Algebra II Winter 2021 8 March

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Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 15 March.

1. Suppose that we have a commutative ring R which has the property that every submodule of every free R-module is free.

   Prove that R is a principal ideal domain. We proved the converse in class.

2. (This problem is worth double) Let L, M, N be modules over a commutative ring R.

   Show that the set $\mathcal{L}(L, M; N)$ of all R-bilinear maps $L \times M \to N$ is an R-module.

   Here, the module structure is induced by the following functions: For all $f, g \in \mathcal{L}(L, M; N)$, $(\ell, m) \in L \times M$, and $r \in R$, we have

   $$(f+g)(\ell, m) \;=\; f(\ell, m) + g(\ell(m) \qquad \text{and} \qquad (rf)(\ell, m) \;=\; rf(\ell, m)\,.$$

   Show that $\mathcal{L}(L, M; N)$ is isomorphic to each of the following three R-modules:
   (a) $\mathrm{Hom}_R(L \otimes_R M, N)$
   (b) $\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$
   (c) $\mathrm{Hom}_R(M, \mathrm{Hom}_R(L, N))$

3. Let R be a principal ideal domian. Suppose that A is a cyclic R module of order $r \in R$. Prove the following.
   (a) If $s \in R$ is relatively prime to r, then $sA = A$ and $A[s] = \{0\}$.
   (b) Suppose that $s \in R$ divides r, and let $t \in R$ be such that $st = r$. Then $sA \simeq R/\langle t \rangle$ and $A[s] \simeq R/\langle s \rangle$.