Foundations of Mathematics — Math 300H Section 970

Twelfth Homework: Due 29 November 2022

1. Let A be a set. Prove that the identity function I_A is a bijection.
2. For each of the following, either give an example of functions $f\colon A \to B$ and $g\colon B \to C$ that satisfy the given properties, or explain why no such example exists. I urge simplicity.
   (a) The function f is a surjection, but the function $g \circ f$ is not a surjection.
   (b) The function f is an injection, but the function $g \circ f$ is not an injection.
   (c) The function g is a surjection, but the function $g \circ f$ is not a surjection.
   (d) The function g is an injection, but the function $g \circ f$ is not an injection.
   (e) The function f is not a surjection, but the function $g \circ f$ is a surjection.
   (f) The function f is not an injection, but the function $g \circ f$ is an injection.
   (g) The function g is not a surjection, but the function $g \circ f$ is a surjection.
   (h) The function g is not an injection, but the function $g \circ f$ is an injection.
3. For functions f, g, and h with domain and codomain $\mathbb{R}$, prove or disprove the following:
   (a) $(g+h) \circ f = (g \circ f) + (h \circ f)$.
   (b) $f \circ (g+h) = (f \circ g) + (f \circ h)$.

   Definition: The sum of two g and h with domain and codomain $\mathbb{R}$ is defined to be the function $g+h$ whose value at a number $x \in \mathbb{R}$ is $g(x) + h(x)$.
4. Let A and B be sets. Recall the definitions of the identity functions $I_A\colon A \to A$ and $I_B\colon B \to B$: For $a \in A$, $I_A(a) = a$ and for $b \in B$, $I_B(b) = b$.

   Let $f\colon A \to B$ be a function. Prove by a direct computation that $f = f \circ I_A$ and that $f = I_B \circ f$.
5. Let A, B, and C be nonempty sets, and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Suppose that $g \circ f\colon A \to C$ is an injection. Prove that f is an injection.

   Give an example of functions f and g with these properties illustrating that g need not be an injection.
6. Let A, B, and C be nonempty sets, and suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Suppose that $g \circ f\colon A \to C$ is a surjection. Prove that g is a surjection.

   Give an example of functions f and g with these properties illustrating that f need not be a surjection.
7. Let A, B, and C be nonempty sets, and suppose that $f\colon A \to B$, $g\colon B \to C$, and $h\colon B \to C$ are functions. For each of the following, prove or disprove:
   (a) If $g \circ f = h \circ f$, then $g = h$.
   (b) If f is one-to-one and $g \circ f = h \circ f$, then $g = h$.