Algebra Autumn 2023
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23 October 2023 **Ninth Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 30 October:

39. The wreath product $S_m \wr S_n$ of symmetric groups is the semidirect product $(S_m)^n \rtimes_\varphi S_n$ where φ is the action of S_n on $(S_m)^n$ permuting the factors of $(S_m)^n$.

    (a) For $(\pi_1, \ldots, \pi_n\,, \omega) \in S_m \wr S_n$ ($\pi_i \in S_n$ and $\omega \in S_n$) define the map from $[m] \times [n]$ to itself by
    $$(\pi_1, \ldots, \pi_n\,, \omega).(i,j) \ := \ (\pi_{\omega(j)}(i)\,, \omega(j))\,.$$
    (Here, $[m] := \{1, \ldots, m\}$ and the same for $[n]$. Show that this defines an action of $S_m \wr S_n$ on $[m] \times [n]$.

    (b) Using this action or any other methods show that $S_2 \wr S_2 \simeq D_8$, the dihedral group with 8 elements.

    (c) This action realizes $S_2 \wr S_3$ as a sugroup of S_6. What are the cycle types of permutations of $S_2 \wr S_3$? For each cycle type, how many elements of $S_2 \wr S_3$ have that cycle type?

40. Prove that the converse to Lagrange's Theorem holds for nilpotent groups. That is, if G is a finite nilpotent group and n divides the order of G, then G has a subgroup of order n.

    Hint: first prove it for p-groups.

41. Prove (without using the Feit-Thompson Theorem) that the following two statements are equivalent:

    (a) Every group of odd order is solvable.

    (b) The only simple groups of odd order are the abelian groups of prime order.

42. Let $H = \mathbb{Z}_3$ and $K = \mathbb{Z}_4$, and consider the homomorphism $\varphi\colon K \to \mathrm{Aut}(\mathbb{Z}_3)$ which sends the generator of $\mathbb{Z}_4$ to multiplication by -1. Show that $H \rtimes_\varphi K$ is a nonabelian group of order 12 that is not isomorphic to either A_4 or D_{12}.

43. Use semidirect products to classify all groups of order 28 up to isomorphism. (There are four isomorphism classes.)

44. Consider the ring $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$ of endomorphisms of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$. Prove that $\mathrm{End}(\mathbb{Z} \oplus \mathbb{Z})$ is noncommutative.

45. Let R be a ring such that for every $a \in R$, there is a unique $b \in R$ such that $aba = a$. Prove that:

    (a) R has no zero divisors.

    (b) $bab = b$, where a, b are as above.

    (c) R is a division ring.