Foundations of Mathematics   Math 300H Section 970

Eleventh Homework: Due 22 November 2022

1. Let H be a nonempty subset of $\mathbb{Z}$. Suppose that the relation R defined on $\mathbb{Z}$ by aRb if $a - b \in H$ is an equivalence relation. Verify the following
   (a) $0 \in H$.
   (b) If $a \in H$, then $-a \in H$.
   (c) If $a, b \in H$, then $a + b \in H$.
2. Let R be a relation defined on the set $\mathbb{N}$ by aRb if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is an equivalence relation.
3. The relation R on $\mathbb{Z}$ defined by aRb if $a^4 \equiv b^4 \mod 8$ is known to be an equivalence relation. Determine the distinct equivalence classes.
4. In $\mathbb{Z}_{11}$, express the following sums and products as $[r]$, where $0 \leq r < 11$.
   (a) $[7] + [5]$ (b) $[7] \cdot [5]$ (c) $[-82] + [207]$ (d) $[-82] \cdot [207]$.
5. Compute the addition and multiplication tables for $\mathbb{Z}_5$.
6. Prove that the multiplication in $\mathbb{Z}_m$, for $m \geq 2$, defined by $[a]_m \cdot [b]_m = [a \cdot b]_m$ is well-defined. (See Result 4.11 in our book.)

   Hint: $ab - cd = ab - cb + cb - cd = (a - c)b + c(b - d)$, the oldest trick in the book.
7. **Proof Analysis.** Which oxf the following is true for the proof below (is the statement true or false, and is the proof correct or incorrect.) ?

   Statement: Every symmetric and transitive relation on a nonempty set A is an equivalence relation.

   Proof: Let R be a symmetric and transitive relation on a nonempty set A. We need only to show that R is reflexive. Let $x \in A$. Let $y \in A$ be such that xRy. As R is symmetric, we have yRx. Now xRy and yRx, so we conclude that xRx, by transitivity. Thus, R is reflexive. □
8. Let A be a nonempty set. Suppose that R is a relation from A to A that is both an equivalence relation and a function. What familiar function is R? Justify your answer.
9. Let ν be the function from $\mathbb{N}$ to $\mathbb{N}$ whose value at a positive integer n is the number of digits in the American English spelling of the number n. For example $\nu(0) = 4$, as '0' is written zero with four letters. Similarly, $\nu(22) = 9$, as twentytwo has nine letters.

   If we restrict the domain of ν to $\{1, 2, \ldots, 20\}$, what is its range?
10. A *real function* is one whose domain and codomain are subsets of $\mathbb{R}$. For each of the following real functions, determine their largest possible domain and their range.
    (a) The function f defined by $f(x) = x/(x^2 - 3x - 2)$.
    (b) The function g defined by $g(x) = \ln(1 - \cos(x))$.