Algebra II Winter 2021 15 February
Frank Sottile Fifth Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 22 February.

1. Recall that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is the group of integers modulo 2. Let $\pi\colon \mathbb{Z} \to \mathbb{Z}_2$ be the canonical surjection. Prove that the induced map $\pi_*\colon \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Z}_2)$ is the zero map and is therefore not a surjection. (You could do worse than compute all the Hom-groups.)

2. For a ring R, let R^{op} be the abelian group R with multiplication $\cdot_{\mathrm{op}}\colon R \times R \to R$ define by $s \cdot_{\mathrm{op}} r := rs$. (This is called the *opposite ring to R*.)

   Prove that there is a ring isomorphism $\mathrm{Hom}_R(R, R) \xrightarrow{\sim} R^{\mathrm{op}}$.

3. Write carefully the proof that the following conditions on a module J over a ring R are equivalent:

   (a) J is injective.

   (b) If $\varphi\colon A \to B$ an injection of R-modules, then $\varphi^*\colon \mathrm{Hom}_R(B, J) \to \mathrm{Hom}_R(A, J)$ is a surjection of abelian groups.

   (c) The functor $\mathrm{Hom}_R(-, J)$ from R-modules to abelian groups is an exact functor.