Foundations of Mathematics Math 300H Section 970

YOUR NAME

Seventh Homework: Due 25 October 2021

Use English when possible. Answers should not just be symbols.

---

1. Disprove the statement: If a, b, and c are distinct positive integers, then $3|(2^a + 2^b + 2^c)$.

   Can you determine for which distinct positive integers a, b, and c, we have that $3|(2^a+2^b+2^c)$?

2. Use a proof by contradiction to show that if a and b are odd integers, then $4 \not| \, (a^2 + b^2)$.

3. Prove that when an irrational number is multiplied by a nonzero rational number, the resulting number is irrational.

4. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

5. Prove that there do not exist three distinct real numbers a, b, and c such that all of the numbers $a + b + c$, ab, ac, bc, and abc are equal.

6. Is the following proposition true or false? (Justify your conclusion with a proof or a counterexample).
   "For all nonnegative real numbers x and y, $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$.

7. Prove that if n is an odd integer, then $7n - 5$ is even by
   (a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.

8. Prove that there exist four distinct positive integers integers such that each integer divides the sum of the remaining three.

   This question should suggest another problem to you. State and solve such a problem.

9. **Polya's Theorem.** *All horses have the same color.*

   We prove that for each natural number n if a herd of horses contains n horses, then all the horses in that herd have the same color.

   Base case: The case with just one horse is trivial. If a herd has only one horse, then clearly all horses in that herd have the same color.

   Inductive step: Let $k \in \mathbb{N}$ and assume that in any herd of k horses, all the horses have the same color. Consider now a herd consisting of k+1 horses.

   First, exclude one horse from the herd. The remaining k horses form a herd, and by induction, they all have the same color. Likewise, exclude some other horse (not the same one who was first removed) and look only at the other k horses. By the same reasoning, these too, must also be of the same color. Therefore, the first horse that was excluded is of the same color as the non-excluded horses, who in turn are of the same color as the other excluded horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color.

   This completes the proof. $\square$

   Analyze this proof of Polya's Theorem: determine if it is correct or not, and explain your analysis.