Algebra Autumn 2013
Frank Sottile
10 September 2013

Third Homework

Hand in to Frank Tuesday 17 September:

15. **20 points**

    Let m be a positive integer. Show that $\mathbb{Z}_m^* := \mathbb{Z}_m \smallsetminus \{0\}$ is a group under multiplication if and only if m is a prime number. Let p be a prime number. Using that $\mathbb{Z}_p$ is a field and in a field a polynomial of degree n has at most n roots, or any other valid facts, prove that $\mathbb{Z}_p^*$ is a cyclic group of order $p-1$. Use this to prove Wilson's Theorem that $(p-1)!$ is congruent to -1 modulo p.

Hand in for the grader Tuesday 17 September:

16. Prove that every finitely generated subgroup of the rational numbers $\mathbb{Q}$ is cyclic. Give an example (with proof) of a subgroup of $\mathbb{Q}$ that is not finitely generated.
17. Let H, K be subgroups of a group G. Show that any right coset of $H \cap K$ is the intersection of a right coset of H with a right coset of K. Use this to prove Poincaré's Theorem that if H and K have finite index, then so does $H \cap K$.
18. Show that the symmetric group S_n may be generated by two elements.
19. List all of the elements of S_4 in cycle notation. Use this to prove that D_{24}, the dihedral group of order 24, is not isomorphic to S_4.
20. Let $SL_2(\mathbb{Z}_3)$ be the group of 2×2 matrices of determinant 1 with entries in the field $\mathbb{Z}_3$ with three elements. Show that $SL_2(\mathbb{Z}_3)$ has order 24, and that it is not isomorphic to S_4. Is it isomorphic to D_{24}?