Algebra Autumn 2023
Frank Sottile
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Thirteenth Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 4 December:

68. Let $\mathbb{F}$ be a field. Show that the subring $\mathbb{F}[[x]][x^{-1}]$ of the quotient field of $\mathbb{F}[[x]]$ is a field. This is the field of formal Laurent series in x.

69. The *nth cyclotomic polynomial* is

$$f_n \ := \ (x^n - 1)/(x-1) = x^{n-1} + \cdots + 1 \ \in \ \mathbb{Z}[x]\,.$$

Use Eisenstein's criterion to show that if p is prime, then $f_p(x+1)$ is irreducible, and deduce that f_p is irreducible.

70. If $c_0, c_1, \ldots, c_n$ are distinct elements of an integral domain D, and $d_0, \ldots, d_n$ are elements of D, then there is at most one polynomial $f \in D[x]$ of degree n such that $f(c_i) = d_i$ for each $i = 0, \ldots, n$.

71. Show that for any ring R and R-module M, $\mathrm{Hom}_R(R, M) \simeq (M, +, 0)$, as abelian groups.

72. Let R be a ring and A be an abelian group. For $r \in R$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$, define $r.f \colon R \to A$ by $(r.f)(x) = f(xr)$ for $x \in R$. Show that this gives $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ the structure of an R-module. (Part of this problem is showing that $r.f \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$.)

73. Let R be a ring and $A, B, M,$ and N be R-modules. Let $f \in \mathrm{Hom}_R(A, M)$ and $g \in \mathrm{Hom}_R(N, B)$. For $\varphi \in \mathrm{Hom}_R(M, N)$, define $f^*(\varphi) := \varphi \circ f$ and $g_*(\varphi) := g \circ \varphi$. Show that these give homomorphisms of abelian groups,

$$f^* \colon \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(A, N) \quad \text{and} \quad g_* \colon \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, B).$$

Show that $f \mapsto f^*$ is a homomorphism of abelian groups $\mathrm{Hom}_R(A, M) \to \mathrm{Hom}_Z(\mathrm{Hom}_R(M, N), \mathrm{Hom}_R(A, N))$.

74. Let M be an R-module. Show that $\mathrm{Hom}_R(M, M)$ is a ring whose product is the composition of functions. It is called the *endomorphism ring* of M, written $\mathrm{End}(M)$.

Show that M is a left $\mathrm{End}(M)$-module under the action by elements $f \in \mathrm{End}(M)$ defined by $f.m = f(m)$, for $m \in M$.

75. An R-module M is *simple* if its only submodules are 0 and M. Prove that every simple R-module is cyclic.

Prove *Schur's Lemma*, that if M is simple and $M \neq 0$, then $\mathrm{End}(M)$ is a division ring.