Honors Multivariate Calculus Math 221H Section 201

Twelfth Homework: Due in recitation: Thursday 16 November 2023

1. A 50kg woman carries a 6kg can of paint up a helical staircase the encircles a silo with a radius of 6m and a height of 30m. How much work does she do against gravity in accomplishing this task?

2. Suppose now that the paint leaks out of the can at a constant rate (and that she trudges at a constant rate) so that 1 kg is left when she gets to the top. How much work does she do against gravity in accomplishing this task?

3. For each of the following vector fields $\mathbf{F}$, determine whether or not it is conservative. If it is, find a potential function f such that $\mathbf{F} = \nabla f$.
   (a) $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j}$ (b) $\mathbf{F}(x, y) = (\arctan(x) + y)\mathbf{i} + (\sin(xy) + x)\mathbf{j}$
   (c) $\mathbf{F}(x, y) = (ye^{xy} + 4x^3y)\mathbf{i} + (xe^{xy} + x^4)\mathbf{j}$

4. Suppose that $\mathbf{F}(x, y, z)$ is a vector function defined on $\mathbb{R}^3 \smallsetminus \{(0, 0, 0)\}$ whose direction at a point is away from the origin and magnitude is proportional to the distance from the origin.

   Show that $\mathbf{F}$ is conservative.

5. More generally, suppose that $g(t)$ is a continuous function of a nonnegative variable t. Show that the vector field $\mathbf{F}(x, y, z) = g(x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ is conservative.

6. Show that each of the following line integrals are independent of path, and then evaluate the integral, either by finding a propitious path or a potential function.

   (a) $\displaystyle\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\, dx \ + e^x \cos y\, dy$ (b) $\displaystyle\int_{(-1,1)}^{(4,2)} \big(y - \frac{1}{x^2}\big)\, dx \ + \ \big(x - \frac{1}{y^2}\big)\, dy$

   (c) $\displaystyle\int_{(0,0,0)}^{(1,1,1)} (6xy^3 + 2z^2)dx \ + \ 9x^2y^2\, dy \ + \ (4xy + 1)dz$
   (Hint: try a piecewise linear path parallel to coordinate axes.)

7. Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, and R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\ , \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}\ , \quad \text{and} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}\ .$$

8. Let $\mathbf{F}(x, y) = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$. Show that $\partial P/\partial y = \partial Q/\partial x$.

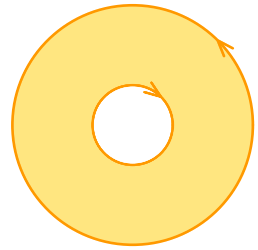
   Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path, for example by computing it along the upper and lower halves of the unit circle, appropriately oriented.

9. Use Green's Theorem to evaluate the integral along the positively oriented curve.

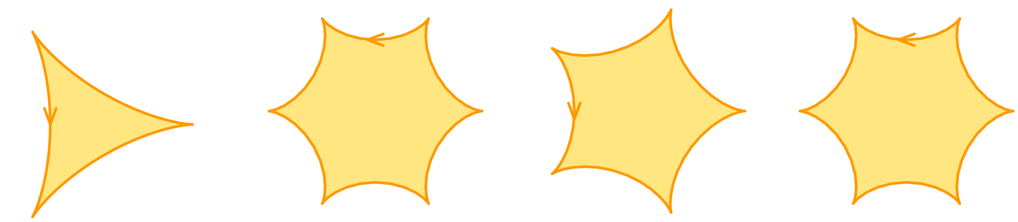
   (a) $\displaystyle\oint_C x^2y\,dx + xy^5\,dy$, where C is the square with vertices $(\pm1, \pm1)$.

   (b) $\displaystyle\oint_C x^2dx + y^2dy$, where C is the curve $x^6 + y^6 = 1$.

   (c) $\displaystyle\oint_C (x^3 - y^3)dx + (x^3 + y^3)dy$, where C is the boundary of the annulus lying between the circles of radius 1 and 3 centered at the origin.



10. Let $k > 2$ be an integer. Find the area of the region bounded by the *k-hypocycloid* with vector equation $\mathbf{r}(t) = \big(a(k-1)\cos\theta + a\cos((k-1)\theta)\big)\mathbf{i} + \big(a(k-1)\sin\theta - a\sin((k-1)\theta)\big)\mathbf{j}$.



$k = 3$, deltoid $k = 4$, astroid $k = 5$, pentoid $k = 6$, hexoid?

11. Let S be a region in the xy-plane with boundary C. Show that its moments M_x and M_y about the x- and y- axes are given by

$$M_x \;=\; -\frac{1}{2}\oint_C y^2\,dx \quad \text{and} \quad M_y \;=\; \frac{1}{2}\oint_C x^2\,dy\,,$$

where S has constant mass density.

12. Use the previous problem to find the centroid of a semicircular region of radius a.

13. (Area of a polygon) Let $v_0 = (x_0, y_0)$, $v_1 = (x_1, y_1)$, $\dots$, $v_n = (x_n, y_n)$ with $v_0 = v_n$ be the vertices of a simple polygon P in the plane, labeled counterclockwise. Show each of the following.

   (a) $\displaystyle\int_C x\,dy \;=\; \frac{1}{2}(x_1 + x_0)(y_1 + y_0)$, where C is the edge v_0v_1.

   (b) The area of P is $\displaystyle\sum_{i=1}^{n} \frac{1}{2}(x_i + x_{i-1})(y_i + y_{i-1})$.

   (c) The area of a polygon whose coordinates are integers (a *lattice polygon*) is always a multiple of $\frac{1}{2}$.

   (d) Check the formula for the polygon with vertices $(2, 0)$, $(2, -2)$, $(6, -2)$, $(6, 0)$, $(10, 4)$, and $(-2, 4)$.