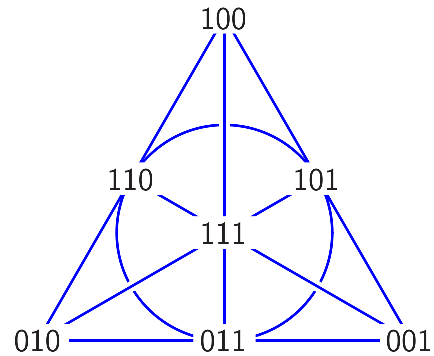
Algebra Autumn 2013
Frank Sottile
19 September 2013

Fourth Homework

Hand in to Frank Tuesday 24 September:

21. Investigate the group $PSL(2,7) \simeq GL(3,2)$, which is also the group of symmetries of the Fano plane. Consider it as a subgroup of S_7, say by identifying $\{100, 110, 101, 010, 011, 001, 111\}$ with $\{1,2,3,4,5,6,7\}$. Find subgroups of orders 8, 3, and 7. Describe a subgroup of this group isomorphic to D_6. Show that none of these four subgroups are normal.
Please use the identification I suggest, so that I can more readily check your work. Also, we may all find it more useful if you give a description of how these groups act on the Fano plane, along with listing their elements.



Hand in to Frank Thursday 26 September:

22. Let G be a group and $C(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$ be its *center*.

    (a) Prove that if $G/C(G)$ is cyclic, then G is abelian.

    (b) Let p be a prime number. Prove that any group of order p^2 is abelian.

Hand in for the grader Tuesday 24 September:

23. Let p be the smallest prime number dividing the order $|G|$ of a finite group G and suppose that G has a subgroup H of index p, $[G : H] = p$. Prove that H is normal in G.

24. Let G be a finite group of order n and let $\varphi : \hookrightarrow S_n$ be the right regular representation of G on itself (the Cayley embedding). Find necessary and sufficient conditions on G so that its image under φ is a subgroup of the alternating group, A_n.

25. Suppose that G and K are groups with respective normal subgroups $H \triangleleft G$ and $L \triangleleft K$. Give examples showing that each of the following statements do not hold for all groups.

    (a) $G \simeq K$ and $H \simeq L$ implies that $G/H \simeq K/L$.

    (b) $G \simeq K$ and $G/H \simeq K/L$ implies that $H \simeq L$.

    (c) $G/H \simeq K/L$ and $H \simeq L$ implies that $G \simeq K$.

26. True or False, with justification. Given a collection of groups $\{H_\alpha \mid \alpha \in I\}$ then the Cartesian product $\prod\{H_\alpha \mid \alpha \in I\}$ is generated by its collection of subgroups $\iota_\alpha(H_\alpha)$ for $\alpha \in I$, where, for $h \in H_\alpha$, the element $\iota_\alpha(h)$ takes value h at α, and is the identity at $\beta \in I \setminus \{\alpha\}$.