ENRICHED SCHUBERT PROBLEMS IN THE GRASSMANNIAN OF LAGRANGIAN SUBSPACES IN 8-SPACE

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Abstract. We explain how to formulate Schubert problems on the Lagrangian Grassmannian, and then describe the three enriched Schubert problems on the Lagrangian Grassmannian $LG(4)$ of isotropic 4-planes in 8-space, and use that to determine their Galois groups. (This is in progress.)

Assume that $\mathbb{K}$ is a field not of characteristic 2. Suppose that $\langle\bullet,\bullet\rangle$ is a nondegenerate alternating form on a finite-dimensional $\mathbb{K}$-vector space V. Nondegenerate means that the map $\varphi\colon V \to V^*$ is an isomorphism from V to its dual space, or rather that, for all $0 \neq v \in V$, there is some $u \in V$ such that $\langle v,u\rangle \neq 0$. Alternating means that for all $u,v \in V$, we have $\langle v,u\rangle = -\langle u,v\rangle$. The existence of such a form implies that V has even dimension $2n$ for some positive integer n.

For a linear subspace $H \subset V$, its *annihilator* is

$$H^{\angle} \;:=\; \{u \in V \mid \langle u,h\rangle = 0 \quad \forall\, h \in H\}\,.$$

A linear subspace $H \subset V$ is *isotropic* if $H \subset H^{\angle}$, so that $\langle h,h'\rangle = 0$ for all $h,h' \in H$. As $\langle\ ,\ \rangle$ is a nondegenerate, we have $\dim H \leq \dim H^{\angle} = 2n - \dim H$, so that $\dim H \leq n$.

An isotropic subspace L of dimension n is necessarily maximal (and vice-versa). We call such a maximal isotropic subspace *Lagrangian*. The set of all Lagrangian subspaces forms the *Lagrangian Grassmannian*, $LG(V)$ or $LG(n)$. This has dimension $\binom{n+1}{2}$. It has a transitive action by the *symplectic group*, which is the group of all linear transformations of V that preserve the form $\langle\bullet,\bullet\rangle$,

$$\mathrm{Sp}(V) \;=\; \mathrm{Sp}(n) \;:=\; \{g \in \mathrm{GL}(V) \mid \langle gv,gu\rangle = \langle v,u\rangle \quad \forall u,v \in V\}\,.$$

A complete flag $F_\bullet\colon \{0\} \subset F_1 \subset \cdots \subset F_{2n-1} \subset V$ ($\dim F_j = j$) of subspaces in V is *isotropic* if $F_i^{\angle} = F_{2n-i}$. Consequently, an isotropic flag is determined by its restriction $F_1 \subset \cdots \subset F_n$ to F_n, which is Lagrangian. Let $\mathbb{F}\ell_n$ be the set of all isotropic flags, which we call the *sympletic flag manifold*. This has a transitive action of the symplectic group $\mathrm{Sp}(n)$ and is an algebraic manifold of dimension n^2.

The *attitude* of a Lagrangian subspace L with respect to an isotropic flag is the sequence $A(L,F_\bullet) := (j \in [2n] \mid L\cap F_{j-1} \subsetneq L \cap F_j)$ of indices of elements of $F_\bullet$ where the dimension of $L \cap F_j$ jumps. As L and $F_\bullet$ are isotropic, any attitude A is necessarily symmetric in that

$$j \in A \iff 2n - j \notin A\,.$$

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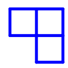
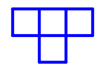
Consequently, A is determined by its intersection with $[n] := \{1,\ldots,n\}$. For reasons that will become clear in the sequel, we define $\lambda(A) := \{n+1-j \mid j \in A\cap[n]\}$. It is an exercise that all subsets λ of $[n]$ may occur. For each $\lambda \subset [n]$ and isotropic flag $F_\bullet$, the set

$$\Omega^\circ_\lambda F_\bullet \;:=\; \{L \in LG(n) \mid \lambda = \lambda(A(L,F_\bullet))\}$$

is isomorphic to the affine space $\mathbb{K}^{\binom{n+1}{2}-|\lambda|}$, where $|\lambda| = \sum_{\ell\in\lambda}\ell$. It is traditional to write such a subset in decreasing order, $\lambda\colon \lambda_1 > \cdots > \lambda_r > 0$, and refer to this as a *(strict) partition*. A strict partition λ is represented by an array of boxes, with λ_i in the ith row. These are not left-justified, but rather the ith row begins in the $i$the position from the left. For example,

$(2,1) \longleftrightarrow$  and $(3,1) \longleftrightarrow$  .

We have just described the Schubert decomposition of $LG(n)$ into Schubert cells. The closure of the Schubert cell $\Omega^\circ_\lambda F_\bullet$ is the *Schubert variety* $\Omega_\lambda F_\bullet$. We have

$$\Omega_\lambda F_\bullet \;=\; \{L \in LG(V) \mid \dim L\cap F_{n+1-\lambda_i} \geq i \quad i = 1,\ldots,r\}\,. \tag{1}$$

(Here, λ has r parts.) *A priori*, there are other conditions for higher-dimensional intersections, but they are implied from these as L and the flag $F_\bullet$ are isotropic.

A *Schubert problem* on $LG(n)$ is a list of strict partitions $\boldsymbol{\lambda} = \lambda^1,\ldots,\lambda^r$ such that $\sum_i |\lambda^i| = \binom{n+1}{2}$. An *instance* of a Schubert problem $\boldsymbol{\lambda}$ is given by a list $F^1_\bullet,\ldots,F^r_\bullet$ of isotropic flags, and its *solutions* are the points in the intersection of Schubert varieties

$$\Omega_{\lambda^1}F^1_\bullet \;\cap\; \Omega_{\lambda^2}F^2_\bullet \;\cap\; \cdots \;\cap\; \Omega_{\lambda^r}F^r_\bullet\,. \tag{2}$$

It is known that there is a dense open set $U \subset (\mathbb{F}\ell_n)^r$ and a nonnegative integer $d = d(\boldsymbol{\lambda})$ such that for all instances $(F^1_\bullet,\ldots,F^r_\bullet) \in U$, either $d(\boldsymbol{\lambda}) = 0$ and the intersection (2) is empty, or $d(\boldsymbol{\lambda}) > 0$ and the intersection (2) is transverse and consists of $d(\boldsymbol{\lambda})$ Lagrangian subspaces. These numbers $d(\boldsymbol{\lambda})$ are known and easy to compute.

1. Formulating Schubert Problems on $LG(n)$

To represent a Schubert problem (2) on a computer, we need to choose coordinates and give defining equations. There exists an ordered basis $e_1,\ldots,e_{2n}$ for $V \simeq \mathbb{K}^{2n}$ such that for any indices $i \leq j$, we have $\langle e_i, e_j\rangle = \delta_{j,2n+1-i}$. Write ω_0 for the $n\times n$ matrix with 1's along its anti-diagonal, in positions $(i, n+1-i)$. It represents the longest permutation in S_n. Let J be the $2n\times 2n$-matrix with 2×2 block structure $\left(\begin{smallmatrix} 0 & \omega_0 \\ -\omega_0 & 0\end{smallmatrix}\right)$. Then, if $u,v \in V$ are expressed in this basis, we have $\langle u,v\rangle = u^T J v$.

With respect to these coordinates, both the standard coordinate flag $E'_\bullet$ and the reverse coordinate flag $E_\bullet$ are isotropic. (For $E'_\bullet$, the i-th subspace E'_i is spanned by the first i basis vetors $e_1,\ldots,e_i$ and the i-th subspace E_i of $E_\bullet$ is spanned by the last i basis vectors $e_{2n+1-i},\ldots,e_{2n}$.)

The Schubert cell $\Omega^\circ_\emptyset E_\bullet$ is dense in $LG(n)$ and its elements are represented as the column span of a matrix $\left(\begin{smallmatrix}\omega_0 \\ X\end{smallmatrix}\right)$, where X is a symmetric matrix. For example, when $n = 2$, we display J and a parametrization of $\Omega^\circ_\emptyset E_\bullet$

$$\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix} \qquad \begin{pmatrix} 0 & 1\\ 1 & 0\\ a & b\\ b & c\end{pmatrix}. \qquad (a,b,c\in\mathbb{K})$$

We may represent an isotropic flag $F_\bullet$ by a $2n\times 2n$ invertible matrix (also written $F_\bullet$) such that F_i is the span of the first i columns of the matrix $F_\bullet$. That $F_\bullet$ is isotropic has the following consequences for the matrix $F_\bullet$. For each $i=1,\dots,2n$, let v_i be the ith column of $F_\bullet$. Then

$$\langle v_i, v_j\rangle \;=\; v_i^T J v_j \;=\; 0 \quad\text{for}\quad i+j\leq 2n\,.$$

That is, the skew-symmetrix matrix $F_\bullet^T J F_\bullet$ has only 0 entries above its anti-diagonal. Observe that there is not neessarily a linear parametrization of the space of isotropic flags.

Steifel coordinates for a subset $Y\subset LG(n)$ are given by a subset Z of $\mathrm{Mat}_{2n\times n}(\mathbb{K})$ consisting of full-rank matrices satisfying $M^TJM = 0_{n\times n}$ for $M\in Z$ such that the map $M\mapsto$ column span$M$ is a birational map from Z to Y. Given an isotropic flag $F_\bullet$ and a strict partition λ, the intersetion $Y\cap X_\lambda F_\bullet$ is represented in a set Z of Stiefel coordinates for Y by rank conditions (*cf.* (1)) on $M\in Z$:

$$\mathrm{rank}\bigl(M\colon F_{n+1-\lambda_i}\bigr) \;\leq\; 2n{+}1{-}\lambda_i - i\,, \qquad \text{for } i=1,\dots,r\,.$$

That is, by the vanishing of all minors of $(M\colon F_{n+1-\lambda_i})$ of size $2n{+}2{-}\lambda_i - i$. Thus we see that typically a Schubert problem is not a complete intersection.

2. Formulation of Schubert problems of interest

In $LG(4)$, there are three enriched Schubert problems

$$\yng(2,1)^2\cdot\yng(3,1) \;=\; 4\,,\quad \yng(2,1)^2\cdot\yng(3)\cdot\yng(1) \;=\; 4\,,\quad\text{and}\quad \yng(2,1)^3\cdot\yng(1) \;=\; 8\,.$$

(These are written multiplicatively and the number for a Schubert problem $\boldsymbol{\lambda}$ is $d(\boldsymbol{\lambda})$.) All three contain $\yng(2,1)^2$. We give Stiefel coordiniates for $\Omega_{21}E_\bullet\cap\Omega_{21}E'_\bullet$:

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 1 & a & b\\ 0 & 0 & 0 & 0 & 1 & 0 & c & d\\ \hline 0 & 1 & -a & -b & 0 & 0 & 0 & 0\\ 1 & 0 & -c & -d & 0 & 0 & 0 & 0\end{array}\right).$$

Also, note that

$$\begin{aligned} \Omega_{sThI}F_\bullet &= \{L\mid \dim L\cap F_2\geq 1 \;\text{ and }\; \dim L\cap F_4\geq 2\}\,,\\ \Omega_{sTh}F_\bullet &= \{L\mid \dim L\cap F_2\geq 1\}\,,\\ \Omega_{sT}F_\bullet &= \{L\mid \dim L\cap F_4\geq 2\}\,,\\ \Omega_{sI}F_\bullet &= \{L\mid \dim L\cap F_4\geq 1\}\,. \end{aligned}$$

3. Preliminary calculations

Using the Frobenius algorithm, we determined that three of the 44 essential Schubert problems on $LG(4)$ are enriched. We have yet to be able to compute an eliminant for a problem with 768 solutions, which is the largest.

Using strict partitions to represent Schubert conditions, these three problems are

$$\ydiagram{2,1+1}^2 \cdot \ydiagram{3,1+1} \;=\; 4\,,\quad \ydiagram{2,1+1}^2 \cdot \ydiagram{3} \cdot \ydiagram{1} \;=\; 4\,,\quad \text{and} \quad \ydiagram{2,1+1}^3 \cdot \ydiagram{1} \;=\; 8\,.$$

Let $V \simeq \mathbb{C}^8$ be a vector space equipped with a nondegenerate alternating form $\langle\bullet,\bullet\rangle$. We call $(V,\langle\bullet,\bullet\rangle)$ a *symplectic vector space*. The annihilator of a linear space H of V is $H^{\angle} := \{v \in V \mid \langle u,v\rangle = 0\ \forall u \in H\}$. As $\langle\bullet,\bullet\rangle$ is nondegenerate, $\dim H + \dim H^{\angle} = \dim V$. A subspace $H \subset V$ is *isotropic* if $H \subset H^{\angle}$. Then the dimension of an isotropic subspace H is at most $\frac{1}{2}\dim V$, and it is *Lagrangian* (maximal isotropic) if $\dim H = \frac{1}{2}\dim V$. Write $LG(V)$ or $LG(4)$ for the space of Lagrangian subspaces of V. This is a ten-dimensional smooth subvariety of $Gr(4,V)$, the Grassmannian of 4-planes in V. We will assume that the reader is familiar with our terminology, as well as the basics of Schubert calculus on $LG(V)$.

Let $L,M \in LG(V)$ be two general Lagrangian subspaces. In particular $L\cap M = \{0\}$ so that the map $L\oplus M \to V$ defined by $u\oplus v \mapsto u+v$ is an isomorphism. For $0 \neq v \in M$ consider the linear function $\Lambda_v\colon L \to \mathbb{C}$ defined by $\Lambda_v(u) = \langle u,v\rangle$. As L is Lagrangian and $L\cap M = \{0\}$, this linear form is nonzero on L. In particular, $v \mapsto \Lambda_v$ identifies M with the linear dual $L^* := \mathrm{Hom}(L,\mathbb{C})$ of L.

Suppose that $N \in LG(V)$ is a third Lagrangian subspace in general position with respect to both L and M. Then the projections π_L and π_M of N to the summands in $L\oplus M \simeq V$ are isomorphisms. This identifies N as the graph of a linear isomorphism

$$\varphi_N \;:=\; \pi_M \circ \pi_L^{-1} \;:\; L \xrightarrow{\;\sim\;} M\,.$$

This linear isomorphism $\varphi_N\colon L \to M \simeq L^*$ induces a nondegenerate bilinear form $(\bullet,\bullet)_N$ on L which is defined for $u,u \in L$ by $(u,u')_N := \langle u, \varphi_N(u')\rangle$.

The bilinear form $(\bullet,\bullet)_N$ is symmetric. Indeed, as N is isotropic, we have that for $u,u' \in L$, $u+\varphi_N(u)$ and $u'+\varphi_N(u')$ lie in N so that

$$\begin{aligned} 0 \;=\; & \langle u+\varphi_N(u)\,,\, u'+\varphi_N(u')\rangle \\ & \qquad =\; \langle u\,,\,u'\rangle \;+\; \langle u\,,\,\varphi_N(u')\rangle \;+\; \langle \varphi_N(u)\,,\,u'\rangle \;+\; \langle \varphi_N(u)\,,\,\varphi_N(u')\rangle\,. \end{aligned}$$

As $u,u' \in L$ and $\varphi_N(u),\varphi_N(u') \in M$, we see that $0 = \langle u,\varphi_N(u')\rangle + \langle\varphi_N(u),u'\rangle$, so that $\langle u,\varphi_N(u')\rangle = \langle u',\varphi_N(u)\rangle$, as $\langle\bullet,\bullet\rangle$ is alternating. Thus $(u,u')_N = (u',u)_N$ is symmetric.

Define $\ydiagram{2,1+1}(L) := \{H \in LG(V) \mid \dim H\cap L \geq 2\}$, which is a Schubert subvariety of codimension three in $LG(V)$. It is the intersection of $LG(V)$ with the Schubert subvariety $\Omega_{\ydiagram{2,2}}(L)$ of the Grassmannian $Gr(4,V)$. Set $X(L,M) := \ydiagram{2,1+1}(L)\cap\ydiagram{2,1+1}(M)$, a Richardson variety. If $H \in X(L,M)$, then $H\cap L \in Gr(2,L)$ and $H\cap M \in Gr(2,M)$. If we set $h := H\cap L$ and $h' := H\cap M$, then $H = h\oplus h'$. As H is isotropic, $\langle h,h'\rangle \equiv 0$, which implies that h' is the annihilator $h^{\perp}$ of h in $M = L^*$.

Following work on the Pieri formula in isotropic Schubert calculus [2] (see also [1]), it is useful to define the union, $Z(L,M)$, of the linear spaces in $X(L,M)$,

$$Z(L,M) \;:=\; \bigcup \{H \;\mid\; H \in X(L,M)\}\,,$$

which we consider to be a subvariety of the projective space $\mathbb{P}(V)$. More formally and working projectively, let

$$C(1,4;V) \;:=\; \{(\ell,H) \mid H \in LG(V) \text{ and } \ell \in \mathbb{P}(H)\}$$

be the symplectic flag variety of isotropic lines lying on Lagrangian subspaces in V. This has projections to projective space $\mathbb{P}(V)$ and to the Lagrangian Grsassmannian.

$$\begin{tikzcd} & C(1,4;V) \arrow[dl, "pr"'] \arrow[dr, "\pi"] & \\ \mathbb{P}(V) & & LG(V) \end{tikzcd}$$

Each realizes $C(1,4;V)$ as a fibre bundle, with $\pi^{-1}(H) = \mathbb{P}(H) \simeq \mathbb{P}^3$ and $pr^{-1}(\ell) = LG(3, \ell^{\angle}/\ell)$. Then $Z(L,M) := pr \circ \pi^{-1}(X(L,M))$. Define

$$Y(L,M) \;:=\; \pi^{-1}(X(L,M)) \subset C(1,4;V)\,.$$

For $0 \neq u \in L$, let $u^{\perp} \subset M$ be its annihilator, which is 3-dimensional. Similarly, for $0 \neq v \in M$, let $v^{\perp} \subset L$ be its annihilator.

Lemma 3.1. *In the coordinates $\{(u,v) \mid u \in L$ and $v \in M\}$ for $\mathbb{P}(V)$, the variety $Z(L,M)$ is the quadratic hypersurface with equation $\langle u,v\rangle = 0$. The map $pr\colon Y(L,M) \to Z(L,M)$ has fibre over a point $(u,v) \in Z(L,M)$ identified with $\mathbb{P}(v^{\perp}/u)$. When u and v are nonzero, this is isomorphic to $\mathbb{P}^1$; otherwise it is isomorphic to $\mathbb{P}^2$.*

Let $(u,v) \in Z(L,M)$ with u and v both nonzero. If we restrict the maps π, pr to $Y(L,M)$, then the set

$$\bigcup\{H \in X(L,M) \mid (u,v) \in H\} \;=\; pr \circ \pi^{-1} \circ \pi \circ pr^{-1}(u,v)\,, \tag{3}$$

is the quadric hypersurface $Z(L,M,u,v) := Z(L,M) \cap \mathbb{P}(v^{\perp} \oplus u^{\perp})$ in $\mathbb{P}(v^{\perp} \oplus u^{\perp}) \simeq \mathbb{P}^5$, and the map between $\pi \circ pr^{-1}(u,v) \subset LG(V)$ and $Z(L,M,u,v)$ is birational away from the exceptional divisor $\mathbb{P}(u + v^{\perp}) \cup \mathbb{P}(u^{\perp} + v)$.

Proof. A Lagrangian subspace $H \in X(L,M)$ has the form $h \oplus h^{\perp}$ for $h \in Gr(2,L)$. Thus if $(u,v) \in H$, then $u \in h$ and $v \in h^{\perp}$, so that $\langle u,v\rangle = 0$, and we have that $u \in h \subset v^{\perp}$. A point $(u,v) \in V$ with $\langle u,v\rangle = 0$ has $u \in v^{\perp} \subset L$. Given any $h \in Gr(2,L)$ with $u \in H \subset v^{\perp}$, we have $v \in h^{\perp}$ so that $(u,v) \in h \oplus h^{\perp} \in X(L,M)$. This shows that $Z(L,M)$ equals the quadratic hypersurface and that the fibre $pr^{-1}(u,v) = \mathbb{P}(v^{\perp}/u)$. Since at most one of u or v may be zero, this is ispmorphic to $\mathbb{P}^2$ if one is zero and $\mathbb{P}^1$ if neither is zero.

For the last statement, note that the set (3) is contained in $Z(L,M) \cap \mathbb{P}(v^{\perp} \oplus u^{\perp})$. Indeed, suppose that $(u,v) \in H$ and $H \in X(L,M)$. Then $H = h \oplus h^{\perp}$ and $u \in h \subset v^{\perp}$ and $v \in h^{\perp} \subset u^{\perp}$. If $(a,b) \in H$, then $a \in v^{\perp}$ and $b \in u^{\perp}$, and $\langle a,b\rangle = 0$.

Let $(a,b) \in Z(L,M) \cap \mathbb{P}(v^{\perp} \oplus u^{\perp})$. Suppose that a is linearly independent of u and b is linearly indepedent of v. As $u,a \in L$, $v,b \in M$, $a \in v^{\perp}$ and $b \in u^{\perp}$, $\operatorname{span}\{a,u\}$ annihilates span$\{b, v\}$, so that span$\{a, b, u, v\}$ is the unique Lagrangian subspace containing these four points. $\square$

The quadric $Z(L, M, u, v)$ is singular; it is the cone over a quadric isomorphic to $\mathbb{P}(v^\perp/u) \times \mathbb{P}(u^\perp/v)$ in a $\mathbb{P}^3$ with vertex $\mathbb{P}(u + v)$, a $\mathbb{P}^1$.

3.1. **The Galois group of $\ydiagram{2,1+1}^2 \cdot \ydiagram{3} \cdot \ydiagram{1} = 4$ is D_4.** Let L, M, and N be general Lagrangian subspaces in V as before, and let m be an isotropic 2-plane, also in general position. Observe that

$$\ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M) \cap \ydiagram{3}(m) \;=\; \pi\bigl(pr^{-1}(m \cap Z(L, M))\bigr)\,. \tag{4}$$

By Lemma 3.1, $Z(L, M)$ is a quadric. Thus it meets m in two points (u, v) and (u', v'), showing that the intersection (4) has two components.

Let W be the component of (4) coming from (u, v). By Lemma 3.1 again, if we restrict π to $Y(L, M)$, then $pr(\pi^{-1}(W)) = Z(L, M) \cap \mathbb{P}(u^\perp \oplus v^\perp) = Z(L, M, u, v)$ is a quadric hypersurface in the $\mathbb{P}^5 \simeq \mathbb{P}(u^\perp \oplus v^\perp)$. Each of the two points of intersection of N with $Z(L, M, u, v)$ gives a solution to the Schubert problem

$$\ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M) \cap \ydiagram{3}(m) \cap \ydiagram{1}(N)\,. \tag{5}$$

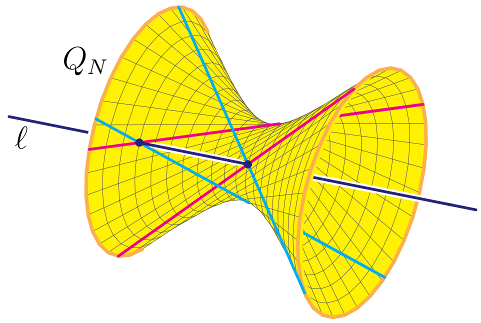
With the other point (u', v') of $m \cap Z(L, M)$, this gives four solutions to the Schubert problem (5). Note that N is spanned by its intersections with $Z(L, M, u, v)$ and $Z(L, M, u', v')$ As its Galois group must preserve the partition coming from the two points (u, v) and (u', v'), it is a subgroup of D_4. We have computed Frobenius elements which show that the Galois group is D_4.

For an alternative proof, note that it is possible to find a monodromy loop that fixes L, M, m (and hence the points (u, v) and (u', v')), as well as the two points $N \cap Z(L, M, u, v)$, but interchanges the other two points $N \cap Z(L, M, u', v')$. Indeed, let $\{x, y\} = N \cap Z(L, M, u, v)$. Then the set of Lagrangian planes containing $h := \text{span}\{x, y\}$ is identified with $LG(h^\angle/h)$, and any two points in $\mathbb{P}(x^\angle) \cap \mathbb{P}(y^\angle)$ that are independent. Fix this. It is important to make these kinds of arguments.

3.2. **The Galois group of $\ydiagram{2,1+1}^2 \cdot \ydiagram{3,1+1} = 4$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$.** Let L, M, N be as before, and consider a Lagrangian subspace $H \in \ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M) \cap \ydiagram{2,1+1}(N)$. As $H \in \ydiagram{2,1+1}(L) \cap \ydiagram{2,1+1}(M)$, it has the form $h \oplus h^\perp$ for $h \in Gr(2, L)$, and it is not hard to see that $h^\perp = \varphi_N(h)$. These together imply that $(h, h)_N \equiv 0$, so that h is an isotropic 2-plane in the linear space $L \simeq \mathbb{C}^4$ equipped with the nondegenerate symmetric form $(\bullet, \bullet)_N$. Let us work in $\mathbb{P}(L)$. Then h lies in one of the two families of lines that rule the quadric surface $Q_N := \{u \in \mathbb{P}(L) \mid (u, u)_N = 0\}$ in $\mathbb{P}(L)$. Now let $\ell \subset L$ be an isotropic 2-plane in L, which is a line in $\mathbb{P}(L)$. This will meet Q in two points, and through each point there will be two lines—one in each ruling. These four solutions h give the four solutions $h \oplus h^\perp$ to the Schubert problem.

The partition of the four solutions by the corresponding points of intersection $\ell \cap Q$ show that the Galois group is a subgroup of D_4. To analyze this further, let p and q be the two points in $\ell \cap Q_N$, and let the four lines on Q_N meeting these points be h_p^1, h_p^2, h_q^1, and h_q^2, with the upper index representing the ruling of Q_N the line lies in and the lower indicating the point of $\ell \cap Q_N$ it meets. However, there are two solution lines h in each ruling and the

Galois group must preserve their intersections. Consequently, the Galois group is the Klein 4-group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.



3.3. The Galois group of $\yng(2,1)^3 \cdot \yng(1) = 8$ is not yet determined.

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