Algebra II Winter 2021 — 1 March

Frank Sottile — **Seventh Homework**

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 8 March.

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1. Let M_R and $_RN$ be right- and left- modules over a ring R and P be an abelian group.
   Prove that the first two (additive) conditions on a balanced map $f\colon M \times N \to P$ imply that $f(0,n) = f(m,0) = 0$ and that $f(-m,n) = -f(m,n) = f(m,-n)$.
   (Remember, $-a$ is the additive inverse of a.)

2. Let A be an abelian group and $m \geq 1$ a positive integer. Show that $A \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \simeq A/mA$. Use this to deduce that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ and to describe the tensor product of any two finitely generated abelian groups.

3. Let A be a module over a commutative ring R. Show that $A \times A^* \to R$ defined by $(a,f) \mapsto f(a)$ is R-bilinear (balanced). Show that $A \otimes_R A^*$ has a natural map to $\mathrm{End}_R(A)$.

4. Let $f\colon M_R \to M'_R$ and $g\colon {_RN} \to {_RN'}$ be R-module homomorphisms. Explain the difference between the induced homomorphism $f \otimes g$ of tensor products of R-modules and the element $f \otimes g$ of the tensor product of abelian groups $\mathrm{Hom}_R(M, M') \otimes \mathrm{Hom}_R(N, N')$

5. Consider an attempt to define an $\mathbb{R}$-linear map
   $$f \;:\; \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \qquad \text{or} \qquad \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}\,,$$
   in either direction given by the formula
   $$f(x \otimes y) \;=\; x \otimes y\,.$$
   In which direction is this map well-defined? Is it then surjective? Is it injective?

6. Let $m > 1$ be an integer. Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$, where $\cdot m$ is the map with image $m\mathbb{Z}$ induced by $1 \mapsto m$.

   Apply the tensor product functor $\_\_ \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ to this exact sequence, determining all groups and maps. Do you obtain an exact sequence ?

7. Let R be a commutative ring and $I, J \subset R$ ideals of R. Prove that there is an R-module isomorphism $R/I \otimes_R R/J \simeq R/(I+J)$.

8. Let $\varphi\colon R \to S$ be a ring homomorphism. Show that this induces on S the structure of an $R{-}R$ bimodule. Let M be a left R-module and show that $S \otimes_R M$ is a left S-module. (This is called *base extension*, and induces a functor from R-mod to S-mod.)