BLOCH DISCRIMINANTS OF DISCRETE PERIODIC OPERATORS

MARGARET H. REGAN, FRANK SOTTILE, AND SIMON TELEN

Abstract. We describe how the qualitative behaviour of critical points of Bloch varieties depends upon parameters for three $\mathbb{Z}^2$-periodic graphs. This includes determining the number and arrangements of critical points and whether or not they are degenerate as the parameters vary. As the critical points are real-algebraic with respect to a non-standard real structure on the complex torus, this is also a first step towards developing computational tools for varieties with a non-standard real structure.

Introduction

Discrete operators on periodic graphs are an approximation to physical models of crystals, e.g., the tight binding approximations in solid state physics [1, 15]. In these models, the spectrum of the operator is the set of energies at which waves may propagate through the medium. After discretizing, the spectra of such operators may be studied from the perspective of (real) algebraic geometry via objects such as Bloch and Fermi varieties associated to it. This point of view was taken in [8], which considered how the behavior of Bloch varieties at extrema depends upon free parameters, observing in Theorem 12 *loc. cit.* that the Bloch variety has degenerate critical points either at almost all parameter values or at almost no parameter values. This algebraic perspective is developed in [23]. The view we take here is that of computational algebraic geometry as well. We aim to describe explicitly how the features of a Bloch variety depend upon parameters. We develop some general theory and work out case studies for several graphs. These graphs include the hexagonal lattice and a graph whose Bloch variety has a curve of degenerate critical points, similar to the example of Filonov and Kachkovskiy [10].

Let Γ be a graph equipped with a free action of $\mathbb{Z}^d$ having finitely many orbits on its edges $\mathcal{E}(\Gamma)$ and vertices $\mathcal{V}(\Gamma)$. We label Γ with parameters $p = (e, V)$ where $e\colon \mathcal{E}(\Gamma) \to \mathbb{R}_+$ and $V\colon \mathcal{V}(\Gamma) \to \mathbb{R}$ are constant on orbits. In Section 1, we recall how these data define a Schrödinger operator H_p on the Hilbert space $\ell_2(\mathcal{V}(\Gamma))$. Applying the Floquet transform (a version of the Fourier transform), H_p is represented by a matrix $H_p(z)$ of Laurent polynomials in the Floquet parameters $z \in \mathbb{T}^d$. Here, $\mathbb{T} \subset \mathbb{C}^\times$ is the group of unit complex numbers, which is the set of unitary characters of $\mathbb{Z}$. For $z \in \mathbb{T}^d$, the Floquet matrix $H_p(z)$ is Hermitian, and so all of its eigenvalues are real. The Bloch variety $\mathcal{B}_p$ is the hypersurface in $\mathbb{T}^d \times \mathbb{R}$ defined by the dispersion polynomial $D_p(z, \lambda) = \det(\lambda \cdot \mathrm{id} - H_p(z)) = 0$. It encodes the interaction between the spectrum of H_p and the representation theory of $\mathbb{Z}^d$.

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We are concerned with the critical points of the coordinate function λ on the Bloch variety, their number, location, multiplicity, etc., as the parameters $p = (e, V)$ vary. The *Bloch discriminant* is the subset of the parameter space $\mathbb{R}^{|\mathcal{E}(\Gamma)|} \times \mathbb{R}^{|\mathcal{V}(\Gamma)|}$ where a qualitative change of the critical points occurs. Its complement consists of connected components, called discriminant chambers with the property that all operators whose parameters line in a single chamber have qualitatively similar arrangements of critical points. A precise definition will be given in Section 1 (Definition 1.6). Describing the Bloch discriminant and the arrangement of critical points for each chamber is an instance of the general problem of determining the geography of parameter space [5, 13].

Besides illuminating the behavior of the family of operators, this question is challenging in the case of Bloch varieties, as the features we are studying (critical points) are real points of $(\mathbb{C}^\times)^d \times \mathbb{C}$ for a non-standard real structure: That is, the usual complex conjugation $(z, \lambda) \mapsto (\overline{z}, \overline{\lambda})$ is replaced by the antiholomorphic involution $(z, \lambda) \mapsto (1/\overline{z}, \overline{\lambda})$. The fixed (real) points of this new structure are $\mathbb{T}^d \times \mathbb{R}$, in contrast to the classical real points $(\mathbb{R}^\times)^d \times \mathbb{R}$. A goal of this paper is to initiate the development of computational and analytical tools for real algebraic geometry of algebraic varieties with a non-standard real structure.

As a first step towards this goal, we propose to study discriminants of algebraic hypersurfaces $f(z) = 0$ in $(\mathbb{C}^\times)^d$ which are stable under the involution $z \mapsto z^{-1}$. The (complexified) Bloch variety has this property, because $H_p(z^{-1}) = H_p(z)^\top$. In analogy with the classical literature on A-discriminants [11], we define *reciprocal A-discriminants* in Section 2. We prove some first properties and show that one can find relations among critical energies and parameters $p = (e, V)$ by specializing the reciprocal A-discriminant.

Bring in relation to other work, and complete this.

1. Bloch varieties

We recall some notation and definitions from [9]. This is standard, and may be found in many sources, e.g. [4, Ch. 4]. A $\mathbb{Z}^d$-periodic graph is a simple graph Γ equipped with a free action of $\mathbb{Z}^d$ that has finitely many orbits on the vertices $\mathcal{V}(\Gamma)$ and edges $\mathcal{E}(\Gamma)$ of Γ. We write the action of $\mathbb{Z}^d$ on $\mathcal{V}(\Gamma)$ additively: the image of $u \in \mathcal{V}(\Gamma)$ under $\alpha \in \mathbb{Z}^d$ is $u+\alpha \in \mathcal{V}(\Gamma)$. We choose a fundamental domain $W \subset \mathcal{V}(\Gamma)$, which is a set of representatives of orbits of $\mathbb{Z}^d$ on $\mathcal{V}(\Gamma)$. Similarly, let E be the set of $\mathbb{Z}^d$-orbits in $\mathcal{E}(\Gamma)$. A labeling of Γ is a pair $p = (e, V)$ of $\mathbb{Z}^d$-periodic functions $e\colon \mathcal{E}(\Gamma) \to \mathbb{R}$ and $V\colon \mathcal{V}(\Gamma) \to \mathbb{R}$. Equivalently, $p = (e, V)$ is a point in $\mathbb{R}^E \times \mathbb{R}^W$. Figure 1 shows (part of) the $\mathbb{Z}^2$-periodic hexagonal (honeycomb) lattice. Its fundamental domain contains two vertices and it has three orbits of edges with labels a, b, and c as shown.

These are parameters for the tight binding model [1, 15], which involves the discrete Schrödinger operator L acting on functions $f\colon \mathcal{V}(\Gamma) \to \mathbb{C}$. For a labeling $p \in \mathbb{R}^E \times \mathbb{R}^W$, the corresponding *discrete Schrödinger operator* H_p acts on $f\colon \mathcal{V}(\Gamma) \to \mathbb{C}$ as follows:

$$H_p(f)(u) \;=\; V(u)f(u) - \sum_{u\sim v} e(u,v)\cdot f(v)\,. \tag{1}$$

Here the sum is over all vertices $v \in \mathcal{V}(\Gamma)$ for which there is an edge between u and v, written $u \sim v$ for short. This is a potential plus a perturbed graph Laplacian. In the

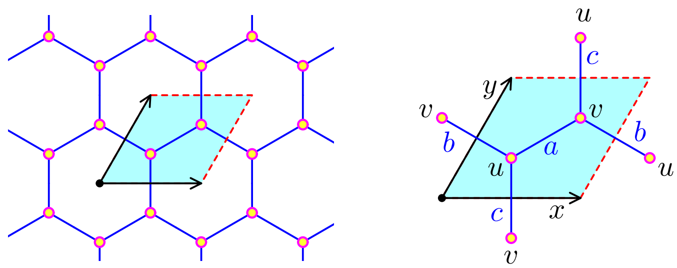


FIGURE 1. Hexagonal (honeycomb) lattice with fundamental domain and a labeling.

standard formulation, the Laplacian contributes a sum of terms $e(u,v)f(u)$ to (1). For the formulation we give, that sum $\sum_v e(u,v)$ is absorbed into the potential $V(u)$, yeilding (1). This gives a self-adjoint operator on the Hilbert space $\ell_2(\Gamma)$ of square summable functions on $\mathcal{V}(\Gamma)$. Its spectrum is a union of intervals in $\mathbb{R}$.

Important features of the spectrum of H_p are revealed by the Floquet transform $f \mapsto \hat{f}$, which is a version of the Fourier transform. References for Floquet theory include [4, 17, 23]. Let $\mathbb{T} \subset \mathbb{C}^\times$ be the group of unit complex numbers, which we identify with the set of unitary characters on $\mathbb{Z}$ via $z(n) = z^n$ for $n \in \mathbb{Z}$ and $z \in \mathbb{T}$. Similarly, $\mathbb{T}^d$ is the group of unitary characters on $\mathbb{Z}^d$. For $\alpha \in \mathbb{Z}^d$ and $z \in \mathbb{T}^d$, we write z^α for $z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. The Floquet transform $\hat{f}$ of a function f on $\mathcal{V}(\Gamma)$ is a function on $\mathbb{T}^d \times \mathcal{V}(\Gamma)$ that is quasi-periodic: $\hat{f}(z, u+\alpha) = z^\alpha \cdot \hat{f}(z,u)$ for $u \in \mathcal{V}(\Gamma)$ and $\alpha \in \mathbb{Z}^d$. Such a quasiperiodic function is determined by its values on $\mathbb{T}^d \times W$, which identifies the space of Floquet transforms with $L^2(\mathbb{T}^d, \mathbb{C}^W) = L^2(\mathbb{T}^d)^W$, the space of square integrable functions on $\mathbb{T}^d$ with values in $\mathbb{C}^W$. In fact, the Floquet transform is a linear isometry between $\ell_2(\Gamma)$ and $L^2(\mathbb{T}^d)^W$.

As the labeling is $\mathbb{Z}^d$-periodic, H_p commutes with the $\mathbb{Z}^d$-action and hence with the Floquet transform. For $u \in W$, we have

$$H_p(\hat{f})(z,u) \;=\; V(u)\hat{f}(z,u) - \sum_{u \sim v+\alpha} e(u, v+\alpha) \cdot z^\alpha \cdot \hat{f}(z,v)\,, \tag{2}$$

where the sum is over $v \in W$ and $\alpha \in \mathbb{Z}^d$ such that u is adjacent to $v+\alpha$. From the expression (2), we see that the operator H_p is represented by a $|W| \times |W|$ *Floquet matrix*, $H_p(z)$, whose entries are Laurent polynomials in z. As $u \sim v+\alpha$ if and only if $v \sim u-\alpha$, and both edges have the same label, the matrix satisfies the identity

$$H_p(z)^T \;=\; H_p(z^{-1})\,. \tag{3}$$

Since $z \in \mathbb{T}^d$, we have $\overline{z} = z^{-1}$, so that the Floquet matrix $H_p(z)$ is Hermitian.

Example 1.1. Writing $z = (x,y)$ for points in $\mathbb{T}^2$, the Floquet matrix for the hexagonal lattice of Figure 1 is

$$H_p(x,y) \;=\; \begin{pmatrix} u & -a-bx^{-1}-cy^{-1} \\ -a-bx-cy & v \end{pmatrix}\,. \tag{4}$$ ◇

We now introduce the geometric objects we want to study. Write $D_p(z,\lambda)$ for the characteristic polynomial $\det(\lambda \cdot \mathrm{id}_{|W|} - H_p(z))$, which is called the *dispersion polynomial*.

Definition 1.2 (Real Bloch variety). The *real Bloch variety* of H_p is the hypersurface

$$\mathcal{B} \;=\; \mathcal{B}_p \;:=\; \{(z,\lambda) \in \mathbb{T}^d \times \mathbb{R} \mid D_p(z,\lambda) = 0\}\,.$$

A level set $\mathcal{F}_p(\lambda) = \{z \in \mathbb{T}^d \mid D_p(z,\lambda) = 0\}$ of the Bloch variety is called a *real Fermi variety*.

A reason this is important is that the *spectrum* $\sigma(H_p)$ of the Schrödinger operator H_p is the projection of the real Bloch variety to the $\mathbb{R}$-axis. Thus the Bloch variety encodes the interaction of the spectrum of H_p with the action of $\mathbb{Z}^d$ on $\ell_2(\Gamma)$.

Example 1.3. Figure 2 shows two Bloch varieties for the honeycomb lattice with zero potential, along with the spectrum of the operator. For display purposes, we represent $\mathbb{T}^2$ by a fundamental domain $[-\frac{\pi}{2}, \frac{3\pi}{2}]^2$ in $\mathbb{R}^2$. On the left the edge parameters are $(a,b,c) = (6,3,2)$, and on the right they are $(a,b,c) = (1,1,1)$, giving the graph Laplacian. $\diamond$

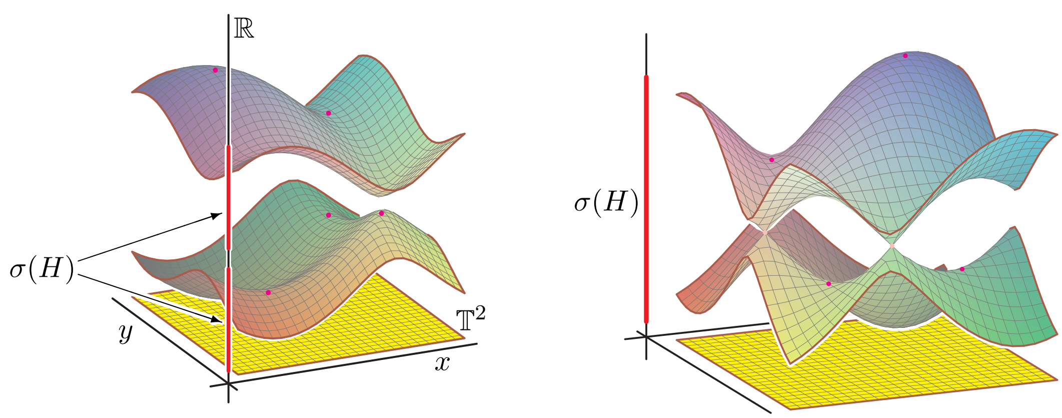


FIGURE 2. Bloch varieties with edge parameters (6,3,2) and (1,1,1).

The terminology *real* in Definition 1.2 deserves some clarification. Although $D_p(z,\lambda)$ has coefficients in $\mathbb{R}$, it is customary in algebraic geometry to study the *complex* points of the hypersurface defined by the equation $D_p(z,\lambda) = 0$. Here is the corresponding definition.

Definition 1.4. The *(complex) Bloch variety* of H_p is the hypersurface

$$\mathcal{B}_p(\mathbb{C}) \;=\; \{(z,\lambda) \in (\mathbb{C}^\times)^d \times \mathbb{C} \mid D_p(z,\lambda) = 0\}\,.$$

The *(complex) Fermi variety* $\mathcal{F}_p(\lambda,\mathbb{C})$ is the level set of the Bloch variety at $\lambda \in \mathbb{C}$. $\diamond$

Such a complexification has conceptual and computational advantages. However, not all complex values of z,λ are physically meaningful. In many applications, one is interested in real solutions. Abstractly, a point (z,λ) is *real* if it is fixed by a given antiholomorphic involution. The most common example of such an involution is taking complex conjugates: $(z,\lambda) \mapsto (\bar{z},\bar{\lambda})$. Since D_p has real coefficients, $\mathcal{B}_p(\mathbb{C})$ is stable under this involution, meaning that (z,λ) belongs to $\mathcal{B}_p(\mathbb{C})$ if and only if $(\bar{z},\bar{\lambda})$ does. The fixed points are the ordinary real points $\mathcal{B}_p(\mathbb{C}) \cap ((\mathbb{R}^\times)^d \times \mathbb{R})$. In spectral theory the meaningful choice of involution is $(z,\lambda) \mapsto (\overline{z^{-1}},\bar{\lambda})$. Since $D_p(z,\lambda) = D_p(z^{-1},\lambda)$, $\mathcal{B}_p(\mathbb{C})$ is stable under this involution as well. Its real points are precisely the points in the real Bloch variety $\mathcal{B}_p$.

The choice of an antiholomorphic involution (real structure) on a complex algebraic variety X determines which points of X are real. Most computational algorithms in real algebraic geometry assume the standard real structure corresponding to complex conjugation. Bloch varieties motivate the need for development of real algebro-geometric tools for natural non-standard real structures, such as that of our involution $(z, \lambda) \mapsto (\overline{z^{-1}}, \bar{\lambda})$.

The coordinate projection $\lambda\colon (\mathbb{C}^\times)^d \times \mathbb{C} \to \mathbb{C}$ restricts to a function on the complex Bloch variety $\mathcal{B}_p(\mathbb{C})$. Taking cues from polynomial optimization, in [9] it is shown that the *critical points* of the function λ are those (z, λ) which satisfy the critical point equations,

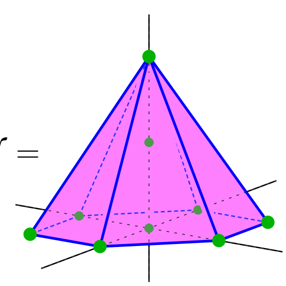
$$D_p \;=\; z_1 \frac{\partial D_p}{\partial z_1} \;=\; \cdots \;=\; z_d \frac{\partial D_p}{\partial z_d} \;=\; 0\,. \tag{5}$$

A motivation to study critical points of λ is the *spectral edges nondegeneracy conjecture* [18, Conj. 5.25], which asserts that for generic values of the parameters $p \in \mathbb{R}^E \times \mathbb{R}^W$, the extrema of the function λ on each branch of the real Bloch variety are nondegeerate. Many important notions in physics depend upon this holding. Other features, such as irreducibility of the Fermi variety [19] are also of interest to mathematical physics.

To express the results of [9], we introduce some objects from geometric combinatorics related to the dispersion polynomial $D_p(z, \lambda)$. The exponents of monomials in $D_p(z, \lambda)$ form a subset of $\mathbb{Z}^d \times \mathbb{N}$, called its *support*. Their convex hull is its *Newton polytope*, $\mathcal{N} = \mathcal{N}(D_p(z, \lambda))$. (We give examples in subsequent sections.) This structure underlies the formulation of the critical point equations (5), while multiplication by $z_i \in \mathbb{C}^\times$ does not change the solution set in $(\mathbb{C}^\times)^d$ and the Newton polytope of $z_i \frac{\partial D_p}{\partial z_i}$ is a subset of the Newton polytope of D_p. A result of [9] states that the number of critical points, counted with multiplicity, is bounded above by the normalized volume of $\mathcal{N} = \mathcal{N}(D_p(z, \lambda))$, that is, the Euclidean volume $\int_{\mathcal{N}} 1 \cdot \mathrm{d}z_1 \cdots \mathrm{d}z_d \, \mathrm{d}\lambda$ multiplied by $(d+1)!$.

Example 1.5. Consider the dispersion polynomial $D_p = \det(\lambda \cdot \mathrm{id} - H_p)$, with H_p as in (4). Using a computer algebra system, such as `Macaulay2` [12], `Oscar` [21], or `Singular` [7], we may check that for general values of the parameters p, the critical point equations (5) have twelve complex solutions. We will show this later. This equals the upper bound given in [9, Cor. 2.5], which is the normalized volume of the Newton polytope of the dispersion polynomial. We display the support of the dispersion polynomial as the columns of a matrix and its Newton polytope, $\mathcal{N}$:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{N} = \tag{6}$$



The base of $\mathcal{N}$ is a hexagon with area 3 and it is a pyramid of height 2. As the base has area 3, this pyramid has Euclidean volume 2 and hence normalized volume $3!\mathrm{vol}(\mathcal{N}) = 12$. $\diamond$

We are now ready to define our main object of study.

Definition 1.6. Let Γ be a $\mathbb{Z}^d$-periodic graph and let $\mathcal{P} \subseteq \mathbb{R}^E \times \mathbb{R}^W$ be a subset of the space of labelings. A *Bloch discriminant* for $(\Gamma, \mathcal{P})$ is a hypersurface $\mathrm{BD}(\Gamma, \mathcal{P}) \subset \mathbb{R}^E \times \mathbb{R}^W$ satisfying the following property: For each connected component of $\mathcal{P} \smallsetminus \mathrm{BD}(\Gamma, \mathcal{P})$, the number of real solutions $(z, \lambda) \in \mathbb{T}^d \times \mathbb{R}$ of (5) is constant for all p in that component.

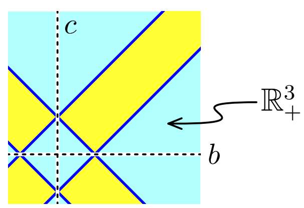
When $\mathcal{P} = \mathbb{R}^E \times \mathbb{R}^W$ is the full space of labelings, we will simply write $\mathrm{BD}_\Gamma = \mathrm{BD}(\Gamma, \mathcal{P})$. The connected components of $\mathcal{P} \smallsetminus \mathrm{BD}(\Gamma, \mathcal{P})$ are called *discriminant chambers*. We will compute a first Bloch discriminant for the hexagonal lattice—summarized below in Theorem 1.7 and expanded on in Section 4.

Theorem 1.7. *Let Γ be the hexagonal lattice of Figure 1. The quartic polynomial*

$$\mathrm{BD}_\Gamma \;=\; (a+b+c)(a+b-c)(a-b+c)(a-b-c)$$

is a Bloch discriminant for Γ.

The sign of this discriminant dictates the critical point structure. For simplicity, let the discriminant polynomial $\mathrm{BD}_\Gamma = \Delta$. It is worth noting that $\Delta < 0$ and $a, b, c > 0$ if and only if a, b, c are the sides of a triangle. We display this in the (b, c)-plane where $a = 1$.



The dotted lines are where b or c vanish, the solid lines are where $\Delta = 0$. The discriminant Δ is negative in the yellow regions and positive in the light blue regions.

Table 1 summarizes the landscape of the critical points based on corresponding parameter values. Note: Maggie is going to fix formatting for this to make it nicer...

	Δ	# nondegenerate critical points	# degenerate critical points	Hessian
$u \neq v$	< 0	12	0	
	> 0	8	0	
	$= 0$	6	1 (mult. 6)	
$u = v$	< 0	8	2 (Dirac)	
	> 0	8	0	
	$= 0$	6	2 (mult. 3)	

TABLE 1. Critical point landscape summary for the hexagonal lattice.

We introduce some tools for studying singular points of Fermi varieties. The λ-coordinate of a critical point is the corresponding *critical energy*.

Lemma 1.8. *The Fermi variety $\mathcal{F}_p(\lambda, \mathbb{C})$ is singular if and only if λ is a critical energy, and its singular points are critical points of the Bloch variety.*

Proof. The Fermi variety $\mathcal{F}_p(\lambda_0, \mathbb{C})$ is singular at z_0 if and only if (z_0, λ_0) satisfies (5). □

As studied in [11], a hypersurface is singular if and only if an appropriate discriminant vanishes at its coefficients. This naturally leads to the topic of our next section.

2. Discriminants of reciprocal polynomials

A Laurent polynomial $f(z)$ is *reciprocal* if it is invariant under inversion: $f(z) = f(z^{-1})$. The dispersion polynomial $D_p(z, \lambda)$, as a polynomial in z, is reciprocal. The hypersurface $V(f)$ of a reciprocal polynomial is invariant under inversion in that $z \in V(f) \Leftrightarrow z^{-1} \in V(f)$. Let us consider a family of reciprocal hypersurfaces defined by

$$f(z,p) \;=\; p_0\left(\frac{z^{a_0}+z^{-a_0}}{2}\right) + p_1\left(\frac{z^{a_1}+z^{-a_1}}{2}\right) + \cdots + p_m\left(\frac{z^{a_m}+z^{-a_m}}{2}\right) \;=\; 0\,. \tag{7}$$

The exponents $a_0, \dots, a_m \in \mathbb{Z}^d$ are distinct and we assume that the span $\mathbb{R}^d$, for otherwise the hypersurface is invariant under a continuous subgroup of $(\mathbb{C}^\times)^d$. As the hypersurface (7) remains unchanged after multiplying the parameters p_i by a nonzero scalar, we consider p as a point in m-dimensional projective space $\mathbb{P}^m$. The *discriminant* $\mathrm{disc}_z(f)$ characterizes values of $p_0, \dots, p_m$ for which the hypersurface (7) has a singular point in $(\mathbb{C}^\times)^d$.

A simple, but useful observation is that the reciprocal structure causes such singular points at the fixed points of $z \mapsto z^{-1}$ under linear conditions on p. Let us denote this set of 2^d fixed points by $\mathrm{CP} := \{(\pm 1, \dots, \pm 1)\} \subset \mathbb{T}^d$. All partial derivatives $\frac{\partial f}{\partial z_j}$ vanish at each point in CP. The fixed points CP are called *corner points* in [3], where an argument using the two real structures is used to show they are critical points of the Bloch variety. The authors of *loc. cit.* also give conditions for the extrema in a band to occur at a corner point.

Lemma 2.1. *If $p \in \mathbb{P}^m$ is such that any of the linear forms $\{f(z^*, p) \mid z^* \in \mathrm{CP}\}$ vanishes, then the hypersurface* (7) *is singular.*

Lemma 2.1 identifies 2^d linear factors of $\mathrm{disc}_z(f)$. All other contributions are captured by the *reciprocal discriminant*. We encode the parametric polynomial $f(z, p)$ by an integer matrix $A \in \mathbb{Z}^{d \times (m+1)}$ whose columns are the exponent tuples $a_0, \dots, a_m$.

Definition 2.2. Let f be as in (7). The *reciprocal A-discriminant variety* ∇_A^{rec} is the Zariski closure of the set

$$\Big\{p \in \mathbb{P}^m \mid f(z,p) = z_1\frac{\partial f}{\partial z_1}(z,p) = \cdots = z_d\frac{\partial f}{\partial z_d}(z,p) = 0,\ \text{for some } z \in (\mathbb{C}^\times)^d \setminus \mathrm{CP}\Big\}\,.$$

Here $f(z, p)$ is the reciprocal polynomial in (7).

Notice that for generic $p \in \nabla_A^{\mathrm{rec}}$, the singular points will not occur at a corner point.

Remark 2.3. The classical theory of *A-discriminants* developed by Gel'fand, Kapranov and Zelevinsky [11] also associates a discriminant to an integer matrix A, which vanishes when a corresponding hypersurface is singular. This starts from a family of Laurent polynomials $g(z, q) \;=\; q_0\, z^{a_0} + q_1\, z^{a_1} + \cdots + q_m\, z^{a_m}$ and asks for which $q \in \mathbb{P}^m$ the hypersurface $g(z, q) = 0$

has a singular point in $(\mathbb{C}^\times)^d$. The *A-discriminant variety* ∇_A is the Zariski closure of the set

$$\left\{q \in \mathbb{P}^m \mid g(z,q) = \frac{\partial g}{\partial z_1}(z,q) = \cdots = \frac{\partial g}{\partial z_d}(z,q) = 0, \text{ for some } z \in (\mathbb{C}^\times)^d\right\}.$$

When this is a hypersurface, its defining polynomial is the A-discriminant, $\Delta_{\tilde{A}}$. We will illustrate this in Example 2.7 below. ⋄

Proposition 2.4. *The reciprocal A-discriminant variety $\nabla_A^{\mathrm{rec}} \subset \mathbb{P}^m$ is irreducible of codimension at least one.*

Proof. Consider the incidence variety

$$Y \ := \ \left\{(z,p) \in ((\mathbb{C}^\times)^d \setminus \mathrm{CP}) \times \mathbb{P}^m \mid f(z,p) = z_1\frac{\partial f}{\partial z_1}(z,p) = \cdots = z_d\frac{\partial f}{\partial z_d}(z,p) = 0\right\}.$$

By definition, the variety ∇_A^{rec} is the closure of the image of the projection $Y \to \mathbb{P}^m$. Hence, it suffices to show that Y is irreducible of dimension $\leq m-1$. For this, consider the other projection $\pi\colon Y \to (\mathbb{C}^\times)^d \setminus \mathrm{CP}$. A fiber $\pi^{-1}(z)$ of that projection is a linear space in $\mathbb{P}^m$.

For this, we assume that the initial column of A is 0 and write the equations for Y in a propitious manner (inspired by the formulation in [20]). Let $\tau := (2, z^{a_1} + z^{-a_1}, \ldots, z^{a_m} + z^{-a_m})$ be monomials in $2f(z,p)$, and let $\tau' := (0, z^{a_1} - z^{-a_1}, \ldots, z^{a_m} - z^{-a_m})$. Writing both as row vectors and the coefficients p as column vectors, we have

$$f \ = \ f(z,p) \ = \ \tau \cdot p\,. \tag{8}$$

Then the vector of toric derivatives $2z_i\partial f/\partial z_i$ is

$$A \cdot \mathrm{Diag}(\tau') \cdot p\,. \tag{9}$$

For a fixed $z \in (\mathbb{C}^\times)^d$ the linear equations $\tau \cdot p = 0$ and $A \cdot \mathrm{Diag}(\tau') \cdot p = 0$ define $\pi^{-1}(z)$. If $z \notin \mathrm{CP}$, then this set of linear equations has rank $d+1$. Indeed, the coefficient of p_0 is 2 in the first equation and 0 in the remaining ones. Our assumption on the exponents is that the matrix A has rank d, and as $z \notin \mathrm{CP}$ no entry of $\mathrm{Diag}(\tau')$ vanishes except for the initial entry corresponding to p_0. This completes the proof. □

Proposition 2.4 justifies the following definition.

Definition 2.5. If ∇_A^{rec} has codimension one, then the *reciprocal A-discriminant* is an irreducible polynomial $\Delta_A^{\mathrm{rec}} \in \mathbb{Z}[p_0, \ldots, p_m]$ with pairwise prime coefficients whose complex variety is ∇_A^{rec}. This polynomial is defined up to multiplication by -1. If ∇_A^{rec} has codimension more than one, then we set $\Delta_A^{\mathrm{rec}} = 1$.

Remark 2.6. It follows from Proposition 2.4 that the reciprocal A-discriminant variety is the projective dual variety X^* of the d-dimensional variety $X \subset \mathbb{P}^m$ parametrized by

$$z \ \longmapsto \ \left(\frac{z^{a_0} + z^{-a_0}}{2} : \cdots : \frac{z^{a_m} + z^{-a_m}}{2}\right).$$

Hence, it is a hypersurface unless X is ruled by linear spaces of positive dimension.

If $a_0 = 0$, then X is the projective closure of the affine variety in $\mathbb{C}^m$ parametrized by $u \mapsto (\cos(a_1 \cdot u), \ldots, \cos(a_m \cdot u))$, which was called a *Chebyshev variety* in [2]. ⋄

We will consider exponents a_i appearing in dispersion polynomials for certain graphs, beginning with the hexagonal lattice.

Example 2.7. Observe that the disersion polynomial $D_p = \det H_p$, with H_p as in (4), belongs to the following family of Laurent polynomials:

$$g(z,q) \;=\; q_0 + q_1\, x + q_2\, y + q_3\, xy^{-1} + q_4 x^{-1} y + q_5 x^{-1} + q_6 y^{-1}\,.$$

The exponents appearing in this polynomial are the columns of the matrix

$$A \;=\; \begin{pmatrix} 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

These are the lattice points in the polygon which is the base of the pyramid shown in (6). The A-discriminant (Remark 2.3) for $g(z,q)$ is homogeneous of degree 12 with 127 terms.

$$\begin{aligned}\Delta_{\tilde{A}} \;=\; & q_0^6 q_1 q_2 q_3 q_4 q_5 q_6 - q_0^5 q_1^2 q_2 q_4 q_5 q_6^2 - q_0^5 q_1^2 q_3 q_4^2 q_5 q_6 - q_0^5 q_1 q_2^2 q_3 q_5^2 q_6 - q_0^5 q_1 q_2 q_3^2 q_4 q_5^2 \\ & - q_0^5 q_1 q_2 q_3 q_4^2 q_6^2 - q_0^5 q_2^2 q_3^2 q_4 q_5 q_6 + q_0^4 q_1^3 q_4^2 q_5 q_6^2 + q_0^4 q_1^2 q_2^2 q_5^2 q_6^2 + \cdots + 16 q_2^2 q_3^4 q_4^4 q_6^2\,.\end{aligned}$$

This polynomial vanishes at those points $q \in \mathbb{P}^6$ for which $g(z,q)$ defines a singular curve in $(\mathbb{C}^\times)^2$. This six-dimensional family of Laurent polynomials contains a three-dimensional reciprocal subfamily obtained by setting

$$q_0 = p_0, \quad 2q_1 = 2q_5 = p_1, \quad 2q_2 = 2q_6 = p_2, \quad 2q_3 = 2q_4 = p_3\,. \tag{10}$$

We obtain the following instance of Equation (7).

$$f(z,p) \;=\; p_0 + p_1\left(\frac{x+x^{-1}}{2}\right) + p_2\left(\frac{y+y^{-1}}{2}\right) + p_3\left(\frac{xy^{-1}+x^{-1}y}{2}\right). \tag{11}$$

Substituting (10) in Δ_A we obtain

$$\frac{1}{256}(p_0+p_1-p_2-p_3)(p_0-p_1+p_2-p_3)(p_0-p_1-p_2+p_3)(p_0+p_1+p_2+p_3)\cdot(\Delta_A^{\mathrm{rec}})^2\,,$$

where $\Delta_A^{\mathrm{rec}} = 2p_0p_1p_2p_3 - p_1^2p_2^2 - p_1^2p_3^2 - p_2^2p_3^2$. The linear factors are predicted by Lemma 2.1. The quartic Δ_A^{rec} is the reciprocal A-discriminant with $A = \left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1\end{smallmatrix}\right)$. It is the defining equation of the dual surface of the unirational Chebyshev surface in $\mathbb{P}^3$ parametrized by

$$(x,y) \;\longmapsto\; \left(1 : \frac{x+x^{-1}}{2} : \frac{y+y^{-1}}{2} : \frac{xy^{-1}+x^{-1}y}{2}\right).$$

This surface is alternatively parametrized by $(1 : \cos(u) : \cos(v) : \cos(u-v))$. The dual surface $\Delta_A^{\mathrm{rec}} = 0$ is parametrized by $(m_{234} : -m_{134} : m_{124} : -m_{123})$, where m_{ijk} are the maximal minors of the following 3×4-matrix:

$$\begin{pmatrix} 1 & \cos(u) & \cos(v) & \cos(u-v) \\ 0 & -\sin(u) & 0 & -\sin(u-v) \\ 0 & 0 & -\sin(v) & \sin(u-v) \end{pmatrix}.$$

Further specializing the A-discriminant to the dispersion polynomial D_p is done by setting

$$p_0 \;=\; (\lambda-u)(\lambda-v) - a^2 - b^2 - c^2\,, \quad p_1 \;=\; -2ab\,, \quad p_2 \;=\; -2ac\,, \quad p_3 \;=\; -2bc\,. \tag{12}$$

These are the coefficients of D_p when represented in the form (7). Here we abbreviated $u_0 = u+a+b+c$ and $v_0 = v+a+b+c$. with this substitution, the reciprocal discriminant Δ_A^{rec} specializes to $16\,a^2b^2c^2\,(\lambda-u)(\lambda-v)$. $\diamond$

We will analyze this discriminant further in Section 4. Example 2.7 illustrates how to associate an $\tilde{A}$-discriminant to the reciprocal A-discriminant: one takes $\tilde{A}$ to be the matrix whose columns are $\pm a_i$, where only one copy of the zero column is retained. The reciprocal A-discriminant is much simpler than that associated $\tilde{A}$-discriminant. In particular, its degree is significantly smaller.

Theorem 2.8. *If the reciprocal A-discriminant is a hypersurface, then we have* $\deg(\Delta_A^{\mathrm{rec}}) \leq \frac{1}{2}(\deg(\Delta_{\tilde{A}}) - 2^d)$.

Proof. sketch We start by showing that the restriction of the A-discriminant to the reciprocal subspace is not identically zero. That is, the polynomial Δ_A does not evaluate to the zero polynomial when substituting q by p as in (10). Let us write $\Delta_A(L_{\mathrm{rec}})$ for this substitution. That is, $L_{\mathrm{rec}} \subset \mathbb{P}^{|A|-1}$ is the linear space of reciprocal polynomials. gap

Note that $\Delta_A(L_{\mathrm{rec}})$ is a polynomial of degree $\deg(\Delta_A)$ in the variables $p_0,\ldots,p_m$. All hyperplanes in Lemma 2.1 are distinct, and $\Delta_A(L_{\mathrm{rec}})$ has 2^d linear factors. By construction, one of the remaining factors of $\Delta_{\tilde{A}}(L_{\mathrm{rec}})$ is Δ_A^{rec}. We claim that, in fact, the factor Δ_A^{rec} appears with an exponent ≥ 2 (as we saw in Example 2.7). Indeed, for a generic point on ∇_A^{rec}, the hypersurface (7) has at least two singular points related by taking reciprocals. This means that the A-discriminant self-intersects along L_{rec}. One way to see this is via the Horn uniformization of ∇_A [14]. $\square$

We illustrate this theorem in the next example.

Example 2.9. We add two more columns to the matrix A from Example 2.7.

$$A \;=\; \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

The associated matrix $\tilde{A}$ has 11 columns. We compute using the software from [22] that the $\tilde{A}$-discriminant has degree 24, and its Newton polytope contains 200249 lattice points. Computing its coefficients is infeasible. However, we may compute the reciprocal discriminant without computing the A-discriminant first. Let

$$f(z,p) \;=\; p_0 + p_1\left(\frac{x+x^{-1}}{2}\right) + \cdots + p_4\left(\frac{xy+x^{-1}y^{-1}}{2}\right) + p_5\left(\frac{x^2+x^{-2}}{2}\right)$$

be the polynomial in (11) plus two new terms. Specializing the A-discriminant to this reciprocal family, we expect four linear factors (Lemma 2.1) multiplied by $(\Delta_A^{\mathrm{rec}})^2$. The exponent 2 here is explained by the fact that for a generic point $p \in \nabla_A^{\mathrm{rec}}$, the hypersurface $V(f)$ has two singular points. Hence, the A-discriminant self-intersects along the reciprocal A-discriminant. From this observation, we learn that Δ_A^{rec} has degree 10. We sample rational points on ∇_A^{rec} via its parametrization, and interpolate to find the irreducible polynomial Δ_A^{rec}. The result has (only) 249 terms. We will relate this to a graph in Section 5. $\diamond$

3. Resultants and sums of squares

Dispersion polynomials are reciprocal in the z-variables. We will see in examples that the specialization of the reciprocal A-discriminant and the factors identified in Lemma 2.1 describe the dependence of the critical energies on the parameters $p = (e, V)$. We will be interested in detecting for which p two or more critical energies coincide. For this, we exploit an additional feature of the dispersion polynomials considered in this article arising as our graphs are bipartite: They are determinants of structured 2×2 matrices.

Proposition 3.1. *Let H be a 2×2-matrix of the following form.*

$$L \;=\; \begin{pmatrix} f(p) & g(z,p) \\ g(z^{-1},p) & h(p) \end{pmatrix}, \tag{13}$$

where f, h are polynomials in $p = (p_1, \ldots, p_n)$, and g is a bivariate Laurent polynomial in z whose exponent vectors span $\mathbb{R}^2$. Let $D = \det(\lambda \mathrm{id}_2 - H)$ be the characteristic polynomial of H. The closure of the projection of the incidence variety

$$\Big\{(\lambda, p, z) \in \mathbb{C} \times \mathbb{C}^n \times (\mathbb{C}^*)^2 \,\Big|\, D \;=\; z_1 \frac{\partial D}{\partial z_1} \;=\; z_2 \frac{\partial D}{\partial z_2} \;=\; 0\Big\}$$

to $\mathbb{C} \times \mathbb{C}^n$ contains the hypersurface $\{(\lambda, p) \mid (\lambda - f(p))(\lambda - h(p)) \prod_{z^* \in \mathrm{CP}} D(p, \lambda, z^*) = 0\}$.

Proposition 3.1 uses the notation $\mathrm{CP} = \{-1, 1\}^2$ for the fixed points of $z \mapsto z^{-1}$. It shows that, for each choice of p, there are at least 10 critical energies, counting multiplicities. The condition on the Newton polygon of g ensures that the system $g(z, p) = g(z^{-1}, p)$ has solutions in $(\mathbb{C}^\times)^2$.

Proof of Proposition 3.1. For a general point on $f(p) = \lambda$, we have $\det H = g(z, p)g(z^{-1}, p)$. The equations $D = z_1 \partial_{z_1} D = z_2 \partial_{z_2} D = 0$ are satisfied at the singular points of the curve defined by $g(z, p)g(z^{-1}, p) = 0$, which include points where the the two branches meet, e.g. points z satisfying $g(z, p) = g(z^{-1}, p) = 0$. The same argument applies for $h(p) = \lambda$. The factors $D(p, \lambda, z^*)$ are essentially those identified in Lemma 2.1. □

Example 3.2. The matrix $H_p(x, y)$ in Example 1.1 has the form (13) with $f(p) = u$, $g(z, p) = -a - bx^{-1} - cy^{-1}$ and $h(p) = v$. The factors $\lambda - u, \lambda - v$ were identified as factors of the specialized reciprocal A-discriminant in Example 2.7. ⋄

A first step in determining parameter values p for which critical energies coincide is to investigate the discriminants and pairwise resultants of the factors in Proposition 3.1, viewed as polynomials in λ.

Proposition 3.3. *Let D, f, g, h, z^* be as in Proposition 3.1. If f, g, h have coefficients in a subfield $K \subset \mathbb{C}$, then the discriminant $\mathrm{disc}_\lambda(D(\lambda, z^*, p))$ of D viewed as a polynomial in λ is a sum of squares of polynomials in p with coefficients in K. More specifically, we have*

$$\mathrm{disc}_\lambda(D(\lambda, z^*, p)) \;=\; (f(p) - h(p))^2 + 4\, g(z^*, p)^2.$$

Proof. This is a specialization of the general form $(a_{11} - a_{22})^2 + 4a_{12}^2$ for the discriminant of a symmetric 2×2-matrix (a_{ij}). □

Proposition 3.4. *Let D, f, g, h be as in Proposition 3.1 and let $z^*, z_1^*, z_2^* \in \mathrm{CP}$.*

(1) Up to sign, the resultant $\mathrm{res}_\lambda(D(p,\lambda,z_1^*), D(p,\lambda,z_2^*))$ *equals* $(g(z^*,p_1)^2 - g(z^*,p_2)^2)^2$.
(2) Up to sign, the resultant $\mathrm{res}_\lambda(D(p,\lambda,z^*), f(p)-\lambda)$ *equals* $g(z^*,p)^2$.

Proof. Writing $g_i = g(z,p_i^*)$ for short, we find using Sylvester's determinant formula that

$$\mathrm{res}_\lambda(D(p,\lambda,z_1^*), D(p,\lambda,z_2^*)) = \det\begin{pmatrix} fh-g_1^2 & 0 & fh-g_2^2 & 0 \\ -f-h & fh-g_1^2 & -f-h & fh-g_2^2 \\ 1 & -f-h & 1 & -f-h \\ 0 & 1 & 0 & 1 \end{pmatrix}$$
$$= \det\begin{pmatrix} fh-g_1^2 & 0 & g_1^2-g_2^2 & 0 \\ -f-h & fh-g_1^2 & 0 & g_1^2-g_2^2 \\ 1 & -f-h & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This proves point (1). A similar 3×3 determinant expansion shows point (2). $\square$

Note that Proposition 3.4 implies that $f(p)$ is a root of $D(p,\lambda,z^*)$ precisely when z^* is a singular solution of $g(z,p) = g(z^{-1},p) = 0$. Check this.

4. The Hexagonal Lattice

Consider the family of operators on the hexagonal lattice of Figure 1. The fundamental domain contains two vertices, hence $|W| = 2$. We write u and v for the value of the potential V at these vertices. There are three orbits of edges, with labels a, b, and c. Writing $z = (x,y)$ to avoid subscripts, the Floquet matrix $H_p(x,y)$ was given in (4), and its dispersion polynomial $D_p(x,y,\lambda)$ is

$$(\lambda-u)(\lambda-v) - (a+bx^{-1}+cy^{-1})(a+bx+cy) \tag{14}$$

We eliminate the variables x and y from the critical point equations (5), obtaining a degree 10 polynomial in λ, whose coefficients depend on $p = (a,b,c,u,v)$. There are six irreducible factors. All of them are predicted by Proposition 3.1:

$$\begin{gathered} \lambda - u \\ \lambda - v \\ (\lambda-u)(\lambda-v)-(a+b+c)^2 \\ (\lambda-u)(\lambda-v)-(a+b-c)^2 \\ (\lambda-u)(\lambda-v)-(a-b+c)^2 \\ (\lambda-u)(\lambda-v)-(a-b-c)^2 \end{gathered} \tag{15}$$

The first two factors arise from the diagonal entries in the Floquet matrix and each of the last four corresponds to a corner point, where $(x,y) = (\pm1,\pm1)$. The roots of these six polynomials in λ are the ten critical energies. The linear factors appear in the reciprocal discriminant $\Delta_A^{\mathrm{rec}} = 16\,a^2b^2c^2(\lambda-u)(\lambda-v)$ from Example 2.7. The quadratic factors are the linear forms from Lemma 2.1 after substituting (12).

For generic values of p, there are ten distinct (complex) critical energies. This fails when a discriminant from Proposition 3.3 or a resultant from Proposition 3.4 vanishes.

Theorem 1.7. *Let Γ be the hexagonal graph from Figure 1. The quartic polynomial* [1]

$$\begin{aligned} \mathrm{BD}_\Gamma &= (u-v)^2(a+b+c)(a+b-c)(a-b+c)(a-b-c) \\ &= a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2 \,. \end{aligned}$$

is a Bloch discriminant for Γ.

Proof. The resultants from Proposition 3.4(1) do not contribute to the Bloch discriminant. The corresponding critical points have the same critical energy, but they lie above different corner points. The resultants from Proposition 3.4(2) are (the squares of) $(a \pm b \pm c)$.

The discriminants from Proposition 3.3 are the sums of squares

$$(u-v)^2 + 4\,(a \pm b \pm c)^2 \,.$$

These vanish only when $u - v = a \pm b \pm c = 0$, which is accounted for by BD_Γ. ☐

The factor $(u-v)^2$ gives two cases dependent on the potentials u and v. However, the analysis of the quadratic factors of (15) does not depend on the potentials. We study this first.

Consider the quadratic factor

$$\lambda^2 - \lambda(u+v) - (a+b+c)^2 \,.$$

Its discriminant is the sum of squares $(u-v)^2 + 4(a+b+c)^2$. The critical energies are

$$\frac{u+v}{2} \pm \sqrt{(u-v)^2 + 4(a+b+c)^2 2} \,.$$

When $u = v$, these become

$$u + \pm 2|a+b+c| \,.$$

They are real, and for $(a,b,c) \in \mathbb{R}^3$, they are distinct and occur at $(x,y) = (1,1)$. Working this out for all four quadratic factors in (15), we find eight real critical energies

$$\frac{u+v}{2} \pm \sqrt{(u-v)^2 + 4(a \pm b \pm c)^2} \,. \tag{16}$$

When (a,b,c) are positive and $b \neq c$, all eight critical energies are distinct. For each of these critical energies λ, the Fermi curve $\mathcal{F}(\lambda)$ has a singularity at the corresponding critical point $(\pm 1, \pm 1, \lambda)$ whose signs correspond to the signs in the term $(a \pm b \pm c)$ in (16). The Newton polytope of the Fermi curve is the hexagon which has one integer point in the interior, so $\mathcal{F}(\lambda)$ has arithmetic genus one and is a rational curve. While we do not pursue this, it is possible to determine the signature of the Hessian at each nondegenerate critical point. Do this! Maggie to fill in...

From here, the results in the study of the linear factors differs depending on if $u \neq v$ or $u = v$. First, we assume $u \neq v$. If $\lambda = u$ or $\lambda = v$, the determinant $D(z, \lambda)$ factors, and the Fermi variety $\mathcal{F}(\lambda)$ is reducible,

$$D(z,\lambda) \;=\; (-a - bx^{-1} - cy^{-1})(-a - bx - cy) \,.$$

By Lemma 1.8, the complex critical points at this critical energy are singular points of the Fermi variety, which are the intersections of the curves defined by the two factors, as each

[1] F: do we not need also a factor of $(u-v)^2$ (needed for sign).

curve is smooth. For general a, b, c, there are two such points. Indeed, let us solve the system

$$-a - bx^{-1} - cy^{-1} \;=\; -a - bx - cy \;=\; 0\,.$$

Multiplying these by cy and $a + bx^{-1}$ respectively, and subtracting the first from the second gives the univariate consequence

$$0 \;=\; abx + (a^2 + b^2 - c^2) + abx^{-1}\,. \tag{17}$$

Substituting its two solutions into $y = -\frac{1}{c}(a + bx)$ gives the two critical points

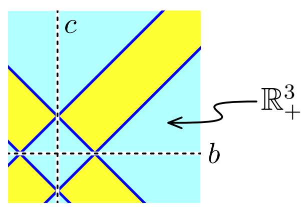
$$(x, y) \;=\; \Bigl(-\frac{a^2 + b^2 - c^2}{2ab} \;\pm\; \frac{1}{2ab}\sqrt{\Delta}\;,\; -\frac{a^2 - b^2 + c^2}{2ac} \;\mp\; \frac{1}{2ac}\sqrt{\Delta}\Bigr)\,, \tag{18}$$

where Δ is the discriminant of the polynomial (17)

$$\Delta \;=\; (a^2 + b^2 - c^2)^2 - 2a^2b^2\,.$$

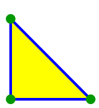
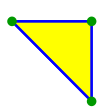
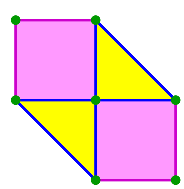
This is equal to $(a-b-c)(c-a-b)(b-a-c)(a+b+c)$, the part of the Bloch discriminant BD_Γ not involving the potentials. The two points (18) are interchanged under the involution $(x, y) \mapsto (x^{-1}, y^{-1})$.

Observe that when $\Delta > 0$, the two points (18) lie in $\mathbb{R}^2$ and are therefore not critical points of the real Bloch variety. When $\Delta < 0$, they both lie on the compact torus $\mathbb{T}^2$ and both are critical points of the real Bloch variety. It is worth noting that $\Delta < 0$ if and only if $|a|, |b|, |c|$ are the sides of a triangle. We display this in the (b, c)-plane where $a = 1$.



The dotted lines are where b or c vanish, the solid lines are where $\Delta = 0$. The discriminant Δ is negative in the yellow regions and positive in the light blue regions.

This number, 2, of solutions is the mixed area of the Newton polygons of the factors, which is the upper bound given by the BKK Theorem [6]. Consequently, the two curves meet transversally at each solution when $\Delta = 0$. Below, we show the Newton polytopes P and $-P$ of each factor, as well as their Minkowski sum $P + (-P)$, which is the hexagonal base of the Newton polytope $\mathcal{N}$ of the dispersion relation shown in (6).

(19) $P \;=$  $\;-P =$  $\;P + (-P) \;=$ 

The mixed area is given by $\mathrm{Area}(P + (-P)) - \mathrm{Area}(P) - \mathrm{Area}(-P) = 3 - \frac{1}{2} - \frac{1}{2} = 2$.

As a, b, c are all nonzero, at most one factor of the discriminant Δ vanishes when $\Delta = 0$. For appropriate signs, this factor is $a \pm b \pm c$. In this case, the two points (18) coincide with the corner point $(\pm 1, \pm 1)$ having the same signs, which is a degenerate critical point. This explains the appearence of Δ as a factor in the Bloch discriminant.

This analysis also determines the spectrum of L, showing that it has a band gap. Indeed, if $\Delta > 0$, so that the critical points from the linear factors of (15) do not lie in $\mathbb{T}^2$, and if we let m, respectively M, be the minimum (respectively maximum) of the quantities

$$\left\{ \sqrt{(u-v)^2 + 4(a \pm b \pm c)^2} \;\middle|\; \text{all choices of } \pm \right\} ,$$

and write μ for $\frac{u+v}{2}$, then the spectrum of H is

$$[\mu - M, \mu - m] \bigcup\, [\mu + m, \mu + M] \ .$$

As $|u - v| < m$, there is a band gap.

If $\Delta < 0$, let us assume that $u > v$. Then as $0 < u - v < m$,

$$u \ < \ \tfrac{u+v}{2} + (u-v) \ < \ \tfrac{u+v}{2} + m\,.$$

We also have $v > \frac{u+v}{2} - m$. Thus the spectrum of H is

$$[\mu - M, v] \bigcup\, [u, \mu + M] \ .$$

This has a band gap of $u - v$.

In either case, the critical point equations have twelve distinct solutions, although not all real. As this equals Kuchnirenko's bound [16] (normalized volume of Newton polytope [9]), all solutions occur with multiplicity 1. By [9, Thm. 5.2], all critical points are nondegenerate and the spectral edges conjecture holds when $u \neq v$ and $\Delta \neq 0$.

When $\Delta = 0$, exactly one of the factors in Δ vanishes and as we remarked there are two degenerate critical points corresponding to that factor at energies u and v. Figure 3 shows three Bloch varieties with their corresponding parameter values.

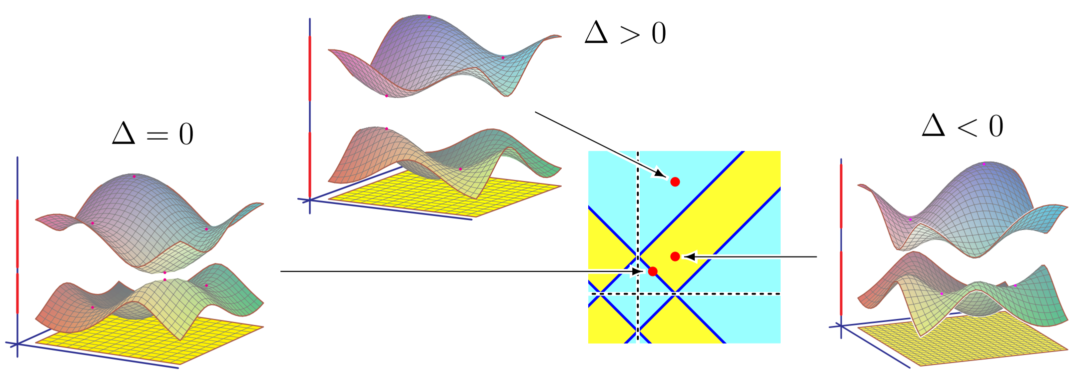


FIGURE 3. Three Bloch varieties with $u \neq v$ at parameters $(1, 0.4, 0.6)$, $(1, 3, 1)$, and $(1, 1, 1)$.

Suppose now that $u = v$. The two linear factors in (15) become equal and the elimination polynomial acquires a square factor. Setting $\kappa := \lambda - u$, the dispersion polynomial becomes

$$\kappa^2 \ - \ (-a - bx^{-1} - cy^{-1})(-a - bx - cy)\,. \tag{20}$$

When $\Delta \neq 0$, the curves defined by the two factors meet transversally at the points (18). Then $w = (-a - bx^{-1} - cy^{-1})$ and $z = (-a - bx - cy)$ give local coordinates at these points.

Δ	# nondegenerate critical points	# degenerate critical points	Hessian
< 0	12	0	
> 0	8	0	
$= 0$	6	1 (mult. 6)	

TABLE 2. Critical point landscape summary for the hexagonal lattice for when $u \neq v$.

In these coordinates, the Bloch variety is defined by $\kappa^2 - wz = 0$, showing that these critical points when $u = v$ are ordinary double point singularities on the complex Bloch variety.

If $\Delta > 0$, then the points (18) lie in $\mathbb{R}^2$ and the corresponding critical points are not on the real Bloch variety. In this case, the real Bloch variety is smooth with eight nondegenerate critical points and a band gap as $|a \pm b \pm c| > 0$. We see this at the top (middle) of Figure 2.

If $\Delta < 0$, then the points (18) lie in $\mathbb{T}^2$. To analyze the singularity, we introduce local coordinates that are fixed under the non-standard real structure (conjugation) giving $\mathbb{T}^2$. Observe that under this conjugation, $\overline{w} = z$. Then $r = \frac{w+z}{2}$ and $s = \frac{w-z}{2\sqrt{-1}}$ are real (fixed under conjugation) local coordinates. In a neighborhood of a critical point the Bloch variety is defined by $\kappa^2 = r^2 + s^2$, showing that it is a double cone, called a *Dirac point*. These may be observed on the right in Figures 2 and 4.

When $\Delta = 0$, the functions w and z no longer give local coordinates at the corresponding critical point, as their gradients are dependent. Thus the singularity (20) is not an ordinary double point, but of higher order. Figure 4 shows three Bloch varieties when $u = v$ with their corresponding parameter values.

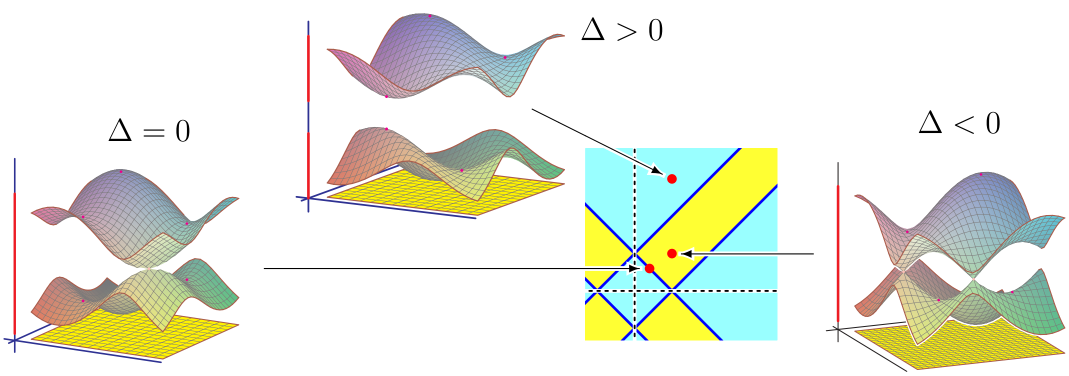


FIGURE 4. Three Bloch varieties with $u = v$ at parameters $(1, 3, 1)$, $(1, 0.4, 0.6)$, and $(1, 1, 1)$.

5. Another bipartite graph

References

[1] N. Ashcroft and N. Mermin. Solid state physics. Cengage Learning, 2022.