1. Let R be a ring with no zero divisors such that for all $r, s \in R$, there are $a, b \in R$, not both zero, such that $ar + bs = 0$. Show that if $R = M \oplus N$ as R-modules, then one of M or N is the 0-module, $\{0\}$. Use this to show that R has the invariant dimension property.

Suppose that $0 \neq m \in M$, so that $M \neq \{0\}$. Let $n \in N$. Then there exist $a, b \in R$ with not both zero, such that $0 = am + bn$, or rather $am = -bn$. Since $am \in M$, $\pm bn \in N$, and $M \cap N = \{0\}$, we must have that $am = 0 = bn$. As $m \neq 0$ and R has no zero divisors, $a = 0$. But then $b \neq 0$, and as $bn = 0$, we conclude that $n = 0$. As $n \in N$ was arbitrary, this implies that $N = \{0\}$.

This was too hard. Let m, n be nonnegative integers. We claim that if a map $\varphi \colon R^n \to R^m$ of free modules is injective, then $n \leq m$. Thus if φ is an isomorphism, then $m = n$.

Suppose by way of contradiction that $n > m$ and φ is injective. We use this to express φ in a normal form, and then deduce that this normal form implies φ is not injective. Note that if $\psi \colon R^m \to R^m$ is injective, then $\ker \varphi = \ker \psi \circ \varphi$. Thus we will freely replace φ by compositions $\psi \circ \varphi$, with $\psi \colon R^m \to R^m$ injective.

We will express ψ as a composition of two types of injective maps. The first is simply a permutation of R^m, which amounts to reordering its basis, and the second is an *elementary map*. Let $X := \{x_1, \ldots, x_m\}$ be the (implied) basis of R^m, $a, b \in R$ and $1 \leq i < j \leq m$. Define $E_{ij}(a, b)$ by

$$E_{ij}(a,b)(x_i) \;=\; x_i + a x_j\,, \quad E_{ij}(a,b)(x_j) \;=\; b x_j\,, \quad E_{ij}(a,b)(x_k) \;=\; x_k\,, \text{ if } k \notin \{i, j\}\,.$$

Let us compute $E_{ij}(a, b)(x)$, wjere $x := r_1 x_1 + \cdots + r_m x_m)$. This is

$$r_1 x_1 + \cdots + r_{j-1} x_{j-1} + (a r_i + b r_j) x_j + r_{j+1} x_{j+1} + \cdots + r_m x_m\,.$$

Thus if $b \neq 0$, then $E_{ij}(a, b)$ is injective, for if x lies in its kernel, then this expression and the independence of X implies that if $k \neq j$, then $r_k = 0$ and thus also $b r_j = 0$. Since $b \neq 0$, we conclude that $r_j = 0$ and thus $x = 0$.

Now let $y_1, \ldots, y_n$ be the implied basis for R^n. Let $k \in [n]$ and let us expand $\varphi(y_k)$ in the basis X,

$$\varphi(y_k) \;=\; f_{1,k} x_1 + f_{2,k} x_2 + \cdots + f_{m,k} x_m\,.$$

Then $F = (f_{i,k})_{i \in [m]}^{k \in [n]}$ is a $m \times n$ matrix with entries in R that represents the map φ in the given bases. Composing φ with a permutation of X gives the matrix obtained by the correspomding permutation of the rows of F. Composing φ with an elementary map $E_{ij}(a, b)$ gives a matrix that is the same as F in rows $\ell \neq j$, but the entry in row j and column ℓ is replaced by $a f_{i,\ell} + b f_{j,\ell}$. That is, the matrix for $E_{ij}(a, b) \circ \varphi$ is the result of an elementary row operation on F.

Lemma. *Let $p < q$ be postive integers and suppose that $B \colon R^q \to R^p$ is a module homomorphism such that its matrix $B = (b_{i,j})$ in the implied bases is upper triangular with non-zero elements along its diaginal. Then B has a non-trivial kernel.*

Proof. Let $\{w_1, \ldots, w_q\}$ and $\{v_1, \ldots, v_p\}$ be the ordered bases for R^q and R^p. Let $r_p, s \in R$. As B is triangular, the coefficient of v_p in $B(r_p w_p + s w_q)$ is $r_p b_{p,p} + s b_{p,q}$. We may choose $r_p, s \in R$, not both zero so that this vanishes. As $b_{p,p} \neq 0$, we have that $s \neq 0$.

Let r_{p-1} and s' be elements of R, and consider the coeffcient of v_{p-1} in $B(r_{p-1} v_{p-1} + s'(r_p w_p + s w_q))$. This is

$$r_{p-1} b_{p-1,p-1} + s'(r_p b_{p-1,1} + s b_{p-1,q})\,.$$

As before, we may choose r_{p-1}, s' with this sum equal equal to zero, and as $b_{p-1,p-1} \neq 0$, $s' \neq 0$. Note that by the previous case, the coefficent of v_p in $B(r_{p-1}v_{p-1} + s'(r_p w_p + s w_q))$ is already 0. Replace w_p by $s'w_p$ and s by $s's$, and observe that we can continue in this fashion to obtain a nonzero element $r_1 w_1 + \cdots + r_p w_p + s w_q$ with $s \neq 0$ in the kernel of B. $\square$

Claim: There is a composition ψ of elementary maps such that the matrix for $\psi \circ \varphi$ is upper triangular.

This will complete the proof.

Since $\psi(y_1) \neq 0$, there is some entry $f_{j,1}$ that is nonzero. Applying a permutation of X, we may assume that $f_{1,1} \neq 0$. For every $j > 1$, our assumptions on R imply that there are $r, s \in R$, not both zero, so that $rf_{1,1} + sf_{j,1} = 0$. We do not have $s = 0$, for then $rf_{1,1} = 0$. As $f_{1,1} \neq 0$, this would imply that $r = 0$, a contradiction. The matrix for $E_{i,j}(r,s) \circ \varphi$ is the same as that for φ, except in row j. Its entry in column 1 of row j is $rf_{1,1} + sf_{j,1} = 0$.

Repeating this for each nonzero entry in the first column below $f_{1,1}$, we may assume that φ is an injection such that the corresponding matrix F has $f_{1,1} \neq 0$, but $f_{j,1} = 0$ for $j > 1$.

Suppose that we have found an injective map $\psi\colon R^m \to R^m$ so that the matrix F for $\psi \circ \varphi$ has the block form

$$F = \begin{pmatrix} T_{p\times p} & A_{p\times(n-p)} \\ 0_{(m-p)\times p} & B_{(m-p)\times(n-p)} \end{pmatrix},$$

where the subscripts indicate the sizes of the blocks, $T_{p\times p}$ is an upper triangular matrix with nonzero entries on its diagonal, and $A_{p\times(n-p)}$ and $B_{(m-p)\times(n-p)}$ are matrices of the indicated dimensions with entries in R.

We claim that the first column of B is nonzero. Indeed, if not, then applying the lemma to the map $g\colon R^{p+1} \to R^p$ given by the upper left $p\times(p+1)$ principal submatrix of F, there is a nonzero element $y = r_1 y_1 + \cdots + r_p y_p + s y_{p+1}$ with $s \neq 0$ such that $g(y) = 0$. Then $\varphi(y) = s(b_{p+1,p+1}x_{p+1} + \cdots + b_{m,p+1}x_m)$, which much be nonzero, as φ is injective. (I am indexing the elements of B by their position in F.) Then the same arguments as before, but on B, show that we may compose φ with an injective map so that the resulting matrix has this block form, where the first column of B has a unique nonzero entry in position $(p+1, p+1)$. Thus F now has a block form with the principal $(p+1)\times(p+1)$ submatrix upper triangular.

Continuing in this fashion completes the proof.

2. Suppose that M is an R-module and that for $i = 1, 2$, we have short exact sequences $0 \to N_i \to P_i \to M \to 0$ with P_1 and P_2 projective. Show that $P_1 \oplus N_2 \simeq P_2 \oplus N_1$ as R-modules.

Thanks to Josh Crouch.

Proof. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r, "i_1"] & P_1 \arrow[r, "f_1"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ 0 \arrow[r] & N_2 \arrow[r, "i_2"] & P_2 \arrow[r, "f_2"] & M \arrow[r] & 0 \end{tikzcd}$$

We will first show that there exist R-module homomorphisms $\phi\colon P_1 \to P_2$ and $\psi\colon N_1 \to N_2$ such that the diagram commutes. Since P_1 is projective, there exists an R-module homomorphism $\phi\colon P_1 \to P_2$ such that $f_2\phi = f_1$. Since $f_1 = 1_M f_1$, the above diagram with ϕ is commutative. Next, since i_1 is injective, there is an isomorphic copy of N_1 in P_1. Identifying N_1 with this isomorphic copy, define $\psi = \phi|_{N_1}$. Since ψ is the restriction of an R-module homomorphism to an R-submodule, ψ is an R-module homomorphism. Now, we will show Im$\psi \subset N_2$. If $n \in N_1$, then $\psi(n) = \phi(n)$. Since $f_1 = f_2\phi$ and the top row is exact, Imi_1 = Ker$f_2\phi$. Thus, $f_2(\phi(i_1(n))) = 0$, so $\phi(i_1(n)) \in$ Kerf_2 = Imi_2. Since we can identify N_1 and $i_1(N_1)$ as well as N_2 and $i_2(N_2)$, we see that $\phi(n) \in N_2$. Thus, $\psi\colon N_1 \to N_2$

is an R-module homomorphism. Since $i_2(\psi(n)) = i_2(\phi(n)) = \phi(n) = \phi(i_1(n))$, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, "i_1"] \arrow[d, "\psi"] & P_1 \arrow[r, "f_1"] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, "1_M"] & 0 \\
0 \arrow[r] & N_2 \arrow[r, "i_2"] & P_2 \arrow[r, "f_2"] & M \arrow[r] & 0
\end{tikzcd}$$

We will now show that

$$0 \to N_1 \xrightarrow{\alpha} P_1 \oplus N_2 \xrightarrow{\beta} P_2 \to 0$$

is exact where $\alpha(n) = (i_1(n), \psi(n))$ and $\beta(x, y) = \phi(x) - i_2(y)$. Note that α and β are R-module homomorphisms since i_1, ψ, ϕ, and i_2 are R-module homomorphisms. Thus, since P_2 is projective, the exactness of the above sequence will imply $P_1 \oplus N_2 \cong P_2 \oplus N_1$ by Theorem 3.4.

If $\alpha(n) = (0, 0)$, then $i_1(n) = 0$. The injectivity of i_1 implies $n = 0$, so α is injective.

Next, consider $\beta(\alpha(n)) = \beta(i_1(n), \psi(n)) = \phi(i_1(n)) - i_2(\psi(n))$. Since the diagram commutes, $\phi i_1 = i_2\psi$, so we see that $\beta(\alpha(n)) = i_2(\psi(n)) - i_2(\psi(n)) = 0$, and this implies $\text{Im}\alpha \subset \text{Ker}\beta$.

Next, suppose $(x, y) \in \text{Ker}\beta$. Then, $\phi(x) - i_2(y) = 0$, so $\phi(x) = i_2(y)$. Thus, $f_2\phi(x) = f_2 i_2(y) = 0$ since the bottom row is exact. This means $x \in \text{Ker} f_2\phi = \text{Ker} f_1 = \text{Im} i_1$. So, $x = i_1(m)$ for some $m \in N_1$. Making a substitution gives $i_2(y) = \phi(i_1(m)) = i_2\psi(m)$. Since i_2 is injective, $y = \psi(m)$. Thus, $\alpha(m) = (i_1(m), \psi(m)) = (x, y)$, so $\text{Ker}\beta \subset \text{Im}\alpha$.

Finally, we must show that β is a surjection. Suppose $q \in P_2$. Then, $f_2(q) = m$ for some $m \in M$. Since f_1 is a surjection, we have $m = f_1(p)$ for some $p \in P_1$. Thus, $f_2(q) = f_1(p) = f_2\phi(p)$, so $f_2(q - \phi(p)) = 0$ since f_2 is an R-module homomorphism. Since the bottom row is exact, it follows that $q - \phi(p) = i_2(n')$ for some $n' \in N_2$, so $q = \phi(p) + i_2(n') = \phi(p) - i_2(-n')$ since i_2 is an R-module homormorphism. Thus, $q = \beta(p, -n')$, so $q \in \text{Im}\beta$. Therefore, the sequence is exact, and $P_1 \oplus N_2 \cong P_2 \oplus N_1$. $\square$