Four Lectures on Toric Varieties

Abstract. This is an outline for Frank Sottile's short course of four lectures at the 2017 CIMPA school on Combinatorial and Computational Algebraic Geometry in Ibadan, Nigeria.

1. Affine Toric Varieties

- For a finite set $\mathcal{A} \subset \mathbb{Z}^n$, the closure of the image of the map $\varphi_{\mathcal{A}} \colon (\mathbb{C}^*)^n \to \mathbb{C}^{\mathcal{A}}$ given by the momomials in $\mathcal{A}$ is the *affine toric variety $X_{\mathcal{A}}$*.
- Describe the ideal $I_{\mathcal{A}}$ of $X_{\mathcal{A}}$ (a *toric ideal*), giving a geometric description of its generators.

2. Toric Varieties in Projective Space

- When $\mathcal{A}$ is homogeneous, $X_{\mathcal{A}}$ is a cone and defines a projective variety in $\mathbb{P}^{\mathcal{A}}$.
- Explain the relation between faces of the cone/polytope and coordinate subvarieties of $X_{\mathcal{A}}$.
- Compute the Hilbert function of $X_{\mathcal{A}}$, relating it to the Ehrhart polynomial of the polytope.
- Deduce Kushnirenko's Theorem.

3. Toric Varieties From Fans

- Begin with a good example. Either the projective plane or the double pillow.
- Discuss cones and fans, as well as polyhedral complexes. (For Kristin's third lecture)
- Construct toric varieties from rational fans, and prove a theorem or two about them.

4. Bernstein's Theorem and mixed volumes

- Develop the theory of mixed volumes of polytopes, proving main properties, including that mixed volume is uniquely determined by symmetry, multilinearity, and normalization.
- Give a proof of Bernstein's Theorem. Follow his proof, showing that the number of isolated solutions in $(\mathbb{C}^\times)^n$ to a sparse system is less than the generic number if and only if some facial system has a solution. This uses some tropical geometry (while Bernstein's proof uses Puiseaux expansions) and initial ideals.

  This implies that the generic root count (which is symmetric in the Newton polytopes) is multilinear. Kushnirenko's Theorem shows that it also satisfies the normalization condition, and is therefore equal to mixed volume.

References

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