Algebra Autumn 2013
Frank Sottile
24 October 2013

Eighth Homework

Hand in to Frank Tuesday 29 October:

44. [20] The wreath product $S_m \wr S_n$ of symmetric groups is the semidirect product $(S_m)^n \rtimes_\varphi S_n$ where φ is the action of S_n on $(S_m)^n$ permuting the factors of $(S_m)^n$.

    (a) For $(\pi_1, \ldots, \pi_n\,,\,\omega) \in S_m \wr S_n$ ($\pi_i \in S_n$ and $\omega \in S_n$) define the map from $[m] \times [n]$ to itself by

    $$(\pi_1, \ldots, \pi_n\,,\,\omega).(i,j) \;:=\; (\pi_{\omega(j)}(i)\,,\,\omega(j))\,.$$

    (Here, $[m] := \{1, \ldots, m\}$ and the same for $[n]$. Show that this defines an action of $S_m \wr S_n$ on $[m] \times [n]$.

    (b) Using this action or any other methods show that $S_2 \wr S_2 \simeq D_8$, the dihedral group with 8 elements.

    (c) This action realizes $S_3 \wr S_2$ as a sugroup of S_6. What are the cycle types of permutations of $S_3 \wr S_2$? For each cycle type, how many elements of $S_3 \wr S_2$ have that cycle type?

Hand in for the grader Tuesday 29 October:

45. Suppose that R is a commutative ring with characteristic p, a prime. Prove that the map $F\colon R \to R$ defined by $F(r) = r^p$ is a ring homomorphism. Hint: You may first need to prove the binomial theorem for this ring.

46. An element x in a ring R is *nilpotent* if there is a positive integer n with $x^n = 0$. Prove that the set $\eta(R)$ of nilpotent elements of a *commutative* ring R forms an ideal, called the *nilradical* of R. Show that $\eta(R/\eta(R)) = \{0\}$.