PHASE LIMIT SETS OF LINEAR SPACES AND DISCRIMINANTS

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ABSTRACT. We show that the closure of the coamoeba of a linear space/hyperplane complement is the union of products of coamoebas of hyperplane complements coming from flags of flats, and relate this to the Bergman fan. Using the Horn-Kapranov parameterization of a reduced discriminant, this gives a description of the phase limit sets of discriminants and duals of toric varieties. When $d = 3$, we show that each 3-dimensional component of the phase limit set of the discriminant is a prism over a discriminant coamoeba in dimension 2, which has a polyhedral description by a result of Nilsson and Passare.

INTRODUCTION

Gelfand, Kapranov, and Zelevinsky [17] defined the amoeba of a subvariety X of a complex torus to be the set of lengths (coordinatewise logarithm of absolute value) of points in X. Previously, Bergman [3] and Bieri and Groves [5] showed that the asymptotic directions of the amoeba form a polyhedral complex in the sphere. The cone over this *logarithmic limit set* is now known as the tropical variety of X [30].

Passare (in a talk in 2004) defined the coamoeba of X to be the set of angles (coordinatewise arguments) of points in X. Its phase limit set [25] captures the asymptotic behavior of the angles in X. This has a combinatorial structure reflected in the tropical variety and the closure of a coamoeba is its union with its phase limit set. Coamoebas independently arose in the study of mirror symmetry and dimer models [13], have found further applications [16] in this area, and were used to study Euler-Mellin integrals [4]. The phase limit set was used to prove a version of Chow's Lemma for subvarieties of the torus [22], and to prove that complements of coamoebas are higher convex [26] in the sense of Gromov [18, Sect. $\frac{1}{2}$]. The key idea is that the phase limit set forms a combinatorial skeleton of the coamoeba.

The intersection of a linear space V in $\mathbb{C}^n$ with the torus $(\mathbb{C}^\times)^n$ is the complement $\mathcal{H}^c := V \setminus \mathcal{H}$ of a central arrangement $\mathcal{H}$ of n hyperplanes in V. We give (Theorem 2.3) a recursive description of the phase limit set of $\mathcal{H}^c$ in terms of the matroid of $\mathcal{H}$. While such a description is expected, as the tropical variety of $\mathcal{H}^c$ is described in terms of its matroid [1, 12, 30], our proof is direct and does not use a description of the tropical variety.

We use this description of the phase limit set of a hyperplane complement to study coamoebas of discriminants, extending some results of [24, 29] on discriminant coamoebas in dimension two to higher dimensions. The Horn-Kapranov parameterization of the reduced discriminant [20] is the composition of an inclusion of a d-dimensional linear space (actually the complement of a hyperplane arrangement) into $(\mathbb{C}^\times)^n$ followed by a homomorphism into $(\mathbb{C}^\times)^d \subset \mathbb{P}^d$. We use the action of the torus of the toric variety to describe the phase limit sets of both its dual variety and of the reduced discriminant in terms of the phase limit set of the linear space. We show that all d-dimensional components of the phase limit set admit a surjection to the closed coamoeba of a discriminant of smaller dimension $d' < d$, and that for some, the fibres are the full compact torus $\mathbb{T}^{d-d'}$. Thus these components are prisms over d'-dimensional closed coamoeba. When $d = 3$, all top-dimensional components are prisms over 2-dimensional closed discriminant coamoebae (Theorem 4.2).

A complete description of the coamoeba of a hyperplane complement remains open and appears to be a difficult problem; earlier versions of this manuscript led to the papers [27, 10, 9], which addressed questions of dimension. The dimension of the coamoeba (and also the amoeba) is determined by the matroid of the arrangement $\mathcal{H}$ [9]. The closure of the coamoeba is a semialgebraic set, as is its algebraic amoeba, and these have only been described for hyperplanes and lines [27]. Lastly, we expect (Conjecture 4.1) that the coamoeba of a discriminant in dimensions three and higher is a subset of its phase limit set. This is an analog of the solidity of discriminant amoebas [28]. Specifically, in dimension two, the discriminant coamoeba is the complement of a zonotope, and when $d = 3$, this conjecture and Theorem 4.2 imply that it is a finite union of prisms over such zonotope complements.

In Section 1, we recall the definitions of amoebas, coamoebas, and tropical varieties, discuss the structure of the phase limit set of a variety, and establish how these objects transform under homomorphisms of tori. In Section 2, we study the phase limit set of a linear space, proving Theorem 2.3 which describes the components of the phase limit set. We also relate this to the Bergman fan of the linear space. In Section 3 we recall the Horn-Kapranov parameterization of a reduced discriminant, which relates the phase limit set of a linear space to that of a discriminant. Corollary 3.7 describes the closed coameoba of the dual variety to a torus in terms of the coameoba of a linear space. Our structure theorems about phase limit sets of reduced discriminants are presented in Section 4

1. Coamoebas and phase limit sets

The compact torus $\mathbb{T} \subset \mathbb{C}$ is the set of unit norm complex numbers. The set $\mathbb{C}^\times$ of nonzero complex numbers is isomorphic to the product $\mathbb{R} \times \mathbb{T}$ under $z \mapsto (\log |z|, z/|z|)$. Thus $(\mathbb{C}^\times)^m \simeq \mathbb{R}^m \times \mathbb{T}^m$, and we let $\mathrm{Log} \colon (\mathbb{C}^\times)^m \to \mathbb{R}^m$ and $\mathrm{Arg} \colon (\mathbb{C}^\times)^m \to \mathbb{T}^m$ be the two projection maps. The *amoeba* $\mathscr{A}(X) \subset \mathbb{R}^m$ and *coamoeba* $c\mathscr{A}(X) \subset \mathbb{T}^m$ of a subvariety $X \subset (\mathbb{C}^\times)^m$ are its images under Log and Arg, respectively. Given an affine variety $V \subset \mathbb{C}^m$, we may also consider its amoeba and coamoeba, which are the amoeba and coamoeba of its intersection $V \cap (\mathbb{C}^\times)^m$ with the complement of the coordinate axes.

While these are images of algebraic varieties under real analytic maps, and thus are subanalytic sets, the amoeba is homeomorphic to a semialgebraic set and the coamoeba is a semialgebraic set [2]—this is explained in [27].

Proposition 1.1 ([19, 23])**.** *The amoeba and coamoeba of a subvariety $X \subset (\mathbb{C}^\times)^m$ have the same dimension as subanalytic subsets of $\mathbb{R}^m$ and $\mathbb{T}^m$, respectively.*

Proof. It suffices to show that the differentials of Log and Arg at $x \in X$ have the same rank. For $z \in (\mathbb{C}^\times)^m$, the kernel of $d_z \operatorname{Arg}$ is the linear subspace $z \cdot \mathbb{R}^m$ of $\mathbb{C}^m = T_z(\mathbb{C}^\times)^m$, the translation of $\mathbb{R}^m$ by z. Similarly, the kernel of $d_z \operatorname{Log}$ is the linear space $z \cdot (\sqrt{-1}\,\mathbb{R}^m) = \sqrt{-1}z \cdot \mathbb{R}^m$. Thus $v \in T_x X$ lies in the kernel of $d_x \operatorname{Log}$ if and only if $\sqrt{-1}v$ lies in the kernel of $d_x \operatorname{Arg}$. The result follows as $T_x X$ is stable under multiplication by $\sqrt{-1}$. $\qquad\square$

Work of Bergman [3] and of Bieri and Groves [5] showed that the set of asymptotic directions of an amoeba $\mathscr{A}(X)$ in $\mathbb{R}^m$ forms a polyhedral complex in the sphere. This *logarithmic limit set* is the collection of all accumulation points of sequences $\{\operatorname{Log}(x_i)/\|\operatorname{Log}(x_i)\| \mid i \in \mathbb{N}\}$ in the sphere where $\{x_i \mid i \in \mathbb{N}\} \subset X$ is unbounded. The cone over the logarithmic limit set is the *tropical variety $\mathscr{T}(X)$* of X [21].

The tropical variety has another concrete geometric interpretation. The integer lattice $\mathbb{Z}^m$ in $\mathbb{R}^m$ is identified with the lattice of cocharacters (one parameter subgroups) of $(\mathbb{C}^\times)^m$. Given a cocharacter $\alpha \colon \mathbb{C}^\times \to (\mathbb{C}^\times)^m$, we define a subvariety of $\mathbb{C}^\times \times (\mathbb{C}^\times)^m$,

$$\mathcal{X}(\alpha) \ := \ \{(t,x) \in \mathbb{C}^\times \times (\mathbb{C}^\times)^m \mid \alpha(t) \cdot x \in X\}\,.$$

Projection to the first coordinate gives a family of varieties $\mathcal{X}(\alpha) \to \mathbb{C}^\times$ over $\mathbb{C}^\times$.

A cocharacter α lies in the tropical variety $\mathscr{T}(X)$ when $\mathcal{X}(\alpha)$ is not closed in $\mathbb{C} \times (\mathbb{C}^\times)^m$, that is, when the scheme-theoretic limit

$$(1.1) \qquad\qquad\qquad\qquad \lim_{t \to 0} \mathcal{X}(\alpha)_t \ := \ \overline{\mathcal{X}(\alpha)}_0$$

is nonempty, where $\mathcal{X}(\alpha)_t$ is the fiber of $\mathcal{X}(\alpha)$ over $t \in \mathbb{C}^\times$ and $\overline{\mathcal{X}(\alpha)}_0$ is the fiber over the point $0 \in \mathbb{C}$ of the closure of $\mathcal{X}(\alpha)$ in $\mathbb{C} \times (\mathbb{C}^\times)^m$. Since the fiber $\mathcal{X}(\alpha)_t$ equals $\alpha(t)^{-1} \cdot X$, our condition that $\mathcal{X}(\alpha)$ is not closed is equivalent to the limit of translates $\alpha(t) \cdot X$ as $t \to \infty$ being nonempty. This limit (1.1) is an *initial scheme* of X and is written $\operatorname{in}_\alpha X$.

Since we realize the initial scheme as the fiber of a variety that is the closure of a family of isomorphic varieties, we may determine its dimension.

Proposition 1.2. *Suppose that X is irreducible of dimension d and that $\alpha \in \mathscr{T}(X)$. Then every irreducible component of $\operatorname{in}_\alpha X$ has dimension d.*

With these definitions, the tropical variety may also be defined as the closure in $\mathbb{R}^m$ of the rational cone over those cocharacters α such that $\operatorname{in}_\alpha X$ is nonempty,

$$\mathscr{T}(X) \ = \ \overline{\bigcup \{\mathbb{Q}_{\geq 0} \cdot \alpha \mid \operatorname{in}_\alpha X \neq \emptyset\}}\,.$$

The tropical variety admits the structure of a rational polyhedral fan [21], but this structure is not canonical. Fix such a fan structure on $\mathscr{T}(X)$. Its cones σ have the property that

$\mathrm{in}_\alpha X = \mathrm{in}_\beta X$ when both α and β lie in the relative interior of σ. (The *relative interior* of a cone σ is the complement of all proper faces of σ.) Write $\mathrm{in}_\sigma X$ for this common initial scheme. Moreover, the torus $\mathbb{C}_\sigma^\times$ generated by the cocharacters in σ acts freely on $\mathrm{in}_\sigma X$ and the orbit space is a subvariety of $(\mathbb{C}^\times)^m/\mathbb{C}_\sigma^\times$. (This may be seen in (1.1).) We also have that if $\tau, \sigma \in \mathscr{T}(X)$ are cones with τ a face of σ, then

$$(1.2) \qquad\qquad\qquad \mathrm{in}_\sigma(\mathrm{in}_\tau X) = \mathrm{in}_\sigma X.$$

While amoebas are always closed subsets of $\mathbb{R}^m$ (the map Log is proper), coamoebas are typically not closed. For an example, consider the coamoeba of the line ℓ in $(\mathbb{C}^\times)^2$ defined by $x+y+1 = 0$. In the fundamental domain $[-\pi, \pi]^2$ for $\mathbb{T}^2$, $c\mathscr{A}(\ell)$ is the complement of the hexagon that is the Minkowski sum of the segments between the origin and the points $(-\pi, 0)$, $(0, -\pi)$, and (π, π), respectively, and the union of the vertices $(\pi, 0)$, $(0, \pi)$, and (π, π), which are the images of the points of ℓ in the three quadrants of $\mathbb{R}^2$ that meet ℓ.

$$(1.3) \qquad\qquad c\mathscr{A}(\ell):$$
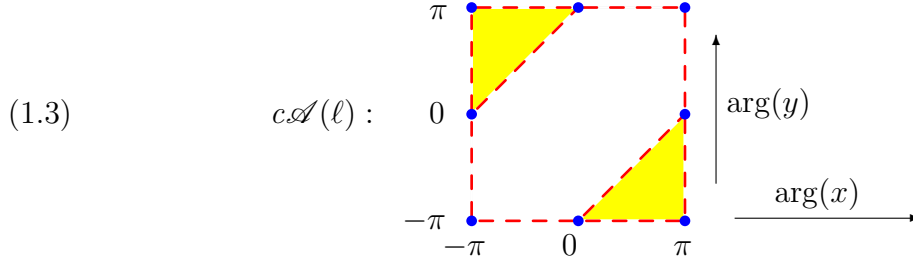
The set-theoretic difference $\overline{c\mathscr{A}(\ell)} \setminus c\mathscr{A}(\ell)$ is most of the three translated subtori

$$(1.4) \qquad (\pi, 0) + (0, \pi) \cdot \mathbb{T}, \quad (0, \pi) + (\pi, 0) \cdot \mathbb{T}, \quad \text{and} \quad (\pi, 0) + (-\pi, -\pi) \cdot \mathbb{T},$$

corresponding to the points where ℓ meets the coordinate lines in $\mathbb{P}^2$. These are the dotted lines in (1.3).

For a subvariety $X \subset (\mathbb{C}^\times)^m$, the *phase limit set* $\mathscr{P}^\infty(X)$ is the collection of all accumulation points of sequences of arguments $\{\mathrm{Arg}(x_i) \mid i \in \mathbb{N}\}$ where $\{x_i \mid i \in \mathbb{N}\} \subset X$ is unbounded. By definition, the closure of the coamoeba is its union with its phase limit set,
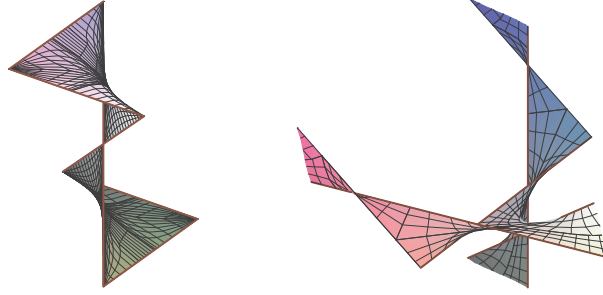
$$\overline{c\mathscr{A}(X)} = c\mathscr{A}(X) \cup \mathscr{P}^\infty(X).$$

We will call $\overline{c\mathscr{A}(X)}$ the *closed coamoeba* of X. The phase limit set of the line ℓ with coamoeba (1.3) consists of the three translated subtori (1.4). Figure 1 shows the coamoebas of two lines in $\mathbb{C}^3$ from [25], in a fundamental domain $[-\pi, \pi]^3$. Their boundaries, which are their phase limit sets, consist of four translated subtori in the directions $(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$, and $(-\pi, -\pi, -\pi)$, which are lines in these pictures[1]. These correspond to the four rays in the tropical variety of these lines. These observations hold in general.

Proposition 1.3 ([25, Theorem 1]). *The phase limit set of a subvariety X of a torus is the union of the coamoebas of its initial schemes,*

$$\mathscr{P}^\infty(X) = \bigcup c\mathscr{A}(\mathrm{in}_\sigma X).$$

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[1]For motion pictures, see `https://franksottile.github.io/research/stories/coAmoeba/index.html`.

FIGURE 1. Coamoebas of lines in $\mathbb{C}^3$.

This union is over all nonminimal cones σ in the tropical variety $\mathcal{T}(X)$ of X.

While this description refers to a particular fan structure on $\mathcal{T}(X)$, the phase limit set does not depend upon the fan structure. We will fix a particular fan structure on $\mathcal{T}(X)$. To simplify discussion, we assume that the minimal linear space contained in $\mathcal{T}(X)$, as a set, is its minimal cone, called its *lineality space*. This is needed when discussing tropical compactifications. Most other statements hold without this restriction.

The phase limit set of X may be understood via the tropical compactification [31] of X given by the fan $\mathcal{T}(X)$ [25, Sect. 4.2]. Let Z be the toric variety associated to the fan $\mathcal{T}(X)$ [14]. For each cone σ of Σ, there is an affine toric variety V_σ (which is $\mathrm{spec}(\mathbb{C}[\sigma^\vee \cap M])$, where M is the character lattice of $(\mathbb{C}^\times)^m$ and $\sigma^\vee$ is the cone dual to σ). The varieties V_σ form a $(\mathbb{C}^\times)^m$-equivariant open cover of Z. Furthermore, Z has a dense orbit and one orbit $\mathcal{O}_\sigma \simeq (\mathbb{C}^\times)^m/\mathbb{C}_\sigma^\times$ for each cone σ of $\mathcal{T}(X)$, where $\mathcal{O}_\sigma$ is the minimal orbit of V_σ and $\mathbb{C}_\sigma^\times$ is the subgroup of $(\mathbb{C}^\times)^m$ spanned by the cocharacters in σ. If $\tau \subset \sigma$ are cones of $\mathcal{T}(X)$, then $\mathcal{O}_\sigma$ lies in the closure of $\mathcal{O}_\tau$. The minimal cone μ of $\mathcal{T}(X)$ is its lineality space, and $X = \mathrm{in}_\mu X$ carries an action of $\mathbb{C}_\mu^\times$. The quotient $X/\mathbb{C}_\mu^\times$ is naturally a subvariety of the dense orbit $\mathcal{O}_\mu$ of Z, and its closure $\overline{X}$ in Z is a complete variety called the *tropical compactification* of X. (This is an actual compactification when $\mu = \{0\}$.)

For each cone σ of $\mathcal{T}(X)$, $\overline{X} \cap \mathcal{O}_\sigma$ is identified with the quotient $\mathrm{in}_\sigma X/\mathbb{C}_\sigma^\times$. Moreover, if $\tau \subset \sigma$ are cones of $\mathcal{T}(X)$, then $\overline{X} \cap \mathcal{O}_\sigma$ lies in the closure of $\overline{X} \cap \mathcal{O}_\tau$. Applying the map Arg, we see that $\mathbb{T}_\sigma := \mathrm{Arg}(\mathbb{C}_\sigma^\times)$ acts on the coamoeba $c\mathcal{A}(\mathrm{in}_\sigma X)$ with quotient

$$ c\mathcal{A}(\overline{X} \cap \mathcal{O}_\sigma) \ \subset \ c\mathcal{A}(\mathcal{O}_\sigma) \ = \ \mathbb{T}^m/\mathbb{T}_\sigma \, . $$

Let $\mathscr{P}_\sigma^\infty(X)$ be the closure of $c\mathcal{A}(\mathrm{in}_\sigma X)$. This has an action of $\mathbb{T}_\sigma$ with quotient the closure of $c\mathcal{A}(\overline{X} \cap \mathcal{O}_\sigma)$. We call $\mathscr{P}_\sigma^\infty(X)$ a *prism* over $c\mathcal{A}(\overline{X} \cap \mathcal{O}_\sigma)$ with fiber $\mathbb{T}_\sigma$. In (1.3), the dotted lines are prisms over points where the line meets the three coordinate axes of $\mathbb{P}^2$.

1.1. **Phase limit sets.** By Proposition 1.3, the phase limit set $\mathscr{P}^\infty(X)$ is the union of the $\mathscr{P}_\sigma^\infty(X)$, for σ not the lineality space μ. These sets $\mathscr{P}_\sigma^\infty(X)$ are the *strata* of the phase limit set $\mathscr{P}^\infty(X)$. If $\tau \subset \sigma$ are cones in $\mathcal{T}(X)$, then $c\mathcal{A}(\mathrm{in}_\sigma X)$ is a subset of $\mathscr{P}_\tau^\infty(X)$ as $\mathrm{in}_\sigma X$

is an initial scheme of $\text{in}_\tau X$ (1.2). A cone ρ of $\mathscr{T}(X)$ is a ray if it is minimal among those properly contain the lineality space μ. We deduce a strengthening of Proposition 1.3.

Lemma 1.4. *With these definitions, we have*

$$(1.5) \qquad \mathscr{P}^\infty(X) \;=\; \bigcup \mathscr{P}_\rho^\infty(X)\,,$$

the union over all rays ρ of $\mathscr{T}(X)$.

Not every ray ρ of $\mathscr{T}(X)$ is needed for the union (1.5). For example, refining the fan of $\mathscr{T}(X)$ may give new rays but does not change $\mathscr{P}^\infty(X)$.

Lemma 1.5. *Let Σ be a fan refining a given tropical fan $\mathscr{T}(X)$, and suppose that $\tau \in \Sigma$ is a cone that is not the lineality space. Then there exists a cone σ of $\mathscr{T}(X)$ such that $\tau \subset \sigma$ and τ meets the relative interior of σ, and $\mathscr{P}_\tau^\infty(X) = \mathscr{P}_\sigma^\infty(X)$.*

Proof. The existence (and uniqueness) of such a cone σ is because Σ refines $\mathscr{T}(X)$. Any cocharacter α that lies in the relative interior of τ also lies in the relative interior of σ. Thus $\text{in}_\tau X = \text{in}_\alpha X = \text{in}_\sigma X$, which implies the result. $\qquad\qquad\square$

The tropical variety $\mathscr{T}(\ell)$ of the line ℓ with coamoeba (1.3) consists of three rays,

$$\mathbb{R}_\geq(1,0)\,, \quad \mathbb{R}_\geq(0,1)\,, \quad \text{and} \quad \mathbb{R}_\geq(-1,-1)\,.$$

The corresponding initial schemes are

$$\text{in}_{(1,0)}\,\ell \;=\; \mathcal{V}(y{+}1)\,, \quad \text{in}_{(0,1)}\,\ell \;=\; \mathcal{V}(x{+}1)\,, \quad \text{in}_{(-1,-1)}\,\ell \;=\; \mathcal{V}(x{+}y)\,.$$

Their coamoebas are the three translated subtori of (1.4), which are prisms over points.

Example 1.6. The closed coamoeba of the plane Π defined by $x + y + z + 1 = 0$ in $\mathbb{C}^3$ is the complement in the fundamental domain $[-\pi, \pi]^3$ of the open zonotope that is the Minkowski sum of the four line segments from the origin to the points

$$(\pi, 0, 0)\,, \; (0, \pi, 0)\,, \; (0, 0, \pi)\,, \; (-\pi, -\pi, -\pi)\,.$$

These four segments are the primitive vectors of the rays in the tropical variety of Π, $\mathscr{T}(\Pi)$. We show the closed coamoeba of Π, this zonotope, and $\mathscr{T}(\Pi)$.
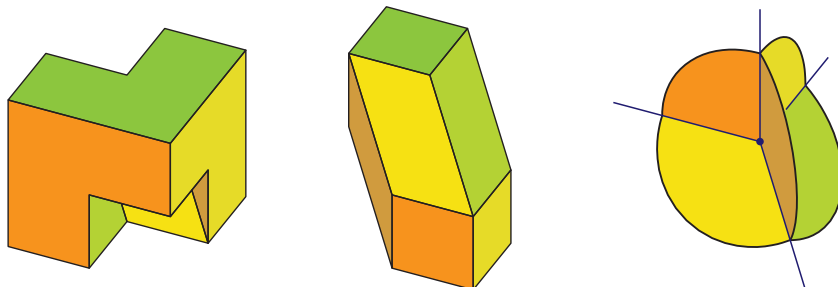
For each ray ρ in $\mathscr{T}(\Pi)$, $\text{in}_\rho\,\Pi$ is irreducible and gives a stratum of the phase limit set $\mathscr{P}^\infty(\Pi)$ which is a prism over the coamoeba of a line in a plane. For example, $\text{in}_{(0,0,1)}\,\Pi = x{+}y{+}1$, so

that the corresponding stratum $\mathscr{P}^\infty_{(0,0,1)}(\Pi)$ (second from the right in Figure 2) is the product of the closure of the coamoeba (1.3) and the subgroup $\mathbb{T}_{(0,0,1)}$, which is a vertical segment in the fundamental domain, and so is three-dimensional. The other three strata are also prisms over coamoebas of lines. Figure 2 shows these strata. To understand the last, project the
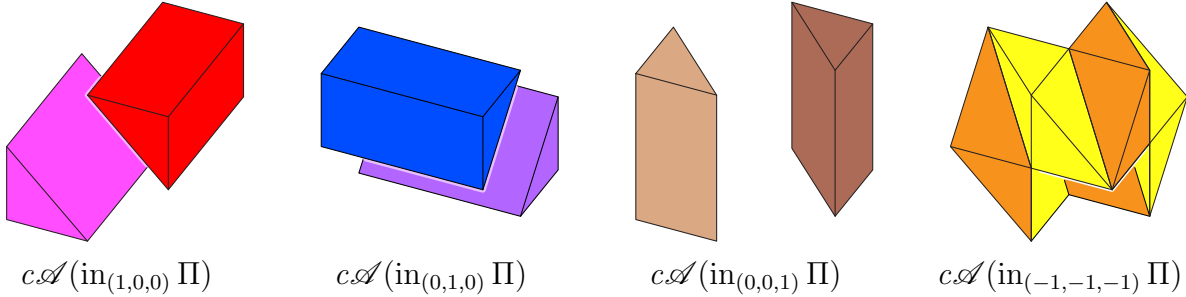

$$c\mathscr{A}\big(\mathrm{in}_{(1,0,0)}\,\Pi\big) \qquad c\mathscr{A}\big(\mathrm{in}_{(0,1,0)}\,\Pi\big) \qquad c\mathscr{A}\big(\mathrm{in}_{(0,0,1)}\,\Pi\big) \qquad c\mathscr{A}\big(\mathrm{in}_{(-1,-1,-1)}\,\Pi\big)$$

FIGURE 2. Strata of the phase limit set of a plane.

cube $[-\pi,\pi]^3$ along the vector $\mathbb{R}(-1,-1,-1)$ to a hexagonal fundamental domain (in place of the square fundamental domain of (1.3)). These strata cover $c\mathscr{A}(\Pi)$, with a general point of $c\mathscr{A}(\Pi)$ lying in two strata. If σ is one of the six two-dimensional cones of $\mathscr{T}(\Pi)$, then $\mathrm{in}_\sigma\,\Pi$ is a translated orbit of the torus $\mathbb{T}_\sigma$ so that $\mathscr{P}^\infty_\sigma(\Pi)$ is two-dimensional. These form the boundaries of the strata in Figure 2. $\diamond$

Example 1.6 illustrates a subtlety in the description of $\mathscr{P}^\infty(X)$. Both the coamoeba and phase limit set of Π are covered by any three top-dimensional strata.

Remark 1.7. This example also illustrates how the coamoeba of the plane Π is a union of polyhedra, as its full-dimensional strata are each prisms over a coamoeba of a line. This plane Π is also a (reduced) discriminant (see Example 3.1). Conjecture 4.1 posits that any discriminant coamoeba is a subset of its phase limit set. By Theorem 4.2, when $d = 3$ each component of the phase limit set a union of polyhedra [24], so Conjecture 4.1 implies that every discriminant coamoeba when $d = 3$ is a finite union of polyhedra. $\diamond$

1.2. **Functoriality of tropical objects.** We explore how these tropical objects behave under maps of tori. A homomorphism $\varphi\colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n$ of tori induces a homomorphism $\varphi\colon \mathbb{T}^m \to \mathbb{T}^n$ and a linear map $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ (we use the same notation for all maps derived from φ). These maps commute with Arg and Log, respectively. We collect some simple results. The first three are well-known; they were used in [8, 29].

Proposition 1.8. *Suppose that $X \subset (\mathbb{C}^\times)^m$ is a subvariety and $\varphi\colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n$ is a group homomorphism. Let Y be the closure of $\varphi(X)$, so that $\varphi\colon X \to Y$ is dominant.*

(1) *The amoeba $\mathscr{A}(Y) \subset \mathbb{R}^n$ is the closure of $\varphi(\mathscr{A}(X))$.*
(2) *The tropical variety $\mathscr{T}(Y)$ equals $\varphi(\mathscr{T}(X))$.*
(3) *The image $\varphi\big(c\mathscr{A}(X)\big) \subset \mathbb{T}^n$ of the coamoeba of X is dense in $c\mathscr{A}(Y)$ and we have*
$$c\mathscr{A}(Y) \subset \varphi\left(\overline{c\mathscr{A}(X)}\right) = \overline{\varphi(c\mathscr{A}(X))}.$$

(4) $\mathscr{P}^\infty(Y) \subset \mathscr{P}^\infty(X)$.

Proof. Claim (1) follows from the continuity of Log and φ as amoebas are closed. As a tropical variety is the cone over the logarithmic limit set, (1) implies (2). Claim (3) is a consequence of the continuity of Arg and φ, and that $\varphi \colon \mathbb{T}^m \to \mathbb{T}^n$ is proper.

For (4), let $\theta \in \mathscr{P}^\infty(Y)$. There exists an unbounded sequence $\{y_i \mid i \in \mathbb{N}\} \subset Y$ such that $\lim_{i\to\infty} \mathrm{Arg}(y_i) = \theta$. Since $\overline{\varphi(X)} = Y$, for all $i \in \mathbb{N}$ there exists a point $x_i \in X$ such that both distances between $\varphi(x_i)$ and y_i and between $\mathrm{Arg}(\varphi(x_i))$ and $\mathrm{Arg}(y_i)$ are at most 2^{-n}. Then the sequence $\{x_i \mid i \in \mathbb{N}\} \subset X$ is unbounded and $\lim_{i\to\infty} \mathrm{Arg}(\varphi(x_i)) = \theta$.

Let $\phi \in \mathbb{T}^m$ be an accumulation point of $\{\mathrm{Arg}(x_i) \mid i \in \mathbb{N}\}$ and replace $\{x_i \mid i \in \mathbb{N}\}$ by a subsequence such that $\lim_{i\to\infty} \mathrm{Arg}(x_i) = \phi$. Then $\phi \in \mathscr{P}^\infty(X)$ and $\varphi(\phi) = \theta$. $\square$

We prove a lemma about images of components of phase limit sets under φ.

Lemma 1.9. *Under the hypotheses of Proposition 1.8, suppose that we have fan structures on $\mathscr{T}(X)$ and $\mathscr{T}(Y)$ such that $\varphi \colon \mathscr{T}(X) \to \mathscr{T}(Y)$ is a map of fans. For any cone τ of $\mathscr{T}(Y)$, the stratum $\mathscr{P}_\tau^\infty(Y)$ of $\mathscr{P}^\infty(Y)$ is the union of images $\varphi(\mathscr{P}_\sigma^\infty(X))$ of strata $\mathscr{P}_\sigma^\infty(X)$ of $\mathscr{P}^\infty(X)$ such that $\varphi(\sigma)$ meets the relative interior of τ.*

Proof. We use tropical compactification. Let Z_X be the toric variety associated to the fan $\mathscr{T}(X)$ and Z_Y the toric variety associated to $\mathscr{T}(Y)$. As $\varphi \colon \mathscr{T}(X) \to \mathscr{T}(Y)$ is a map of fans, it induces a map $\varphi \colon Z_X \to Z_Y$ that is equivariant with respect to $\varphi \colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n$.

As φ is a map of fans, whenever σ is a cone of $\mathscr{T}(X)$ and τ is a cone of $\mathscr{T}(Y)$ such that $\varphi(\sigma) \subset \tau$, if $V_\sigma \subset Z_X$ and $V_\tau \subset Z_Y$ are the corresponding open sets of the tropical compactifications, then $\varphi(V_\sigma) \subset V_\tau$. When $\varphi(\sigma)$ meets the relative interior of τ this is refined to $\varphi(\mathcal{O}_\sigma) \subsetneq \mathcal{O}_\tau$ and we have that $\varphi(\mathbb{C}_\sigma^\times) \subset \mathbb{C}_\tau^\times$.

In particular, if μ is the lineality space of $\mathscr{T}(X)$ and ν the lineality space of $\mathscr{T}(Y)$, then $\varphi(\mu) \subset \nu$ and $\varphi(\mu)$ meets the relative interior of ν (which is ν). Thus the map $\varphi \colon (\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^n$ of tori induces a map $\varphi \colon (\mathbb{C}^\times)^m/\mathbb{C}_\mu^\times \to (\mathbb{C}^\times)^n/\mathbb{C}_\nu^\times$, which is the map $\varphi \colon \mathcal{O}_\mu \to \mathcal{O}_\nu$ of dense orbits of Z_X and Z_Y. Consequently, we have an induced map $\varphi \colon X/\mathbb{C}_\mu^\times \to Y/\mathbb{C}_\nu^\times$, which is dominant. The tropical compactifications $\overline{X}$ and $\overline{Y}$ are the closures of these quotients in Z_X and Z_Y, respectively. Thus $\varphi \colon \overline{X} \to \overline{Y}$ is a surjection.

Let τ be a cone of $\mathscr{T}(Y)$. By Lemma 1.8(2), $\varphi \colon \mathscr{T}(X) \to \mathscr{T}(Y)$ is a surjection, so there is a cone σ of $\mathscr{T}(X)$ such that $\varphi(\sigma) \subset \tau$ and $\varphi(\sigma)$ meets the relative interior of τ. Thus $\varphi(\mathcal{O}_\sigma) \subset \mathcal{O}_\tau$. Consequently, $\varphi(\overline{X} \cap \mathcal{O}_\sigma) \subset \overline{Y} \cap \mathcal{O}_\tau$. We have that $\mathrm{in}_\sigma X/\mathbb{C}_\sigma^\times = \overline{X} \cap \mathcal{O}_\sigma$ and $\mathrm{in}_\tau Y/\mathbb{C}_\tau^\times = \overline{Y} \cap \mathcal{O}_\tau$, so that $\varphi(\mathrm{in}_\sigma X/\mathbb{C}_\sigma^\times) \subset \mathrm{in}_\tau Y/\mathbb{C}_\tau^\times$. Since $\varphi(\mathbb{C}_\sigma^\times) \subset \mathbb{C}_\tau^\times$, we conclude that $\varphi(\mathrm{in}_\sigma X) \subset \mathrm{in}_\tau Y$. Since $\varphi \colon \overline{X} \to \overline{Y}$ is surjective, $\overline{Y} \cap \mathcal{O}_\tau$ is the union of the images $\varphi(\overline{X} \cap \mathcal{O}_\sigma)$ for such cones σ of $\mathscr{T}(X)$, and therefore $\mathrm{in}_\tau Y$ is the union of the images $\varphi(\mathrm{in}_\sigma X)$ for such cones σ. Applying Arg and taking closure completes the proof. $\square$

We finish with a technical lemma that is needed for our description of the phase limit set of a discriminant.

Lemma 1.10. *Under the hypotheses of Lemma 1.8, suppose that ρ is a ray of $\mathscr{T}(Y)$ that is not the image of any ray of $\mathscr{T}(X)$. Let $\varphi^*(\rho)$ be the set of cones σ of $\mathscr{T}(X)$ such that ρ meets the relative interior of σ. Then*

$$(1.6) \qquad \mathscr{P}_\rho^\infty(Y) \;\subset\; \bigcup_{\sigma \in \varphi^*(\rho)} \varphi(\mathscr{P}_\sigma^\infty(X)).$$

Proof. Let Σ_X and Σ_Y be the given fan structures on $\mathscr{T}(X)$ and $\mathscr{T}(Y)$. While φ may not be a map of fans (the image of a cone in Σ_X need not be contained in a single cone of $\mathscr{T}(Y)$), there are refinements of the given fans so that φ becomes a map of fans. Let us equip $\mathscr{T}(X)$ and $\mathscr{T}(Y)$ with such refinements.

The ray ρ remains a ray of $\mathscr{T}(Y)$. If a cone τ of $\mathscr{T}(X)$ has the property that $\varphi(\tau)$ meets the relative interior of ρ, then there is a cone σ of $\varphi^*(\rho)$ such that τ meets the relative interior of σ. Also, if $\sigma \in \varphi^*(\rho)$, then there is a cone τ of $\mathscr{T}(X)$ that meets the relative interior of σ and has the property that $\varphi(\tau)$ meets the relative interior of ρ. Since, for any such pair of cones τ and σ, $\mathscr{P}_\tau^\infty(X) = \mathscr{P}_\sigma^\infty(X)$, it is no loss to prove (1.6) for the refinements $\mathscr{T}(X)$ and $\mathscr{T}(Y)$. But this follows by Lemma 1.9. $\qquad \square$

2. THE PHASE LIMIT SET OF A LINEAR SPACE

A *multiset* is a set with some elements possibly repeated, so that $\{1, 2, 1\} = \{1, 1, 2\}$ and $\{1, 2\}$ are distinct multisubsets of $\mathbb{Z}$. Let V be a complex vector space with dual space V^* and let $B \subset V^*$ be a finite spanning multiset of nonzero vectors, which are linear forms on V. We use B and its subsets as indexing sets, so that $(\mathbb{C}^\times)^B$ is the torus $(\mathbb{C}^\times)^{|B|}$ whose coordinates are indexed by the elements of B and $\mathbb{T}^B$ is its compact subtorus. We extend this indexing to subsets of B. Consider the injective map $\lambda_B \colon V \to \mathbb{C}^B$ where $\lambda_B(v) = (\langle b, v \rangle \mid b \in B)$, whose image is a linear subspace. The tropicalization of its image (actually of $\lambda_B(V) \cap (\mathbb{C}^\times)^B$) is the *Bergman fan* of the matroid associated to B [1, 12]. We first describe the phase limit set of this image and then discuss its relation to the Bergman fan.

2.1. Coamoebas of hyperplane complements.
Let $\mathcal{H}_B$ or $\mathcal{H}$ be the central arrangement of hyperplanes defined by the vectors in B. We write $\mathcal{H}_B^c$ or $\mathcal{H}^c$ for the image of the hyperplane complement $V \smallsetminus \mathcal{H}_B$ in $(\mathbb{C}^\times)^B$ under the map λ_B.

The set $F \subset B$ of forms that vanish at a given point $x \in V$ is a *flat* of B. These are the coordinates of $\lambda_B(x)$ in $\mathbb{C}^B$ which vanish. Elements of the flat F vanish along the intersection L of the hyperplanes in $\mathcal{H}$ defined by forms in F. We also call L a *flat* of $\mathcal{H}$ (corresponding to F). The forms in F define a hyperplane arrangement $\mathcal{H}/L$ in V/L, called the *contraction* of $\mathcal{H}$ by L. Restricting the forms $F^c := B \smallsetminus F$ not in F (the hyperplanes not containing L) to L gives a hyperplane arrangement $\mathcal{H}|_L$ in L, called the *restriction* of $\mathcal{H}$ to L.

For a flat L of $\mathcal{H}$ and the set $F \subset B$ of forms that vanish along L, we have injective maps $\lambda_F \colon V/L \to \mathbb{C}^F$ and $\lambda_{F^c} \colon L \to \mathbb{C}^{F^c}$. As with $\mathcal{H}^c$, we consider the image $(\mathcal{H}/L)^c$ of $V/L \smallsetminus \mathcal{H}/L$ in $(\mathbb{C}^\times)^F$ under λ_F and the image $(\mathcal{H}|_L)^c$ of $L \smallsetminus \mathcal{H}|_L$ in $(\mathbb{C}^\times)^{F^c}$ under λ_{F^c}.

Example 2.1. Suppose that $V = \mathbb{C}^3$ and let $B = \{b_1, \ldots, b_6\}$ consist of the six row vectors of the matrix in Figure 3. These define a central hyperplane arrangement in $\mathbb{C}^3$ whose

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 2 & 0 \\
-2 & -1 & -2 \\
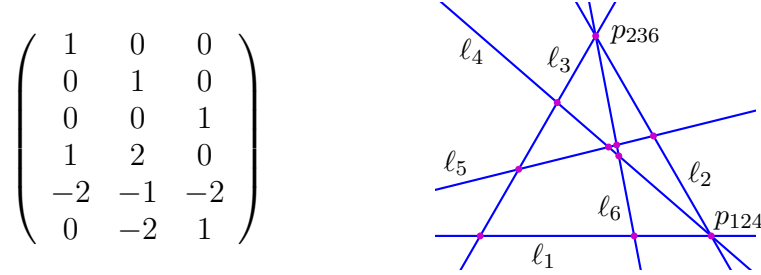0 & -2 & 1
\end{pmatrix}
$$

FIGURE 3. A vector configuration and corresponding line arrangement.

corresponding line arrangement is shown in an affine patch of $\mathbb{P}^2$ in Figure 3. The line ℓ_i is defined by the vector b_i of the ith row of the matrix. The remaining flats of $\mathcal{H}$ are their intersections, which are points in this picture. Nine of the points are incident to two lines, while two (p_{124}, p_{236}) are incident to three lines. The point $\ell_1 \cap \ell_5$ is not shown.

Consider the two flats, ℓ_1 and p_{124}. Viewed projectively, both $\mathcal{H}/\ell_1$ and $\mathcal{H}|_{p_{124}}$ are points but $\mathcal{H}|_{\ell_1}$ and $\mathcal{H}/p_{124}$ are hyperplane (point) arrangements on a $\mathbb{P}^1$:

$$
\mathcal{H}|_{\ell_1} \;=\; \overset{\displaystyle \ell_5 \quad \ell_3 \quad \ell_6 \quad \ell_2=\ell_4}{\text{────────}} \qquad\qquad \mathcal{H}/p_{124} \;=\; \overset{\displaystyle \ell_1 \quad \ell_4 \quad \ell_2}{\text{────────}} \qquad\qquad \diamond
$$

Let us now study the phase limit set $\mathscr{P}^\infty(\mathcal{H}^c)$ of the hyperplane complement $\mathcal{H}^c$. Let $\theta \in \mathscr{P}^\infty(\mathcal{H}^c)$. Then there is an unbounded sequence $\{z_i \mid i \in \mathbb{N}\} \subset \mathcal{H}^c$ such that $\lim_{i \to \infty} \mathrm{Arg}(z_i) = \theta$. As λ_B is an injection, there is a unique unbounded sequence $\{x_i \mid i \in \mathbb{N}\} \subset V \smallsetminus \mathcal{H}$ such that $\lambda_B(x_i) = z_i$ for every i. Since λ_B is linear, $\lambda_B(x_i/\|x_i\|) = z_i/\|x_i\|$ and $\mathrm{Arg}(z_i) = \mathrm{Arg}(z_i/\|x_i\|)$. (For $\|\cdot\|$, choose an inner product on V.) Thus we may assume that the sequence $\{x_i \mid i \in \mathbb{N}\}$ lies in the unit sphere of V. As the sphere is compact, we may further assume that $\{x_i \mid i \in \mathbb{N}\}$ converges to some point x in the sphere, and thus in $V \smallsetminus \{0\}$. Note that x may lie in $\mathcal{H}$. We say that $\theta \in \mathscr{P}^\infty(\mathcal{H}^c)$ *corresponds* to the point x.

We describe the set of points θ that correspond to $x \in V \smallsetminus \{0\}$. To that end, let $\{x_i \mid i \in \mathbb{N}\}$ be a sequence in $V \smallsetminus \mathcal{H}$ that converges to x such that $\{\mathrm{Arg}(\lambda_B(x_i)) \mid i \in \mathbb{N}\} \subset c\mathscr{A}(\mathcal{H}^c)$ converges to a point $\theta \in \mathbb{T}^B$. For each i, set $y_i := x_i - x$ and replace each x_i by $x + y_i$. Then $y_i \to 0$ and $x + y_i \in V \smallsetminus \mathcal{H}$ for every i. Let $F \subset B$ be the set of linear forms in B that vanish at x, which is a flat of B, and let $F^c := B \smallsetminus F$ be those linear forms that do not vanish at x. Writing $\mathbb{C}^B = \mathbb{C}^F \times \mathbb{C}^{F^c}$, $\mathbb{T}^B = \mathbb{T}^F \times \mathbb{T}^{F^c}$, and $\lambda_B = \lambda_F \times \lambda_{F^c}$, we have

$$
\mathrm{Arg}(\lambda_B(x + y_i)) \;=\; \mathrm{Arg}(\lambda_F(y_i)) \;\times\; \mathrm{Arg}(\lambda_{F^c}(x + y_i)) \;\in\; \mathbb{T}^F \times \mathbb{T}^{F^c}.
$$

(As $\mathbb{T}^F \times \mathbb{T}^{F^c} = \mathbb{T}^B$, we interpret this and similar assertions to mean that each factor on the left of the '$\in$' lies in the corresponding factor on the right.) Since $y_i \to 0$, but $x \neq 0$ is fixed, in the limit the second factor becomes $\mathrm{Arg}(\lambda_{F^c}(x))$. For the limit of the first factor, observe that multiplying y_i by any positive scalar does not change $\mathrm{Arg}(\lambda_F(y_i))$, and also that

$\lambda_F(y_i)$ depends not on y_i, but on y_i modulo the linear subspace L of V where the forms in F vanish. This linear space is the minimal flat of $\mathcal{H}$ containing x, so that $x \in L \smallsetminus \mathcal{H}|_L$ and $\mathrm{Arg}(\lambda_F(y_i)) \in \mathrm{Arg}((\mathcal{H}/L)^c)$.

Taking limits and considering all sequences $\{x_i \mid i \in \mathbb{N}\} \subset V \smallsetminus \mathcal{H}$ that converge to x gives the following.

Lemma 2.2. *The set of points of $\mathscr{P}^\infty(\mathcal{H}_B^c)$ that correspond to a point $x \in V$ is the product*

$$\overline{c\mathscr{A}((\mathcal{H}/L)^c)} \ \times \ \{\mathrm{Arg}(\lambda_{F^c}(x))\} \ \subset \ \mathbb{T}^F \times \mathbb{T}^{F^c} \, ,$$

where L is the minimal flat of $\mathcal{H}$ containing x and F^c are the forms that do not vanish at x. For a flat L of $\mathcal{H}$, the union of these sets for points $x \in L \smallsetminus (\mathcal{H}|_L)^c$ is

$$(2.1) \qquad\qquad \overline{c\mathscr{A}((\mathcal{H}/L)^c)} \ \times \ c\mathscr{A}((\mathcal{H}|_L)^c) \ \subset \ \mathbb{T}^F \times \mathbb{T}^{F^c} \, .$$

We use Lemma 2.2 to give a description of the phase limit set of a linear space $V \subset \mathbb{C}^n$.

Theorem 2.3. *Let $V \subset \mathbb{C}^n$ be a linear space that does not lie in any coordinate plane, $\mathcal{H} \subset V$ the central arrangement of hyperplanes given by the coordinates of $\mathbb{C}^n$, and $\mathcal{H}^c := V \smallsetminus \mathcal{H} = V \cap (\mathbb{C}^\times)^n$ its complement. Then, the phase limit set of $\mathcal{H}^c$ in $\mathbb{T}^n$ is*

$$(2.2) \qquad\qquad \mathscr{P}^\infty(\mathcal{H}^c) \ = \ \bigcup_L \left(\overline{c\mathscr{A}((\mathcal{H}/L)^c)} \times c\mathscr{A}((\mathcal{H}|_L)^c) \right) ,$$

the union over all nonzero proper flats L of $\mathcal{H}$, where $\mathcal{H}/L$ is the contraction of $\mathcal{H}$ by L and $\mathcal{H}|_L$ is the restriction of $\mathcal{H}$ to L.

Proof. The discussion preceding the statement of Lemma 2.2 shows that every point θ in the closure of the coamoeba of the hyperplane complement $\mathcal{H}^c$ corresponds to a point $x \in V$. The set of points corresponding to x depends upon the minimal flat L of $\mathcal{H}$ containing x, so that $x \in L \smallsetminus \mathcal{H}|_L$, and (2.1) describes all points θ corresponding to points in a given set $L \smallsetminus \mathcal{H}|_L$. Thus the closure of the coamoeba of the hyperplane complement is the union of the sets (2.1) over all flats, but this is the statement of Theorem 2.3. $\qquad\square$

We refine the statement of Theorem 2.3. Consider the first factor in the term of the decomposition (2.2) corresponding to a flat L, $\overline{c\mathscr{A}((\mathcal{H}/L)^c)}$, which is the union of the coamoeba of $(\mathcal{H}/L)^c$ with its phase limit set, by Proposition 1.3. By Theorem 2.3 this is the union over all nonzero flats Λ of $\mathcal{H}/L$ of a similar product, and the resulting term of the decomposition (2.2) of $\overline{c\mathscr{A}(\mathcal{H}^c)}$ gives the product

$$\overline{c\mathscr{A}((\mathcal{H}/\Lambda)^c)} \ \times \ c\mathscr{A}(((\mathcal{H}/L)|_\Lambda)^c) \ \times \ c\mathscr{A}((\mathcal{H}|_L)^c) \, .$$

(We have $(\mathcal{H}/L)/\Lambda = \mathcal{H}/\Lambda$.) These terms are indexed by flags of flats $\{0\} \neq L \subsetneq \Lambda$ of $\mathcal{H}$. Notice that when Λ is a hyperplane, $(\mathcal{H}/\Lambda)^c = \mathbb{C}^n/\Lambda \smallsetminus \Lambda/\Lambda \simeq \mathbb{C}^\times$, whose coamoeba is isomorphic to $\mathbb{T}$ and is closed.

Continuing, we express the closure of the coamoeba as a union of products of coamoebas of restrictions of contractions. Let

$$(2.3) \qquad\qquad \mathcal{L} \ = \ (\{0\} = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_k \subsetneq L_{k+1} = V)$$

be a flag of flats of $\mathcal{H}$. If $F_i \subset B$ is the set of linear forms in B that vanish along L_i, then

$$(2.4) \qquad \mathcal{F}(\mathcal{L}) \;=\; (B = F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_k \supsetneq F_{k+1} = \emptyset) \,,$$

is a flag of flats of B.

Let $\mathcal{L}$ be a flag of flats of $\mathcal{H}$ with associated flag of flats $\mathcal{F}(\mathcal{L})$ (2.4) of B. Then $(\mathcal{H}/L_{i-1})|_{L_i}$ is the hyperplane arrangement in L_i/L_{i-1} induced by $F_{i-1} \smallsetminus F_i$. Let $((\mathcal{H}/L_{i-1})|_{L_i})^c$ be the image of its complement in $(\mathbb{C}^\times)^{F_{i-1} \smallsetminus F_i}$ under the map $\lambda_{F_{i-1} \smallsetminus F_i}$. Define

$$(2.5) \qquad \mathcal{H}(\mathcal{L})^c \;:=\; \prod_{i=1}^{k+1} \big((\mathcal{H}/L_{i-1})|_{L_i}\big)^c \,.$$

Then its coamoeba in $\mathbb{T}^B$ is

$$c\mathscr{A}(\mathcal{H}(\mathcal{L})^c) \;=\; \prod_{i=1}^{k+1} c\mathscr{A}\big(((\mathcal{H}/L_{i-1})|_{L_i})^c\big) \;\subset\; \prod_{i=1}^{k+1} \mathbb{T}^{F_{i-1} \smallsetminus F_i} \;\subseteq\; \mathbb{T}^B \,.$$

With these definitions, we give a refined version of the decomposition in Theorem 2.3.

Corollary 2.4. *As* $\overline{c\mathscr{A}(\mathcal{H}^c)} = c\mathscr{A}(\mathcal{H}^c) \cup \mathscr{P}^\infty(\mathcal{H}^c)$, *we have*

$$\overline{c\mathscr{A}(\mathcal{H}^c)} \;=\; \bigcup_{\mathcal{L}} c\mathscr{A}(\mathcal{H}(\mathcal{L})^c) \,,$$

the union over all flags $\mathcal{L}$ of flats in $\mathcal{H}$.

The closure of $c\mathscr{A}(\mathcal{H}(\mathcal{L})^c)$ is the stratum of $\overline{c\mathscr{A}(\mathcal{H}^c)}$ corresponding to the flag of flats $\mathcal{L}$. We will show that this terminology is consistent with that in Section 1.

2.2. **Relation to the tropical variety.** The tropicalization $\mathscr{T}(\mathcal{H}_B^c)$ of $\mathcal{H}_B^c = \mathcal{H}^c$ depends only on the matroid of B [30]. We relate Corollary 2.4 to the structure of $\mathscr{T}(\mathcal{H}_B^c)$.

A basis of B is a subset E of B that forms a basis for V^*. Given a function $\omega \colon B \to \mathbb{R}$, written $\omega \in \mathbb{R}^B$, a subset $S \subset B$ has a weight, which is the sum of the values of ω at the elements of S. Let B_ω be the set of bases of B of maximal weight[2]. This forms a matroid on the ground set B. A vector in B is a loop in B_ω if it does not lie in any basis in B_ω. When B_ω has no loops, ω induces a flag of flats $\mathcal{L}(\omega)$ (2.3) of $\mathcal{H}$ and a corresponding flag of flats $\mathcal{F}(\omega)$ (2.4) of B with the following property:

$$(2.6) \qquad \omega \text{ is constant on each } F_{i-1} \smallsetminus F_i \text{ with } \omega|_{F_i \smallsetminus F_{i+1}} > \omega|_{F_{i-1} \smallsetminus F_i}.$$

Conversely, given any flag of flats $\mathcal{L}$ (2.3) of $\mathcal{H}$ with corresponding flag $\mathcal{F}(\mathcal{L})$ (2.4) of flats of B, for any weight $\omega \in \mathbb{R}^B$ satisfying (2.6), we have $\mathcal{L} = \mathcal{L}(\omega)$ and B_ω has no loops. Furthermore, these ω with $\mathcal{L}(\omega) = \mathcal{L}$ form the relative interior of a nonempty cone $\sigma_\mathcal{L} \subset \mathbb{R}^b$.

Let $\mathscr{T}(B)$ be the subset of $\mathbb{R}^B$ consisting of those ω such that B_ω contains no loops. This has a lineality space containing the vector $\mathbb{1}_B := (1, \ldots, 1)$, as $B_\omega = B_{\omega + \mathbb{1}_B}$ for any $\omega \in \mathbb{R}^B$. This set $\mathscr{T}(B)$ coincides with the tropical variety of the linear space $\lambda_B(V) \cap (\mathbb{C}^\times)^B = \mathcal{H}_B^c$,

---

[2]This convention, while opposite to [1], is necessary for compatibility with Section 1.

and it admits several fan structures, which are studied in [1, 12]. In the *fine subdivision* of $\mathcal{T}(B)$ from [1] each cone has the form $\sigma_{\mathcal{L}}$, for some flag of flats $\mathcal{L}$ of $\mathcal{H}_B$.

We relate this to the varieties $\mathcal{H}(\mathcal{L})^c$ (2.5).

Lemma 2.5. *Let $\omega \in \mathbb{Z}^B$ be a weight lying in the relative interior of the cone $\sigma_{\mathcal{L}}$. Then*

$$\mathrm{in}_\omega \, \mathcal{H}_B^c \;=\; \mathcal{H}(\mathcal{L})^c \,.$$

This lemma gives interpretations for every term in the decomposition of $\overline{c\mathscr{A}(\mathcal{H}_B^c)}$ of Corollary 2.4 in terms of cones in the fine subdivision of $\mathcal{T}(B)$, and thus the corresponding terms from Proposition 1.3 and the strata of $\mathscr{P}^\infty(\mathcal{H}_B^c)$.

Corollary 2.6. *For a cone $\sigma_{\mathcal{L}}$ in the fine subdivision of $\mathcal{T}(B)$, we have $\mathrm{in}_{\sigma_{\mathcal{L}}} \, \mathcal{H}_B^c = \mathcal{H}(\mathcal{L})^c$. When $\mathcal{L}$ is not the flag $\{0\} \subset V$, we have $\mathscr{P}^\infty_{\sigma_{\mathcal{L}}}(\mathcal{H}_B^c) = \overline{c\mathscr{A}(\mathcal{H}(\mathcal{L})^c)}$. The closure of each term of the decomposition of the closure of the coamoeba in Corollary 2.4 is a stratum of $\mathscr{P}^\infty(\mathcal{H}_B^c)$ for a cone in the fine subdivision of $\mathcal{T}(B)$.*

Definition 2.7. We simplify our notation, writing $\mathrm{in}_{\mathcal{L}} \, H_B^c$ for $\mathrm{in}_{\sigma_{\mathcal{L}}} \, \mathcal{H}_B^c = \mathcal{H}(\mathcal{L})^c$, and when $\mathcal{L}$ is $\{0\} \subsetneq L \subsetneq V$ for a flat L, we write $\mathrm{in}_L \, \mathcal{H}_B^c$ for $\mathrm{in}_{\mathcal{L}} \, H_B^c$. We define $\mathscr{P}^\infty_{\mathcal{L}}(\mathcal{H}_B^c)$ to be the stratum $\mathscr{P}^\infty_{\sigma_{\mathcal{L}}}(\mathcal{H}_B^c)$ and write $\mathscr{P}^\infty_L(\mathcal{H}_B^c)$ when $\mathcal{L}$ is the flag $\{0\} \subsetneq L \subsetneq V$.

Proof of Lemma 2.5. Suppose first that $k = 1$ in (2.3), so that the flag of flats $\mathcal{L}$ is $\{0\} \subsetneq L \subsetneq V$. Set $F := \{b \in B \mid b|_L \equiv 0\}$ and $F^c := B \smallsetminus F$. Then the weight ω is constant on each of F and F^c with $a := w|_F > w|_{F^c} =: b$.

For $x \in V \smallsetminus \mathcal{H}_B$ and $t \in \mathbb{C}^\times$, we have

$$\omega(t) \cdot \lambda_B(x) \;=\; (t^a \lambda_F(x) \,, \, t^b \lambda_{F^c}(x)) \;\in\; (\mathbb{C}^\times)^F \times (\mathbb{C}^\times)^{F^c} \,.$$

Fix a splitting of the map $V \twoheadrightarrow V/L$ and use it to write $V = V/L \oplus L$. We may write a point $x \in V$ as $x = (y, z)$ with $y \in V/L$ and $z \in L$. Then

$$\omega(t) \cdot \lambda_B(x) \;=\; (t^a \lambda_F(y) \,, \, t^b \lambda_{F^c}(y) + t^b \lambda_{F^c}(z)) \,,$$

as $\lambda_F(L) = \{0\}$ and $z \in L$.

Define an action ν of $\mathbb{C}^\times$ on V by $\nu(t).x = \nu(t).(y, z) = (t^{-a} y, t^{-b} z)$ for $t \in \mathbb{C}^\times$. Then

$$\omega(t) \cdot \lambda_B(\nu(t).x) \;=\; (\lambda_F(y), t^{b-a} \lambda_{F^c}(y) + \lambda_{F^c}(z)) \,.$$

As $b - a < 0$,

$$\lim_{t \to \infty} \omega(t) \cdot \lambda_B(\nu(t).x) \;=\; (\lambda_F(y), \lambda_{F^c}(z)) \,.$$

Since $\mathrm{in}_\omega \, \mathcal{H}_B^c = \lim_{t \to \infty} \omega(t) \cdot \mathcal{H}_B^c$ and ν is a linear action of $\mathbb{C}^\times$ on V, we have

$$\mathrm{in}_\omega \, \mathcal{H}_B^c \;=\; \lim_{t \to \infty} \omega(t) \cdot \lambda_B(V \smallsetminus \mathcal{H}_B) \;=\; \lambda_F(V/L \smallsetminus \mathcal{H}/L) \times \lambda_{F^c}(L \smallsetminus \mathcal{H}|_L) \;=\; \mathcal{H}(\mathcal{L})^c \,.$$

The case of a general flag of flats follows by an induction on k. $\qquad\square$

There is a coarser subdivision of $\mathcal{T}(B)$ which gives the *Bergman fan*, written $\mathcal{B}er(B)$. Two weights $\omega, \nu \in \mathbb{R}^B$ are *equivalent* if they induce the same matroid, $B_\omega = B_\nu$. Equivalence classes of weights form the relative interiors of the cones of the Bergman fan of B.

Example 2.8. Consider the Bergman fan for the matroid of Example 2.1. Let $e_1, \ldots, e_6$ be the coordinate basis of $\mathbb{R}^B = \mathbb{R}^6$ with e_i corresponding to b_i, that is, $e_i(b_j) = \delta_{ij}$. We describe the structure of $\mathcal{B}er(B)$ modulo $\mathbb{R}\mathbb{1}_B$. For $I \subset \{1, \ldots, 6\}$ let $e_I \in \mathbb{R}^B$ be the sum of e_i for $i \in I$ and set $B_I := B_{e_I}$. Observe that

$$B_{12} \;=\; \{\{b_1, b_2, b_3\}, \; \{b_1, b_2, b_5\}, \; \{b_1, b_2, b_6\}\} \,.$$

Since b_4 lies in no basis, it is a loop in B_{12}. Notice that B_{124} and B_{236} are loopless; for example $\{b_2, b_3, b_4\}$ is a basis for B_{124}, as are the three bases in B_{12} (and five others). Further observe that for $i = 1, \ldots, 6$, B_i consists of every basis that contains b_i and it also has no loops. In fact the eight vectors $\omega \in \{e_{124}, e_{236}, e_1, \ldots, e_6\}$ each give a ray of $\mathcal{B}er(B)/\mathbb{R}\mathbb{1}_B$.

This fan $\mathcal{B}er(B)/\mathbb{R}\mathbb{1}_B$ is pure two-dimensional, and it has fifteen maximal cones that occur in two types. One type are spanned by e_i, e_j for $i \neq j$ and not both in $\{1, 2, 5\}$ or in $\{2, 3, 6\}$ and these correspond to the nine flats $p_{ij} = \ell_i \cap \ell_j$ of the line arrangement. Write σ_{ij} for these nine cones. The second type consists of the six cones $\tau_{i,ijk}$ spanned by e_i and e_{ijk}, for $\{i, j, k\}$ either $\{1, 2, 4\}$ or $\{2, 3, 6\}$.

If we intersect $\mathcal{B}er(B)/\mathbb{R}\mathbb{1}_B$ with the sphere in $\mathbb{R}^6/\mathbb{R}\mathbb{1}_B$, we obtain a graph whose vertices correspond to rays of $\mathcal{B}er(B)$ and edges to maximal cones. We show this graph in Figure 4. In Example 4.9 we will show a projection of this fan into $\mathbb{R}^3$ where $e_i \mapsto b_i$.   ◇
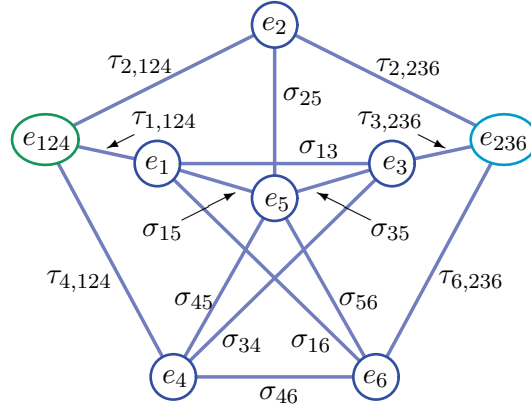

FIGURE 4. The Bergman fan as an graph.

By Lemma 1.4, the strata which cover $\mathscr{P}^\infty(\mathcal{H}_B^c)$ are among those indexed by the rays of the Bergman fan. We will follow [12] to describe the rays of the Bergman fan. We begin by introducing some more structure of matroids.

Definition 2.9. A finite spanning set $B \subset V^*$ of nonzero vectors is *disconnected* if there is a nontrivial direct sum decomposition $V^* = V_1^* \oplus V_2^*$ ($V_i^* \neq \{0\}$) with $B = (B \cap V_1^*) \cup (B \cap V_2^*)$. Otherwise B is *connected*.   ◇

This terminology refers to the connectivity of the exchange graph of the matroid of B. This is a graph with vertex set B that has an edge between b and b' if there exists a basis

E of B containing b such that $E \smallsetminus \{b\} \cup \{b'\}$ is also a basis. The vertex set of a connected component of the exchange graph of B forms a *connected component* of B.

Given a finite spanning set $B \subset V^*$ of nonzero vectors, let $B_1, \ldots, B_r$ be the connected components of B. If for each i we write V_i^* for the linear span of B_i, then we have the direct sum decomposition $V^* = V_1^* \oplus \cdots \oplus V_r^*$ and $B = B_1 \sqcup \cdots \sqcup B_r$. Let $c(B) := r$ be the number of components of B. If we set $V_i := \bigcap \{(V_j^*)^\perp \mid i \neq j\}$, then these subspaces of V are in direct sum and $V = V_1 \oplus \cdots \oplus V_r$. For each i, B_i defines a hyperplane arrangement $\mathcal{H}_{B_i}$ or $\mathcal{H}_i$ in V_i with complement $\mathcal{H}_i^c \subset (\mathbb{C}^\times)^{B_i}$.

We give some consequences of this definition.

Lemma 2.10. *Suppose that $B \subset V^*$ is a finite spanning set of nonzero vectors with connected components $B_1, \ldots, B_r$ and corresponding decompositions $V^* = V_1^* \oplus \cdots \oplus V_r^*$ and $V = V_1 \oplus \cdots \oplus V_r$ as above.*

    (1) *A set $E \subset B$ is a basis for B if and only if for each i, $E_i := E \cap B_i$ is a basis for B_i.*

    (2) *The map $\lambda_B \colon V \to \mathbb{C}^B$ is the product (direct sum) of maps $\lambda_{B_i} \colon V_i \to \mathbb{C}^{B_i}$. We have the product decomposition of the hyperplane complement*

$$\mathcal{H}_B^c = \mathcal{H}_{B_1}^c \times \mathcal{H}_{B_2}^c \times \cdots \times \mathcal{H}_{B_r}^c,$$

    and a similar product decompositions for its coamoeba, amoeba, and Bergman fan.

The decomposition of $\mathcal{H}_B^c$ from Lemma 2.10(2) is because $\mathcal{H}_B^c = \mathcal{H}(\mathcal{L})^c$, where $\mathcal{L}$ is any of the $r!$ flags of flats in which each flat L_j is a direct sum of some j of the V_i. In fact, any flag $\mathcal{L}$ that coarsens such a flag is also trivial in that $\mathcal{H}(\mathcal{L})^c = \mathcal{H}_B^c$. A consequence of Lemma 2.10 is that to understand $\mathcal{H}_B^c$ and its tropical objects it suffices to restrict ourselves to the case when $B \subset V^*$ forms a connected matroid.

Suppose that $B \subset V^*$ is a connected matroid. A flat L of $\mathcal{H}$ is a *flacet* if and only if the flat $F(L) := \{b \in B \mid b|_L \equiv 0\}$ and its complement $F(L)^c := B \smallsetminus F(L)$ are connected matroids. For the vector configuration/line arrangement of Example 2.1, the flacets are the lines (hyperplanes) and the two points p_{124} and p_{236}. No other point $p_{ij} = \ell_i \cap \ell_j$ is a flacet, as $B_{p_{ij}} = \{b_i, b_j\}$ is a basis for $(V/p_{ij})^*$, showing that it is disconnected. The lines ℓ are flacets because each line contains more than two points, for otherwise $B \smallsetminus F(\ell)$ is a basis for L^*, where L is the 2-plane corresponding to ℓ.

Given a flacet L, let $\omega_L \in \mathbb{R}^B$ be the indicator function of $F(L)$. This is the function $\omega_L \colon B \to \mathbb{R}$ whose value at $b \in B$ is 1 if $b|_L \equiv 0$ and 0 otherwise. The purpose of this definition is the following.

Proposition 2.11 ([12, Proposition 2.6])**.** *The rays of the Bergman fan $\mathcal{B}er(B)$ of a connected matroid are generated by the indicator functions of its flacets.*

In [12], the Bergman fan was studied as a subfan of the normal fan of the matroid polytope. Its rays define facets, hence the term flacets. By Lemma 1.4, we have the following.

Corollary 2.12. *When B is connected, the phase limit set of $\mathcal{H}_B^c$ is the union of the strata $\mathscr{P}_{\mathcal{L}}^\infty(\mathcal{H}_B^c)$ for flags $\mathcal{L} \colon \{0\} \subset L \subset V$, where L is a flacet of $\mathcal{H}_B$.*

3. The reduced discriminant

We recall some fundamental properties of discriminants. Much of this is found in [17, 20]. Let $A \subset \mathbb{Z}^m$ be a multiset of n vectors that span $\mathbb{Z}^m$ and lie in an affine hyperplane. If u is the primitive integer vector normal to that hyperplane, then $\langle u, a \rangle = 1$ for all $a \in A$, as A spans $\mathbb{Z}^m$. We adopt the notational convention that coordinates in $\mathbb{Z}^n$, $(\mathbb{C}^\times)^n$, and $\mathbb{C}^n$ are indexed by elements of A, and we write $\mathbb{Z}^A$, $(\mathbb{C}^\times)^A$, and $\mathbb{C}^A$ for these spaces.

The surjective map of free abelian groups $\mathbb{Z}^A \to \mathbb{Z}^m$ defined by

$$(3.1) \qquad\qquad \lambda_A \,:\, \mathbb{Z}^A \ni (c_a \mid a \in A) \;\longmapsto\; \sum_{a \in A} c_a \cdot a \;\in\; \mathbb{Z}^m$$

has kernel a free abelian group of rank $d := n - m$. Choose an isomorphism λ_B between $\mathbb{Z}^d$ and this kernel. This is given by n vectors $B = \{b_a \mid a \in A\}$ in the dual of $\mathbb{Z}^d$,

$$\lambda_B \,:\, \mathbb{Z}^d \ni y \;\longmapsto\; (\langle b_a, y \rangle \mid a \in A) \,.$$

This collection B is a *Gale dual* to A. Composing with λ_A, we obtain

$$\forall y \in \mathbb{Z}^d \qquad 0 \;=\; \sum_{a \in A} \langle b_a, y \rangle \cdot a \,.$$

Pairing this with the primitive vector u such that $\langle u, a \rangle = 1$ for all $a \in A$ gives

$$0 \;=\; \sum_{a \in A} \langle b_a, y \rangle \;=\; \Big\langle \sum_{a \in A} b_a, y \Big\rangle ,$$

for all $y \in \mathbb{Z}^d$, which implies that $0 = \sum_{a \in A} b_a$.

Example 3.1. ($m = 1$) Let A be the multisubset $\{1, \ldots, 1\}$ of $\mathbb{Z}$. The map $\lambda_A \colon \mathbb{Z}^{d+1} \to \mathbb{Z}$ is the sum of the coordinates, $\lambda_A(c_a \mid a \in A) = \sum_{a \in A} c_a$. A Gale dual B to A is given by the rows of the $(d+1) \times d$ matrix

$$(3.2) \qquad\qquad \begin{pmatrix} I_d \\ -1 \;\cdots\; -1 \end{pmatrix} ,$$

where I_d is the $d \times d$ identity matrix.

More interestingly, let A be the columns of the matrix shown in Figure 5. This lies in the

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & 1 & 2 \end{pmatrix}$$
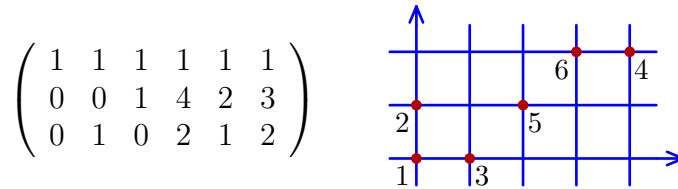

FIGURE 5. Vectors in A and their affine configuration.

affine hyperplane where the first coordinate is 1. Figure 5 shows the points of A in that affine hyperplane. This is Gale dual to the configuration B of Example 2.1.   ◇

An element $b \in \mathbb{Z}^d$ gives a group homomorphism $\mathbb{C}^\times \to (\mathbb{C}^\times)^d$ that sends $t \in \mathbb{C}^\times$ to $t^b := (t^{b_1}, \dots, t^{b_d})$. Let $\varphi_B \colon (\mathbb{C}^\times)^A \to (\mathbb{C}^\times)^d$ be the group homomorphism defined by

$$\varphi_B(z_a \mid a \in A) := \prod_{a \in A} z_a^{b_a} \in (\mathbb{C}^\times)^d.$$

An element $a \in \mathbb{Z}^m$ gives the monomial $x^a := x_1^{a_1} \cdots x_m^{a_m}$, which is a character of $(\mathbb{C}^\times)^m$. The characters in A give a group homomorphism $(\mathbb{C}^\times)^m \to (\mathbb{C}^\times)^A$ defined by

(3.3) $$\varphi_A \colon (\mathbb{C}^\times)^m \ni x \longmapsto (x^a \mid a \in A) \in (\mathbb{C}^\times)^A.$$

These maps $\{1\} \to (\mathbb{C}^\times)^m \xrightarrow{\varphi_A} (\mathbb{C}^\times)^A \xrightarrow{\varphi_B} (\mathbb{C}^\times)^d$ form an exact sequence of groups which is obtained from the exact sequence $\mathbb{Z}^d \xrightarrow{\lambda_B} \mathbb{Z}^A \xrightarrow{\lambda_A} \mathbb{Z}^m \to \{0\}$ by applying $\mathrm{Hom}(\mathbb{C}^\times, \bullet)$.

Let Y_A be the image of $(\mathbb{C}^\times)^m$ in $(\mathbb{C}^\times)^A$ under φ_A. The affine toric variety X_A is its closure in $\mathbb{C}^A$. Writing $\check{\mathbb{C}}^A$ for the dual space of hyperplanes in $\mathbb{C}^A$, the *dual variety* $X_A^* \subset \check{\mathbb{C}}^A$ is the closure of the set of hyperplanes that contain a tangent space of Y_A. When X_A^* is a hypersurface, its defining equation is the A-discriminant, and A is *nondefective*. There are several characterizations of nondefective configurations [6, 11, 15] whose equivalence is shown explicitly in [7]. By [6, Thm. 18], the configuration A is nondefective if and only if there is a complete flag of flats $\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{d-1} \subsetneq B$ such that for all $j = 1, \dots, d-1$

(3.4) $$\sum_{b \in F_j} b \quad \text{does not lie in the span of } F_{j-1}.$$

Call such a flag of flats in B and its corresponding flag $\mathcal{H}_B$ is a *non-splitting flag*. Observe that a non-splitting flag contains a flat of each possible rank $1, \dots, d-1$. A flat L of $\mathcal{H}_B$ that lies in a non-splitting flag is a *non-splitting flat*.

The group $(\mathbb{C}^\times)^m$ acts on both X_A and X_A^* via the homomorphism φ_A. While this has a single dense orbit in X_A, its orbits in X_A^* are more interesting. The closure in $\mathbb{C}^d$ of the image D_B of $X_A^* \cap (\mathbb{C}^\times)^A$ under the map φ_B is a space of orbits of $(\mathbb{C}^\times)^m$ on X_A^*. This is the *reduced discriminant* of A. When A is nondefective, it is a hypersurface.

When A is a pyramid, X_A^* is contained in a coordinate hyperplane, B contains a zero vector and A is defective. We assume that A is not a pyramid. Let $Y_A^* \subset X_A^*$ denote the dense open stratum consisting of all hyperplanes which do not contain any coordinate axis, and which contain the tangent space of a point of Y_A.

The maps $\lambda_A \colon \mathbb{Z}^A \twoheadrightarrow \mathbb{Z}^m$ and $\lambda_B \colon \mathbb{Z}^d \hookrightarrow \mathbb{Z}^A$ of free abelian groups extend to linear maps $\lambda_A \colon \mathbb{C}^A \twoheadrightarrow \mathbb{C}^m$ and $\lambda_B \colon \mathbb{C}^d \hookrightarrow \mathbb{C}^A$, and this agrees with the notation of Section 2. Write $\mathcal{H}_B$ for the central hyperplane arrangement in $\mathbb{C}^d$ obtained from pulling back the coordinate hyperplanes of $\mathbb{C}^A$ along λ_B. Its hyperplanes are defined by the vectors in B. Write $\mathcal{H}_B^c$ for the image $\lambda_B(\mathbb{C}^d \smallsetminus \mathcal{H}_B) \subset (\mathbb{C}^\times)^A$ of its complement.

Proposition 3.2. *The dual variety X_A^* is parameterized by $(\mathbb{C}^\times)^m \times \mathcal{H}_B^c$ via*

(3.5) $$(x, y) \longmapsto (x^a \cdot \langle b_a, y \rangle \mid a \in A).$$

The reduced discriminant D_B is parameterized by the Horn–Kapranov uniformization ψ, which is the composition

$$(3.6) \qquad \psi \,:\, (\mathbb{C}^d \setminus \mathcal{H}_B) \xrightarrow{\;\lambda_B\;} (\mathbb{C}^\times)^A \xrightarrow{\;\varphi_B\;} (\mathbb{C}^\times)^d \,.$$

That is, the image of the Horn–Kapranov parameterization is a dense subset of D_B. It has dimension at most $d-1$.

Proof. We derive the parameterization (3.5) of X_A^*. The incidence variety

$$\Gamma \;:=\; \{(z, H) \in Y_A \times \check{\mathbb{C}}^A \mid T_z Y_A \subset H\}\,,$$

has projections to Y_A and X_A^*. The torus $(\mathbb{C}^\times)^m$ acts on $\check{\mathbb{C}}^A$, on X_A^*, and on Γ.

Let $\mathbb{1} \in (\mathbb{C}^\times)^A$ be the identity of the torus. Thus Γ is determined by its fiber $\Gamma_{\mathbb{1}}$ over $\mathbb{1} \in Y_A$, which is the set of hyperplanes containing the tangent space $T_{\mathbb{1}} Y_A$. Differentiating (3.3) shows that this tangent space is the image of $\mathbb{C}^m$ under the linear map

$$d_{\mathbb{1}}\varphi_A \,:\, \mathbb{C}^m \ni x \;\longmapsto\; (\langle x, a\rangle \mid a \in A)\,.$$

A hyperplane (given by the coefficients $(c_a \mid a \in A)$ of its defining linear form) lies in $\Gamma_{\mathbb{1}}$ if and only if its pullback to $\mathbb{C}^m$ along this map $d_{\mathbb{1}}\varphi_A$ vanishes. This pullback is simply the map λ_A (3.1), and so the space of linear forms vanishing on $T_{\mathbb{1}} X_A$ is $\lambda_B(\mathbb{C}^d)$. Thus

$$\Gamma \;=\; \{(\varphi_A(x), \varphi_A(x) \cdot \lambda_B(y)) \mid x \in (\mathbb{C}^\times)^m \ \text{ and } \ y \in \mathbb{C}^d\}\,.$$

Projecting to the second coordinate parameterizes X_A^*, and gives (3.5).

To find the the image of X_A^* under φ_B, we apply φ_B to the parameterization (3.5), obtaining

$$(3.7) \qquad \prod_{a \in A} (x^a \cdot \langle b_a, y\rangle)^{b_a} \;=\; \prod_{a \in A} (x^a)^{b_a} \cdot \prod_{a \in A} \langle b_a, y\rangle^{b_a} \;=\; \prod_{a \in A} \langle b_a, y\rangle^{b_a}\,,$$

as $\mathbb{1} = \prod_{a \in A} (x^a)^{b_a}$. This is because it is the image of $(\mathbb{C}^\times)^m$ in $(\mathbb{C}^\times)^d$ in the exact sequence $(\mathbb{C}^\times)^m \xrightarrow{\varphi_A} (\mathbb{C}^\times)^A \xrightarrow{\varphi_B} (\mathbb{C}^\times)^d$.

As $\sum_a b_a = 0$, the map φ_B is homogeneous of degree zero, so that for $x \in \lambda_B(\mathbb{C}^d) \cap (\mathbb{C}^\times)^A$ and $t \in \mathbb{C}^\times$, we have $t \cdot x \in \lambda_B(\mathbb{C}^d) \cap (\mathbb{C}^\times)^A$ and $\varphi_B(x) = \varphi_B(t \cdot x)$. Thus the image $\varphi_B(\lambda_B(\mathbb{C}^d) \cap (\mathbb{C}^\times)^A)$ has dimension at most $d-1$. $\qquad\square$

Remark 3.3. By (3.5), X_A^* is the closure of all translations of $\mathcal{H}_B^c$ by the torus $(\mathbb{C}^\times)^m$ acting on $(\mathbb{C}^\times)^A$ through the map φ_A. By (3.6), the reduced discriminant D_B is the image of the linear space $\lambda_B(V \setminus \mathcal{H}_B) = \mathcal{H}_B^c$ under the map φ_B. $\qquad\diamond$

Example 3.4. We consider the reduced discriminants from Example 3.1. When $A = \{1, \ldots, 1\}$ and B consists of the rows of the matrix (3.2), the map ψ (where defined) is

$$\psi \,:\, (y_1, \ldots, y_d) \;\longmapsto\; \left(-\frac{y_1}{y_1 + \cdots + y_d}, \ldots, -\frac{y_d}{y_1 + \cdots + y_d}\right) \;\in\; (\mathbb{C}^\times)^d$$

The image is (a dense subset of) the hyperplane $x_1 + \cdots + x_d + 1 = 0$. Thus the line with coamoeba (1.3) and the plane of Example 1.6 are reduced discriminants.

For the configuration from Example 2.1, $\psi(x, y, z)$ is

$$\left(\frac{x(x+2y)}{(-2x-y-2z)^2} \,,\, \frac{y(x+2y)^2}{(-2x-y-2z)(-2y+z)^2} \,,\, \frac{z(-2y+z)}{(-2x-y-2z)^2} \right) .$$

This is the hypersurface in $\mathbb{C}^3$ defined by

$$-432p^6 - 1024p^5q^2 - 1152p^5q + 864p^5r + 216p^5 + 1280p^4q^2r + 768p^4q^2 + 1584p^4qr$$
$$+832p^4q - 432p^4r^2 + 216p^4r - 27p^4 - 40p^3q^2r^2 + 512p^3q^2r - 192p^3q^2 + 1584p^3qr^2 + 532p^3qr$$
(3.8) $\quad -200p^3q + 4000p^2q^3r^2 - 40p^2q^2r^3 + 5038p^2q^2r^2 - 208p^2q^2r + 16p^2q^2 - 1152p^2qr^3 + 832p^2qr^2$
$$-200p^2qr + 16p^2q + 500pq^3r^3 - 200pq^3r^2 + 1280pq^2r^4 + 512pq^2r^3 - 208pq^2r^2$$
$$+3125q^4r^4 + 4000q^3r^4 - 200q^3r^3 + 16q^3r^2 - 1024q^2r^5 + 768q^2r^4 - 192q^2r^3 + 16q^2r^2 \qquad \diamond$$

A hypersurface $Y = \mathcal{V}(f) \subset (\mathbb{C}^\times)^d$ has a map γ to $\mathbb{P}^{d-1}$ called the *logarithmic Gauss map*,

$$Y \ni y \;\longmapsto\; \gamma(y) \;:=\; \left[y_1 \tfrac{\partial f}{\partial y_1} \,,\, y_2 \tfrac{\partial f}{\partial y_2} \,,\, \ldots \,,\, y_d \tfrac{\partial f}{\partial y_d} \right] \;\in\; \mathbb{P}^{d-1} .$$

Observe that as $\sum_a b_a = 0$, the map φ_B defines a rational map from the projective space $\mathbb{P}^A := \mathbb{P}(\mathbb{C}^A)$ to $\mathbb{C}^d$. As λ_B defines a linear injection $\mathbb{P}^{d-1} \hookrightarrow \mathbb{P}^A$, the Horn-Kapranov parameterization ψ is also a rational map defined on $\mathbb{P}^{d-1}$.

Proposition 3.5 (Kapranov [20]). *When A is nondefective, the map $\psi \colon \mathbb{P}^{d-1} \,-\to \mathbb{C}^d$ parameterizing the reduced discriminant D_B is birational with inverse given by the logarithmic Gauss map. This property—that the logarithmic Gauss map is birational—characterizes reduced discriminants of nondefective configurations.*

Remark 3.6. The *parallel-equivalence class* of $b \in B$ is $[b] := \{c \in B \mid c \text{ is parallel to } b\}$. Let $\eta \in \mathbb{Z}^d$ be a primitive vector parallel to b. For each $c \in [b]$, let q_c be the nonzero integer such that $c = q_c \cdot \eta$. For $y \in (\mathbb{C}^d \smallsetminus \mathcal{H}_B)$, consider the contribution of vectors parallel to b to the Horn-Kapranov parameterization (3.6),

$$\prod_{c \in [b]} \langle c, y \rangle^c \;=\; \prod_{c \in [b]} (q_c \cdot \langle \eta, y \rangle)^{q_c \cdot \eta} \;=\; \left(\prod_{c \in [b]} q_c^{q_c} \right)^\eta \cdot \langle \eta, y \rangle^{\eta \cdot \sum q_c} .$$

If $0 = \sum q_c$, so that the sum of the vectors parallel to b is 0, then this contribution is the element $(\prod_{c \in [b]} q_c^{q_c})^\eta \in (\mathbb{Z}^\times)^d$. Otherwise, it is a nonconstant function of y. $\qquad \diamond$

Set $\mathbb{T}_A := \varphi_A(\mathbb{T}^m) \subset \mathbb{T}^A$. We describe the closed coamoebas of X_A^* and D_B.

Corollary 3.7. *The closed coamoeba of the dual variety X_A^* is*

$$\overline{c\mathscr{A}(X_A^* \cap (\mathbb{C}^\times)^A)} \;=\; \mathbb{T}_A \cdot \overline{c\mathscr{A}(\mathcal{H}_B^c)} ,$$

and we have $\overline{c\mathscr{A}(D_B)} = \varphi_B(\overline{c\mathscr{A}(\mathcal{H}_B^c)})$ *and* $\mathscr{P}^\infty(D_B) \subset \varphi_B(\mathscr{P}^\infty(\mathcal{H}_B^c))$.

Proof. By Remark 3.3, X_A^* is the closure of $\varphi_A(\mathbb{C}^\times)^m \cdot \mathcal{H}_B^c \subset (\mathbb{C}^\times)^A$. Thus the same holds for their coamoebas and closed coamoebas, which is the first statement. Proposition 1.8(3) and (4) applied to the map $\varphi_B \colon X_A^* \to D_B$ implies the remaining statements. $\qquad \square$

Corollary 3.8. *Suppose that A is non-defective. When $d = 1$ the reduced discriminant and its coamoeba are points. For $d > 1$, the coamoeba discriminant has dimension d in $\mathbb{T}^d$.*

Proof. As A is non-defective, D_B is an irreducible hypersurface in $(\mathbb{C}^\times)^d$. Irreducible hypersurfaces in $\mathbb{P}^1$ are points, which proves the first statement. Suppose that $d > 1$. Either both the amoeba and coamoeba of D_B are full-dimensional, or the defining equation of D_B is an irreducible binomial, $y^a - cy^b = 0$. (This is a consequence of the theorem on the dimension of an amoeba in [10].) Suppose that D_B is defined by $y^a - cy^b$. In $(\mathbb{C}^\times)^d$ this becomes $y^\alpha = c$, where $\alpha = a - b$. Since $y_i \frac{\partial}{\partial y_i} y^\alpha = \alpha_i y^\alpha = c\alpha_i$, the logarithmic Gauss map on D_B is constant, and therefore D_B is not a reduced discriminant when $d > 1$, by Proposition 3.5.   □

4. Phase limit set of discriminants

Let D_B be a reduced discriminant arising from a nondefective configuration $A \subset \mathbb{Z}^m$ with Gale dual $B \subset \mathbb{Z}^d$. When $d \geq 2$ the discriminant coamoeba $c\mathscr{A}(D_B)$ has full dimension d, by Corollary 3.8. When $d = 2$, this coamoeba and its phase limit set were described and shown to be essentially polyhedral objects [24, 29]. This is explained in Example 4.14.

By Corollary 3.7, we have $\mathscr{P}^\infty(D_B) = \varphi_B(\mathscr{P}^\infty(\mathcal{H}_B^c))$. Theorem 2.3 and its Corollary 2.4 describe the components of $\mathscr{P}^\infty(\mathcal{H}_B^c)$; we study their images under φ_B to understand $\mathscr{P}^\infty(D_B)$. Our main results are Theorems 4.2, 4.8, and 4.13.

We assume that $d > 2$. By Theorem 4.8, the phase limit set $\mathscr{P}^\infty(D_B)$ has d-dimensional strata that are prisms over discriminant coamoebas corresponding to non-splitting flats of $\mathcal{H}_B$. In a different vein, Theorem 4.13 identifies the d-dimensional components of the phase limit set $\mathscr{P}^\infty(D_B)$ as the union of strata $\mathscr{P}_L^\infty(D_B)$ corresponding to certain essential flacets (see below) L of $\mathcal{H}_B$. We do not know how to relate non-splitting flats and essential flacets, except when $d = 3$, which gives Theorem 4.2. These results generalize what we observed for the plane in $\mathbb{C}^3$ in Example 1.6. That example and some computations suggest the following conjecture, which is an analog of the solidity of discriminant amoebas [28].

Conjecture 4.1. *Let $A \subset \mathbb{Z}^m$ be nondefective with Gale dual $B \subset \mathbb{Z}^d$. When $d > 2$, the closure of the discriminant coamoeba $c\mathscr{A}(D_B)$ equals the phase limit set of D_B.*

In particular, we conjecture that the discriminant coamoeba is a subset of its phase limit set. In Remark 1.7 this was noted for the plane Π of Example 1.6, which is a reduced discriminant as shown in Example 3.4. A restatement of this conjecture is that the fiber above a regular value of the argument map on D_B is unbounded. When $d = 3$, we have an appealing structure theorem, which we prove in Section 4.2.

Theorem 4.2. *When $d = 3$ the phase limit set of the discriminant D_B is the union of prisms over closed coamoebas of discriminants $D_{B|_H}$, for H a hyperplane of $\mathcal{H}_B$. If in addition Conjecture 4.1 holds, the closed discriminant coamoeba is this union of prisms.*

Remark 4.3. The prisms in Theorem 4.2 all have a precise description using the constructions in Section 4.2 and the results of [24, 29]. Let $H \in \mathcal{H}_B$ be a non-splitting hyperplane with $b \in B$ an element vanishing on H. Let $B|_H$ be the multiset of non-zero elements in the image

of B in $\mathbb{Z}^3/\mathbb{Z}b \simeq \mathbb{Z}^2$. By Lemma 4.5, $D_{B|H}$ is a reduced discriminant. We have the map $pr\colon \mathbb{T}^3 \to \mathbb{T}^3/\mathbb{T}_b =: \mathbb{T}_H \simeq \mathbb{T}^2$ and the closed coamoeba of $D_{B|H}$ is a subset of $\mathbb{T}_H$. Then the corresponding component of $\mathscr{P}^\infty(D_B)$ is $pr_H^{-1}(\overline{c\mathscr{A}(D_{B|H})})$.

Each discriminant coamoeba $\overline{c\mathscr{A}(D_{B|H})}$ has a description as a polyhedral object [24, 29]. We illustrate this in Example 4.14. $\diamond$

Our results use an understanding of strata of the phase limit set $\mathscr{P}^\infty(D_B)$ in terms of the matroid of B. This comes from two sources. First, our understanding of $\mathscr{P}^\infty(\mathcal{H}_B^c)$ from Section 2 and the dominant map $\varphi_B\colon \mathcal{H}_B^c \to D_B$ leads to Theorem 4.8 about the structure of strata of $\mathscr{P}^\infty(D_B)$ corresponding to non-splitting flats of $\mathcal{H}_B$. Second, by Lemma 1.4, $\mathscr{P}^\infty(D_B)$ is the union of strata corresponding to rays of the tropical variety $\mathscr{T}(D_B)$. We observe that not all rays of $\mathscr{T}(D_B)$ are needed and note that we only have a partial understanding of $\mathscr{T}(D_B)$: As it is the image of the Bergman fan under the map $\varphi_B\colon \mathbb{R}^B \to \mathbb{R}^d$, this map may be neither injective nor surjective on rays. This leads to Theorem 4.13, about rays that contribute d-dimensional strata to $\mathscr{P}^\infty(D_B)$.

Recall from Definition 2.7 that for a flag of flats $\mathcal{L}$ there is a cone $\sigma_\mathcal{L}$ in the fine subdivision of $\mathscr{T}(B)$ [1] whose corresponding initial scheme $\operatorname{in}_\mathcal{L} \mathcal{H}_B^c$ is the set $\mathcal{H}(\mathcal{L})^c$ (2.5). Its closed coamoeba is the component $\mathscr{P}_\mathcal{L}^\infty(\mathcal{H}_B^c)$ of the phase limit set of $\mathcal{H}_B^c$. We consider the image in $\mathbb{T}^d$ under φ_B of the coamoeba of such a component. For a flat L, let $\mathscr{P}_L^\infty(D_B)$ be the closed coamoeba of $\varphi_B(\operatorname{in}_L \mathcal{H}_B^c)$, and for a flag of flats $\mathcal{L}$, let $\mathscr{P}_\mathcal{L}^\infty(D_B)$ be the closed coamoeba of $\varphi_B(\operatorname{in}_\mathcal{L} \mathcal{H}_B^c)$. By Proposition 1.8(4), these are images of the corresponding components of $\mathscr{P}^\infty(\mathcal{H}_B^c)$ under the map φ_B. More precisely,

$$\mathscr{P}_L^\infty(D_B) \;=\; \varphi_B(\mathscr{P}_L^\infty(\mathcal{H}_B^c)) \qquad \text{and} \qquad \mathscr{P}_\mathcal{L}^\infty(D_B) \;=\; \varphi_B(\mathscr{P}_\mathcal{L}^\infty(\mathcal{H}_B^c))\,.$$

We show in Theorem 4.8 that if L is a non-splitting flat of dimension at least two, then $\mathscr{P}_L^\infty(D_B)$ is a prism over the closed coamoeba of a reduced discriminant corresponding to the restriction $B|_L$ of B to L, and thus has dimension d. A flacet L of $\mathcal{H}$ is *essential* if $\dim L > 1$ and $0 \neq \varphi_B(\omega_L) = \sum\{b \mid b|_L \equiv 0\}$, as $\omega_L \in \mathbb{R}^B$ is the indicator function of those b that vanish on L. Theorem 4.13 asserts that the d-dimensional components of $\mathscr{P}^\infty(D_B)$ consist of the strata $\mathscr{P}_L^\infty(D_B)$ for essential flacets L.

In Section 4.1 we give a coordinate-free description of these objects, describe the structure of $\mathscr{P}_L^\infty(D_B)$, and prove Theorem 4.8. We then study the tropical discriminant, its rays, and corresponding strata in Section 4.2, where we also prove Theorems 4.2 and 4.13. The results of this section will be illustrated on our running example in Example 4.14.

4.1. **Components of $\mathscr{P}_L^\infty(D_B)$.** As in Section 3, let $A \subset \mathbb{Z}^m$ be $m+d$ vectors which span $\mathbb{Z}^m$ and lie on an affine hyperplane. They determine a map $\lambda_A\colon \mathbb{Z}^{m+d} \twoheadrightarrow \mathbb{Z}^m$ whose kernel N is a free abelian group of rank d. The coordinate functions on $\mathbb{Z}^{m+d}$ define the embedding $N \hookrightarrow \mathbb{Z}^{m+d}$ and are given by $m+d$ vectors B that span the dual $M := \operatorname{Hom}(N, \mathbb{Z})$ to N. This set B is a Gale dual to A. Elements of both A and B correspond to the coordinates of $\mathbb{Z}^{m+d}$ and are in a canonical one-to-one correspondence.

The torus $\mathbb{C}_N^\times := N \otimes_{\mathbb{Z}} \mathbb{C}^\times$ has N as its group of cocharacters, $\mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}_N^\times)$, and M as its group of characters, $\mathrm{Hom}(\mathbb{C}_N^\times, \mathbb{C}^\times)$. The groups M and N are naturally dual free abelian groups. The d-dimensional vector spaces $\mathbb{C}_N := N \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathbb{C}_M := M \otimes_{\mathbb{Z}} \mathbb{C}$ are then dual vector spaces. For these tensor products of abelian groups, an integer $k \in \mathbb{Z}$ acts on $\mathbb{C}^\times$ via exponentiation, $k.t := t^k$ for $t \in \mathbb{C}^\times$ and on $\mathbb{C}$ via multiplication, $k.\xi := k \cdot \xi$ for $\xi \in \mathbb{C}$.

The vectors in B induce a hyperplane arrangement $\mathcal{H}_B$ in $\mathbb{C}_N$ (this is V in the notation of Section 2). Evaluation at elements of B induces a linear injection $\lambda_B \colon \mathbb{C}_N \hookrightarrow \mathbb{C}^B$ with $\mathbb{C}_N \smallsetminus \mathcal{H}_B = \lambda_B^{-1}((\mathbb{C}^\times)^B)$. Let $\mathcal{H}_B^c := \lambda_B(\mathbb{C}_N \smallsetminus \mathcal{H}_B) \subset (\mathbb{C}^\times)^B$ be the image in $(\mathbb{C}^\times)^B$ of the hyperplane complement $\mathbb{C}_N \smallsetminus \mathcal{H}_B$. The multiset $B \subset M$ gives the homomorphism

$$\varphi_B \ : \ (\mathbb{C}^\times)^B \ni (z_b \mid b \in B) \longrightarrow \prod_{b \in B} z_b^b \in \mathbb{C}_M^\times \,,$$

as M is the group of cocharacters of $\mathbb{C}_M^\times \simeq (\mathbb{C}^\times)^d$. Then $D_B \subset \mathbb{C}_M^\times$ is the closure of $\varphi_B(\mathcal{H}_B^c)$.

Remark 4.4. Given any multiset B of nonzero vectors spanning M, these definitions of $\lambda_B, \varphi_B, \mathcal{H}_B$, and D_B make sense. Call D_B the *generalized discriminant* associated to B. If $\sum \{b \mid b \in B\} = 0$, then any Gale dual A to B lies in an affine hyperplane, so D_B is a discriminant. If A is nondefective configuration then D_B is a hypersurface.   ◇

Let $L \neq \{0\}$ be a flat of the arrangement $\mathcal{H}_B$. We study the component of $\mathscr{P}^\infty(D_B)$ corresponding to the image of $\mathrm{in}_L \mathcal{H}_B^c$. By Lemma 2.5, $\mathrm{in}_L \mathcal{H}_B^c$ is a product of hyperplane complements in L and in $\mathbb{C}_N/L$. We use this to study its image under φ_B and the coamoeba of that image. The precise statements are Corollary 4.7 and Theorem 4.8 below.

Let $F := \{b \in B \mid b|_L \equiv 0\}$ be the flat of the matroid B corresponding to L. Then $L = \mathcal{V}(F)$. The flat F spans a rational linear subspace $\mathbb{Q}_F \subset \mathbb{Q}_M$ with corresponding saturated subgroup $L^\perp := \mathbb{Q}_F \cap M \subset M$. Note that $L^\perp$ is the saturation of the $\mathbb{Z}$-span of F in M, and that $F = L^\perp \cap B$. Set $F^c := B \smallsetminus F$.

The partition $B = F \coprod F^c$ induces the identification

(4.1) $(\mathbb{C}^\times)^B \ = \ (\mathbb{C}^\times)^F \times (\mathbb{C}^\times)^{F^c} \,.$

We study how this interacts with the map $\varphi_B \colon (\mathbb{C}^\times)^B \to \mathbb{C}_M^\times$. Note that the inclusion $L^\perp \subset M$ gives the short exact sequence $L^\perp \hookrightarrow M \twoheadrightarrow M/L^\perp$, which induces a short exact sequence of tori $\mathbb{C}_{L^\perp}^* \hookrightarrow \mathbb{C}_M^* \twoheadrightarrow \mathbb{C}_{M/L^\perp}^*$. This is the second row of a commutative diagram with surjective vertical maps and whose top row is split (by (4.1))

(4.2)
$$\begin{array}{ccccccccc}
\mathbb{1} & \longrightarrow & (\mathbb{C}^\times)^F & \longrightarrow & (\mathbb{C}^\times)^B & \longrightarrow & (\mathbb{C}^\times)^{F^c} & \longrightarrow & \mathbb{1} \\
& & \downarrow{\varphi_F} & & \downarrow{\varphi_B} & & \downarrow{\varphi_{F^c}} & & \\
\mathbb{1} & \longrightarrow & \mathbb{C}_{L^\perp}^\times & \longrightarrow & \mathbb{C}_M^\times & \stackrel{pr}{\longrightarrow} & \mathbb{C}_{M/L^\perp}^\times & \longrightarrow & \mathbb{1}
\end{array} \ .$$

Here, $\varphi_F = \varphi_B|_{(\mathbb{C}^\times)^F}$ and $\varphi_{F^c} := pr \circ \varphi_B|_{(\mathbb{C}^\times)^{F^c}}$.

As in Section 2.1, F induces a hyperplane arrangement $\mathcal{H}/L$ in the quotient space $\mathbb{C}_N/L$. The linear map $\lambda_F \colon \mathbb{C}_N \to \mathbb{C}^F \subset \mathbb{C}^B$ factors through $\mathbb{C}_N/L$ and induces an embedding of

the complement $\mathbb{C}_N / L \smallsetminus \mathcal{H}/L$ into $(\mathbb{C}^\times)^F$. Write $(\mathcal{H}/L)^c$ for its image in $(\mathbb{C}^\times)^F$. Similarly, F^c induces the hyperplane arrangement $\mathcal{H}|_L$ in L. Restricting the linear map $\lambda_{F^c} \colon \mathbb{C}_N \to \mathbb{C}^{F^c} \subset \mathbb{C}^B$ to $L \subset \mathbb{C}_N$ embeds the complement $L \smallsetminus \mathcal{H}|_L$ into $(\mathbb{C}^\times)^{F^c}$. Write $(\mathcal{H}|_L)^c$ for its image in $(\mathbb{C}^\times)^{F^c}$.

By Lemma 2.5 applied to the flag $\{0\} \subsetneq L \subsetneq \mathbb{C}_N$, we have $\mathrm{in}_L \mathcal{H}_B^c = (\mathcal{H}/L)^c \times (\mathcal{H}|_L)^c$, which is compatible with the splitting (4.1). The image $\varphi_B(c\mathscr{A}(\mathrm{in}_L \mathcal{H}_B^c))$ of the coamoeba of $\mathrm{in}_L \mathcal{H}_B^c$ under the map φ_B is a dense subset of $c\mathscr{A}(\varphi_B(\mathrm{in}_L \mathcal{H}_B^c))$, by Proposition 1.8(3). Let $\mathscr{P}_L^\infty(D_B)$ be the common closure. Then

$$\mathscr{P}_L^\infty(D_B) \;=\; \overline{\varphi_B\big(c\mathscr{A}(\mathcal{H}/L)^c \times c\mathscr{A}(\mathcal{H}|_L)^c\big)}.$$

We call $\mathscr{P}_L^\infty(D_B)$ the stratum of the phase limit set $\mathscr{P}^\infty(D_B)$ corresponding to L.

In the group $(\mathbb{C}^\times)^B = (\mathbb{C}^\times)^F \times (\mathbb{C}^\times)^{F^c}$, $\mathrm{in}_L \mathcal{H}_B^c$ is the product of sets

$$\mathrm{in}_L \mathcal{H}_B^c \;=\; (\mathcal{H}/L)^c \times (\mathcal{H}|_L)^c \;=\; \big((\mathcal{H}/L)^c \times \mathbb{1}\big) \,\cdot\, \big(\mathbb{1} \times (\mathcal{H}|_L)^c\big).$$

We use this and the commutative diagram (4.2) to study the image $\varphi_B(\mathrm{in}_L \mathcal{H}_B^c)$ in $\mathbb{C}_M^\times$. As the span of $F \subset M$ has saturation $L^\perp$ in M, the definitions of $\varphi_\bullet$ and $\lambda_\bullet$ imply that

$$\varphi_B\big((\mathcal{H}/L)^c \times \mathbb{1}\big) \;=\; \varphi_F\big((\mathcal{H}/L)^c\big) \;=\; D_F \;\subset\; \mathbb{C}_{L^\perp}^\times,$$

where D_F is the generalized discriminant associated to F. It is generalized as we do not know if F is Gale dual to a nondefective configuration.

Let us write $B|_L$ for the image of F^c in $M/L^\perp$. This is the multiset of nonzero vectors in the image of B in the dual of L (the restriction of B to L), and $B|_L$ induces the hyperplane arrangement $\mathcal{H}|_L$. The image of $\varphi_B(\mathbb{1} \times (\mathcal{H}|_L)^c) \subset \mathbb{C}_M^\times$ under the map $pr \colon \mathbb{C}_M^\times \to \mathbb{C}_{M/L^\perp}^\times$ has closure the generalized discriminant $D_{B|_L}$ associated to $B|_L$. This is also the closure of $\varphi_{F^c}(\mathcal{H}|_L)^c$, where φ_{F^c} is the rightmost vertical map in (4.2).

Lemma 4.5. *The image of $\varphi_B\big((\mathcal{H}/L)^c \times \mathbb{1}\big)$ in $\mathbb{C}_{L^\perp}^\times \subset \mathbb{C}_M^\times$ is the generalized discriminant D_F and the image under pr of $\varphi_B\big(\mathbb{1} \times (\mathcal{H}|_L)^c\big)$ in $\mathbb{C}_{M/L^\perp}^\times$ is the discriminant $D_{B|_L}$. If L is non-splitting, then $D_{B|_L}$ is a reduced discriminant.*

Proof. The identifications of D_F and $D_{B|_L}$ were made before the statement of the lemma, except for the observation that $D_{B|_L}$ is a discriminant and is reduced when L is non-splitting. First, note that $B|_L$ is Gale dual to the subset $A|_L$ consisting of vectors corresponding to F^c. As B spans M, its image $\{\bar{b} \mid \bar{b} \in B|_L\}$ spans $M/L^\perp$. Also, as $\sum\{b \mid b \in B\} = 0$ its image also has sum zero, and thus $D_{B|_L}$ is a discriminant.

Suppose that L is a non-splitting flat. Then L lies in a non-splitting flag $\mathcal{L}$ of flats satisfying (3.4) that is a witness for A to be nondefective. The image of the flag $\mathcal{L}$ in $M/L^\perp$ (its restriction to L) gives a non-splitting flag as it satisfies the condition (3.4). Thus $A|_L$ is nondefective and $D_{B|_L}$ is a reduced discriminant. $\qquad\square$

We study the map $pr \colon \varphi_B(\mathrm{in}_L \mathcal{H}_B^c) \to D_{B|_L}$ that is the restriction of $pr \colon \mathbb{C}_M^\times \to \mathbb{C}_{M/L^\perp}^\times$.

Lemma 4.6. *Under the map $pr\colon \mathbb{C}^{\times}_M \to \mathbb{C}^{\times}_{M/L^{\perp}}$, the fiber of $\varphi_B(\mathrm{in}_L \mathcal{H}^c_B)$ over a point $z \in D_{B|_L}$ is the union of translates of D_F by $\varphi_B(\mathbb{1} \times \{y\})$ for those $y \in (\mathcal{H}|_L)^c$ with image z.*

Proof. Observe that

$$\varphi_B(\mathrm{in}_L \mathcal{H}^c_B) \;=\; \varphi_B\big((\mathcal{H}/L)^c \times (\mathcal{H}|_L)^c\big) \;=\; D_F \,\cdot\, \varphi_B\big(\mathbb{1} \times (\mathcal{H}|_L)^c\big)\,.$$

As D_F lies in the kernel $\mathbb{C}^{\times}_{L^{\perp}}$ of pr (4.2), we see that $pr^{-1}(z) \cap \varphi_B(\mathrm{in}_L \mathcal{H}^c_B)$ is the translation of D_F by points $y \in pr^{-1}(z) \cap \varphi_B\big(\mathbb{1} \times (\mathcal{H}|_L)^c\big)$. This proves the lemma. $\qquad\square$

The set $\varphi_B(c\mathscr{A}(\mathrm{in}_L \mathcal{H}^c_B)) \subset \mathbb{T}_M$ is a dense subset of the stratum $\mathscr{P}^{\infty}_L(D_B)$ of the phase limit set of D_B corresponding to the flat L of $\mathcal{H}_B$. The commutative diagram (4.2) of exact sequences of tori restricts to a commutative diagram of exact sequence of compact subtori. Lemmas 4.5 and 4.6 have the following consequence for coamoebas.

Corollary 4.7. *Let L be a flat of $\mathcal{H}_B$. Then $\overline{\varphi_B(c\mathscr{A}((\mathcal{H}/L)^c))} = \overline{c\mathscr{A}(D_F)}$ and*

$$\overline{pr(\varphi_B(c\mathscr{A}(\mathrm{in}_L \mathcal{H}^c_B)))} \;=\; \overline{c\mathscr{A}(D_{B|_L})} \;\subset\; \mathbb{T}_{M/L^{\perp}}\,.$$

The fiber of pr over $\theta \in c\mathscr{A}(D_{B|_L})$ is the union of the translates $c\mathscr{A}(D_F) \cdot \vartheta$ for all $\vartheta \in \varphi_B(\mathbb{1} \times c\mathscr{A}(\mathcal{H}|_L)^c)$ with $pr(\vartheta) = \theta$.

We show that $\mathscr{P}^{\infty}_L(D_B)$ is a prism when L is non-splitting and $\dim L > 1$.

Theorem 4.8. *Suppose that A is nondefective with reduced discriminant D_B. Let L be a non-splitting flat of $\mathcal{H}_B$. Then the stratum $\mathscr{P}^{\infty}_L(D_B)$ is the prism over the closed coamoeba of $D_{B|_L}$. When $\dim L > 1$, this has dimension d.*

As we show in Lemma 4.12 below, if $\dim L = 1$, then $\dim \mathscr{P}^{\infty}_L(D_B) < d$.

Proof. First suppose that L is a hyperplane of $\mathcal{H}_B$. Then the corresponding flat F of B is the parallel-equivalence class $[b]$ of any $b \in B$ that defines L, and $L^{\perp} \simeq \mathbb{Z}$. Since L is a hyperplane, $(\mathcal{H}/L)^c = (\mathbb{C}_N/L)^{\times} \simeq \mathbb{C}^{\times}$. As L is non-splitting, $0 \neq \sum_{c \in [b]} c$, and by Remark 3.6 φ_F is not a constant map, which shows that $D_F = \mathbb{C}^{\times}_{L^{\perp}} \simeq \mathbb{C}^{\times}$. Thus $c\mathscr{A}(D_F) = \mathbb{T}_{L^{\perp}}$ and by Corollary 4.7, $\mathscr{P}^{\infty}_L(D_B) = pr^{-1}(\overline{c\mathscr{A}(D_{B|_L})})$ is a prism. Consequently, when $\dim L > 1$, so that $d > 2$, this has dimension $1 + \dim c\mathscr{A}(D_{B|_L}) = 1 + (d-1) = d$.

Suppose now that L is not a hyperplane. Let H be the hyperplane of $\mathcal{H}_B$ in some non-splitting flag containing L. Then H is a non-splitting hyperplane. Consider the flag $\mathcal{L}\colon \{0\} \subset L \subset H \subset \mathbb{C}_N$. Since $\mathrm{in}_{\mathcal{L}} \mathcal{H}^c_B$ is an initial scheme of both $\mathrm{in}_L \mathcal{H}_B$ and $\mathrm{in}_H \mathcal{H}^c_B$, we have the containment of phase limit sets, $\mathscr{P}^{\infty}_{\mathcal{L}}(D_B) \subset \mathscr{P}^{\infty}_L(D_B)$ and $\mathscr{P}^{\infty}_{\mathcal{L}}(D_B) \subset \mathscr{P}^{\infty}_H(D_B)$. We will use induction on the dimension d and the case of the hyperplane H to show that $\mathscr{P}^{\infty}_{\mathcal{L}}(D_B)$ is the prism over the closure of $c\mathscr{A}(D_{B|_L})$, which implies that $\mathscr{P}^{\infty}_L(D_B) = \mathscr{P}^{\infty}_{\mathcal{L}}(D_B)$ and is thus also the same prism. The assertion on dimension follows from Corollary 3.8.

Let $G \subset F \subset B$ be the flag of flats in B corresponding to $L \subset H$. We have that $G = [b]$, the parallel-equivalence class of any element $b \in B$ that defines the hyperplane H. Then $B|_H$ is the image of $B \smallsetminus G$ in $M/H^{\perp}$. Restricting a non-splitting flag containing L and H to H shows that L is a non-splitting flat of the hyperplane arrangement $\mathcal{H}_{B|_H}$. Note that $B|_L = (B|_H)|_L$.

Applying the induction hypothesis, $\mathscr{P}_L^\infty(D_{B|H})$ is the prism in $\mathbb{T}_{M/H^\perp}$ over closed coamoeba of $D_{B|_L}$ (which is a subset 0f $\mathbb{T}_{M/L^\perp}$), and if $\dim L > 1$, then it has dimension $d-1$.

By Lemma 2.5 and (2.5), and again using that $B|_L = (B|_H)|_L$,

$$\mathrm{in}_{\mathcal{L}}\,\mathcal{H}_B^c \;=\; (\mathcal{H}_B/H)^c \times \big((\mathcal{H}_{B|_H})/L\big)^c \times \big(\mathcal{H}_{B|_L}\big)^c\,,$$

which is $(\mathcal{H}_B/H)^c \times \mathrm{in}_L(\mathcal{H}_{B|_H})^c$. Since $\mathscr{P}_{\mathcal{L}}^\infty(D_B)$ is the closure of $c\mathscr{A}(\varphi_B(\mathrm{in}_{\mathcal{L}}\,\mathcal{H}_B^c))$, we see that the image of $\mathscr{P}_{\mathcal{L}}^\infty(D_B) \subset \mathbb{T}_M$ in $\mathbb{T}_{M/H^\perp}$ is $\mathscr{P}_L^\infty(D_{B|_H})$. By Corollary 4.7, the fiber contains a translate of $c\mathscr{A}(D_G) = \mathbb{T}_{H^\perp} \simeq \mathbb{T}$, as $D_G = \mathbb{C}_{H^\perp}^\times \simeq \mathbb{C}^\times$. Thus

$$\mathscr{P}_{\mathcal{L}}^\infty(D_B) \;=\; pr^{-1}\big(\mathscr{P}_L^\infty(D_{B|_H})\big)\,,$$

so that it is the prism over $\mathscr{P}_L^\infty(D_{B|_H})$, which is itself the prism over the closed coamoeba of $D_{(B|_H)|_L} = D_{B|_L}$. Since a prism over a prism is a prism, we conclude that $\mathscr{P}_{\mathcal{L}}^\infty(D_B)$ is the prism over the closed coamoeba of $D_{B|_L}$, which completes the proof of the theorem. $\qquad\square$

4.2. **The phase limit set of a reduced discriminant.** The tropical variety of D_B is the tropical discriminant $\mathscr{T}(D_B)$ [8]. This is the image of the Bergman fan $\mathcal{B}er(B)$ under the linear map $\varphi_B \colon \mathbb{R}^B \to \mathbb{R}_M$ given by

$$\varphi_B \;:\; (x_b \mid b \in B) \;\longmapsto\; \sum x_b\, b\,.$$

Since $\sum\{b \mid b \in B\} = 0$, the subspace $\mathbb{R}\mathbb{1}_B$ of the lineality space of $\mathcal{B}er(B)$ lies in the kernel of φ_B so that the map to the tropical discriminant factors through $\mathcal{B}er(B)/\mathbb{R}\mathbb{1}_B$.

By Lemma 1.4, $\mathscr{P}^\infty(D_B)$ is the union of strata arising from initial schemes of D_B given by the rays in any fan structure on the tropical discriminant. Before studying the tropical discriminant in more detail, we consider our running example.

Example 4.9. Figure 6 shows two views of the tropical discriminant $\mathscr{T}(D_B) \subset \mathbb{R}^3$ for the vectors of Example 2.1. Rays are labeled by spanning vectors. Among them are the six rays
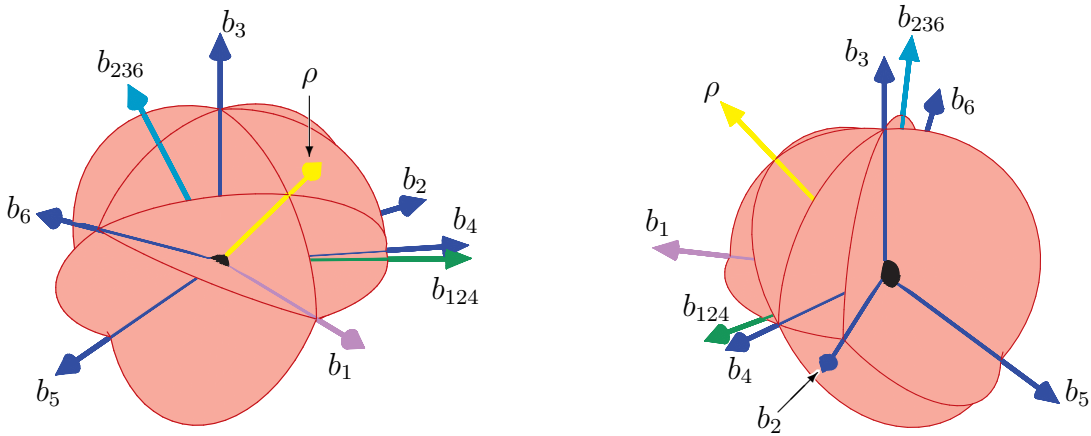

FIGURE 6. Two views of the tropical discriminant $\mathscr{T}(D_B)$.

$\mathbb{R}_{\geq 0} b_i$ for $i = 1, \ldots, 6$ corresponding to the vectors b_i, which are the images of the rays $\mathbb{R}_{\geq 0} e_i$

of $\mathcal{B}er(B)$. Recall from Figure 4 that the Bergman fan $\mathcal{B}er(B)$ has two more rays spanned by the vectors e_{124} and e_{236}. Write b_{124} and b_{236} for their images under φ_B. Notice that

$$b_{124} := \varphi_B(e_{124}) = b_1 + b_2 + b_4 = (2,3,0),$$

and the cones from $\mathcal{B}er(B)$ meeting e_{124} (spanned by e_{124} together with one of e_1, e_2, and e_4) map onto the two coplanar cones spanned by b_1, b_4 and by b_4, b_2, with b_{124} lying in the first. Similarly, the three cones of $\mathcal{B}er(B)$ meeting e_{236} have coplanar image with b_{236} lying in the cone spanned by b_3 and b_6.

Any fan structure on $\mathcal{T}(D_B)$ contains the ray ρ through the point $(1,0,1)$, which is the intersection of the images of the cones σ_{13} and σ_{46}. We will see that coamoebas of initial schemes for ρ, b_{124}, and b_{236} all lie in the union of other components of $\mathcal{P}^\infty(D_B)$.   ◇

Let us consider fan structures on $\mathcal{T}(D_B)$ such that for every cone σ of $\mathcal{B}er(B)$, $\varphi_B(\sigma)$ is a union of cones of $\mathcal{T}(D_B)$. Fix one such fan structure on $\mathcal{T}(D_B)$, and let ρ be one of its rays. Either ρ is the image of a ray of $\mathcal{B}er(B)$ or it is not. Recall that the rays of $\mathcal{B}er(B)$ correspond to flacets of $\mathcal{H}_B$. The ray corresponding to a flacet L is generated by the indicator function $\omega_L \in \mathbb{R}^B$ of L. Let $b_L := \varphi_B(\omega_L)$, which is the sum of the elements b of B that vanish along L. When $b_L \neq 0$, the image of this ray of $\mathcal{B}er(B)$ is a ray of $\mathcal{T}(D_B)$. A consequence of this discussion is the following lemma.

Lemma 4.10. *Rays ρ of $\mathcal{T}(D_B)$ are one of the following two types.*
  (1) *$\rho = \mathbb{R}_{\geq 0} b_L$, where L is a flacet of $\mathcal{H}_B$ with $b_L \neq 0$.*
  (2) *ρ is not the image of a ray of $\mathcal{B}er(B)$.*

Rays ρ of $\mathcal{T}(D_B)$ have type (1) or (2), according to this classification.

Lemma 4.11. *If a ray ρ of $\mathcal{T}(D_B)$ has type (2), then the component $\mathcal{P}_\rho^\infty(D_B)$ lies in the union of components $\mathcal{P}_\tau^\infty(D_B)$ for τ a ray of type (1).*

Proof. Let σ be a cone of $\mathcal{B}er(B)$ such that $\rho \subset \varphi_B(\sigma)$. Then σ is not a ray, as ρ has type (2). Let $\tau \subset \sigma$ be a ray of σ. Then τ corresponds to a flacet L of $\mathcal{H}_B$. As $\mathrm{in}_\sigma \mathcal{H}_B^c$ is an initial scheme of $\mathrm{in}_\tau \mathcal{H}_B^c = \mathrm{in}_L \mathcal{H}_B^c$, we have $c\mathcal{A}(\mathrm{in}_\sigma \mathcal{H}_B^c) \subset \overline{c\mathcal{A}(\mathrm{in}_L \mathcal{H}_B^c)}$, and thus $\mathcal{P}_\sigma^\infty(\mathcal{H}_B^c) \subset \mathcal{P}_L^\infty(\mathcal{H}_B^c)$. Applying φ_B, shows that

$$(4.3) \qquad\qquad \varphi_B(\mathcal{P}_\sigma^\infty(\mathcal{H}_B^c)) \subset \varphi_B(\mathcal{P}_L^\infty(\mathcal{H}_B^c)).$$

Suppose that the flacet L corresponding to τ has $\varphi_B(\omega_L) = 0$. Since ω_L spans τ, $\varphi_B(\tau) = 0$. As $\rho \subset \varphi_B(\sigma)$, and the cone σ is generated by its rays, it has a ray τ with $\varphi_B(\tau) \neq 0$. If L is the flacet corresponding to the ray τ, then we have (4.3).

Lemma 1.10 applied to the homomorphism φ_B implies that $\mathcal{P}_\rho^\infty(D_B)$ is the union of images $\varphi_B(\mathcal{P}_\sigma^\infty(\mathcal{H}_B^c))$ for cones σ of $\mathcal{B}er(B)$ such that $\rho \subset \varphi_B(\sigma)$. Each such cone σ has a ray corresponding to a flacet L with $\varphi_B(\omega_L) \neq 0$, and for this flacet we have (4.3). This completes the proof.   □

Lemma 4.12. *Let L be a one-dimensional flat of $\mathcal{H}_B$. Then the stratum $\mathcal{P}_L^\infty(D_B)$ corresponding to L has dimension at most $d-1$.*

Proof. By Corollary 4.7, if $pr\colon \mathbb{C}_M^\times \to \mathbb{C}_{M/L^\perp}^\times \simeq \mathbb{C}^\times$, then $pr(\mathscr{P}_L^\infty(D_B)) \subset c\mathscr{A}(D_{B|L})$. As L is 1-dimensional, $D_{B|L}$ and its coamoeba are points. Thus $\mathscr{P}_L^\infty(D_B)$ is a subset of a fiber of pr and thus has dimension at most $d-1$. $\qquad\qquad\square$

A flacet L is essential if $\dim L > 1$ and $0 \neq b_L = \varphi_B(\omega_L)$. Let $\mathrm{top}(D_B)$ be the union of d-dimensional components of $\mathscr{P}^\infty(D_B)$. We have proven the following result.

Theorem 4.13. *When D_B is a reduced discriminant, we have* $\mathrm{top}(D_B) = \bigcup \mathscr{P}_L^\infty(D_B)$, *the union over essential flacets L of $\mathcal{H}_B$.*

Proof of Theorem 4.2. When $d = 3$, the essential flacets of $\mathcal{H}_B$ are hyperplanes H with $b_H \neq 0$. All are non-splitting. Theorem 4.2 follows from Theorems 4.13 and 4.8. $\qquad\square$

Example 4.14. We end with our running example, illustrating some results of this section. Figure 7 shows three views of the Newton polytope of the discriminant (3.8). Each cone of the
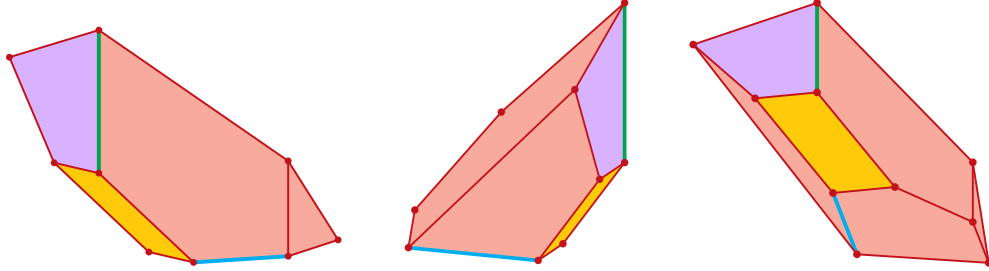

FIGURE 7. Three views of the Newton polytope of the discriminant.

tropical discriminant of Figure 6 consists of inner normals to a face of the Newton polytope, and faces corresponding to the rays b_1, b_{124}, b_{236}, and ρ are color-coded to correspond to Figure 6. An initial scheme of the discriminant is defined by the restriction of the discriminant to the corresponding face, which we call the *initial form*. We give these for these four rays.

$$
\begin{aligned}
\mathrm{in}_{b_1} D_B &= q^2 r^2 (3125 q^2 r^2 - 1024 r^3 + 4000 q r^2 + 768 r^2 - 200 q r - 192 r + 16 q + 16) \\
\mathrm{in}_{b_{124}} D_B &= 16 q^2 r^2 - 1024 q^2 r^5 = 16 q^2 r^2 (1 - 64 r^3) \\
\mathrm{in}_{b_{236}} D_B &= 16 p^2 q^2 - 1024 p^5 q^2 = 16 p^2 q^2 (1 - 64 p^3) \\
\mathrm{in}_{\rho} D_B &= 16 q^3 r^2 + 16 q^2 r^2 + 16 p^2 q^2 + 16 p^2 q = 16 q (q+1)(q r^2 + p^2)
\end{aligned}
$$

Rays b_{124} and b_{236} are images of rays of $\mathcal{B}er(B)$ corresponding to one-dimensional flacets (points p_{124} and p_{236} in Figure 3), which are not essential. Their initial forms are binomials, which implies that their coamoebas are 2-dimensional. This illustrates Lemma 4.12.

The ray ρ occurs as the intersection of the images $\varphi_B(\sigma_{13})$ and $\varphi_B(\sigma_{46})$ of two 2-dimensional cones of $\mathcal{B}er(B)$, subdividing both. The ray ρ exposes the parallelogram facet of the Newton polytope of D_B. Its edges are exposed by the subdivided cones, with the two parts of each exposing parallel edges, and the corresponding initial forms for the parallel edges differ only by a monomial factor. The initial form $\mathrm{in}_{\rho} D_B$ factors (up to monomials) as the product of

forms for each pair of parallel edges. As $\text{in}_\rho D_B$ is the product of binomials, its coamoeba is two-dimensional. This illustrates Lemma 4.11, particularly its proof.

Consider $\text{in}_{b_1} D_B$. Let $H \in \mathcal{H}_B$ be the hyperplane defined by b_1. Then $B|_H$ consists of the rows of the matrix below. We also give the parametrization $\psi \colon \mathbb{C}^2 \smallsetminus \mathcal{H}_{B|_H} \to (\mathbb{C}^\times)^2$ of $D_{B|_F}$.
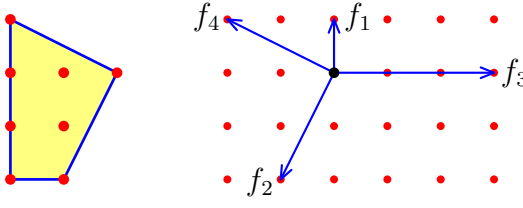
$$B|_H \;=\; B/b_1 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ -1 & -2 \\ -2 & 1 \end{pmatrix} \qquad (y,z) \;\overset{\psi}{\longmapsto}\; \left(\frac{y(2y)^2}{(-y-2z)(-2y+z)^2}, \; \frac{z(-2y+z)}{(-y-2z)^2} \right) .$$

In coordinates q, r for $\mathbb{C}^2$, the reduced discriminant $D_{B|_H}$ has implicit equation

$$D_{B|_H} \;=\; 3125q^2r^2 - 1024r^3 + 4000qr^2 + 768r^2 - 200qr - 192r + 16q + 16 \,,$$

which is the non-monomial factor of $\text{in}_{b_1} D_B$.

Consider the Newton polytope P_H of $D_{B|_H}$ and the configuration F_H, where we replace parallel vectors of $B|_H$ by their sum. By Remark 3.6, this does not change the coamoeba.

(4.4)

While $0 = f_1 + \cdots + f_4$, note that there are other integer linear relations they satisfy,

$$6f_1 + 3f_2 + f_3 + 0 \cdot f_4 \;=\; 0 \cdot f_1 + 3f_2 + 5f_3 + 6f_4 \;=\; 0 \,.$$

Let $d_H = 6$, which is the length (area) of the convex hull of any Gale dual of F_H; this is the difference of the minimum and maximum coefficient in the these two relations.

The coamoeba of a discriminant when $d = 3$ is described in [8, 29]. We explain this for the coamoeba of D_{F_H}. It is most propitious to view this in the universal cover of $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, where the coamoeba is the complement of a zonotope in a covering of $\mathbb{T}^2$.

We let Z_H be the zonotope of F_H—this is the sum of intervals $[\mathbf{0}, \pi b]$ for $b \in F_H$. Its normal fan is the hyperplane arrangement $\mathcal{H}_{F_H}$. We display it as an oriented cycle.

(4.5)

The closed coamoeba $c\mathscr{A}(D_{F_H})$ is the closure of the image of $(\mathbb{C} \smallsetminus \mathbb{R}) \subset \mathbb{P}^1$ under the map $\text{Arg} \circ \psi$, which has no critical points in $\mathbb{C} \smallsetminus \mathbb{R}$. As with the zonotope Z_H, it is best to lift this to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$. Let $c\mathscr{A}_H^+$ be the closure of the image of the upper half plane

under the lift of $\mathrm{Arg} \circ \psi$, and $c\mathscr{A}_H^-$ the same for the lower half plane. We consider this as an oriented cycle.

We describe $c\mathscr{A}_H^+$ by describing its oriented boundary. Let v be a vertex of Z_H whose incoming oriented edge is πf, for some $f \in F_H$. In (4.5), the four choices for v are the marked vertices. Each vector of F_H spans a line, and let $f_1 = f, f_2, \ldots, f_r$ be the ordering of the vectors of F_H corresponding to the order in which the lines they support are encountered moving clockwise from f. In our example, we choose v to the northeasternmost vertex of Z_H—for this choice we get the ordering of F_H indicated in (4.4). Then the oriented boundary of $c\mathscr{A}_H^+$ is, in order

$$v, \; v - \pi f_1, \; v - \pi f_1 - \pi f_2, \; \ldots, \; v - \pi f_1 - \cdots - f_{r-1}, \; v.$$

Let $c\mathscr{A}_H^-$ be the reflection of $c\mathscr{A}_H^+$ in the origin. We show $c\mathscr{A}_H^+$ and the union of the cycles Z_H, $c\mathscr{A}_H^+$, and $c\mathscr{A}_H^i$.
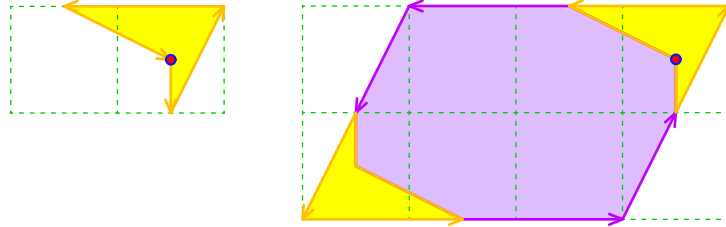
Observe that $Z_H \cup c\mathscr{A}^+ \cup c\mathscr{A}_H^-$ covers $d_H = 6$ fundamental domains of $\mathbb{T}^2$ in $\mathbb{R}^2$. We summarize the major results of [8, 29] about the coamoeba discriminant.

(1) The closed coamoeba of $D_{B|_H}$ is the image of $c\mathscr{A}_H^+ \cup c\mathscr{A}_H^-$ in $\mathbb{T}^2$.
(2) The oriented cycle $Z_H \cup c\mathscr{A}^+ \cup c\mathscr{A}_H^-$ pushes forward to $\mathbb{T}^2$ as the cycle $d_H[\mathbb{T}^2]$.   ◇

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