Galois Groups of Schubert Problems

Wisconsin Colloquium

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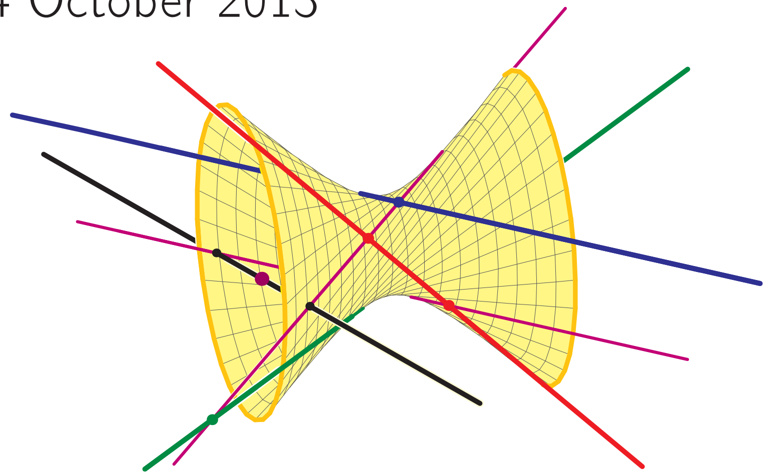

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Work with: Anton Leykin, Jon Hauenstein,

Christopher Brooks, Abraham Martín del Campo,

Jacob White, Zach Maril, and Aaron Moore, and others.

Galois theory and the Schubert calculus

Galois theory originated from understanding certain symmetries of polynomials. Later, Galois groups came to be understood as encoding all the symmetries of field extensions. Today, it is a pillar of number theory.

Galois groups also appear in geometry, particular enumerative geometry. This last aspect is not well-developed, because of its subtlety and because Galois groups are very hard to determine.

I will describe a project to shed more light on Galois groups in enumerative geometry. It is focussed on Galois groups in the *Schubert calculus*, a well-studied class of geometric problems involving linear subspaces.

It is best to begin with examples.

The Problem of Four Lines

What are the lines m_i meeting four general lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4?

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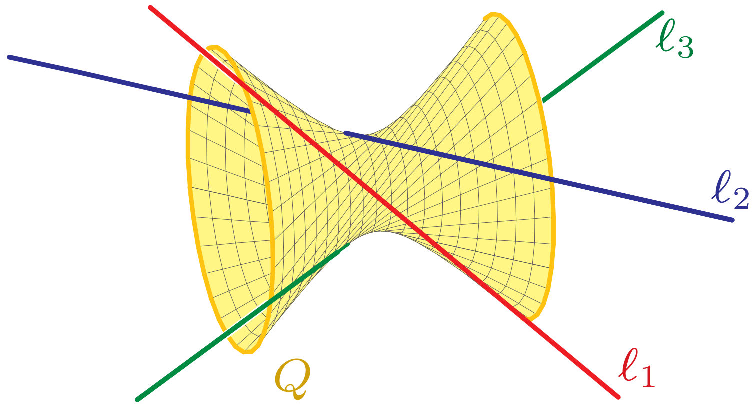

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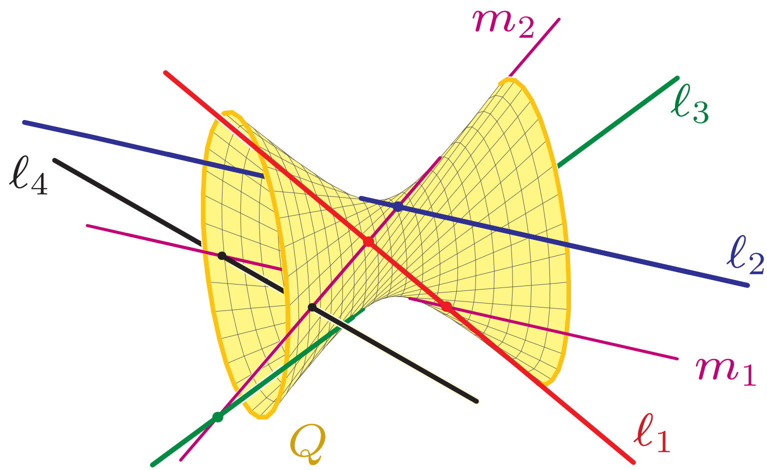

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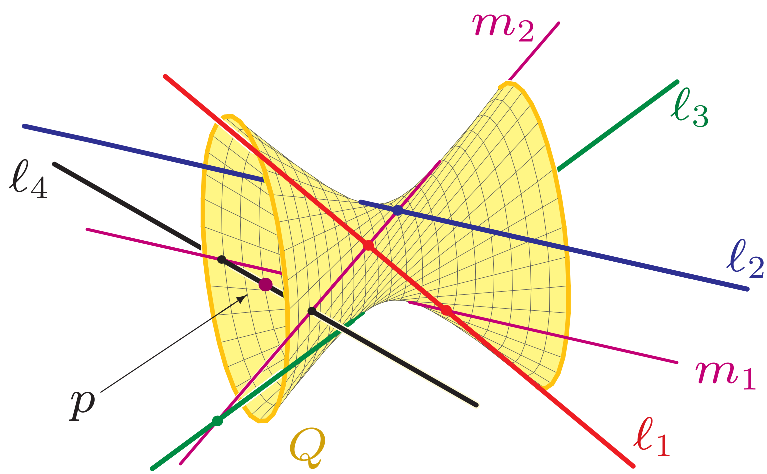

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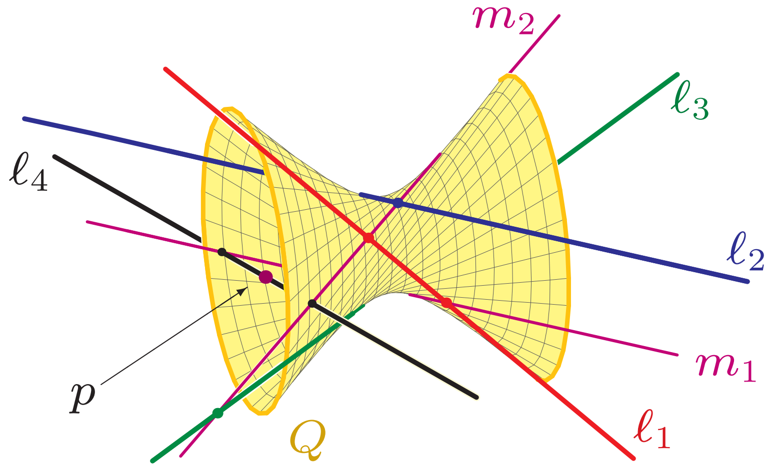


This shows that

The Galois group of the problem of four lines is the symmetric group $\mathcal{S}_2$.

A Problem with Exceptional Geometry

Q: What 4-planes H in $\mathbb{C}^8$ meet four general 4-planes K_1, K_2, K_3, K_4 in a 2-dimensional subspace of each?

Auxiliary problem: There are four (h_1, h_2, h_3, h_4) 2-planes in $\mathbb{C}^8$ meeting each of K_1, K_2, K_3, K_4. Schematically, $\ydiagram{3}^4 = 4$.

Fact: All solutions H to our problem have the form $H_{i,j} = \langle h_i, h_j \rangle$ for $1 \leq i < j \leq 4$. Schematically, $\ydiagram{2,2}^4 = 6$.

It follows that the Galois group of $\ydiagram{2,2}^4 = 6$ equals the Galois group of $\ydiagram{3}^4 = 4$, which is the symmetric group $\mathcal{S}_4$. This action is imprimitive.

This problem $\ydiagram{2,2}^4 = 6$ also has exceptional reality: If K_1, K_2, K_3, K_4 are real, then either two or six of the $H_{i,j}$ are real, and never four or zero.

Galois Groups of Enumerative Problems

In 1870, Jordan explained how *algebraic* Galois groups arise naturally from problems in enumerative geometry; earlier (1851), Hermite showed that such an algebraic Galois group coincides with a geometric monodromy group.

This Galois group of a geometric problem is a subtle invariant. When it is *deficient* (i.e. not the full symmetric group), the geometric problem has some exceptional, intrinsic structure.

Hermite's observation, work of Vakil, and some number theory together with modern computational tools give several methods to determine Galois groups, at least experimentally.

I will describe a project to study Galois groups for problems coming from the Schubert calculus using numerical algebraic geometry, symbolic computation, combinatorics, and more traditional methods.

Some Theory

A degree e dominant map $E \xrightarrow{\pi} B$ of equidimensional irreducible varieties (up to codimension one, $E \to B$ is a covering space of degree e) $\rightsquigarrow$ degree e extension of fields of rational functions $\pi^* K(B) \subset K(E)$. Define the Galois group $\mathrm{Gal}(E/B) \subset \mathcal{S}_e$ to be the Galois group of the Galois closure of this extension.

Hermite's Theorem. (Work over $\mathbb{C}$.) Restricting $E \to B$ to open subsets over which π is a covering space, $E' \to B'$, the Galois group is equal to the monodromy group of deck transformations.

This is the group of permutations of a fixed fiber induced by analytically continuing the fiber over loops in the base.

Point de départ: Such monodromy permutations are readily and reliably computed using methods of numerical algebraic geometry.

Enumerative Geometry

"Enumerative Geometry is the art of determining the number e of geometric figures x having specified positions with respect to other, fixed figures b."
— Hermann Cäser Hannibal Schubert, 1879.

B := configuration space of the fixed figures, and X := the space of the figures x we count. Then $E \subset X \times B$ consists of pairs (x, b) where $x \in X$ has given position with respect to $b \in B$.

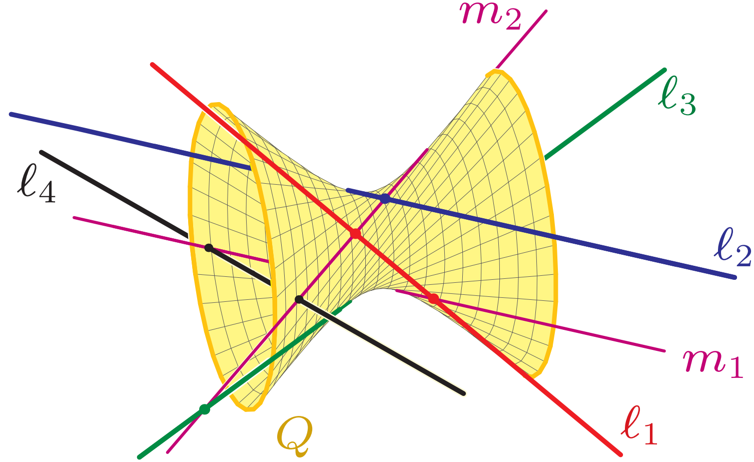
The projection $E \to B$ is a degree e cover outside of some discriminant locus, and the *Galois group of the enumerative problem* is $\mathrm{Gal}(E/B)$.

In the problem of four lines, B = four-tuples of lines, X = lines, and E consists of 5-tuples $(m, \ell_1, \ell_2, \ell_3, \ell_4)$ with m meeting each ℓ_i. We showed that this has Galois group the symmetric group $\mathcal{S}_2$.

Schubert Problems

The Schubert calculus is an algorithmic method promulgated by Schubert to solve a wide class of problems in enumerative geometry.

Schubert problems are problems from enumerative geometry involving linear subspaces of a vector space incident upon other linear spaces, such as the problem of four lines, and the problems $\square\!\square\!\square^{\,4} = 4$ and $\boxplus^{\,4} = 6$.



As there are many millions of computable Schubert problems, many with their own unique geometry, they provide a rich and convenient laboratory for studying Galois groups of geometric problems.

Proof-of-concept computation

Leykin and I used off-the-shelf numerical homotopy continuation software to compute Galois groups of some Schubert problems formulated as the intersection of a skew Schubert variety with Schubert hypersurfaces.

In every case, we found monodromy permutations generating the full symmetric group (determined by `Gap`). This included one Schubert problem with $e = 17,589$ solutions.

We conjectured that problems of this type will always have the full symmetric group as Galois group.

As a first step, White and I showed these Galois groups are 2-transitive.

The bottleneck to studying more general problems numerically is that we need numerical methods to solve *one* instance of the problem.

Numerical Project

Recent work, including certified continuation (Beltrán and Leykin), Littlewood-Richardson homotopies (Vakil, Verschelde, and S.), regeneration (Hauenstein), implementation of Pieri and of Littlewood-Richardson homotopies (Martín del Campo and Leykin) and new algorithms in the works will enable the reliable numerical computation of Galois groups of more general problems.

We plan to use a supercomputer whose day job is calculus instruction to investigate many of the millions of accessible and computable Schubert problems. Our intention is to build a library of Schubert problems (expected to be very few) whose Galois groups are deficient.

These data would be used to generate conjectures, leading to proofs about Galois groups of Schubert problems, as well as showcase the possibilities of numerical computation.

Vakil's Alternating Lemma

Suppose $S \subset B$ has a dense set of regular values of $E \to B$. Then

$$\mathrm{Gal}(E|_S/S) \hookrightarrow \mathrm{Gal}(E/B)\,.$$

Common in enumerative geometry are geometric degenerations

$$X \cap Y \rightsquigarrow W \cup Z$$

which give natural families $S \subset B$ such that

$$E|_S \simeq F \bigcup G \quad \text{where } F \to S \text{ and } G \to S$$

are child problems for W & Z of degrees f, g, with $f + g = e$.

Vakil's Alternating Criteria.

(1) If $f \neq g$ and both $\mathrm{Gal}(F/S)$ and $\mathrm{Gal}(G/S)$ contain the alternating groups A_f and A_g, then $\mathrm{Gal}(E/B)$ contains the alternating group A_e.

(2) If $\mathrm{Gal}(E/B)$ is two-transitive, then we only need $(f, g) \neq (6, 6)$.

Application of Vakil's Criterion

Theorem. (Brooks, Martín del Campo, S.) *The Galois group of any Schubert problem involving 2-planes in $\mathbb{C}^n$ is at least alternating.*

This used Vakil's Criterion (1), a degeneration of Schubert, and some very involved estimates of integrals coming from Weyl's character formula.

By Vakil's Criterion (2), to show high-transitivity (S_e or A_e), we often only need 2-transitivity. All known Galois groups of Schubert problems are either at least alternating or fail to be 2-transitive (in fact are imprimitive).

White and I are studying 2-transitivity.

Theorem. *Every Schubert problem involving 3-planes in $\mathbb{C}^n$ is 2-transitive.*

Every special Schubert problem (partition a single row) is 2-transitive.

$\rightsquigarrow$ The proof suggests that not 2-transitive implies imprimitive.

Vakil's Criteria II

Vakil's geometric Littlewood-Richardson rule, his criteria, and some 2-transitivity give an algorithm that can show a Schubert problem has at least alternating mondromy.

Python code written by Brooks is being modified by Maril and Moore to implement this algorithm. There are serious computer-science challenges to overcome.

Our goal is to use it to test all Schubert problems on all small Grassmannians (many billions to hundreds of billion Schubert problems), and get a second library of Schubert problems with deficient Galois groups.

Specialization Lemma

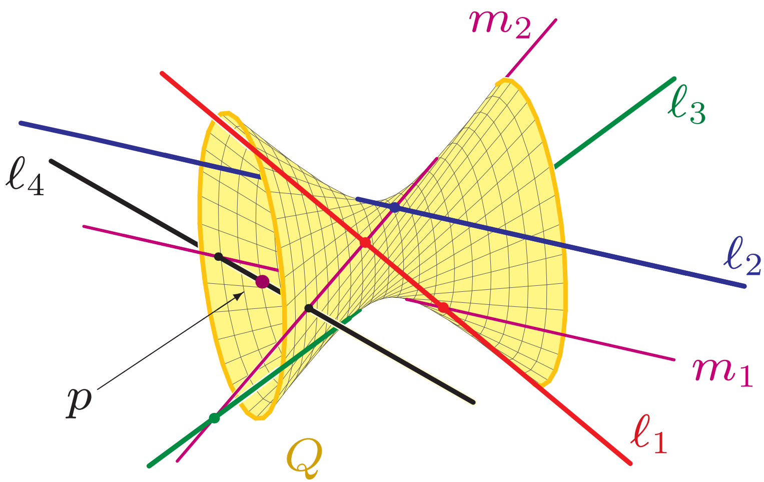
Given $\pi\colon E \to B$ with B rational, the fiber $\pi^{-1}(y)$ above a $\mathbb{Q}$-rational point $y \in B(\mathbb{Q})$ has a minimal polynomial $p_y(t) \in \mathbb{Q}[t]$. In this situation, the algebraic Galois group of $p_y(t)$ is a subgroup of $\mathrm{Gal}(E/B)$.

This can be applied to Schubert problems (and many other geometric problems). Working modulo a prime, the minimal polynomial of such fibers are easy to compute when $e \lesssim 500$. The degrees of its irreducible factors give the cycle type of a Frobenius element in the Galois group.

This quickly determines the Galois group when it is the full symmetric group, and allows the estimation of the Galois group when it is not.

Using Vakil's criteria and this method, we have nearly determined the Galois groups of all Schubert problems involving 4-planes in $\mathbb{C}^8$ and $\mathbb{C}^9$. What we obtain suggests the possibility of classifying all deficient Schubert problems and identifying their Galois groups.

Thank You!



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