

A COMBINATORIAL PROOF THAT SCHUBERT VS. SCHUR COEFFICIENTS ARE NONNEGATIVE

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ABSTRACT. We give a combinatorial proof that the product of a Schubert polynomial by a Schur polynomial is a nonnegative sum of Schubert polynomials. Our proof uses Assaf's theory of dual equivalence to show that a quasisymmetric function of Bergeron and Sottile is Schur-positive. By a geometric comparison theorem of Buch and Mihalcea, this implies the nonnegativity of Gromov-Witten invariants of the Grassmannian.

Dedicated to the memory of Alain Lascoux

INTRODUCTION

A Littlewood-Richardson coefficient is the multiplicity of an irreducible representation of the general linear group in a tensor product of two irreducible representations, and is thus a nonnegative integer. Littlewood and Richardson conjectured a formula for these coefficients in 1934 [25], which was proven in the 1970's by Thomas [33] and Schützenberger [31]. Since Littlewood-Richardson coefficients may be defined combinatorially as the coefficients of Schur functions in the expansion of a product of two Schur functions, these proofs of the Littlewood-Richardson rule furnish combinatorial proofs of the nonnegativity of Schur structure constants.

The Littlewood-Richardson coefficients are also the structure constants for expressing products in the cohomology of a Grassmannian in terms of its basis of Schubert classes. Independent of the Littlewood-Richardson rule, these Schubert structure constants are known to be nonnegative integers through geometric arguments. The integral cohomology ring of any flag manifold has a Schubert basis and again geometry implies that the corresponding Schubert structure constants are nonnegative integers. These cohomology rings and their Schubert bases have combinatorial models, and it remains an open problem to give a combinatorial proof that the Schubert structure constants are nonnegative.

We give such a combinatorial proof of nonnegativity for a class of Schubert structure constants in the classical flag manifold. These are the constants that occur in the product of an arbitrary Schubert class by one pulled back from a Grassmannian projection. They are defined combinatorially as the product of a Schubert polynomial [22] by a Schur symmetric polynomial; we call them *Schubert vs. Schur coefficients*. As the Schubert

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polynomials form a basis for the ring of polynomials in z_1, z_2, \dots , these coefficients determine its structure as a module over polynomials symmetric in z_1, \dots, z_k , for any k . These coefficients were studied by Bergeron and Sottile [4, 6, 7], who defined a quasisymmetric generating function associated to intervals in the Grassmannian-Bruhat order (a partial order on the symmetric group), showed that this quasisymmetric function is symmetric, and that the Schubert vs. Schur coefficients are the coefficients of Schur functions in this symmetric function. This work relied upon the Pieri formula [32], which has since been given combinatorial proofs [24, 29]. Consequently, a combinatorial proof that these quasisymmetric functions are Schur-positive gives a combinatorial proof of nonnegativity of the Schubert vs. Schur coefficients.

Another important question of combinatorial positivity concerned the Macdonald polynomials [27]. These are symmetric functions with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in variables q, t , which were conjectured by Macdonald to have coefficients in $\mathbb{N}[q, t]$ (polynomials in q, t with nonnegative integer coefficients) when expanded in the Schur basis. Five nearly simultaneous proofs in the mid 1990's showed that these coefficients were in $\mathbb{Z}[q, t]$ [12, 13, 18, 19, 30]. While positivity was proven by Haiman [17] using algebraic geometry and representation theory, a combinatorial proof took longer. A breakthrough came when Macdonald polynomials were given a combinatorial definition as a quasisymmetric function with coefficients in $\mathbb{N}[q, t]$ [15]. That work also showed them to have a positive expansion in terms of LLT polynomials [21]. The Macdonald positivity conjecture was finally given a combinatorial proof by Assaf [2], who introduced a new technique—dual equivalence graphs—for proving Schur-positivity of quasisymmetric functions and applied it to the LLT polynomials. Dual equivalence not only shows Schur-positivity, but it also gives a (admittedly complicated) combinatorial formula for the Schur coefficients in the quasisymmetric function.

When the quasisymmetric function comes from descents in a collection of words the theory of dual equivalence may be recast in terms of a family of involutions on the words [1]. The critical condition for a dual equivalence only needs to be verified locally on all subwords of length up to five and some of length six. The symmetric function of Bergeron and Sottile is the quasisymmetric function associated to descents on a collection of words that are themselves saturated chains in intervals of the Grassmannian-Bruhat order. The results of [6] lead to a unique family of involutions which immediately satisfy most properties of a dual equivalence for these chains. All that remains is to verify the local conditions on chains of length up to six. There are only finitely many chains of a given length, up to an equivalence, and thus we may verify this local condition on a computer. This shows that the symmetric function of Bergeron and Sottile is Schur-positive and gives a combinatorial proof that the Schubert vs. Schur constants are nonnegative.

Identifying the fundamental class of a Schubert variety with a Schur function identifies the homology of the Grassmannian with a linear subspace in the algebra of symmetric functions. Under this identification, each symmetric function of Bergeron and Sottile is the fundamental cycle of the image in the Grassmannian of a Richardson variety (intersection of two Schubert varieties) under the projection from the flag variety. These Richardson images are also known intrinsically as positroid varieties [20]. Buch, Chaput,

Mihalcea, and Perrin [9] showed that each structure constant in the quantum cohomology of the Grassmannian [8] (quantum Littlewood-Richardson numbers) naturally arises when expressing a certain projected Richardson class in the Schubert basis of the homology of the Grassmannian, and is thus naturally a Schubert vs. Schur coefficient.

Our combinatorial proof of positivity of Schubert vs. Schur coefficients does not yield a combinatorial proof of positivity of the quantum Littlewood-Richardson coefficients (the comparison theorem in [9] is geometric), but it does give a (complicated) combinatorial formula for those coefficients. Recent work of Buch, Kresch, Purbhoo, and Tamvakis [10] giving a combinatorial formula for the structure constants of two-step flag manifolds also gives a combinatorial formula for the quantum Littlewood-Richardson coefficients as each is equal to some Schubert structure constant on a two-step flag manifold [11]. This however does not give a combinatorial proof of nonnegativity, as the comparison theorem in [11] is geometric.

This paper is organized as follows. Section 1 develops background material of quasisymmetric functions, Assaf's dual equivalence, and the work of Bergeron and Sottile. We also state our main theorem (Theorem 1.11) in Section 1. In Section 2 we show that there is at most one dual equivalence on the set of chains in intervals in the Grassmannian-Bruhat order satisfying some natural properties (Lemma 2.1), and then we prove that this is a dual equivalence (Theorem 2.5), which implies our main result.

1. QUASISYMMETRIC FUNCTIONS, DUAL EQUIVALENCE, AND SCHUBERT COEFFICIENTS

We collect here background on quasisymmetric functions, recall the salient parts of Assaf's theory of dual equivalence, define the symmetric function of Bergeron and Sottile, and connect it to the Schubert vs. Schur coefficients.

1.1. Symmetric functions and tableaux. The algebra Λ of symmetric functions is freely generated by the complete symmetric functions h_1, h_2, \dots , where h_m is the formal sum of all monomials of degree m in the countably many variables x_1, x_2, \dots . Thus Λ has a basis of monomials in the h_m . More interesting is its basis of Schur functions s_λ , which are indexed by partitions λ and are generating functions for Young tableaux.

A *partition* λ is a weakly decreasing finite sequence of integers $\lambda: \lambda_1 \geq \dots \geq \lambda_k \geq 0$. Write $|\lambda|$ for the sum $\lambda_1 + \dots + \lambda_k$. We represent a partition λ by its *Young diagram*, which is the left-justified array of boxes with λ_i boxes at height i . Thus

$$(3, 2) \longleftrightarrow \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \quad \text{and} \quad (5, 4, 2, 1) \longleftrightarrow \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}.$$

A *Young tableau* is a filling of λ with positive integers that weakly increase across each row and strictly increase up each column. It is *standard* if the integers are $1, 2, \dots, |\lambda|$. Here are four Young tableaux of shape $(4, 3, 2)$. Only the last is standard.

$$(1.1) \quad \begin{array}{|c|c|} \hline 4 & 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 2 & 5 & 8 \\ \hline 1 & 1 & 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 7 & 9 \\ \hline 3 & 6 & 8 \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array}$$

To a tableau T of shape λ , we associate a monomial

$$x^T := \prod_{i \in T} x_i,$$

the product is over all entries i of T . The tableaux in (1.1) give the monomials

$$x_1^2 x_2^3 x_3^2 x_4^2, \quad x_1^3 x_2^2 x_3 x_4^2 x_5, \quad x_1^2 x_2 x_4^2 x_5 x_7 x_8^2, \quad \text{and} \quad x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9,$$

respectively. The Schur function s_λ is the generating function for tableaux of shape λ ,

$$s_\lambda = \sum_T x^T,$$

the formal sum over all tableaux T of shape λ .

1.2. Quasisymmetric functions. Gessel's quasisymmetric functions arise as generating functions for enumerative combinatorial invariants [14]. A formal power series $F(x)$ in countably many variables x_1, x_2, \dots having bounded degree is *quasisymmetric* if for any list of positive integers $(\alpha_1, \dots, \alpha_n)$ and increasing sequence of positive integers $i_1 < \dots < i_n$, the coefficient of the monomial

$$x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_n}^{\alpha_n}$$

in $F(x)$ does not depend upon the choice of $i_1 < \dots < i_n$. For example,

$$2x_1^2 x_2 + 2x_1^2 x_3 + 2x_2^2 x_3 + \cdots - x_1 x_2 x_3 - x_1 x_2 x_4 - x_1 x_3 x_4 - x_2 x_3 x_4 - \cdots$$

is quasisymmetric.

A fundamental quasisymmetric function $Q_D(x)$ is given by a positive integer n and a subset D of $[n-1] := \{1, \dots, n-1\}$ and defined to be

$$Q_D(x) := \sum_{\substack{i_1 \leq \dots \leq i_n \\ j \in D \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

The degree n is implicit in our notation. These form a basis of quasisymmetric functions.

Given a set \mathcal{C} of combinatorial objects and a map Des from \mathcal{C} to the subsets of $[n-1]$, we define the quasisymmetric generating function of $(\mathcal{C}, \text{Des})$,

$$K_{(\mathcal{C}, \text{Des})} = K_{\mathcal{C}} := \sum_{c \in \mathcal{C}} Q_{\text{Des}(c)}(x).$$

A source for this is when \mathcal{C} is a set of (saturated) chains in a labeled poset of rank n . Here, a labeled poset is a finite ranked poset P together with an integer label on each cover of P . Given a chain c in P , the sequence of labels of its covers is a word w of length n , and we let $\text{Des}(c) := \{i \in [n-1] \mid w_i > w_{i+1}\}$, the descent set of w .

A map $f: P \rightarrow Q$ of finite labeled posets is a *label-equivalence* if f is an isomorphism of posets and if the labels in P occur in the same relative order as the labels in Q . That is, if for any two covers $u \lessdot v$ and $x \lessdot y$ in P with labels α and β , respectively, if α and β are the labels of the corresponding covers $f(u) \lessdot f(v)$ and $f(x) \lessdot f(y)$ in Q , then $\alpha \leq \beta$ if and only if $\alpha \leq \beta$. Label-equivalent posets have identical quasisymmetric functions.

Example 1.1. Partitions are partially ordered by containment of their Young diagrams and the resulting poset is Young's lattice. A cover is given by adding a box in row i and column j and is labeled with $j-i$ to obtain a labeled poset. Standard tableaux of shape λ correspond to saturated chains from \emptyset to λ : the box containing the integer n is the box corresponding to the n th cover in the chain. Figure 1 shows part of Young's lattice.

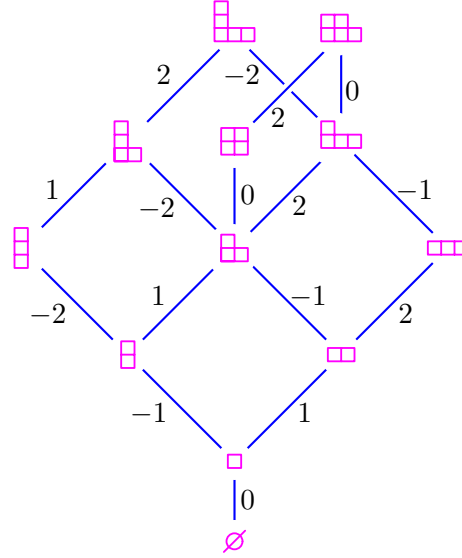


FIGURE 1. Part of Young's Lattice

We display the five Young tableaux of shape $(3, 2)$. Below each, we give the sequence of labels in the corresponding chain, writing \bar{a} for $-a$ and placing a dot at each descent.

$$(1.2) \quad \begin{array}{ccccc} \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 & 5 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 & 5 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 5 \\ \hline 1 & 3 & 4 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 3 & 5 \\ \hline 1 & 2 & 4 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \\ 01\bar{1}02 & 0\bar{1}102 & 0\bar{1}120 & 01\bar{1}20 & 012\bar{1}0 \end{array}$$

We display the six Young tableaux of shape $(3, 1, 1)$, together with the sequence of labels in the corresponding chain and descent sets.

$$(1.3) \quad \begin{array}{ccccc} \begin{array}{|c|} \hline 5 \\ \hline 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array} & \begin{array}{|c|} \hline 5 \\ \hline 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array} & \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 & 2 & 5 \\ \hline \end{array} & \begin{array}{|c|} \hline 5 \\ \hline 2 \\ \hline 1 & 3 & 4 \\ \hline \end{array} & \begin{array}{|c|} \hline 4 \\ \hline 2 \\ \hline 1 & 3 & 5 \\ \hline \end{array} & \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 4 & 5 \\ \hline \end{array} \\ 012\bar{1}\bar{2} & 01\bar{1}2\bar{2} & 01\bar{1}22 & 0\bar{1}12\bar{2} & 0\bar{1}1\bar{2}2 & 0\bar{1}\bar{2}12 \end{array}$$

A standard young tableau T has a *descent at i* when $i+1$ is above i in T , equivalently when $i+1$ is weakly left of i . (These are the descents we observe in the previous two examples.) The quasisymmetric function associated to a partition λ is

$$K_\lambda := \sum_T Q_{\text{Des}(T)},$$

the sum over all standard Young tableaux T where $\text{Des}(T)$ is the descent set of the chain in Young's lattice corresponding to T . Gessel [14] showed that $K_\lambda = s_\lambda$, the Schur function

associated to λ . From (1.2) and (1.3), we have

$$\begin{aligned} s_{(3,2)} &= Q_{\{3\}} + Q_{\{2,4\}} + Q_{\{1,4\}} + Q_{\{1,3\}} + Q_{\{2\}} \quad \text{and} \\ s_{(3,1,1)} &= Q_{\{3,4\}} + Q_{\{2,4\}} + Q_{\{2,3\}} + Q_{\{1,4\}} + Q_{\{1,3\}} + Q_{\{1,2\}}. \end{aligned}$$

Example 1.2. Consider the interval $[e, (145326)]_{\leq}$ in the Grassmannian-Bruhat order [6]. An edge is a transposition $t_{ab} = (a, b)$ with $a < b$, which is given the label b . This poset

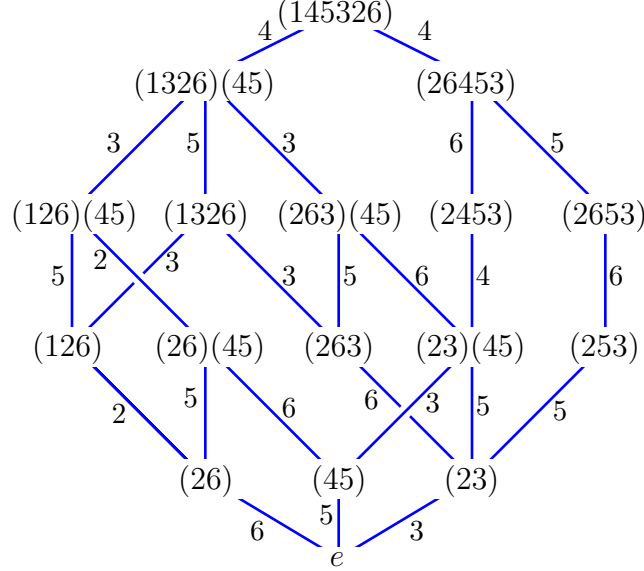


FIGURE 2. A labeled poset

has eleven chains. Here is the rightmost

$$e \xrightarrow{3} (23) \xrightarrow{5} (253) \xrightarrow{6} (2653) \xrightarrow{5} (26453) \xrightarrow{4} (145326).$$

Its sequence of labels is 35654, which has descent set $\{3, 4\}$. We list the labels of the chains in this poset, placing a dot under each descent.

$$(1.4) \quad \begin{aligned} &356\dot{3}4, \quad 36\dot{3}5\dot{4}, \quad 6\dot{2}35\dot{4}, \quad 6\dot{2}5\dot{3}4, \quad 56\dot{2}34, \\ &356\dot{5}4, \quad 35\dot{4}6\dot{4}, \quad 36\dot{5}3\dot{4}, \quad 5\dot{3}46\dot{4}, \quad 5\dot{3}6\dot{3}4, \quad 6\dot{5}2\dot{3}4. \end{aligned}$$

Thus the quasisymmetric function of this interval is

$$Q_{\{3,4\}} + 2Q_{\{2,4\}} + Q_{\{2,3\}} + 2Q_{\{1,4\}} + Q_{\{3\}} + 2Q_{\{1,3\}} + Q_{\{1,2\}} + Q_{\{2\}},$$

which is $s_{(3,2)} + s_{(3,1,1)}$. \square

1.3. Dual Equivalence. Assaf's theory of dual equivalence [1, 2] is a general framework for proving Schur-positivity of quasisymmetric generating functions.

Definition 1.3. Let \mathcal{C} be a finite set, n a positive integer, and Des a map from \mathcal{C} to the subsets of $[n-1]$ where, for $c \in \mathcal{C}$, if $i \in \text{Des}(c)$, we say that c has a *descent at i* . A *strong dual equivalence* for \mathcal{C} (or for $(\mathcal{C}, \text{Des})$) is a collection $\{\varphi_i \mid i = 2, \dots, n-1\}$ of involutions on \mathcal{C} which satisfy the following conditions.

- (i) The fixed points of the involution φ_i are those $c \in \mathcal{C}$ which either have a descent at both $i-1$ and i or do not have a descent at either $i-1$ or at i .
- (ii) For elements $c \in \mathcal{C}$ with $c \neq \varphi_i(c)$, so that c has exactly one descent in $\{i-1, i\}$,
 - (a) c and $\varphi_i(c)$ have the same descents, except possibly in $\{i-2, i-1, i, i+1\}$.
 - (b) For each $j \in \{i-1, i\}$ exactly one of c and $\varphi_i(c)$ has a descent at j .
 - (c) If exactly one of c and $\varphi_i(c)$ has a descent at $i-2$, then $c \neq \varphi_{i-1}(c)$.
 - (d) If exactly one of c and $\varphi_i(c)$ has a descent at $i+1$, then $c \neq \varphi_{i+1}(c)$.
- (iii) If $|i-j| \geq 3$, then φ_i and φ_j commute.
- (iv) For any $i < j \leq i+3$, if $b = \varphi_{i_\ell} \circ \cdots \circ \varphi_{i_1}(c)$ for indices i_1, \dots, i_ℓ in the interval $[i, j]$ with $\ell > 0$, then there exist indices j_1, \dots, j_m in the interval $[i, j]$ with $m > 0$ and at most one $j_k = j$ such that $b = \varphi_{j_m} \circ \cdots \circ \varphi_{j_1}(c)$.

When \mathcal{C} is the set of standard Young tableaux T with shape λ and $\text{Des}(T)$ the descent set defined after Example 1.1, Haiman's notion of dual equivalence [16] gives a strong dual equivalence on \mathcal{C} . Let T be a standard tableau with entries $1, \dots, n$. For each $i = 2, \dots, n-1$ define $\varphi_i(T)$ by the relative positions of $i-1$, i , and $i+1$ in T . If they are in order left-to-right, so that T has no descents in position $\{i-1, i\}$, we set $\varphi_i(T) = T$. If $i+1$ is in a row above i , which is in a row above $i-1$, then T has a descent at both $i-1$ and i , and we also set $\varphi_i(T) = T$. Otherwise, either i is above both $i-1$ and $i+1$ or else it is below or to the right of both. In either case, let $\varphi_i(T)$ be the tableau obtained from T by switching i with whichever of $i-1$ or $i+1$ that is further away from it in T .

We illustrate this for the partition $(3, 2)$, displaying the descent set below each tableau.

$$(1.5) \quad \begin{array}{ccccc} \begin{array}{|c|c|c|} \hline 3 & 4 & \\ \hline 1 & 2 & 5 \\ \hline \end{array} & \xleftrightarrow[\varphi_2]{\varphi_3} & \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array} & \xleftrightarrow{\varphi_4} & \begin{array}{|c|c|c|} \hline 2 & 5 & \\ \hline 1 & 3 & 4 \\ \hline \end{array} & \xleftrightarrow{\varphi_2} & \begin{array}{|c|c|c|} \hline 3 & 5 & \\ \hline 1 & 2 & 4 \\ \hline \end{array} & \xleftrightarrow[\varphi_3]{\varphi_4} & \begin{array}{|c|c|c|} \hline 4 & 5 & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \\ \{2\} & & \{1, 3\} & & \{1, 4\} & & \{2, 4\} & & \{3\} \end{array}$$

Strong dual equivalence is the relation \sim on \mathcal{C} generated by $c \sim \varphi_i(c)$, for $c \in \mathcal{C}$ and $i = 2, \dots, n-1$, when the φ_i satisfy the conditions of Definition 1.3. A main result of [2] is that if \mathcal{A} is a strong dual equivalence class, then there is a partition λ such that

$$(1.6) \quad K_{\mathcal{A}} = \sum_{a \in \mathcal{A}} Q_{\text{Des}(a)} = s_{\lambda}.$$

This leads to a combinatorial formula (which we do not state) for the Schur coefficients $c_{\mathcal{C}}^{\lambda}$ defined by the identity

$$K_{(\mathcal{C}, \text{Des})} = \sum_{\lambda} c_{\mathcal{C}}^{\lambda} s_{\lambda},$$

where $(\mathcal{C}, \text{Des})$ as above admits a strong dual equivalence structure.

Condition (iv) for a strong dual equivalence is hard to satisfy and to check. Assaf introduced a weaker notion which implies that the quasisymmetric generating function of each equivalence class is Schur-positive, but not necessarily equal to a single Schur function.

Suppose that we have a set \mathcal{C} and a notion of descent Des for elements of \mathcal{C} and involutions φ_i for $i = 2, \dots, n-1$ as above. Given $2 \leq i < j \leq n-1$, we may restrict Des and the equivalence relation \sim to the interval $[i, j]$ as follows. For $c \in \mathcal{C}$, define $\text{Des}_{(i,j)}(c) \subset \{1, \dots, j-i+2\}$ by first intersecting $\text{Des}(c)$ with the interval $[i-1, j]$ and then

subtracting $i-2$ from each element. Similarly, $\sim_{(i,j)}$ is the coarsest equivalence relation on \mathcal{C} in which $c \sim_{(i,j)} \varphi_k(c)$ where $i \leq k \leq j$. Write $[c]_{(i,j)}$ for the $\sim_{(i,j)}$ -equivalence class containing $c \in \mathcal{C}$.

We need the following rather technical definition. A list c_1, \dots, c_{2r} of distinct elements of \mathcal{C} is a *flat i -chain* if it satisfies the following conditions:

- (1) No c_j is fixed by either φ_{i-2} or φ_i and each of $c_3, c_5, \dots, c_{2r-1}$ is fixed by φ_{i-1} ,
- (2) for each $j = 1, \dots, r$, we have that $\varphi_i(c_{2j-1}) = c_{2j}$, and
- (3) for each $j = 1, \dots, r-1$ we have that c_{2j+1} is equal to $(\varphi_{i-2} \circ \varphi_{i-1})^t \circ \varphi_{i-2}(c_{2j})$, for some $t \geq 0$.

Definition 1.4. Suppose that we have a family of combinatorial objects \mathcal{C} , an integer n , and a descent statistic Des as before. A family of involutions φ_i for $2 \leq i \leq n-1$ on \mathcal{C} is a *dual equivalence* if it satisfies Conditions (i), (ii), and (iii) of Definition 1.3 and if the following three conditions hold.

- (iv.a) For any $i < j \leq i+2$ and $c \in \mathcal{C}$, the restricted generating function

$$\sum_{a \in [c]_{(i,j)}} Q_{\text{Des}_{(i,j)}(a)}$$

is symmetric and Schur-positive.

- (iv.b) For every $3 \leq i \leq n-1$ and $c \in \mathcal{C}$ that is not fixed by φ_i for which neither c nor $\varphi_i(c)$ is fixed by φ_{i-1} or by φ_{i+1} , the quasisymmetric functions $Q_{[c]_{(i-1,i)}}$ and $Q_{[c]_{(i,i+1)}}$ are equal.
- (iv.c) For each $3 < i < n-1$ and every flat i -chain c_1, \dots, c_{2r} , if for some j with $1 < j < r$ we have that neither c_{2j-1} nor c_{2j} is fixed by φ_{i+1} , then either none of c_1, \dots, c_{2j} are fixed by φ_{i+1} or none of $c_{2j-1}, c_{2j}, \dots, c_{2r}$ are fixed by φ_{i+1} .

We also require the symmetric statement given by reversal, replacing i by $n-i$.

We state the main result of [1].

Theorem 1.5 ([1], Theorem 5.3). *If $\{\varphi_i \mid 1 < i < n\}$ is a dual equivalence for $(\mathcal{C}, \text{Des})$, then the generating function $K_{(\mathcal{C}, \text{Des})}$ is symmetric and Schur-positive.*

Remark 1.6. The proof in [1] invokes an algorithm to transform a dual equivalence into a strong dual equivalence, yielding an explicit, albeit complicated, combinatorial formula for the Schur coefficients in $K_{(\mathcal{C}, \text{Des})}$ based on the description given for the Schur coefficients of a strong dual equivalence. Conditions (iv.b) and (iv.c) are needed for the transformation algorithm. They will be explained in Subsection 2.6 in simple graphical terms.

Remark 1.7. These conditions for a dual equivalence are local in that they depend only on the involutions φ_i, φ_j for $|i-j| \leq 3$. This will enable us to reduce their verification to a computer check.

This definition of a dual equivalence is seemingly more restrictive than given in [1]; there, the second assertion about reversal symmetry in (iv.c) was not invoked. While this additional condition can be shown to follow from the others, that is not necessary for our purposes, for Theorem 1.5 (whose proof did not use this symmetry assertion) remains valid.

Remark 1.8. Given a set $(\mathcal{C}, \text{Des})$ with \mathcal{C} finite, $\text{Des}(c) \subset [n-1]$ for $c \in \mathcal{C}$, and involutions φ_i on \mathcal{C} for $2 \leq i \leq n-1$ that form a dual equivalence, we may construct three additional dual equivalences as follows. Let ω be the map that takes a subset of $[n-1]$ to its complement. Thus $i \in \omega(D) \Leftrightarrow i \notin D$. The definition of a dual equivalence implies that the involutions φ_i form a dual equivalence for $(\mathcal{C}, \omega \circ \text{Des})$.

Let ρ be the involution on subsets D of $[n-1]$ that reverses a subset


$$\rho(D) := \{n-j \mid j \in D\}.$$

If we define $\psi_i := \varphi_{n+1-i}$ for $2 \leq i \leq n-1$, then the definition of a dual equivalence implies that these involutions ψ_i form a dual equivalence for $(\mathcal{C}, \rho \circ \text{Des})$. This requires the symmetry assertion from (iv.c).

We may compose these to get a third additional dual equivalence: the involutions ψ_i form a dual equivalence for $(\mathcal{C}, \rho \circ \omega \circ \text{Des})$.

Given a finite set $(\mathcal{C}, \text{Des})$ equipped with a notion of descent $\text{Des}(c) \subset [n-1]$ for $c \in \mathcal{C}$ and a collection of involutions φ_i on \mathcal{C} for $2 \leq i \leq n-1$, construct a colored graph $\mathcal{G}_{\mathcal{C}}$ with vertex set \mathcal{C} which has an edge of color i between c and $\varphi_i(c)$ when $c \neq \varphi_i(c)$. When the involutions φ_i form a dual equivalence, $\mathcal{G}_{\mathcal{C}}$ is a *dual equivalence graph*.

For example, here are the dual equivalence graphs for the partitions $(3, 2)$ (from (1.5)) and $(3, 1, 1)$, which give the symmetric functions $s_{(3,2)}$ and $s_{(3,1,1)}$, respectively.

(1.7) 

Interchanging the labels $2 \leftrightarrow 4$ is an automorphism of both graphs. This may be understood from the interplay of the involution ρ with quasisymmetric functions.

Consider two involutions on the algebra of quasisymmetric functions. The first, ρ , comes from reversing the alphabet. If $X = x_1 < x_2 < \dots$ and $\rho(X) := x_1 > x_2 > \dots$ is its reversal, then

$$\rho(Q_D)(X) := Q_D(\rho(X)) = Q_{\rho(D)}(X).$$

(Here, we are using that the definition of quasisymmetric function relies on a total ordering of the variables.) Reversal restricts to the identity on symmetric functions. Thus if $(\mathcal{C}, \text{Des})$ admits a dual equivalence structure then

$$K_{(\mathcal{C}, \text{Des})} = \rho(K_{(\mathcal{C}, \text{Des})}) = K_{(\mathcal{C}, \rho \circ \text{Des})},$$

where the first equality is because $K_{(\mathcal{C}, \text{Des})}$ is symmetric and the second follows from the definitions of ρ .

Similarly, the complementation map ω on subsets D of $[n-1]$ induces an involution, also written ω , on quasisymmetric functions,

$$\omega(Q_D) := Q_{\omega(D)}.$$

This restricts to the fundamental involution on the algebra of symmetric functions,

$$\omega(s_\lambda) = s_{\lambda'},$$

where λ' is the matrix transpose (interchanging rows with columns) of the partition λ .

We suppressed the descent sets in these graphs (1.7). This results in only a mild ambiguity.

Lemma 1.9. *The edges in a connected component of a graph $\mathcal{G}_{\mathcal{C}}$ given by involutions that satisfy Conditions (i) and (ii.b) determine the descent sets of the vertices, up to a global application of the complementation map ω .*

Proof. Let v be a vertex in a connected component of $\mathcal{G}_{\mathcal{C}}$. Define a subset $D(v) \subset [n-1]$ as follows. First, let $1 \in D(v)$. Then $2 \in D(v)$ if and only if v has no incident edge labeled 2. This ensures that Condition (i) of Definition 1.3 holds. For each of $i = 3, \dots, n-1$ note that knowing if $i-1 \in D(v)$ and whether or not v has an incident edge with label i determines whether or not $i \in D(v)$, by Condition (i). Then Conditions (i) and (ii.b) also determine $D(w)$ for any vertex w adjacent to v . In this manner, we determine sets $D(w)$ for all vertices in the component of $\mathcal{G}_{\mathcal{C}}$ containing v .

Had we assumed that $1 \notin D(v)$, we would have obtained the sets $\omega(D(w))$, the complement in $[n-1]$ of the sets $D(w)$. \square

Remark 1.10. Given a labeled graph \mathcal{G} on a set $(\mathcal{C}, \text{Des})$, its reversal is the labeled graph $\rho\mathcal{G}$ on the set $(\mathcal{C}, \rho \circ \text{Des})$ where we replace an edge label of i by $n - i$. Write $\omega\mathcal{G}$ for the graph obtained from \mathcal{G} by replacing Des by $\omega \circ \text{Des}$.

We say that two labeled graphs \mathcal{G} and \mathcal{G}' are $\{\omega, \rho\}$ -equivalent if, as labeled graphs whose vertices are descent sets, we have

$$\mathcal{G}' \in \{\mathcal{G}, \omega\mathcal{G}, \rho\mathcal{G}, \rho\omega\mathcal{G} = \omega\rho\mathcal{G}\}.$$

If $\mathcal{G}, \mathcal{G}'$ are equivalent graphs, then one is a dual equivalence graph if and only if the other one is.

1.4. Schubert vs. Schur constants. Let S_{∞} be the set of permutations w of $\{1, 2, \dots\}$ with $\{i \mid w(i) \neq i\}$ finite, which is the union of all finite symmetric groups. The length $\ell(w)$ of a permutation $w \in S_{\infty}$ is its number of inversions, $\{i < j \mid w(i) > w(j)\}$. We write t_{ij} for the transposition interchanging the numbers $i < j$.

Lascoux and Schützenberger [22] defined the Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[z_1, z_2, \dots]$, which are indexed by permutations $w \in S_{\infty}$. The polynomial \mathfrak{S}_w is homogeneous of degree $\ell(w)$, and if w has no descents after position n , then $\mathfrak{S}_w \in \mathbb{Z}[z_1, \dots, z_n]$. Schubert polynomials form a basis for the free \mathbb{Z} -module of all polynomials, which contains all Schur symmetric polynomials $s_{\lambda}(z_1, \dots, z_k)$ for all partitions λ with $\lambda_{k+1} = 0$ and all k . In particular, if $v(\lambda, k)$ is the permutation with a unique descent at position k and values $i + \lambda_{k+1-i}$ at $1 \leq i \leq k$, then $\mathfrak{S}_{v(\lambda, k)} = s_{\lambda}(z_1, \dots, z_k)$. See [26, 27] for more on Schubert and Schur polynomials.

Lascoux and Schützenberger showed that Schubert polynomials are polynomial representatives of Schubert classes in the cohomology of the flag manifold. Thus the integral structure constants $c_{u,v}^w \in \mathbb{Z}$ defined by the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c_{u,v}^w \mathfrak{S}_w$$

are nonnegative. We are concerned here with the *Schubert vs. Schur constants* $c_{u,v(\lambda,k)}^w$ which are defined by the identity

$$(1.8) \quad \mathfrak{S}_u \cdot s_\lambda(z_1, \dots, z_k) = \mathfrak{S}_u \cdot \mathfrak{S}_{v(\lambda,k)} = \sum_w c_{u,v(\lambda,k)}^w \mathfrak{S}_w.$$

We use abstract dual equivalence to give a combinatorial proof that these constants $c_{u,v(\lambda,k)}^w$ are nonnegative.

Theorem 1.11. *For any permutations $u, w \in S_\infty$, positive integer k , and partition λ , we have $c_{u,v(\lambda,k)}^w \geq 0$.*

This follows from Theorem 2.5 in Section 2.

Nonnegativity is known combinatorially in some cases. When $u = v(\mu, k)$ is also Grassmannian with descent k , then $c_{v(\mu,k),v(\lambda,k)}^w = 0$ unless $w = v(\nu, k)$, and in that case it equals the Littlewood-Richardson coefficient $c_{\mu,\lambda}^\nu$. Monk's formula (proven using geometry by Monk [28] and combinatorially by Lascoux and Schützenberger [23]) is

$$(1.9) \quad \mathfrak{S}_u \cdot (z_1 + \dots + z_k) = \mathfrak{S}_u \cdot \mathfrak{S}_{t_{k,k+1}} = \sum \mathfrak{S}_{u'},$$

the sum over all $u' = ut_{ij}$ where $\ell(u') = \ell(u) + 1$ and $i \leq k < j$. The indices of summation in this formula define the cover relation in the *k-Bruhat order*, \leq_k , that is,

$$u \leq_k ut_{ij} \quad \text{if and only if} \quad \ell(ut_{ij}) = \ell(u) + 1 \quad \text{and} \quad i \leq k < j.$$

Given a cover $u \leq_k w = ut_{ij}$ in the k -Bruhat order, let t_{ab} with $a < b$ be the transposition such that $w = t_{ab}u$. Label the cover $u \leq_k t_{ab}u$ with the integer b . This gives the k -Bruhat order the structure of a labeled poset. Write $u \xrightarrow{b} w$ to indicate that $u \leq_k w$ where wu^{-1} is a transposition t_{ab} with $a < b$ so that the cover is labeled with b . Bergeron and Sottile [5] considered the quasisymmetric generating function for intervals in this labeled poset,

$$K_{[u,w]_k} := \sum_{\text{chains } c \text{ in } [u,w]_k} Q_{\text{Des}(c)}.$$

Using the Pieri formula (see below) and the Jacobi-Trudy formula, they showed that this is the symmetric generating function of the Schubert vs. Schur coefficients,

$$(1.10) \quad K_{[u,w]_k} = \sum_{|\lambda|=\ell(w)-\ell(u)} c_{u,v(\lambda,k)}^w s_\lambda.$$

Our proof of Theorem 1.11 will involve putting the structure of a dual equivalence on the set of labeled chains in an interval in the k -Bruhat order. When u, w are Grassmannian with descent k and correspond to the partitions μ, λ , respectively, then the interval $[u, w]_k$ is isomorphic to the interval $[\mu, \lambda]$ in Young's lattice, if we shift the labels by $-k$.

We explain the formula (1.10) and develop some combinatorics of the k -Bruhat order. Recall [27, I.5, Example 2] that we have

$$(z_1 + \dots + z_k)^m = \sum_{|\lambda|=m} f^\lambda s_\lambda(z_1, \dots, z_k),$$

the sum over all partitions λ of m where f^λ is the number of standard Young tableaux of shape λ . Multiplying this by \mathfrak{S}_u , expanding using (1.8) and (1.9), and then equating coefficients of \mathfrak{S}_w gives the following proposition.

Proposition 1.12 (Prop. 1.1.1 [4]). *The number of saturated chains in the k -Bruhat order from u to w is equal to*

$$\sum_{\lambda} f^{\lambda} c_{u,v(\lambda,k)}^w.$$

In particular, $c_{u,v(\lambda,k)}^w = 0$ unless $u \leq_k w$. This suggests that a description of the constants in terms of chains in the k -Bruhat order should exist. Our proof of Theorem 1.11 using abstract dual equivalence provides such a description.

The complete homogeneous symmetric polynomial $h_m(z_1, \dots, z_k)$ is the sum of all monomials of degree m in z_1, \dots, z_k . It is the Schur polynomial $s_{(m)}(z_1, \dots, z_k)$ where (m) is the partition with only one part and it has size m . The Pieri formula gives the rule for multiplication of a Schubert polynomial by $h_m(z_1, \dots, z_k)$. It is

$$\mathfrak{S}_u \cdot h_m(z_1, \dots, z_k) = \sum_{\gamma} \mathfrak{S}_{\text{end}(\gamma)},$$

the sum over all saturated chains γ in the k -Bruhat order of length m starting at u ,

$$\gamma : u \xrightarrow{b_1} w_1 \xrightarrow{b_2} \dots \xrightarrow{b_m} w_m =: \text{end}(\gamma)$$

with increasing labels, $b_1 < b_2 < \dots < b_m$. Call this an *increasing chain* of length m .

An equivalent formula was conjectured by Lascoux and Schützenberger in [22], where they suggested an inductive proof. The above formulation of the Pieri formula was conjectured in [3], and proven using geometry in [32]. Postnikov gave a combinatorial proof [29] and later another combinatorial proof [24] was given following the suggestion of Lascoux and Schützenberger.

The Pieri formula implies a formula for multiplying a Schubert polynomial by a product of complete homogeneous symmetric polynomials in terms of chains whose labels have descents in a subset. Through the Jacobi-Trudy formula, it implies a (non-positive) combinatorial formula for the constants $c_{u,v(\lambda,k)}^w$ in terms of chains in $[u, w]_k$ [7]. There, the symmetric function on the right hand side of (1.10) was defined, and in [5] the quasisymmetric function $K_{[u,w]_k}$ was defined and the equality (1.10) was proven.

1.5. Grassmannian-Bruhat order on S_∞ . The Grassmannian-Bruhat order \preceq was introduced and studied in [4, §3]. It is a ranked labeled poset on the infinite symmetric group S_∞ which has the following defining property: If $u \leq_k w$ and we set $\zeta := wu^{-1}$, then the map $\eta \mapsto \eta u$ is an isomorphism of labeled posets,

$$[u, w]_k \xrightarrow{\sim} [e, \zeta]_{\preceq}.$$

Consequently, the quasisymmetric function $K_{[u,w]_k}$ and therefore the constants $c_{u,v(\lambda,k)}^w$ depend only upon ζ and λ . We say that ζ has *rank n* if the poset $[e, \zeta]_{\preceq}$ has rank n .

For example, the labeled poset of Example 1.2 is isomorphic to $[142635, 456123]_3$. The computation in that example shows that $K_{(1,4,5,3,2,6)} = s_{32} + s_{311}$ and therefore

$$c_{142635, v((3,2),3)}^{456123} = c_{142635, v((3,1,1),3)}^{456123} = 1 \quad \text{and} \quad c_{142635, v((2,2,1),3)}^{456123} = 0.$$

The Grassmannian-Bruhat order has a more direct definition, given in Theorem 3.1.5(ii) of [4]. For $\zeta \in S_\infty$, define $\text{up}(\zeta) := \{a \mid a < \zeta(a)\}$ and $\text{dw}(\zeta) := \{b \mid b > \zeta(b)\}$. Let $\eta, \zeta \in S_\infty$. Then $\eta \preceq \zeta$ if and only if the following hold.

- (1) For all $a \in \text{up}(\zeta)$, we have $a \leq \eta(a) \leq \zeta(a)$,
- (2) for all $b \in \text{dw}(\zeta)$, we have $b \geq \eta(b) \geq \zeta(b)$, and
- (3) For all $a < b$ with either $a, b \in \text{up}(\zeta)$ or $a, b \in \text{dw}(\zeta)$, if $\zeta(a) < \zeta(b)$, then $\eta(a) < \eta(b)$.

The labeled structure of \preceq is inherited from the k -Bruhat order as follows. When $\eta \prec \zeta$ is a cover, we have a transposition $t_{ab} = \zeta \eta^{-1}$ with $a < b$, and the cover is labeled with $\max\{a, b\}$. Saturated chains in an interval $[e, \zeta]_{\preceq}$ in the Grassmannian-Bruhat order are therefore represented by sequences $(t_{a_1 b_1}, \dots, t_{a_n b_n})$ of transpositions, where, if we set $\eta_i := t_{a_i b_i} \cdots t_{a_1 b_1}$ for each $i = 1, \dots, n$ with $e = \eta_0$, then we have $\eta_{i-1} \prec \eta_i$ for each $i = 1, \dots, n$ and $\eta_n = \zeta$.

There are several important properties that can be deduced from this definition. One is *invariance under relabeling*. Suppose that $I = \{i_1 < i_2 < \dots\} \subset \mathbb{N}$ is a set of integers. This induces an inclusion of $S_{|I|}$ into S_∞ ,

$$\iota_I(\zeta)(j) = \begin{cases} j & \text{if } j \notin I \\ i_{\zeta(k)} & \text{if } j = i_k \in I \end{cases}.$$

For example, if $\zeta = (1, 4, 2)(3, 5)$ and $I = \{1, 3, 5, 7, 9, \dots\}$, then $\iota_I(\zeta) = (1, 7, 3)(5, 9)$. Then $\eta \preceq \zeta$ if and only if $\iota_I(\eta) \preceq \iota_I(\zeta)$. The following properties of the Grassmannian-Bruhat order from Theorem 3.2.3 of [4] follow directly from its definition.

Proposition 1.13. *The Grassmannian-Bruhat order \preceq on S_∞ has the following properties.*

- (1) If $\eta \preceq \zeta$, then $\xi \mapsto \xi \eta^{-1}$ is an isomorphism of labeled posets $[\eta, \zeta]_{\preceq} \xrightarrow{\sim} [e, \zeta \eta^{-1}]_{\preceq}$.
- (2) For every $I \subset \mathbb{N}$ the map $\iota_I: S_{|I|} \rightarrow S_\infty$ is an injection of labeled posets such that if $\eta \in S_{|I|}$ then $[e, \eta]_{\preceq}$ is label-equivalent to $[e, \iota(\eta)]_{\preceq}$.
- (3) The map $\eta \mapsto \eta \zeta^{-1}$ induces an order-reversing but label-preserving isomorphism between $[e, \eta]_{\preceq}$ and $[e, \eta^{-1}]_{\preceq}$.

Remark 1.14. This result has consequences for the study of chains in the Grassmannian-Bruhat order, and ultimately the quasisymmetric function $K_{[u,w]_k}$. By (1), if

$$\eta \prec t_{a_1 b_1} \eta \prec t_{a_2 b_2} t_{a_1 b_1} \eta \prec \cdots \prec t_{a_n b_n} \cdots t_{a_1 b_1} \eta = \zeta$$

is a (saturated) chain in the interval $[\eta, \zeta]_{\preceq}$, then

$$(1.11) \quad e \prec t_{a_1 b_1} \prec t_{a_2 b_2} t_{a_1 b_1} \prec \cdots \prec t_{a_n b_n} \cdots t_{a_2 b_2} t_{a_1 b_1} = \zeta \eta^{-1} =: \xi$$

is a chain in the interval $[e, \xi]_{\preceq}$.

Statement (3) implies that (1.11) is a chain in $[e, \xi]_{\preceq}$ if and only if

$$e \prec t_{a_n b_n} \prec \cdots \prec t_{a_2 b_2} \cdots t_{a_n b_n} \prec t_{a_1 b_1} t_{a_2 b_2} \cdots t_{a_n b_n} = \xi^{-1}$$

is a chain in $[e, \xi^{-1}]_{\leq}$. Let c be the chain (1.11) in $[e, \xi]_{\leq}$ and c^\vee the corresponding reversed chain in $[e, \xi^{-1}]_{\leq}$. Observe that the sequence of labels is reversed in passing from c to c^\vee , and therefore $\text{Des}(c^\vee) = \rho\omega\text{Des}(c)$. That is, $i \in \text{Des}(c) \Leftrightarrow n-i \notin \text{Des}(c^\vee)$.

Chains in the Grassmannian-Bruhat order were studied in [6]. It was written in terms of a monoid for that order analogous to the nil-Coxeter monoid for the weak order on the symmetric group. We summarize its results for chains in the Grassmannian-Bruhat order.

Proposition 1.15. *Let $c := (t_{a_1b_1}, \dots, t_{a_nb_n})$ be the sequence of transpositions in a chain in the interval $[e, \zeta]_{\leq}$ of the Grassmannian-Bruhat order of rank n . Any other chain in this interval is obtained from c by a sequence of substitutions that replace a subchain of length two or three by an equivalent subchain according to one of the following rules.*

$$(1.12) \quad \begin{array}{lll} (i) & (t_{\beta\gamma}, t_{\gamma\delta}, t_{\alpha\gamma}) & \leftrightarrow (t_{\beta\delta}, t_{\alpha\beta}, t_{\beta\gamma}) \quad \text{if } \alpha < \beta < \gamma < \delta \\ (ii) & (t_{\alpha\gamma}, t_{\gamma\delta}, t_{\beta\gamma}) & \leftrightarrow (t_{\beta\gamma}, t_{\alpha\beta}, t_{\beta\delta}) \quad \text{if } \alpha < \beta < \gamma < \delta \\ (iii) & (t_{\alpha\beta}, t_{\gamma\delta}) & \leftrightarrow (t_{\gamma\delta}, t_{\alpha\beta}) \quad \text{if } \beta < \gamma \text{ or } \alpha < \gamma < \delta < \beta \end{array}$$

Moreover, any sequence of such replacements results in a chain in the interval $[e, \zeta]_{\leq}$.

No chain in $[e, \zeta]_{\leq}$ contains a subsequence having the following forms,

$$(1.13) \quad \begin{array}{lll} (iv) & t_{\alpha\gamma}, t_{\beta\delta} \text{ or } t_{\beta\delta}, t_{\alpha\gamma} & \text{for } \alpha \leq \beta < \gamma \leq \delta \\ (v) & t_{\beta\gamma}, t_{\alpha\beta}, t_{\beta\gamma} \text{ or } t_{\alpha\beta}, t_{\beta\gamma}, t_{\alpha\beta} & \text{for } \alpha < \beta < \gamma. \end{array}$$

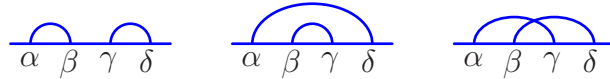
Finally, a sequence $(t_{a_1b_1}, \dots, t_{a_nb_n})$ of transpositions is a chain in the Grassmannian-Bruhat order if and only if for any sequence of substitutions using (i), (ii), or (iii), no subsequence of form (iv) or (v) is ever encountered.

Observe that the reversal of chains (sending a chain in $[e, \zeta]_{\leq}$ to the corresponding chain in $[e, \zeta^{-1}]_{\leq}$) fixes the substitution (iii) and interchanges (i) with (ii).

Remark 1.16. We express the substitution (iii) in another form. Call transpositions satisfying the conditions for (iii) **disjoint**. There are exactly two pairs of disjoint transpositions on a four element set $\{\alpha < \beta < \gamma < \delta\}$, namely

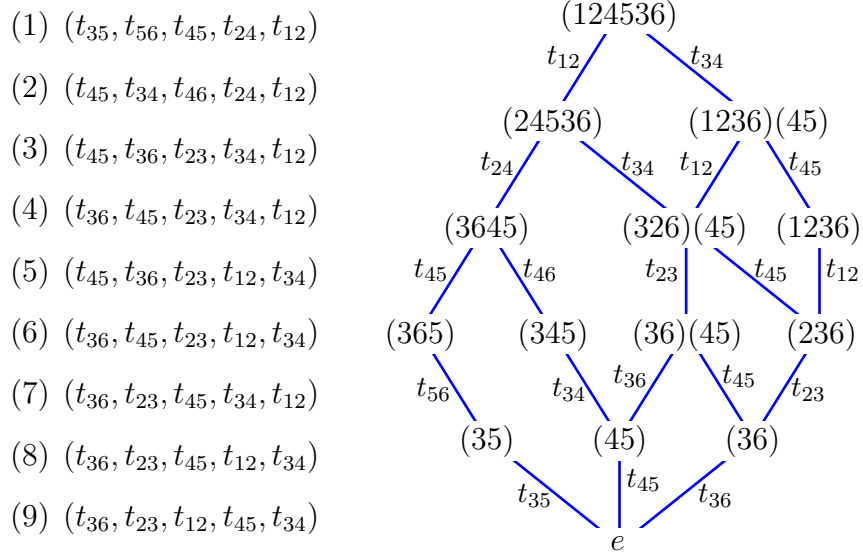
$$(t_{\alpha\beta}, t_{\gamma\delta}) \quad \text{and} \quad (t_{\alpha\delta}, t_{\beta\gamma}).$$

These correspond to the two noncrossing matchings on this set. The one crossing matching gives rise to the excluded subsequence (1.13)(iv). We display these three below.

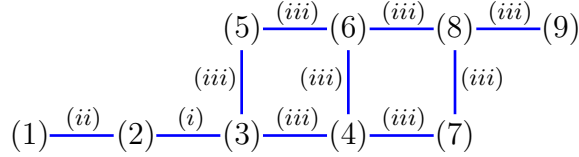


The pairs of transpositions $(t_{\alpha\beta}, t_{\beta\gamma})$ and $(t_{\beta\gamma}, t_{\alpha\beta})$ with $\alpha < \beta < \gamma$ are **connected** and they each give a valid chain of length 2. By (1.13)(iv), only connected or disjoint pairs of transpositions form valid chains of length 2.

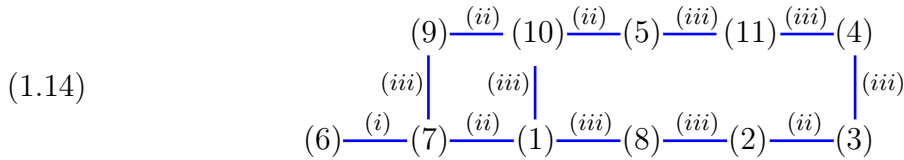
A set of n pairwise disjoint transpositions with indices drawn from a set of size $2n$ corresponds to a noncrossing complete matching on this set, and there are therefore C_n of these, where C_n is the n th Catalan number. Given such a matching, all $n!$ orderings of its n transpositions obtained using substitution (1.12)(iii) give valid chains.

FIGURE 3. The interval $[e, (124536)]_{\leq}$ and its nine chains.

Example 1.17. The interval $[e, (124536)]_{\leq}$ is displayed in Figure 3 with its nine chains. By Proposition 1.15, these chains are connected by substitutions of type (i), (ii), and (iii). We display the graph of these substitutions, with the numbering from Figure 3.



The interval $[e, (145326)]_{\leq}$ of Example 1.2 has eleven chains. If we number them 1—11 according to the order in which their labels appear in (1.4), then we have the following graph of substitutions, which may be read from the poset of Figure 2.



2. A DUAL EQUIVALENCE FOR $K_{[e, \zeta]}_{\leq}$

We prove Theorem 1.11 by putting the structure of a dual equivalence on the set of labeled chains in an interval of the Grassmannian-Bruhat order. For this, we will use the substitutions of Proposition 1.15 to define involutions φ_i on chains in the Grassmannian-Bruhat order, which we will show satisfy the conditions for a dual equivalence. For the last three conditions, (iv.a), (iv.b), and (iv.c), we use Proposition 1.13 (2) to reduce this to a finite verification that we complete with the help of a computer.

Let c be a chain in the Grassmannian-Bruhat order of length n . Using the substitutions (i), (ii), and (iii) of Proposition 1.15 to define $\varphi_i(c)$ ensures that it is a chain in the same

interval as c . To ensure that the involutions φ_i satisfy Condition (iii) of a dual equivalence, we require that φ_i acts *locally* on c in that the chains c and $\varphi_i(c)$ agree, except possibly at positions $i-1, i, i+1$. We ask for a *uniform* definition of these involutions in that φ_i and φ_j have the same definition, once we translate indices by $j-i$, and that this definition is invariant under relabeling. We explain this. If $I \subset \mathbb{N}$ is a subset and c a chain in $[e, \zeta]_{\preceq}$, write $\iota_I(c)$ for the corresponding chain in $[e, \iota_I(\zeta)]_{\preceq}$. If for every I , we have,

$$\iota_I(\varphi_i(c)) = \varphi_i(\iota_I(c)),$$

then φ_i is *invariant under relabeling*. It follows that local and uniform involutions are determined by their action on chains of length three. An involution $\varphi_i(c)$ is *reversible* if $\varphi_i(c^\vee) = \varphi_i(c)^\vee$, where c^\vee is the reversal of the chain c .

Lemma 2.1. *There is a unique dual equivalence on chains in the Grassmannian-Bruhat order of length three for which $\varphi_i(c)$ is obtained from c by a minimal number of substitutions (i), (ii), and (iii) of Proposition 1.15. It is given by the following three rules.*

$$\begin{array}{ll} \text{(A)} & \begin{array}{l} \varphi_2: (t_{\beta\gamma}, t_{\alpha\beta}, t_{\beta\delta}) \longleftrightarrow (t_{\alpha\gamma}, t_{\gamma\delta}, t_{\beta\gamma}) \\ \varphi_2: (t_{\beta\delta}, t_{\alpha\beta}, t_{\beta\gamma}) \longleftrightarrow (t_{\beta\gamma}, t_{\gamma\delta}, t_{\alpha\gamma}) \end{array} & \text{if } \alpha < \beta < \gamma < \delta \\ \text{(B)} & \begin{array}{l} \varphi_2: (t_{b\beta}, t_{a\alpha}, t_{c\gamma}) \longleftrightarrow (t_{b\beta}, t_{c\gamma}, t_{a\alpha}) \\ \varphi_2: (t_{c\gamma}, t_{a\alpha}, t_{b\beta}) \longleftrightarrow (t_{a\alpha}, t_{c\gamma}, t_{b\beta}) \end{array} & \begin{array}{l} \text{if } \alpha < \beta < \gamma \\ \text{and } \{a, \alpha\}, \{c, \gamma\} \text{ are disjoint} \end{array} \\ \text{(C)} & \begin{array}{l} \varphi_2: (t_{pq}, t_{\alpha\beta}, t_{\beta\gamma}) \longleftrightarrow (t_{\alpha\beta}, t_{\beta\gamma}, t_{pq}) \\ \varphi_2: (t_{\beta\gamma}, t_{\alpha\beta}, t_{pq}) \longleftrightarrow (t_{pq}, t_{\beta\gamma}, t_{\alpha\beta}) \end{array} & \text{if } \alpha < \beta < p < q < \gamma \end{array}$$

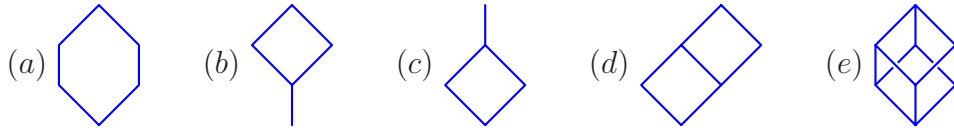
This is uniform and reversible.

Proof. A dual equivalence on the set C_3 of chains of length three in the Grassmannian-Bruhat order is given by an involution φ_2 on C_3 satisfying Conditions (i) and (ii.b) of Definition 1.3, for the other conditions are automatically satisfied on chains of length three. That is, if c is a chain of length three in an interval $[e, \zeta]_{\preceq}$, then $\varphi_2(c)$ is another chain of length three in $[e, \zeta]_{\preceq}$, and we have

$$\begin{array}{ll} \text{(i)} & \varphi_2(c) = c \iff \text{Des}(c) = \emptyset \text{ or } \text{Des}(c) = \{1, 2\}. \\ \text{(ii.b)} & \varphi_2(c) \neq c \iff \text{Des}(c) = \{1\} \text{ and } \text{Des}(\varphi_2(c)) = \{2\} \text{ or vice-versa.} \end{array}$$

In both cases, the quasisymmetric function of the chains matched by φ_2 is symmetric, so Condition (iv.a) is satisfied and the other conditions are vacuous for chains of length three.

Ignoring labels, there are exactly five isomorphism classes of intervals in the Grassmannian-Bruhat order of rank three.



Intervals of type (a) come from a substitution (i) or (ii) from Proposition 1.15. Observe that if $\alpha < \beta < \gamma < \delta$, then

$$\text{Des}(t_{\alpha\gamma}, t_{\gamma\delta}, t_{\beta\gamma}) = \{2\} \quad \text{while} \quad \text{Des}(t_{\beta\gamma}, t_{\alpha\beta}, t_{\beta\delta}) = \{1\}$$

in the case of (i), and similarly for (ii). It follows that φ_2 must interchange the two chains in intervals of type (a), which gives the rule **(A)**.

An interval of type (b) will have two chains, necessarily of the form $(t_{pq}, t_{\alpha\beta}, t_{\gamma\delta})$ and $(t_{pq}, t_{\gamma\delta}, t_{\alpha\beta})$ where $t_{\alpha\beta}$ and $t_{\gamma\delta}$ are disjoint (so the diamond is formed by a substitution (iii)), but neither is disjoint from t_{pq} . To distinguish these chains, suppose that $\beta < \delta$. By the Prohibition (1.13)(iv), $p \in \{\beta, \delta\}$ and $q \in \{\alpha, \gamma\}$. As $p < q$, the only possibility is that $p = \beta$ and $q = \gamma$, and $\alpha < \beta < \gamma < \delta$. Then the descent sets of the chains are $\text{Des}(t_{\beta\gamma}, t_{\alpha\beta}, t_{\gamma\delta}) = \{1\}$ and $\text{Des}(t_{\beta\gamma}, t_{\gamma\delta}, t_{\alpha\beta}) = \{2\}$, and we must have

$$\varphi_2 : (t_{\beta\gamma}, t_{\alpha\beta}, t_{\gamma\delta}) \longleftrightarrow (t_{\beta\gamma}, t_{\gamma\delta}, t_{\alpha\beta}).$$

Similarly, there is one choice for the chains in an interval of type (c), and we must have

$$\varphi_2 : (t_{\gamma\delta}, t_{\alpha\beta}, t_{\beta\gamma}) \longleftrightarrow (t_{\alpha\beta}, t_{\gamma\delta}, t_{\beta\gamma}).$$

These are both included in rule **(B)**.

An interval of type (d) has one transposition disjoint from the other two (both diamonds are formed by substitutions (iii)), but the other two are connected (do not commute), and have either the form $t_{\alpha\beta}, t_{\beta\gamma}$ or the form $t_{\beta\gamma}, t_{\alpha\beta}$ with $\alpha < \beta < \gamma$. We consider chains whose connected permutations have the first type.

The third transposition is t_{pq} with one of the five inequalities holding:

$$\begin{aligned} (1) \ q < \alpha, \quad (2) \ \alpha < p < q < \beta, \quad (3) \ \beta < p < q < \gamma, \\ (4) \ \gamma < p, \quad \text{or} \quad (5) \ p < \alpha, \ \gamma < q. \end{aligned}$$

The three chains in this interval will be $(t_{pq}, t_{\alpha\beta}, t_{\beta\gamma})$, $(t_{\alpha\beta}, t_{pq}, t_{\beta\gamma})$, and $(t_{\alpha\beta}, t_{\beta\gamma}, t_{pq})$, with each obtained from the previous by one application of the substitution (iii). For each of the five types of chains, the three descent sets are as follows, respectively

$$\begin{aligned} (1) \ \emptyset, \{1\}, \{2\}, \quad (2) \ \emptyset, \{1\}, \{2\}, \quad (3) \ \{1\}, \emptyset, \{2\}, \\ (4) \ \{1\}, \{2\}, \emptyset, \quad \text{or} \quad (5) \ \{1\}, \{2\}, \emptyset. \end{aligned}$$

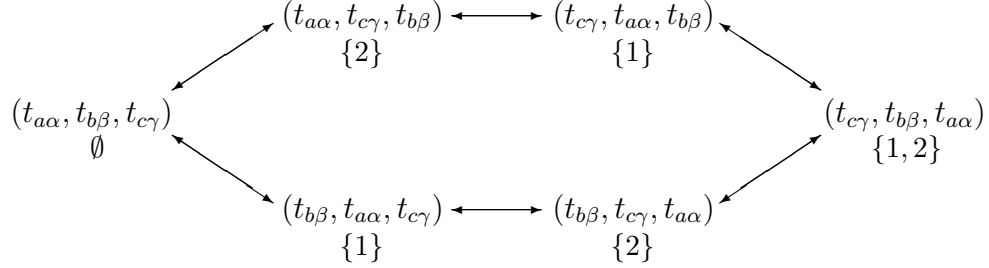
For all types except (3), the chains with descent sets $\{1\}$ and $\{2\}$ are obtained from each other by a single substitution (iii), and these are again covered by rule **(B)**. For the remaining type (3), φ_2 must involve two applications of the substitution (iii), so that

$$\varphi_2 : (t_{pq}, t_{\alpha\beta}, t_{\beta\gamma}) \longleftrightarrow (t_{\alpha\beta}, t_{\beta\gamma}, t_{pq}),$$

and this is covered by rule **(C)**. The case when the connected permutations are $t_{\beta\gamma}, t_{\alpha\beta}$ is identical upon reversing the chain.

In the remaining case of an interval of type (e), the three transpositions are disjoint. Writing them as $t_{a\alpha}, t_{b\beta}$, and $t_{c\gamma}$ with $\alpha < \beta < \gamma$, the six chains are related by substitutions (iii), and they form a hexagon (as in the weak order on S_3). Below we display the

chains, their descent sets, and indicate the substitutions (iii) by two-headed arrows.



As we seek a rule for φ_2 which uses a minimal number of the substitutions (i), (ii), and (iii) from Proposition 1.15, we must have that φ_2 is the involution given by the two horizontal arrows, which is again included in rule **(B)**. \square

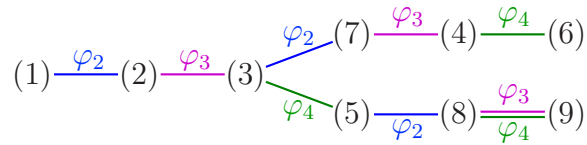
Consequently, there is at most one dual equivalence on chains in the Grassmannian-Bruhat order that is local and uniform and given by a minimal number of substitutions, and that dual equivalence is reversible.

Definition 2.2. Suppose that ζ is a permutation of rank n . Define involutions φ_i for $2 \leq i \leq n-1$ on chains in $[e, \zeta]_{\leq}$ where $\varphi_i(c) = c$ either if the chain c has a descent in both positions $i-1$ and i or c has no descents in those positions. If c has exactly one descent in positions $i-1$ and i , then $\varphi_i(c)$ is obtained from c by applying the involution φ_2 of Lemma 2.1 to the transpositions in c at positions $i-1$, i , and $i+1$. These involutions φ_i act locally and are uniform and reversible, and are the unique such involutions for which c and $\varphi_i(c)$ differ by a minimal number of substitutions of Proposition 1.15.

Remark 2.3. Any interval in Young's lattice is naturally an interval in the Grassmannian-Bruhat order. For such an interval, all substitutions have type (iii) and only rule **(B)** applies. In this case, it is just the Knuth relation, $(\beta, \alpha, \gamma) \leftrightarrow (\beta, \gamma, \alpha)$ or $(\gamma, \alpha, \beta) \leftrightarrow (\alpha, \gamma, \beta)$, where $\alpha < \beta < \gamma$, which gives Haiman's dual equivalence.

Whenever rule **(B)** applies, it is simply the Knuth relation on the labels of the chains.

Example 2.4. Figure 4 displays how the involutions φ_2 , φ_3 , and φ_4 act on the nine chains of length five from Figure 3 in Example 1.17. We indicate the descent set by placing dots at the descents, and indicate which involution φ_i and rule **(A)**, **(B)**, or **(C)** applies to each edge. Below we write the graph, labeling the vertices by the numbers of the corresponding chains of Figure 3 and the edges with φ_i .



For the eleven chains in the interval $[e, (1, 4, 5, 3, 2, 6)]_{\leq}$ of Example 1.2 we have the following graph, where the vertex labels (1)–(11) are the same as in (1.14), and the

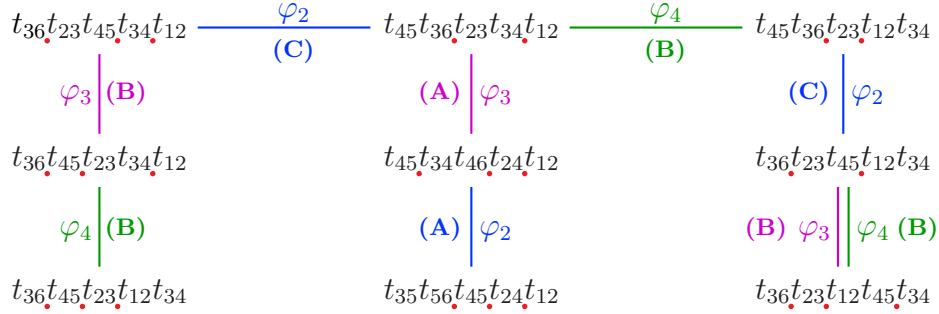
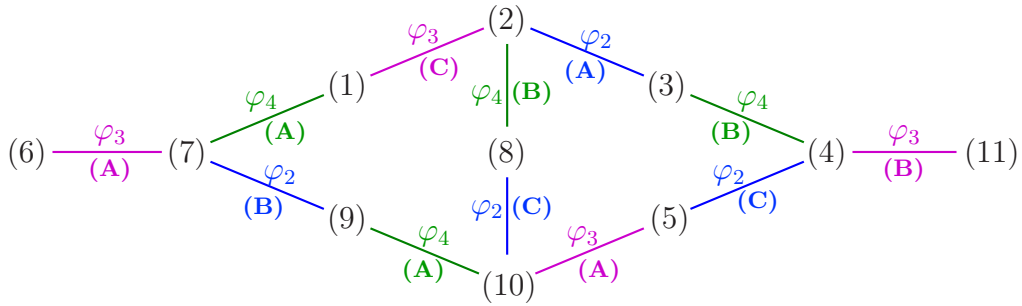


FIGURE 4. The graph from the chains in Figure 3.

edges are labeled with the involution φ_i and rule (A), (B), or (C) for that edge.



Our main Theorem 1.11 is a consequence of the following result.

Theorem 2.5. *The involutions φ_i for $2 \leq i \leq n-1$ of Definition 2.2 form a dual equivalence on the set of chains in an interval $[e, \zeta]_{\leq}$ of rank n in the Grassmannian-Bruhat order.*

Proof. Conditions (i), (ii.a), (ii.b), and (iii) of Definition 1.3 for a dual equivalence hold immediately by the definition of φ_i and Lemma 2.1.

For Condition (ii.c), suppose that exactly one of the chains c and $\varphi_i(c)$ has a descent at position $i-2$. Then $c \neq \varphi_i(c)$ and exactly one of c and $\varphi_i(c)$ has a descent at position $i-1$, by Condition (ii.b). Without any loss, assume that c has a descent at position $i-1$ and thus it has no descent at position i . Then, in the positions $i-1, i, i+1$, the chain c is one of the triples in the first column in the rules (A), (B), (C) used to define φ_i . Since exactly one of c and $\varphi_i(c)$ has a descent at position $i-2$, that one must be $\varphi_i(c)$, as the label on the transposition of c in position $i-1$ is at least as large as the corresponding label for $\varphi_i(c)$. It follows that in positions $i-2$ and $i-1$, c has a descent only at $i-1$ and $\varphi_i(c)$ has a descent only at $i-2$. Therefore, by Property (i), $\varphi_{i-1}(c) \neq c$ and $\varphi_{i-1}(\varphi_i(c)) \neq \varphi_i(c)$, which shows (ii.c). The Condition (ii.d) follows by reversing the chains and using the reversibility of the involutions φ_i .

Consider now the remaining conditions (iv.a), (iv.b), and (iv.c) of Definition 1.4. These conditions may be verified by checking all subchains of all chains, where the subchains have lengths up to five for (iv.a) and (iv.b), and those of length six for (iv.c). By Proposition 1.13(1), this is equivalent to checking all chains in intervals $[e, \zeta]_{\leq}$ in the

Grassmannian-Bruhat order of ranks up to five and six, as the involutions φ_i are local. As the involutions are uniform, this is a finite set of intervals. Indeed, if ζ is a permutation of rank six, then ζ is the product of six transpositions, and these transpositions involve at most twelve different numbers. Thus $\zeta = \iota_I(\eta)$ for some permutation η in S_{12} .

Thus we may complete the proof by generating all chains in the Grassmannian-Bruhat order of lengths up to six, up to the equivalence ζ is equivalent to $\iota_I(\eta)$. We have done just that and have written software that generates the chains and verifies Conditions (iv.a), (iv.b), and (iv.c). There are 1236 equivalence classes of chains of length four, 29400 equivalence classes of chains of length five, and 881934 equivalence classes of chains of length six. Given this set of equivalence classes of chains of length n for $n = 4, 5, 6$, the software determines the involutions φ_i for $i = 2, \dots, n-1$. From these data, the software determines all connected colored graphs \mathcal{G} whose vertices are this set of chains where c and $\varphi_i(c)$ are connected by an edge of color i if $c \neq \varphi_i(c)$. Then these graphs are partitioned into isomorphism classes, where the isomorphism respects the edge labels and descent sets of the vertices. Finally, the conditions (iv.a), (iv.b), and (iv.c) are checked on representatives of these isomorphism classes. Thus these graphs are all dual equivalence graphs, which completes the proof. \square

The software for these tasks is available on the web[†], as well as documentation, the sets of chains, connected dual equivalence graphs, and isomorphism class representatives. Some of this verification may be done by hand or inspection and it is not necessary to generate all chains of length six to verify (iv.c). We discuss this more in detail, including giving all dual equivalence graphs of chains of lengths four and five, as well as one for chains of length six in the subsections that follow. The numbers of chains, graphs, and classes of graphs is displayed in Table 1.

TABLE 1. Numbers of chains and graphs

	Numbers			
n	Chains	dual equivalence graphs	Isomorphism Classes	$\{\omega, \rho\}$ -classes
4	1236	499	7	4
5	29500	5948	28	12
6	881934	82294	178	73

2.1. Chains of pairwise disjoint transpositions. As we noted in Remark 1.16, a valid chain of length n in the Grassmannian Bruhat order consisting of pairwise disjoint transpositions is equivalent to one in S_{2n} . These are in turn given by one of $n!$ orderings of the pairs in a complete non-crossing matching on $[2n]$, and the number of such matchings is the Catalan number C_n . These are in bijection with parenthesizations of n objects, and thus with the vertices of the associahedron and with plane binary trees. Thus up to equivalence there are $n!C_n$ chains in the Grassmannian Bruhat order consisting of pairwise disjoint transpositions.

[†]<http://www.math.tamu.edu/~sottile/research/pages/positivity/>

As observed in Remark 2.3, the involutions on such a chain (which come from rule **(B)**) amount to applications of the Knuth relations on the labels of a chain. Consequently, the colored graph \mathcal{G} constructed from an interval $[e, \zeta]_{\leq}$ where $\zeta \in S_{2n}$ has rank n and one chain (hence all chains) is composed of pairwise disjoint transpositions, is isomorphic to the colored graph \mathcal{G}_n constructed from the symmetric group S_n coming from Knuth equivalences. Since \mathcal{G}_n is a strong dual equivalence (each component corresponds to all the Young tableaux of a given shape under Haiman's dual equivalence), we conclude that \mathcal{G} is a strong dual equivalence graph.

Thus in the proof of Theorem 2.5 we did not need a computer to verify the conditions for these chains of disjoint transpositions. Table 2 shows the number of chains of rank n for $n = 3, 4, 5, 6$ together with the numbers that are composed of pairwise disjoint transpositions.

TABLE 2. Numbers of disjoint chains

n	3	4	5	6
$n!C_n$	30	336	5040	95040
Number of Chains	70	1236	29500	881934

By Proposition 1.13 there is a bijection $c \leftrightarrow c^\vee$, given by reversal, between chains in $[e, \zeta]_{\leq}$ and chains in $[e, \zeta^{-1}]_{\leq}$. Since $\text{Des}(c^\vee) = \rho\omega\text{Des}(c)$, for every graph \mathcal{G} coming from a set of chains in the Grassmannian-Bruhat order, we have also the graph $\rho\omega\mathcal{G}$ coming from reversal of the chains giving \mathcal{G} . Thus we will always obtain both quasisymmetric functions $K_{\mathcal{G}}$ and $K_{\rho\omega\mathcal{G}} = \omega K_{\mathcal{G}}$.

2.2. Dual equivalence graphs from chains of length four. There are four $\{\rho, \omega\}$ -classes of dual equivalence graphs coming from chains of length 4.



The first three are strong dual equivalence graphs, and these four give the symmetric functions

$$s_{(4)}, s_{(2,2)}, s_{(3,1)}, \text{ and } s_{(2,2)} + s_{(3,1)},$$

respectively, where we write only one of $K_{\mathcal{G}}$ and $K_{\rho\omega\mathcal{G}} = \omega K_{\mathcal{G}}$ when the two functions differ.

2.3. Dual equivalence graphs from chains of length five. There are twelve $\{\rho, \omega\}$ -classes of dual equivalence graphs coming from chains of length five, which we give in Figure 5. The first four are all strong dual equivalence graphs. For each of these dual equivalence graphs \mathcal{G} , the pair of symmetric functions $K_{\mathcal{G}}$ and $K_{\omega\rho\mathcal{G}} = \omega K_{\mathcal{G}}$ depend only upon the number of nodes and are given in Table 3, where we write only one of the two functions $K_{\mathcal{G}}$ and $K_{\rho\omega\mathcal{G}}$. This census for $n = 4$ and $n = 5$ demonstrates Condition (iv.a) of local Schur positivity.

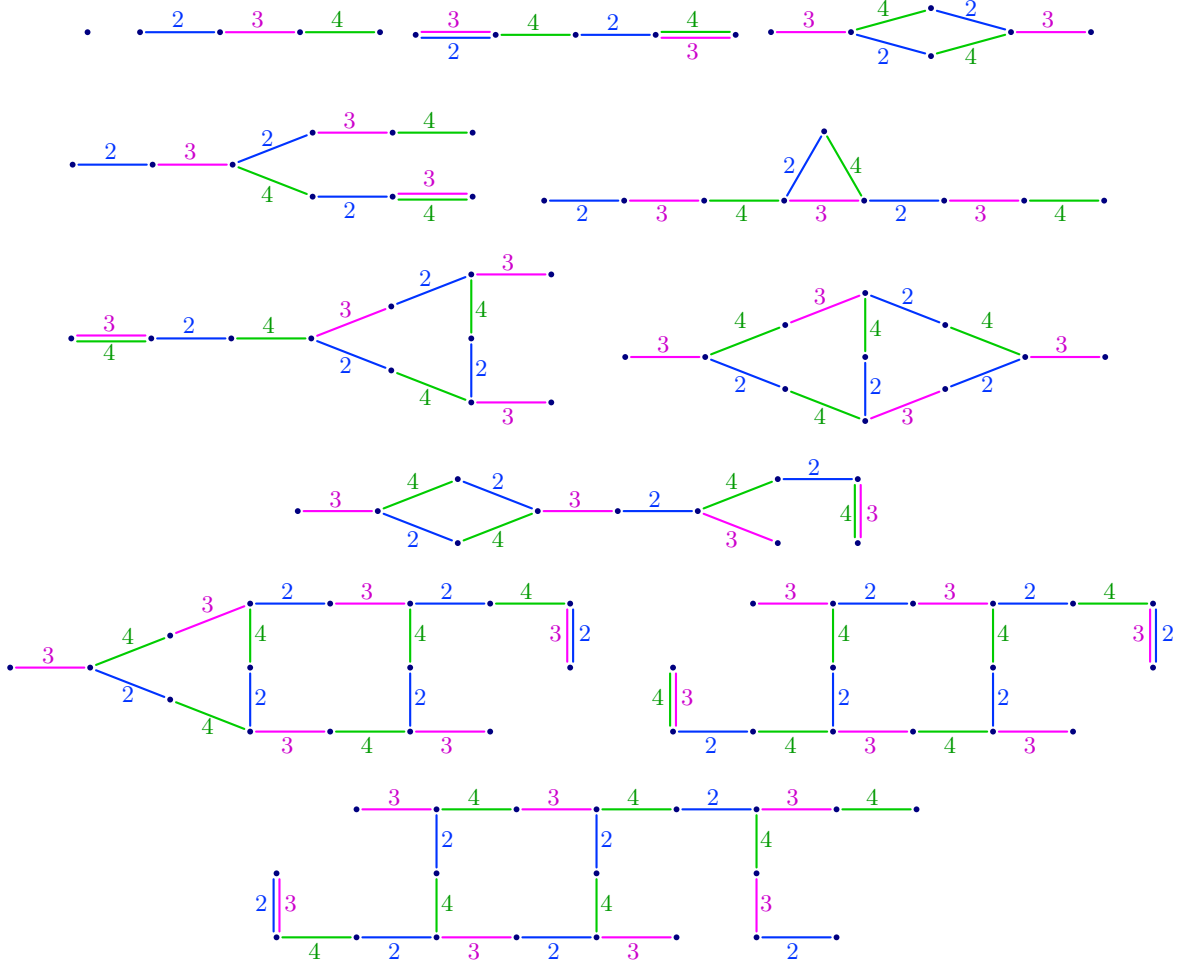


FIGURE 5. Dual equivalence graphs in the Grassmannian Bruhat order for $n = 5$

TABLE 3. Symmetric functions of graphs vs. number of vertices

Vertices	1	4	5	6	9	11
$K_{\mathcal{G}}$	$s_{(5)}$	$s_{(4,1)}$	$s_{(3,2)}$	$s_{(3,1,1)}$	$s_{(4,1)} + s_{(3,2)}$	$s_{(3,2)} + s_{(3,1,1)}$
Vertices	16			20		
$K_{\mathcal{G}}$	$s_{(2,2,1)} + s_{(3,1,1)} + s_{(3,2)}$			$s_{(2,2,1)} + s_{(3,1,1)} + s_{(3,2)} + s_{(4,1)}$		

2.4. Dual equivalence graphs from chains of length six. Table 4 gives the numbers of graphs from chains of length six by the numbers of vertices, as well as the numbers of isomorphism classes and the different symmetric functions.

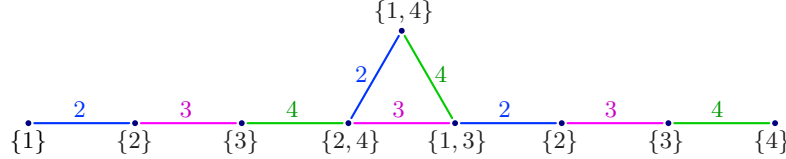
2.5. Condition (iv.b). Condition (iv.b) may be verified by inspection of the dual equivalence graphs when $n = 5$. For this, we have $i = 3$, and we first need an edge with label 3 (so that $c \neq \varphi_i(c)$) for which both endpoints admit one edge with label 2 and one edge

TABLE 4. Graphs and symmetric functions for $n=6$

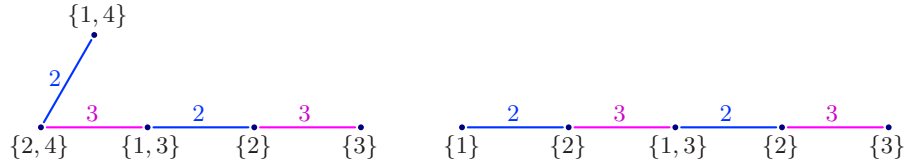
vertices	1	5	9	10	14	16	19	21	26	35
graphs	1806	20922	18594	19828	796	16134	786	414	948	738
$\{\rho, \omega\}$ -classes	1	2	1	1	4	1	7	3	8	6
Symm. Fns.	2	4	2	2	4	1	4	2	2	2

vertices	37	40	42	45	47	54	56	59	61	66	75	80	91	96
graphs	14	408	48	22	254	53	370	12	54	28	10	48	4	4
$\{\rho, \omega\}$ -classes	1	10	1	2	3	2	6	1	6	3	1	1	1	1
Symm. Fns.	2	4	2	4	2	3	4	2	6	2	2	1	2	1

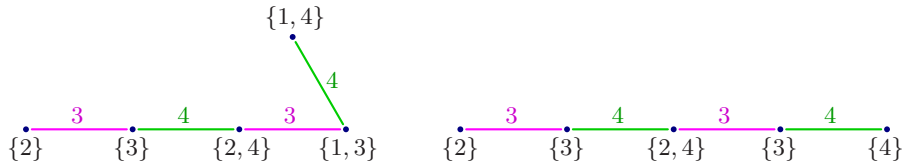
with label 4. Among the 12 dual equivalence graphs, this only occurs for the middle edge in the graph \mathcal{G} on nine vertices containing a triangle. We display this graph, labeling the vertices with their descent sets.



Note that $\rho\mathcal{G} = \mathcal{G}$. Condition (iv.b) considers the connected subgraph containing this middle edge and any edges with labels 2 and 3, as well as that containing this middle edge and any edges with labels 3 and 4. Each consists of five vertices, and for the first, we restrict each descent set to $\{1, 2, 3\}$ and for the second we restrict each descent set to $\{2, 3, 4\}$, and subtract one. Condition (iv.b) is that the quasisymmetric functions of each of these restricted subgraphs are equal. Here is the subgraph with edge labels 2 and 3, and the restricted graph.



Here is the subgraph with edge labels 3 and 4, and the restricted graph.

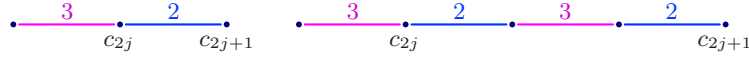


Observe that if we subtract 1 from all descents and edge labels in the second restricted graph, we obtain the first restricted graph. This shows that Condition (iv.b) holds. We note that we could also deduce this fact for \mathcal{G} using that $\rho\mathcal{G} = \mathcal{G}$.

2.6. Condition (iv.c). By Proposition 1.13, the reversal of a chain in the interval $[e, \zeta]_{\leq}$ gives a chain in $[e, \zeta^{-1}]_{\leq}$ whose edge labels are the reverse of the original chain. Thus if \mathcal{G} is the colored graph constructed from labeled chains in $[e, \zeta]_{\leq}$, then $\rho\omega\mathcal{G}$ is the colored

graph constructed from labeled chains in $[e, \zeta^{-1}]_{\leq}$. Thus \mathcal{G} satisfies both conditions (iv.c) (the one given in detail *and* its reversal) if and only if both \mathcal{G} and $\rho\omega\mathcal{G}$ satisfy the same one of these two conditions. Thus we will only check the condition given in detail, for all graphs from chains of length $n = 6$.

This condition concerns 5-edges incident to flat 4-chains in a graph \mathcal{G} . A flat 4-chain in is a list c_1, c_2, \dots, c_{2r} of distinct vertices of \mathcal{G} where, for each $j = 1, \dots, r$ the vertices c_{2j-1} and c_{2j} are connected by a 4-edge. Furthermore, for each $j = 1, \dots, r-1$, the vertex c_{2j} is connected to c_{2j+1} by an alternating sequence of 2- and 3- edges, beginning and ending with a 2-edge, and c_{2j+1} is *not* incident to a 3-edge. From our classification of graphs for chains of length 4 in Subsection 2.2, these connections are one of two types. We display them below, showing the 2- and 3- edges between c_{2j} and c_{2j+1} , as well as those incident on c_{2j} and c_{2j-1} .



As c_{2j+1} is not incident to any 3-edge, we must have that c_{2j} is incident to a 3-edge.

Then Condition (iv.c) asserts that for every flat 4-chain c_1, \dots, c_{2r} , if we have that both c_{2j-1} and c_{2j} admit a 5-edge for some $1 < j < r$, then either each of the first $2j$ vertices $c_1, \dots, c_{2j-1}, c_{2j}$ admit a 5-edge, or else each of the last $2r-2j+2$ vertices $c_{2j-1}, c_{2j}, \dots, c_{2r}$ admit a 5-edge. This condition is vacuous except for $r \geq 3$.

Figure 6 shows a dual equivalence graph \mathcal{G} with a flat 4-chain. This graph with 21 vertices comes from the chain $(t_{27}, t_{45}, t_{12}, t_{34}, t_{56}, t_{45})$ in the interval $[e, (1, 2, 7)(3, 5, 4, 6)]_{\leq}$,

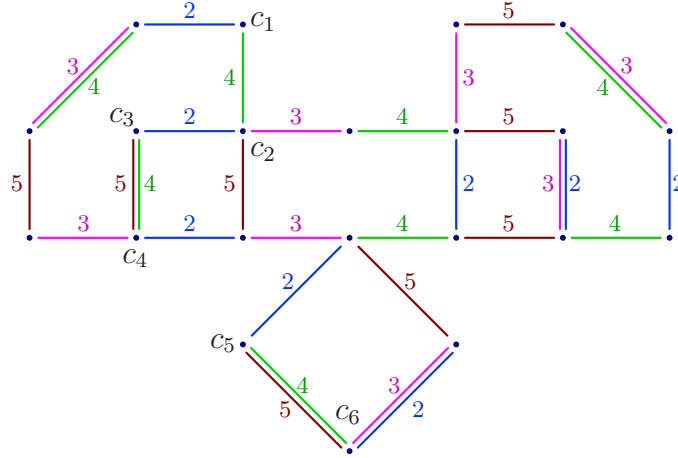
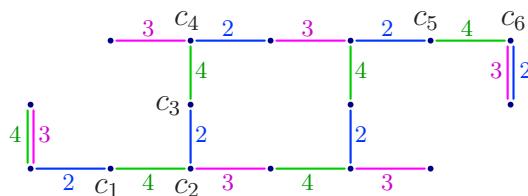


FIGURE 6. A dual equivalence graph with a flat 4-chain

which corresponds to the lower left vertex of \mathcal{G} . Since flat 4-chains involve only 2-, 3-, and 4-edges, and $\varphi_i(c)$ has the same sixth transposition as c for $i = 2, 3, 4$, a flat 4-chain in a graph with $n = 6$ comes by appending the same sixth transposition to all chains of vertices in a flat 4-chain in a graph with $n = 5$. For example, the flat 4-chain in the graph

of Figure 6 comes from the flat 4-chain in the graph with $n = 5$ depicted below.



Thus it is possible to check Condition (iv.c) by first generating all flat 4-chains in graphs for $n = 5$, and then considering all possible ways to append a transposition to all the chains in the Grassmannian-Bruhat order appearing in those flat 4-chains.

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