Math 629, Homework #12 5 April 2018

14.5.2 Using formulas for $\sin 2\theta$ and $\cos 2\theta$, or otherwise, show that

$$\tan 2\theta \;=\; \tfrac{2\tan\theta}{1-\tan^2\theta}\ ,$$

and check that this follows from the formula

$$(y+\sqrt{-1})(x-\sqrt{-1})^n \;=\; D(y-\sqrt{-1})(x+\sqrt{-1})^n\,, \qquad (*)$$

for $D=-1$, but not for $D=1$.

While all double angle formulas are standard topics in trigonometry, they are easy to derive from those for sine and cosine, $\sin 2\theta = 2\cos\theta\sin\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, so that

$$\tan 2\theta \;=\; \tfrac{\sin 2\theta}{\cos 2\theta} \;=\; \tfrac{2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta} \;=\; \tfrac{2\tan\theta}{1-\tan^2\theta}\ ,$$

this last step follows upon division of numerator and denominator by $\cos^2\theta$.

When $n=2$ in the formula $(*)$, we get $(y+\sqrt{-1})(x^2-2x\sqrt{-1}-1) \;=\; D(y-\sqrt{-1})(x^2+2x\sqrt{-1}-1)$. This simplifies to $(1-D)\big(y(x^2-1)+2x\big)-(1+D)\sqrt{-1}\big(2xy-(x^2-1)\big)=0$. Setting $D=1$ and dividing by $-(1+D)\sqrt{-1}=-2\sqrt{-1}$ gives $2xy-(x^2-1)=0$, so that $y=(x^2-1)/2x$. Setting $D=-1$ and dividing by 2 gives $y=2x/(1-x^2)$, as desired.

14.5.3 Use this formula to express $\tan 4\theta$ in terms of $\tan 2\theta$, and thus in terms of $\tan\theta$.

Following this suggestion gives

$$\tan 4\theta \;=\; \tfrac{2\tan 2\theta}{1-\tan^2 2\theta} \;=\; \frac{2\frac{2\tan\theta}{1-\tan^2\theta}}{1-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)^2} \;=\; \tfrac{4\tan\theta(1-\tan^2\theta)}{(1-\tan^2\theta)^2-4\tan^2\theta} \;=\; \tfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\ .$$

For the penultimate step, multiply the numerator and denominator by $(1-\tan^2\theta)^2$.

14.5.4 Check that this result follows from $(*)$ when $n=4$ and $D=-1$.

It is worthwhile to first study $(*)$. Take the difference of both sides of $(*)$ and expand using the binomial theorem $(a+b)^n=\sum_{k=0}n\binom{n}{k}b^ka^{n-k}$ to get

$$y\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\big((-\sqrt{-1})^k-D(\sqrt{-1})^k\big) \;+\; \sqrt{-1}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\big((-\sqrt{-1})^k+D\sqrt{-1}^k\big)\ .$$

Simplifying, we obtain

$$y\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(\sqrt{-1})^k((-1)^k-D) \;+\; \sqrt{-1}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(\sqrt{-1})^k((-1)^k+D)\ .$$

When $D=1$ even terms in the left sum and odd terms in the right sum vanish (and there is a common factor of $2\sqrt{-1}$), and when $D=-1$, it is the other sets of terms that vanish and there is a common factor of 2.

When $D=-1$ and $n=4$, taking into account the terms that vanish, dividing by 2 and a factor of -1 in the second term, this becomes

$$y(x^4+6x^2(\sqrt{-1})^2+(\sqrt{-1})^4) \;-\; (4x^3(\sqrt{-1})^2+4x(\sqrt{-1})^4)\ .$$

Setting equal to zero, solving for y, and then substituting $y=\tan 4\theta$ and $x=\tan\theta$, we get the formula of the previous exercise.

15.2.1 Show that the projective completion of the curve $Y = X^2$ is topologically a sphere by considering the parametrization $Y = t^2$ and $X = t$, where t ranges over the points of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, that is, show that the mapping is injective and continuous.

We use the model of $\mathbb{CP}^2$ as the collection of 1-dimensional linear subspaces of $\mathbb{C}^3$, placing the point (x, y) of the complex plane $\mathbb{C}^2$ as the line through the point $(x, y, 1)$ of $\mathbb{C}^3$.

Note that the line through $(t, t^2, 1)$ is the line through $(\frac{1}{t}, 1, \frac{1}{t^2}) = (s, 1, s^2)$, where $s = \frac{1}{t}$. We use that t is a coordinate for $\mathbb{CP}^1 \smallsetminus \{\infty\}$ and $s = \frac{1}{t}$ is a coordinate for $\mathbb{CP}^1 \smallsetminus \{0\}$. This observation about the lines through $(t, t^2, 1)$ and $(s, 1, s^2)$ shows that this map is continuous (we freely use that functions defined by polynomials are continuous).

As for injectivity, the only minor issue is the point at infinity, which corresponds to the Y-axis, the line through $(0, 1, 0)$. Since for points $t \neq \infty$, the image is the line through $(t, t^2, 1)$, none of which have $Z = 0$, and so none of which are the Y-axis. This shows that the map $\mathbb{CP}^1 \mapsto \mathbb{CP}^2$ is injective.

15.2.2 Show the same for the curve $Y^2 = X^3$, using the parametrization $Y = t^3$ and $X = t^2$.

Note that the intersection of the line $Y = tX$ for $t \in \mathbb{C}$ with the curve $Y^2 = X^3$ is the point (t^2, t^3) of the parametrization. Looking at the line through $(t^2, t^3, 1)$ as $t \to \infty$ (via the substitution $s = \frac{1}{t}$), this is the line through $(s, 1, s^3)$. At $s = 0$, this is corresponds to the point at infinity on the vertical line in the (X, Y)-plane, which shows injectivity (and identifies the curve with the set $\mathbb{CP}^1$ of lines through the origin—the same identification as in the previous problem).

15.2.3 Now consider the curve $Y^2 = X^2(X + 1)$. Again, consider the intersection of this curve with the line through the origin $Y = tX$. This maps the pencil $\mathbb{CP}^1$ of lines through the origin to the curve and is one-to-one for $t \neq \pm 1$, but that both points are mapped to the origin.

We already showed that this map from the pencil $\mathbb{CP}^1$ of lines through the origin to the curve is given by (t^2-1, t^3-t). Since these lines are dijoint away from the origin, the map is one-to-one except for $t = \pm 1$, as both points are sent ot the origin. This includes at $t = \infty$. It is continuous as it is given by polynomials. Since both points $t = 1$ and $t = -1$ are sent to the origin, this identifies the curve $Y^2 = X^2(X + 1)$ with the Riemann sphere $\mathbb{CP}^1$ where the two points $t = 1$ and $t = -1$ are identified.

16.1.1 Assuming that the power series for e^y is valid also for $y = \sqrt{-1}x$, show that

$$e^{\sqrt{-1}x} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \sqrt{-1}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \tag{1}$$

First, this formula is valid because the series for the exponential function $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$ is absolutely convergent for any $y \in \mathbb{C}$ by, for example, the root test: $\lim_{n\to\infty} |\frac{y^n}{n!}|^{1/n} = \lim_{n\to\infty} \frac{|y|}{(n!)^{1/n}} = 0$.

Given this, if we consider a typical term $\frac{(\sqrt{-1})^n x^n}{n!}$ in the expansion of $e^{\sqrt{-1}x}$ then when $n = 2k$ is even we have $(-1)^k \frac{x^{2k}}{(2k)!}$ and when $n = 2k+1$ is odd, we have $\sqrt{-1}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$. As this series is absolutely convergent, we may rearrange the terms, collecting the even terms together and the odd terms together, to obtain (1).

16.1.2 Differentiating the series for the sine function, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, or by other means, deduce Euler's formula $e^{\sqrt{-1}x} = \cos x + \sqrt{-1}\sin x$.

This differentiation can be done by eyeballing the two series for sine and cosine. It also holds by Theorems in Calculus 2. Even more simply, the power series for $\cos x$ and $\sin x$ are also standard in Calculus 2, and so (1) immediately implies Euler's formula $e^{\sqrt{-1}x} = \cos x + \sqrt{-1}\sin x$. This is most fun in the form $e^{\sqrt{-1}\pi} + 1 = 0$: five mathematical constants in one equation!