Foundations of Mathematics Math 300H Section 970

Ninth Homework: Due 8 November 2022

Definition: The *Fibonacci sequence* $\{f_n \mid n \geq 1\}$ is defined by $f_1 = f_2 = 1$ and for $n \geq 2$, $f_{n+1} = f_n + f_{n-1}$.
Read: Chapters 7 and 8 in Fourth edition. (Chapter 8 in fourth edition is Chapter 7 in third. The new Chapter 7 in Fourth edition is a review of proof techniques.)

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1. Look up the term *Pythagrean triple* (it is in our book). Investigate the following

   Conjecture. *For each natural number n, the numbers $f_n f_{n+3}$, $2f_{n+1}f_{n+2}$, and $(f_{n+1}^2 + f_{n+2}^2)$ form a Pythagorean triple.*

   If true, provide a proof, and if false, a counterexample.

2. Find (and prove) a formula for f_{n+5} in terms of f_n and f_{n+1}. Use this formula to give a proof by induction that for all $n \in \mathbb{N}$ the Fibonacci number f_{5n} is a multiple of f_n.

3. Explore the relation between f_k and f_{kn} for small values of k, and make a conjecture.

4. Find natural numbers a and b such that $a^2 + b^2 = 10$, and then natural numbers c and d such that $c^2 + d^2 = 10^2$.

   Prove the following statement by mathematical induction: For every natural number n, there are natural numbers x and y such that $x^2 + y^2 = 10^n$.

5. Suppose that a and b are integers such that $a + b$ is even. Prove that there exist integers x and y such that $x^2 - y^2 = ab$.

6. Prove or disprove.

   (a) Let A, B, C, and D be sets with $A \subseteq C$ and $B \subseteq D$. If A and B are disjoint, then C and D are disjoint.

   (b) Every even integer can be expressed as the sum of two odd integers.

7. Prove or disprove.

   (a) There is a real number solution of the equation $x^4 + x^2 + 1 = 0$.

   (b) There exist positive integers a and b such that $a^2 - b^2 = 101$.

8. Evaluate the proof of the following statement.

   Statement. Let $x, y, z \in \mathbb{Z}$ be such that $3x + 5y = 7z$. If at least one of x, y, or z is odd, then at least one of x, y, or z is even.

   Proof. Let $x, y, z \in \mathbb{Z}$ be such that $3x + 5y = 7z$. Assume, to the contrary, that none of x, y, or z is odd andthat none of x, y, or z is even. This is impossible. $\square$