Chapter V

Orthogonality

Recall that

$$P_n = \mathbf{Z}[x_1, \ldots, x_n],$$

$$\Lambda_n = \mathbf{Z}[x_1, \ldots, x_n]^{S_n}$$

where $x_1, \ldots, x_n$ are independent indeterminates.

(5.1) *P_n is a free Λ_n-module of rank $n!$ with basis*

$$B_n = \{x^\alpha : 0 \le \alpha_i \le i-1,\ 1 \le i \le n\}.$$

Proof: by induction on n. The result is trivally true when $n = 1$, so assume that $n > 1$ and that P_{n-1} is a free Λ_{n-1}-module with basis B_{n-1}. Since $P_n = P_{n-1}[x_n]$, it follows that P_n is a free $\Lambda_{n-1}[x_n]$-module with basis B_{n-1}. Now

$$\Lambda_{n-1}[x_n] = \Lambda_n[x_n],$$

because the identities

$$e_r(x_1, \ldots, x_n) = \sum_{s=0}^{r} (-x_n)^s e_{r-s}(x_1, \ldots, x_n)$$

show that $\Lambda_{n-1} \subset \Lambda_n[x_n]$, and on the other hand it is clear that $\Lambda_n \subset \Lambda_{n-1}[x_n]$. Hence P_n is a free $\Lambda_n[x_n]$-module with basis B_{n-1}.

To complete the proof it remains to show that $\Lambda_n[x_n]$ is a free Λ_n-module with basis $1, x_n, \ldots, x_n^{n-1}$. Since $\prod_{i=1}^{n}(x_n - x_i) = 0$, we have

$$x_n^n = e_1 x_n^{n-1} - e_2 x_n^{n-2} + \ldots + (-1)^{n-1} e_n,$$

from which it follws that the x_n^{n-i} $(1 \le i \le n)$ generate $\Lambda_n[x_n]$ as a Λ_n-module. On the other hand, if we have a relation of linear dependence

$$\sum_{i=1}^{n} f_i x_n^{n-i} = 0$$

with coefficients $f_i \in \Lambda_n$, then we have also

$$\sum_{i=1}^{n} f_i x_j^{n-i} = 0$$

for $j = 1, 2, \ldots, n$, and since

$$det(x_j^{n-i}) = \prod_{i<j} (x_i - x_j) \neq 0,$$

it follows that $f_1 = \cdots = f_n = 0.\|$

As before, let $\delta = (n-1, n-2, \cdots, 1, 0)$. By reversing the order of $x_1, \ldots, x_n$ in (5.1) it follows that

(5.1′) *The monomials* $x^\alpha, \alpha \subset \delta$(*i.e.,* $0 \leq \alpha_i \leq n-i$ *for* $1 \leq i \leq n$) *form a* Λ_n*-basis of* $P_n.\|$

We define a scalar product on P_n, with values in Λ_n, by the rule

$$< f, g >= \partial_{w_0}(fg) \qquad (f, g \in P_n) \tag{5.2}$$

where w_0 is the longest element of S_n. Since ∂_{w_0} is Λ_n-linear, so is the scalar product.

(5.3) *Let* $w \in S_n$ *and* $f, g \in P_n$. *Then*

(i) $< \partial_w f, g >=< f, \partial_{w^{-1}} g >$

(ii) $< wf, g >= \epsilon(w) < f, w^{-1} g >$.

where $\epsilon(w) = (-1)^{\ell(w)}$ *is the sign of* w.

Proof: (i) It is enough to show that $< \partial_i f, g >=< f, \partial_i g >$ for $i \leq i \leq n-1$. We have

$$< \partial_i f, g > = \partial_{w_0}((\partial_i f)g) = \partial_{w_0 s_i} \partial_i((\partial_i f)g)$$
$$= \partial_{w_0 s_i}((\partial_i f)(\partial_i g))$$

because $\partial_i f$ is symmetrical in x_i and x_{i+1}. The last expression is symmetrical in f and g, hence $< \partial_i f, g >=< \partial_i g, f >=< f, \partial_i g >$ as required.

(ii) Again it is enough to show that $< s_i f, g >= - < f, s_i g >$. We have

$$< s_i f, g >= \partial_{w_0}((s_i f)g) = \partial_{w_0 s_i} \partial_i s_i(f(s_i g))$$

and since $\partial_i s_i = -\partial_i$ this is equal to

$$-\partial_{w_0 s_i} \partial_i(f(s_i g)) = -\partial_{w_0}(f(s_i g)) = - < f, s_i g > .\|$$

(5.4) *Let* $u, v \in S_n$ *be such that* $\ell(u) + \ell(v) = \binom{n}{2}$. *Then*

$$< \mathfrak{S}_u, \mathfrak{S}_v >= \begin{cases} 1 & \textit{if } v = w_0 u, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: We have

$$< \mathfrak{S}_u, \mathfrak{S}_v > = < \partial_{u^{-1}w_0} x^\delta, \mathfrak{S}_v >$$
$$= < x^\delta, \partial_{w_0 u} \mathfrak{S}_v >$$

by (5.3). Also $\ell(w_0 u) = \ell(w_0) - \ell(u) = \ell(v)$, hence

$$\partial_{w_0 u} \mathfrak{S}_v = \begin{cases} 1 & \text{if } v = w_0 u, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$< \mathfrak{S}_u, \mathfrak{S}_v > = \begin{cases} 0 & \text{if } v \neq w_0 u, \\ < x^\delta, 1 > = \partial_{w_0}(x^\delta) = 1 & \text{if } v = w_0 u. \| \end{cases}$$

(5.5) *Let* $u, v \in S_n$. *Then*

$$< w_0 \mathfrak{S}_u, \mathfrak{S}_{v w_0} > = \epsilon(v) \delta_{uv}.$$

Proof: We have

$$< w_0 \mathfrak{S}_u, \mathfrak{S}_{v w_0} > = < w_0 \mathfrak{S}_u, \partial_{w_0 v^{-1} w_0} x^\delta >$$
$$= < \partial_{w_0 v w_0}(w_0 \mathfrak{S}_u), x^\delta >$$
$$= \epsilon(v) < w_0 \partial_v \mathfrak{S}_u, x^\delta >$$

by (5.3) and (2.12). By (4.2) the scalar product is therefore zero unless $\ell(u) - \ell(v) = \ell(uv^{-1})$, and then it is equal to $\epsilon(v) < w_0 \mathfrak{S}_{uv^{-1}}, x^\delta >$. Now $\mathfrak{S}_{uv^{-1}}$ is a linear combination of monomials x^α such that $\alpha \subset \delta$ and $|\alpha| = \ell(u) - \ell(v)$. Hence $w_0(\mathfrak{S}_{uv^{-1}}) x^\delta$ is a sum of monomials x^β where

$$\beta = w_0 \alpha + \delta \subset w_0 \delta + \delta = (n-1, \ldots, n-1).$$

Now $\partial_{w_0} x^\beta = 0$ unless all the components β_i of β are distinct; since $0 \leq \beta_i \leq n-1$ for each i, it follows that $\partial_{w_0} x^\beta = 0$ unless $\beta = w\delta$ for some $w \in S_n$, and in that case

$$w_0 \alpha = \beta - \delta = w\delta - \delta$$

must have all its components ≥ 0. So the only possibility that gives a nonzero scalar product is $w = 1, \alpha = 0, u = v$, and in that case

$$< w_0 \mathfrak{S}_u, \mathfrak{S}_{v w_0} > = \epsilon(v) < 1, x^\delta >$$
$$= \epsilon(v) \partial_{w_0}(x^\delta) = \epsilon(v). \|$$

(5.6) *The Schubert polynomials* $\mathfrak{S}_w, w \in S_n$, *form a* Λ_n-*basis of* P_n.

Proof: Let $u, v \in S_n$ and let

$$w_0 \mathfrak{S}_u = \sum_{\alpha \subset \delta} a_{u\alpha} x^\alpha, \tag{1}$$

(2) $$\epsilon(v)\mathfrak{S}_{vw_0} = \sum_{\beta\subset\delta} b_{v\beta} x^\beta,$$

with coefficients $a_{u\alpha}, b_{v\beta} \in \Lambda_n$. Let $c_{\alpha\beta} =< x^\alpha, x^\beta >$. Then from (5.5) we have

$$\sum_{\alpha,\beta} a_{u\alpha} c_{\alpha\beta} b_{v\beta} = \delta_{uv},$$

or in matrix terms

(3) $$ACB^t = 1$$

where $A = (a_{u\alpha}), B = (b_{v\beta})$ and $C = (c_{\alpha\beta})$ are square matrices of size $n!$, with coefficients in Λ_n. From (3) it follows that each of A, B, C has determinant ± 1; hence the equations (2) can be solved for $x^\beta, \beta \subset \delta$, as Λ_n-linear combinations of the Schubert polynomials $\mathfrak{S}_w, w \in S_n$. Since by (5.1′) the x^β from a Λ_n-basis of P_n, so also do the $\mathfrak{S}_w$.‖

We have

(5.7) $$< f, g >= \sum_{w\in S_n} \epsilon(w)\partial_w(w_0 f)\partial_{ww_0}(g)$$

for all $f, g \in P_n$.

Proof: Let $\Phi(f, g)$ denote the right-hand side of (5.7). We claim first that

(1) $$\Phi(f, g) \in \Lambda_n.$$

For this it is enough to show that $\partial_i \Phi = 0$ for $1 \le i \le n-1$. Let

$$A_i = \{w \in S_n : \ell(s_i w) > \ell(w)\},$$

then S_n is the disjoint union of A and $s_i A$, and $s_i A = A w_0$. Hence

$$\Phi(f, g) = \sum_{w\in A_i} \epsilon(w)\{\partial_w(w_0 f)\partial_i(\partial_{s_i w w_0} g) - \partial_i \partial_w(w_0 f)(\partial_{s_i w w_0} g)\}.$$

Since for all $\phi, \psi \in P_n$ we have

$$\partial_i(\phi\partial_i\psi - (\partial_i\phi)\psi) = (\partial_i\phi)(\partial_i\psi) - (\partial_i\phi)(\partial_i\psi) = 0,$$

it follows that $\partial_i\Phi(f, g) = 0$ for all i as required.

Next, since each operator ∂_w is Λ_n-linear, it follows that $\Phi(f, g)$ is Λ_n-linear in each argument. By (5.6) it is therefore enough to verify (5.7) when $f = w_0\mathfrak{S}_u$ and $g = \mathfrak{S}_{vw_0}$, where $u, v \in S_n$. We have then

$$\Phi(w_0\mathfrak{S}_u, \mathfrak{S}_{vw_0}) = \sum_{w\in S_n} \epsilon(w)\partial_{w^{-1}}(\mathfrak{S}_u)\partial_{w^{-1}w_0}(\mathfrak{S}_{vw_0})$$

which by (4.2) is equal to

$$\sum_w \epsilon(w)\mathfrak{S}_{uw}\mathfrak{S}_{vw} \tag{2}$$

summed over $w \in S_n$ such that

$$\ell(uw) = \ell(u) - \ell(w^{-1}) = \ell(u) - \ell(w)$$

and

$$\ell(vw) = \ell(vw_0) - \ell(w^{-1}w_0) = \ell(w) - \ell(v).$$

Hence the polynomial (2) is (i) symmetric in $x_1, ..., x_n$ (by (1) above), (ii) independent of x_n, (iii) homogeneous of degree $\ell(u) - \ell(v)$. Hence it vanishes unless $\ell(u) = \ell(v)$ and $u = w^{-1} = v$, in which case it is equal to $\epsilon(w) = \epsilon(v)$. Hence

$$\Phi(w_0\mathfrak{S}_u, \mathfrak{S}_{vw_0}) = \epsilon(v)\delta_{uv} =< w_0\mathfrak{S}_u, \mathfrak{S}_{vw_0} >$$

by (5.5). This completes the proof of (5.7).‖

Now let $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots, y_n)$ be two sequences of independent variables, and let

$$\Delta = \Delta(x, y) = \prod_{i+j \leq n} (x_i - y_j) \tag{5.8}$$

(the "semiresultant"). We have

$$\Delta(wx, x) = \begin{cases} 0 & \text{if } w \neq w_0, \\ \epsilon(w_0)a_\delta(x) & \text{if } w = w_0. \end{cases} \tag{5.9}$$

For

$$\Delta(wx, x) = \prod_{i+j \leq n} (x_{w(i)} - x_j)$$

is non-zero if and only if $w(i) \neq j$ whenever $i + j \leq n$, that is to say if and only if $w \neq w_0$; and

$$\begin{aligned} \Delta(w_0x, x) &= \prod_{i+j \leq n} (x_{n+1-i} - x_j) \\ &= \prod_{j<k} (x_k - x_j) = \epsilon(w_0)a_\delta(x).\| \end{aligned}$$

The polynomial $\Delta(x, y)$ is a linear combination of the monomials $x^\alpha, \alpha \subset \delta$, with coefficients in $\mathbf{Z}[y_1, \ldots, y_n] = P_n(y)$, hence by (4.11) can be written uniquely in the form

$$\Delta(x, y) = \sum_{w \in S_n} \mathfrak{S}_w(x)T_w(y)$$

with $T_w(y) \in P_n(y)$. By (5.5) we have

$$T_w(y) =< \Delta(x,y), w_0\mathfrak{S}_{ww_0}(-x) >_x$$

where the suffix x means that the scalar product is taken in the x variables. Hence

$$\begin{aligned} T_w(y) &= \partial_{w_0}(\Delta(x,y)w_0(\mathfrak{S}_{ww_0}(-x))) \\ &= a_\delta(x)^{-1}\sum_{v\in S_n}\epsilon(v)\Delta(vx,y)vw_0(\mathfrak{S}_{ww_0}(-x)) \end{aligned} \tag{1}$$

by (2.10), where $v \in S_n$ acts by permuting the x_i.

Now this expression (1) must be independent of $x_1, \ldots, x_n$. Hence we may set $x_i = y_i$ ($1 \le i \le n$). But then (5.9) shows that the only non-zero term in the sum (1) is that corresponding to $v = w_0$, and we obtain

$$T_w(y) = \mathfrak{S}_{ww_0}(-y).$$

Hence we have proved

(5.10) ("Cauchy formula")

$$\Delta(x,y) = \sum_{w\in S_n}\mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(-y). \quad \|$$

Remark. Let $n = r + s$ where $r, s \ge 1$, and regard $S_r \times S_s$ as a subgroup of S_n, with S_r permuting $1, 2, \ldots, r$ and S_s permuting $r+1, \ldots, r+s$. Let $w_0^{(r)}, w_0^{(s)}$ be the longest elements of S_r, S_s respectively, and let $u = w_0^{(r)} \times w_0^{(s)}$. If $w \in S_n$, we have $\partial_u\mathfrak{S}_w = \mathfrak{S}_{wu}$ if $\ell(wu) = \ell(w) - \ell(u)$, that is to say if wu is Grassmannian (with its only descent at r), and $\partial_u\mathfrak{S}_w = 0$ otherwise. Hence by applying ∂_u to the x-variables in (5.10) we obtain

$$\partial_u\Delta(x,y) = \sum_{v\in G_{rs}}\mathfrak{S}_v(x)\mathfrak{S}_{vuw_0}(-y)$$

where $G_{r,s} \subset S_n$ is the set of Grassmannian permutations v with descent at r (i.e. $v(i) < v(i+i)$ if $i \ne r$). On the other hand, it is easily verified that

$$\partial_u\Delta(x,y) = \prod_{i=1}^{r}\prod_{j=1}^{s}(x_i - y_j)$$

and that $v' = vuw_0$ is the permutation

$$(v(r+1), \ldots, v(r+s), v(1), \ldots, v(r))$$

hence is also Grassmannian, with descent at s.

The shape of v is

$$\lambda = \lambda(v) = (v(r) - r, \ldots, v(2) - 2, v(1) - 1)$$

and the shape of v' is say

$$\mu' = \lambda(v') = (v(r+s) - s, \ldots, v(r+2) - 2, v(r+1) - 1).$$

The relation between these two partitions is

$$\mu_i = s - \lambda_{r+1-i} \qquad (1 \leq i \leq r)$$

that is to say λ is the complement, say $\hat{\mu}$, of μ in the rectangle (s^r) with r rows and s columns. Hence, replacing each y_j by $-y_j$, we obtain from (5.10) by operating with ∂_u on both sides and using (4.8)

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(x_i + y_j) = \sum s_{\hat{\mu}}(x)s_{\mu'}(y) \tag{5.11}$$

summed over all $\mu \subset (s^r)$, where $\hat{\mu}$ is the complement of μ in (s^r). This is one version of the usual Cauchy identity [M, Chapter I ,(4.3)$'$].

Let $(\mathfrak{S}^w)_{w\in S_n}$ be the Λ_n-basis of P_n dual to the basis $(\mathfrak{S}_w)$ relative to the scalar product (5.2). By (5.3) and (5.5) we have

$$< \mathfrak{S}_u, w_0\mathfrak{S}_{vw_0} >= \epsilon(vw_0)\delta_{uv}$$

or equivalently

$$< \mathfrak{S}_u(x), w_0\mathfrak{S}_{vw_0}(-x) >= \delta_{uv}$$

which shows that

$$\mathfrak{S}^w(x) = w_0\mathfrak{S}_{ww_0}(-x) \tag{5.12}$$

for all $w \in S_n$. From (5.10) it follows that

$$\Delta(x,y) = \sum_{w\in S_n} \mathfrak{S}_w(x)w_0\mathfrak{S}^w(y)$$

or equivalently

$$\prod_{1\leq i<j\leq n}(x_i - y_j) = \sum_{w\in S_n} \mathfrak{S}_w(x)\mathfrak{S}^w(y). \tag{5.13}$$

Let $(x_\beta)_{\beta\subset\delta}$ be the basis dual to $(x^\alpha)_{\alpha\subset\delta}$. If

$$\mathfrak{S}_u = \sum a_{u\alpha} x^\alpha,$$
$$\mathfrak{S}^v = \sum b_{v\beta} x_\beta,$$

then by taking scalar products we have

$$\sum_\alpha a_{u\alpha} b_{v\beta} = \delta_{uv}$$

and therefore also

$$\sum_w a_{w\alpha} b_{w\beta} = \delta_{\alpha\beta},$$

so that

$$\begin{aligned}\sum_{w\in S_n} \mathfrak{S}_w(x)\mathfrak{S}^w(y) &= \sum_{\alpha,\beta}(\sum_w a_{w\alpha}b_{w\beta})x^\alpha y_\beta \\ &= \sum_\alpha x^\alpha y_\alpha.\end{aligned}$$

From (5.13) it follows that y_α is the coefficient of x^α in $\prod_{i<j}(x_i - y_j)$, and hence we find

$$x_\alpha = (-1)^{|\beta|} \prod_{i=1}^{n-1} e_{\beta_i}(x_{i+1},\ldots,x_n) \tag{5.14}$$

where $\beta = \delta - \alpha$.‖

Let

$$C(x,y) = \epsilon(w_0)\Delta(w_0 x, y) = \prod_{i<j}(y_i - x_j).$$

If $f(x) \in H_n$ (4.11), let $f(y)$ denote the polynomial in $y_1,\ldots,y_n$ obtained by replacing each x_i by y_i. Then we have

$$< f(x), C(x,y) >_x = f(y), \tag{5.15}$$

where as before the suffix x means that the scalar product is taken in the x variables. In other words, $C(x,y)$ is a "reproducing kernel" for the scalar product.

Proof: From (5.13) we have

$$C(x,y) = \sum_{w\in S_n} \epsilon(w_0)\mathfrak{S}_w(w_0x)\mathfrak{S}_{ww_0}(-y).$$

Hence by (5.5)

$$\begin{aligned}< C(x,y), \mathfrak{S}_{ww_0}(x) >_x &= \epsilon(ww_0)\mathfrak{S}_{ww_0}(-y) \\ &= \mathfrak{S}_{ww_0}(y).\end{aligned}$$

Hence (5.15) is true for all Schubert polynomials $\mathfrak{S}_u, u \in S_n$. Since the scalar product is Λ_n-linear it follows from (5.6) that (5.15) is true for all $f \in H_n$.‖

Let θ_{yx} be the homomorphism that replaces each y_i by x_i. Then (5.15) can be restated in the form

$$\theta_{yx} < f(x), C(x,y) >_x = f(x) \tag{5.15$'$}$$

for all $f \in H_n$.

Now let $z = [z_1, \ldots, z_n]$ be a third set of variables and consider

$$< C(x,y), \partial_u v^{-1} C(x,z) >_x \tag{1}$$

for $u, v \in S_n$, where ∂_u and v^{-1} act on the x variables. By (5.3) this is equal to

$$\epsilon(v) < C(x,z), v\partial_{u^{-1}} C(x,y) >_x \tag{2}$$

and by (5.15′) we have

$$\theta_{yx} < C(x,y), \partial_u v^{-1} C(x,z) >_x = \partial_u v^{-1} C(x,z), \tag{3}$$

$$\theta_{zx} < C(x,z), v\partial_{u^{-1}} C(x,y) >_x = v\partial_{u^{-1}} C(x,y). \tag{4}$$

Since θ_{yx} and θ_{zx} commute, it follows from (1)-(4) that

$$\begin{aligned}\theta_{yx} v\partial_{u^{-1}} C(x,y) &= \epsilon(v)\theta_{zx}\partial_u v^{-1} C(x,z)\\ &= \epsilon(v)\theta_{yx}\partial_u v^{-1} C(x,y).\end{aligned}$$

Hence we have

$$\theta(v\partial_{u^{-1}} w_0 \Delta) = \epsilon(v)\theta(\partial_u v^{-1} w_0 \Delta) \tag{5.16}$$

for all $u, v \in S_n$, where $\Delta = \Delta(x,y)$ and $\theta = \theta_{yx}$.

Let E_n denote the algebra of operators ϕ of the form

$$\phi = \sum_{w \in S_n} \phi_w w,$$

with coefficients $\phi_w \in Q_n = \mathbf{Q}(x_1, \ldots, x_n)$. For such a ϕ we have

$$\phi_w = \epsilon(w_0) a_\delta^{-1} \theta(\phi(w^{-1} w_0 \Delta)) \tag{5.17}$$

for all $w \in S_n$, where ϕ and $w^{-1}w_0$ act on the x variables in Δ.

For $\theta(\phi(w^{-1}w_0\Delta)) = \sum_{u\in S_n} \phi_u \theta(uw^{-1}w_0\Delta)$, and by (5.8) $\theta(uw^{-1}w_0\Delta) = \Delta(uw^{-1}w_0x, x)$ is zero if $u \neq w$, and is equal to $\epsilon(w_0)a_\delta$ if $u = w$.

Let $u \in S_n$, and let $(a_1, \ldots, a_p)$ be a reduced word for u, so that $\partial_u = \partial_{a_1} \cdots \partial_{a_p}$. Since $\partial_a = (x_a - x_{a+1})^{-1}(1 - s_a)$ for each $a \geq 1$, it follows that we may write

$$\partial_u = \epsilon(w_0)a_\delta^{-1} \sum_{v\leq u} \alpha_{uv} v, \tag{5.18}$$

where $v \leq u$ means that v is of the form $s_{b_1} \ldots s_{b_q}$, where $(b_1, \ldots, b_q)$ is a subword of $(a_1, \ldots, a_p)$.

The coefficients α_{uv} in (5.18) are polynomials, for it follows from (5.16) and (5.17) that

$$\begin{aligned} \alpha_{uv} &= \theta(\partial_u(v^{-1}w_0\Delta)) \\ &= \epsilon(v)\theta(v\partial_{u^{-1}}w_0\Delta). \end{aligned} \tag{5.19}$$

(5.20) *For all* $f \in P_n$ *we have*

$$\theta(\partial_u(\Delta f)) = \begin{cases} w_0 f & \text{if } u = w_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: From (5.18) we have

$$\theta(\partial_u(\Delta f)) = a_\delta^{-1} \sum_{v\leq u} \alpha_{uv} v(f)\theta(v\Delta).$$

By (5.9) this is zero if $u \neq w_0$, and if $u = w_0$ then by (2.10)

$$\begin{aligned} \theta(\partial_{w_0}(\Delta f)) &= a_\delta^{-1} \sum_{w\in S_n} \epsilon(w)w(f)\theta(w\Delta) \\ &= a_\delta^{-1}\epsilon(w_0)w_0(f)\epsilon(w_0)a_\delta = w_0(f) \end{aligned}$$

by (5.9) again.‖

The matrix of coefficients (α_{uv}) in (5.18) is triangular with respect to the ordering $\leq$, and one sees easily that the diagonal entries α_{uu} are non-zero (they are products in which each factor is of the form $x_i - x_j$). Hence we may invert the equations (5.18), say

$$u = \sum_{v\leq u} \beta_{uv}\partial_v \tag{5.21}$$

and thus we can express any $\phi \in E_n$ as a linear combination of the operators ∂_w. Explicitly, we have

$$\phi = \sum_{w\in S_n} \theta(\phi(\partial_{w^{-1}w_0}\Delta))\partial_w. \tag{5.22}$$

Proof: By linearity we may assume that $\phi = f\partial_u$ with $f \in Q_n$. Then

$$\theta(\phi(\partial_{w^{-1}w_0}\Delta)) = f\theta(\partial_u\partial_{w^{-1}w_0}\Delta).$$

Now by (4.2) $\partial_u\partial_{w^{-1}w_0}$ is either zero or equal to $\partial_{uw^{-1}w_0}$, and by (5.20) $\theta(\partial_{uw^{-1}w_0}\Delta)$ is zero if $w \neq u$, and is equal to 1 if $w = u$. Hence the right-hand side of (5.22) is equal to $f\partial_u = \phi$, as required.‖

In particular, it follows from (5.22) and (5.21) that

(5.23) $$\beta_{uv} = \theta(u\partial_{v^{-1}w_0}\Delta),$$

hence is a polynomial.

The coefficients α_{uv}, β_{uv} in (5.18) and (5.23) satisfy the following relations:

(5.24) (i) $\beta_{uv} = \epsilon(uv)\alpha_{vw_0,uw_0}$,

(ii) $\alpha_{u^{-1}v^{-1}} = v^{-1}(\alpha_{uv})$,

(iii) $\alpha_{\overline{u},\overline{v}} = \epsilon(uw_0)w_0(\alpha_{uv})$,

for all $u, v \in S_n$, *where* $\overline{u} = w_0uw_0, \overline{v} = w_0vw_0$.

Proof: (i) By (5.23) and (2.12) we have

$$\begin{aligned}\beta_{uv} &= \epsilon(v^{-1}w_0)\theta(uw_0\partial_{w_0v^{-1}}w_0\Delta) \\ &= \epsilon(v^{-1}w_0)\epsilon(uw_0)\theta(\partial_{vw_0}w_0u^{-1}w_0\Delta) && \text{by (5.16)} \\ &= \epsilon(uv)\alpha_{vw_0,uw_0}. && \text{by (5.19).}\end{aligned}$$

(ii) From (5.18) we have

$$\begin{aligned}\theta(v\partial_{u^{-1}}w_0\Delta) &= \epsilon(w_0)v(a_\delta^{-1})\sum_w v(\alpha_{u^{-1},w^{-1}})\theta(vw^{-1}w_0\Delta) \\ &= \epsilon(v)v(\alpha_{u^{-1},v^{-1}}) && \text{by (5.9),}\end{aligned}$$

and likewise

$$\begin{aligned}\theta(\partial_u v^{-1}w_0\Delta) &= \frac{\epsilon(w_0)}{a_\delta}\sum_w \alpha_{uw}\theta(wv^{-1}w_0\Delta) \\ &= \alpha_{uv}\end{aligned}$$

again by (5.9). Hence (ii) follows from (5.16).

(iii) Since $\partial_{\overline{u}} = \epsilon(u)w_0\partial_u w_0$ (2.12) we have

$$\begin{aligned}\sum_v \alpha_{\overline{uv}}\overline{v} &= \epsilon(uw_0)w_0(\sum_v \alpha_{uv}v)w_0 \\ &= \epsilon(uw_0)\sum_v w_0(\alpha_{uv})\overline{v}\end{aligned}$$

and hence $\alpha_{\overline{uv}} = \epsilon(uw_0)w_0(\alpha_{uv}).\|$

(5.25) *Let* E'_n *be the subalgebra of operators* $\phi \in E_n$ *such that* $\phi(P_n) \subset P_n$. *Then* E'_n *is a free* P_n*-module with basis* $(\partial_w)_{w \in S_n}$.

Proof: If $\phi = \sum_{w \in S_n} \phi_w \partial_w \in E'_n$, then by (5.22)

$$\phi_w = \theta(\phi(\partial_{w^{-1}w_0}\Delta)) \in P_n.$$

On the other hand, the ∂_w are a Q_n-basis of E_n, and hence are linearly independent over P_n.$\|$

Chapter VI

Double Schubert Polynomials

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ be two sequences of independent indeterminates, and recall (5.8) that

$$\Delta(x, y) = \prod_{i+j \leq n} (x_i - y_j).$$

For each $w \in S_n$, we define the *double Schubert polynomial* $\mathfrak{S}_w(x, y)$ to be

$$\mathfrak{S}_w(x, y) = \partial_{w^{-1}w_0} \Delta(x, y) \tag{6.1}$$

where $\partial_{w^{-1}w_0}$ acts on the x variables.

Since $\Delta(x, 0) = x^\delta$ we have

$$\mathfrak{S}_w(x, 0) = \mathfrak{S}_w(x), \tag{6.2}$$

the (single) Schubert polynomial indexed by w.

From the Cauchy formula (5.10) we have

$$\mathfrak{S}_w(x, y) = \sum_{v \in S_n} \partial_{w^{-1}w_0} \mathfrak{S}_{vw_0}(x) \mathfrak{S}_v(-y)$$

and by (4.2)

$$\partial_{w^{-1}w_0} \mathfrak{S}_{vw_0}(x) = \mathfrak{S}_{vw}(x)$$

if $\ell(vw) = \ell(vw_0) - \ell(w^{-1}w_0)$, i.e. if $\ell(vw) = \ell(w) - \ell(v)$, and

$$\partial_{w^{-1}w_0} \mathfrak{S}_{vw_0}(x) = 0$$

otherwise. Hence

$$\mathfrak{S}_w(x, y) = \sum_{u,v} \mathfrak{S}_u(x) \mathfrak{S}_v(-y) \tag{6.3}$$

summed over all $u, v \in S_n$ *such that* $w = v^{-1}u$ *and* $\ell(w) = \ell(u) + \ell(v)$.

From (6.3) it follows that $\mathfrak{S}_w(x, y)$ is a homogeneous polynomial of degree $\ell(w)$ in $x_1, \ldots, x_{n-1}$, $y_1, \ldots, y_{n-1}$. We have

(6.4) (i) $\mathfrak{S}_{w_0}(x, y) = \Delta(x, y)$,

(ii) $\mathfrak{S}_1(x, y) = 1$,

(iii) $\mathfrak{S}_{w^{-1}}(x, y) = \mathfrak{S}_w(-y, -x) = \epsilon(w)\mathfrak{S}_w(y, x)$ *for all* $w \in S_n$,

(iv) $\mathfrak{S}_w(x, x) = 0$ *for all* $w \in S_n$ *except* $w = 1$.

Proof: (i) is immediate from the definition (6.1).

(ii) and (iii) follow from (6.3).

(iv) follows from (5.20), since $\mathfrak{S}_w(x, x) = \theta(\partial_{w^{-1}w_0}\Delta) = 0$ if $w \neq 1$.‖

(6.5) (Stability) *If* $m > n$ *and* i *is the embedding of* S_n in S_m, *then*

$$\mathfrak{S}_{i(w)}(x, y) = \mathfrak{S}_w(x, y)$$

for all $w \in S_n$.

Proof: This again follows from (6.3) and the stability of the single Schubert polynomials (4.5).‖

From (6.5) it follows that the double Schubert polynomials $\mathfrak{S}_w(x, y)$ are well defined for all permutations $w \in S_\infty$.

For any commutative ring K, let $K(S_\infty)$ denote the K-module of all functions on S_∞ with values in K. We define a multiplication in $K(S_\infty)$ as follows: for $f, g \in K(S_\infty)$,

$$(fg)(w) = \sum_{u,v} f(u)g(v)$$

summed over all $u, v \in S_\infty$ such that $uv = w$ and $\ell(u) + \ell(v) = \ell(w)$. For this multiplication, $K(S_\infty)$ is an associative (but not commutative) ring, with identity element $\underline{\underline{1}}$, the characteristic function of the identity permutation 1. It carries an involution $f \mapsto f^*$, defined by

$$f^*(w) = f(w^{-1})$$

which satisfies

$$(fg)^* = g^* f^*$$

for all $f, g \in K(S_\infty)$.

(6.6) Let $f, g \in K(S_\infty)$.

(i) *If $fg = f$ and $f(1)$ is not a zero divisor in K, then $g = \underline{\underline{1}}$.*

(ii) *If $fg = \underline{\underline{1}}$, then $gf = \underline{\underline{1}}$.*

(iii) *f is a unit (i.e. invertible) in $K(S_\infty)$ if and only if $f(1)$ is a unit in K.*

Proof: (i) We have $f(1) = f(1)g(1)$ and hence $g(1) = 1$. We shall show by induction on $\ell(w)$ that $g(w) = 0$ for all $w \neq 1$. So let $r > 0$ and assume that $g(v) = 0$ for all $v \in S_\infty$ such that $1 \leq \ell(v) \leq r-1$. Let w be a permutation of length r. We have

$$f(w) = (fg)(w) = f(w)g(1) + f(1)g(w) + \sum_{u,v} f(u)g(v) \tag{1}$$

where the sum on the right is over $u, v \in S_\infty$ such that $u \neq 1, v \neq 1, uv = w$ and $\ell(u) + \ell(v) = \ell(w)$, so that $1 \leq \ell(v) \leq r-1$ and therefore $g(v) = 0$. Hence (1) reduces to $f(1)g(w) = 0$ and therefore $g(w) = 0$ as required.

(ii) We have $f(1)g(1) = 1$ so that $f(1)$ is a unit in K. Also $f(gf) = (fg)f = f$, whence $gf = \underline{\underline{1}}$ by (i) above.

(iii) Suppose f is a unit in $K(S_\infty)$, with inverse g. Since $fg = \underline{\underline{1}}$ we have $f(1)g(1) = 1$, whence $f(1)$ is an unit in K.

Conversely, if $f(1)$ is an unit in K we construct an inverse g of f as follows. We define $g(1) = f(1)^{-1}$ and proceed to define $g(w)$ by induction on $\ell(w)$. Assume that $g(v)$ has been defined for all v such that $\ell(v) < \ell(w)$ and set

$$g(w) = -f(1)^{-1} \sum_{u,v} f(u)g(v)$$

summed over u, v such that $uv = w, v \neq w$ and $\ell(u) + \ell(v) = \ell(w)$. This definition gives $(fg)(w) = 0$ as required.‖

Now let $\mathfrak{S}(x)$ (resp. $\mathfrak{S}(x,y)$) be the function on S_∞ whose value at a permutation w is $\mathfrak{S}_w(x)$ (resp. $\mathfrak{S}_w(x,y)$). (The coefficient ring K is now the ring $\mathbf{Z}[x,y]$ of polynomials in the $x's$ and $y's$.) Since $\mathfrak{S}_1(x) = \mathfrak{S}_1(x,y) = 1$, it follows from (6.6)(iii) that $\mathfrak{S}(x)$ and $\mathfrak{S}(x,y)$ are units in $K(S_\infty)$.

(6.7) (i) $\mathfrak{S}(x,0) = \mathfrak{S}(x)$,

(ii) $\mathfrak{S}(x,x) = \underline{\underline{1}}$,

(iii) $\mathfrak{S}(x,y)^* = \mathfrak{S}(-y,-x)$,

(iv) $\mathfrak{S}(x)^{-1} = \mathfrak{S}(0,x)$,

(v) $\mathfrak{S}(x)^* = \mathfrak{S}(-x)^{-1}$,

(vi) $\mathfrak{S}(x,y) = \mathfrak{S}(y)^{-1}\mathfrak{S}(x) = \mathfrak{S}(y,x)^{-1}$.

Proof: (i)-(iii) follow directly from (6.2) and (6.4).

From (6.3) and (6.4) we have

$$\mathfrak{S}_w(x,y) = \sum_{u,v} \mathfrak{S}_{u^{-1}}(-y)\mathfrak{S}_v(x) = \sum_{u,v} \mathfrak{S}_u(0,y)\mathfrak{S}_v(x)$$

summed over $u, v \in S_\infty$ such that $uv = w$ and $\ell(u) + \ell(v) = \ell(w)$. In other words,

$$\mathfrak{S}(x,y) = \mathfrak{S}(0,y)\mathfrak{S}(x). \tag{1}$$

In particular, when $y = x$ we obtain $\mathfrak{S}(0,x)\mathfrak{S}(x) = \mathfrak{S}(x,x) = \underline{\underline{1}}$ by (ii) above, and hence $\mathfrak{S}(0,x) = \mathfrak{S}(x)^{-1}$. This establishes (iv); part(v) now follows from (iv) and (iii), and (vi) from (iv) and (1) above. ‖

From (6.7) (vi) we have

$$\mathfrak{S}(x) = \mathfrak{S}(y)\mathfrak{S}(x,y)$$

or explicitly

$$\mathfrak{S}_w(x) = \sum_{u,v} \mathfrak{S}_u(y)\mathfrak{S}_v(x,y)$$

summed over u, v such that $uv = w$ and $\ell(u) + \ell(v) = \ell(w)$, so that $u = wv^{-1}$ and $\mathfrak{S}_u = \partial_v \mathfrak{S}_w$ by (4.2). Hence

$$\mathfrak{S}_w(x) = \sum_{v} \mathfrak{S}_v(x,y)\partial_v \mathfrak{S}_w(y)$$

(where the operators ∂_v act on the y variables). The sum here may be taken over all permutations v, since $\partial_v \mathfrak{S}_w = 0$ unless $\ell(wv^{-1}) = \ell(w) - \ell(v)$. By linearity and (4.13) it follows that

(6.8) (Interpolation Formula) *For all* $f \in P_n = \mathbf{Z}[x_1, \ldots, x_n]$ *we have*

$$f(x) = \sum_{w} \mathfrak{S}_w(x,y)\partial_w f(y)$$

summed over permutations $w \in S^{(n)}$.‖

(The reason for the restriction to $S^{(n)}$ in the summation is that if $w \notin S^{(n)}$ we shall have $w(m) > w(m+1)$ for some $m > n$, and hence $\partial_w = \partial_v \partial_m$ where $v = ws_m$; but $\partial_m f = 0$ for all $f \in P_n$, since $m > n$, and therefore $\partial_w f = 0$.)

Remarks. 1. By setting each $y_i = 0$ in (6.8) we regain (4.14).

2. When $n = 1$, the sum is over $S^{(1)}$, which consists of the permutations $w_p = s_p s_{p-1} \ldots s_1$ ($p \geq 0$); w_p is dominant, of shape (p), so that (see (6.15) below) $\mathfrak{S}_{w_p}(x,y) = (x - y_1) \cdots (x - y_p)$. Hence the case $n = 1$ of (6.8) is *Newton's interpolation formula*

$$f(x) = \sum_{p \geq 0} (x - y_1) \cdots (x - y_p) f_p(y_1, \ldots, y_{p+1})$$

where $f_p = \partial_p \partial_{p-1} \cdots \partial_1 f$, or explicitly

$$f_p(y_1, \ldots, y_{p+1}) = \sum_{i=1}^{p+1} \frac{f(y_i)}{\prod_{j \neq i}(y_i - y_j)}.$$

For any integer r, let $\mathfrak{S}_w(x, r)$ denote the polynomial obtained from $\mathfrak{S}_w(x, y)$ by setting $y_1 = y_2 = \cdots = r$. Since

$$\begin{aligned} \mathfrak{S}_{w_0}(x, r) = \Delta(x, r) &= \prod_{i=1}^{n-1} (x_i - r)^{n-i} \\ &= \mathfrak{S}_{w_0}(x - r) \end{aligned}$$

where $x - r$ means $(x_1 - r, x_2 - r, \ldots)$, it follows from the definitions (6.1) and (4.1) that

$$\mathfrak{S}_w(x, r) = \mathfrak{S}_w(x - r)$$

for all permutations w. Hence, by (6.7)(vi),

$$\mathfrak{S}(x - r) = \mathfrak{S}(r)^{-1} \mathfrak{S}(x)$$

and in particular, for all integers q,

$$\mathfrak{S}(q - r) = \mathfrak{S}(r)^{-1} \mathfrak{S}(q)$$

from which it follows that

$$\mathfrak{S}(r) = \mathfrak{S}(1)^r \tag{6.9}$$

for all $r \in \mathbf{Z}$.

Since $\mathfrak{S}_w(x)$ is a sum of monomials with positive integral coefficients (4.17), $\mathfrak{S}_w(1)$ is the number of monomials in $\mathfrak{S}_w(x)$ (each monomial counted the number of times it occurs). By homogeneity, we have

$$\mathfrak{S}_w(r) = r^{\ell(w)} \mathfrak{S}_w(1). \tag{6.10}$$

From (6.7)(v) and (6.9) we obtain

$$\mathfrak{S}(1)^* = \mathfrak{S}(-1)^{-1} = \mathfrak{S}(1)$$

so that we have another proof of the fact (4.30) that $\mathfrak{S}_w(1) = \mathfrak{S}_{w^{-1}}(1)$.

Now consider the function $F = \mathfrak{S}(1) - \underline{\underline{1}}$, whose value at $w \in S_\infty$ is

$$F(w) = \begin{cases} \text{number of monomials in } \mathfrak{S}_w, & \text{if } w \neq 1, \\ 0, & \text{if } w = 1. \end{cases}$$

For each positive integer p we have

$$\begin{aligned} F^p &= (\mathfrak{S}(1) - \underline{\underline{1}})^p \\ &= \sum_{r=0}^{p} (-1)^r \binom{p}{r} \mathfrak{S}(1)^r \\ &= \sum_{r=0}^{p} (-1)^r \binom{p}{r} \mathfrak{S}(r) \qquad (1) \end{aligned}$$

by (6.9). The value of (1) at a permutation w of length p is by (6.10) equal to

$$\left(\sum_{r=0}^{p} (-1)^r \binom{p}{r} r^p \right) \mathfrak{S}_w(1)$$

which is equal to $p!\mathfrak{S}_w(1)$ (consider the coefficient of t^p in $(e^t - 1)^p$). On the other hand, $F^p(w)$ is by definition equal to

$$\sum_{w_1,\ldots,w_p} F(w_1) \cdots F(w_p) \qquad (2)$$

summed over all sequences $(w_1, \ldots, w_p)$ of permutations such that $w_1 \ldots w_p = w, \ell(w_1) + \cdots + \ell(w_p) = \ell(w) = p$, and $w_i \neq 1$ for $1 \leq i \leq p$. It follows that each w_i has length 1, hence $w_i = s_{a_i}$ say, and that $(a_1, \ldots, a_p)$ is a reduced word for w. Since

$$\mathfrak{S}_{s_a} = x_1 + \cdots + x_a$$

by (4.4), we have $F(w_i) = \mathfrak{S}_{s_{a_i}}(1) = a_i$, and hence the sum (2) is equal to $\sum a_1 a_2 \cdots a_p$ summed over all $(a_1, \ldots, a_p) \in R(w)$.

We have therefore proved that

(6.11) *The number of monomials in* $\mathfrak{S}_w$ *is*

$$\mathfrak{S}_w(1) = \frac{1}{p!} \sum a_1 a_2 \cdots a_p$$

summed over all $(a_1, \ldots, a_p) \in R(w)$, *where* $p = \ell(w)$.‖

Remarks. 1. The reduced words for $1_m \times w (m \geq 1)$ are $(m + a_1, \ldots, m + a_p)$ where $(a_1, \ldots, a_p) \in R(w)$. Hence from (6.11) and homogeneity we have

$$\mathfrak{S}_{1_m \times w}\left(\frac{1}{m}\right) = \frac{1}{p!} \sum \left(1 + \frac{a_1}{m}\right) \cdots \left(1 + \frac{a_p}{m}\right)$$

summed over $R(w)$ as before. Letting $m \to \infty$, we deduce that

$$\text{Card } R(w) = p! \lim_{m \to \infty} \mathfrak{S}_{1_m \times w}\left(\frac{1}{m}\right). \qquad (6.12)$$

2. If w is dominant of length p, then $\mathfrak{S}_w$ is a monomial by (4.7), and hence in this case

$$\sum_{R(w)} a_1 \dots a_p = p!$$

3. Suppose that w is vexillary of length p. Then by (4.9) we have

$$\mathfrak{S}_w = s_\lambda(X_{\phi_1}, \dots, X_{\phi_r})$$

where λ is the shape of w and $\phi = (\phi_1, \dots, \phi_r)$ the flag of w. Hence

$$\mathfrak{S}_{1_m \times w} = s_\lambda(X_{\phi_1+m}, \dots, X_{\phi_r+m})$$

for each $m \geq 1$. If we now set each $x_i = \frac{1}{m}$ and then let $m \to \infty$, we shall obtain in the limit the Schur function s_λ for the series e^t ([M], Ch. I, §3, Ex. 5), which is equal to $h(\lambda)^{-1}$, where $h(\lambda)$ is the product of the hook-lengths of λ. Hence it follows from (6.12) that if w is vexillary of length p, then

$$\text{(6.13)} \qquad \operatorname{Card} R(w) = \frac{p!}{h(\lambda)}$$

where λ is the shape of w. In other words, the number of reduced words for a vexillary permutation of length p and shape $\lambda \vdash p$ is equal to the degree of the irreducible representation of S_p indexed by λ.

4. It seems likely that there is a q-analogue of (6.11). Some experimental evidence suggests the following conjecture:

$$(6.11_q?) \qquad \mathfrak{S}_w(1, q, q^2, \dots) = \sum q^{\phi(\boldsymbol{a})} \frac{(1-q^{a_1}) \cdots (1-q^{a_p})}{(1-q) \cdots (1-q^p)}$$

summed as in (6.11) over all reduced words $\boldsymbol{a} = (a_1, \dots, a_p)$ for w, where

$$\phi(\boldsymbol{a}) = \sum \{i : a_i < a_{i+1}\}.$$

When w is vexillary the double Schubert polynomial $\mathfrak{S}_w(x, y)$ can be expressed as a multi-Schur function, just as in the case of (single) Schubert polynomials (Chap. IV). We consider first the case of a dominant permutation:

(6.14) *If w is dominant of shape λ, then*

$$\mathfrak{S}_w(x, y) = \prod_{(i,j) \in \lambda} (x_i - y_j)$$
$$= s_\lambda(X_1 - Y_{\lambda_1}, \dots, X_m - Y_{\lambda_m})$$

where $m = \ell(\lambda)$ *and* $X_i = x_1 + \cdots + x_i, Y_i = y_1 + \cdots + y_i$ for all $i \geq 1$.

Proof: As in (4.6) we proceed by descending induction on $\ell(w), w \in S_n$. The result is true for $w = w_0$, since w_0 is dominant of shape δ and

$$\mathfrak{S}_{w_0}(x,y) = \Delta(x,y) = \prod_{(i,j)\in\delta} (x_i - y_j).$$

Suppose $w \neq w_0$ is dominant of shape λ. Then $\lambda \subset \delta$ (and $\lambda \neq \delta$). Let $r \geq 0$ be the largest integer such that $\lambda'_i = n - i$ for $1 \leq i \leq r$, and let $a = \lambda'_{r+1} + 1 \leq n - r - 1$. Then ws_a is dominant, $\ell(ws_a) = \ell(w) + 1$, and $\lambda(ws_a) = \lambda(w) + \epsilon_a$, and therefore

$$\begin{aligned}\mathfrak{S}_w(x,y) &= \partial_a \mathfrak{S}_{ws_a}(x,y)\\ &= \partial_a((x_a - y_{r+1}) \prod_{(i,j)\in\lambda} (x_i - y_j))\end{aligned}$$

by the inductive hypothesis; since $\lambda_a = \lambda_{a+1}$ it follows that

$$\mathfrak{S}_w(x,y) = \prod_{(i,j)\in\lambda} (x_i - y_j)$$

which is equal to $s_\lambda(X_1 - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_m})$ by (3.5).‖

(6.15) *If* w *is Grassmannian of shape* λ *then*

$$\mathfrak{S}_w(x,y) = s_\lambda(X_m - Y_{\lambda_1+m-1}, \cdots, X_m - Y_{\lambda_m}).$$

Proof: This follows from (6.14) just as (4.8) follows from (4.7).‖

Finally, let w be vexillary with shape

$$\lambda(w) = (p_1^{m_1}, \ldots, p_k^{m_k})$$

and flag

$$\phi(w) = (f_1^{m_1}, \ldots, f_k^{m_k})$$

as in Chapter IV. Then w^{-1} is also vexillary, with shape

$$\lambda(w^{-1}) = \lambda(w)' = (q_1^{n_1}, \ldots, q_k^{n_k})$$

the conjugate of $\lambda(w)$, and flag

$$\phi(w^{-1}) = (g_1^{n_1}, \ldots, g_k^{n_k})$$

where by (1.41)

$$g_i + q_i = f_{k+1-i} + p_{k+1-i} \qquad (1 \leq i \leq k).$$

With this notation recalled, we have

(6.16) $$\mathfrak{S}_w(x,y) = s_\lambda((X_{f_1} - Y_{g_k})^{m_1}, \ldots, (X_{f_k} - Y_{g_1})^{m_k}).$$

Proof: The proof is essentially the same as that of (4.9) (which is the case $y = 0$). By (4.10) the dominant permutation w_k constructed from w in the proof of (4.9) has shape

$$\mu = (g_k^{m_1}, g_{k-1}^{m_2}, \ldots, g_1^{m_k})$$

and therefore by (6.15) we have

$$\mathfrak{S}_{w_k}(x,y) = s_\mu(X'_1, \ldots, X'_m)$$

where $m = m_1 + \cdots + m_k = \ell(\lambda)$ and the sequence $(X'_1, \ldots, X'_m)$ is obtained by subtracting the sequence $((Y_{g_k})^{m_1}, \ldots, (Y_{g_1})^{m_k})$ term by term from the sequence $(X_1, \ldots, X_m)$. Hence the same argument as in (4.9) establishes (6.17).‖

Remark. From (6.16) and (6.4)(iii) we obtain

$$s_\lambda(Z_1^{m_1}, \ldots, Z_k^{m_k}) = (-1)^{|\lambda|} s_{\lambda'}((-Z_k)^{n_1}, \ldots, (-Z_1)^{n_k})$$

where $Z_i = X_{f_i} - Y_{g_{k+i-1}}$ so that (if $rk\ (x_i) = \ rk\ (y_i) = 1$ for each $i \geq 1$)

$$\begin{aligned} rk\ (Z_{i+1} - Z_i) &= f_{i+1} - f_i + g_{k+1-i} - g_{k-i} \\ &= m_{i+1} - n_{k+1-i} \end{aligned}$$

by (1.41). Hence (6.4)(iii) reduces to the duality theorm (3.8″) (with $\mu = 0$) when w is vexillary.

Let τ_x (resp. τ_y) be the shift operator (4.21) acting on the x (resp. y) variables. Then we have

(6.17) $$\tau_x^r \tau_y^r \mathfrak{S}_w(x,y) = \mathfrak{S}_{1_r \times w}(x,y)$$

for all $r \geq 1$ and all permutations w.

Proof: By (6.3) and (4.21) we have

$$\tau_x^r \tau_y^r \mathfrak{S}_w(x,y) = \sum_{u,v} \epsilon(v) \mathfrak{S}_{1_r \times u}(x) \mathfrak{S}_{1_r \times v}(y)$$

summed over u, v such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$. By (6.3) again, the right-hand side is equal to $\mathfrak{S}_{1_r \times w}(x,y)$. ‖

In particular, suppose that w is vexillary. With the notation of (6.16), the flag of $1_r \times w$ (resp. $1_r \times w^{-1}$) is obtained from that of w (resp. w^{-1}) by replacing each f_i by $f_i + r$ (resp. each g_i by $g_i + r$). Hence by (6.16) we have

$$\mathfrak{S}_{1_r \times w}(x, y) = s_\lambda((X_{f_1+r} - Y_{g_k+r})^{m_1}, \ldots, (X_{f_k+r} - Y_{g_1+r})^{m_k})$$

and hence

$$\rho_r^{(x)} \rho_r^{(y)} \mathfrak{S}_{1_r \times w}(x, y) = s_\lambda(X_r - Y_r) \tag{6.18}$$

for all $r \geq 1$, where $\rho_r^{(x)}$ (resp. $\rho_r^{(y)}$) is the homomorphism ρ_r of (4.25) acting on the x (resp. y) variables.

(6.19) *Let π_x (resp. π_y) denote $\pi_{w_0^{(r)}}$ acting on the x (resp. y) variables. Then if w is vexillary of shape λ, we have*

$$\pi_x \pi_y \mathfrak{S}_w(x, y) = s_\lambda(X_r - Y_r).$$

Proof: By (4.24) we have $\pi_x = \rho_r^{(x)} \tau_x^r$ and $\pi_y = \rho_y^{(r)} \tau_y^r$. Hence (6.19) follows from (6.17) and (6.18).‖

In particular, suppose that w is dominant of shape λ, so that by (6.14)

$$\mathfrak{S}_w(x, y) = \prod_{(i,j) \in \lambda} (x_i - y_j) = f_\lambda(x, y) \text{ say.}$$

In this case (6.19) gives

$$\pi_x \pi_y f_\lambda(x, y) = s_\lambda(X_r - Y_r)$$

for all $r \geq 1$, which is Sergeev's formula (3.12′).

Chapter VII

Schubert Polynomials (2)

Recall the decomposition (4.17) of a Schubert polynomial $\mathfrak{S}_w$:

$$\mathfrak{S}_w(x_1, x_2, \ldots) = \sum_{u,v} d_{uv}^w \mathfrak{S}_u(x_1, \ldots, x_m)\mathfrak{S}_v(x_{m+1}, x_{m+2}, \ldots)$$

Our first aim in this Chapter will be to give a method for calculating the coefficients d_{uv}^w. We shall then apply our results to the calculation of Card $(R(w))$, the number of reduced decompositions $w = s_{a_1} \cdots s_{a_p}$ (where $p = \ell(w)$) of a permutation w.

For this purpose, we introduce the operators $\partial_i^*,\ i \geq 1$, defined by

(7.1)
$$\partial_i^* \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Remarks. 1. If ω is the (linear) involution defined by $\omega(\mathfrak{S}_w) = \mathfrak{S}_{w^{-1}}$ for each permutation w, it follows from (4.2) that $\partial_i^* = \omega\partial_i\omega$. Hence we may define $\partial_w^* = \omega\partial_w\omega$ for any permutation w, and we have $\partial_w^* = \partial_{a_1}^* \cdots \partial_{a_p}^*$ whenever $(a_1, \ldots, a_p)$ is a reduced word for w.

2. If $w \in S_n$ we have $\partial_i^*\mathfrak{S}_w = 0$ for all $i > n$, because $\partial_i^*\mathfrak{S}_w = \omega\partial_i\mathfrak{S}_{w^{-1}}$, which is zero because $w^{-1}(i) < w^{-1}(i+1)$.

(7.2) *∂_i^* commutes with ∂_j for all* $i, j \geq 1$.

Proof: We have

$$\partial_i^*\partial_j\mathfrak{S}_w = \begin{cases} \partial_i^*\mathfrak{S}_{ws_j} = \mathfrak{S}_{s_i w s_j} & \text{if } \ell(s_i w s_j) = \ell(w) - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Likewise

$$\partial_j\partial_i^*\mathfrak{S}_w = \begin{cases} \partial_j\mathfrak{S}_{s_i w} = \mathfrak{S}_{s_i w s_j} & \text{if } \ell(s_i w s_j) = \ell(w) - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\partial_i^*\partial_j - \partial_j\partial_i^*$ vanishes on each Schubert polynomial $\mathfrak{S}_w$, and therefore vanishes identically.‖

(7.3) *Let* $w_0 = w_0^{(n)}$ *be the longest element of* S_n*. Then for* $r = 1, 2, \ldots, n-1$ *we have*

$$(1 + t\partial^*_{n-r}) \cdots (1 + t\partial^*_{n-1})\mathfrak{S}_{w_0} = (1 + t\partial_1) \cdots (1 + t\partial_r)\mathfrak{S}_{w_0}$$

as polynomials in $t, x_1, x_2, \ldots$.

Proof: The coefficient of t^p $(1 \le p \le r)$ on the left-hand side is

$$\sum \partial^*_{a_1} \cdots \partial^*_{a_p} \mathfrak{S}_{w_0} \tag{1}$$

summed over all reduced sequences $(a_1, \ldots, a_p)$ satisfying

$$n - r \le a_1 \le \ldots \le a_p \le n-1.$$

Let $b_i = n - a_{p+1-i}$ for all $1 \le i \le p$, so that

$$1 \le b_1 < \ldots < b_p \le r. \tag{2}$$

Let $w = s_{a_p} \cdots s_{a_1}$, so that $w_0 w w_0 = s_{b_1} \cdots s_{b_p}$. Then

$$\begin{aligned} \partial^*_{a_1} \cdots \partial^*_{a_p} \mathfrak{S}_{w_0} &= \mathfrak{S}_{w^{-1}w_0} = \partial_{w_0 w w_0} \mathfrak{S}_{w_0} \\ &= \partial_{b_1} \cdots \partial_{b_p} \mathfrak{S}_{w_0}. \end{aligned}$$

Hence (1) is equal to

$$\sum \partial_{b_1} \cdots \partial_{b_p} \mathfrak{S}_{w_0}$$

summed over all reduced sequences $(b_1, \ldots, b_p)$ satisfying (2), which is the coefficient of t^p on the right hand side of (7.2).‖

Next, we have

$$\mathfrak{S}_{1 \times w_0}(t, x_1, \ldots, x_{n-1}) = (1 + t\partial_1) \cdots (1 + t\partial_{n-1})\mathfrak{S}_{w_0}(x_1, \ldots, x_{n-1}). \tag{7.4}$$

Proof: By (4.22) we have to show that

$$(1 + t\partial_1) \cdots (1 + t\partial_{n-1}) s_\delta(X_1, \ldots, X_{n-1}) = s_\delta(t + X_1, \ldots, t + X_{n-1})$$

where $X_i = x_1 + \cdots + x_i$ for each $i \ge 1$, and $\delta = \delta_n$. For this it is enough to show that

$$(1 + t\partial_i) s_\delta(X_1, \ldots, X_i, t + X_{i+1}, \ldots, t + X_{n-1}) = s_\delta(X_1, \ldots, X_{i-1}, t + X_i, \ldots, t + X_{n-1}) \tag{1}$$

for $i = 1, 2, \ldots, n-1$.

Both sides of (1) are determinants with $n-1$ rows and columns which agree in all rows except the i^{th} row. On the left-hand side, the elements of the i^{th} row are by (3.10)

$$h_k(X_i) + th_{k-1}(X_{i+1})$$

and on the right-hand side they are $h_k(t+X_i)$, where k runs form $n-2i+1$ to $2n-2i-1$ in each case.

Now we have

$$\begin{aligned} h_k(X_i) + th_{k-1}(X_{i+1}) &= h_k(t+X_i) - th_{k-1}(t+X_i) + th_{k-1}(t+X_{i+1}) - t^2h_{k-2}(t+X_{i+1}) \\ &= h_k(t+X_i) - t(t-x_{i+1})h_{k-2}(t+X_{i+1}) \end{aligned}$$

Hence if we add $t(t-x_{i+1})$ times the $(i+1)^{\text{th}}$ row to the i^{th} row in the determinant on the left-hand side, we shall obtain the right-hand side of (1).‖

For each $r \geq 1$, let

$$\Phi_r(t) = t^r(1+t\partial^*_{r+1})(1+t\partial^*_{r+2})\cdots$$

For each permutation w, we have $(1+t\partial^*_j)\mathfrak{S}_w = \mathfrak{S}_w$ for all sufficiently large j by (7.1), so that $\Phi_r(t)\mathfrak{S}_w$ is a polynomial in t (and $x_1, x_2, \ldots$). With this notation, we have

(7.5) $$\partial_1\partial_2\cdots\partial_{n-r+1}(x_1^n x_2^{n-1}\cdots x_n) = \Phi_{r-1}(x_1)\mathfrak{S}_{w_0^{(n)}}(x_2, x_3, \ldots)$$

Proof: Let $s = n-r+1$ and

$$a = x_2^{s-1}x_3^{s-2}\cdots x_s, \quad b = x_{s+2}^{r-2}x_{s+3}^{r-3}\cdots x_n, \quad c = (x_2\cdots x_{s+1})^{r-1}$$

so that $abc = x_2^{n-1}x_3^{n-2}\cdots x_n$. Hence

$$\begin{aligned} \partial_1\partial_2\cdots\partial_n(x_1^n x_2^{n-1}\cdots x_n) &= x_1^{r-1}bc\partial_1\cdots\partial_s(x_1^s x_2^{s-1}\cdots x_s) \\ &= x_1^{r-1}bc\mathfrak{S}_{1\times w_0^{(s)}}(x_1,\ldots,x_s) && \text{by (4.21)} \\ &= x_1^{r-1}bc(1+x_1\partial_2)\cdots(1+x_1\partial_s)a && \text{by (7.4)} \\ &= x_1^{r-1}(1+x_1\partial_2)\cdots(1+x_1\partial_s)abc \\ &= x_1^{r-1}(1+x_1\partial_2)\cdots(1+x_1\partial_s)\mathfrak{S}_{w_0^{(n)}}(x_1,\ldots,x_n) \\ &= x_1^{r-1}(1+x_1\partial^*_r)\cdots(1+x_1\partial^*_{n-1})\mathfrak{S}_{w_0^{(n)}}(x_2,\ldots,x_n) && \text{by (7.3).‖} \end{aligned}$$

Let w be any permutation. If $w(1) = r$, then $s_1\cdots s_{r-1}w(1) = 1$, so that we may write

$$s_1\cdots s_{r-1}w = 1\times w_1$$

where w_1 is defined by

$$w_1(i) = \begin{cases} w(i+1) & \text{if } w(i+1) < r, \\ w(i+1)-1 & \text{if } w(i+1) > r. \end{cases}$$

If the code of w is $(c_1, c_2, \ldots)$ (so that $c_1 = r-1$), the code of w_1 is $(c_2, c_3, \ldots)$. With this notation we have

$$\mathfrak{S}_w(x_1, x_2, \ldots) = \Phi_{r-1}(x_1)\mathfrak{S}_{w_1}(x_2, x_3, \ldots) \tag{7.6}$$

Proof: Suppose that $w \in S_{n+1}$. Then

$$\begin{aligned} w_0^{(n+1)} w &= w_0^{(n+1)} s_{r-1} \cdots s_1 (1 \times w_1) \\ &= s_{n-r+2} \cdots s_n w_0^{(n+1)} (1 \times w_0^{(n)})(1 \times w_0^{(n)} w_1) \\ &= s_{n-r+1} \cdots s_1 (1 \times w_0^{(n)} w_1) \end{aligned}$$

since $w_0^{(n+1)}(1 \times w_0^{(n)}) = s_n s_{n-1} \cdots s_1$. Hence

$$\begin{aligned} \mathfrak{S}_w(x_1, \ldots, x_n) &= \partial_{w^{-1} w_0^{(n+1)}} (x_1^n x_2^{n-1} \cdots x_n) \\ &= \partial_{1 \times w_1^{-1} w_0^{(n)}} \partial_1 \cdots \partial_{n-r+1} (x_1^n \cdots x_n) \\ &= \partial_{1 \times w_1^{-1} w_0^{(n)}} \Phi_{r-1}(x_1) \mathfrak{S}_{w_0^{(n)}}(x_2, x_3, \ldots, x_n) && \text{by (7.5)} \\ &= \Phi_{r-1}(x_1) \partial_{1 \times w_1^{-1} w_0^{(n)}} \mathfrak{S}_{w_0^{(n)}}(x_2, x_3, \ldots, x_n) && \text{by (7.2)} \\ &= \Phi_{r-1}(x_1) \mathfrak{S}_{w_1}(x_2, x_3, \ldots). \| \end{aligned}$$

Remark. The right-hand side of (7.6) is a sum of terms of the form $x_1^p \mathfrak{S}_u(x_2, x_3, \ldots)$. By applying (7.6) to each $\mathfrak{S}_u$, and so on, we can decompose $\mathfrak{S}_w$ into a sum of monomials, and thus we have another proof of the fact (4.17) that $\mathfrak{S}_w$ is a polynomial in $x_1, x_2, \ldots$ with positive integer coefficients.

Next, let $m \geq 1$ and assume that the permutation w statisfies

$$w(1) > w(2) > \cdots > w(m).$$

Define a partition $\mu = \mu(w, m)$ of length $\leq m$ by

$$\mu_i = w(i) - (m+1-i) \qquad (1 \leq i \leq m).$$

If $w \in S_{m+n}$ we have $\mu_1 \leq n$, hence $\mu \subset (n^m)$.

Also let

$$\Phi_\mu(x_1, \ldots, x_m) = \Phi_{\mu_m}(x_m) \cdots \Phi_{\mu_2}(x_2) \Phi_{\mu_1}(x_1)$$

and let w_m be the permutation whose code is $(c_{m+1}, c_{m+2}, \ldots)$, where $(c_1, c_2, \ldots)$ is the code of w.

With this notation established, we have

$$\mathfrak{S}_w(x) = x^{\delta_m}\Phi_\mu(x_1, \ldots, x_m)\mathfrak{S}_{w_m}(x_{m+1}, x_{m+2}, \ldots). \tag{7.7}$$

Proof: We proceed by induction on m; the case $m = 1$ is (7.6). From (7.6) we have

$$\begin{aligned}\mathfrak{S}_w(x) &= \Phi_{\mu_1+m-1}(x_1)\mathfrak{S}_{w_1}(x_2, x_3, \ldots)\\ &= \sum_u x_1^{\mu_1+m+p-1}\mathfrak{S}_{uw_1}(x_2, x_3, \ldots)\end{aligned}$$

summed over all $u = s_{a_1} \cdots s_{a_p}$, where

$$c_1(w) + 1 = \mu_1 + m \le a_1 < \cdots < a_p$$

and $\ell(uw_1) = \ell(w_1) - p$. The code of uw_1 satisfies $c_i(uw_1) = c_i(w_1)$ for $1 \le i \le m-1$, and hence

$$(uw_1)_{m-1} = s_{a_1-m+1} \cdots s_{a_p-m+1}w_m.$$

It follows that

$$\sum_u x_1^{\mu_1+m+p-1}\mathfrak{S}_{(uw_1)_{m-1}}(x_{m+1}, x_{m+2}, \ldots) = x_1^{m-1}\Phi_{\mu_1}(x_1)\mathfrak{S}_{w_m}(x_{m+1}, x_{m+2}, \ldots)$$

and therefore, by the inductive hypothesis,

$$\begin{aligned}\mathfrak{S}_w(x) &= \sum_u x_1^{\mu_1+m+p-1}x_2^{m-2} \cdots x_{m-1}\Phi_{\mu_m}(x_m) \cdots \Phi_{\mu_2}(x_2)\mathfrak{S}_{(uw_1)_{m-1}}(x_{m+1}, x_{m+2}, \ldots)\\ &= x_1^{m-1}x_2^{m-2} \cdots x_{m-1}\Phi_{\mu_m}(x_m) \cdots \Phi_{\mu_1}(x_1)\mathfrak{S}_{w_m}(x_{m+1}, x_{m+2}, \ldots).\|\end{aligned}$$

Finally, for any permutation w, let v be the unique element of S_m such that $wv(1) > \cdots > wv(m)$, and let $\mu = \mu(wv, m)$. We have $\ell(wv) = \ell(w) + \ell(v)$ and $(wv)_m = w_m$, so that by (7.7)

$$\mathfrak{S}_{wv}(x) = x^{\delta_m}\Phi_\mu(x_1, \ldots, x_m)\mathfrak{S}_{w_m}(x_{m+1}, x_{m+2}, \cdots).$$

Hence

$$\begin{aligned}\mathfrak{S}_w(x) &= \partial_v\mathfrak{S}_{wv}(x)\\ &= \partial_v(x^{\delta_m}\Phi_\mu(x_1, \ldots, x_m))\mathfrak{S}_{w_m}(x_{m+1}, x_{m+2}, \ldots).\end{aligned} \tag{7.8}$$

Now by (4.14), for any polynomial $f \in P_m$, we have

$$f = \sum_{u \in S^{(m)}} \eta(\partial_u f)S_u$$

where $S^{(m)}$ consists of the permutations whose codes have length $\leq m$, and $\eta(\partial_u f)$ is the constant term of the polynomial $\partial_u f$. Applying this to (7.8), we obtain our final result:

(7.9) $$\mathfrak{S}_w(x) = \sum_u \mathfrak{S}_u(x_1,\ldots,x_m)\eta(\partial_{uv}(x^{\delta_m}\Phi_\mu(x_1,\ldots,x_m)))\mathfrak{S}_{w_m}(x_{m+1},x_{m+2},\ldots)$$

summed over all $u \in S^{(m)}$ *such that* $\ell(uv) = \ell(u)+\ell(v)$.‖

For each such u, the constant term $\eta(\partial_{uv}(x^{\delta_m}\Phi(x_1,\ldots,x_m)))$ is a polynomial in the (non-commuting) operators ∂_i^* with integer coefficients. Hence (7.9) gives a decomposition of the Schubert polynomial $\mathfrak{S}_w(x)$ of the form

(7.10) $$\mathfrak{S}_w(x) = \sum_{u,v} d_{uv}^w \mathfrak{S}_u(y)\mathfrak{S}_v(z),$$

where $y = (x_1,\ldots,x_m)$ and $z = (x_{m+1},x_{m+2},\ldots)$. If $w \in S^{(m+n)}$, so that $\mathfrak{S}_w(x) \in P_{m+n}$, then $u \in S^{(m)}$ and $v \in S^{(n)}$ in this sum. From (4.18) we know that the coefficients d_{uv}^w in (7.10) are ≥ 0.

In particular, if we apply (7.7) to a permutation of the form $w_0^{(m)} \times w$, we shall obtain

(1) $$\mathfrak{S}_{w_0^{(m)}\times w}(x) = x^{\delta_m}\Phi_0(x_1,\ldots,x_m)\mathfrak{S}_w(x_{m+1},x_{m+2},\ldots).$$

On the other hand, by (4.6) we have

(2) $$\mathfrak{S}_{w_0^{(m)}\times w} = \mathfrak{S}_{w_0^{(m)}}\mathfrak{S}_{1_m\times w}$$

and comparison of (1) and (2) gives

(7.12) $$\mathfrak{S}_{1_m\times w}(x) = \Phi_0(x_1,\ldots,x_m)\mathfrak{S}_w(x_{m+1},x_{m+2},\ldots).$$

By (4.3), $\mathfrak{S}_{1_m\times w}$ is symmetrical in $x_1,\ldots,x_m$. Hence so is the operator $\Phi_0(x_1,\ldots,x_m)$, and we may therefore write Φ_0 in the form

(7.13) $$\Phi_0(x_1,\ldots,x_m) = \sum_{\lambda,v} \alpha_m(\lambda,v)s_\lambda(x_1,\ldots,x_m)\partial_v^*$$

summed over partitions λ of length $\leq m$ and permutations v, with integral coefficients $\alpha_m(\lambda,v)$. From (7.12) and (7.13) we have

(7.14) $$\mathfrak{S}_{1_m\times w} = \sum_{\lambda,v} \alpha_m(\lambda,v)s_\lambda(x_1,\ldots,x_m)\mathfrak{S}_{vw}(x_{m+1},x_{m+2},\ldots)$$

summed over λ of length $\leq m$ and v such that $\ell(vw) = \ell(w)-\ell(v)$. The Schur functions occuring here are precisely the Schubert polynomials $\mathfrak{S}_u$, where u is Grassmannian with descent at m. Hence, by (4.18), (7.15) *The coefficients* $\alpha_m(\lambda, v)$ *in* (7.13) *are* ≥ 0.‖

Since $\Phi_0(x_1, \ldots, x_m, 0) = \Phi_0(x_1, \ldots, x_m)$ and $s_\lambda(x_1, \ldots, x_m, 0) = s_\lambda(x_1, \ldots, x_m)$ if $\ell(\lambda) \leq m$, it follows from (7.13) that

$$\alpha_{m+1}(\lambda, v) = \alpha_m(\lambda, v) = \alpha(\lambda, v) \text{ say} \tag{7.16}$$

for all partitions λ such that $\ell(\lambda) \leq m$.

We may also calculate the operator $\Phi_0(x_1, \ldots, x_m)$ as follows. For each integer $p \geq 1$ and each subset D of $\{1, 2, \ldots, p-1\}$ let

$$Q_{D,p}(x_1, \ldots, x_m) = \sum x_{u_1} \cdots x_{u_p}$$

summed over all sequences $(u_1, \ldots, u_p)$ such that $1 \leq u_1 \leq \cdots \leq u_p \leq m$ and $u_i < u_{i+1}$ whenever $i \in D$. Then $Q_{D,p}(x_1, \ldots, x_m)$ is a homogeneous polynomial of degree p, and is zero if $m \leq \mathrm{Card}(D)$.

Now let $\boldsymbol{a} = (a_1, \ldots, a_p)$ be a reduced word, so that $\ell(s_{a_1} \cdots s_{a_p}) = p$. The *descent set* of $\boldsymbol{a}$ is

$$D(\boldsymbol{a}) = \{i : a_i > a_{i+1}\}.$$

We now define, for each permutation w,

$$F_w(x_1, \ldots, x_m) = \sum_{\boldsymbol{a} \in R(w)} Q_{D(\boldsymbol{a}),\ell(w)}(x_1, \ldots, x_m),$$

a homogeneous polynomial of degree $\ell(w)$.

With these definitions we have

$$\Phi_0(x_1, \ldots, x_m) = \sum_w F_w(x_1, \ldots, x_m)\partial_w^*. \tag{7.17}$$

Proof: Let $\boldsymbol{a} = (a_1, \ldots, a_p)$ be a reduced word. Since

$$\Phi_0(x_i) = (1 + x_i\partial_1^*)(1 + x_i\partial_2^*) \cdots$$

it is clear from the definitions that the coefficient of $\partial_a^* = \partial_{a_1}^* \cdots \partial_{a_p}^*$ in $\Phi_0(x_1, \ldots, x_m) = \prod_{i=1}^m \Phi_0(x_i)$ is just $Q_{D(\boldsymbol{a}),p}(x_1, \ldots, x_m)$. Hence

$$\begin{aligned}\Phi_0(x_1, \ldots, x_m) &= \sum_a Q_{D(\boldsymbol{a}),p}(x_1, \ldots, x_m)\partial_{\boldsymbol{a}}^* \\ &= \sum_w F_w(x_1, \ldots, x_m)\partial_w^*.\|\end{aligned}$$

Comparison of (7.17) and (7.13) now shows that $F_w(x_1,\ldots,x_m)$ is a symmetric polynomial in $x_1,\ldots,x_m$, and that

$$F_w(x_1,\ldots,x_m) = \sum_\lambda \alpha_m(\lambda,w)s_\lambda(x_1,\ldots,x_m) = \rho_m(\mathfrak{S}_{1_m\times w^{-1}}). \tag{7.18}$$

The sum in (7.18) is over partitions λ such that $\ell(\lambda)\le m$ and $|\lambda| = \ell(w)$. By (7.16) we have

$$F_w(x_1,\ldots,x_m,0) = F_w(x_1,\ldots,x_m)$$

and therefore we have a well defined symmetric function $F_w\in\Lambda$, such that $\rho_m(F_w) = F_w(x_1,\ldots,x_m)$ for all $m\ge 0$: namely

$$F_w = \sum_\lambda \alpha(\lambda,w)s_\lambda \tag{7.19}$$

where the sum is over partitions λ of $\ell(w)$, and $\alpha(\lambda,w) = \alpha_m(\lambda,w)$ for any $m\ge\ell(\lambda)$.

Since the coefficient of $x_1\cdots x_p$ in $Q_{D,p}(x_1,\ldots,x_m)$ is 1 if $m\ge p$, it follows that the coefficient of $x_1\cdots x_p$ (where $p=\ell(w)$) in $F_w(x_1,\ldots,x_m)$ is equal to Card$(R(w))$ whenever $m\ge\ell(w)$. On the other hand, the coefficient of $x_1\cdots x_p$ in a Schur function s_λ, where $|\lambda|=p$, is equal to f^λ, the number of standard tableaux of shape λ, or equivalently the degree of the irreducible representation χ^λ of S_p indexed by the partition λ ([M], Ch.I, §7). It follows therefore from (7.19) that

$$\text{Card } R(w) = \sum_{|\lambda|=\ell(w)} \alpha(\lambda,w)f^\lambda.\| \tag{7.20}$$

Remark. Since the coefficients $\alpha(\lambda,w)$ are ≥ 0 by (7.15), the number of reduced words for w is always equal to the degree of an (in general reducible) representation of the symmetric group $S_{\ell(w)}$. It is therefore natural to ask whether there is a "natural" action of this symmetric group on the **Z**-span (or perhaps **Q**-span) of the set $R(w)$, with character $\sum\limits_\lambda \alpha(\lambda,w)\chi^\lambda$.

We shall conclude with some properties of the symmetric functions F_w and the coefficients $\alpha(\lambda,w)$.

(7.21) *Let* $u\in S_m, v\in S_n$. *Then*

$$F_{u\times v}(x) = F_u(x)F_v(x).$$

Proof: By (7.18), we have for any N,

$$\begin{aligned} F_{u\times v}(x_1,\ldots,x_N) &= \rho_N(S_{1_N\times u^{-1}\times v^{-1}}) \\ &= \rho_N(S_{1_N\times u^{-1}}S_{1_{m+N}\times v^{-1}}) && \text{by (4.6)} \\ &= \rho_N(S_{1_N\times u^{-1}})\rho_N(\rho_{m+N}(S_{1_{m+N}\times v^{-1}})) \\ &= F_u(x_1,\ldots,x_N)F_v(x_1,\ldots,x_N).\| \end{aligned}$$

(7.22) *Let* $w \in S_n$ *and let* $\overline{w} = w_0 w w_0$, *where* w_0 *is the longest element of* S_n. *Then*

$$F_{w^{-1}} = F_{\overline{w}} = \omega F_w$$

where ω *is the involution that interchanges* s_λ and $s_{\lambda'}$. *In other words*

$$\alpha(\lambda, w^{-1}) = \alpha(\lambda, \overline{w}) = \alpha(\lambda', w)$$

for all partitions λ.

For the proof of (7.22) we require a lemma. If t is a standard tableau of shape λ, the *descent set* $D(t)$ of t is the set of i such that $i+1$ lies in a lower row than i in the tableau t. We have

(7.23) $$s_\lambda = \sum_t Q_{D(t),p}$$

where the sum is over the standard tableaux of shape λ, *and* $p = |\lambda|$.

Proof: In the notation of [M, Ch. I, §5], s_λ is the sum of monomials x^T where T runs through the (column-strict) tableaux of shape λ. Each such tableau T determines a standard tableau t, as follows. If a square in the j^{th} column of the diagram of λ is occupied by the number i, replace i by the pair (i,j). Since T is column-strict the pairs (i,j) so obtained are all distinct. If we now order them lexiographically, (so that (i,j) precedes $\lambda(i',j')$ if and only if either $i < i'$, or $i = i'$ and $j < j'$) and relabel them as $1, 2, \ldots, p$, we have a standard tableau t : say $T \to t$. It follows easily that $\sum_{T \to t} x^T = Q_{D(t),p}$, which proves the lemma.‖

If D is any subset of $\{1, 2, \ldots, p-1\}$, let $\overline{D}$ denote the complementary subset, and let $D^* = \{p - i : i \in D\}$. From the definition of $Q_{D,p}$ we have

(1) $$Q_{D,p}(x_m, x_{m-1}, \ldots, x_1) = Q_{D^*,p}(x_1, \ldots, x_m).$$

If $\boldsymbol{a} = (a_1, \ldots, a_p) \in R(w)$, let $\overline{\boldsymbol{a}} = (n - a_1, \ldots, n - a_p)$ and $\boldsymbol{a}^* = (n - a_p, \ldots, n - a_1)$. Then we have

(2) $$\overline{\boldsymbol{a}} \in R(\overline{w}), \qquad \boldsymbol{a}^* \in R(w^*),$$

where $w^* = (\overline{w})^{-1} = w_0 w^{-1} w_0$. Also

(3) $$D(\overline{\boldsymbol{a}}) = \overline{D(\boldsymbol{a})}, \qquad D(\boldsymbol{a}^*) = D(\boldsymbol{a})^*.$$

Moreover, it t is a standard tableau we have

(4) $$D(t') = \overline{D(t)}$$

where t' is the transpose of t, obtained by reflecting t in the main diagonal. For $i \in D(t)$ if and only if $i+1$ does *not* lie in a later column than i in the tableau t, that is to say if and only if $i \notin D(t')$.

Since F_w is symmetric, it follows from (1),(2), and (3) that

$$F_w(x_1, \ldots, x_m) = F_w(x_m, \ldots, x_1) = F_{w^*}(x_1, \ldots, x_m)$$

and hence by (7.16) that $F_w = F_{w^*}$.

From (7.23) and (4) above we have

$$\omega s_\lambda = s_{\lambda'} = \sum_{t \in St(\lambda)} Q_{\overline{D(t)},p}$$

for all partitions λ of p, where $St(\lambda)$ is the set of standard tableaux of shape λ, and hence it follows from (2) and (3) and the definition of F_w that $\omega F_w = F_{\overline{w}}$. Hence

$$\omega F_{w^{-1}} = F_{w^*} = F_w,$$

which completes the proof of (7.22).‖

(7.24) (i) $\alpha(\mu, w) = 0$ *unless* $\lambda(w^{-1}) \leq \mu \leq \lambda(w)'$.

(ii) $\alpha(\mu, w) = 1$ *if* $\mu = \lambda(w^{-1})$ *or* $\mu = \lambda(w)'$.

(iii) *w is vexillary if and only if F_w is a Schur function.*

Proof: (i) Suppose $\alpha(\mu, w) \neq 0$. Then the monomial x^μ occurs in F_w, and hence there is a reduced word $(a_1, \ldots, a_p)$ for w such that

$$a_1 < \cdots < a_{\mu_1}, \quad a_{\mu_1+1} < \cdots < a_{\mu_1+\mu_2}, \cdots. \tag{1}$$

By (1.14) the code of w is

$$c(w) = \sum_{i=1}^{p} s_{a_p} \cdots s_{a_{i+1}}(\epsilon_{a_i}). \tag{2}$$

If $w^{(1)} = s_{a_p} \cdots s_{a_{\mu_1}+1}$, the sum of the first μ_1 terms of this series is

$$w^{(1)}(\epsilon_{a_{\mu_1}} + s_{a_{\mu_1}}(\epsilon_{a_{\mu_1-1}}) + \cdots + s_{a_{\mu_1}} \cdots s_{a_2}(\epsilon_{a_1})),$$

and since $a_1 < \ldots < a_{\mu_1}$ this is equal to

$$w^{(1)}(\epsilon_{a_{\mu_1}} + \epsilon_{a_{\mu_1-1}} + \cdots + \epsilon_{a_1}) = V_1 \text{ say,} \tag{3}$$

where V_1 is a (0,1) vector (i.e., a vector with each component 0 or 1) of weight μ_1. Likewise the sum of the next block of μ_2 terms of the series (2) is a (0,1) vector V_2 of weight μ_2, and so on. Hence

$$c(w) = V_1 + \cdots + V_m$$

where $m = \ell(\lambda)$, and each V_i is a (0,1) vector of weight μ_i. Let V be the (0,1) matrix whose i^{th} row is V_i, for $i = 1, 2, \ldots, m$. Then V has row sums $\mu_1, \ldots, \mu_m$ and column sums $c_1(w), c_2(w), \ldots$ As in the proof of (1.26) it follows that $\mu \le \lambda(w)'$. Since $\alpha(\mu, w) = \alpha(\mu', w^{-1})$ by (7.23), the same argument applied to μ' and w^{-1} gives $\mu' \le \lambda(w^{-1})'$ i.e., $\lambda(w^{-1}) \le \mu$.

(ii) Suppose now that $\mu = \lambda(w)'$. Then there is only one (0,1) matrix V with row sums μ_i and column sums c_i. Its first row V_1 is $\sum \epsilon_j$ summed over j such that $c_j \neq 0$, i.e. such that there exists $k > j$ with $w(k) < w(j)$. From (3) it follows that

$$wV_1 = \sum_{i=1}^{\mu_1} \epsilon_{a_i+1}$$

and therefore $a_1 + 1, \ldots, a_{\mu_1} + 1$ are the terms of the sequence w that have a smaller element somewhere to the right, in increasing order of magnitude. Hence a_1 has no smaller elements to the right of it, and therefore lies to the right of $a_1 + 1$, so that $\ell(s_{a_1} w) = \ell(w) - 1$. The same argument shows that $\ell(s_{a_2} s_{a_1} w) = \ell(s_{a_1} w) - 1$ and so on. Hence if $w_1 = s_{a_{\mu_1}} \cdots s_{a_1} w$ we have $\ell(w_1) = \ell(w) - \mu_1$, and $\lambda(w_1') = (\mu_2, \mu_3, \ldots)$. It follows by induction on $\ell(\mu)$ that the word $(a_1, \ldots, a_p)$ determined by the matrix V is reduced, and hence $\alpha(\mu, w) = 1$ when $\mu = \lambda(w)'$. By (7.23) it follows that $\alpha(\mu, w) = 1$ when $\mu = \lambda(w^{-1})$.

(iii) This follows immediately from (i) and (ii), and the characterization (1.27) of vexillary permutations.$\|$

Appendix

Schubert varieties

Let V be a vector space of dimension n over a field K, and let $(e_1, \dots, e_n)$ be a basis of V, fixed once and for all. A *flag* in V is a sequence $\boldsymbol{U} = (U_i)_{0 \leq i \leq n}$ of subspaces of V such that

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = V$$

with strict inclusions at each stage, so that $\dim U_i = i$ for each i. In particular, if V_i is the subspace of V spanned by $e_1, \dots, e_i$, then $\boldsymbol{V} = (V_i)_{0 \leq i \leq n}$ is a flag in V, called the *standard* flag.

The set $F = F(V)$ of flags in V is called the *flag manifold* of V.

Let G be the group of all automorphisims of the vector space V. Since we have fixed a basis of V, we may identify G with the general linear group $GL_n(k)$: if $g \in G$ and

$$ge_j = \sum_{i=1}^{n} g_{ij} e_i \qquad (1 \leq j \leq n)$$

then g is identified with the matrix (g_{ij}).

The group G acts on F: if $\boldsymbol{U} = (U_i)$ and $g \in G$, then $g\boldsymbol{U}$ is the flag (gU_i). Let B be the subgroup of G that fixes the standard flag $\boldsymbol{V}$. Then $g \in B$ if and only if ge_j is a linear combination of $e_1, \dots, e_j$, for $1 \leq j \leq n$, that is to say if and only if $g_{ij} = 0$ whenever $i > j$, so that B is the group of upper triangular matrices in $GL_n(k)$.

A *basis* of a flag $\boldsymbol{U} = (U_i)$ is a sequence $(u_1, \dots, u_n)$ in V such that $u_i \in U_i - U_{i-1}$ for $1 \leq i \leq n$, or equivalently such that $u_1, \dots, u_i$ is a basis of U_i for each i. Given such a basis of $\boldsymbol{U}$, there is a unique $g \in G$ such that $ge_i = u_i$ for each i, and we have $\boldsymbol{U} = g\boldsymbol{V}$. Hence G acts transitively on the flag manifold F, and the mapping $g\boldsymbol{V} \mapsto gB$ is a bijection of F onto the coset space G/B.

For a flag $\boldsymbol{U} = (U_i)$, let

$$E_i = E_i(\boldsymbol{U}) = \{j : 1 \leq j \leq n \text{ and } U_i \cap V_j \neq U_i \cap V_{j-1}\}$$

for $0 \le i \le n$. Then $(E_0, \ldots, E_n)$ is a 'flag of sets', i.e. we have

(A.1) (i) $\mathrm{Card}(E_i) = i$ *for* $0 \le i \le n$,

(ii)$E_{i-1} \subset E_i$ *for* $1 \le i \le n$.

Proof: (i) Fix i and let $d_j = \dim\,(U_i \cap V_j)$. Since

$$\frac{U_i \cap V_j}{U_i \cap V_{j-1}} = \frac{U_i \cap V_j}{(U_i \cap V_j) \cap V_{j-1}} \cong \frac{(U_i \cap V_j) + V_{j-1}}{V_{j-1}} \subset \frac{V_j}{V_{j-1}}$$

it follows that $d_j - d_{j-1} = 0$ or 1. Since $d_0 = 0$ and $d_n = i$, there are therefore i jumps in the sequence $(d_0, d_1, \ldots, d_n)$, which proves (i).

(ii) Suppose that $j \notin E_i$, so that $U_i \cap V_j = U_i \cap V_{j-1}$. Intersecting with U_{i-1}, we see that $j \notin E_{i-1}$. Hence $E_{i-1} \subset E_i$.‖

From (A.1) it follows that that each $\boldsymbol{U} \in F$ determines a permutation $w \in S_n$ as follows : $w(i)$ is the unique element of $E_i - E_{i-1}$, for $i = 1, 2, \ldots, n$. Let $\phi : F \to S_n$ denote the mapping so defined.

The symmetric group acts on V by permuting the basis elements e_i:

$$w(e_i) = e_{w(i)}$$

for $w \in S_n$ and $1 \le i \le n$. Hence we may regard S_n as a subgroup of G.

(A.2) *Let* $\boldsymbol{U} \in F, w \in S_n$. *Then* $\phi(\boldsymbol{U}) = w$ *if and only if* $\boldsymbol{U} = bw\boldsymbol{V}$ *for some* $b \in B$.

Proof: Suppose $\phi(\boldsymbol{U}) = w$. Then for $i = 1, \ldots, n$ we have

$$U_i \cap V_{w(i)} \supset U_i \cap V_{w(i)-1} \tag{1}$$

and

$$U_{i-1} \cap V_{w(i)} = U_{i-1} \cap V_{w(i)-1} \tag{2}$$

By virtue of (1) we can choose $u_i \in U_i$ of the form

$$u_i = e_{w(i)} + \text{lower terms} \tag{3}$$

where by 'lower terms' is meant a linear combination of $e_1, \ldots, e_{w(i)-1}$; and $u_i \notin U_{i-1}$ by virtue of (2).

By rewriting (3) in the form

$$u_{w^{-1}(j)} = e_j + \text{lower terms} \qquad (1 \le j \le n)$$

we see that there exists $b \in B$ such that $u_{w^{-1}(j)} = be_j$ for all j, or equivalently

$$u_i = be_{w(i)} = bwe_i.$$

Hence $\boldsymbol{U} = bw\boldsymbol{V}$ as required.

For the converse it is enough to show that (i) $\phi(w\boldsymbol{V}) = w$ and (ii) $\phi(b\boldsymbol{U}) = \phi(\boldsymbol{U})$ for all $b \in B$ and $\boldsymbol{U} \in F$. As to (i), $wV_i \cap V_j$ is spanned by the basis vectors $e_{w(k)}$ such that $k \leq i$ and $w(k) \leq j$, and therefore $wV_i \cap V_j \neq wV_i \cap V_{j-1}$ if and only if $j = w(k)$ for some $k \leq i$. Thus the set $E_i(w\boldsymbol{V})$ consists of $w(1), \dots, w(i)$, which establishes (i). Finally as to (ii), we have $bU_i \cap V_j = b(U_i \cap V_j)$ if $b \in B$, so that $E_i(b\boldsymbol{U}) = E_i(\boldsymbol{U})$ and hence $\phi(b\boldsymbol{V}) = \phi(\boldsymbol{U})$ as required.‖

From (A2) we have immediately

(A3) (Bruhat decomposition) *G is the disjoint union of the double cosets BwB, $w \in S_n$.*‖

For each $w \in S_n$, let

$$C_w = (BwB)/B \subset G/B = F.$$

The subsets C_w are the *Schubert cells* in the flag manifold F. By (A.3), F is the disjoint union of the C_w.

Let $\boldsymbol{U} \in F$. Then $\boldsymbol{U} \in C_w$ if and only if $\boldsymbol{U}$ has a basis $(u_1, \dots, u_n)$ such that $u_i \in V_{w(i)} - V_{w(i)-1}$ for each i. We may normalize the u_i by taking

$$u_i = e_{w(i)} + \text{lower terms}.$$

We can then subtract from u_i suitable multiples of the u_k for which $k < i$ and $w(k) < w(i)$, so as to make the coefficient of $e_{w(k)}$ in u_i zero for each such k. Then u_i is replaced by a vector of the form

$$e_{w(i)} + \sum_j a_{ij} e_j$$

where the sum is over $j < w(i)$ such that $j \neq w(k)$ for any $k < i$, i.e., such that $j < w(i)$ and $w^{-1}(j) > i$, or equivalently $(i, j) \in D(w)$, the diagram of w.

(A.4) *Let $\boldsymbol{U} \in F$. Then $\boldsymbol{U} \in C_w$ if and only if $\boldsymbol{U}$ has a basis $(u_1, \dots, u_n)$ of the form*

$$u_i = e_{w(i)} + \sum_j a_{ij} e_j$$

where the sum is over all j in the i^{th} row of the diagram of w, and the coefficients a_{ij} are arbitrary elements of the field K. Moreover, the a_{ij} are uniquely determined by the flag $\boldsymbol{U}$, and the mapping $C_w \to K^{D(w)}$ so defined is a bijection.

Proof: Clearly each "matrix" $a = (a_{ij})$ of shape $D(w)$ determines a basis $(u_1, \ldots, u_n)$ of V as above, and hence a flag $\boldsymbol{U} \in C_w$. If $a^* = (a^*_{ij})$ determines $(u^*_1, \ldots, u^*_n)$ and the same flag $\boldsymbol{U}$, then each u^*_i must be expressible as

$$u^*_i = u_i + \sum_{j<i} c_{ij} u_j,$$

and from the form of u^*_i and the u_j it follows that $u^*_i = u_i$ for each i, and hence $a^* = a$.‖

Since Card $D(w) = \ell(w)$ it follows from (A.4) that the Schubert cell C_w is isomorphic to affine space of dimension $\ell(w)$.

Let $\boldsymbol{U} \in F$ and let $(u_1, \ldots, u_n)$ be any basis of $\boldsymbol{U}$. Since $u_1, \ldots, u_i$ is a basis for U_i for each $i = 1, \ldots, n-1$, the flag $\boldsymbol{U}$ determines each of the exterior products $u_1 \wedge \cdots \wedge u_i \in \Lambda^i(V)$ up to a nonzero scalar multiple, and hence $\boldsymbol{U}$ determines the vector

$$u_1 \otimes (u_1 \wedge u_2) \otimes \cdots \otimes (u_1 \wedge \cdots \wedge u_{n-1}) \in E \tag{1}$$

up to a nonzero scalar multiple, where $E = V \otimes \Lambda^2 V \otimes \cdots \otimes \Lambda^{n-1} V$. If $P(E)$ denotes the projective space of E (i.e. the space whose points are the lines in E), we have an injective mapping

$$\pi : F \mapsto P(E)$$

(the *Plücker embedding*) for which $\pi(\boldsymbol{U})$ is the line in E generated by the vector (1).

Assume from now on that the field K is the field of complex numbers. Then the embedding π realizes the flag manifold F as a complex projective algebraic variety, which is smooth because F has a transitive group of automorphisms (namely G). Each Schubert cell C_w is a locally closed subvariety of F, isomorphic to affine space of dimension $\ell(w)$.

For each $w \in S_n$ let

$$X_w = \overline{C}_w$$

be the closure of C_w in F. The X_w are the *Schubert varieties* in F, and a flag $\boldsymbol{U}$ lies in X_w if and only if $\boldsymbol{U}$ has a basis $(u_1, \ldots, u_n)$ such that $u_i \in V_{w(i)}$ for each i. Each X_w is in fact a union of Schubert cells C_v : if $(a_1, \ldots, a_p)$ is a reduced word for w, then $C_v \subset X_w$ if and only if v is of the form $s_{b_1} \cdots s_{b_q}$ where $(b_1, \ldots, b_q)$ is a subsequence of $(a_1, \ldots, a_p)$, that is to say if and only if $v \leq w$ in the Bruhat order. In particular, $X_1 = C_1$ is the single point $\boldsymbol{V} \in F$. At the other extreme, if w_0 is the longest element of S_n, then X_{w_0} is the whole of F, and the dimension of F is $\ell(w_0) = \frac{1}{2}n(n-1)$.

Let $H^*(F; \mathbf{Z})$ be the cohomology ring (with integral coefficients) of the projective variety F. Each closed subvariety X of F determines an element $[X] \in H^*(F; \mathbf{Z})$, and cup-product in $H^*(F; \mathbf{Z})$

corresponds, roughly speaking, to intersection of subvarieties. In particular, for each $w \in S_n$, we have a cohomology class $[X_w] \in H^*(F; \mathbf{Z})$, and it is a consequence of the cell decomposition (A.3) of F that the $[X_w]$ form a $\mathbf{Z}$-basis of $H^*(F; \mathbf{Z})$. In particular, $[X_{w_0}]$ is the identity element.

The connection between the classes $[X_w]$ and the Schubert polynomials $\mathfrak{S}_w (w \in S_n)$ is given by

(A.5) *There is a surjective ring homomorphism*

$$\alpha : \mathbf{Z}[x_1, \ldots, x_n] \to H^*(F; \mathbf{Z})$$

such that

$$\alpha(\mathfrak{S}_w) = [X_{w_0 w}]$$

for each $w \in S_n$.

Proof: Let us temporarily write

$$\sigma_w = [X_{w_0 w}]$$

for $w \in S_n$. Monk [Mo] proved that for all $w \in S_n$ and $r = 1, \ldots, n-1$

(1) $$\sigma_w \cdot \sigma_{s_r} = \sum_t \sigma_{wt}$$

where the sum on the right hand side is over all transpositions $t = t_{ij}$ such that $i \leq r < j \leq n$ and $\ell(wt) = \ell(w) + 1$, as in (4.15″).

Define $\xi, \ldots, \xi_n \in H^*(F; \mathbf{Z})$ by

$$\xi_1 = \sigma_1$$

$$\xi_i = \sigma_i - \sigma_{i-1} \qquad (2 \leq i \leq n-1)$$

$$\xi_n = -\sigma_{n-1}$$

From (1) we deduce the counterpart of (4.16): if r is the last descent of w (so that $r \leq n-1$), then we have

(2) $$\sigma_w = \sigma_v \xi_r + \sum_{w'} \sigma_{w'}$$

where v, w' are as in (4.16). Now iteration of (4.16) will ultimately express $\mathfrak{S}_w$ as a sum of monomials, i.e. as a polynomial in $x_1, \ldots, x_{n-1}$; and iteration of (2) will express σ_w as the *same* polynomial in $\xi_1, \ldots, \xi_{n-1}$. Hence if we define $\alpha : P_n \mapsto H^*(F; \mathbf{Z})$ by $\alpha(x_i) = \xi_i$ $(1 \leq i \leq n)$, we have $\sigma_w = \alpha(\mathfrak{S}_w)$ for all $w \in S_n$, and the proof of (A.5) is complete. $\|$

In fact the kernel of the homomorphism α is generated by the elementary symmetric functions $e_1, \dots, e_n$ of the x's.

We shall draw one consequence of (A.5) that we have not succeeded in deriving directly from the definition (4.1) of the Schubert polynomials. Since the $\sigma_w, w \in S_n$, form a $\mathbf{Z}$-basis of $H^*(F;\mathbf{Z})$, any product $\sigma_u\sigma_v (u, v \in S_n)$ is uniquely a linear combination of the σ_w, and it follows from intersection theory on F that the coefficient of σ_w in $\sigma_u\sigma_v$ is a non-negative integer. From this we deduce

(A.6) *Let u, v be permutations, and write $\mathfrak{S}_u\mathfrak{S}_v$ as an integral linear combination of the $\mathfrak{S}_w$, say*

$$\mathfrak{S}_u\mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w. \tag{1}$$

Then the coefficients c_{uv}^w are non-negative.

We have only to choose n sufficiently large so that u, v and all the permutations w such that $c_{uv}^w \neq 0$ lie in S_n, and then apply the homomorphism α of (A.5).

Remark. The coefficients c_{uv}^w in (A.6) are zero unless

(a) $\ell(w) = \ell(u) + \ell(v)$,

(b) $u \le w$ and $v \le w$.

For $\mathfrak{S}_u\mathfrak{S}_v$ is homogeneous of degree $\ell(u) + \ell(v)$, which gives condition (a). Also we have

$$\begin{aligned} c_{uv}^w &= \partial_w(\mathfrak{S}_u\mathfrak{S}_v) \\ &= \sum_{v_1 \le w} v_1 \partial_{w/v_1}(\mathfrak{S}_u)\partial_{v_1}(\mathfrak{S}_v) \end{aligned}$$

by (2.17), and the only possible nonzero term in this sum is that corresponding to $v_1 = v$. Hence if $c_{uv}^w \neq 0$ we must have $v \le w$, and by symmetry also $u \le w$.

Notes and References

Chapter I. The notion of the *diagram* of a permutation w is ascribed to J. Riguet in [LS1]. The *code* of w is the Lehmer code, familiar to computer scientists. Vexillary permutations were introduced in [LS1] and enumerated in [LS4], though from a somewhat different point of view from that in the text.

Chapter II. Divided differences, in the context of an arbitrary root system, were introduced independently by Bernstein, Gelfand and Gelfand [BGG] and Demazure [D]. Both these papers establish (2.5), (2.10) and (2.13) in this more general context.

Chapter III. Multi-Schur functions were introduced, and the duality theorm (3.8) proved, by Lascoux [L1]. The proof of Sergeev's formula (3.12) is also due to Lascoux (private communication).

Chapter IV. Schubert polynomials, like divided differences, are defined in the context of an arbitrary root system in [BGG] and in [D]. What is special to the root systems of type A is the stability property (4.5), which ensures that the Schubert polynomial $\mathfrak{S}_w$ is well-defined for all permutations $w \in S_\infty$. Propositions (4.7), (4.8) and (4.9) are stated without proof in various places in [LS1]-[LS7] but as far as I am aware the only published proof of (4.9) is that of M. Wachs [W], which is different from the proof in the text. Proposition (4.15), appropriately modified, is valid for any root system, and in this more general form will be found in [BGG] and [D].

Chapter V. The scalar product (5.2) is introduced in [LS7]. The symmetry properties (5.23) of the coefficient matrices $(\alpha_{uv}), (\beta_{uv})$ are indicated in [LS6].

Chapter VI. Double Schubert polynomials were introduced in [L2]. For the interpolation forumla (6.8), see [LS5]. The generalization (6.20) of Sergeev's formula (3.12) is due to Lascoux (private communication).

Chapter VII. This chapter is mostly an amplification of [LS2]. Propositions (7.21)-(7.24) are due to Stanley [S].

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