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Enumerative Real Algebraic Geometry

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1. Introduction

Consider the following question.

Question 1.1. *Find a priori information about the number of real solutions to a structured system of real polynomial equations*

$$0 \;=\; f_1 \;=\; f_2 \;=\; \cdots \;=\; f_N, \qquad \text{where each } f_i \in \mathbb{R}[x_1,\ldots,x_n]\,. \tag{1.1}$$

Two well-defined classes of structured polynomial systems have been studied from this point of view—sparse systems, where the structure is encoded by the monomials in the polynomials f_i—and geometric systems, where the structure comes from geometry. This second class consists of polynomial formulations of enumerative geometric problems, and in this case Question 1.1 is the motivating question of enumerative real algebraic geometry, the subject of this survey. Treating both sparse polynomial systems and enumerative geometry together in the context of Question 1.1 gives useful insight.

Given a system of polynomial equations (1.1) with d complex solutions, we know the following easy facts about its number ρ of real solutions.

$$\begin{array}{l} \text{We have } \rho \le d, \text{ and} \\ \text{since } d-\rho \text{ an even integer, } \rho \;\ge\; \begin{cases} 0 & \text{if } d \text{ is even} \\ 1 & \text{if } d \text{ is odd}\,. \end{cases} \end{array} \tag{1.2}$$

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In a surprising number of cases, much better information than this is known.

Structured systems occur in families of systems sharing the same structure. The common structure determines the number d of complex solutions to a general member of the family. We assume throughout that a general system in any family has only simple solutions in that each complex solution occurs without multiplicity.

Given such a family of structured systems whose general member has d complex solutions, perhaps the ultimate answer to our motivating question is to determine exactly which numbers ρ of real solutions can occur and also which systems have a given number of real solutions. Because this level of knowledge may be unattainable, we will be satisfied with less knowledge.

For example, are the trivial bounds given in (1.2) sharp? That is, do there exist systems attaining the maximal and minimal number of real solutions allowed by (1.2)? If these bounds are not sharp, do there exist better sharp bounds? Perhaps we are unable to determine sharp bounds, but can exhibit systems in a family with many (or few) real solutions. This gives lower bounds on the maximum number of real solutions to a system in a family (or upper bounds on the minimum number).

These answers have two parts: bounds and constructions. We shall see that bounds (or other limitations) often come from topological considerations. On the other hand, the constructions often come by deformations from/to a degenerate situation. In enumerative geometry, this is the classical technique of special position, while for sparse systems, it is Viro's method of toric deformations.

In Section 2 we discuss sparse polynomial systems from the point of view of Question 1.1. The heart of this survey begins in Section 3, where we discuss some of the myriad examples of enumerative geometric problems that have been studied from this perspective. In particular, for many enumerative problems the upper bound (1.2) is sharp. In Section 4, we concentrate on enumerative problems from the Schubert calculus, where much work has been done on this question of real solutions. Section 5 is devoted to a conjecture of Shapiro and Shapiro, whose study has led to many recent results in this area. Finally, in Section 6 we describe new ideas of Eremenko and Gabrielov giving lower bounds better than (1.2) for some enumerative problems.

2. Sparse Polynomial Systems

Perhaps the most obvious structure of a system of n polynomials in n variables

$$f_1(x_1,\ldots,x_n) \;=\; f_2(x_1,\ldots,x_n) \;=\; \cdots \;=\; f_n(x_1,\ldots,x_n) \;=\; 0\,, \tag{2.1}$$

is the list of total degrees of the polynomials $f_1, f_2, \ldots, f_n$. For such a system, we have the degree or Bézout upper bound, which is a consequence of the refined Bézout Theorem of Fulton and MacPherson [**18**, §1.23].

THEOREM 2.1 (Bézout Bound). *The system* (2.1)*, where the polynomial f_i has total degree $d_i := \deg(f_i)$, has at most $d_1 \cdot d_2 \cdots d_n$ isolated complex solutions.*

The Bézout bound on the number of real solutions is sharp. For example, if

$$f_i \;:=\; (x_i-1)(x_i-2)\cdots(x_i-d_i) \quad i=1,\ldots,n\,, \tag{2.2}$$

then the system (2.1) has $d_1 \cdot d_2 \cdots d_n$ real solutions. The reader is invited to construct systems with the minimum possible number (0 or 1) of real solutions.

A system of polynomial equations with only simple solutions, but with fewer solutions than the Bézout bound is called deficient. For example, fewer monomials in the polynomials lead to fewer solutions. We make this idea more precise. A monomial (or rather its exponent vector) is a point of $\mathbb{N}^n$. The convex hull of the monomials in f is its *Newton polytope*, $New(f)$, a polytope with integral vertices. The terms of f are indexed by the lattice points, $New(f) \cap \mathbb{N}^n$, in the Newton polytope of f.

Figure 1 displays the monomials (dots) and Newton polytopes of the polynomials

$$\begin{aligned} f &= 1 + x - y + xyz^4\,, \\ g &= 1 + x - y + 3z - z^3 + 2z^4\,. \end{aligned}$$

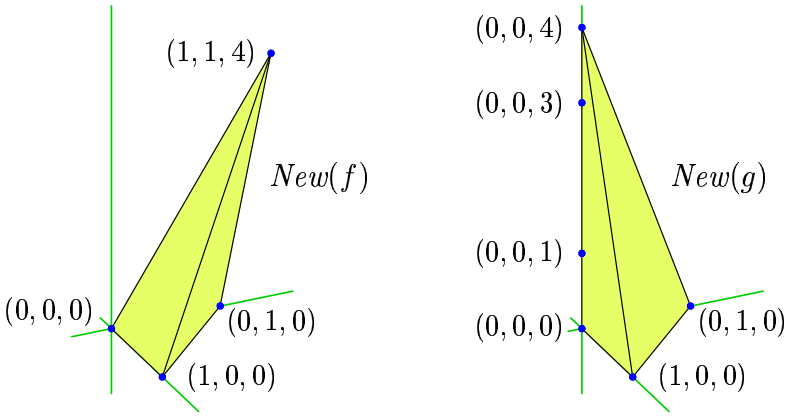


FIGURE 1. Newton Polytopes

Given polytopes $P_1, P_2, \ldots, P_m \subset \mathbb{R}^n$, their Minkowski sum is the pointwise sum

$$P_1 + P_2 + \cdots + P_m \;=\; \{p_1 + p_2 + \cdots + p_m \mid p_i \in P_i,\ i = 1, 2, \ldots, m\}\,.$$

The mixed volume, $MV(P_1, P_2, \ldots, P_m)$, of a collection of m polytopes is

$$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\cdots\frac{\partial}{\partial t_m}\Big[\text{Volume of } \; t_1P_1 + t_2P_2 + \cdots + t_mP_m\Big]\,.$$

When there are n equal polytopes, the mixed volume is $n!\mathrm{Vol}(P)$, the normalized volume of the common Newton polytope P, and when each P_i is a line segment of length d_i in the ith coordinate (so that $P_i = New(f_i)$, where f_i is the polynomial given in (2.2)), the mixed volume is the Bézout number $d_1 d_2 \cdots d_n$.

Given a list $(P_1, P_2, \ldots, P_n)$ of polytopes in $\mathbb{R}^n$ with vertices in the integral lattice $\mathbb{N}^n$, a *sparse polynomial system* with this structure is a system of polynomials (2.1) with $New(f_i) = P_i$. These sparse systems may have trivial solutions where some coordinates vanish. Thus we only consider solutions in $(\mathbb{C}^\times)^n$. We have the following basic result on the number of solutions to such a sparse system of polynomials.

THEOREM 2.2 (BKK bound). *A sparse polynomial system* (2.1) *with structure* $(P_1, P_2, \ldots, P_n)$ *has at most* $MV(P_1, P_2, \ldots, P_n)$ *isolated solutions in* $(\mathbb{C}^\times)^n$. *If the system is generic given its structure, then it has exactly this number of solutions in* $(\mathbb{C}^\times)^n$.

This result was developed in a series of papers by Kouchnirenko [**43**], Bernstein [**1**], and Khovanskii [**38**]. For simplicity of exposition, we will largely restrict ourselves to the case when the polynomials all have the same Newton polytope P.

Given a polytope P with vertices in the integral lattice, what are the possible numbers of real solutions to systems with that Newton polytope? We shall focus on understanding $\rho(P)$, the maximum number of real solutions to a sparse system with Newton polytope P. The following example serves as an introduction to this question.

EXAMPLE 2.3. Let d be a positive integer, and set

$$\begin{aligned} P_d &:= \text{Convex hull } \{(0,0,0),\ (1,0,0),\ (0,1,0)\,,\ (1,1,d)\}\,, \quad \text{and} \\ Q_d &:= \text{Convex hull } \{(0,0,0),\ (1,0,0),\ (0,1,0)\,,\ (0,0,d)\}\,. \end{aligned}$$

(Figure 1 shows P_4 and Q_4.) These tetrahedrons each have normalized volume d.

A general sparse system with Newton polytope P_d

$$\begin{aligned} A_1xyz^d + B_1x + C_1y + D_1 &= 0 \\ A_2xyz^d + B_2x + C_2y + D_2 &= 0 \\ A_3xyz^d + B_3x + C_3y + D_3 &= 0 \end{aligned}$$

is equivalent to a system of the form

$$z^d \ = \ \alpha, \quad x \ = \ \beta, \quad y \ = \ \gamma\,,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Thus the original system has d complex solutions, but only 0, 1, or 2 real solutions, so $\rho(P_d)$ is 1 or 2, depending upon the parity of d.

A general sparse system with Newton polytope Q_d is 3 polynomials of the form

$$Ax + By + C(z)\,,$$

where $C(z)$ has degree d. This is equivalent to a system of the form

$$f(z) \ = \ 0, \quad x \ = \ g(z), \quad y \ = \ h(z)\,,$$

where $f(z), g(z)$, and $h(z)$ are real polynomials with $\deg(f) = d$ but $\deg(g) = \deg(h) = d-1$. In this case, the general system again has d complex solutions, but there are systems with any number of real solutions, so $\rho(Q_d) = d$.

2.1. Polyhedral homotopy algorithm. The polyhedral homotopy algorithm of Huber and Sturmfels [**31**] deforms the sparse system (2.1) into a system where the number of solutions is evident. It gives an effective demonstration of the BKK bound and is based upon Sturmfels's generalization [**81**] of Viro's method for constructing real varieties with controlled topology [**85**].

EXAMPLE 2.4. Suppose P is a n-simplex which meets the integral lattice only at its vertices. Translating one vertex to the origin, the others are linearly independent. (Translating corresponds to division by a monomial.) Let M be the $n \times n$ integral matrix whose columns are these vertices. Multiplying M by an invertible integral matrix and taking the resulting columns to be a basis for $\mathbb{R}^n$ corresponds to a multiplicative change of coordinates

$$(x_1, x_2, \ldots, x_n) \ \longmapsto \ (y^{m_1}, y^{m_2}, \ldots, y^{m_n})\,,$$

with $y_1, y_2, \ldots, y_n \in (\mathbb{C}^\times)^n$ and $m_1, m_2, \ldots, m_n$ linearly independent integer vectors. Doing this for the Smith normal form of M transforms a polynomial f with Newton polytope P into a polynomial of the form

$$c_0 + c_1 y_1^{d_1} + c_2 y_2^{d_2} + \cdots + c_n y_n^{d_n}\,, \tag{2.3}$$

where $d_1 \mid d_2 \mid \cdots \mid d_n$ and $d_1 d_2 \cdots d_n = n!\mathrm{Vol}(P)$, the normalized volume of P. (These d_i are the invariant factors of the integral matrix M.) A system consisting of n general polynomials of the form (2.3) is equivalent to one of the form

$$y_1^{d_1} \;=\; \alpha_1\,, \quad y_2^{d_2} \;=\; \alpha_2\,, \quad \ldots\,, \quad y_2^{d_n} \;=\; \alpha_n\,.$$

Thus a general system whose Newton polytope is P has $n!\mathrm{Vol}(P)$ simple complex solutions.

The combinatorial structure underlying the homotopy algorithm of Huber and Sturmfels is that of a regular triangulation, which is a special case of a regular subdivision. A regular subdivision P_w of a lattice polytope P is given by a lifting function

$$w\colon P \cap \mathbb{N}^n \;\longrightarrow\; \mathbb{Q}_{\geq 0}$$

as follows. Set

$$Q \;:=\; \text{Convex hull } \{(a, w(a)) \mid a \in P \cap \mathbb{N}^n\} \;\subset\; \mathbb{R}^{n+1}$$

This lifted polytope Q has distinguished lower facets, those facets whose inward pointing normal vector has positive last coordinate. Forgetting the last coordinate projects these lower facets into $\mathbb{R}^n$ (and hence P). Their totality gives the *regular subdivision* P_w of P. When all the lattice points in $P \cap \mathbb{N}^n$ lift to vertices of lower facets of Q, and these lower facets are all simplices, then P_w is a *regular triangulation* of P.

In Figure 2 the triangulation on the left is regular and the triangulation on the right is not regular. Consider a hypothetical lifting function w for the triangulation on the right. We assume that w takes the value 0 at the three interior vertices. The clockwise neighbour of any vertex of the big triangle must be lifted higher than that vertex. (Consider the figure they form with the parallel edge of the interior triangle.) Since the edge of the big triangle is lifted to a concave broken path, this implies that each vertex is lower than its clockwise neighbour, which is impossible, except in some M.C. Escher woodcuts.

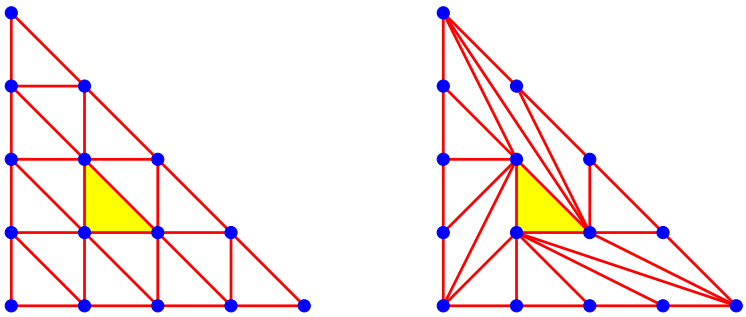


FIGURE 2. Regular and non-regular triangulations

Crucial to this algorithm are polynomial systems derived from the original system and the regular subdivision P_w of P by a lifting function w. Given a polynomial f with Newton polytope P and a face F of the subdivision P_w, consider the sum of terms of f

with exponent vector in F, restricting f to the face F. Doing this for each polynomial f_i in our original system, we obtain the facial subsystem of (2.1) given by F.

We continue with the algorithm of Huber and Sturmfels. Given a lifting function $w\colon P\cap\mathbb{N}^n \to \mathbb{Q}_{\geq 0}$ and a polynomial f with Newton polytope P

$$f(x) \;=\; \sum_{a\in P\cap\mathbb{N}^n} c_a x^a\,,$$

we multiply the term with exponent a by $t^{w(a)}$ to obtain

$$f(x;t) \;:=\; \sum_{a\in P\cap\mathbb{N}^n} c_a x^a t^{w(a)}\,.$$

Modifying all polynomials in the original sparse system in this way gives the *lifted system*, a family of sparse systems depending upon the parameter t.

$$f_1(x;t) \;=\; f_2(x;t) \;=\; \cdots \;=\; f_n(x;t) \;=\; 0\,, \tag{2.4}$$

Solutions to this system are algebraic functions $t\mapsto x(t)$ in the parameter t. In a neighborhood of 0 in the complex plane, each branch is expressed as a Puiseaux series

$$\begin{aligned} x(t) \;&=\; (x_1(t), x_2(t), \ldots, x_n(t))\,, \qquad \text{with}\\ x_i(t) \;&=\; t^{u_i}y_i + \text{ higher order terms in } t\,, \end{aligned}$$

where the y_i are non-zero constants and the exponents $u_i\in\mathbb{Q}$. Substituting this expression into (2.4), we obtain

$$0 \;=\; \sum_{a\in P\cap\mathbb{N}^n} c_a y^a t^{u\cdot a + w(a)} + \text{ higher order terms in } t\,.$$

The exponent of t is the dot product $(u,1)\cdot(a,w(a))$. Thus the terms of lowest order in t correspond to points $(a,w(a))$ in the lifted polytope where the exponent vector $(u,1)$ achieves its minimum—a face in the lower hull of the lifted polytope. Let F be the corresponding face of the regular subdivision P_w. Removing the common factor $t^{u\cdot a+w(a)}$ from these lowest order terms gives the restriction of f to the face F. Thus the coefficients y_i of the leading terms of the Puiseaux expansion are solutions to the facial subsystem of the original system (2.1) given by the face F selected by the initial Puiseaux exponents.

Suppose the original system is general in the sense that each of its facial subsystems given by faces F of the regular subdivision has solutions only if $\mathrm{Vol}(F) > 0$. Consider substituting the Puiseaux series into the polynomial system (2.4) and then taking the limit as t approaches 0. The resulting polynomial system for the leading coefficients y of the Puiseaux series are solutions to the corresponding facial system. By previous assumption, this has no solutions unless F is a facet of the subdivision P_w. In that case, the inward pointing normal vector $(u,1)$ to the corresponding facet of the lifted polytope gives the initial exponent u of the Puiseaux expansion.

The full Puiseaux series can be reconstructed from its initial terms (see [**31**] for details) giving the 1-1 correspondence

$$\left\{\begin{matrix}\text{Solutions to facial subsystems}\\ \text{of (2.1) given by facets of } P_w\end{matrix}\right\} \;\Longleftrightarrow\; \left\{\begin{matrix}\text{Branches of the algebraic}\\ \text{function } t\mapsto x(t) \text{ near } t=0\end{matrix}\right\}$$

This is the number of solutions to (2.4) for general $t\in\mathbb{C}$, so the number of solutions to the original sparse system equals the number of solutions to all the facial subsystems given by facets of P_w.

Now suppose the regular subdivision P_w is a regular triangulation and each facial subsystem is general. Then the facets F of P_w are simplices with no interior vertices. Since general sparse systems whose Newton polytope is such a simplex have exactly $n!\mathrm{Vol}(F)$ solutions, the number of solutions to facial subsystems given by facets of P_w is exactly the sum of the normalized volumes of all facets of P_w, which is the normalized volume of P.

2.2. Real solutions to sparse polynomial systems. How many real solutions are there to a sparse polynomial system (2.1) with common Newton polytope P? By Example 2.4, if P is the convex hull of $v_0, v_1, \ldots, v_n$, a simplex with no interior lattice points, then a general system (2.1) is equivalent to the system of binomials

$$y_1^{d_1} \;=\; \alpha_1, \quad y_2^{d_2} \;=\; \alpha_2, \quad \ldots, \quad y_n^{d_n} \;=\; \alpha_n\,, \tag{2.5}$$

with each $a_i \neq 0$. Here, $d_1 \mid d_2 \mid \cdots \mid d_n$ are the invariant factors of the matrix whose ith column is $v_i - v_0$ and $d_1 d_2 \cdots d_n$ is the normalized volume of P. Following Sturmfels [**80**], let $e(P)$ be the number of these invariant factors which are even. If $e(P) = 0$, so that d is odd, then P is an odd cell.

PROPOSITION 2.5. *The polynomial system* (2.5) *has* $2^{e(P)}$ *real solutions if* $a_i > 0$ *whenever* d_i *is even, and* 0 *real solutions otherwise. In particular, if* P *is an odd cell, then there is one real solution.*

THEOREM 2.6 (Sturmfels [**80**, Corollary 2.3]). *The maximum number* $\rho(P)$ *of real solutions to a sparse polynomial system* (2.1) *with common Newton polytope* P *satisfies*

$$\#\text{ odd cells in } P_w \;\leq\; \rho(P)\,,$$

for any regular triangulation P_w *of* P.

PROOF. In the limit as $t \to 0$, the lifted system (2.4) given by a regular triangulation P_w of P becomes the disjunction of facial subsystems of (2.1), one for each facet F of P_w. Thus the number of real solutions in the limit is a constructive lower bound for $\rho(P)$.

By Proposition 2.5, the number of odd cells in P_w is a lower bound on the number of real solutions to the facial subsystem given by the facets of P_w. □

Proposition 2.5 also gives an upper bound on the limiting number of real solutions ρ of the lifted system (2.4) as $t \to 0$ (due to Sturmfels [**80**, Theorem 2.2].)

$$\rho \;\leq \sum_{F \text{ a facet of } P_w} 2^{e(F)}\,. \tag{2.6}$$

More sophisticated accounting of the possible signs of the coefficients of facial subsystems improves this bound. This accounting is accomplished in [**54**] and [**35**], leading to a combinatorial upper bound for such limiting systems. Itenberg and Roy [**35**] show there is a system (2.1) for which this upper bound is attained, and thus obtain

$$\text{Combinatorial upper bound for limiting systems} \;\leq\; \rho(P)\,.$$

Itenberg and Roy then conjectured that this combinatorial bound was in fact the global bound, that is, they conjectured that the maximal number of real solutions occurs in limiting systems. They also gave a similar bound for $\rho_+(P)$, the number of solutions with

positive coordinates. This was too optimistic, for Li and Wang [**48**] found a remarkably simple counterexample to this conjecture of Itenberg and Roy[†]

$$y - x - 1 \;=\; 200 - 100y^3 + 900x^3 - x^3y^3 \;=\; 0\,.$$

This system has 3 positive solutions

$$(0.317659, 1.317659), \quad (.659995, 1.659995), \quad \text{and} \quad (8.12058, 9.12058)\,,$$

but the combinatorial upper bound is 2. Thus systems with the maximal number of real solutions cannot in general be constructed with these limiting techniques.

We still have more questions than answers. Among the questions are:

(1) Improve this lower bound for $\rho(P)$ (or for $\rho_+(P)$) of Itenberg and Roy.
(2) Find new general methods to construct systems with many real solutions.
(3) Determine which polytopes P have the property that $\rho(P) < n!\mathrm{Vol}(P)$, that is, not all the solutions can be real.

2.3. Fewnomial bounds. These definitions, constructions, and results apply to what have come to be known as fewnomial systems. A fewnomial is a polynomial f with *few* mo*nomials*—the monomials of f are members of some set $\mathcal{A}$ not necessarily equal to the lattice points within its convex hull. In particular, the results of Section 2.2 give lower bounds on the maximum number of real solutions $\rho(\mathcal{A})$ to a system of fewnomials whose monomials are from $\mathcal{A}$. Here, we use a regular triangulation Δ_w of the point set $\mathcal{A}$ induced from a lifting function $w\colon \mathcal{A} \to \mathbb{Q}$.

When $n = 1$, consider the binomial (a fewnomial) system

$$x^d - 1 \;=\; 0\,.$$

This has d complex solutions. Similarly, the number of complex solutions to a general fewnomial system is equal to its BKK bound. The above binomial system has either 1 or 2 real solutions, and so we see that the number of real solutions to a fewnomial system should be less than its BKK bound. The question is: How much less?

Khovanskii [**38, 39**] established a very general result concerning systems where each f_i is a *polynomial* function of the monomials x^a for $a \in \mathcal{A}$. He proves that the number of real solutions to such a system is at most $2^n 2^{\binom{N}{2}}(n+1)^N$, where $N = \#\mathcal{A}$, the number of monomials in $\mathcal{A}$. When the polynomial functions are linear, they are polynomials with monomials from $\mathcal{A}$, and hence we have Khovanskii's fewnomial bound.

$$\rho(\mathcal{A}) \;\leq\; 2^n \cdot 2^{\binom{N}{2}}(n+1)^N\,.$$

While this bound seems outrageously large, it does not depend upon the volume of the convex hull of $\mathcal{A}$, but rather on the algebraic complexity—the ambient dimension n and the size N of $\mathcal{A}$. That such a bound exists at all was revolutionary.

We compare this complexity bound to the combinatorial upper bound (2.6) on the number of real solutions to a lifted fewnomial system (2.4) in the limit as $t \to 0$. The invariant $e(F)$ of a facet of the regular triangulation Δ_w of $\mathcal{A}$ is at most n. Thus

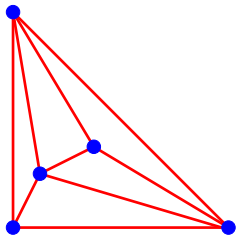
$$\rho \;\leq\; 2^n \cdot \#\text{facets of } \Delta_w \;\left(\leq\; 2^n \cdot \binom{N}{n+1}\right)\,,$$

[†]Strictly speaking, this counterexample is to their more general conjecture concerning *mixed systems* which are systems of polynomials with possibly different Newton polytopes.

as a facet of Δ_w involves $n+1$ points of $\mathcal{A}$. This bound is typically much lower than Khovanskii's bound. For example, consider two trinomials in two variables. Here $n=2$, and after multiplying each equation by a suitable monomial, we have $\#\mathcal{A}=5$. Thus we have Khovanskii's fewnomial bound

$$\rho(\mathcal{A}) \;\leq\; 2^2\cdot 2^{\binom{5}{2}}\cdot 3^5 \;=\; 995,328\,.$$

In contrast, a triangulation of 5 points in the plane has at most 5 simplices



and so the bound ρ for limiting lifted systems is

$$\rho \;\leq\; 2^2\cdot 5 \;=\; 20\,.$$

It may be more feasible to look for bounds on $\rho_+(\mathcal{A})$, the number of solutions with positive coordinates. Kouchnirenko made[†] the following conjecture.

CONJECTURE 2.7 (Kouchnirenko (see [**5**, p. 300])). *A system of n polynomials in n variables with only simple solutions whose ith polynomial has m_i monomials has at most*

$$(m_1-1)(m_2-1)\cdots(m_n-1)$$

solutions with positive coordinates.

For a system of two trinomials in 2 variables, Kouchnirenko's conjecture asserts that $\rho_+\leq 4$. In 2000, Haas [**24**] gave the counterexample

$$\begin{aligned} x^{108}+1.1y^{54}-1.1y &= 0\\ y^{108}+1.1x^{54}-1.1x &= 0,\end{aligned}$$

a system of two trinomials with 5 solutions having positive coordinates. Although Kouchnirenko's conjecture is false, the question remains: Is the true value for $\rho_+(\mathcal{A})$ (or $\rho(\mathcal{A})$) closer to Khovanskii's bound or to the number in Kouchnirenko's conjecture? Recent work suggests that it is the latter.

THEOREM 2.8 (Li, Rojas, and Wang [**47**]). *A system of 2 polynomials in 2 variables*

$$f_1(x,y) \;=\; f_2(x,y) \;=\; 0\,,$$

where f_1 has 3 terms and f_2 has m terms, and every solution is isolated, has at most

$$2\cdot(2^{m-1}-1)\,,$$

solutions with positive coordinates.

When $m=3$, so we have two trinomials in 2 variables, this gives a bound of 6. There is presently no known example with more that 5 positive solutions. Does Theorem 2.8 give the correct value of 6, or is Haas's construction with 5 solutions the best possible?

[†]Apparently, Kouchnirenko did not believe this bound. Nonetheless, the conjecture and its attribution have passed into folklore.

REMARK 2.9. Khovanskii's bound for the number of real solutions with positive coordinates holds also for systems of power functions, where the exponents of monomials are arbitrary real numbers. One might suppose that this added generality is the source of the large size of his bound and its apparent lack of tightness for polynomial systems. However, Napoletani [**53**] has shown that the complexity bounds are the same for both polynomials and for power functions.

3. Enumerative Real Algebraic Geometry

In his treatise on enumerative geometry, Schubert [**62**] declared enumerative geometry to be concerned with all questions of the following form: How many geometric figures of a fixed type satisfy certain conditions? This includes problems as diverse as the number of lines on a cubic surface (27) and the number of lines meeting four fixed lines in 3-space (see Section 4 for the answer). These are archetypes for two distinct classes of enumerative geometric problems. For the purpose of this survey, we ignore the first class (except in Section 3.1) and concentrate on the second. Specifically, we consider conditions imposed by geometric figures that may be moved independently into general position.

Enumerative geometry (in the broad sense) had a great flourishing in the 19th century in the hands of Chasles, Schubert, Halphen, Zeuthen, and others. (The survey of Kleiman [**41**] is a good historical source.) At that time, it had long been recognized that it was necessary to work over the complex numbers to ensure the existence of solutions. We know of only a handful of cases where the number of real solutions was considered. (We discuss some in Section 3.1.) Asking how many solutions can be real is the motivating question of enumerative real algebraic geometry and an analog of Question 1.1.

QUESTION 3.1. *In a given enumerative geometric problem, if the general figures are chosen to be real, how many of the solution figures can be real?*

For example, how many of the lines on a real cubic surface can be real? (Answer: all 27.) How many of the lines meeting four given real lines can be real? (Answer: all can.) This question was posed by Fulton [**19**, p. 55]: "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

This problem is fundamentally hard. Of the geometric figures satisfying real conditions, some will be real while the rest will occur in complex conjugate pairs, and the number which are real will depend subtly on the configuration of the conditions. Despite this difficulty, this is an important question often asked in applications.

One fruitful variant has been whether it is possible that *all* solution figures can be real. We call an enumerative problem *fully real* if there are general real conditions for which all solution figures are real. That is, if the upper bound of (1.2) is sharp. In light of the situation for sparse polynomial systems, it is quite surprising that there are no known enumerative problems which are not fully real. For this it is important that the conditions are imposed by general figures. A related question is whether the opposite situation can occur: Is it possible to have no (or only one) real solutions? We give some examples in Section 4.2.4. We shall see in Section 6 that there are many enumerative problems whose number of real solutions is always at least 2.

In the above passage, Fulton [**19**] goes on to ask: "For example, how many of the 3264 conics tangent to five general conics can be real?" He answered this question in the affirmative in 1986, but did not publish that result. Later, Ronga, Tognoli, and Vust [**60**]

gave a careful argument that all 3264 can be real. This example is very striking, both for the number, 3264, and because this problem of conics has long been an important testing ground for ideas in enumerative geometry.

One difficulty with enumerative real algebraic geometry is that there are few techniques or theorems with a wide range of applicability. Sometimes a direct calculation suffices (Sections 3.3 and 3.4) or more commonly real solutions are constructed by deforming from a special configuration, as in the homotopy algorithm of Huber and Sturmfels in Section 2.1. Attempts to formalize this method include 'Schubert induction' [**74**]; this is presented in the proof of Theorem 4.6, and used to establish the other results of Section 4. Another formalization is the notion of 'real effective algebraic equivalence' [**69**]. That (together with Theorem 4.4) gives such results as [**69**, Theorem 18]: The enumerative problem of $\binom{2n-2}{n-1}n^{2n-3}$ codimension 2 planes meeting $2n-2$ rational normal curves is fully real and [**67**, Corollary 5]: The enumerative problem of 11,010,048 2-planes in $\mathbb{P}^5$ meeting 9 general Veronese surfaces is fully real.

We devote the rest of this section to a description of some enumerative problems in the context of Question 3.1.

3.1. Enumerative problems not involving general conditions. We give a brief tour of some enumerative problems that do not involve general conditions, but nonetheless raise some interesting questions regarding real solutions.

Schläfli (see the survey of Coxeter [**11**]) showed there are 4 possibilities for the number of real lines on a real cubic surface: 27, 15, 7, or 3. Recent developments in enumerative geometry (mirror symmetry) have led to the solution of a large class of similar enumerative problems involving, among other things, the number of rational curves of a fixed degree on a Calabi-Yau threefold. (See the book of Cox and Katz [**10**].) For example, on a general quintic hypersurface in $\mathbb{P}^4$ there are 2875 lines [**26**], 609,250 conics [**36**], and 371,206,375 twisted cubics. The number of twisted cubics and higher degree rational curves was computed in the seminal paper of Candelas, de la Ossa, Green, and Parkes [**6**]. How many of the curves can be real in problems of this type? For example, how many real lines can there be on a real quintic hypersurface in $\mathbb{P}^4$?

A real homogeneous polynomial $f(x)$ is positive semi-definite (psd) if $f(x) \geq 0$ whenever x is real. Hilbert [**27**] proved that a psd ternary quartic is a sum of three squares of real quadratic forms. In fact, a general quartic is a sum of three squares of *complex* quadratic forms in 63 different ways [**86**]. Powers and Reznick [**56**] studied the question of how many ways one may represent a ternary quartic as a real sum or difference of three squares. In every instance, they found that 15 of the 63 ways involved real quadratic forms. Is it true that a general psd quartic is a sum or a differrence of three real squares in exactly 15 different ways? In how many ways is it a sum of three squares?[†]

A general plane curve C of degree d has $3d(d-2)$ flexes. These are the points on C where the Hessian determinant of the form defining C vanishes. Since the Hessian determinant has degree $3(d-2)$, we expect there to be $3d(d-2)$ such points. This involves intersecting the curve with its Hessian curve, and *not* with a general curve of degree $3(d-2)$. A real smooth plane cubic has 3 of its 9 flexes real. Zeuthen [**89**] found that at most 8 of the 24 flexes of a real plane quartic can be real. An example of a plane

[†]With Powers, Reznick, and Scheiderer, we have shown that the answers to these two questions are 15 and 8, respectively [**57**].

quartic with 8 real flexes is provided by the Hilbert quartic [**28**], which is defined by

$$(x^2+2y^2-z^2)(2x^2+y^2-z^2)+z^4/100 \;=\; 0\,.$$

We display this curve in Figure 3, marking the flexes with dots. Klein [**42**] later showed

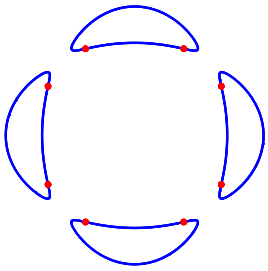


FIGURE 3. Hilbert's quartic: a plane quartic with 8 flexes

that a general real plane curve has at most $1/3$ of its flexes real.

Harnack [**25**] proved that a smooth real algebraic curve of genus g has at most $g+1$ topological components, and he constructed real algebraic curves of genus g with $g+1$ components. In particular, a plane curve of degree d has genus $g=(d-1)(d-2)/2$ and there are real plane curves of degree d with $g+1$ components. (An example is provided by Hilbert's quartic, which has genus 3.) Finer topological questions than enumerating the components leads to (part of) Hilbert's 16th problem [**29**], which asks for the determination of the topological types of smooth projectively embedded real algebraic varieties.

A variant concerns rational plane curves of degree d. A general rational plane curve of degree d has $3(d-2)$ flexes and $g=(d-1)(d-2)/2$ nodes. Theorem 4.2 implies that there exist real rational plane curves of degree d with all $3(d-2)$ flexes real, which we call maximally inflected curves. See Section 5.1 for the connection. Such curves have at most $g-d+2$ of their nodes real, and there exist curves with the extreme values of 0 and of $g-d+2$ real nodes [**37**]. For example, a rational quartic ($d=4$) has 6 flexes and $g=3$ nodes. If all 6 flexes are real, then at most one node is real. Figure 4 shows maximally inflected quartics with and 0 and 1 nodes. The flexes are marked by dots.

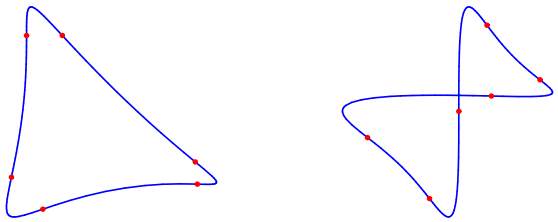


FIGURE 4. Rational quartics with 6 real flexes

Recently, Huisman asked and answered a new question about real curves. A component X of a real algebraic curve is a *psuedoline* if its homology class $[X]$ in $H^1(\mathbb{P}^n_{\mathbb{R}},\mathbb{Z}/2\mathbb{Z})$ is non-zero, and an *oval* otherwise.

QUESTION 3.2. *Given a smooth (irreducible over $\mathbb{C}$) real algebraic curve C in $\mathbb{P}^n$ of genus g and degree c, how many real hypersurfaces of degree d are tangent to at least s components of C with order of tangency at least m?*

Let ν be that number, when there are finitely many such hypersurfaces.

THEOREM 3.3 (Huisman [**33**, Theorem 3.1]). *When $s = g$ and $gm = cd$, and the restriction*

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(C, \mathcal{O}(d))$$

is an isomorphism, then ν is finite. Moreover, ν is non-zero if and only if

(1) *Both m and d are odd and C consists of exactly g psuedolines, or*
(2) *m is even and either d is even or all components of C are ovals.*

In case (1), $\nu = m^g$, *and in case* (2),

$$\nu = \begin{cases} (g+1)m^g & \text{if } C \text{ has } g+1 \text{ components} \\ m^g & \text{if } C \text{ has } g \text{ components.} \end{cases}$$

It is notable that this problem can only be stated over the real numbers.

3.2. The Stewart-Gough platform. The position of a rigid body in $\mathbb{R}^3$ has 6 degrees of freedom. This is exploited in robotics, giving rise to the Stewart-Gough platform [**23, 79**]. Specifically, suppose we have 6 fixed points $A_1, A_2, \ldots, A_6$ in space and 6 points $B_1, B_2, \ldots, B_6$ on a rigid body B (the framework of Figure 5). The body is

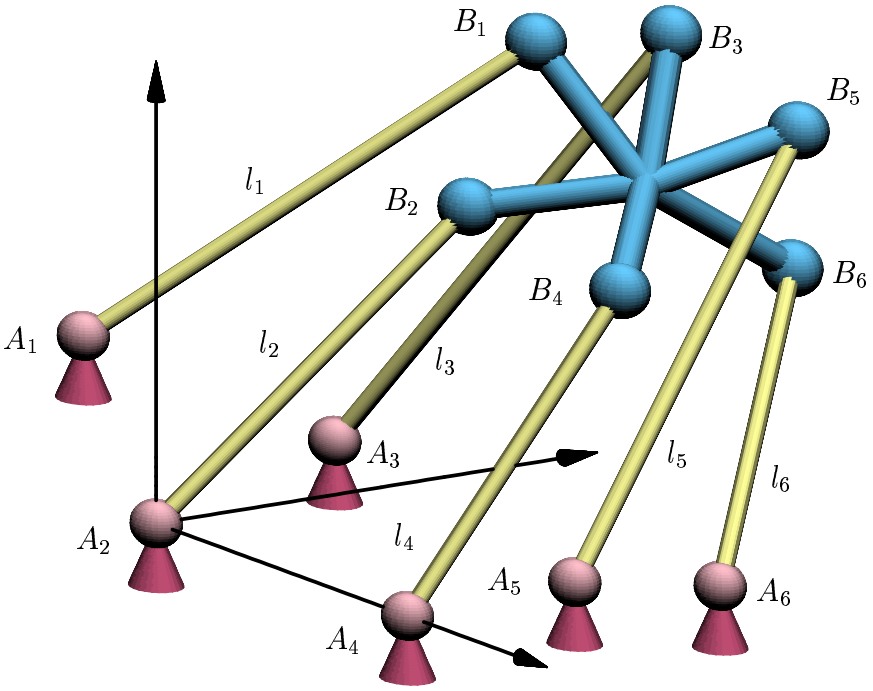


FIGURE 5. A Stewart Platform

controlled by varying each distance l_i between the fixed point A_i and the point B_i on B. This may be accomplished by attaching rigid actuators to spherical joints located at the points B_i, or by suspending the platform from cables.

Given a position of the body B in space, the distances $l_1, l_2, \ldots, l_6$ are uniquely determined. A fundamental problem is the inverse question: Given a platform (positions of the A_i fixed and the relative positions of the B_i specified) and a sextuple of distances $l_1, l_2, \ldots, l_6$, what is the position of the platform in space?

It had long been understood that several positions were possible for a given sextuple of lengths. An early work in 1897 showed there could be as many as 16 different positions [**4**]. This leads to the following enumerative problem.

QUESTION 3.4. *For a given Stewart platform, how many (complex) positions are there for a generic choice of the distances $l_1, l_2, \ldots, l_6$? How many of these can be real?*

In the early 1990's, several approaches (numerical experimentation [**58**], intersection theory [**61**], Gröbner bases [**46**], resultants [**51**], and algebra [**52**]) each showed that there are 40 complex positions of a general Stewart platform. The obviously practical question of how many positions could be real remained open until 1998, when Dietmaier introduced a novel method to find a value of the distances $l_1, l_2, \ldots, l_6$ with all 40 positions real.

THEOREM 3.5 (Dietmaier [**14**]). *All 40 positions can be real.*

Dietmaier's method will find future applications to other problems of this kind. He began with a formulation of the problem as a system of equations depending upon the distances $l_1, l_2, \ldots, l_6$. An initial solution for a given instance of the distances gave 6 real solutions and 17 pairs of complex conjugate solutions. He then used an ingenious algorithm to vary the distances in search of a configuration with all 40 solutions real.

This algorithm systematically varies the distances with the intention of increasing the number of real solutions. It proceeds in two stages. In the first stage, a pair of complex conjugate solutions $(x, \overline{x})$ are driven together, eventually creating a double solution, while at the same time the existing real solutions are kept bounded away from one another. At the formation of a double (necessarily real) solution, the distances are further incremented to create two new nearby real solutions (x_1, x_2), which are then driven apart in the second stage. This procedure is repeated again with another pair of complex conjugate solutions, and *et cetera*. Figure 6 ilustrates the two stages.

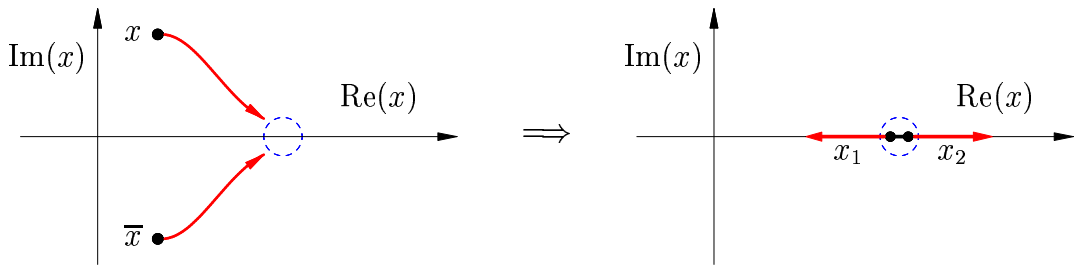


FIGURE 6. The two stages of Dietmaier's algorithm.

In each stage, Dietmaier accomplishes the given task (eg. colliding conjugate solutions) by linearizing the system at the current solutions and then solving a linear program for the optimal increment of the distances for the given goal. Changing the distances, he uses Newton's method beginning with the current solutions to find soutions for the new set of

distances, and then repeats this procedure until the goal is acheived (eg. the conjugate pair collides). This is an application of numerical homotopy continuation [**84**].

While there is no guarantee that this method will even successfully collide two conjugate solutions, Dietmaier uses it to find a sextuple of distances with all 40 solutions real. While at each step the solutions are only numerical approximations to the actual solutions, the condition number N guarantees the existence of a genuine solution within $1/N$ of each approximate solution. Since the approximate real solutions were separated by more than $2/N$, the requirement that non-real solutions occur in complex conjugate pairs forced these genuine solutions to be real.

3.3. Real rational cubics through 8 points in $\mathbb{P}^2_{\mathbb{R}}$. There are 12 singular (rational) cubic curves containing 8 general points in the plane.

THEOREM 3.6 (Degtyarev and Kharlamov [**12**, Proposition 4.7.3]). *Given* 8 *general points in* $\mathbb{P}^2_{\mathbb{R}}$*, there are at least* 8 *singular real cubics containing them, and there are choices of the* 8 *points for which all* 12 *singular cubics are real.*

PROOF. Since a cubic equation in the plane has 10 coefficients, the space of cubics is identified with $\mathbb{P}^9$. The condition for a plane cubic to contain a given point is linear in these coefficients. Given 8 general points, these linear equations are independent and so there is a pencil ($\mathbb{P}^1$) of cubics containing 8 general points in $\mathbb{P}^2$.

Two cubics P and Q in this pencil meet transversally in 9 points. Since curves in the pencil are given by $aP+bQ$ for $[a,b] \in \mathbb{P}^1$, any two curves in the pencil meet transversally in these 9 points. Let Z be $\mathbb{P}^2$ blown up at these same 9 points. We have a map

$$f \,:\, Z \;\longrightarrow\; \mathbb{P}^1\,,$$

where $f^{-1}([a,b])$ is the cubic curve defined by the polynomial $aP+bQ$.

Consider the Euler characteristic $\chi(Z)$ of Z first over $\mathbb{C}$ and then over $\mathbb{R}$. Blowing up a smooth point on a surface replaces it with a $\mathbb{P}^1_{\mathbb{C}}$ and thus increases the Euler characteristic by 1. Since $\chi(\mathbb{P}^2_{\mathbb{C}}) = 3$, we see that $\chi(Z) = 12$. The general fiber of f is a smooth plane cubic which is homeomorphic to $S^1 \times S^1$, and so has Euler characteristric 0. Thus only the singular fibers of f contribute to the Euler characteristic of Z. Assume that the 8 points are in general position so there are only nodal cubics in the pencil. A nodal cubic has Euler characteristic 1. Thus there are 12 singular fibers of f and hence 12 singular cubics meeting 8 general points in $\mathbb{P}^2_{\mathbb{C}}$.

Consider now the Euler characteristic of $Z_{\mathbb{R}}$. Blowing up a smooth point on a real surface replaces the point by $\mathbb{P}^1_{\mathbb{R}} = S^1$, and hence decreases the Euler characterstic by 1. Since $\chi(\mathbb{P}^2_{\mathbb{R}}) = 1$, we have $\chi(Z_{\mathbb{R}}) = 1-9 = -8$. A nonsingular real cubic is homeomorphic to either one or two disjoint copies of S^1, and hence has Euler characteristic 0. Again the Euler characteristic of $Z_{\mathbb{R}}$ is carried by its singular fibers. There are two nodal real cubics; either the node has two real branches or two complex conjugate branches so that the singular point is isolated. Call these curves real nodal and complex nodal, respectively. They are displayed in Figure 7. The real nodal curve is homoemorphic to a figure 8 and has Euler characteristic -1, while the complex nodal curve is the union of a S^1 with a point and so has Euler characteristic 1.

Among the singular fibers, we have

$$\begin{aligned} -8 &= \#\text{complex nodal} - \#\text{real nodal}\,, \quad \text{with} \\ 12 &\geq \#\text{complex nodal} + \#\text{real nodal}\,. \end{aligned}$$

FIGURE 7. A real nodal and complex nodal curve

Thus there are at least 8 real nodal curves containing 8 general points in $\mathbb{P}^2_{\mathbb{R}}$. The pencil of cubics containing the 2 complex nodal cubics of Figure 8 has 10 real nodal cubics. Thus there are 12 real rational cubics containing any 8 of the 12 points in Figure 8.

$$\begin{aligned} \frac{(y+9x-28)^2}{4} &= (x-1)x^2 \\ \frac{(x+10y-28)^2}{4} &= y(y+1)^2 \end{aligned}$$

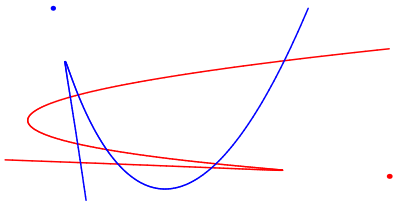


FIGURE 8. Complex nodal curves meeting in 9 points

□

REMARK 3.7. This classical problem of 12 plane cubics containing 8 points generalizes to the problem of enumerating rational plane curves of degree d containing $3d-1$ points. Let N_d be the number of such curves, which satisfies the recursion [**20**]

$$N_d \;=\; \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right] .$$

The values $N_1 = N_2 = 1$ are trivially fully real, and we have just seen that $N_3 = 12$ is fully real. The next case of $N_4 = 620$ (computed by Zeuthen [**88**]) seems quite challenging.

REMARK 3.8. The most interesting feature of Theorem 3.6 is the existence of a lower bound on the number of real solutions, which is a new phenomenon. In Section 6 we shall see evidence that this may be a pervasive feature of this field.

3.4. Common tangent lines to $2n-2$ spheres in $\mathbb{R}^n$. Consider the following.

QUESTION 3.9. *How many common tangent lines are there to $2n-2$ spheres in $\mathbb{R}^n$?*

For example, when $n=3$, how many common tangent lines are there to four spheres in $\mathbb{R}^3$? Despite its simplicity, this question does not seem to have been asked classically, but rather arose in discrete and computational geometry[†]. The case $n=3$ was solved by Macdonald, Pach, and Theobald [**49**] and the general case more recently[§] [**78**].

[†]The question of the maximal number of (real) common tangents to 4 balls was first formulated by David Larman [**45**] at DIMACS in 1990.

[§]This was solved during the DIMACS Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science.

THEOREM 3.10. *$2n-2$ general spheres in $\mathbb{R}^n$ ($n \geq 3$) have $3 \cdot 2^{n-1}$ complex common tangent lines, and there are $2n-2$ such spheres with all common tangent lines real.*

Figure 9 displays a configuration of 4 spheres in $\mathbb{R}^3$ with 12 real common tangents. The number $2n-2$ is the dimension of the space of lines in $\mathbb{R}^n$ and is necessary for there

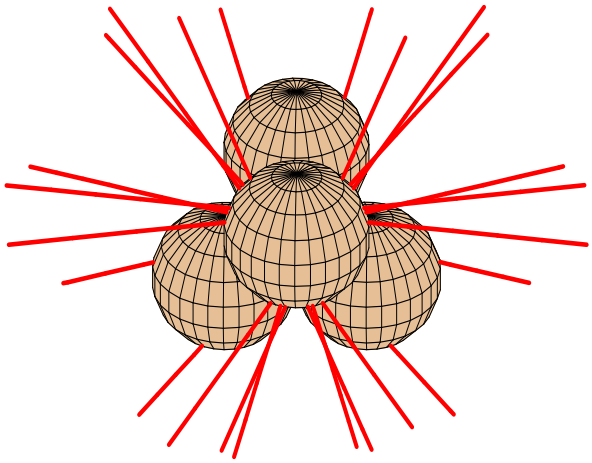


FIGURE 9. Four spheres with 12 common tangents

to be finitely many common tangents.

Represent a line in $\mathbb{R}^n$ by a point p on the line and its direction vector $v \in \mathbb{P}^{n-1}$. Imposing the condition

$$p \cdot v \;=\; 0\,, \tag{3.1}$$

makes this representation unique. Here $\cdot$ is the usual Euclidean dot product. Write v^2 for $v \cdot v$. A line (p, v) is tangent to the sphere with center c and radius r when

$$v^2p^2 - 2v^2p \cdot c + v^2c^2 - (v \cdot c)^2 - r^2v^2 \;=\; 0\,.$$

Suppose we have $2n-2$ spheres with centers $c_1, c_2, \ldots, c_{2n-2}$ and corresponding radii $r_1, r_2, \ldots, r_{2n-2}$. Without any loss of generality, we may assume that the last sphere is centered at the origin and has radius r. Then its equation is

$$v^2p^2 - v^2r^2 \;=\; 0\,. \tag{3.2}$$

Subtracting this from the equations for the other spheres, we obtain the equations

$$2v^2p \cdot c_i \;=\; v^2c_i^2 - (v \cdot c_i)^2 - v^2(r_i^2 - r^2) \tag{3.3}$$

for $i = 1, 2, \ldots, 2n-3$.

These last equations are linear in p. If the centers $c_1, c_2, \ldots, c_n$ are linearly independent (which they are, by our assumption on generality), then we use the corresponding equations to solve for v^2p as a homogeneous quadratic in v. Substituting this into the equations (3.1) and (3.2) gives a cubic and a quartic in v, and substituting the expression for v^2p into (3.3) for $i = n+1, n+2, \ldots, 2n-3$ gives $n-3$ quadratics in v. By Bézout's Theorem, if there are finitely many complex solutions to these equations, their number is bounded by $3 \cdot 4 \cdot 2^{n-3} = 3 \cdot 2^{n-1}$.

This upper bound is attained with all solutions real. Suppose that the spheres all have the same radius, r, and the first four have centers

$$\begin{aligned} c_1 &:= (\ 1,\ \ 1,\ \ 1,\ 0,\ldots,0)\,,\\ c_2 &:= (\ 1,-1,-1,\ 0,\ldots,0)\,,\\ c_3 &:= (-1,\ \ 1,-1,\ 0,\ldots,0)\,,\\ c_4 &:= (-1,-1,\ \ 1,\ 0,\ldots,0)\,, \end{aligned}$$

and subsequent centers are at the points $\pm a e_j$ for $j = 4, 5, \ldots, n$, where $e_1, e_2, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. Let $\gamma := a^2(n-1)/(a^2+n-3)$, which is positive.

THEOREM 3.11 ([**78**, Theorem 5]). *When*

$$a\,r\,(r^2-3)\,(3-\gamma)\,(a^2-2)\,(r^2-\gamma)\,\left((3-\gamma)^2+4\gamma-4r^2\right)\ \neq\ 0\,,$$

there are exactly $3\cdot 2^{n-1}$ *complex lines tangent to the spheres. If we have*

(a) $\frac{1}{4}(3-\gamma)^2+\gamma\ >\ r^2\ >\ \gamma$ *and*

(b) $\frac{n-1}{n-4}+2\ >\ a^2\ >\ 2\,,$

then all the $3\cdot 2^{n-1}$ *lines are in fact real.*

Theorem 3.10 is false when $n = 2$. There are 4 lines tangent to 2 circles in the plane, and all may be real. The argument given for Theorem 3.10 fails because the centers of the spheres do not affinely span $\mathbb{R}^2$. This case of $n = 2$ does generalize, though.

THEOREM 3.12 (Megyesi [**50**]). *Four unit spheres in* $\mathbb{R}^3$ *whose centers are coplanar but otherwise general have* 12 *common complex tangents. At most* 8 *of these* 12 *are real.*

REMARK 3.13. This problem of common tangents to 4 spheres with equal radii and coplanar centers gives an example of an enumerative geometric problem that is not fully real. We do not feel this contradicts the observation that there are no enumerative problems not known to be fully real, as the spheres are not sufficiently general.

4. Schubert Calculus

The classical Schubert calculus of enumerative geometry is concerned with questions of enumerating linear subspaces of a vector space or projective space that satisfy incidence conditions imposed by other linear subspaces. A non-trivial instance is the question posed at the beginning of Section 3.

QUESTION 4.1. *How many lines in space meet four general lines* ℓ_1, ℓ_2, ℓ_3, *and* ℓ_4*?*

Three pairwise skew lines ℓ_1, ℓ_2, and ℓ_3 lie on a unique smooth quadric surface Q. There are two families of lines that foliate Q—one family includes ℓ_1, ℓ_2, and ℓ_3 and the other consists of the lines meeting each of ℓ_1, ℓ_2, and ℓ_3. The fourth line ℓ_4 meets Q in two points, and each of these points determines a line in the second family. Thus there are 2 lines μ_1, μ_2 in space that meet general lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4. Figure 10 shows this configuration. Figure 10 also shows how the two lines can be real—if ℓ_4 meets Q in two real points. (The two lines are complex when ℓ_4 meets Q in two complex conjugate points.)

The classical Schubert calculus is a vast generalization of this problem of four lines. In the 1980's Robert Speiser suggested to Fulton that the classical Schubert calculus may

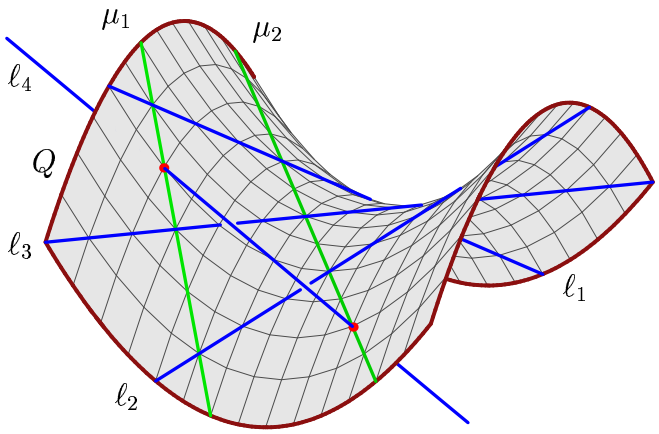


FIGURE 10. The two lines meeting four general lines in space.

be a good testing ground for Question 3.1. This was also considered by Chiavacci and Escamilla-Castillo [**8**]. We will discuss increasingly more general versions of the Schubert calculus, and the status of Question 3.1 for each.

Consider first more general problems involving lines. The space of lines in $\mathbb{P}^n$ is a smooth projective variety of dimension $2n-2$ called the Grassmannian of lines in $\mathbb{P}^n$. The set of lines meeting a linear subspace L of dimension $n-1-l$ has codimension l in the Grassmannian. Thus given general linear subspaces $L_1, L_2, \ldots, L_s$ of $\mathbb{P}^n$ with $\dim L_i = n-1-l_i$ where $l_1 + l_2 + \cdots + l_s = 2n-2$, we expect (and there are indeed) finitely many lines in $\mathbb{P}^n$ meeting each linear subspace $L_1, L_2, \ldots, L_s$. Schubert [**64**] discovered algorithms for computing this number $d(l_1, l_2, \ldots, l_s)$ of lines. For example, if each $l_i = 1$, so that $s = 2n-2$, then this number is the nth Catalan number[†]

$$C_n \;=\; \frac{1}{n}\binom{2n-2}{n-1}. \tag{4.1}$$

Enumerative problems of lines in $\mathbb{P}^n$ meeting general linear subspaces furnished the first infinite family of non-trivial enumerative problems known to be fully real.

THEOREM 4.2 ([**68**, Theorem C]). *Given positive integers $l_1, l_2, \ldots, l_s$ with $l_1 + l_2 + \cdots + l_s = 2n-2$, there exist linear subspaces $L_1, L_2, \ldots, L_s$ of $\mathbb{RP}^n$ with $\dim L_i = n+1-l_i$ such that there are exactly $d(l_1, l_2, \ldots, l_s)$ complex lines meeting each subspace L_i, and each of these lines are real.*

4.1. The special Schubert calculus. More generally, we may ask how many linear subspaces of a fixed dimension meet general linear subspaces. We formulate this question in terms of linear subspaces of a vector space.

The set of k-dimensional subspaces (k-planes) of an n-dimensional vector space forms the *Grassmannian of k-planes in n-space*, $\mathrm{Gr}(k,n)$, a smooth projective variety of dimension $k(n-k)$. Those k-planes meeting a linear subspace L of dimension $n-k+1-l$ non-trivially (that is, the intersection has positive dimension) form the special Schubert subvariety $\Omega(L)$ of $\mathrm{Gr}(k,n)$ which has codimension l. The special Schubert calculus is concerned with the following question.

[†]This indexing of the Catalan numbers is shifted from that of some other authors.

QUESTION 4.3. *Given general linear subspaces $L_1, L_2, \ldots, L_s$ of $\mathbb{C}^n$ with $\dim L_i = n-k+1-l_i$ where $l_1+l_2+\cdots+l_s = k(n-k)$, how many k-planes K meet each subspace L_i non-trivially, that is, satisfy*

$$K \cap L_i \neq \{0\} \qquad i = 1, 2, \ldots, s\,? \tag{4.2}$$

The condition (4.2) is expressed in the global geometry of $\mathrm{Gr}(k,n)$ as the number of points in the intersection of the special Schubert varieties

$$\Omega(L_1) \cap \Omega(L_2) \cap \cdots \cap \Omega(L_s)\,, \tag{4.3}$$

when the intersection is transverse. (A general theorem of Kleiman [**40**] guarantees transversality when the L_i are in general position, and also implies transversality for the other intersections considered in this section.)

There are algorithms due to Schubert [**65**] (when each $l_i = 1$) and Pieri [**55**] to compute the expected number of solutions. When each $l_i = 1$, Schubert [**63**] showed that the number of solutions is equal to

$$d(n,k) \;:=\; \frac{1!\,2!\,\cdots\,(k-1)!\cdot[k\,(n-k)]!}{(n-k)!\,(n-k+1)!\cdots(n-1)!}\,.$$

A line in $\mathbb{P}^n$ is a 2-plane in $(n+1)$-space and two linear subspaces in $\mathbb{P}^n$ meet if and only if the corresponding linear subspaces in $(n+1)$-space have a non-trivial intersection. Thus the problem of lines in projective space corresponds to the case $k=2$ of the special Schubert calculus. While the geometric problem generalizes easily from $k=2$ to arbitrary values of k, the proof of Theorem 4.2 does not. There is, however, a relatively simple argument that this special Schubert calculus is fully real.

THEOREM 4.4 ([**71**, Theorem 1]). *Suppose $n > k > 0$ and $l_1, l_2, \ldots, l_s$ are positive integers with $l_1+l_2+\cdots+l_s = k(n-k)$. Then there are linear subspaces $L_1, L_2, \ldots, L_s$ of $\mathbb{R}^n$ in general position with $\dim L_i = n-k+1-l_i$ such that each of the a priori complex k-planes K satisfying (4.2) are in fact real.*

We present an elementary proof of this result in the important special case when each $l_i = 1$ so that the conditions are simple, meaning each $\Omega(L_i)$ has codimension 1. This proof generalizes to show that some other classes of enumerative problems in the Schubert calculus are fully real (see Sections 4.2.2 and 4.2.3). This generalization constructs sufficiently many real solutions using a limiting argument, as in Section 2.2. Just as the arguments of Section 2.2 were linked to the homotopy algorithms of Huber and Sturmfels, the proof of Theorem 4.4 leads to numerical homotopy methods for solving these problems [**30, 32**].

We develop further geometric properties of Grassmann varieties. The kth exterior power of the embedding $K \to \mathbb{C}^n$ of a k-plane K into $\mathbb{C}^n$ gives the embedding

$$\mathbb{C} \;\simeq\; \wedge^k K \;\longrightarrow\; \wedge^k \mathbb{C}^n\,, \tag{4.4}$$

whose image is a line in $\wedge^k\mathbb{C}^n$ and thus a point in the projective space $\mathbb{P}(\wedge^k\mathbb{C}^n) \simeq \mathbb{P}^{\binom{n}{k}-1}$. This point determines the k-plane K uniquely. The *Plücker embedding* is the resulting projective embedding of the Grassmannian

$$\mathrm{Gr}(k,n) \;\longrightarrow\; \mathbb{P}^{\binom{n}{k}-1}\,.$$

The $\binom{n}{k}$ homogeneous Plücker coordinates for the Grassmannian in this embedding are realized concretely as follows. Represent a k-plane K as the row space of a $k \times n$

matrix, also written K. A maximal minor of $K = (a_{ij})$ is the determinant of a $k \times k$ submatrix of K: Given a choice of columns $\alpha\colon 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$, set

$$p_\alpha(K) \;:=\; \det \begin{bmatrix} a_{1,\alpha_1} & a_{1,\alpha_2} & \cdots & a_{1,\alpha_k} \\ a_{2,\alpha_1} & a_{2,\alpha_2} & \cdots & a_{2,\alpha_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,\alpha_1} & a_{k,\alpha_2} & \cdots & a_{k,\alpha_k} \end{bmatrix} .$$

The vector $(p_\alpha(K))$ of $\binom{n}{k}$ maximal minors of K defines the map (4.4) giving *Plücker coordinates* for K. Let $\binom{[n]}{k}$ be the collection of these indices of Plücker coordinates.

The indices $\binom{[n]}{k}$ have a natural Bruhat order

$$\beta \leq \alpha \iff \beta_j \leq \alpha_j \quad \text{for} \quad j = 1, 2, \ldots, k\,.$$

The Schubert variety Ω_α is

$$\Omega_\alpha \;=\; \{K \in \mathrm{Gr}(k,n) \;\mid\; p_\beta(K) = 0 \text{ for } \beta \not\leq \alpha\}\,. \tag{4.5}$$

This has dimension $|\alpha| := \sum_j (\alpha_j - j)$.

The relevance of the Plücker embedding to Question 4.3 when $l_i = 1$ is seen as follows. Let L be a $(n-k)$-plane, represented as the row space of a $(n-k)$ by n matrix, also written L. Then a general k-plane K meets L non-trivially if and only if

$$\det \begin{bmatrix} K \\ L \end{bmatrix} \;=\; 0\,.$$

Laplace expansion along the rows of K gives

$$0 \;=\; \det \begin{bmatrix} K \\ L \end{bmatrix} \;=\; \sum_{\alpha \in \binom{[n]}{k}} p_\alpha(K) L_\alpha\,, \tag{4.6}$$

where L_α is the appropriately signed minor of L given by the columns complementary to α. Hence the set $\Omega(L)$ of k-planes that meet the $(n-k)$-plane L non-trivially is a hyperplane section of the Grassmannian in its Plücker embedding.

Thus the set of k-planes meeting $k(n-k)$ general $(n-k)$-planes non-trivially is a complementary linear section of the Grassmannian, and so the number $d(k,n)$ of such k-planes is the degree of the Grassmannian in its Plücker embedding. More generally, if $\alpha \in \binom{[n]}{k}$ and $L_1, L_2, \ldots, L_{|\alpha|}$ are general $(n-k)$-planes, then the number of points in the intersection

$$\Omega_\alpha \cap \Omega(L_1) \cap \Omega(L_2) \cap \cdots \cap \Omega(L_{|\alpha|}) \tag{4.7}$$

is the degree $d(\alpha)$ of the Schubert variety Ω_α, which we now compute.

An intersection $X \cap Y$ is *generically transverse* if X and Y meet transversally along an open subset of every component of $X \cap Y$. When $\beta, \alpha \in \binom{[n]}{k}$ satisfy $\beta < \alpha$ but there is no index γ with $\beta < \gamma < \alpha$, then we write $\beta \lessdot \alpha$. The following fact is elementary and due to Schubert.

THEOREM 4.5. *Let $\alpha \in \binom{[n]}{k}$ and set H_α to be the hyperplane defined by $p_\alpha = 0$. Then*

$$\Omega_\alpha \cap H_\alpha \;=\; \bigcup_{\beta \lessdot \alpha} \Omega_\beta\,,$$

and the intersection is generically transverse.

In fact this intersection is transverse along $\Omega_\beta^\circ := \Omega_\beta - \bigcup_{\delta<\beta}\Omega_\delta$. We obtain the recursion for the degree $d(\alpha)$ of the Schubert variety Ω_α

$$d(\alpha) \;=\; \sum_{\beta\lessdot\alpha} d(\beta)\,.$$

Since the minimal Schubert variety is a point (which has degree 1), this gives a conceptual formula for $d(\alpha)$. Let $\hat{0} = (1,2,\ldots,k)$ be the minimal element in the Bruhat order.

(4.8) $\qquad d(\alpha) \;=\;$ the number of saturated chains in the Bruhat order from $\hat{0}$ to α.

Figure 11 displays both the Bruhat order for $k=3$ and $n=6$ (on the left) and the degrees of the corresponding Schubert varieties (on the right).

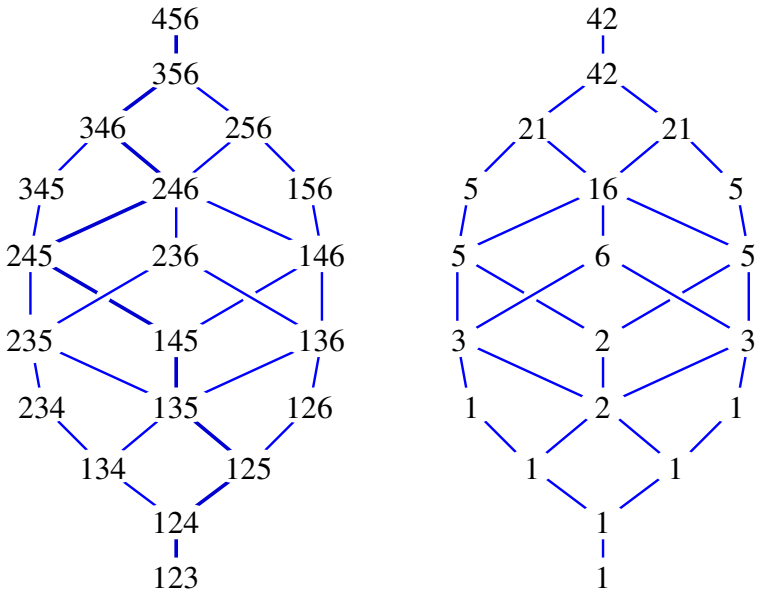


FIGURE 11. Bruhat order and degrees of Schubert varieties, $k=3$, $n=6$

Consider the action of the non-zero real numbers $\mathbb{R}^\times$ on $\mathbb{R}^n$

(4.9) $$t.e_j \;:=\; t^j\cdot e_j\,,$$

where $t\in\mathbb{R}^\times$ and $e_1,e_2,\ldots,e_n$ is a basis for $\mathbb{R}^n$ (corresponding to the rows of the $n\times n$ identity matrix). Let L be a $(n-k)$-plane. By (4.6), the equation for a k-plane K to meet $t.L$ non-trivially is

$$0 \;=\; \sum_\beta t^{\binom{n+1}{2}-|\beta|}L_\beta\, p_\beta(K)\,.$$

For $K\in\Omega_\alpha$, the sum is over $\beta\le\alpha$ by (4.5). Removing the common factor $t^{\binom{n+1}{2}-|\alpha|}$ gives

(4.10) $$0 \;=\; \sum_{\beta\le\alpha} t^{|\alpha|-|\beta|}L_\beta\, p_\beta(K)\,.$$

The case $l_i=1$ of Theorem 4.4 is implied by the case $\alpha=(n{-}k{+}1,\ldots,n{-}1,n)=\hat{1}$ ($\Omega_{\hat{1}}=\mathrm{Gr}(k,n)$) of the following theorem.

THEOREM 4.6 ([**76**, Theorem 4.2]). *Let $L \subset \mathbb{R}^n$ be a $(n-k)$-plane, none of whose Plücker coordinates vanishes. Then there exist real numbers $t_1, t_2, \ldots, t_{k(n-k)} \in \mathbb{R}^\times$ such that for every $\alpha \in \binom{[n]}{k}$ the intersection*

$$\Omega_\alpha \cap \Omega(t_1.L) \cap \Omega(t_2.L) \cap \cdots \cap \Omega(t_{|\alpha|}.L)$$

is transverse (so it contains $d(\alpha)$ points) with all points real.

PROOF. We induct on m to construct numbers $t_1, t_2, \ldots, t_{k(n-k)} \in \mathbb{R}^\times$ having the property that, for all $\alpha \in \binom{[n]}{k}$ with $|\alpha| = m$,

$$\Omega_\alpha \cap \Omega(t_1.L) \cap \Omega(t_2.L) \cap \cdots \cap \Omega(t_m.L)$$

is transverse (over $\mathbb{C}$) and each of its $d(\alpha)$ points are real.

The case $m = 0$ is trivial, as $|\alpha| = 0$ implies that $\alpha = \hat{0}$ and $\Omega_{\hat{0}}$ is a single (real) point in $\mathrm{Gr}(k,n)$. Suppose we have constructed $t_1, t_2, \ldots, t_m \in \mathbb{R}^\times$ with the above properties.

Let $\alpha \in \binom{[n]}{k}$ with $|\alpha| = m+1$ and consider

$$\left(\sum_{\beta \lessdot \alpha} \Omega_\beta\right) \cap (\Omega(t_1.L) \cap \cdots \cap \Omega(t_m.L)) \;=\; \sum_{\beta \lessdot \alpha} \Omega_\beta \cap \Omega(t_1.L) \cap \cdots \cap \Omega(t_m.L)\,. \tag{4.11}$$

Each summand in the second sum is transverse (over $\mathbb{C}$) and consists of $d(\beta)$ real points. The intersection on the left will be transverse and consist of $d(\alpha) = \sum_{\beta \lessdot \alpha} d(\beta)$ real points only if no two summands in the second sum share a point in common.

If two summands β, γ with $\beta \neq \gamma$ share a point, then

$$\Omega_\beta \cap \Omega_\gamma \cap \Omega(t_1.L) \cap \Omega(t_2.L) \cap \cdots \cap \Omega(t_m.L) \tag{4.12}$$

is non-empty. By (4.5), $\Omega_\beta \cap \Omega_\gamma$ is contained in a union of Schubert varieties of dimension less than m. By (4.10) the condition for a fixed k-plane $K \in \Omega_\delta$ to meet $t.L$ non-trivially is a polynomial of degree at most $|\delta|$ in t. Thus a fixed k-plane $K \in \Omega_\delta$ lies in at most $|\delta|$ distinct $(n-k)$-planes in the family $t.L$. Hence the intersection (4.12) is empty, and so the summands in the second sum of (4.11) are disjoint. This argument also shows that the points in the summand indexed by β lie in $\Omega_\beta^\circ = \Omega_\beta - \bigcup_{\delta < \beta} \Omega_\delta$.

For $t \in \mathbb{R}$, let $Z_t \subset \Omega_\alpha$ be the set of k-planes K satisfying the polynomial (4.10). For $t \neq 0$, we have $Z_t = \Omega_\alpha \cap \Omega(t.L)$. Since L has no vanishing Plücker coordinates, the constant term of that polynomial is $p_\alpha(K)$, and so by Theorem 4.5,

$$Z_0 \;=\; \Omega_\alpha \cap H_\alpha \;=\; \sum_{\beta \lessdot \alpha} \Omega_\beta\,.$$

Thus Z_0 meets

$$\Omega(t_1.L) \cap \Omega(t_2.L) \cap \cdots \cap \Omega(t_m.L) \tag{4.13}$$

transversally (over $\mathbb{C}$) in $d(\alpha)$ real points. We see that there exists $\epsilon_\alpha > 0$ such that if $0 < t \leq \epsilon_\alpha$, then Z_t meets (4.13) transversally (over $\mathbb{C}$) in $d(\alpha)$ real points. Since the intersection defining Z_0 is transverse along each Ω_β° for $\beta \lessdot \alpha$, we may assume that for $0 < t \leq \epsilon_\alpha$,

$$\Omega_\alpha \cap \Omega(t.L) \cap \Omega(t_1.L) \cap \cdots \cap \Omega(t_m.L)$$

is transverse (over $\mathbb{C}$) and consists of $d(\alpha)$ real points.

Let $t_{m+1} = \min\{\epsilon_\alpha \colon |\alpha| = m+1\}$. Then for any $\alpha \in \binom{[n]}{k}$ with $|\alpha| = m+1$,

$$\Omega_\alpha \cap \Omega(t_1.L) \cap \cdots \cap \Omega(t_m.L) \cap \Omega(t_{m+1}.L)$$

is transverse (over $\mathbb{C}$) and consists of $d(\alpha)$ real points. $\square$

4.2. Further extensions of the Schubert calculus. This special Schubert calculus admits further extensions, some of which are known to be fully real. Interestingly, some problems involving the Lagrangian Grassmannian 4.2.4 are known to be *fully unreal*, a new phenomenon. Further investigation is encouraged; in particular, the Lagrangian Schubert calculus may yield the first class of enumerative which are *not* fully real.

4.2.1. *General Schubert calculus.* A *flag* in n-space is a sequence of linear subspaces $F_\bullet\colon F_1 \subset F_2 \subset \cdots \subset F_n$ with $\dim F_i = i$. Given $\alpha \in \binom{[n]}{k}$, the Schubert condition of type α on a k-plane K imposed by the flag $F_\bullet$ is

$$\dim K \cap F_{n+1-\alpha_j} \;\geq\; k+1-j \qquad \text{for} \quad j = 1, 2, \ldots, n\,. \tag{4.14}$$

The Schubert variety $X_\alpha F_\bullet \subset \mathrm{Gr}(k,n)$ is the set of all k-planes K satisfying (4.14).

We relate this to the definitions of Section 4.1. Let $e_1, e_2, \ldots, e_n$ be a basis for $\mathbb{R}^n$ and for its complexification $\mathbb{C}^n$. Defining F_i to be the linear span of $e_1, e_2, \ldots, e_i$ gives the flag $F_\bullet$. Then the Schubert variety $X_\alpha F_\bullet$ is $\Omega_{\alpha^\vee}$, where $\alpha^\vee_j = n+1-\alpha_{k+1-j}$ for each j, and so the codimension of $X_\alpha F_\bullet$ is $|\alpha|$. A special Schubert condition is when $\alpha = (1, \ldots, k{-}1, k{+}l)$ so that $X_\alpha F_\bullet = \Omega(F_{n-k+1-l})$.

The general problem of the classical Schubert calculus of enumerative geometry asks, given $\alpha^1, \alpha^2, \ldots, \alpha^s \in \binom{[n]}{k}$ with $|\alpha^1| + |\alpha^2| + \cdots + |\alpha^s| = k(n-k)$ and general flags $F_\bullet{}^1, F_\bullet{}^2, \ldots, F_\bullet{}^s \subset \mathbb{C}^n$, how many points are there in the intersection[†]

$$X_{\alpha^1} F_\bullet{}^1 \cap X_{\alpha^2} F_\bullet{}^2 \cap \cdots \cap X_{\alpha^s} F_\bullet{}^s\,? \tag{4.15}$$

There are algorithms due to Pieri [**55**] and Giambelli [**21**] to compute these numbers. Other than the case when the α^i are indices of special Schubert varieties, it remains open whether the general Schubert calculus is fully real. (See [**73**] and [**70**] for some cases.)[§]

4.2.2. *Quantum Schubert calculus.* The space $\mathcal{M}^q_{k,n}$ of degree q maps $M\colon \mathbb{P}^1 \to \mathrm{Gr}(k,n)$ is a smooth quasi-projective variety [**9**]. A point $t \in \mathbb{P}^1$ and a Schubert variety $X_\alpha F_\bullet$ together impose a quantum Schubert condition on maps $M \in \mathcal{M}^q_{k,n}$,

$$M(t) \;\in\; X_\alpha F_\bullet\,.$$

The set of such maps has codimension $|\alpha|$ in $\mathcal{M}^q_{k,n}$. The quantum Schubert calculus of enumerative geometry asks the following question.

QUESTION 4.7. *Given $\alpha^1, \alpha^2, \ldots, \alpha^s \in \binom{[n]}{k}$ with $|\alpha^1| + |\alpha^2| + \cdots + |\alpha^s| = \dim \mathcal{M}^q_{k,n} = qn + k(n-k)$, how many maps $M \in \mathcal{M}^q_{k,n}$ satisfy*

$$M(t_i) \;\in\; X_{\alpha^i} F_\bullet{}^i \qquad \textit{for } i = 1, 2, \ldots, s\,,$$

where $t_1, t_2, \ldots, t_s \in \mathbb{P}^1$ are general and $F_\bullet{}^1, F_\bullet{}^2, \ldots, F_\bullet{}^s$ are general flags?

Algorithms to compute this number were proposed by Vafa [**82**] and Intriligator [**34**] and proven by Siebert and Tian [**66**] and by Bertram [**3**]. A simple quantum Schubert condition defines a subvariety of codimension 1,

$$M(t) \;\cap\; L \;\neq\; \{0\}\,, \tag{4.16}$$

[†]In this survey flags are general when the corresponding intersection is transverse.

[§]While this survey was in review, Ravi Vakil communicated to the author a proof that the classical Schubert calculus is fully real [**83**].

where L is a $(n-k)$-plane. Let $d(q;k,n)$ be the number of maps $M \in \mathcal{M}^q_{k,n}$ satisfying $\dim \mathcal{M}^q_{k,n}$–many general simple quantum Schubert conditions (4.16). A combinatorial formula for this number was given by Ravi, Rosenthal, and Wang [**59**]. For a survey of this particular enumerative problem and its importance to linear systems theory, see [**75**].

THEOREM 4.8 ([**72**, Theorem 1.1]). *Let $q \geq 0$ and $n > k > 0$ and set $N := \dim \mathcal{M}_{k,n}$. Then there exist $t_1, t_2, \ldots, t_N \in \mathbb{RP}^1$ and $(n-k)$-planes $L_1, L_2, \ldots, L_N \subset \mathbb{R}^n$ such that there are exactly $d(q;k,n)$ maps $M \in \mathcal{M}^q_{k,n}(\mathbb{C})$ satisfying*

$$M(t_i) \cap L_i \ \neq \ \{0\} \qquad \text{for } i = 1, 2, \ldots, N\,,$$

and all of them are real.

As with the classical Schubert calculus, the question of whether the general quantum Schubert calculus is fully real remains open.

4.2.3. *Schubert calculus of flags.* Let $\mathbf{a} := 0 < a_1 < \cdots < a_r < a_{r+1} = n$ be a sequence of integers. The manifold of partial flags in n-space (or the flag manifold) $\mathbb{F}\ell_{\mathbf{a}}$ is the collection of partial flags of subspaces

$$E_\bullet \ : \ E_{a_1} \subset E_{a_2} \subset \cdots \subset E_{a_r}$$

in n-space, where $\dim E_{a_i} = a_i$. A complete flag $F_\bullet$ is when $\mathbf{a} = 1, 2, \ldots, n-1$.

The Schubert varieties of $\mathbb{F}\ell_{\mathbf{a}}$ are indexed by permutations $w \in \mathcal{S}_n$, the symmetric group on n letters, whose descents only occur at positions in $\mathbf{a}$. That is, $w(i) > w(i+1)$ implies that $i \in \{a_1, a_2, \ldots, a_r\}$. Let $I_{\mathbf{a}}$ be this set of indices. For a complete flag $F_\bullet$ and $w \in I_{\mathbf{a}}$, $\mathbb{F}\ell_{\mathbf{a}}$ has a Schubert variety

$$X_w F_\bullet \ := \ \left\{ E_\bullet \in \mathbb{F}\ell_{\mathbf{a}} \mid \dim E_{a_j} \cap F_i \geq \#\{l \leq a_j \mid w(l) \leq i\} \right\}\,.$$

The codimension of $X_w F_\bullet$ is $\ell(w) := \#\{i < j \mid w(i) > w(j)\}$. We state the general question in the Schubert calculus of enumerative geometry for flags.

QUESTION 4.9. *Given permutations $w_1, w_2, \ldots, w_s \in I_{\mathbf{a}}$ with $\ell(w_1) + \ell(w_2) + \cdots + \ell(w_s) = \dim \mathbb{F}\ell_{\mathbf{a}}$ and general flags $F_\bullet^{\,1}, F_\bullet^{\,2}, \ldots, F_\bullet^{\,s}$, what is the number of points in the intersection*

$$X_{w_1} F_\bullet^{\,1} \cap X_{w_2} F_\bullet^{\,2} \cap \cdots \cap X_{w_s} F_\bullet^{\,s} \ ?$$

There are algorithms [**2, 13**] for computing this number and the numbers for the Lagrangian Schubert calculus in Section 4.2.4. When $w = (i, i+1)$, $X_w F_\bullet$ is the simple Schubert variety, written $X_i F_\bullet$,

$$X_i F_\bullet \ = \ X_{(i,i+1)} F_\bullet \ = \ \{E_\bullet \in \mathbb{F}\ell_n \mid E_i \cap F_{n-i} \neq \{0\}\}\,. \tag{4.17}$$

THEOREM 4.10 ([**74**, Corollary 2.2]). *Given a list $i_1, i_2, \ldots, i_N$ ($N = \dim \mathbb{F}\ell_{\mathbf{a}}$) of numbers with $i_j \in \mathbf{a}$, there exist real flags $F_\bullet^{\,1}, F_\bullet^{\,2}, \ldots, F_\bullet^{\,N}$ such that the intersection of simple Schubert varieties*

$$X_{i_1} F_\bullet^{\,1} \cap X_{i_2} F_\bullet^{\,2} \cap \cdots \cap X_{i_N} F_\bullet^{\,N}$$

is transverse and consists only of real flags.

The case of $\mathbf{a} = 2 < n-2$ of this theorem was proven earlier [**69**, Theorem 13]. It remains open whether the general Schubert calculus of flags is fully real.

4.2.4. *Unreality in the Lagrangian Schubert calculus.* Let V be a $2n$-dimensional vector space equipped with the alternating bilinear form

$$\left\langle \sum x_i e_i \,,\ \sum y_j e_j \right\rangle \ := \ \sum_{i=1}^{n} x_i y_{2n+1-i} - y_i x_{2n+1-i} \,. \tag{4.18}$$

A subspace $H \subset V$ is *isotropic* if the restriction of the form to H is identically zero, $\langle H, H\rangle \equiv 0$. The dimension of an isotropic subspace is at most n and *Lagrangian subspaces* are isotropic subspaces with this maximal dimension. The Lagrangian Grassmannian $LG(n)$ is the set of Lagrangian subspaces of V, an algebraic manifold of dimension $\binom{n+1}{2}$.

A flag $F_\bullet$ in V is *isotropic* if F_n is Lagrangian and $\langle F_i, F_{2n-i}\rangle \equiv 0$ for all $i = 1, 2, \ldots, 2n-1$. Given an isotropic flag $F_\bullet$, the Lagrangian Grassmannian has Schubert varieties $\Psi_\lambda F_\bullet$ indexed by decreasing sequences $\lambda\colon n \geq \lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$ of positive integers, called strict partitions. (Here l can be any integer between 0 and n). The codimension of $\Psi_\lambda F_\bullet$ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. The Lagrangian Schubert calculus asks for the number of points in a transverse zero-dimensional intersection of Schubert varieties.

The *simple Schubert variety* $\Psi_1 F_\bullet$ consists of those Lagrangian subspaces meeting the Lagrangian subspace F_n non-trivially. The simple Lagrangian Schubert calculus is fully unreal.

THEOREM 4.11. *There exist isotropic real flags* $F_\bullet^{\,1}, F_\bullet^{\,2}, \ldots, F_\bullet^{\,\binom{n+1}{2}}$ *such that the intersection of Schubert varieties*

$$\Psi_1 F_\bullet^{\,1} \cap \Psi_1 F_\bullet^{\,2} \cap \cdots \cap \Psi_1 F_\bullet^{\,\binom{n+1}{2}}$$

is transverse with **no** *real points.*

REMARK 4.12. In [**74**, Theorem 4.2] flags were given so that the intersection had no real points, and it was not known if that intersection was transverse. Perturbing those flags slightly (so that the intersection becomes transverse) gives the above result.

We do not know if these (or many other) enumerative problems in the Lagrangian Schubert calculus are fully real. Experimentation[†] suggests that the situation is complicated. Briefly, while many other enumerative problems in the Lagrangian Schubert calculus are fully unreal, there are a few which are fully real. For example, there exist 2 real isotropic 2-planes and 2 real isotropic 3-planes such that all 4 Lagrangian subspaces meeting each of these are real (see Theorem 5.13).

The problems of Theorem 4.11 may give examples of enumerative problems that are not fully real. Experimental evidence suggests however that this may be unlikely. For example, (case $n = 3$ of Theorem 4.11) there are 16 Lagrangian subspaces in $\mathbb{C}^6$ having non-trivial intersection with 6 general Lagrangian subspaces. We computed 30,000 random instances of this enumerative problem, and found two examples of 6 real Lagrangian subspaces such that all 16 Lagrangian subspaces meeting them are real. Table 1 shows the number of these 30,000 systems having a given number of real solutions.

[†]This will be reported in the forthcoming article [**77**].

Number of real solutions	0	2	4	6	8	10	12	14	16
Frequency	4983	8176	9314	5027	1978	445	67	8	2

TABLE 1. Frequency with given number of real solutions.

5. The Conjecture of Shapiro and Shapiro

The results of Section 4 were inspired by a remarkable conjecture of Boris Shapiro and Michael Shapiro. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be the rational normal curve

$$\gamma(t) \;=\; (1,\, t,\, t^2,\, \ldots,\, t^{n-1}) \;=\; \sum_{i=1}^{n} t^{i-1} e_i\,.$$

For $t \in \mathbb{R}$, define the flag $F_\bullet(t)$ by any of the three equivalent ways

$$F_i(t) \;=\; \text{linear span of } \gamma(t),\, \frac{d}{dt}\gamma(t),\, \ldots,\, \frac{d^{i-1}}{dt^{i-1}}\gamma(t)\,, \tag{5.1}$$

$$\;=\; \text{row space} \begin{bmatrix} 1 & t & t^2 & & \cdots & & t^{n-1} \\ 0 & 1 & 2t & & \cdots & & (n{-}1)t^{n-2} \\ \vdots & & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 0 & 1 & it & \cdots & \binom{n-1}{i-1} t^{n-i} \end{bmatrix}, \tag{5.2}$$

$$\;=\; i\text{-plane osculating rational normal curve } \gamma \text{ at } \gamma(t)\,. \tag{5.3}$$

This makes sense for $t \in \mathbb{C}$ and is extended to $t = \infty \in \mathbb{P}^1$ by setting $F_i(\infty)$ to be the row space of the last i rows of the $n \times n$ identity matrix.

CONJECTURE 5.1 (Shapiro-Shapiro). *Let $\alpha^1, \alpha^2, \ldots, \alpha^s \in \binom{[n]}{k}$ be such that $|\alpha^1| + |\alpha^2| + \ldots + |\alpha^s| = k(n-k)$. Then, for every distinct $t_1, t_2, \ldots, t_s \in \mathbb{R}$, the intersection of Schubert varieties*

$$X_{\alpha^1} F_\bullet(t_1) \cap X_{\alpha^2} F_\bullet(t_2) \cap \cdots \cap X_{\alpha^s} F_\bullet(t_s) \tag{5.4}$$

is (a) *transverse, and* (b) *consists only of real points.*

Eisenbud's and Harris's dimensional transversality result [**15**, Theorem 2.3] guarantees that the intersection (5.4) is zero-dimensional. Not only does Conjecture 5.1 state that the classical Schubert calculus is fully real, but it also proposes flags witnessing this full reality. This conjecture has been central to subsequent developments in the real Schubert calculus and it has direct connections to other parts of mathematics, including linear systems theory and linear series on $\mathbb{P}^1$ (see Remark 5.8). The article [**73**] and the web page [**70**] give a more complete discussion.

One aspect of this conjecture which we relate is the following.

THEOREM 5.2 ([**73**, Theorem 3.3]). *For a given k and n, the general case of Conjecture 5.1 follows from the special case when each Schubert condition is simple, that is, when each $\alpha^i = (1, 2, \ldots, k{-}1, k{+}1)$.*

Consider these osculating flags $F_\bullet(t)$ in more detail. If the ith row of the matrix (5.2) is multiplied by t^i (which does not affect its row space when $t \neq 0$), then the entry in

position i,j is $\binom{j-1}{i-1}t^j$, and so we have

$$F_i(t) \;=\; t.F_i(0)\,,$$

where $t.F_i(0)$ is given by the action (4.9) of $\mathbb{R}^\times$ on $\mathbb{R}^n$. The αth Plücker coordinate of $F_i(0)$ is

$$p_\alpha(F_i(0)) \;=\; \prod_{j<l} \frac{\alpha_j-\alpha_l}{j-l}\,, \tag{5.5}$$

which is non-vanishing. Thus Theorem 4.6 has the following corollary.

THEOREM 5.3 ([**71**, Theorem 1]). *There exist $t_1,t_2,\ldots,t_{k(n-k)}\in\mathbb{R}$ such that there are exactly $d(k,n)$ k-planes meeting each $(n-k)$-plane $F_{n-k}(t_i)$ non-trivially, and all are real. Equivalently, if $\alpha=(1,2,\ldots,k-1,k+1)$ so that $|\alpha|=1$, then the intersection of Schubert varieties*

$$X_\alpha F_\bullet(t_1)\cap X_\alpha F_\bullet(t_2)\cap\cdots\cap X_\alpha F_\bullet(t_{k(n-k)}) \tag{5.6}$$

is transverse with all points real.

This establishes a weak form of Conjecture 5.1 for simple Schubert conditions, replacing the quantifier *for all* $t_i\in\mathbb{R}$ by *there exists* $t_i\in\mathbb{R}$.

If the parameters t_i in (5.6) vary, then the number of real points in that intersection could change, but only if two points first collide (prior to spawning a complex conjugate pair of solutions). This is the reverse of the progression in Dietmaier's algorithm, as displayed in Figure 6. This situation cannot occur if the intersection remains transverse. Together with Theorems 5.2 and 5.3 we deduce the following result.

THEOREM 5.4 ([**71**, Theorem 6]). *Part* (a) *of Conjecture 5.1 for simple conditions implies part* (b) *for arbitrary Schubert conditions.*

If $t>0$, then by (5.5) the Plücker coordinates of $F_i(t)$ are strictly positive. An upper triangular $n\times n$ matrix g is *totally positive* if every $i\times i$ subdeterminant of g is non-negative, and vanishes only if that subdeterminant vanishes on all upper triangular matrices. For example, when $t>0$ and $i=n$, the matrix (5.2) is totally positive. Write $\Gamma(t)$ for this matrix. It has the form $e^{t\eta}$ where η is the principal nilpotent matrix $\Gamma'(0)$. Observe that if $t_1<t_2<\cdots<t_s$, then

$$F_\bullet(t_i) \;=\; \Gamma(t_i-t_{i-1})\cdot F_\bullet(t_{i-1})$$

Conjecture 5.1 has a more general version involving totally positive matrices.

CONJECTURE 5.5 (Shapiro-Shapiro [**73**, Conjecture 4.1]). *Let $\alpha^1,\alpha^2,\ldots,\alpha^s\in\binom{[n]}{k}$ be such that $|\alpha^1|+|\alpha^2|+\ldots+|\alpha^s|=k(n-k)$ and suppose $g_2,g_3,\ldots,g_s$ are totally positive matrices. Given any real flag $F_\bullet{}^1$, define $F_\bullet{}^i$ for $i>1$ by $F_\bullet{}^i:=g_i\cdot F_\bullet{}^{i-1}$, the intersection of Schubert varieties*

$$X_{\alpha^1}F_\bullet{}^1\;\cap\; X_{\alpha^2}F_\bullet{}^2\;\cap\;\cdots\;\cap\; X_{\alpha^s}F_\bullet{}^s$$

is transverse with all points real.

There is some experimental evidence for this version of Conjecture 5.1. Subsequent conjectures involving osculating flags have versions involving totally positive matrices. We leave their statements to the reader, they will be explored more fully in [**77**].

5.1. Rational functions with real critical points. By far the strongest evidence for Conjecture 5.1 is that it is true when k or $n-k$ is equal to 2.

THEOREM 5.6 (Eremenko and Gabrielov [**16**]). *Conjecture 5.1 is true when one of k or $n-k$ is 2.*

This is a consequence of a theorem about rational functions with real critical points. A rational function is an algebraic map $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$. Two rational functions φ_1 and φ_2 are equivalent if $\varphi_1 = g \circ \varphi_2$, where g is a fractional linear transformation.

THEOREM 5.7 ([**16**]). *If all the critical points of a rational function are real, then it is equivalent to a real rational function.*

Consider the composition

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d \dashrightarrow \mathbb{P}^1\,, \tag{5.7}$$

where the first map is the rational normal curve

$$\gamma\,:\,[s,t] \longmapsto [s^d,\, s^{d-1}t,\, \ldots,\, st^{d-1},\, t^d]\,,$$

and the second is a linear projection

$$[x_0, x_1, \ldots, x_d] \longmapsto [\Lambda_1(x), \Lambda_2(x)]\,,$$

where Λ_1 and Λ_2 are independent linear forms. Let $E \subset \mathbb{P}^d$ be the center of this projection, the linear subspace where $\Lambda_1 = \Lambda_2 = 0$. When E is disjoint from the rational normal curve, this composition defines a rational function of degree d, and all such rational functions occur in this manner. In fact, equivalence classes of rational maps are exactly those maps with the same center of projection. A rational function φ has a critical point at $t \in \mathbb{P}^1$ if $d\varphi$ vanishes at t. If we consider the composition (5.7), this implies that the center E meets the line tangent to the rational normal curve $\gamma(\mathbb{P}^1)$ at $\gamma(t)$.

Goldberg [**22**] asked (and answered) the question: how many equivalence classes of rational functions of degree d have a given set of $2d-2$ critical points? Reasoning as above, she reduced this to the problem of determining the number of codimension 2 planes E in $\mathbb{P}^d$ meet $2d-2$ given tangents to the rational normal curve. Formulating this in the dual projective space, we recover the problem of the Introduction to Section 4: Determine the number of lines in $\mathbb{P}^d$ meeting $2d-2$ general codimension 2 planes. The answer is the dth Catalan number, $C_d = \frac{1}{d}\binom{2d-1}{d-1}$ (4.1).

Consider the above description in the affine cone over $\mathbb{P}^d$. The centers $E \in \mathrm{Gr}(d-1, d+1)$ giving a rational function of degree d with critical point at $t \in \mathbb{C} \subset \mathbb{P}^1$ are points in the Schubert variety $X_\alpha F_\bullet(t)$, where α is the simple Schubert condition of Theorem 5.2 and $F_\bullet(t)$ is the flag of subspaces osculating the rational normal curve $\gamma(\mathbb{P}^1)$ at $\gamma(t)$. An equivalence class of rational functions contains a real rational function when the common center E is real. In this way, Theorem 5.7 implies Conjecture 5.1 when $n-k=2$ and each $|\alpha^i| = 1$. By Theorem 5.2, this implies the full conjecture when $n-k=2$. Working in the dual space, we deduce Conjecture 5.1 when $k=2$.

Gabrielov and Eremenko prove Theorem 5.7 by showing there exist C_d distinct real rational functions with critical points at a given set of $2d-2$ real numbers. Let $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a real rational function of degree d with only real critical points. Then $\varphi^{-1}(\mathbb{P}^1_{\mathbb{R}}) \subset \mathbb{P}^1_{\mathbb{C}}$ is an embedded graph containing $\mathbb{P}^1_{\mathbb{R}}$ which is stable under complex conjugation and whose vertices have valence 4 and are at the critical points. There are exactly C_d isotopy classes

of such graphs. For each isotopy class Γ and collection of $2d-2$ distinct real points, they essentially construct a rational function φ of degree d having these critical points with $\varphi^{-1}(\mathbb{P}^1_{\mathbb{R}}) \in \Gamma$. A key point involves degenerate rational maps with fewer than $2d-2$ critical points. It may be interesting to relate this to the degenerations in the proof of Theorem 4.6.

REMARK 5.8. If we project the rational normal curve from a center E of codimension $m+1$, then the image of the projection is $\mathbb{P}^m$ and the composition is a parameterized rational curve in $\mathbb{P}^m$. Post-composition by an element of $\mathrm{Aut}(\mathbb{P}^m)$ defines an equivalence relation on such maps and equivalence classes are determined by the centers of projection. These centers are codimension $m+1$ linear series of degree d on $\mathbb{P}^1$. Such a linear series is *ramified* at a point $t \in \mathbb{P}^1$ when the center E meets the m-plane osculating the rational normal curve. A rational curve/linear series is *maximally inflected* if the ramification is at real points. We just considered the case $d = 1$ of rational functions with real critical points, and the case $d = 2$ was discussed in Section 3.1. The existence of maximally inflected curves with simple ramification is a consequence of Theorem 5.3, and Conjecture 5.1 predicts this is a rich class of real curves in $\mathbb{P}^m$. This connection between linear series and the Schubert calculus originated in work of Castelnuovo [**7**].

5.2. Generalizations of Conjecture 5.1. The Grassmannian, flag manifolds, and Lagrangian Grassmannian are examples of flag varieties G/P where G is a reductive algebraic group and P a parabolic subgroup. These flag varieties have Schubert varieties and the most general form of the Schubert calculus involves zero-dimensional intersections of these Schubert varieties. Likewise, these flag varieties have real forms (given by split[†] forms of G and P) and there is a generalization of Conjecture 5.1 for these real forms. This generalization is false, but in a very interesting way. We describe what is known about this general conjecture for the flag manifolds and the Lagrangian Grassmannian, and give conjectures describing what we believe to be true.

5.2.1. *The manifold of partial flags.* Let $\mathbf{a} := 0 < a_1 < a_2 < \cdots < a_r < a_{r+1} = n$ be a sequence of integers. The straightforward generalization (which was its original form and which is false) of Conjecture 5.1 for the flag manifold $\mathbb{F}\ell_{\mathbf{a}}$ declares that a zero-dimensional intersection of Schubert varieties given by flags osculating the real rational normal curve consists only of real points.

For $\alpha \in \binom{[n]}{a_i}$, we have the *Grassmannian Schubert variety*

$$Y_{\alpha,i}F_\bullet \ := \ \{E_\bullet \in \mathbb{F}\ell_{\mathbf{a}} \mid E_{a_i} \in X_\alpha F_\bullet \subset \mathrm{Gr}(a_i, n)\}\,.$$

For example, if $|\alpha| = 1$, then $Y_{\alpha,i}F_\bullet$ is the simple Schubert variety $X_i F_\bullet$ of Section 4.2.3.

EXAMPLE 5.9. Let $\mathbf{a} = 2 < 3 < 5 = n$ so that $\mathbb{F}\ell_{\mathbf{a}}$ is the manifold of partial flags $E_2 \subset E_3$ in 5-space. This flag manifold has two simple Schubert varieties

$$\begin{aligned} X_2F_\bullet &= \{E_2 \subset E_3 \mid E_2 \cap F_3 \neq \{0\}\} \\ X_3F_\bullet &= \{E_2 \subset E_3 \mid E_3 \cap F_2 \neq \{0\}\} \end{aligned}$$

A calculation [**74**, Example 2.5] shows that

$$\begin{aligned} X_2F_\bullet(-8) \cap X_3F_\bullet(-4) \cap X_2F_\bullet(-2) \cap X_3F_\bullet(-1) \cap \\ X_2F_\bullet(1) \cap X_3F_\bullet(2) \cap X_2F_\bullet(4) \cap X_3F_\bullet(8) \end{aligned} \tag{5.8}$$

[†]Split is a technical term: $\mathbb{R}^\times \subset \mathbb{C}^\times$ is a split form of GL_1, but $S^1 \subset \mathbb{C}^\times$ is not.

is transverse with *none* of its 12 points real.

Thus the straightforward generalization of Conjecture 5.1 is completely false. On the other hand, if $t_1 < t_2 < \cdots < t_8$ are any of the 24,310 subsets of eight numbers from

$$\{-6,-5,-4,-3,-2,-1,1,2,3,5,7,11,13,17,19,23,29\}\,,$$

then the intersection

$$\begin{aligned} X_2F_\bullet(t_1) \cap X_2F_\bullet(t_2) \cap X_2F_\bullet(t_3) \cap X_2F_\bullet(t_4) \\ \cap X_3F_\bullet(t_5) \cap X_3F_\bullet(t_6) \cap X_3F_\bullet(t_7) \cap X_3F_\bullet(t_8) \end{aligned} \tag{5.9}$$

is transverse with *all* of its 12 points real [**74**, Example 2.5].

We have used more than 2×10^7 seconds of CPU time investigating this problem of intersections of Schubert varieties in manifolds of partial flags given by flags osculating the rational normal curve, and a picture is emerging of what to expect, at least for Grassmannian Schubert varieties.

Suppose we have a list of indices of Grassmannian Schubert varieties

$$(\alpha^1, i_1),\ (\alpha^2, i_2),\ \ldots,\ (\alpha^s, i_s), \qquad \text{with} \quad \alpha^j \in \binom{[n]}{i_j}$$

where $i_j \in \{a_1, \ldots, a_r\}$ and $|\alpha^1|+|\alpha^2|+\cdots+|\alpha^s| = \dim \mathbb{F}\ell_{\mathbf{a}}$. Call such a list *Grassmannian Schubert data* for $\mathbb{F}\ell_{\mathbf{a}}$. Consider an intersection of Grassmannian Schubert varieties

$$Y_{\alpha^1,i_1}F_\bullet(t_1) \cap Y_{\alpha^2,i_2}F_\bullet(t_2) \cap \cdots \cap Y_{\alpha^s,i_s}F_\bullet(t_s)\,, \tag{5.10}$$

where $t_1 < t_2 < \cdots < t_s$ are distinct real numbers and $F_\bullet(t)$ is the flag osculating the rational normal curve γ at $\gamma(t)$.

CONJECTURE 5.10. *Let* $\mathbf{a} = 1 < a_1 < a_2 < \cdots < a_r < a_{r+1} = n$ *and*

$$(\alpha^1, i_1),\ (\alpha^2, i_2),\ \ldots,\ (\alpha^s, i_s), \qquad \textit{with} \quad \alpha^j \in \binom{[n]}{i_j}$$

be Grassmannian Schubert data for $\mathbb{F}\ell_{\mathbf{a}}$.

(1) *For every choice of real numbers* $t_1 < t_2 < \cdots < t_s$, *the intersection* (5.10) *is* (a) *transverse with* (b) *with all points of intersection real if the indices* i_j *are in order:* $i_1 \le i_2 \le \cdots \le i_s$ *or else* $i_1 \ge i_2 \ge \cdots \ge i_s$.

(2) *If the indices* i_j *are not in order, nor is any cyclic permutation of the indices, then there exist real numbers* $t_1 < t_2 < \cdots < t_s$ *such that the intersection* (5.10) *is transverse with* **all** *points real, and there exist real numbers* $t_1 < t_2 < \cdots < t_s$ *such that the intersection is transverse with* **not all** *points real.*

In particular, enumerative problems involving Grassmannian Schubert varieties on $\mathbb{F}\ell_{\mathbf{a}}$ *are fully real.*

REMARK 5.11.

(1) When each $|\alpha^i| = 1$, there are real numbers $t_1 < t_2 < \ldots < t_s$ such that the intersection (5.10) is transverse with all points real [**74**, Corollary 2.2]. Thus the number of real points in the intersection (5.10) is expected to vary when the the indices i_j are not in order.

(2) When $r = 1$, $\mathbb{F}\ell_{\mathbf{a}}$ is the Grassmannian $\mathrm{Gr}(a_1, n)$ and the condition on the indices being ordered is empty, so this case of Conjecture 5.10 reduces to Conjecture 5.1.

(3) There is considerable evidence for this conjecture when $r = 2$. Part (2) is true for every set of Grassmannian Schubert data in $\mathbb{F}\ell_{2<3}\mathbb{C}^5$ and all except one such set in $\mathbb{F}\ell_{2<4}\mathbb{C}^6$[†]. Many instances of these same enumerative problems with ordered partitions have been computed, and in each instance of (1) the intersection (5.10) is transverse with all points real.
(4) As for Conjecture 5.1, (a) implies (b) in Part (1) of Conjecture 5.10.
(5) We have tested no instances of the intersection (5.10) with $r > 2$, so the truth may differ from the exact statement of Conjecture 5.10.
(6) Conjecture 5.10 has nothing to say when the Schubert data are not Grassmannian.

5.2.2. *The Lagrangian Grassmannian.* As in Section 4.2.4, let V be a $2n$-dimensional vector space equipped with the alternating form $\langle\cdot,\cdot\rangle$ defined by (4.18). Set

$$\gamma(t) \;:=\; \left(1,\, t,\, \frac{t^2}{2},\, \ldots,\, \frac{t^n}{n!},\, -\frac{t^{n+1}}{(n+1)!},\, \frac{t^{n+2}}{(n+2)!},\, \ldots,\, (-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}\right). \tag{5.11}$$

The flag $F_\bullet(t)$ of subspaces osculating this rational normal curve is isotropic for all $t \in \mathbb{C}$.

Given a strict partition $\lambda\colon \lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$ and an isotropic flag $F_\bullet$, set

$$\Psi_\lambda F_\bullet \;:=\; \{H \in LG(n) \mid H \cap F_{n+1-\lambda_i} \geq i \text{ for } i = 1, \ldots, l\}.$$

The codimension of this Schubert variety is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. For example,

$$\Psi_a F_\bullet = \{H \mid H \cap F_{n+1-a} \neq \{0\}\}.$$

With these definitions, we state the version of Theorem 4.11 proven in [**74**].

THEOREM 5.12 ([**74**, Theorem 4.2]). *Given a strict partition λ, let $N := \binom{n+1}{2} - |\lambda| = \dim \Psi_\lambda F_\bullet$. If $N > 1$, then there exist distinct real numbers $t_1, t_2, \ldots, t_N$ such that*

$$\Psi_\lambda F_\bullet(0) \cap \Psi_1 F_\bullet(t_1) \cap \Psi_1 F_\bullet(t_2) \cap \cdots \cap \Psi_1 F_\bullet(t_N)$$

is zero-dimensional with **no** *points real.*

Thus the generalization of Conjecture 5.1 is badly false for the Lagrangian Grassmannian. On the other hand it holds for some enumerative problems in the Lagrangian Grassmannian.

THEOREM 5.13 ([**77**]). *For any distinct real numbers t_1, t_2, t_3, t_4, the intersection of Schubert varieties in $LG(3)$*

$$\Psi_1 F_\bullet(t_1) \cap \Psi_1 F_\bullet(t_2) \cap \Psi_2 F_\bullet(t_3) \cap \Psi_2 F_\bullet(t_4)$$

is transverse with **all** *points real.*

These two results illustrate a dichotomy that is emerging from experimentation. Call a list of strict partitions $\lambda^1, \lambda^2, \ldots, \lambda^s$ with $|\lambda^1| + |\lambda^2| + \cdots + |\lambda^s| = \binom{n+1}{2}$ Lagrangian Schubert data. In every instance we have computed of a zero-dimensional intersection of Lagrangian Schubert varieties whose flags osculate the rational normal curve (5.11), the intersection has been transverse with either all points real or no points real. Most interestingly, the outcome—all real or no real—has depended only upon the Lagrangian Schubert data of the intersection.

[†]The one exception is due to our inability to compute any instances.

CONJECTURE 5.14. *Given Lagrangian Schubert data $\lambda^1, \lambda^2, \ldots, \lambda^s$ and distinct real numbers $t_1, t_2, \ldots, t_s$, the intersection*

$$\Psi_{\lambda^1} F_\bullet(t_1) \cap \Psi_2 \lambda^2 F_\bullet(t_2) \cap \cdots \cap \Psi_{\lambda^s} F_\bullet(t_s)$$

is transverse with either

(a) *all points real, or*
(b) *no points real,*

and the outcome (a) *or* (b) *depends only upon the list $\lambda^1, \lambda^2, \ldots, \lambda^s$.*

We do not have a good idea what distinguishes the Lagrangian Schubert data giving all points real from the data giving no points real. Further experimentation is needed.

5.2.3. *Further generalizations of Conjecture 5.1.* The status of the generalizations of Conjecture 5.1 to other flag varieties is almost completely unknown. There is one flag variety, the Orthogonal Grassmannian, for which much is known. In particular, the analog of Theorem 5.3 (involving codimension 1 Schubert varieties given by isotropic flags osculating the rational normal curve) holds for the Orthogonal Grassmannian [**74**, Corollary 3.3]. There has also been a significant amount of computer experimentation testing cases of the obvious generalization of Conjecture 5.1 for the orthogonal Grassmannian, and in each, all points of intersection were found to be real. Lastly, we remark that we do not know of any reasonable version of Conjecture 5.1 for the quantum Schubert calculus.

6. Lower Bounds in the Schubert calculus

In Section 3.3, we saw that of the 12 rational cubics meeting 8 real points in the plane, at least 8 were real. This was the first instance of a non-trivial lower bound on the number of real solutions to a problem in enumerative geometry. Recent work of Eremenko and Gabrielov shows this phenomenon is pervasive in the Schubert calculus.

Recall from Section 4.1 that the k-planes in $\mathbb{C}^n$ meeting $k(n-k)$ general $(n-k)$-planes non-trivially is a complementary dimensional linear section of the Grassmannian,

$$\Lambda \cap \mathrm{Gr}(k,n)\,,$$

where Λ has codimension $k(n-k)$ in Plücker space. The number of such k-planes is the degree of $\mathrm{Gr}(k,n)$. This is also the degree of the linear projection

$$\pi_E \;:\; \mathrm{Gr}(k,n) \;\hookrightarrow\; \mathbb{P}^{\binom{n}{k}-1} \;\dashrightarrow\; \mathbb{P}^{k(n-k)}$$

with center of projection a plane E with codimension $k(n-k)+1$ disjoint from $\mathrm{Gr}(k,n)$. The connection between these two definitions of degree is that when $E \subset \Lambda$, $\pi_E(\Lambda)$ is a point $x \in \mathbb{P}^{k(n-k)}$ and

$$\Lambda \cap \mathrm{Gr}(k,n) \;=\; \pi_E^{-1}(x)\,.$$

Since complex manifolds are canonically oriented, the degree of such a linear projection is just the number of points in the inverse image of a regular value x.

An important such linear projection is the Wronski map. Let $L(t)$ be the $(n-k)$-plane osculating the rational normal curve (5.1). By (4.6) and the discussion following Theorem 5.2, the equation for a k-plane K to meet $L(t)$ non-trivially is

$$\sum t^{\binom{n+1}{2}-\binom{n-k+1}{2}-|\alpha|} L_\alpha p_\alpha(K)\,, \tag{6.1}$$

where $p_\alpha(K)$ is the αth Plücker coordinate of K and L_α is the appropriately signed Plücker coordinate of $L(0)$ complementary to α. The association of a k-plane K to the polynomial (6.1) is the *Wronski map*

$$\pi_W \ : \ \mathrm{Gr}(k,n) \ \longrightarrow \ \mathbb{P}^{k(n-k)}\,,$$

where $\mathbb{P}^{k(n-k)}$ is the space of polynomials of degree at most $k(n-k)$, modulo scalars. This is a linear projection as the coefficients in (6.1) are linear in the Plücker coordinates. If f has distinct roots $t_1, t_2, \ldots, t_{k(n-k)}$, then $\pi_W^{-1}(f)$ is the set of k-planes meeting each of $L(t_1), L(t_2), \ldots, L(t_{k(n-k)})$ non-trivially. Observe that $\pi_W^{-1}(f)$ is given by real linear equations on the Grassmannian if and only if f has real coefficients, which includes the case when the t_i are all real (the situation of the Shapiro Conjecture 5.1).

To see why this is called the Wronski map, consider $\mathrm{Gr}(k,n)$ as the set of k-planes in the space of polynomials of degree at most $n-1$. Given a k-plane

$$K \ = \ \text{linear span}\{f_1, f_2, \ldots, f_k\}\,,$$

the Wronski determinant of K is

$$W(K) \ = \ \det \begin{bmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix}\,,$$

a polynomial of degree at most $k(n-k)$, well-defined modulo scalars. Under a choice of coordinates given by the coefficients of a polynomial, $W(K) = \pi_W(K)$.

Since real manifolds are not necessarily orientable, the degree of a map is a $\mathbb{Z}/2\mathbb{Z}$-valued invariant. However, Kronecker [**44**] defined the degree of a regular map $\mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$, which he called the *characteristic*, and his definition makes sense for many maps $f\colon X \to Y$ of (not necessarily orientable) compact manifolds.

First suppose that X is oriented. For a regular value $y \in Y$ of f, define

$$\mathrm{char}\ f \ := \ \left| \sum_{x \in f^{-1}(y)} \mathrm{sgn}\det df_x \right|\,,$$

using local coordinates in X consistent with its orientation and any local coordinate near $y \in Y$. The sum is well-defined up to multiplication by ± 1. This number is independent of choices, if Y is connected.

When X may not be orientable, let $\widetilde{X}$ be the space of orientations of X, which is canonically oriented, and similarly let $\widetilde{Y}$ be the space of orientations of Y. Then $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ are 2 to 1 coverings of X and Y, with covering group $\mathbb{Z}/2\mathbb{Z}$. The map f is *orientable* if it has a lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ that is equivariant with respect to the covering group $\mathbb{Z}/2\mathbb{Z}$. Define the characteristic of an orientable map f to be $\mathrm{char}\widetilde{f}$. This is well-defined, even if $\widetilde{Y}$ consists of 2 components. This characteristic satisfies a fundamental inequality.

PROPOSITION 6.1. *Let $f\colon X \to Y$ be an orientable map. Then, for every regular value $y \in Y$,*

$$\#f^{-1}(y) \ \geq \ \mathrm{char} f\,.$$

We call this characteristic the *real degree* of the map f.

Consider this notion for the Wronski map of the real Grassmannian $\mathrm{Gr}_{\mathbb{R}}(k,n)$. The Grassmannian has dimension $k(n-k)$ and is orientable if and only if n is even. Eremenko and Gabrielov [**17**] compute the degree of the Wronski map. To state their result, we introduce some additional combinatorics. Recall the interpretation (4.8)

$$\text{degree } \mathrm{Gr}(k,n) \;=\; \#\text{ chains in Bruhat order from } \hat{0} \text{ to } \hat{1}\,,$$

where $\hat{1} = (n{-}k{+}1,\ldots,n{-}1,n)$ is the top element in the Bruhat order. We introduce a statistic on these chains. Each cover $\alpha \lessdot \beta$ in the Bruhat order has a unique index i with

$$\alpha_i \;<\; \beta_i \;=\; \alpha_i + 1 \qquad \text{but} \qquad \alpha_j \;=\; \beta_j \quad \text{for } j \;\neq\; i\,.$$

The word $w(\mathrm{ч})$ of a chain $\mathrm{ч}\colon \hat{0} \lessdot \alpha^1 \lessdot \alpha^2 \lessdot \cdots \lessdot \alpha^{k(n-k)} = \hat{1}$ is the sequence of indices $(i_1, i_2, \ldots, i_{k(n-k)})$ of these indices for covers in ч. An inversion in such a word is a pair $j < l$ with $i_j < i_l$, and the weight, $\omega(\mathrm{ч})$ is the number of inversions in the word of the chain ч. For instance the chain in the Bruhat order of $\mathrm{Gr}(3,6)$ highlighted in Figure 11

$$123 \lessdot 124 \lessdot 125 \lessdot 135 \lessdot 145 \lessdot 245 \lessdot 246 \lessdot 346 \lessdot 356 \lessdot 456\,,$$

has word 332213121 and length 5. Define the inversion polynomial

$$I(k,n)(q) \;:=\; \sum q^{\omega(\mathrm{ч})}\,,$$

the sum over all chains ч in the Bruhat order from $\hat{0}$ to $\hat{1}$.

THEOREM 6.2 (Eremenko and Gabrielov [**17**, Theorem 1]). *The characteristic of the Wronski map* $\pi_W\colon \mathrm{Gr}_{\mathbb{R}}(k,n) \to \mathbb{P}^{k(n-k)}_{\mathbb{R}}$ *is* $|I(k,n)(-1)|$.

They prove this by an induction reminiscent of that used in the proof of Theorem 4.6. In the course of their proof, they construct a polynomial f with all roots real having $d(k,n)$ (=degree of the Grassmannian) real points in $\pi_W^{-1}(f)$ (one for each chain in the Bruhat order) and show that

$$\det d\pi_W \;=\; (-1)^{\omega(\mathrm{ч})}\,,$$

at the point in $\pi_W^{-1}(f)$ corresponding to the chain ч.

White [**87**] studied the statistic $|I(k,n)(-1)|$ and showed that it equals zero if and only if n is even, and that $|I(2,2n)(-1)| = C_n$. The main result of Eremenko and Gabrielov is the following, which gives a lower bound for the number of real solutions in some cases of the Shapiro conjecture.

COROLLARY 6.3 (Eremenko and Gabrielov [**17**, Corollary 2]). *Let* $L_1, L_2, \ldots, L_{k(n-k)}$ *be codimension* $(n-k)$*-planes in* $\mathbb{R}^n$ *osculating the rational normal curve at* $k(n-k)$ *general real points. Then number* ρ *of real* k*-planes* K *meeting each* L_i *non-trivially satisfies*

$$|I(k,n)(-1)| \;\leq\; \rho \;\leq\; I(n,k)(1)\,.$$

In particular, when n *is odd this number* ρ *is non-zero.*

As we remarked before, this lower bound holds if the osculating planes are no longer required to be real, but that the set $\{t_1, t_2, \ldots, t_{k(n-k)}\}$ of points of osculation are real (the roots of a real polynomial).

Consider now a linear projection π_E different from the Wronski map. If we move the center E, the degree of π_E does not change as long as E does not meet the Grassmannian. Thus if E is in the same component of the space of codimension $k(n-k)+1$ planes not meeting the Grassmannian as is the center of the Wronski projection π_W, then Corollary 6.3 applies to $\pi_E^{-1}(x)$ for real x.

These new ideas of Eremenko and Gabrielov, particularly their notion of real degree, greatly increase our understanding of the real Schubert calculus. We emphasize that this is an important start in the search for lower bounds to other enumerative problems. Not all enumerative problems involve a linear projection of a submanifold of projective space. In fact, of the other enumerative problems we have considered in the Schubert calculus, only those of Theorem 4.11 on the Lagrangian Grassmannian involve linear projections, and they have a lower bound of 0. There has been very little experimental work investigating lower bounds. We describe some of it in the final section.

6.1. Lower bounds in the Schubert calculus for flags? Conjecture 5.10 speculates that a zero-dimensional intersection of Grassmannian Schubert varieties in a flag manifold $\mathbb{F}\ell_{\mathbf{a}}$ has only real points, when the Schubert varieties are given by flags osculating the rational normal curve at points and when the indices of the Grassmannian Schubet data are ordered. If the indices cannot be ordered, then we conjectured (and have found experimentally) that there is a selection of osculating flags with all points of intersection real, as well as a selection with not all points real.

Table 6.1 shows the results of computing 160,000 instances of the intersection of 4

Necklace	Number of Real Solutions						
	0	2	4	6	8	10	12
22223333	0	0	0	0	0	0	20000
22322333	0	0	9	1677	3835	6247	8232
22233233	0	0	67	3015	6683	4822	5413
22332233	0	0	136	1533	7045	5261	6025
22323323	0	0	303	2090	6014	7690	3903
22323233	0	37	1944	4367	6160	4634	2858
22232333	0	195	1476	1776	3628	4546	8379
23232323	251	929	5740	3168	5420	2828	1662

simple Schubert varieties $X_2F_\bullet(t_i)$ and 4 simple Schubert varieties $X_3F_\bullet(t_i)$ in the manifold $\mathbb{F}\ell_{2<3}$ of partial flags $E_2 \subset E_3$ in 5-space. Here the numbers $t_1 < t_2 < \cdots < t_8$ were 20,000 random subsets of 8 numbers between 1 and 80, chosen using Maple's random number generator. For each choice, we considered the 8 possibilities of orderings of the indices, and for each system we computed the number of real solutions out of 12. Observe that the apparent lower bound on the number of real solutions depends on ordering of the indices of the Schubert data. Other calculations we have done reinforce this observation.

We conclude by reminding the reader of the observation in Section 3 that the number of real solutions to a problem in enumerative goemetry depends subtly on the configuration of the conditions.

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