EULERIAN COMBINATORIAL HOPF ALGEBRAS AND THE GENERALIZED DEHN-SOMMERVILLE RELATIONS

M. AGUIAR, N. BERGERON, AND F. SOTTILE

ABSTRACT. A *Combinatorial Hopf algebra* is graded connected Hopf algebra over a field $\Bbbk$ equipped with a multiplicative linear functional $\zeta\colon \mathcal{H} \to \Bbbk$. The terminal object in the category of combinatorial Hopf algebras is the algebra of quasi-symmetric functions, which explains the ubiquity of quasi-symmetric functions as generating functions in combinatorics. The Möbius function of a combinatorial Hopf algebra gives natural Dehn-Sommerville relations on the Hopf algebra, and a combinatorial Hopf algebra satisfying these relations is eulerian. A combinatorial Hopf algebra has a unique maximal eulerian subalgebra. The maximal eulerian subalgebra of the algebra of quasi-symmetric functions is the peak algebra of Stembridge.

1. Introduction

A *Combinatorial Hopf algebra* is a pair $(\mathcal{H}, \zeta)$ where $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ is a graded connected Hopf algebra over a field $\Bbbk$ whose characteristic is not 2 such that $\dim(\mathcal{H}_n)$ is finite for all $n \geq 0$ and $\zeta\colon \mathcal{H} \to \Bbbk$ is a multiplicative functional, called its *zeta function*. A morphism $\Psi\colon (\mathcal{H}, \zeta) \to (\mathcal{H}', \zeta')$ of combinatorial Hopf algebras is a graded Hopf morphism such that $\zeta = \zeta' \circ \Psi$. The terminal object in the category of combinatorial Hopf algebras is the algebra $\mathcal{Q}Sym$ of quasi-symmetric functions, with a natural structure map $h\colon \mathcal{Q}Sym \to \Bbbk$. This theory provides a natural framework for combinatorial invariants encoded via quasi-symmetric generating functions.

The algebra $\mathcal{Q}Sym$ of quasi-symmetric functions was introduced by Stanley [23] and Gessel [14] as a source of generating functions for P-partitions. The algebra Sym of symmetric functions is a subalgebra of $\mathcal{Q}Sym$. The work of Malvenuto and Reutenauer [20] and Gelfand *et. al.* [13] uncovered a more profound relationship between Sym and $\mathcal{Q}Sym$: the graded Hopf dual of $\mathcal{Q}Sym$ is the Hopf algebra $\mathcal{N}Sym$ of non-commutative symmetric functions. Malvenuto and Reutenauer introduced a self-dual Hopf algebra $\mathfrak{S}Sym$ of permutations [19, 20] that contains $\mathcal{N}Sym$ as a Hopf subalgebra and has $\mathcal{Q}Sym$ as a Hopf quotient. Subsequently, Loday and Ronco [17] introduced a self-dual Hopf algebra $\mathcal{Y}Sym$ with these same properties that is itself both a sub- and quotient Hopf algebra of $\mathfrak{S}Sym$.

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Joni and Rota [16] made the fundamental observation that many discrete structures give rise to natural Hopf algebras whose comultiplications encode the disassembly of those structures (see also [22]). A first link between these Hopf algebras and $\mathcal{Q}Sym$ was shown by Ehrenborg [12], whose flag f-vector quasi-symmetric function of a graded poset gave a Hopf morphism from a Hopf algebra of graded posets to $\mathcal{Q}Sym$. This theory was augmented in [8] where it was shown that a quasi-symmetric function associated to an edge-labeled poset similarly gives a Hopf morphism. Such quasi-symmetric functions encode the structure of the cohomology of a flag manifold as a module over the ring of symmetric functions [7, 9]. These results were later unified via graded representations of $\mathcal{N}Sym$ on the linear span $\Bbbk P$ of a graded poset P [5, 6]. In Example 3.2, we show how this construction is equivalent to the general result for combinatorial Hopf algebras. Finally, the notion of *eulerian subalgebra* for infinitesimal Hopf algebras was developed in [1]. There, these morphisms to $\mathcal{Q}Sym$ were explained as $\mathcal{Q}Sym$ also has a structure of an infinitesimal Hopf algebra for which it is the terminal object in the category of infinitesimal Hopf algebras. Here, we unify this previous work, adapting the ideas of [1] to graded Hopf algebras to develop the notion of a combinatorial Hopf algebra.

We first recall the definitions of the Hopf algebra $\mathcal{Q}Sym$ and its graded dual $\mathcal{N}Sym$ in Section 2. In Section 3, we introduce the multiplicative functional $h\colon \mathcal{Q}Sym \to \Bbbk$, endowing $\mathcal{Q}Sym$ with the structure of a combinatorial Hopf algebra. We show (Theorem 3.1) the following universal property.

For any combinatorial Hopf algebra $(\mathcal{H},\zeta)$, there is a unique morphism of combinatorial Hopf algebras $\Psi\colon \mathcal{H} \to \mathcal{Q}Sym$.

Thus $(\mathcal{Q}Sym, h)$ is the terminal object in the category of combinatorial Hopf algebras.

The zeta function ζ of a combinatorial Hopf algebra $(\mathcal{H},\zeta)$ is invertible under convolution, with inverse the Möbius function of $(\mathcal{H},\zeta)$. In Section 4 we show that there is a unique maximal combinatorial Hopf subalgebra $(E(\mathcal{H}),\zeta)$ of $(\mathcal{H},\zeta)$ with the property that for any homogeneous element $x \in E(\mathcal{H})_n$, we have $\mu(x) = (-1)^n\zeta(x)$. We call $E(\mathcal{H})$ the *eulerian subalgebra* of $(\mathcal{H},\zeta)$. This possesses two important properties. First, let $\xi\colon \mathcal{H} \to \Bbbk$ be the map defined by $\xi(x) = \mu(x) - (-1)^n\zeta(x)$, for homogeneous elements $x \in \mathcal{H}$ of degree n. Then $x \in E$ if and only if

$$(Id \otimes \xi \otimes Id)\Delta^2(x) \;=\; 0\,.$$

These are generalized Dehn-Sommerville relations on $(\mathcal{H},\zeta)$. For $(\mathcal{Q}Sym, h)$, they are precisely the relations of Bayer and Billera [4]. Second, given a morphism $\Psi\colon (\mathcal{H},\zeta) \to (\mathcal{H}',\zeta')$ of combinatorial Hopf algebras, we have $\Psi(E(\mathcal{H})) \subseteq E(\mathcal{H}')$. Thus the map $(\mathcal{H},\zeta) \mapsto (E(\mathcal{H}),\zeta)$ is a retraction from the category of combinatorial Hopf algebras onto its full subcategory of eulerian combinatorial Hopf algebras.

In Section 5 we show that the eulerian subalgebra of $(\mathcal{Q}Sym, h)$ is the peak Hopf algebra Π of Stembridge [5, 25]. This uses a functional $q\colon \mathcal{Q}Sym \to \Bbbk$ for which $(\mathcal{Q}Sym, q)$ is eulerian. The universal property of $(\mathcal{Q}Sym, h)$ defines a unique morphism of combinatorial Hopf algebras $\Theta\colon \mathcal{Q}Sym \to \mathcal{Q}Sym$, and thus $\Theta\colon \mathcal{Q}Sym \to \Pi$. We identify this Hopf morphism as the map defined by Stembridge in [25].

In the remaining sections, we describe the eulerian subalgebras of the Malvenuto-Reutenauer Hopf algebra of permutations [19, 2], the algebras of commutative and non-commutative symmetric functions, and the Loday-Ronco Hopf algebra of planar binary trees [17].

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2. The Hopf algebras $\mathcal{QS}ym$ and $\mathcal{NS}ym$

We recall the Hopf structure of $\mathcal{QS}ym$ and its graded dual $\mathcal{NS}ym$. For more details, see [13, 19, 20, 24]. For a survey of their Hopf-algebraic properties, see [15]

A composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of a positive integer n, written $\alpha \vDash n$, is an ordered list of positive integers whose sum is n. Compositions of n are in one-to-one correspondence with subsets of $\{1, 2, \dots, n-1\}$. If $A = \{a_1, a_2, \dots, a_{k-1}\} \subseteq \{1, 2, \dots, n-1\}$, where $a_1 < a_2 < \cdots < a_{k-1}$, then A corresponds to the composition $\alpha = [a_1 - a_0, a_2 - a_1, \dots, a_k - a_{k-1}]$, where $a_0 = 0$ and $a_k = n$. We denote the set corresponding to a given composition α by $D(\alpha)$. For compositions α and β we say that α is a *refinement* of β if $D(\beta) \subseteq D(\alpha)$ and denote this by $\beta \preccurlyeq \alpha$.

The *monomial quasi-symmetric function* M_α indexed by a composition α is

$$M_\alpha \ := \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}\,, \tag{2.1}$$

where $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$. We set $M_\emptyset = 1$, where $\emptyset$ denotes the unique (empty) composition of 0. The *fundamental quasi-symmetric function* F_α is

$$F_\alpha = \sum_{\alpha \preccurlyeq \beta} M_\beta. \tag{2.2}$$

Let $\mathcal{QS}ym$ be the $\Bbbk$-vector space spanned by all quasi-symmetric functions. The families $\{M_\alpha\}$ and $\{F_\alpha\}$ of quasi-symmetric functions each give bases of $\mathcal{QS}ym$.

We introduce some notation for the multiplication formula in $\mathcal{QS}ym$. Let $\mathfrak{S}_n$ be the symmetric group on n elements. Given a permutation $w = (w_1, w_2, \dots, w_n) \in \mathfrak{S}_n$, we let $d(w) = [\alpha_1, \alpha_2, \dots, \alpha_k]$ be the descent composition of w. That is,

$$D(d(w)) \ = \ \{i \mid w_i > w_{i+1}\}\,.$$

For example, if $w = (3, 6, 1, 4, 2, 7, 5)$, then $d(w) = [2, 2, 2, 1]$. Given $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_m$, set $u \times v := (u_1 \dots, u_n, v_1{+}n, \dots, v_m{+}n) \in \mathfrak{S}_{n+m}$. Consider the set of permutations with at most one descent in position n,

$$\mathfrak{S}^{(n,m)} = \bigl\{\sigma \in \mathfrak{S}_{n+m} \bigm| \sigma(1) < \cdots < \sigma(n),\ \sigma(n+1) < \cdots < \sigma(n+m)\bigr\}\,. \tag{2.3}$$

For any two permutations u and w such that $d(w) = \alpha \vDash n$ and $d(u) = \beta \vDash m$ (such u and v always exist), we have

$$F_\alpha F_\beta \ = \sum_{\sigma \in \mathfrak{S}^{(n,m)}} F_{d((u \times v)\sigma^{-1})}\ . \tag{2.4}$$

Hence $\mathcal{QS}ym$ is closed under multiplication. Thus we obtain a graded algebra $\mathcal{QS}ym = \bigoplus_{n \geq 0} \mathcal{QS}ym_n$ where $\mathcal{QS}ym_n$ is the homogeneous component of degree n

spanned by $\{M_\alpha\}_{\alpha\models n}$. Then $\dim \mathcal{Q}Sym_0 = 1$ and $\dim \mathcal{Q}Sym_n = 2^{n-1}$, the number of compositions of n.

We put a Hopf algebra structure on $\mathcal{Q}Sym$ with the comultiplication

$$\Delta(M_\alpha) \ = \ \sum_{\alpha=\beta\,.\,\gamma} M_\beta \otimes M_\gamma, \tag{2.5}$$

where $\beta\,.\,\gamma$ is the concatenation of compositions β and γ. For example, $\Delta(M_{32}) = 1 \otimes M_{32} + M_3 \otimes M_2 + M_{32} \otimes 1$. If $w \in \mathfrak{S}_n$ is a permutation with $d(w) = \alpha$, then

$$\Delta(F_\alpha) \ = \ \sum_{w=u\,.\,v} F_{d(\mathrm{st}(u))} \otimes F_{d(\mathrm{st}(v))}, \tag{2.6}$$

where $u\,.\,v$ is the concatenation of the sequences u and v, and $\mathrm{st}(u)$ is the unique permutation with the same inversion set as the sequence u. The counit is given by $\epsilon\colon \mathcal{Q}Sym \to \Bbbk$ where $\epsilon(\psi)$ is the coefficient of M_0 in ψ.

Since $\mathcal{Q}Sym$ is a graded bialgebra, that is both graded and cograded, it has a unique antipode $S\colon \mathcal{Q}Sym \to \mathcal{Q}Sym$, by the formula of Milnor and Moore [21]. Let $w = (w_1, w_2, \ldots, w_n)$ be any permutation with $d(w) = \alpha$, then $S(F_\alpha) = (-1)^n F_{d(w_n,\ldots,w_2,w_1)}$.

The graded Hopf dual of $\mathcal{Q}Sym$ is identified with the graded Hopf algebra *NSym* of non-commutative symmetric functions [13]. Let $NSym = \Bbbk\langle h_1, h_2, h_3, \ldots\rangle$ be the non-commutative algebra freely generated by infinitely many non-commuting variables $\{h_1, h_2, \ldots\}$. We let $\deg(h_n) = n$ and define

$$\Delta(h_n) \ := \ \sum_{i+j=n} h_i \otimes h_j \tag{2.7}$$

where $h_0 = 1$. The antipode is again given by the formula of Milnor and Moore [21]. With these structures, *NSym* is a graded non-commutative Hopf algebra.

Write $\mathcal{H}_n$ for the homogeneous component of degree n of a given graded algebra $\mathcal{H}$. Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ be a graded Hopf algebra such that $\dim(\mathcal{H}_n)$ is finite. This has two dual Hopf algebras. The graded dual Hopf algebra is $\mathcal{H}^* := \bigoplus_{n\geq 0} (\mathcal{H}_n)^*$, where $\mathcal{H}_n^* = (\mathcal{H}_n)^* = \big\{\phi\colon \mathcal{H}_n \to \Bbbk \,|\, \phi \textit{ is } \Bbbk\textit{-linear}\big\}$. Then $(\mathcal{H}^*)^* = \mathcal{H}$. The linear dual Hopf algebra $\overline{\mathcal{H}}^*$ is the set of all functionals $\big\{\phi\colon \mathcal{H} \to \Bbbk \,|\, \phi \textit{ is } \Bbbk\textit{-linear}\big\}$. The nth homogeneous component ψ_n of a linear map $\psi \in \overline{\mathcal{H}}^*$ is $\psi \circ \pi_n$, where $\pi_n\colon \mathcal{H} \to \mathcal{H}_n$ is the canonical projection. While $\overline{\mathcal{H}}^*$ is in general not graded, each element ψ is the formal sum $\sum_{n\geq 0} \psi_n$ of its homogeneous components, as $\overline{\mathcal{H}}^*$ is a completion of $\mathcal{H}^*$. The product $\circledast$ in both $\mathcal{H}^*$ and $\overline{\mathcal{H}}^*$ is given by convolution

$$\phi \circledast \psi \ := \ m_\Bbbk(\phi \otimes \psi)\Delta_{\mathcal{H}}\,, \tag{2.8}$$

where $m_\Bbbk$ is the multiplication in $\Bbbk$ and $\Delta_{\mathcal{H}}$ is the comultiplication of $\mathcal{H}$.

A linear basis for *NSym* is given by $\{h_\alpha\}_{\alpha\models n\geq 0}$ where $h_{[\alpha_1,\alpha_2,\ldots,\alpha_k]} = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_k}$. A basis for the graded dual Hopf algebra *NSym* is given by the dual basis $\{h_\alpha^*\}$, where $h_\alpha^*(h_\beta) = \delta_{\alpha,\beta}$. Then $h_\alpha^* \mapsto M_\alpha$ induces an isomorphism of graded Hopf algebras $NSym^* \xrightarrow{\sim} \mathcal{Q}Sym$. The basis of *NSym* dual to the basis $\{F_\alpha\}$ of $\mathcal{Q}Sym$ consists of the ribbon non-commutative symmetric functions, R_α.

The commutator ideal $\mathcal{I} = \langle h_i h_j - h_j h_i \rangle$ in *NSym* is a Hopf ideal and the quotient $\mathit{NSym}/\mathcal{I}$ is isomorphic to *Sym*, the self-dual Hopf algebra of symmetric functions. The quotient map is the abelianization map ab. By duality, we have the inclusion $\mathit{Sym} \cong \mathit{Sym}^* \hookrightarrow \mathcal{Q}\mathit{Sym}$, and thus the composition

$$\mathit{NSym} \xrightarrow{ab} \mathit{Sym} \hookrightarrow \mathcal{Q}\mathit{Sym}\,. \tag{2.9}$$

3. The category of combinatorial Hopf algebras and its terminal object $(\mathcal{Q}\mathit{Sym}, h)$

A combinatorial Hopf algebra is a pair $(\mathcal{H}, \zeta)$ where $\mathcal{H}$ is a graded connected Hopf algebra over a field $\Bbbk$ such that $\dim(\mathcal{H}_n)$ is finite and $\zeta \in \overline{\mathcal{H}}^*$ is multiplicative, $\zeta(ab) = \zeta(a)\zeta(b)$. Since $\zeta(ab) = \Delta\zeta(a \otimes b)$ and $\zeta(a)\zeta(b) = (\zeta \otimes \zeta)(a \otimes b)$, we see that $\zeta \in \overline{\mathcal{H}}^*$ is multiplicative of and only if it is group-like,

$$\Delta(\zeta) \;=\; \zeta \otimes \zeta\,. \tag{3.1}$$

A morphism $\Psi\colon \mathcal{H} \to \mathcal{H}'$ of graded Hopf algebras is a morphism of combinatorial Hopf algebras $\Psi\colon (\mathcal{H}, \zeta) \to (\mathcal{H}', \zeta')$ if the following diagram commutes

$$\begin{tikzcd} \mathcal{H} \arrow[rr, "\Psi"] \arrow[dr, "\zeta"'] & & \mathcal{H}' \arrow[dl, "\zeta'"] \\ & \Bbbk & \end{tikzcd}$$

That is, if $\zeta = \zeta' \circ \Psi$. Combinatorial Hopf algebras over a field $\Bbbk$, together with their morphisms, form the category of combinatorial Hopf algebras over $\Bbbk$.

Returning to $\mathcal{Q}\mathit{Sym}$, we define $h\colon \mathcal{Q}\mathit{Sym} \to \Bbbk$ by

$$h(M_\alpha) \;=\; \begin{cases} 1 & \text{if } \alpha = [n] \textit{ or } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We thus have $h = \sum_{n \geq 0} h_n$ in $\overline{\mathcal{Q}\mathit{Sym}}^*$ where $h_n \in \mathit{NSym}$. It is easy to check that h is group-like (satisfies (3.1)). Thus $(\mathcal{Q}\mathit{Sym}, h)$ is a combinatorial Hopf algebra. By triangularity, we also have $h(F_\alpha) = 1$ only if $\alpha = [n]$ or $\alpha = 0$, and 0 otherwise.

$\mathcal{Q}\mathit{Sym}$ is the terminal object in this category of combinatorial Hopf algebras.

Theorem 3.1. *For any combinatorial Hopf algebra $(\mathcal{H}, \zeta)$, there exists a unique morphism of combinatorial Hopf algebras $\Psi\colon \mathcal{H} \to \mathcal{Q}\mathit{Sym}$.*

Proof. Suppose that $\Psi\colon \mathcal{H} \to \mathcal{Q}\mathit{Sym}$ is a graded linear map satisfying $\zeta = h \circ \Psi$. That is, $\zeta = \Psi^*(h)$, where Ψ^* is the dual morphism $\Psi^*\colon \overline{\mathcal{Q}\mathit{Sym}}^* \to \overline{\mathcal{H}}^*$. Since this restricts to a map $\Psi^*\colon \mathcal{Q}\mathit{Sym}^* \to \mathcal{H}^*$ of graded duals, $\Psi^*(h_n) = \zeta_n$, where ζ_n is the nth homogeneous component of ζ. Then Ψ^* is an algebra map as $\mathcal{Q}\mathit{Sym}^*$ is the free algebra generated by the h_n and so Ψ a morphism of coalgebras.

This shows that any graded linear map $\Psi\colon \mathcal{H} \to \mathcal{Q}\mathit{Sym}$ satisfying $\zeta = h \circ \Psi$ must be the morphism of graded coalgebras satisfying $\Psi^*(h_n) = \zeta_n$. It only remains to show that this map is an algebra map. We deduce this from uniqueness. Consider the following two commutative diagrams where the vertical maps are algebra maps

$$
\begin{tikzcd}
\mathcal{H}^{\otimes 2} \arrow[r, "m"] \arrow[dr, "\zeta^{\otimes 2}"'] & \mathcal{H} \arrow[r, "\Psi"] \arrow[d, "\zeta"'] & \mathcal{Q}Sym \arrow[dl, "h"] \\
 & \Bbbk &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}^{\otimes 2} \arrow[r, "\Psi^{\otimes 2}"] \arrow[dr, "\zeta^{\otimes 2}"'] & \mathcal{Q}Sym^{\otimes 2} \arrow[r, "m"] \arrow[d, "h^{\otimes 2}"'] & \mathcal{Q}Sym \arrow[dl, "h"] \\
 & \Bbbk &
\end{tikzcd}
$$

The maps $\Psi \circ m$ and $m \circ (\Psi \otimes \Psi)$ are each morphisms of graded coalgebras making the exterior triangles commute. By uniqueness, we have $\Psi \circ m = m \circ (\Psi \otimes \Psi)$, showing that Ψ is a morphism of algebras. ∎

Example 3.2. In [5, 6, 8] Hopf morphisms from Hopf algebras of posets and graphs to $\mathcal{Q}Sym$ were constructed using Pieri operations. These Hopf morphisms arise from our universal property (Theorem 3.1). We recall the construction for a graded poset P. Let $\mathcal{H}P$ be the $\Bbbk$-algebra freely generated by the intervals $[x, y]$ of P, modulo identifying all singleton intervals $[x, x]$ with the unit 1 and empty intervals with zero. Then $\mathcal{H}P$ has a grading induced by the rank of an interval of P, and it is connected. It has a coalgebra structure induced by $\Delta([x, y]) = \sum_{z\in[x,y]} [x, z] \otimes [z, y]$, and is thus a graded connected Hopf algebra.

Since $\mathcal{H}P$ is the algebra freely generated by all non-trivial intervals of P, assigning arbitrary values to $\zeta([x, y])$ gives an algebra map to $\Bbbk$. This is equivalent to defining a Pieri operation in the sense of [6] since a zeta function ζ determines a right action on the linear span $\Bbbk P$ of the poset,

$$x.h_n = \sum_{\substack{x\leq y \\ n=\mathrm{rank}([x,y])}} \zeta([x, y]) \cdot y \,, \tag{3.3}$$

and conversely such a right action determines ζ.

Given such a zeta function we have a combinatorial Hopf algebra $(\mathcal{H}P, \zeta)$. From the universal properties of $(\mathcal{Q}Sym, h)$ we get two morphisms

$$\Psi^* \colon NSym \to \mathcal{H}P^* \qquad \text{and} \qquad \Psi \colon \mathcal{H}P \to \mathcal{Q}Sym.$$

the first gives a graded right action of $NSym$ on $\Bbbk P$ via (3.3). The second morphism is the generating function for the enumerative combinatorial invariants associated to this right $NSym$-structure on $\Bbbk P$, as constructed in [6].

We consider one example from this theory. (For more, see [6] and [1].) Suppose that we have a graded poset P and set $\zeta([x, y]) = 1$ for all intervals, giving the classical zeta function of posets. The induced map

$$\Psi([x, y]) \;=\; \sum_{\alpha\vDash\mathrm{rank}([x,y])} f_\alpha([x, y]) M_\alpha \;=\; \sum_{\alpha\vDash\mathrm{rank}([x,y])} c_\alpha([x, y]) F_\alpha$$

is such that $f_\alpha([x, y])$ is the classical flag f-vector of $[x, y]$ and $c_\alpha([x, y])$ is the classical flag h-vector. This map was introduced by Ehrenborg [12].

4. Eulerian combinatorial Hopf algebras

We introduce the notions of Möbius function and *eulerian* combinatorial Hopf subalgebra. The counit ϵ of a graded Hopf algebra $\mathcal{H}$ is the unit of $\overline{\mathcal{H}}^*$ under the convolution multiplication. Any non-zero group-like element ζ of $\overline{\mathcal{H}}^*$ has inverse $\zeta^{-1} = \zeta \circ S$, where S is the antipode of $\mathcal{H}$. Indeed

$$\begin{aligned} \zeta \circledast (\zeta \circ S) &= m_{\Bbbk} \circ (\zeta \otimes \zeta) \circ (Id \otimes S) \circ \Delta \;=\; m_{\Bbbk} \circ \Delta(\zeta) \circ (Id \otimes S) \circ \Delta \\ &= \zeta \circ m_{\mathcal{H}} \circ (Id \otimes S) \circ \Delta \;=\; \zeta \circ u \circ \epsilon \;=\; \epsilon\,, \end{aligned}$$

where $m_{\mathcal{H}}$ is the multiplication of $\mathcal{H}$ and $u\colon \Bbbk \to \mathcal{H}$ is the unit map, so $f \circ u = Id_{\Bbbk}$.

The *Möbius function* of a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ is this inverse of its zeta function, $\mu := \zeta^{-1} = \zeta \circ S$. Note that μ is also group-like.

Example 4.1. Let P be a graded poset with classical zeta function $\zeta([x,y]) := 1$ for all $x < y \in P$. Consider the combinatorial Hopf algebra $(\mathcal{H}P, \zeta)$. Then $\mu = \zeta \circ S$ is the classical Möbius function on P. Indeed, from $\mu * \zeta = \epsilon$ we deduce the defining relations $\mu([x,x]) = 1$ and $\sum_{x \leq z \leq y} \mu([x,z]) = 0$. The poset P is *eulerian* if $\mu([x,y]) = -1^{\mathrm{rank}([x,y])}$. This can be written as

$$\mu([x,y]) \;=\; (-1)^{\mathrm{rank}([x,y])} \zeta([x,y])\,.$$

We generalize this notion to any combinatorial Hopf algebra $(\mathcal{H}, \zeta)$.

Definition 4.2. A combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ is *eulerian* if $x \in \mathcal{H}_n$ implies that $\mu(x) = (-1)^n \zeta(x)$.

For example, if P is an eulerian poset with classical zeta function ζ, then $(\mathcal{H}P, \zeta)$ is an eulerian combinatorial Hopf algebra. Given a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$, we construct its maximal combinatorial Hopf subalgebra $(E(\mathcal{H}), \zeta)$ that is eulerian as follows. Consider the map $\xi\colon \mathcal{H} \to \Bbbk$ defined for $x \in \mathcal{H}_n$ by

$$\xi(x) \;=\; \mu(x) - (-1)^n \zeta(x)\,.$$

Any eulerian combinatorial Hopf subalgebra of $(\mathcal{H}, \zeta)$ must be a subset of

$$\widetilde{E} \;:=\; \ker \xi \;=\; \bigoplus_{n \geq 0} \ker(\xi_n\colon \mathcal{H}_n \to \Bbbk)\,.$$

For $x \in \mathcal{H}_n$ and $y \in \mathcal{H}_m$ we have $\xi(xy) = \mu(x)\xi(y) + (-1)^m \xi(x)\zeta(y)$, hence if both $x, y \in \widetilde{E}$ then $xy \in \widetilde{E}$. Since $\xi(1) = 0$, we have $1 \in \widetilde{E}$. Thus $\widetilde{E}$ is a graded subalgebra of $\mathcal{H}$, but not necessarily a Hopf subalgebra.

Definition-Proposition 4.3. *Let $(\mathcal{H}, \zeta)$ be a combinatorial Hopf algebra and let $E(\mathcal{H})$ be the largest coalgebra of $\mathcal{H}$ contained in $\widetilde{E} = \ker \xi$. Then*

(a) *$E(\mathcal{H})$ is a graded Hopf subalgebra of $\mathcal{H}$, and $(E(\mathcal{H}), \zeta)$ is eulerian. We call $(E(\mathcal{H}), \zeta)$ the* maximal eulerian subalgebra *of $(\mathcal{H}, \zeta)$.*

(b) *An element $x \in \mathcal{H}$ lies in $E(\mathcal{H})$ if and only if*

$$(Id \otimes \xi \otimes Id)\Delta^2(x) \;=\; 0\,. \tag{4.1}$$

(c) *Let* $\Psi\colon (\mathcal{H},\zeta)\to(\mathcal{H}',\zeta')$ *be a morphism of combinatorial Hopf algebras with* $(E(\mathcal{H}),\zeta)$ *and* $(E(\mathcal{H}'),\zeta')$ *the corresponding eulerian subalgebras. Then we have* $\Psi(E)\subseteq E(\mathcal{H}')$.

Proof. The proofs are from [1]. We reproduce them here for completeness.

For (a), if C and D are coalgebras contained in the algebra $\widetilde{E}\subseteq\mathcal{H}$, then $C\cdot D$ is also a coalgebra contained in $\widetilde{E}$. By the maximality of E, we have

$$E\cdot E\ \subseteq\ E\,,$$

showing that E is a graded bialgebra. The general formula due to Milnor and Moore [21] for the antipode of a graded bialgebra shows that E has an antipode which must agree with that of $\mathcal{H}$.

For (b), the ideal generated by a linear subspace V of $\mathcal{H}$ is given by $m^2(\mathcal{H}\otimes V\otimes\mathcal{H})$. Dually, the coideal cogenerated by the kernel $\widetilde{E}$ of ξ is given by (4.1).

For (c), as Ψ is a map of combinatorial Hopf algebras, we have $\zeta=\zeta'\circ\Psi$ and $\Psi\circ S=S'\circ\Psi$ hence $\mu=\mu'\circ\Psi$ and so $\xi=\xi'\circ\Psi$. Thus, for $x\in E(\mathcal{H})$,

$$(Id\otimes\xi'\otimes Id)(\Delta')^2\big(\Psi(x)\big)\ =\ (\Psi\otimes Id\otimes\Psi)(Id\otimes\xi\otimes Id)\Delta^2(x)\ =\ 0\,.$$

We conclude that $\Psi(x)\in E(\mathcal{H}')$. ∎

Definition-Proposition 4.3 (c)says that the association of a combinatorial Hopf algebra to its maximal eulerian subalgebra is a functor from the category of combinatorial Hopf algebras to its subcategory of eulerian combinatorial Hopf algebras.

Remark 4.4. The relations (4.1) are generalized Dehn-Sommerville relations for the maximal eulerian subalgebra $E(\mathcal{H})$ of the pair $(\mathcal{H},\zeta)$. Let ξ_n be the nth homogeneous component of $\xi\in\overline{\mathcal{H}}^*$. Then the dual of $E(\mathcal{H})$ is

$$E(\mathcal{H})^*\ =\ \mathcal{H}^*/\langle\xi_n\mid n\geq 0\rangle\,, \tag{4.2}$$

where $\langle\xi_n\mid n\geq 0\rangle$ is the ideal of $\mathcal{H}^*$ generated by $\{\xi_n\mid n\geq 0\}$. This is the *Dehn-Sommerville ideal* of $(\mathcal{H},\zeta)$. Given a morphism $\Psi\colon\mathcal{H}\to\mathcal{H}'$ of combinatorial Hopf algebras, the Dehn-Sommerville ideal of $(\mathcal{H},\zeta)$ is generated by $\Psi^*(I')$, where I' is the Dehn-Sommerville ideal of $(\mathcal{H}',\zeta')$.

5. The maximal eulerian subalgebra of $(\mathcal{Q}Sym,h)$

By the universal property of $(\mathcal{Q}Sym,h)$, its maximal eulerian subalgebra is the terminal object in the category of eulerian combinatorial Hopf algebras. We shall see that it is exactly the peak algebra Π introduced by Stembridge in [25].

To this end, let us consider the pair $(\mathcal{Q}Sym,q)$ where

$$q(F_\alpha)\ :=\ \begin{cases}1 & \textit{if } \alpha=0,\\ 2 & \textit{if } \alpha=[1,1,\ldots,1,n-k] \textit{ and } 0\leq k\leq n-1,\\ 0 & \textit{otherwise.}\end{cases}$$

Proposition 5.1. *The pair* $(\mathcal{Q}Sym,q)$ *is eulerian.*

Proof. In *NSym* we have $e_k = (-1)^k S(h_k)$ [13] and $e_k h_\ell = R_{[1^k,\ell]} + R_{[1^{k-1},\ell+1]}$. Thus

$$q \;=\; \Big(\sum_{k\geq 0} e_k\Big)\Big(\sum_{\ell\geq 0} h_\ell\Big)\,, \tag{5.1}$$

and $\Delta(q) = q\otimes q$. We conclude that $(\mathcal{Q}Sym, q)$ is a combinatorial Hopf algebra. Its Möbius function is $\mu = q\circ S_{\mathcal{Q}Sym} = S_{NSym}(q)$. Using (5.1), we have

$$\mu \;=\; S(q) \;=\; \Big(\sum_{\ell\geq 0}(-1)^\ell e_\ell\Big)\Big(\sum_{k\geq 0}(-1)^k h_k\Big)\,.$$

Thus $q(f) = (-1)^n\mu(f)$ for $f\in\mathcal{Q}Sym_n$, and so $(\mathcal{Q}Sym, q)$ is eulerian. ∎

Remark 5.2. Comparing (5.1) to the formula [18, III.8.1] for the Schur Q-functions $q_n(x)$, we see that the components of q are the non-commutative analogues of the Schur Q-functions $q_n(x)$, and thus validating our choice of notation q.

By Theorem 3.1, there is a unique morphism of combinatorial Hopf algebras $\Theta\colon \mathcal{Q}Sym\to\mathcal{Q}Sym$ where Θ is characterized by $q = h\circ\Theta$. Moreover, since $(\mathcal{Q}Sym, q)$ is eulerian, the image of Θ lies inside the maximal eulerian subalgebra of $(\mathcal{Q}Sym, h)$.

We recall the Hopf-algebraic description [5] of Stembridge's peak algebra [25]. A *peak composition* is a composition $\alpha = [\alpha_1,\alpha_2,\ldots,\alpha_k]$ with $\alpha_i\geq 2$ for $i\leq k-1$. Given a peak composition α, set

$$\theta_\alpha \;:=\; \sum_{\beta\in B(\alpha)} F_\beta\,, \tag{5.2}$$

where

$$B(\alpha) \;:=\; \{\beta \mid i\in D(\alpha) \implies \#\{i{-}1,\, i\}\cap D(\beta) = 1\}\,. \tag{5.3}$$

The peak algebra Π is spanned by the functions θ_α indexed by peak compositions.

Let $\widetilde{\Theta}$ be the $\Bbbk$-linear map from $\mathcal{Q}Sym$ to Π defined by $\widetilde{\Theta}(F_\alpha) = 2^{\ell(\alpha)}\theta_{\Lambda(\alpha)}$, where $\Lambda(\alpha)$ is the composition formed from $\alpha = [\alpha_1,\alpha_2,\ldots,\alpha_k]$ by adding together adjacent components $\alpha_i,\alpha_{i+1},\ldots,\alpha_{i+j}$ where $\alpha_{i+l} = 1$ for $l = 0,\ldots,j-1$, and either $\alpha_{i+j}\neq 1$, or $i+j = k$. This linear map is in fact a morphism of graded Hopf algebra [5]. Since $h\circ\widetilde{\Theta}(F_\alpha) = q(F_\alpha)$ we have that $\widetilde{\Theta} = \Theta$.

Theorem 5.3. *(Π, h) is the maximal eulerian subalgebra of $(\mathcal{Q}Sym, h)$.*

Proof. The set of compositions of n is totally ordered by the reverse lexicographic order (revlex), defined as follows. Given compositions $\alpha = [\alpha_1,\alpha_2,\ldots,\alpha_j]$ and $\beta = [\beta_1,\beta_2,\ldots,\beta_k]$ of n, we say that $\alpha > \beta$ if, for some t, we have $\alpha_{j+1-t} > \beta_{k+1-t}$, but for $1\leq i\leq t$, $\alpha_{j+1-i} = \beta_{k+1-i}$. Thus $[3,4] > [4,3]$. Note that α is the leading (greatest in revlex) composition in $B(\alpha)$ and hence F_α is the leading term of θ_α.

The discussion preceding the statement of the theorem shows that $\Pi\subseteq\widetilde{\Pi}$, the maximal eulerian subalgebra of $(\mathcal{Q}Sym, h)$. We follow the subduction algorithm to prove that $\widetilde{\Pi}\subseteq\Pi$, arguing inductively on the degree and leading term of a homogeneous element

$$f \;=\; \sum_{\beta\models n} c_\beta F_\beta \;\in\; \widetilde{\Pi}_n\,.$$

First, recall that if $\beta \vDash n$, then

$$\xi(F_\beta) \;=\; \mu(F_\beta) - (-1)^n h(F_\beta) \;=\; \begin{cases} (-1)^{n+1} & \text{if } \beta = [n], \\ (-1)^n & \text{if } \beta = [1,1,\ldots,1], \\ 0 & \text{otherwise.} \end{cases}$$

We leave the reader to use this and (4.1) to show that $\widetilde{\Pi}_n = \Pi_n$ for $n = 0, 1$, and 2.

Suppose that $n > 2$ and $\nu = [\nu_1, \nu_2, \ldots, \nu_k]$ indexes the leading term of f, so that it is the revlex largest index of a term $c_\alpha F_\alpha$ appearing in f. Assuming that $\widetilde{\Pi}_m = \Pi_m$ for all $m < n$, we will show that ν is a peak composition and thus $f - c_\nu \theta_\nu \in \widetilde{\Pi}_n$ has a smaller leading term, or is zero. Induction on the revlex order of compositions of n, and then on n will complete the proof.

Since $[n]$ is a peak composition, we may assume that $\ell := \nu_k < n$. Consider the component g of

$$\Delta(f) \;\in\; \bigoplus_{i+j=n} \widetilde{\Pi}_i \otimes \widetilde{\Pi}_j$$

lying in $\widetilde{\Pi}_{n-\ell} \otimes \widetilde{\Pi}_\ell$. Since $0 < \ell < n$, our inductive hypothesis implies that $\widetilde{\Pi}_{n-\ell} \otimes \widetilde{\Pi}_\ell = \Pi_{n-\ell} \otimes \Pi_\ell$, and so there are elements $d_{\delta,\gamma} \in \Bbbk$ for $\delta \vDash (n-\ell)$ and $\gamma \vDash \ell$ such that

$$g \;=\; \sum_{\delta,\gamma} d_{\delta,\gamma}\, \theta_\delta \otimes \theta_\gamma \;\in\; \Pi_{n-\ell} \otimes \Pi_\ell\,.$$

Since $\nu_k = \ell$, this has a summand with $\gamma = [\ell]$. Let $\delta \vDash (n-\ell)$ be the maximal in revlex composition such that $d_{\delta,[\ell]} \neq 0$. Then the expression (5.2) for θ_δ implies that $F_\delta \otimes F_{[\ell]}$ appears in g and hence $\Delta(f)$, and δ is the maximal composition with this property. By our choice of ν, only F_ν may contribute to $F_\delta \otimes F_{[\ell]}$ in $\Delta(f)$, hence $\delta = [\nu_1, \nu_2, \ldots, \nu_{k-1}]$ is a peak composition. But then $\nu = \delta.\ell$ is also a peak composition, which completes the proof. ∎

The Dehn-Sommerville ideal of $(\mathcal{Q}Sym, h)$ is generated by the differences $R_n - R_{[1,1,\ldots,1]} \in\in NSym$. In characteristic zero, Billera and Liu [10] show that this ideal is generated by the Euler relations

$$X_{2n} \;=\; 2h_{2n} + \sum_{i=1}^{2n-1} (-1)^i h_i h_{2n-i}\,. \tag{5.4}$$

From this we deduce that $\dim(\Pi_n) = f_{n-1}$ where f_k is the kth Fibonacci number.

6. The maximal eulerian subalgebras of Sym and $NSym$, and a commutative diagram

The Hopf algebras Sym of symmetric functions and $NSym$ of noncommutative symmetric functions are also combinatorial Hopf algebras, where we simply pull back the structure map $h\colon \mathcal{Q}Sym \to \Bbbk$ along the composition

$$NSym \xrightarrow{ab} Sym \hookrightarrow \mathcal{Q}Sym\,. \tag{6.1}$$

Since Sym is self-dual and $NSym$ and $\mathcal{Q}Sym$ are a dual pair of Hopf algebras, the pullbacks of the component $h_n \in NSym$ of h are simply its images under the above maps. The image of h_n in *Sym* is the complete homogeneous symmetric function $h_n(x)$ and the image of $h_n(x)$ in $\mathcal{Q}Sym$ is F_n. We deduce the following theorem.

Theorem 6.1. *Let*

$$h(x) \ := \ \sum_{n\geq 0} h_n(x) \ \in \ \overline{Sym}^* \qquad \textit{and} \qquad F \ := \ \sum_{n\geq 0} F_n \ \in \ \overline{NSym}^* .$$

Then $(Sym, h(x))$ *and* $(NSym, F)$ *are combinatorial Hopf algebras with the maps of* (6.1) *morphisms of combinatorial Hopf algebras.*

Let $(S\Pi, h(x))$ and $(N\Pi, F)$ be the maximal eulerian subalgebras of $(Sym, h(x))$ and $(NSym, F)$, respectively. To construct the analogs of the maps Θ, we first give *Sym* and *NSym* the structure of an eulerian combinatorial Hopf algebra. For this, we pull back the structure map $q\colon \mathcal{Q}Sym \to \Bbbk$ of the eulerian combinatorial Hopf algebra $(\mathcal{Q}Sym, q)$ of Section 5. As for h, we consider the images of the components $q_n \in NSym_n$ along the composition (6.1). A noted in Remark 5.2, the image of q_n in $\mathfrak{S}$ is the Schur Q-function $q_n(x)$. From Proposition 3.2 of [11], the image of $q_n(x)$ in $\mathcal{Q}Sym$ is θ_n. Set $\theta := \sum_{n\geq 0}\theta_n$ and $q(x) := \sum_{n\geq 0} h_n(x)$.

Theorem 6.2. $(NSym, \theta)$ *and* $(\mathfrak{S}, q(x))$ *are eulerian combinatorial Hopf algebras.*

I have a little bit more; I think that I may be able to describe the maps Θ and the factoring through the maximal eulerian subalgebras. Maybe if I have time on Thursday morning, I'll add this and resend it.
—**F**

7. The maximal eulerian subalgebra of a cofree coalgebra

We will describe the maximal eulerian subalgebra of two other combinatorial Hopf algebras that are cofree as coalgebras. A first step is to identify the Hilbert series of such an algebra, which we do via its dual, which is a free graded algebra.

Suppose that $A = \Bbbk\langle \mathbf{S}\rangle$ is a free graded algebra with grading induced by a grading on the generators $\mathbf{S}$ of A, where there are only finitely many elements of $\mathbf{S}$ of any degree, and $\mathbf{S}$ has no elements of degree 0. Writing $A(x)$ for the Hilbert series of A,

$$A(t) \ = \ \sum_{n\geq 0} \dim A_n t^n ,$$

we have

$$A(t) \ = \ \frac{1}{1-\mathbf{S}(t)} ,$$

where $\mathbf{S}(t) = \sum_{s\in\mathbf{S}} t^{\deg(s)}$.

Suppose now that we have another free graded algebra $B = \Bbbk\langle\mathbf{T}\rangle$, where $\mathbf{T}$ is a set of homogeneous generators, and we define $B(t)$ and $\mathbf{T}(t)$ analogously to $A(t)$ and $\mathbf{S}(t)$. The free product $A*B$ of A and B is the graded algebra $\Bbbk\langle\mathbf{S}\cup\mathbf{T}\rangle$, with Hilbert series

$$\big(A*B\big)(t) \ = \ \frac{1}{1-\big(\mathbf{S}(t)+\mathbf{T}(t)\big)} \ = \ \frac{1}{\dfrac{1}{A(t)}+\dfrac{1}{B(t)}-1} . \tag{7.1}$$

We also remark that if I and J are ideals of A and B respectively, then

$$A * B \Big/ \langle I \cup J \rangle \;\cong\; (A/I) * (B/J)\,. \tag{7.2}$$

We use this to determine the Hilbert series of certain combinatorial Hopf algebras. Let $(\mathcal{H}, \zeta)$ be a combinatorial Hopf algebra that is a cofree as a coalgebra. Thus $\mathcal{H}^* = \Bbbk\langle \mathbf{S}\rangle$ is a free algebra. Suppose that we have a second graded connected Hopf algebra $\mathcal{K}$ also cofree with $\mathcal{K}^* = \Bbbk\langle \mathbf{S}\cup\mathbf{T}\rangle = \mathcal{H}^* * \Bbbk\langle \mathbf{T}\rangle$. The inclusion $\mathcal{H}^* \subseteq \mathcal{K}^*$ induces a surjective graded Hopf morphism $\Psi\colon \mathcal{K} \to \mathcal{H}$. Since $\Psi^*(\zeta) = \zeta \in \overline{\mathcal{H}}^* \subseteq \overline{\mathcal{K}}^*$, we have that $(\mathcal{K}, \zeta)$ is a combinatorial Hopf algebra, and Ψ is a morphism of combinatorial Hopf algebras. In this setting, let $(E(\mathcal{H}), \zeta)$ and $(E(\mathcal{K}), \zeta)$ be the respective maximal eulerian subalgebras. Let $I \subset \mathcal{H}^*$ be the Dehn-Sommerville ideal of $(\mathcal{H}, \zeta)$, and let J be the ideal of $\mathcal{K}^*$ generated by I. From (7.2) and (4.2) we have

$$E(\mathcal{K})^* \;\cong\; \mathcal{K}^*/J \;\cong\; \big(\mathcal{H}^*/I\big) * \Bbbk\langle \mathbf{T}\rangle \;\cong\; E(\mathcal{H})^* * \Bbbk\langle \mathbf{T}\rangle\,. \tag{7.3}$$

In particular we use (7.1) to deduce that the Hilbert series of $E(\mathcal{H})$ and $F(\mathcal{K})$ (which are the same as those of their graded duals) are related by

$$\frac{1}{E(\mathcal{K})(t)} \;=\; \frac{1}{E(\mathcal{H})(t)} - \mathbf{T}(t) \;=\; \frac{1}{E(\mathcal{H})(t)} + \frac{1}{\mathcal{K}(t)} - \frac{1}{\mathcal{H}(t)}\,,$$

or rather more elegantly,

$$\frac{1}{E(\mathcal{K})(t)} - \frac{1}{\mathcal{K}(t)} \;=\; \frac{1}{E(\mathcal{H})(t)} - \frac{1}{\mathcal{H}(t)}\,. \tag{7.4}$$

This situation is of particular interest for $(\mathcal{Q}Sym, h)$, as it is a cofree coalgebra ($\mathcal{Q}Sym^* \cong NSym = \Bbbk\langle h_1, h_2, \ldots\rangle$). In Section 5 we described its maximal eulerian subalgebra (Π, h). The corresponding Hilbert series are

$$\mathcal{Q}Sym(t) \;=\; \frac{1-t}{1-2t} \qquad\text{and}\qquad \Pi(t) \;=\; \frac{1-t^2}{1-t-t^2}\,. \tag{7.5}$$

The first series comes from $NSym = \Bbbk\langle \mathbf{S}\rangle$ where $\mathbf{S} = \{h_1, h_2, \ldots\}$ and so $\mathbf{S}(t) = \frac{t}{1-t}$. The second series follows from $\dim(\Pi_{n+1}) = F_n$, the nth Fibonacci number.

In the following sections, we use this to describe the maximal eulerian subalgebras and corresponding Hilbert series for two other combinatorial Hopf algebras $(\mathfrak{S}Sym, h)$ and $(\mathcal{Y}Sym, h)$ that are also cofree and contain $NSym$.

8. The maximal eulerian subalgebra of $(\mathfrak{S}Sym, h)$

The Malvenuto-Reutenauer Hopf algebra $\mathfrak{S}Sym$ of permutations [19, 20] is self-dual, contains $NSym$, and projects onto $\mathcal{Q}Sym$. Let $\mathfrak{S}_n$ be the symmetric group on n elements and define $\mathfrak{S}Sym$ to be the vector space over $\Bbbk$ with basis $\{\mathcal{F}_u \mid u \in \mathfrak{S}_n,\ n \geq 0\}$. For $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_m$, define the product $\mathcal{F}_u \cdot \mathcal{F}_v$ in $\mathfrak{S}Sym$ by

$$\mathcal{F}_u \cdot \mathcal{F}_v \;=\; \sum_{\sigma \in \mathfrak{S}^{(n,m)}} \mathcal{F}_{(u\times v)\sigma^{-1}}\,. \tag{8.1}$$

The comultiplication is

$$\Delta(\mathcal{F}_w) \;=\; \sum_{w=u\,.\,v} \mathcal{F}_{\mathrm{st}(u)} \otimes \mathcal{F}_{\mathrm{st}(v)}. \tag{8.2}$$

By (2.4) and (2.6), the map $D\colon \mathfrak{S}\mathit{Sym} \to \mathcal{Q}\mathit{Sym}$ defined by $D(\mathcal{F}_u) = F_{d(u)}$ is a morphism of graded bialgebras (and hence a morphism of graded Hopf algebras). The isomorphism $\mathfrak{S}\mathit{Sym} \cong \mathfrak{S}\mathit{Sym}^*$ is given by $\mathcal{F}_u \mapsto \mathcal{F}^*_{u^{-1}}$.

Further structures of $\mathfrak{S}\mathit{Sym}$ are revealed by the $\mathcal{M}$ basis of [2]. For $u \in \mathfrak{S}_n$ define $\mathcal{M}_u = \sum_{u\leq v} \mu_{\mathfrak{S}_n}(u,v)\mathcal{F}_v$ where $\mu_{\mathfrak{S}_n}$ is the (poset) Möbius function of the weak order on $\mathfrak{S}_n$. By Möbius inversion, $\mathcal{F}_u = \sum_{u\leq v} \mathcal{M}_v$. We use this basis to describe $\mathfrak{S}\mathit{Sym}^*$ as a free algebra.

A permutation $u \in \mathfrak{S}_n$ has a *global descent* at position k if $i \leq k < j$ implies that $u(i) > u(j)$. For example, the permutation $u = (4,6,5,2,1,3)$ has a global descent at position 3. A permutation u splits uniquely as a concatenation $u = v_1\,.\,v_2\,\ldots\,v_k$ where $\mathrm{st}(v_i)$ has no global descents and for each $1 \leq i \leq k-1$, u has a global descent between the end of v_i and the beginning of v_{i+1}. We call $\mathrm{st}(v_1),\ldots,\mathrm{st}(v_k)$ the *global descent components* of u. For example, with $u = (4,6,5,2,1,3) = (4,6,5)\,.\,(2,1,3)$ the global descent components are $(1,3,2)$ and $(2,1,3)$.

This basis has many remarkable properties related to global descents. For instance,

$$D(\mathcal{M}_u) \;=\; \begin{cases} M_{d(u)} & \text{if all global descent components are identities} \\ 0 & \text{otherwise.} \end{cases}$$

Also $\mathfrak{S}\mathit{Sym}^* = \Bbbk\langle \mathcal{M}^*_u \mid u \text{ has no global descents}\rangle$ and the multiplication in $\mathfrak{S}\mathit{Sym}^*$ is

$$\mathcal{M}^*_u \cdot \mathcal{M}^*_v \;=\; \mathcal{M}^*_{(u+m)\,.\,v}\,, \tag{8.3}$$

where $v \in \mathfrak{S}_m$ and $(u{+}m)$ is the sequence obtained by adding m to each value of u, so that $(u{+}m)\,.\,v$ has a global descent between $(u{+}m)$ and v. The embedding $\mathit{NSym} \hookrightarrow \mathfrak{S}\mathit{Sym}^*$ is given by $h_n \mapsto \mathcal{M}^*_{id_n}$. The map $D\colon \mathfrak{S}\mathit{Sym} \to \mathcal{Q}\mathit{Sym}$ is dual to that inclusion. Thus $(\mathfrak{S}\mathit{Sym}, h)$ is a combinatorial Hopf algebra and D is a morphism of combinatorial Hopf algebras.

Let $\mathbf{S} = \{\mathcal{M}^*_{id_n} \mid n \geq 1\}$ and $\mathbf{T} = \{\mathcal{M}^*_u \mid u \neq id \text{ has no global descents}\}$. Then $\mathit{NSym} \cong \Bbbk\langle\mathbf{S}\rangle$ and $\mathfrak{S}\mathit{Sym}^* = F\langle \mathbf{S} \cup \mathbf{T}\rangle$. By (7.3), the maximal eulerian subalgebra $(\mathfrak{S}\Pi, h)$ of $(\mathfrak{S}\mathit{Sym}, h)$ is

$$\mathfrak{S}\Pi^* \;=\; \Pi^* * \Bbbk\langle \mathbf{T}\rangle\,, \tag{8.4}$$

and its Hilbert series is (by (7.4) and (7.5)):

$$\frac{1}{\mathfrak{S}\Pi^*(x)} \;=\; \frac{1}{\mathfrak{S}\mathit{Sym}(x)} + \frac{x^2}{1-x^2}\,. \tag{8.5}$$

Here, $\mathfrak{S}\mathit{Sym}(x) = \sum_{n\geq 0} n!x^n$. One can give an explicit basis for $\mathfrak{S}\Pi$ using suitable dualization, the explicit basis (5.2) of Π, and the realization of $\mathfrak{S}\mathit{Sym}^*$ as a free graded algebra. We also remark that the Dehn-Sommerville ideal of $(\mathfrak{S}\mathit{Sym}, h)$ has the same generators (Dehn-Sommerville relations) as the Dehn-Sommerville ideal of $(\mathcal{Q}\mathit{Sym}, h)$. The generators of Billera and Liu (5.4) are again particularly nice.

9. The Maximal Eulerian subalgebra of $(\mathcal{Y}Sym, h)$

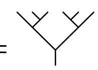
The Loday-Ronco Hopf algebra $\mathcal{Y}Sym$ of planar binary trees was introduced in [17]. We follow the treatment in [3].

The set Y_n of planar binary trees with n internal vertices is in bijection with 132-avoiding permutations in $\mathfrak{S}_n$. (A permutation w is 132-avoiding if we do not have $i < j < k$ with $w_j > w_k > w_i$.) The number of such trees is the Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$. We identify such a tree with the corresponding permutation. Let $\tau\colon \mathfrak{S}_n \to Y_n$ be the order-preserving mapping such that $\tau(w)$ is the minimal 132-avoiding permutation greater than w. We describe τ recursively. For $w \in \mathfrak{S}_n$, let $j = w^{-1}(n)$ be the position of n in w. Then

$$\tau(w) \;:=\; \big(\tau(\mathrm{st}(w_1,\ldots,w_{j-1})) + n{-}j\big)\,.\,n\,.\,\tau(\mathrm{st}(w_{j+1},\ldots,w_n))\,,$$

where, '$.$' denotes concatenation of sequences and $\tau(\mathrm{st}(w_1,\ldots,w_{j-1})) + n{-}j$ is the sequence obtained by adding $n{-}j$ to each value of $\tau(\mathrm{st}(w_1,\ldots,w_{j-1}))$.

This recursive algorithm mimics one to construct a planar binary tree from a permutation. Set $\tau(\emptyset) = |$. Then, given a permutation w as above, we graft $\tau(\mathrm{st}(w_1,\ldots,w_{j-1}))$ onto the left branch of  $= \tau(\mathrm{st}(n))$ and $\tau(\mathrm{st}(w_{j+1},\ldots,w_n))$ onto its right branch. Thus

$\tau(21) =$ , $\quad \tau(12) =$ , $\quad$ and $\quad \tau(21534) =$  .

Let $\mathcal{Y}Sym$ be the vector space over $\Bbbk$ with basis $\{\mathbf{F}_t \mid t \in Y_n, n \geq 0\}$. For $t \in Y_n$ and $s \in Y_m$, we define the multiplication in $\mathcal{Y}Sym$ by

$$\mathbf{F}_t \cdot \mathbf{F}_s \;=\; \sum_{\sigma \in \mathfrak{S}^{(n,m)}} \mathbf{F}_{\tau\left((t\times s)\sigma^{-1}\right)}\,, \tag{9.1}$$

where we regard a tree as a permutation. The comultiplication is given by

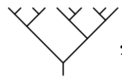
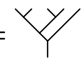
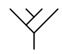
$$\Delta(\mathbf{F}_t) \;=\; \sum_{t=u\,.\,v} \mathbf{F}_{\tau(\mathrm{st}(u))} \otimes \mathbf{F}_{\tau(\mathrm{st}(v))}\,. \tag{9.2}$$

Comparing these formulas with the corresponding formulas (8.1) and (8.2) for $\mathfrak{S}Sym$ and (2.4) and (2.6) for $\mathcal{Q}Sym$, we see that the map $D\colon \mathfrak{S}Sym \to \mathcal{Q}Sym$ of graded Hopf algebras factors through $\mathcal{Y}Sym$:

$$\mathfrak{S}Sym \xrightarrow{\;T\;} \mathcal{Y}Sym \xrightarrow{\;D\;} \mathcal{Q}Sym$$

where $T(\mathcal{F}_u) = \mathbf{F}_{\tau(u)}$ and D is the linear map $D(\mathbf{F}_t) = F_{d(t)}$, where $d(t)$ is the descent composition of a tree t, regarded as a permutation. As before, the pullback of the structure map h of the combinatorial Hopf algebra $(\mathcal{Q}Sym, h)$ equips $\mathcal{Y}Sym$ with the structure of a combinatorial Hopf algebra so that these maps are morphisms of combinatorial Hopf algebras.

Further structure of the Hopf algebra $\mathcal{Y}Sym$ is revealed through a second basis introduced in [3]. For $t \in Y_n$, define $\mathbf{M}_t := \sum_{t\leq s} \mu_{Y_n}(t,s)\mathbf{F}_s$, where μ_{Y_n} is the (poset) Möbius function of the order on Y_n. By Möbius inversion, $\mathbf{F}_t = \sum_{t\leq s} \mathbf{M}_s$. It is with this basis that one can explicitly describe $\mathcal{Y}Sym^*$ as a free algebra. As with permutations, we may decompose trees into global descent components. Given a tree t regarded as a permutation, its global descent components as a permutation are themselves trees. In fact, they have a very nice geometric description. For example, the permutation 786943512 has global descent components $\mathrm{st}(7869) = 2314$, $\mathrm{st}(435) = 213$, and 12, and we have

$\tau(786943512) =$ , $\tau(2314) =$ , $\tau(213) =$ , $\tau(12) =$ .

The global descents of a tree record the branchings along its main right branch, and the global descent components are given by cutting the right branch in between each branching.

This basis has many remarkable properties related to global descents. For instance,

$$T(\mathcal{M}_u) = \begin{cases} \mathbf{M}_{\tau(u)} & \text{if all global descent components are trees} \\ 0 & \text{otherwise.} \end{cases}$$

$$D(\mathbf{M}_t) = \begin{cases} M_{d(t)} & \text{if all global descent components are identities} \\ 0 & \text{otherwise.} \end{cases}$$

Also $\mathcal{Y}Sym^* = \Bbbk\langle \mathbf{M}^*_t \,|\, t \text{ has no global descents}\rangle$ and the multiplication in $\mathcal{Y}Sym^*$ is

$$\mathbf{M}^*_t \cdot \mathbf{M}^*_s = \mathbf{M}^*_{(t+m)\,.\,s}\,, \tag{9.3}$$

where $s \in Y_m$ and $(t+m)$ is the sequence obtained by adding m to each value of t, so that $(t+m)\,.\,s$ has a global descent between $(t+m)$ and s. We have the embedding $NSym \hookrightarrow \mathcal{Y}Sym^*$ given by $h_n \mapsto \mathbf{M}^*_{id_n}$.

We now apply the techniques of Section 7. Let $\mathbf{S} = \{\mathbf{M}^*_{id_n} \,|\, n \geq 1\}$ and $\mathbf{T} = \{\mathbf{M}^*_u \,|\, t \neq id \text{has no global descent}\}$. Then $NSym \cong k\langle \mathbf{S}\rangle$ and $\mathcal{Y}Sym^* = k\langle \mathbf{S} \cup \mathbf{T}\rangle$. By (7.3), the eulerian subalgebra $(\mathcal{Y}\Pi, h)$ of $(\mathcal{Y}Sym, h)$ satisfies

$$\mathcal{Y}\Pi^* = \Pi^* * k\langle \mathbf{T}\rangle\,, \tag{9.4}$$

and the Hilbert series is (by (7.4) and (7.5)):

$$\frac{1}{\mathcal{Y}\Pi^*(x)} = \frac{1}{\mathcal{Y}Sym(x)} + \frac{x^2}{1-x^2}\,. \tag{9.5}$$

Here, $\mathcal{Y}Sym(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$. One can give an explicit basis for $\mathcal{Y}\Pi$ using suitable dualization, the explicit basis (5.2) of Π, and the realization of $\mathcal{Y}Sym^*$ as a free graded algebra. We also remark that the Dehn-Sommerville ideal of $(\mathcal{Y}Sym, h)$ has the same generators (Dehn-Sommerville relations) as the Dehn-Sommerville ideal of $(\mathcal{Q}Sym, h)$. The generators of Billera and Liu (5.4), are again particularly nice.

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(Marcelo Aguiar) Department of Mathematics, Texas A&M University, College Station, Texas, 77843, USA

E-mail address, Marcelo Aguiar: `maguiar@math.tamu.edu`

URL, Marcelo Aguiar: `http://www.math.tamu.edu/~maguiar`

(Nantel Bergeron) Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA

E-mail address, Nantel Bergeron: `bergeron@mathstat.yorku.ca`

URL, Nantel Bergeron: `http://www.math.yorku.ca/bergeron`

(Frank Sottile) Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA

E-mail address, Frank Sottile: `sottile@math.umass.edu`

URL, Frank Sottile: `http ://www.math.umass.edu/~sottile`