Trigonometric addition formulae

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Suppose that x and y are real numbers. We all learned the (in)famous sum formulae for sine and cosine,

$$(1) \qquad \begin{array}{rcl} \sin(x+y) &=& \sin x\cos y + \cos x \sin y \qquad \text{and} \\ \cos(x+y) &=& \cos x \cos y - \sin x \sin y\,, \end{array}$$

but we mostly do not recall how to prove these formulae.

My personal favourite proof of this fact is geometric, using similar triangles to directly show (1). Suppose that we start by marking off distances y and then x anti-clockwise along the unit circle. Then $\sin(x+y)$ and $\cos(x+y)$ are the legs of the triangle in Figure 1. The

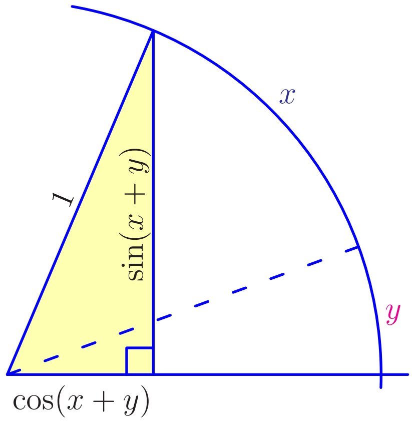
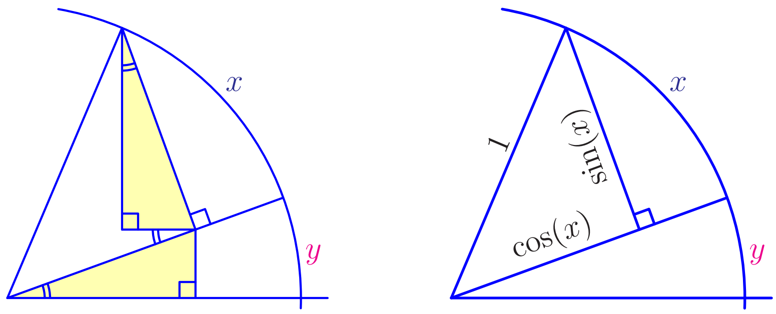


Figure 1. The goal.

key to the formula is to write these legs as the sum and differences of legs from the two triangles in the picture on the left below, whose hypotenuses are indicated on the right.



(We used the definition of $\cos(x)$ and $\sin(x)$ for this.) The one subtle point in this argument is that the three indicated angles are all equal to y. (Can you prove this?) We will use this now. The upper triangle with hypotenuse $\sin(x)$ has vertical leg $\sin(x)\cos(y)$ and horizontal leg $\sin(x)\sin(y)$—this is because y is the angle at its apex. The lower triangle with hypotenuse $\cos(x)$ has vertical leg $\cos(x)\sin(y)$ and horizontal leg $\cos(x)\cos(y)$. This is shown in Figure 2 on the next page.

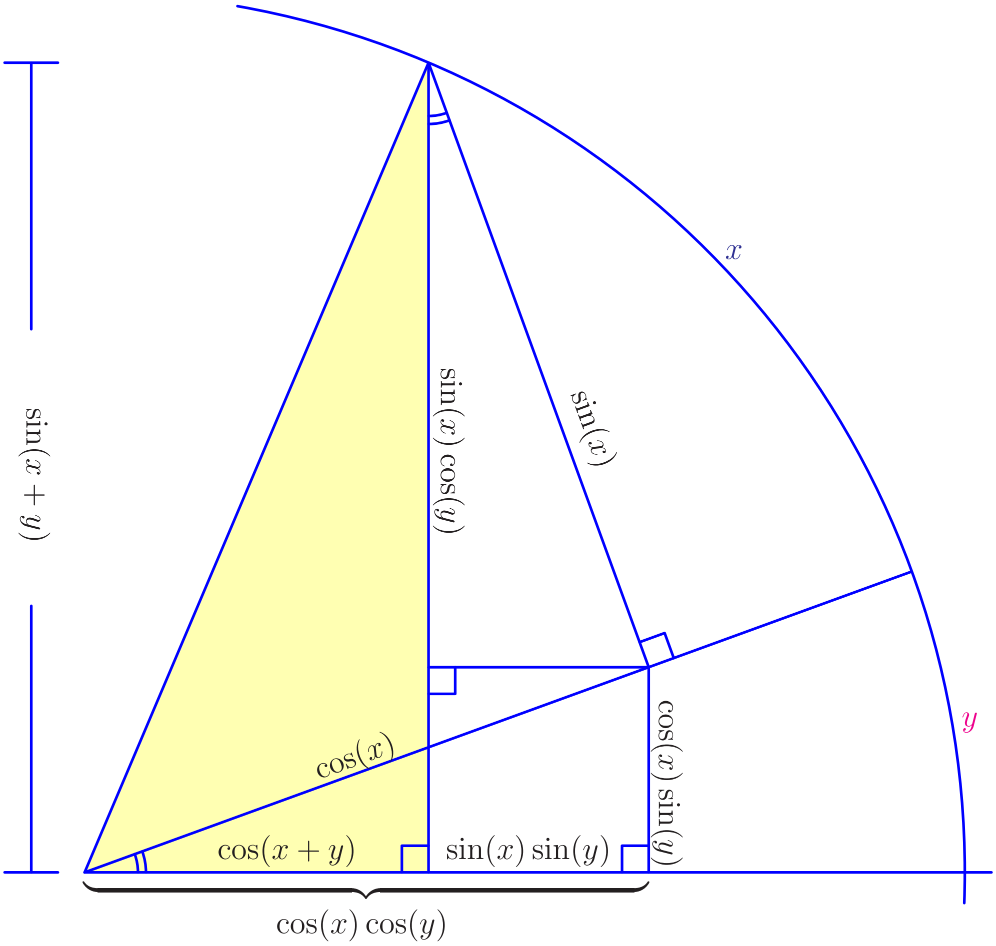


FIGURE 2. Putting it all together.

This shows that

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \qquad \text{and} \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\,, \end{aligned}$$

as promised.