Algebra II Winter 2021 8 February
Frank Sottile Fourth Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 15 February.

1. An *idempotent e* in a ring R is an element $e \in R$ such that $e^2 = e$. Show that if e is an idempotent, then Re is a projective left R-module, and that eR is a projective right R-module. (Hint: show that $1-e$ is also an idempotent.)

2. Suppose that R is a principal ideal domain and F is free R-module of finite rank. Show that any R-submodule $P \subset F$ is free.

   Deduce that finitely generated projective R-modules are free.

3. Prove that the following conditions on a ring R are equivalent.
   (a) Every R-module is projective.
   (b) Every short exact sequence of R-modules splits.
   (c) Every R-module is injective.

4. Prove that a direct product $\prod\{J_i \mid i \in I\}$ of R-modules $\{J_i \mid i \in I\}$ is an injective R-module if and only if each factor J_i is an injective R-module.

5. A left R-module J is injective if for every left ideal I of R and R-module homomorphism $f\colon I \to J$, there is an element $a \in J$ with $f(r) = ra$ for $r \in I$.

6. Injective (divisible) abelian groups. Prove the following.
   (a) For each prime $p \in \mathbb{Z}$, the group $\mathbb{Z}(p^\infty)$ (see exercise I.1.10 in Hungerford) is divisible and torsion.
   (b) No nonzero finite abelian group is divisible.

7. More injective abelian groups. Prove the following.
   (a) No nonzero free abelian group is divisible.
   (b) The group $\mathbb{Q}$ of rational numbers is a divisible abelian group.

8. Let D be a torsion-free divisible abelian group ($\mathbb{Z}$-module). Show that D is naturally a $\mathbb{Q}$-module (vector space over $\mathbb{Q}$).

9. Suppose that M is an R-module and that for $i = 1, 2$, we have short exact sequences $0 \to N_i \to P_i \to M \to 0$ with P_1 and P_2 projective. Show that $P_1 \oplus N_2 \simeq P_2 \oplus N_1$ as R-modules.