 10 days in all, including one for travel.

Participant	Airfare	Hotel/food	total
Chandrajit Bajaj	500	270	770
Tobias Beck	1000	900	1900
Ron Goldman	500	270	770
Brendan Hassett	500	270	770
Amit Khetan	500	900	1400
Kęstutis Karčiauskas	1500	900	2400
Miriam Lucian	350	270	620
Martin Peternell	1000	900	1900
Jorg Peters	500	270	770
Josef Schicho	1000	900	1900
Ivan Soprounov	500	270	770
Ming Zhang	500	270	770
	500	270	770
	500	270	770
			16,280

The last two lines are because we expect to invite some additional (as yet unidentified) junior scientists, and want to budget for this activity. This budget exceeds the $15,000 we are requesting in this proposal. While some of the difference will be made up by inevitable changes in people's plans, the MSRI will cover the difference.

8. Biographical Sketch and Relevant Publications: Frank Sottile

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Citizenship: U.S.A.

Professional Preparation

Michigan State University, Honors B.S. in Physics, 1985.
University of Cambridge, CPGS, Maths Tripos Part III, with distinction, 1986.
University of Chicago, S.M. Mathematics, 1989.
University of Chicago, Ph.D. in Mathematics, 1994.

Appointments

Clay Mathematics Institute Senior Researcher, January-June 2004.
Assistant Professor, University of Massachusetts at Amherst, since 1999.
Van Vleck Assistant Professor, University of Wisconsin at Madison, 1999-2000.
MSRI Postdoctoral Fellow, Autumn 1998.
MSRI Postdoctoral Fellow, 1996-1997.
Term-Limited Assistant Professor, University of Toronto, 1994-1998.

Thesis Advisor: William Fulton

Postdoctoral Advisor: Bernd Sturmfels

Graduate Students: Jim Ruffo (current), Mariana Periera (current)

Postdoctoral Advisees: Greg Warrington (2000-3), Evgenia Soprunova (current)

Collaborators in past four years:

M. Aguiar, Texas A & M University.
N. Bergeron, York University, Toronto, Canada.
T. Braden, University of Massachusetts, Amherst.
Hervé Brönniman, Brooklyn Polytechnic University.
A. Buch, Arhus University, Denmark.
Olivier Devillers, INRIA, Sophia Antipholos, France.
H. Everett, INRIA, Nancy, France.
V. Kharlamov, IRMA Strasbourg, France.
S. Lazard, INRIA, Nancy, France.
C. Lenart, SUNY Albany.
G. Megyesi, UMIST, Manchester, England.
S. Robinson, Rutgers University.
E. Soprunova, University of Massachusetts.
B. Sturmfels, University of California, Berkeley.
T. Theobald, Teschnische Universität München, Germany.
S. Whitesides, McGill University.
S. van Willigenburg, University of British Columbia.
A. Yong, University of California, Berkeley.

Publications most related to project

(1) *Lines tangent to $2n-2$ spheres in $\mathbb{R}^n$*, with T. Theobald, Trans. Amer. Math. Soc., **354**, (2002), 4815–4829.
(2) *Maximally inflected real rational curves*, with V. Kharlamov. Moscow Mathematics Journal, to appear.
(3) *Common transversals and tangents to two lines and two quadrics in $\mathbb{P}^3$*, with G. Megyesi and T. Theobald. Discrete and Computational Geometry, to appear.
(4) *Toric ideals, real toric varieties, and the moment map*, for Proc. Algebraic Geometry & Geometric Modeling 2002, Vilnius, Lithuania, Contemporary Mathematics, AMS, to appear.
(5) *The envelope of lines meeting a fixed line that are tangent to two spheres*, with G. Megyesi.

Other Significant Publications

(1) *Enumerative geometry for the real Grassmannian of lines in projective space*, Duke Math. J., **87** (1997), 59–85.
(2) *Real rational curves in Grassmannians*, J. Amer. Math. Soc., **13** (2000), 333–341.
(3) *Intersection theory on spherical varieties*, with W. Fulton, R. MacPherson, and B. Sturmfels, J. Alg. Geom., **4** (1995), 181–193.
(4) *Pieri's formula for flag manifolds and Schubert polynomials*, Ann. de l'Institut Fourier, **46** (1996), 89–110.
(5) *Schubert polynomials, the Bruhat order, and the geometry of flag manifolds*, with N. Bergeron, Duke Math. J., **94** (1998), 273–423.

9. Current and Pending Support of the PI

National Security Agency, Conference Support, 2004
'Applications of Real Algebraic Geometry', 12-16 April 2004
Estimated Amount: $15,000
Awarded.

National Science Foundation, DMS-0134860, CAREER Award, 2002-2007,
'Computation, Combinatorics, and Reality in Algebraic Geometry,
with Applications',
Estimated Amount: $ 344,577.
Expiration Date: July 31, 2007.

National Security Agency, Conference Support, 2002-2004, MDA904-01-1-0125
'Discrete Mathematics in New England Conferences',
Estimated Amount: $16,400.
Expiration Date: August 31 2004.