Algebra II Winter 2021 20 January
Frank Sottile Eleventh Homework

Write your answers neatly, in complete sentences. Start each problem on a new page (this makes it easier for Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

1. Suppose that $S \subset R$ is a multiplicatively closed subset of an integral domain R that does not contain 0. Prove that if R is a principal ideal domain, then so is $R[S^{-1}]$, and we have the same implication for unique factorization domains.

2. Let R be an integral domain, and for each maximal ideal $\mathfrak{m}$ of R, show that the localization $R_{\mathfrak{m}}$ is a subring of the quotient field of R.

3. Continuing the previous problem, show that the intersection of the rings $R_{\mathfrak{m}}$, as $\mathfrak{m}$ ranges over all maximal ideals of R, is R itself.

4. Show that the equation $x^2 + 1 = 0$ has infinitely many solutions in Hamilton's Quaternions, $\mathbb{H}$, which is $\mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$, where $ij = k$, $ji = -k$, etc. These are defined in the Example on page 117 of my copy of Hungerford in Section III.1.

5. Let F be a field, and consider the ring of formal power series $R := \mathbb{F}[[x]]$ in one variable. Show that $f \in R$ is a unit if and only if it has a nonzero constant term. Use this to show that the only ideals in R are $\{\langle x^n \rangle \mid n \in \mathbb{N}\}$.

6. Continuing the previous problem, show that the subring $\mathbb{F}[[x]][x^{-1}]$ of the quotient field of $\mathbb{F}[[x]]$ is a field. This is the field of formal Laurent series in x.

7. (a) If D is an integral domain and c is an irreducible element in D, show that $D[x]$ is not a principal ideal domain. (Hint: consider the ideal generated by x and c.)

   (b) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

   (c) If $\mathbb{F}$ is a field and $n \geq 2$, show that $\mathbb{F}[x_1, \ldots, x_n]$ is not a principal ideal domain. (Hint: show that x_1 is irreducible in $\mathbb{F}[x_1, \ldots, x_{n-1}]$.)