SOFTWARE FOR THE GALE TRANSFORM OF FEWNOMIAL SYSTEMS AND A DESCARTES RULE FOR FEWNOMIALS

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ABSTRACT. We give a Descartes'-like bound on the number of positive solutions to a system of fewnomials that holds when its exponent vectors are not in convex position and a sign condition is satisfied. This was discovered while developing algorithms and software for computing the Gale transform of a fewnomial system, which is our main goal. This software is a component of a package we are developing for Khovanskii-Rolle continuation, which is a numerical algorithm to compute the real solutions to a system of fewnomials.

Determining the *real* solutions to a system of polynomial equations is a challenging problem with real-world applications. Typically, few of the complex solutions are real and even fewer satisfy meaningful constraints such as positivity. Methods based on symbolic computation [22] or numerical homotopy continuation [4, 23] first find all complex solutions. It is an active area of research to develop methods that find only the meaningful real solutions. Among these are exclusion methods [11, 12], methods based on semidefinite programming [17], the critical point method [13, 21], and one proposed by two of us [5] that is based on the Khovanskii-Rolle Theorem [16] from the theory of fewnomials.

A system of polynomial equations with few monomials is a *fewnomial system*. Such a system enjoys bounds on its number of real solutions independent of its number of complex solutions [1, 8, 16]. Khovanskii-Rolle continuation is a symbolic-numeric method that can find all positive real solutions of a fewnomial system without first finding all complex solutions. In fact, it is the first algorithm for computing positive solutions with complexity depending primarily on the fewnomial bound on the number of real solutions, rather than the number of complex solutions or the ambient dimension. The core algorithm was described in [5], along with a proof of concept implementation.

Khovanskii-Rolle continuation rests upon the fundamental notion of *Gale duality for polynomial systems*, which is a scheme-theoretic isomorphism between complete intersections of Laurent polynomials in an algebraic torus and complete intersections of master

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functions in the complement of a hyperplane arrangement [9]. The support of the polynomials annihilates the weights of the master functions, whence the term Gale duality.

The Khovanskii-Rolle continuation algorithm finds all real solutions to a system of master functions. Computing approximations to the real solutions of a fewnomial system F requires that F be converted into its dual Gale system G which is solved using Khovanskii-Rolle continuation, then the numerical approximations to solutions of G are converted back to numerical approximations to solutions of the original fewnomial system F. Symbolic choices made in constructing the dual system affect the performance of the Khovanskii-Rolle continuation algorithm.

One such choice is that of a basis for the nullspace of the matrix of exponent vectors of the monomials of F. When the nullspace meets the positive orthant, a bound lower than the fewnomial bound holds. Choosing a positive basis element results in far fewer paths to be followed in the Khovanskii-Rolle continuation algorithm. This happens when the exponent vectors of the original polynomial system are not in convex position and a sign condition holds on the coefficient matrix. Such sign conditions are also found in recent work giving very strong bounds on positive solutions [6, 7, 10, 20] and are considered to be multivariate versions of Descartes' rule of signs.

In addition to this new Descartes'-like bound, we make explicit the algorithms for both the creation of G from F and the computation of solutions of F from solutions of G. These steps were only described existentially in [9]. We have implemented these algorithms in a software package `galeDuality`, which is available at each authors' website. This will be the front end for a software package that we are developing for Khovanskii-Rolle continuation.

Section 1 gives a description of Gale duality and sketches Khovanskii-Rolle continuation, illustrating it through an extended example. This material from [5, 9] is presented in a condensed form to give background both for our Decartes' bound and for our algorithms and software. In Section 2, we present our Descartes'-like bound for fewnomial systems and establish a key lemma about the location of points to be tracked in the algorithm. Explicit pseudocode for using Gale duality to compute master functions (Algorithm 3.1) and to transform approximations to solutions of the Gale system into approximations to solutions of the fewnomial system (Algorithm 3.3) is provided in Section 3, along with details for portions of these algorithms. One subroutine of Algorithm 3.1 requiring more care is covered in Subsection 3.3. Finally, a brief description of our software package is provided in Section 4.

1. Khovanskii-Rolle continuation and the Gale transform

Gale duality transforms a system of Laurent polynomials into a system of rational master functions, which are in a form that allows us to bound their number of positive real solutions and compute them. We begin with an example of Gale duality for polynomial systems before presenting Gale duality in general and sketching the Khovanskii-Rolle continuation algorithm, including making precise some assumptions and hypotheses of the algorithm. For a complete treatment, see [5, 9].

1.1. **An example of Gale duality.** Consider the system of polynomials,

$$(1.1) \quad \begin{aligned} v^2w^3 - 11uvw^3 - 33uv^2w + 4v^2w + 15u^2v + 7 &= 0, \\ v^2w^3 \qquad\qquad + 5uv^2w - 4v^2w - 3u^2v + 1 &= 0, \\ v^2w^3 - 11uvw^3 - 31uv^2w + 2v^2w + 13u^2v + 8 &= 0. \end{aligned}$$

This system has 20 complex solutions. Six are real and three are positive,

$$(1.194, 0.374, 1.231), \ (0.431, 0.797, 0.972), \ (0.613, 0.788, 0.850).$$

Solving the equations in (1.1) for the monomials v^2w^3, v^2w, and uvw^3, gives

$$(1.2) \quad \begin{aligned} v^2w^3 &= 1 - u^2v - uv^2w, \\ v^2w &= \tfrac{1}{2} - u^2v + uv^2w, \quad \text{and} \\ uvw^3 &= \tfrac{10}{11}(1 + u^2v - 3uv^2w). \end{aligned}$$

Since

$$\begin{aligned} \left(uv^2w\right)^3 \cdot \left(v^2w^3\right) &= u^3v^8w^6 = \left(u^2v\right) \cdot \left(v^2w\right)^3 \cdot \left(uvw^3\right) \quad \text{and} \\ \left(u^2v\right)^2 \cdot \left(v^2w^3\right)^3 &= u^4v^8w^9 = \left(uv^2w\right)^2 \cdot \left(v^2w\right) \cdot \left(uvw^3\right)^2, \end{aligned}$$

we may substitute the expressions on the right hand sides of (1.2) for the monomials v^2w^3, v^2w, and uvw^3 in these expressions to obtain the system

$$\begin{aligned} \left(uv^2w\right)^3 \cdot \left(1 - u^2v - uv^2w\right) &= \left(u^2v\right) \cdot \left(\tfrac{1}{2} - u^2v + uv^2w\right)^3 \cdot \left(\tfrac{10}{11}(1 + u^2v - 3uv^2w)\right) \\ \left(u^2v\right)^2 \cdot \left(1 - u^2v - uv^2w\right)^3 &= \left(uv^2w\right)^2 \cdot \left(\tfrac{1}{2} - u^2v + uv^2w\right) \cdot \left(\tfrac{10}{11}(1 + u^2v - 3uv^2w)\right)^2. \end{aligned}$$

Writing x for u^2v and y for uv^2w and solving for 0, these become

$$(1.3) \quad \begin{aligned} g &:= \quad y^3(1 - x - y) \ - \ x(\tfrac{1}{2} - x + y)^3 \left(\tfrac{10}{11}(1 + x - 3y)\right) &= 0, \quad \text{and} \\ f &:= \ x^2(1 - x - y)^3 \ - \ y^2(\tfrac{1}{2} - x + y)\left(\tfrac{10}{11}(1 + x - 3y)\right)^2 &= 0. \end{aligned}$$

This system also has 20 solutions with six real, outside of the five lines where the degree one factors vanish. Figure 1 shows the curves the equations (1.3) define and the five lines. Three solutions lie in the pentagon. They correspond to the positive solutions of (1.1).

This transformation from the original polynomial system (1.1) to the system (1.3) is called *Gale duality for polynomial systems*. The second system has three forms which are equivalent in the pentagon. We first rewrite (1.3) as a system of rational master functions

$$(1.4) \quad \frac{x^2(1 - x - y)^3}{y^2(\tfrac{1}{2} - x + y)\left(\tfrac{10}{11}(1 + x - 3y)\right)^2} = \frac{y^3(1 - x - y)}{x(\tfrac{1}{2} - x + y)^3\left(\tfrac{10}{11}(1 + x - 3y)\right)} = 1.$$

Then we take logarithms to obtain the last form.

$$(1.5) \quad \begin{aligned} 2\log(x) + 3\log(1 - x - y) - 2\log(y) - \log(\tfrac{1}{2} - x + y) - 2\log\left(\tfrac{10}{11}(1 + x - 3y)\right) &= 0 \\ 3\log(y) + \log(1 - x - y) - \log(x) - 3\log(\tfrac{1}{2} - x + y) - \log\left(\tfrac{10}{11}(1 + x - 3y)\right) &= 0 \end{aligned}$$
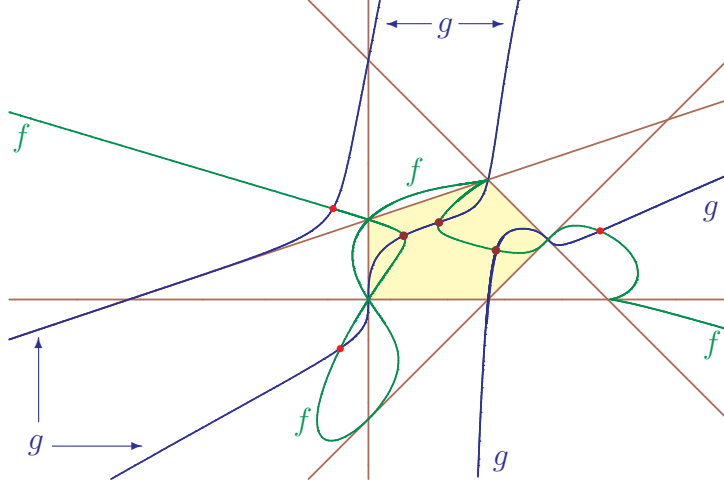
FIGURE 1. Curves defined by [**?**].

1.2. **Geometry of Gale duality.** Let $\mathbb{R}^{\times} = \mathbb{R}/\{0\}$ be the nonzero real numbers. Suppose that $F: (\mathbb{R}^{\times})^n \to \mathbb{R}^n$ is given by Laurent polynomials that have a total of $n+\ell$ distinct monomials, none of which is a constant. For $b \in \mathbb{R}^n$, the system $F(x) = b$ of Laurent polynomials has an equivalent expression. Let $\mathcal{A} := \{a_1, \ldots, a_{n+\ell}\} \subset \mathbb{Z}^n$ be the set of $n+\ell$ exponents of the monomials in F, which we call the *support* of F. Consider the map

$$\begin{aligned} \varphi_{\mathcal{A}} : \ (\mathbb{R}^{\times})^n \ &\longrightarrow \ (\mathbb{R}^{\times})^{n+\ell} \ \subset \ \mathbb{R}^{n+\ell} \\ x \ &\longmapsto \ (x^{a_1}, x^{a_2}, \ldots, x^{a_{n+\ell}}) \,. \end{aligned}$$

(1.6)

Affine-linear forms $L(z) = \lambda$ pull back to polynomials with support $\mathcal{A}$, and so $F(x) = b$ is the pullback of a codimension n (dimension ℓ) affine linear space H of $\mathbb{R}^{n+\ell}$ defined by $C(z) = b$, where $C \in \mathbb{R}^{n \times (n+\ell)}$ is the coefficient matrix of F. We define S_F to be the set of solutions of $F(x) = b$. Then,

$$(1.7) \qquad\qquad \varphi_{\mathcal{A}}(S_F) \ = \ \varphi_{\mathcal{A}}((\mathbb{R}^{\times})^n) \cap H \,.$$

Let $\psi: \mathbb{R}^{\ell} \to H$ be any affine isomorphism. This is given by a point $\lambda = \psi(0) \in H$ and $n+\ell$ linear forms $L_1, \ldots, L_{n+\ell}$ on $\mathbb{R}^{\ell}$ which span the dual space of $\mathbb{R}^{\ell}$,

$$(1.8) \qquad \psi(y) \ = \ (L_1(y) + \lambda_1, L_2(y) + \lambda_2, \ldots, L_{n+\ell}(y) + \lambda_{n+\ell}) \ \in \ H, \quad \text{for } y \in \mathbb{R}^{\ell}.$$

The forms $L_1, \ldots, L_{n+\ell}$ span the kernel of C, $C(L_1(y), \ldots, L_{n+\ell}(y)) = 0$. Define

$$S_G := \psi^{-1}(\varphi_{\mathcal{A}}(S_F)) \subset \mathbb{R}^{\ell}.$$

This set (which is isomorphic to S_F as a scheme) satisfies a simple system of equations in $\mathbb{R}^{\ell}$.

Any integer linear relation with coefficients $\beta_i \in \mathbb{Z}$ among the exponents in $\mathcal{A}$,

$$0 \ = \ \sum_{i=1}^{n+\ell} \beta_i a_i \,,$$

gives a monomial equation that is satisfied on $\varphi_{\mathcal{A}}((\mathbb{R}^{\times})^n)$ in $(\mathbb{R}^{\times})^{n+\ell}$,

$$(1.9) \qquad \prod_{i=1}^{n+\ell} z_i^{\beta_i} \;=\; 1\,.$$

This pulls back under ψ to an equation involving a rational master function,

$$(1.10) \qquad \prod_{i=1}^{n+\ell} (L_i(y) + \lambda_i)^{\beta_i} \;=\; 1\,,$$

which is defined in the complement of the hyperplanes given by the affine forms $L_i(y) + \lambda_i$.

Writing elements of $\mathcal{A}$ as column vectors yields an $n \times (n+\ell)$ integer matrix, also written $\mathcal{A}$. An element of the integer nullspace of $\mathcal{A}$ gives a monomial equation (1.9) vanishing on $\varphi_{\mathcal{A}}((\mathbb{R}^{\times})^n)$. Choosing a $\mathbb{Z}$-basis for this nullspace gives a set G of ℓ equations involving master functions (1.10), and these define the set S_G. Assuming there are finitely many solutions to $F(x) = b$ in $(\mathbb{C}^{\times})^n$, these ℓ monomial equations (1.9) that cut out the torus $\varphi_{\mathcal{A}}((\mathbb{C}^{\times})^n)$ in $(\mathbb{C}^{\times})^{n+\ell}$ restrict to a complete intersection in H and any r of these master functions defines a subset in the hyperplane complement of codimension r. We call this set G of ℓ master functions a *Gale system* dual to the system $F(x) = b$ of Laurent polynomials. We summarize some results in [8, 9].

Proposition 1.1. *Let $F(x) = b$ be a system of n Laurent polynomials on $(\mathbb{R}^{\times})^n$ with support $\mathcal{A}$ consisting of $n+\ell$ nonzero exponents of monomials expressed as a matrix $\mathcal{A} \in \mathbb{Z}^{n \times (n+\ell)}$ and finitely many zeroes in $(\mathbb{C}^{\times})^n$. Necessarily the exponent vectors span a full rank sublattice $\mathbb{Z}\mathcal{A}$ of $\mathbb{Z}^n$. Let $\varphi_{\mathcal{A}}$ be the map (1.6) and $C \in \mathbb{R}^{n \times (n+\ell)}$ be the coefficient matrix of F. The kernel of $\varphi_{\mathcal{A}}$ is the Pontryagin dual to $\mathbb{Z}^n / \mathbb{Z}\mathcal{A}$.*

A Gale system dual to $F(x) = b$ is given by a choice of a basis $\mathcal{B} = \{\beta^{(1)}, \ldots, \beta^{(\ell)}\} \subset \mathbb{Z}^{n+\ell}$ for the integer nullspace of $\mathcal{A}$, a choice of a basis $L := \{L_1, \ldots, L_{n+\ell}\}$ for the kernel of C^T, and a choice of a point $\lambda \in \mathbb{R}^{n+\ell}$ with $C^T \lambda = b$. Considering each L_i as a linear map on $\mathbb{R}^{\ell}$, the system of master functions Gale dual to the Laurent system $F(x) = b$ is

$$(1.11) \qquad \prod_{i=1}^{n+\ell} (L_i(y) + \lambda_i)^{\beta_i^{(j)}} \;=\; 1 \qquad j = 1, \ldots, \ell\,.$$

When $\mathbb{Z}\mathcal{A}$ has odd index in $\mathbb{Z}^n$, the solutions to $F(x) = b$ in $(\mathbb{R}^{\times})^n$ are scheme-theoretically isomorphic to the solutions to (1.11) in the complement of the hyperplane arrangement defined by $L_i(y) = -\lambda_i$ for $i = 1, \ldots, n+\ell$. For any $\mathcal{A}$ with $\mathbb{Z}\mathcal{A}$ of full rank, the solutions to $F(x) = b$ in $(\mathbb{R}_{>0})^n$ correspond to solutions to (1.11) where $L_i(y) > -\lambda_i$, for $i = 1, \ldots, n+\ell$. These functions (1.11) form a regular sequence in the hyperplane complement.

The effect of the choices of $\mathcal{B}$, L, and λ on the Gale system and its solutions may be deduced from Proposition 1.1.

Corollary 1.2. *Changing the basis $\mathcal{B}$ of the integer nullspace of $\mathcal{A}$ will change the Gale system, but not its solutions in $\mathbb{R}^{\ell}$ or the hyperplane arrangement. Changing the basis L and the point λ has the effect of an affine-linear change of coordinates in $\mathbb{R}^{\ell}$.*

Remark 1.3. There are two other forms for the Gale system. First, we may take logarithms of the equations (1.11) to obtain the logarithmic form,

$$(1.12) \qquad \phi_j(y) := \sum_{i=1}^{n+\ell} \beta_i^{(j)} \log(L_i(y) + \lambda_i) = 0 \qquad j = 1, \ldots, \ell.$$

This only makes sense in the *positive chamber* of the hyperplane complement,

$$\Delta := \{y \in \mathbb{R}^\ell \mid L_i(y) > -\lambda_i \text{ for all } i = 1, \ldots, n+\ell\}.$$

To obtain the other form, set $s^+ := \max\{s, 0\}$ and $s^- := \max\{-s, 0\}$, for $s \in \mathbb{Z}$. Note that $s = s^+ - s^-$. With this notation, we may clear the denominators of (1.11) to get a polynomial form of the Gale dual system,

$$(1.13) \qquad \prod_{i=1}^{n+\ell}(L_i(y) + \lambda_i)^{(\beta_i^{(j)})^+} - \prod_{i=1}^{n+\ell}(L_i(y) + \lambda_i)^{(\beta_i^{(j)})^-} = 0 \qquad j = 1, \ldots, \ell.$$

In the complement of the hyperplanes $L_i(x) + \lambda_i = 0$, this has the same solutions as (1.11).

Remark 1.4. It is no loss of generality to assume that this positive chamber Δ is bounded. Indeed, suppose that Δ is unbounded. This occurs if and only if the coefficient matrix C annihilates a positive vector. Since $L_1, \ldots, L_{n+\ell}$ span the dual space to $\mathbb{R}^\ell$, Δ is strictly convex and therefore has a bounded face, P. Then there is an affine form, $a_0 + L(y)$ where L is a linear function on $\mathbb{R}^\ell$ and $a_0 > 0$, that is strictly positive on Δ and such that P is the set of points of Δ where this polynomial achieves its minimum value on Δ. Dividing by a_0, we may assume that the constant term is 1.

Consider the projective coordinate change $y \mapsto \overline{y}$, where

$$(1.14) \qquad \overline{y}_j := y_j \cdot \frac{1}{1 + L(y)} \qquad \text{for } j = 1, \ldots, \ell.$$

We have

$$y_j = \overline{y}_j \cdot \frac{1}{1 - L(\overline{y})} \qquad \text{for } j = 1, \ldots, \ell.$$

Note that $(1 + L(y))(1 - L(\overline{y})) = 1$.

The coordinate change (1.14) manifests itself on affine forms as follows. If $p(y)$ is a affine form, then

$$(1.15) \qquad p(y) = \overline{p}(\overline{y}) \cdot \frac{1}{1 - L(\overline{y})} \qquad \text{and} \qquad \overline{p}(\overline{y}) = p(y) \cdot \frac{1}{1 + L(y)},$$

where $\overline{p}(\overline{y}) := p(\overline{y}) - p(0)L(\overline{y})$. If $\Lambda_i(y) := L_i(y) + \lambda_i$ is one of the affine forms defining Δ, then $\overline{\Lambda}_i(\overline{y}) = L_i(\overline{y}) + \lambda_i(1 - L(\overline{y}))$. Under the projective transformation (1.14), the polyhedron Δ is transformed into $\overline{\Delta}$, where

$$\overline{\Delta} := \{\overline{y} \in \mathbb{R}^\ell \mid 1 - L(\overline{y}) > 0 \text{ and } \overline{\Lambda}_i(\overline{y}) > 0 \text{ for } i = 1, \ldots, n+\ell\}.$$

Set $\overline{\Lambda}_0(\overline{y}) := 1 - L(\overline{y})$ and $\beta_0^{(j)} := -\sum_{i=1}^{\ell+n} \beta_i^{(j)}$.

Proposition 1.5 ([5], Prop. 2.3). *Under the coordinate change* (1.14), *the system of master functions* (1.11) *on Δ is transformed into the system*

$$\prod_{i=0}^{\ell+n} \overline{\Lambda}_i(\overline{y})^{\beta_i^{(j)}} \;=\; 1 \qquad for \quad j = 1, \ldots, \ell,$$

on the bounded polyhedron $\overline{\Delta}$.

This coordinate transformation $y \mapsto \overline{y}$ is a concrete and explicit projective transformation sending the unbounded polyhedron Δ to a bounded polyhedron $\overline{\Delta}$. We will henceforth assume that Δ is bounded.
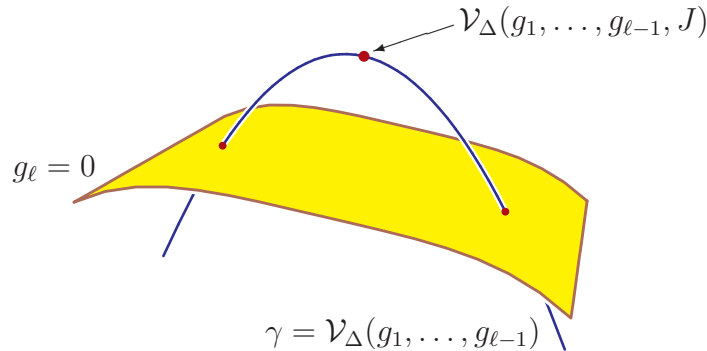
1.3. **Khovanskii-Rolle continuation.** Khovanskii-Rolle continuation is a numerical algorithm introduced in [5] to compute the solutions to a Gale system in the positive chamber. It is based on the Khovanskii-Rolle Theorem and the proof of the fewnomial bound [8] which bounds the number of solutions to a Gale system.

We first state the Khovanskii-Rolle Theorem. Write $\mathcal{V}_\Delta(f_1, \ldots, f_k)$ for the common zeroes in Δ of functions $f_1, \ldots, f_k$ defined on Δ, counted with multiplicity and $J(f_1, \ldots, f_\ell)$ for the Jacobian (determinant), $\det(\partial f_i / \partial y_j)$, of $f_1, \ldots, f_\ell$. For a curve γ in Δ, let $\mathrm{ubc}_\Delta(\gamma)$ be the number of unbounded components of γ in Δ (those that meet its boundary). This is one-half the number of points of intersection of γ with the boundary of Δ.

Theorem 1.6 (Khovanskii-Rolle [16]). *Let $g_1, \ldots, g_\ell$ be smooth functions defined on a polyhedral domain $\Delta \subset \mathbb{R}^\ell$ with finitely many common zeroes and suppose $\gamma := \mathcal{V}_\Delta(g_1, \ldots, g_{\ell-1})$ is a smooth curve. Let J be the Jacobian of $g_1, \ldots, g_\ell$. Then*

(1.16) $$|\mathcal{V}_\Delta(g_1, \ldots, g_\ell)| \;\leq\; \mathrm{ubc}_\Delta(\gamma) \;+\; |\mathcal{V}_\Delta(g_1, \ldots, g_{\ell-1}, J)|.$$

The main idea is that along an arc of the curve γ, the Jacobian vanishes at least once between any two consecutive zeroes of g_ℓ.

If we let S be the points of γ where $J = 0$ and T be the points where γ meets the boundary of Δ, then every zero of g_ℓ on γ lies in a unique interval of $\gamma \smallsetminus (S \cup T)$. Thus, the Khovanskii-Rolle Theorem leads to the following continuation algorithm, which is Algorithm 2.2 in [5].

Algorithm 1.7. *Using arclength continuation to follow γ starting at points of $S \cup T$, where γ is traced in both directions from points of S and is traced into the interior of the*

polyhedron Δ from points of T, will recover all points of γ where $g_\ell = 0$, each at least twice.

We apply this to a Gale system. Let $\phi_1, \ldots, \phi_\ell$ be the functions in the logarithmic form of a Gale system (1.12). Set $\gamma_\ell := \mathcal{V}(\phi_1, \ldots, \phi_{\ell-1})$. Then γ_ℓ is an algebraic curve in Δ, as logarithmic functions ϕ_i define the same sets as the polynomial functions (1.13), and these form a complete intersection. Let J_ℓ be the Jacobian determinant of $\phi_1, \ldots, \phi_\ell$. Given the points $S_{\ell-1} := \mathcal{V}(\phi_1, \ldots, \phi_{\ell-1}, J_\ell)$ where J_ℓ vanishes on γ_ℓ and the points T_ℓ where γ_ℓ meets the boundary of Δ, we may compute the solutions $S_\ell := \mathcal{V}_\Delta(\phi_1, \ldots, \phi_\ell)$ using arclength continuation along γ_ℓ, by Algorithm 1.7.

A genericity assumption on the coefficient matrix C and the exponents $\mathcal{B}$ implies that T_ℓ is a subset of the vertices of Δ. By Lemma 3.4 (1) of [8], $\widetilde{J_\ell} := J_\ell \cdot \prod_i (L_i(y) + \lambda_i)$ is a polynomial of degree n. Thus we may compute the solutions S_ℓ to the Gale system G, given knowledge of the set T_ℓ and the set $S_{\ell-1}$ in which the transcendental function ϕ_ℓ defining S_ℓ is replaced by the low degree polynomial $\widetilde{J_\ell}$. The Khovanskii-Rolle algorithm [5] and fewnomial bound [8] iterate this procedure. Before explaining them, we discuss our genericity assumptions.

The exponent vectors $\mathcal{B} = \{\beta^{(1)}, \ldots, \beta^{(\ell)}\}$ are the columns of a $(n+\ell+1) \times \ell$-matrix, also written $\mathcal{B}$, whose rows correspond to the affine functions defining Δ. For a face P of Δ and any $j = 1, \ldots, \ell$, let $\mathcal{B}_{P,j}$ be the submatrix of $\mathcal{B}$ consisting of the first j columns of the rows of $\mathcal{B}$ corresponding to the affine forms vanishing on P. We assumed Δ is bounded, which may be achieved by a projective change of coordinates, as explained in Remark 1.4.

Assumption 1.8. *We will make the following assumptions.*

(1) *The affine forms $L_i(y) + \lambda_i$ for $i = 1, \ldots, n+\ell+1$ are in general position in that exactly ℓ of them vanish at each vertex of Δ.*
(2) *For every face P of Δ, the square matrix $\mathcal{B}_{P,\mathrm{codim}(P)}$ annihilates no nonzero nonnegative row vector.*
(3) *The algebraic varieties γ_j defined below are curves.*

Assumption (1) is implied by the assumption that the coefficients of the system $F(x) = b$ are general, which is mild, as we already assumed that the system had only isolated zeroes. A consequence is that Δ is a simple polyhedron. That is, any face P of codimension r lies on exactly r facets, and thus exactly r affine forms $L_i(y) + \lambda_i$ vanish on P.

Assumptions (2) and (3) depend upon the exponent vectors $\mathcal{A}$ of the monomials in F, as well as the polyhedron Δ (which comes from C and b). As explained in [8], these assumptions hold on a Zariski open subset of the spaces of matrices C and $\mathcal{B}$ (when $\mathcal{B}$ is allowed to have real entries).

We sketch the steps of Khovanskii-Rolle continuation under these assumptions. For each $j = \ell, \ell-1, \ldots, 1$, set J_j to be the Jacobian of $\phi_1, \ldots, \phi_j, J_{j+1}, \ldots, J_\ell$ and define γ_j to be $\mathcal{V}_\Delta(\phi_1, \ldots, \phi_{j-1}, J_{j+1}, \ldots, J_\ell)$. By Assumption 1.8(3), this is a curve. Let T_j be the points where γ_j meets $\partial\Delta$. Finally, for $j = 0, \ldots, \ell$ set $S_j := \mathcal{V}_\Delta(\phi_1, \ldots, \phi_j, J_{j+1}, \ldots, J_\ell)$. Then S_ℓ is the set of solutions to the Gale system.

By Algorithm 1.7, arclength continuation along γ_j beginning from points of T_j and S_{j-1} will compute all points of S_j. Thus, the solutions S_ℓ of the Gale system in Δ may be computed iteratively from S_0 and the sets $T_1, \ldots, T_\ell$. Computing these sets is feasible, for by Lemma 3.4 of [8] and Corollary 2.3, we have

(1) $\widetilde{J_i} = J_i \cdot \left(\prod_i (L_i(y) + \lambda_i) \right)^{2^{\ell-j}}$ is a polynomial of degree $2^{\ell-j} \cdot n$.

(2) The points of T_j all lie on some faces of Δ of dimension $\ell-j$ and are the points of those faces where the polynomials $\widetilde{J_j}, \ldots, \widetilde{J_\ell}$ vanish.

For $j = 2, \ldots, \ell$, let $\mu_j := \mathcal{V}_\Delta(\phi_1, \ldots, \phi_{j-1})$. The faces of Δ in (2) lie in the closure of the set μ_j and are determined by $\mathcal{B}$ (Corollary 2.3 gives the precise statement). This set μ_j is algebraic, as the transcendental functions ϕ_i define the same set in Δ as do the polynomials (1.13). Furthermore, μ_j has dimension $\ell-j+1$. Indeed, the rational function versions (1.11) of the $\phi_i(y)$ form a regular sequence in the hyperplane complement, so their common zero set μ_j has expected dimension $\ell-(j-1)$. The assumption that γ_j is a curve is that the Jacobians $J_{j+1}, \ldots, J_\ell$ form a complete intersection on μ_j.

1.4. **Our running example.** Let us consider this for the example of Subsection 1.1. The first two assumptions hold. All polygons are simple, and no entry and no minor vanishes in the matrix of exponents,

$$\mathcal{B} = \begin{bmatrix} -1 & -3 & 1 & -1 & 3 \\ 2 & -1 & 3 & -2 & -2 \end{bmatrix}^T .$$

The Jacobian of the logarithmic system (1.5) is the rational function,

$$J_2 := \frac{2x^3 - 16x^2y + 12xy^2 + 6y^3 - \frac{31}{2}x^2 + 26xy - \frac{53}{2}y^2 + \frac{9}{2}x + \frac{15}{2}y - 2}{xy(1 - x - y)(\frac{1}{2} - x + y)(1 + x - 3y)} ,$$

whose denominator is the product of the linear factors defining the lines in Figure 1. The denominator does not vanish in the interior of the pentagon. Taking the numerator and multiplying by 2 gives the cubic polynomial,

$$\widetilde{J_2} := 4x^3 - 32x^2y + 24y^2x + 12y^3 - 31x^2 + 52xy - 53y^2 + 9x + 15y - 4 .$$

The Jacobian J_1 of g and J_2 has denominator the product of the squares of the forms defining the lines, and its numerator is the sextic

$$-56x^6 + 464x^5y - 456x^4y^2 - 1792x^3y^3 + 1896x^2y^4 + 336xy^5 - 72y^6 + 640x^5 - 1960x^4y$$
$$+ 456x^3y^2 + 3096x^2y^3 - 5176xy^4 + 480y^5 - 482x^4 + 2248x^3y - 2416x^2y^2 + 4256xy^3 + 250y^4$$
$$+ 61x^3 - 401x^2y - 1769xy^2 - 491y^3 - 10x^2 + 664xy + 278y^2 - 81x - 101y + 16 .$$

The system $J_1 = J_2 = 0$ has 18 solutions with 10 real, but it has only two solutions S_0 in the pentagon. The curve γ_1 is defined by the vanishing of the Jacobian J_2. The set T_1 of points where it meets the boundary of the pentagon consists of two points.

In Figure 2, we illustrate the configuration of the sets $S_0, S_1, S_2, T_1,$ and T_2 in the pentagon of Figure 1, along with the curves defined by J_1, J_2 (written γ_1), g (γ_2), and f. The middle picture shows that arclength continuation along γ_1 in both directions from
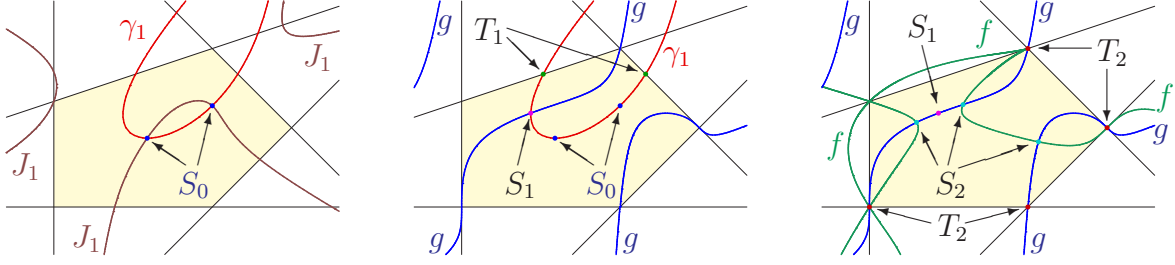
FIGURE 2. Solving the Gale system with Khovanskii-Rolle continuation

each point of S_0 and into the pentagon from each point of T_1 will find the point S_1 twice (the other four continuations terminate when they either reach a point of S_0 or exit the pentagon). The picture on the right shows that arclength continuation along γ_2 in both directions from the point of S_1 and into the pentagon from the four points of T_2 will reach the three points of S_2, twice each. These points are

$$(1.17) \qquad (0.29557, 0.32316), \ (0.14846, 0.26681), \ (0.53346, 0.20563).$$

Remark 1.9. In Figure 2, we see that some of the points T_2 are singular points of $f = 0$. In general the points T_1 are smooth points on the curve γ_1, but the other points T_j with $j > 1$ may be singular points of the curves γ_j they lie upon. In Section 2, we will discuss when it is possible to eliminate tracking from these possibly singular points. A future publication [2] detailing methods (and software) will explore strategies for arclength continuation along γ_j from these singular points of the boundary.

Remark 1.10. Having computed approximations to the points (1.17), we do not yet have solutions to the original polynomial system. By Gale duality, if S_G is the set of solutions to the Gale system, then $\varphi_{\mathcal{A}}^{-1}(\psi(S_G))$ is the set of solutions to the original polynomial system. Suppose S_G^* is a set of approximations to solutions of the Gale system. The difficulty is that points of $\psi(S_G^*)$ likely do not lie in $\varphi_{\mathcal{A}}((\mathbb{R}^\times)^n)$, and so $\varphi_{\mathcal{A}}^{-1}(\psi(S_G^*))$ is undefined. In Subsection 3.2, we give a method to overcome this obstruction.

2. A DESCARTES' BOUND FOR FEWNOMIALS

We give a refinement to the upper bound of $\frac{e^2+3}{4}2^{\binom{\ell}{2}}n^\ell$ for the number of solutions to a Gale system that was discovered while developing our software. This refinement involves sign conditions and gives a Descartes'-like bound for fewnomials. We then discuss how this refinement affects the running of the Khovanskii-Rolle algorithm.

Theorem 2.1. *Suppose that $\Phi(x) = 0$ is a system of n Laurent polynomials in n variables involving $n+\ell+1$ monomials. If the exponent vector of one monomial lies in the relative interior of the convex hull of the exponent vectors of the other monomials and if the coefficient matrix of these other monomials does not have a positive vector in its kernel, then this system has at most*

$$\left(1 + 2^{-\ell}\left(1 + \tfrac{\ell}{n}\right)\right)2^{\binom{\ell}{2}}n^\ell$$

nondegenerate positive solutions.

We will give a proof of Theorem 2.1 as part of our discussion on our improvement to the fewnomial bound. We first derive the fewnomial bound originally given in [8], for we need to modify that derivation to obtain Theorem 2.1.

As explained at the end of Subsection 1.3, the solutions S_ℓ to the Gale system are found by first computing $S_0 := \mathcal{V}_\Delta(\widetilde{J}_1, \ldots, \widetilde{J}_\ell)$ and certain sets $T_1, \ldots, T_\ell$ where T_j consists of the points of the boundary of Δ that meet the curve γ_j. By Corollary 2.3 below, T_j consists of the points of certain faces of Δ of dimension $\ell - j$ where $\widetilde{J}_{j+1}, \ldots, \widetilde{J}_\ell$ vanish. As $\widetilde{J}_i$ is a polynomial of degree $2^{\ell-i}n$, we may replace faces of Δ by their affine span and use Bézout's Theorem to estimate the cardinality of these sets,

$$
\begin{aligned}
|S_0| &\leq \prod_{i=1}^{\ell} \deg(\widetilde{J}_i) = 2^{\binom{\ell}{2}} n^\ell \\
|T_j| &\leq f_{\ell-j}(\Delta) \cdot \prod_{i=j+1}^{\ell} \deg(\widetilde{J}_i) = f_{\ell-j}(\Delta) 2^{\binom{\ell-j}{2}} n^{\ell-j} \,,
\end{aligned}
$$
(2.1)

where $f_i(\Delta)$ is the number of i-dimensional faces of Δ. In fact, there is a tighter estimate for $|T_j|$ given below (2.4).

From the Khovanskii-Rolle Theorem, we have $|S_i| \leq |S_{i-1}| + \frac{1}{2}|T_i|$, and so

$$
\begin{aligned}
|S_\ell| &\leq |S_0| + \tfrac{1}{2}\big(|T_1| + \cdots + |T_\ell|\big) \\
&\leq 2^{\binom{\ell}{2}} n^\ell + \frac{1}{2} \sum_{j=1}^{\ell} f_{\ell-j}(\Delta) 2^{\binom{\ell-j}{2}} n^{\ell-j} \\
&< 2^{\binom{\ell}{2}} n^\ell + \frac{1}{2} \sum_{j=1}^{\ell} \binom{n+\ell+1}{j} 2^{\binom{\ell-j}{2}} n^{\ell-j} \\
&\leq 2^{\binom{\ell}{2}} n^\ell + \frac{1}{2} \sum_{j=1}^{\ell} \frac{2^{j-1}}{j!} 2^{\binom{\ell}{2}} n^\ell \quad < \quad \frac{e^2+3}{4} 2^{\binom{\ell}{2}} n^\ell \,.
\end{aligned}
$$
(2.2)

The estimate of the last row is Lemma 3.5 in [8].

Any difference in the two sides of any inequality (its *slack*) propagates through to the final bound. One clear way that the estimates (2.1) could have slack would be if a system $\widetilde{J}_{j+1}, \ldots, \widetilde{J}_\ell$ on the complex affine span of a face did not have all of its solutions lying in that face. In practice, we have found that very few of the solutions lie on the desired face of Δ. While this seems impossible to predict or control, the Khovanskii-Rolle continuation algorithm always takes advantage of this source of slack.

Another source of slack is our estimates for $|T_j|$, which depend on the number of certain faces of Δ, as not all faces of codimension j can contain points of T_j. We investigate the location of the points T_j. For each $j = 2, \ldots, \ell$, we defined $\mu_j \subset \Delta$ to be closure of the points in the interior of Δ given by

$$
\phi_1(y) = \cdots = \phi_{j-1}(y) = 0 \,.
$$
(2.3)

This set is algebraic and has dimension $\ell - j + 1$.

Lemma 2.2. *Suppose that Assumption* 1.8 (1) *holds. If P is a face of Δ whose interior meets μ_j, then the submatrix $\mathcal{B}_{P,j-1}$ of $\mathcal{B}$ consisting of the first $j-1$ columns of rows corresponding to the facets of Δ containing P annihilates a nonzero nonnegative row vector. When this occurs, we have $P \subset \mu_j$.*

By annihilating a nonzero nonegative row vector, we mean there is a nonzero vector $v = (v_1, \ldots, v_r)$ with $v_i \geq 0$ for all i such that $v\mathcal{B}_{P,j-1} = 0$. Here, r is the number of facets containing P.

Proof. Let p be a point lying in the interior of P. After an affine change of coordinates and a reordering, we may assume that $p = 0$ and the affine forms defining Δ are

$$y_1, \; y_2, \; \ldots, \; y_r, \; \Lambda_{r+1}(y), \; \ldots, \; \Lambda_{n+\ell+1}(y),$$

where $\Lambda_a(0) > 0$ for $a > r$. That is, points of P have their first r coordinates equal to 0.

Then μ_j is defined on the interior of Δ by the rational functions

$$y_1^{\beta_1^{(i)}} \cdots y_r^{\beta_r^{(i)}} \cdot \prod_{a=r+1}^{n+\ell+1} \Lambda_a(y)^{\beta_a^{(i)}} \; = \; 1 \qquad \text{for} \quad i = 1, \ldots, j-1.$$

After an analytic change of coordinates in the neighborhood of the origin p, we may assume that these become monomial,

$$y_1^{\beta_1^{(i)}} \cdots y_r^{\beta_r^{(i)}} \; = \; 1 \qquad \text{for} \quad i = 1, \ldots, j-1.$$

The solution set meets the origin p if and only if it contains a monomial curve of the form

$$(t^{\alpha_1}, \; t^{\alpha_2}, \; \ldots, \; t^{\alpha_r})$$

with each exponent α_i nonnegative and not all are zero. But then $\alpha = (\alpha_1, \ldots, \alpha_r)$ satisfies

$$\sum_{t=1}^{r} \beta_t(i)\alpha_t \; = \; 0 \qquad \text{for } i = 1, \ldots, j-1,$$

so that α is a nonzero nonnegative row vector annihilated by $\mathcal{B}_{P,j-1}$.

The last statement follows as these arguments may be reversed. $\qquad\square$

Corollary 2.3. *Suppose that Assumption* 1.8 *holds. Then $\mu_j \cap \partial\Delta$ is the union of those codimension j faces P of Δ where $\mathcal{B}_{P,j-1}$ annihilates a nonzero nonnegative row vector.*

The points T_j are the points of $\mu_j \cap \partial\Delta$ where the polynomials $\widetilde{J}_{j+1}, \ldots, \widetilde{J}_{\ell}$ vanish. By Corollary 2.3, this is the restriction of those $\ell-j$ polynomials to a collection of faces, each of dimension $\ell-j$.

Proof. Since μ_j has dimension $\ell-j+1$, its intersection with the boundary of Δ has dimension $\ell-j$. If this intersection meets the relative interior of a face P of Δ of codimension r, then the matrix $\mathcal{B}_{P,j-1}$ annihilates a nonzero nonnegative row vector, by Lemma 2.2.

If $r < j$, then $\mathcal{B}_{P,r}$ is a submatrix of $\mathcal{B}_{P,j-1}$, and so $\mathcal{B}_{P,r}$ annihilates a nonzero nonnegative row vector and contradicts our assumption. Thus $r \geq j$, and we see that the rest of the corollary follows from Lemma 2.2. $\qquad\square$

Corollary 2.3 restricts the faces on which points of T_j can lie and implies the following tighter estimate,

$$(2.4) \qquad |T_j| \ \leq \ M_{\ell-j}(\Delta, \mathcal{B}) \cdot 2^{\ell-j} n^{\ell-j} \qquad \text{for} \quad j > 1\,,$$

where $M_{\ell-j}(\Delta, \mathcal{B})$ counts the faces P of Δ with codimension j for which $\mathcal{B}_{P,j-1}$ annihilates a nonzero nonnegative row vector.

Proof of Theorem 2.1. Let $\Phi(x) = 0$ be a system of n Laurent polynomials in n variables involving $n+\ell+1$ monomials. Let x^a be a monomial in Φ whose exponent vector a lies in the interior of the convex hull of the other exponent vectors. We may replace Φ by $x^{-a}\Phi$ to obtain a system of Laurent polynomials with the same zeroes in $(\mathbb{R}^{\times})^n$ having a constant term whose exponent 0 lies in the interior of the convex hull of the exponent vectors of its remaining monomials.

Rewrite $\Phi(x) = 0$ as $F(x) = b$ where F has $n+\ell$ monomials with exponent vectors $\mathcal{A}$, and the origin lies in the interior of the convex hull of $\mathcal{A}$. Thus the origin is a nonnegative integer combination of the vectors in $\mathcal{A}$, with every vector in $\mathcal{A}$ participating in this expression. Consequently, this expression is a positive vector in the integer nullspace of $\mathcal{A}$, and we may assume it is the first column of the matrix $\mathcal{B}$ of exponents for the Gale system. This implies that no $j \times (j-1)$ submatrix of the first $j-1$ columns of $\mathcal{B}$ has a positive row vector in its kernel, and so $M_{\ell-j}(\Delta, \mathcal{B}) = 0$, when $j > 1$.

If the positive chamber Δ is bounded, then it has at most $n+\ell$ facets and $T_2, \ldots, T_\ell$ are empty. This gives the estimate

$$
\begin{aligned}
|S_\ell| \ &\leq \ |S_0| + \tfrac{1}{2}|T_1| \ \leq \ 2^{\binom{\ell}{2}} n^\ell + (n+\ell) 2^{\binom{\ell-1}{2}-1} n^{\ell-1} \\
&= \ \Big(1 + 2^{-\ell}\big(1 + \tfrac{\ell}{n}\big)\Big) 2^{\binom{\ell}{2}} n^\ell\,.
\end{aligned}
$$
(2.5)

If Δ is unbounded, then, after the transformation of Remark 1.4, Δ possibly has $n+\ell+1$ facets and the first column of $\mathcal{B}$ has exactly one negative entry.

To complete the proof, suppose further that the coefficient matrix C of those other exponent vectors has no positive vector in its kernel. This is equivalent to Δ being bounded, and we have the bound (2.5). $\qquad\square$

The sets $T_2, \ldots, T_\ell$ are empty under the hypotheses of Theorem 2.1 and when the first column of $\mathcal{B}$ has every entry positive. In this case, Assumption 1.8 (2) on $\mathcal{B}$ is satisfied.

We noted in Remark 1.9 that the points T_j for $j \geq 2$ will typically be singular points of the curve γ_j as μ_j is singular along the boundary of Δ. Minimizing the number of points T_j (or eliminating them altogether as in Theorem 2.1), reduces the difficult path tracking required to track from these singular points T_j, thereby improving the performance of the Khovanskii-Rolle continuation algorithm.

3. Algorithmic details for the Gale transform

Obtaining the solutions of a fewnomial system by Khovanskii-Rolle continuation is a 3-step process.

(1) Apply the Gale transform to a fewnomial system $F(x) = b$, producing a dual Gale system $G(y) = 1$ of master functions.

(2) Use Khovanskii-Rolle continuation to find approximations to elements of S_G which are the real solutions of $G(y) = 1$ in the positive chamber $\Delta \subset \mathbb{R}^\ell$. These approximations form the set S_G^*.

(3) For each $s^* \in S_G^*$, transform s^* to t^*, a numerical approximation to some $t \in S_F$. Set $S_F^* := \{t^* \mid t^*$ is an approximation to $t \in S_F\}$.

A fewnomial system may be solved using Gale duality by other root-finding methods in place of Khovanskii-Rolle continuation in Step 2. We provide pseudocode for the Gale transform of Step 1 (Algorithm 3.1) and for the transformation of the approximate solutions S_G^* of the Gale system G to the approximate solutions S_F^* of the fewnomial system F in Step 3 (Algorithm 3.3). Two steps of Algorithm 3.1 require extra attention; they are detailed in the final two subsections.

3.1. **Converting a fewnomial system to a system of master functions.** Using the notation of Section 1, we give pseudocode for the basic algorithm for converting a system $F(x) = b$ of fewnomials into a system of master functions $G(y) = 1$.

Algorithm 3.1.
Input: ℓ, n, an $n \times (n+\ell)$ integer matrix $\mathcal{A}$, an $n \times (n+\ell)$ coefficient matrix C, and target column vector b.
Output: A system $G(y) = 1$ of ℓ master functions in ℓ variables that is Gale dual to the system $F(x) = b$.

1) Choose ℓ monomials and reorder the columns of C so that these monomials are the last ℓ columns.
2) Put $[C \mid b]$ in echelon form, obtaining a matrix $[-I_n \mid L \mid \lambda]$.
3) From the rows of L and entries of λ form the affine forms $L_i(y) + \lambda_i$.
4) Choose ℓ integral basis vectors β_i of the nullspace of $\mathcal{A}$ to construct $\mathcal{B} = \{\beta^{(1)}, \ldots, \beta^{(\ell)}\}$.
5) Use $\mathcal{B}$ and the forms $L_i(y) + \lambda_i$ to form the Gale dual system $G(y) = 1$ as in (1.11).

We explain some of details in the design and implementation of each step of Algorithm 3.1, and then illustrate it with an example.

1) The effect of this choice is an affine change of coordinates in $\mathbb{R}^\ell$, which affects the shape of the positive chamber $\Delta \subset \mathbb{R}^\ell$ and the performance of the Khovanskii-Rolle continuation when no scaling of the Gale system functions occur. With a simple scaling routine, discussed in Section 4, the parameterization of the monomials appears to not matter when $\ell = 2$.
2) We simply use Gaussian elimination with partial pivoting.
3) If the space $\mathbb{R}^{n+\ell}$ has coordinates $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^\ell$, the echelon form $[-I_n \mid L \mid \lambda]$ gives a parameterization of the set $C(z, y) = b$ by the affine forms $z_i = L_i(y) + \lambda_i$ for $i = 1, \ldots, n$.
4) We use the LLL algorithm [18] to compute an initial basis $\mathcal{B}$ and then find vectors in the row space with few negative entries, a heuristic to minimize $M_{\ell-j}(\Delta, \mathcal{B})$ (2.4), which is the number of codimension j faces of Δ on which the points T_j may lie. This step is the focus of Section 3.3.

5) If we set $L_{n+j}(y) := y_j$ and $\lambda_{n+j} := 0$ for $j = 1, \ldots, \ell$, then the vectors in $\mathcal{B}$ and affine functions $L_i(y) + \lambda_i$ for $i = 1, \ldots, n+\ell$ together give the Gale system $G(y) = 1$ as in (1.11) in Proposition 1.1.

Example 3.2. Consider the fewnomial system, where $n = 5$ and $\ell = 2$

$$
\begin{array}{rl}
-a^{-1}b^2c^2d + & \left(\tfrac{1}{2}b^2c + 2b^{-4}c^{-7}d^{-5}e^{-1} - 1\right) = 0, \\
-ac + & \tfrac{1}{2}\left(\tfrac{1}{4}b^2c - b^{-4}c^{-7}d^{-5}e^{-1} + 1\right) = 0, \\
-bc^4d^4 + & \tfrac{1}{4}\left(-\tfrac{1}{4}b^2c - 3b^{-4}c^{-7}d^{-5}e^{-1} + 6\right) = 0, \\
-d + & \tfrac{1}{2}\left(-\tfrac{3}{4}b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} + 8\right) = 0, \\
-e + & \left(-\tfrac{1}{2}b^2c + 2b^{-4}c^{-7}d^{-5}e^{-1} + 3\right) = 0,
\end{array}
$$

(3.1)

with support $\{a^{-1}b^2c^2d,\, ac,\, bc^4d^4,\, d,\, e,\, b^2c,\, b^{-4}c^{-7}d^{-5}e^{-1}\}$ and thus matrix of exponents

$$
\mathcal{A} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 2 & -4 \\
2 & 1 & 4 & 0 & 0 & 1 & -7 \\
1 & 0 & 4 & 1 & 0 & 0 & -5 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}
$$

and coefficient matrix

$$
[C \mid b] = \left[\begin{array}{ccccccc|c}
-1 & 0 & 0 & 0 & 0 & 1/2 & 2 & -1 \\
0 & -1 & 0 & 0 & 0 & 1/8 & -1/2 & 1/2 \\
0 & 0 & -1 & 0 & 0 & -1/16 & -3/4 & 3/2 \\
0 & 0 & 0 & -1 & 0 & -3/8 & -1 & 4 \\
0 & 0 & 0 & 0 & -1 & -1/2 & 2 & 3
\end{array}\right]
$$

where the first seven columns correspond to the seven monomials in $\mathcal{A}$ and the final column corresponds to the constants.

We presented this fewnomial system with a coefficient matrix in echelon form for the given order of the monomials, so Steps 1 and 2 of Algorithm 3.1 are complete. If we set $s := b^2c$ and $t := b^{-4}c^{-7}d^{-5}e^{-1}$, the rows of this echelon matrix express each of the first five monomials as affine functions of the parameters s and t, so that, for example, $a^{-1}b^2c^2d = \tfrac{1}{2}s + 2t - 1$. We have the seven ($= 5 + 2$) affine forms,

$$
\begin{array}{rcl}
\Lambda_1(s,t) & = & \tfrac{1}{2}s + 2t - 1 \\
\Lambda_2(s,t) & = & \tfrac{1}{8}s - \tfrac{1}{2}t + \tfrac{1}{2} \\
\Lambda_3(s,t) & = & -\tfrac{1}{16}s - \tfrac{3}{4}t + \tfrac{3}{2} \\
\Lambda_4(s,t) & = & -\tfrac{3}{8}s - t + 4 \\
\Lambda_5(s,t) & = & -\tfrac{1}{2}s + 2t + 3 \\
\Lambda_6(s,t) & = & s \\
\Lambda_7(s,t) & = & t
\end{array}
$$

(3.2)

Given a vector $\beta = [\beta_1, \ldots, \beta_7]^T$ in the nullspace of $\mathcal{A}$, we have $\prod_{i=1}^{7}(x^{a_i})^{\beta_i} = 1$, where a_i is the ith exponent vector in $\mathcal{A}$. Under the substitution $x^{a_i} = \Lambda_i(s, t)$, we obtain $\prod_{i=1}^{7} \Lambda_i(s, t)^{\beta_i} = 1$. One choice of basis $\mathcal{B}$ for the nullspace of $\mathcal{A}$ is

$$\{[4, 4, 2, 3, 3, 1, 3]^T, \ [-1, -1, 2, -2, 1, 2, 1]^T\}.$$

This gives the system of master functions (written in the polynomial form (1.13))

$$
\begin{aligned}
(3.3) \qquad g(s,t) &= (s{+}4t{-}2)^4(s{-}4t{+}4)^4(s{+}12t{-}24)^2(3s{+}8t{-}32)^3(s{-}4t{-}6)^3st^3 \\
&\qquad -68719476736, \\
f(s,t) &= -2(s{+}12t{-}24)^2(s{-}4t{-}6)s^2t - (s{+}4t{-}2)(s{-}4t{+}4)(3s{+}8t{-}32)^2.
\end{aligned}
$$

Figure 3 displays the algebraic curves in $\mathbb{R}^2$ defined by these polynomials, together with the lines defined by the affine forms $\Lambda_i(s, t) = 0$ for $i = 1, \ldots, 7$.
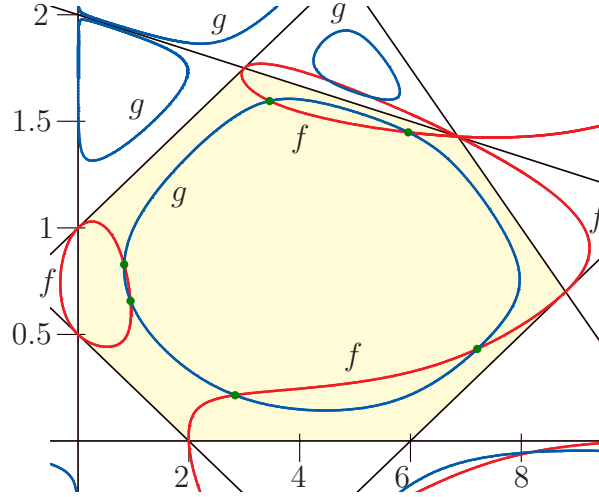

FIGURE 3. Master function curves in the septagon

The original system (3.1) has 95 complex solutions with 11 real and six in the positive orthant. The hypotheses of Theorem 2.1 hold; with $n = 5$ and $\ell = 2$, its improved bound is 87 by Theorem 2.1 and 67 using (2.2). These hypotheses imply that the curve $g = 0$ does not meet the boundary of the septagon Δ.

3.2. **Transforming Gale system solutions to fewnomial solutions.** The output of Khovanskii-Rolle continuation is a set S_G^* of approximations to solutions S_G of the Gale system in the positive chamber Δ. Converting these points of Δ to points S_F^* of $\mathbb{R}_>^n$ that are approximations to solutions S_F of the fewnomial system is problematic, for there is no natural map between points of Δ and $\mathbb{R}_>^n$, except on the solutions to the two systems. We present a simple method to accomplish this conversion.

Let $s^* \in S_G^*$ and suppose that s is the point of S_G that s^* approximates. Applying the map ψ (1.8) of Subsection 1.2 whose components are the affine functions $L_i(y) + \lambda_i$ gives points $\psi(s^*)$ and $\psi(s)$ in $\mathbb{R}_>^{n+\ell}$ that lie on the affine plane H. The point $\psi(s)$ lies on

$\varphi_{\mathcal{A}}(\mathbb{R}_>^n)$, but the point $\psi(s^*)$ does not lie on $\varphi_{\mathcal{A}}(\mathbb{R}_>^n)$ (unless $s^* \in S_G$), as the former is a solution of the Gale system, while the latter typically is not.

To remedy this, choose a square submatrix $\mathcal{A}'$ of rank n of the matrix $\mathcal{A}$ of exponents. Projecting $\mathbb{R}^{n+\ell}$ to those coordinates indexed by the columns of $\mathcal{A}'$ gives a map $\pi_{\mathcal{A}'} \colon \mathbb{R}^{n+\ell} \to \mathbb{R}^n$. The composition

$$\mathbb{R}_>^n \xrightarrow{\varphi_{\mathcal{A}}} \mathbb{R}_>^{n+\ell} \xrightarrow{\pi_{\mathcal{A}'}} \mathbb{R}_>^n$$

is the map $\varphi_{\mathcal{A}'}$, which is invertible on $\mathbb{R}_>^n$. We set

$$t^* := \varphi_{\mathcal{A}'}^{-1} \circ \pi_{\mathcal{A}'} \circ \psi(s^*) .$$

Since $t = \varphi_{\mathcal{A}'}^{-1} \circ \pi_{\mathcal{A}'} \circ \psi(s) = \varphi_{\mathcal{A}}^{-1} \circ \psi(s)$ is the solution to the fewnomial system, and these maps are differentiable, we expect that t^* is close to t. We formalize this transformation.

Algorithm 3.3.
Input: The exponent matrix $\mathcal{A}$ and affine functions $L_i(y) + \lambda_i$ from Algorithm 3.1, and the set $S_G^* = \{s_i^* \mid i = 1, ..., N\}$ of approximations to solutions of the Gale system G in the positive chamber Δ.
Output: A set $S_F^* = \{t_i^* \mid i = 1, ..., N\}$ of approximations to positive solutions of the original fewnomial system F.

Choose a full rank $n \times n$ submatrix $\mathcal{A}'$ of the matrix $\mathcal{A}$ and compute $(\mathcal{A}')^{-1}$.
for $i := 1$ to N **do**
   a) Evaluate the map ψ at the point s_i^* to get the point $z_i^* \in \mathbb{R}_>^{n+\ell}$,

$$z_i^* := (L_1(s_i^*) + \lambda_1, \ldots, L_{n+\ell}(s_i^*) + \lambda_{n+\ell}) .$$

   b) Project z_i^* to $\mathbb{R}_>^n$ using $\pi_{\mathcal{A}'}$, set $v_i^* := \pi_{\mathcal{A}'}(z_i^*)$.
   c) Let $w_i^* \in \mathbb{R}^n$ be the coordinatewise logarithm of v_i^*.
   d) Set $x_i^* := (w_i^*(\mathcal{A}')^{-1})^T$ and let t_i^* be the coordinatewise exponentiation of x_i^*.
end for

Note that the homeomorphism $\varphi_{\mathcal{A}'} \colon \mathbb{R}_>^n \to \mathbb{R}_>^n$ becomes the linear isomorphism induced by $x \mapsto \mathcal{A}'x$ in logarithmic coordinates. Steps c) and d) invert this transformation.

Even if s_i^* is certified to be an approximate solution to the Gale system with associated solution s_i (i.e. $s_i^* \to s_i$ under Newton iterations, doubling the number of significant digits with each iteration), it need not be the case that t_i^* is an approximate solution to the fewnomial system with associated solution $t_i := \varphi_{\mathcal{A}}^{-1} \circ \psi(s_i)$. However, we may check this using, for example, the software `alphaCertified` [14]. If t_i^* is not certified to be an approximate solution, then we could refine s_i^* to be closer to s_i and reapply the transformation. Our software has the functionality to do this, either producing a soft certificate (all calculations done in floating point arithmetic) or a hard certificate (calculations done with rational arithmetic), which is a proof that t_i^* is an approximate solution.

Example 3.4. An approximation to a solution of $f(s,t) = g(s,t) = 0$ from Example 3.2 lying in the positive chamber is $s_1^* = (0.94884808, 0.65721633)$. We follow the steps of Algorithm 3.3 to recover a solution of the original fewnomial system.

We first choose an invertible submatrix of $\mathcal{A}$. The first $5(= n)$ columns of A suffice as these are linearly independent. Evaluating the map ψ at s_1^* gives the point

$$z_1^* \;=\; \begin{bmatrix} 0.78885671 \\ 0.28999784 \\ 0.94778474 \\ 2.98696564 \\ 3.84000863 \\ 0.94884808 \\ 0.65721633 \end{bmatrix}.$$

Projecting to the first 5 coordinates of z_1^* and taking coordinatewise logarithms yields

$$w_1^* \;=\; \begin{bmatrix} -0.23717058 \\ -1.23788180 \\ -0.05362787 \\ 1.09425803 \\ 1.34547461 \end{bmatrix}.$$

Proceeding to the next step, we evaluate

$$x_1^* \;=\; (w_1^*(\mathcal{A}'^{-1}))^T \;=\; \begin{bmatrix} 0.020520123 \\ 0.60294767 \\ -1.25840192 \\ 1.09425803 \\ 1.34547461 \end{bmatrix}.$$

Finally,

$$t_1^* \;=\; \exp(x_1^*) \;=\; \begin{bmatrix} 1.0207321 \\ 1.8274977 \\ 0.28410769 \\ 2.9869656 \\ 3.8400086 \end{bmatrix}.$$

We carry out Algorithm 3.3 with s_1^* starting with 135 bits of precision for the input and truncate the output to an approximation that can be stored in one 64-bit floating point number.

Using `alphaCertified` [14], we perform two Newton iterations on this approximation, doubling the precision each time. We obtain a soft-certification of this refined approximation, and then convert the numerical approximation to its rational form and obtain a hard certification that proves that t_1^* converges to t_1.

3.3. **Heuristic for the nullspace basis.** A choice in the Gale transform that affects the efficiency of numerical tracking in the Khovanskii-Rolle algorithm is the selection of the basis $\mathcal{B}$ of the annihilator of $\mathcal{A}$.

Observe that the points of T_j for $j \geq 2$ are often singular points of the curve γ_j. For efficiency and numerical stability, we want to avoid tracking near these points. Corollary 2.3 implies that the curve γ_j may approach a codimension j face P of Δ only when $\mathcal{B}_{P,j-1}$ annihilates a nonzero nonnegative row vector. By Corollary 1.2, changing the basis

elements $\beta^{(j)}$ of $\mathcal{B}$ will change the Gale system as well as the curves γ_j, without changing the solutions. Ideally, we would use this freedom to choose the basis $\mathcal{B}$ of the annihilator of $\mathcal{A}$ to minimize the number of faces of P on which points T_j for $j \geq 2$ could lie.

The number of possible choices for $\mathcal{B}$ up to patterns for the signs of the coordinates suffers from a combinatorial explosion [15] as ℓ increases. Consequently, this optimization problem may become infeasible for ℓ large. Instead, we find elements of the annihilator of $\mathcal{A}$ that have the fewest number of negative coordinates. We expect that this choice will tend to minimize the number of faces on which the points T_j could lie. In the case of Theorem 2.1, this choice achieves the minimization. When there is a positive vector annihilating $\mathcal{A}$, we may choose $\mathcal{B}$ to consist only of positive vectors, and so no submatrix annihilates any nonzero nonnegative row vector. By Lemma 2.2, no face of P will contain points of T_j.

Once a basis for the annihiliator of $\mathcal{A}$ is known, the algorithm proceeds differently depending upon the size of ℓ. For small ℓ, we can enumerate all possible patterns for the signs of the coordinates of vectors in the annihilitor of $\mathcal{A}$, and then choose a basis $\mathcal{B}$ consistsing of vectors having the fewest number of negative coordinates. When ℓ is large, we take random integer linear combinations of a basis of the annihilator of $\mathcal{A}$ and choose of these generated vectors those with the minimum number of negative entries.

This heuristic does not always work in parctice, as we observed in our test suite described below. One problem is when a point of some S_j—a solution to an intermediate problem—lies close to the boundary of Δ. Such solutions may be numerically ill-conditioned, and the subroutine in the Gale duality software package that tracks the curves γ_j near the boundary of Δ (the monomial tracker described in [5, § 4.3]) may miss such ill-conditioned solutions. A future publication [2] will focus on the design and performance of the monomial tracker subroutine.

4. Software

The algorithms of Section 3 are implemented in a software package, `galeDuality` [3], which is available at each author's website and written in `C++`. This will be the front end for a package we are developing for Khovanskii-Rolle continuation. We describe a test suite (4.1) to evaluate the proposed heuristic in Section 3.3, and some implementation details of the package (4.2).

4.1. **Test suite.** As noted in Sections 1 and 2, Khovanskii-Rolle continuation finds the solutions S_ℓ to a Gale system in a polyhedron Δ using arclength continuation along curves γ_j starting at points $T_0 := S_0, T_1, \ldots, T_\ell$ where T_j are solutions to $\ell - j$ Jacobian determinants (having Bézout number $2^{\binom{\ell-j}{2}} n^{\ell-j}$) in certain codimension j faces of Δ. The set T_0 lies in the interior of the polyhedron Δ and points of T_1 could lie in every facet, but the other T_j lie in only a subset of the faces of codimension j. The heuristic of Subsection 3.3 is intended to minimize or eliminate the number of faces contributing to T_j for $j \geq 2$.

To test this heuristic, we created a test suite of Gale systems for each pair (ℓ, n) with $\ell = 2$ and $n = 2, \ldots, 13$.

For each n, we generated an $(n+2)$-gon by successively selecting random rational points in an annulus until we had $n+2$ points in convex position. These points formed the vertices of the $(n+2)$-gon. The edges of this polygon then give $n+2$ affine forms with relatively prime integer coefficients. To create a Gale system, we generated two vectors of exponents whose components correspond to the affine forms. Both had integer components with absolute values selected uniformly at random in the range $1, \ldots, 10$. One vector, β^+, had all components positive, but the other, $\beta^\pm$, had signs alternating as much as possible traversing the polygon cyclically. This was intended to emulate the configuration of the master function curves in Figure 3.

Our test suite consists of 100 random Gale systems for each n. For each random system, we conducted two types of runs of Khovanskii-Rolle continuation, one for each ordering of the exponent vectors β^+ and $\beta^\pm$ as columns in $\mathcal{B}$. When β^+ is the first column of $\mathcal{B}$, the set T_2 is empty, by Theorem 2.1, but when $\beta^\pm$ is the first column, T_2 consists of the vertices defined by affine forms such that the components of $\beta^\pm$ had differing signs. Our proxy to test for the affect of the heuristic in Section 3.3 was to compare the average of 10 running times for these two different choices.

For any fixed n, in a head-to-head comparison of the average run-time of Khovanskii-Rolle continuation for the two different choices, choosing β^+ as the first column of $\mathcal{B}$ outperformed $\beta^\pm$ for $> 92\%$ of the systems in the test suite. For the majority of the instances in which the choice of $\beta^\pm$ outperformed β^+, the difference between the run times was insignificant.

For some systems, the choice of β^+ causes the Khovanskii-Rolle continuation software to not find solutions near the boundary of Δ that an equivalent choice of $\beta^\pm$ was able to compute (and also certify). This is due to the inherent numerical instability of the monomial tracker subroutine that approximates the γ_j curves near the boundary of Δ. Improvements to the monomial tracker is the subject of a future article [2].

Timing of the Khovanskii-Rolle continuation algorithm, information on the number of points found in T_j, and the number of real solutions for each of the systems in the test suite is archived with our software, `galeDuality` [3].

4.2. **galeDuality.** Our software package, `galeDuality`, is an open-source program written in `C++` and accepts as input either a fewnomial system or a Gale system. When given a fewnomial system, the software symbolically computes the Gale transform and saves the Gale system to a file, using the algorithms of Section 3, for arbitrary ℓ. Similarly, when given a Gale system, `galeDuality` will transform the system of master functions into its dual fewnomial system and save the fewnomial system to a file.

If the polyhedron Δ is unbounded, the software applies the projective transformation of Proposition 1.5. When $\ell = 2$, `galeDuality` calls the maple script of [5] to solve the Gale system using Khovanskii-Rolle continuation.

After the Khovanskii-Rolle continuation computation is finished, `galeDuality` will convert the approximations to solutions of the Gale system back to approximations to solutions of the fewnomial system using Algorithm 3.3, and then call `alphaCertified`, if desired by the user, to guarantee that the approximations converge to solutions. If

the user wishes, `galeDuality` will also certify the approximations in Δ found by the Khovanskii-Rolle continuation algorithm.

Finally, software and a user manual with more detailed information is available online from each of the authors' websites.

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