Algebra Autumn 2013
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3 September 2013

Homework

Hand in to Frank Thursday 5 September:

7. T/F, with reason: Every group has maximal cyclic subgroups.
8. Let G be an abelian group. Show that $T := \{x \in G \;:\; |x| < \infty\}$ is a subgroup.

Hand in for the grader Tuesday 10 September:

9. T/F, with reason. For any group G, the set T of torsion elements of Problem 8 is a subgroup.
10. A group G is abelian if and only if the map $G \to G$ given by $a \mapsto a^{-1}$ is an automorphism.
11. Let Q be the group of complex matrices generated by

$$A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ,$$

    where $i^2 = -1$. Show that Q is a nonabelian group of order 8, called the *quaternion group*. (Hint: Show that $BA = A^3B$, and $A^4 = B^4 = I$, where I is the identity matrix.)
12. The *dihedral group D_{2n}* is the group of symmetries of the regular n-gon in the plane. Show that this group has order $2n$, and that it is generated by two elements ρ and σ where $\rho^2 = \sigma^2 = e$ and $\rho\sigma$ has order n. Identify these elements and their product as explicit symetries of the n-gon. *Your answer will be incorrect if you use any other definition of a dihedral group without first proving that it satisfies the definition given above.*
13. Is $D_8 \simeq Q$?
14. Let $S \subset G$ be a subset of a group G and define the relation $\sim$ by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if S is a subgroup of G.
15. The *center* of a group G is the set $C(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$. For $g \in G$, the *centralizer of g* is the set $C_G(g) := \{a \in G \mid ag = ga\}$. Prove that $C(G)$ and $C_G(g)$ are subgroups of G.