Convex Hulls of Pairs of Circles

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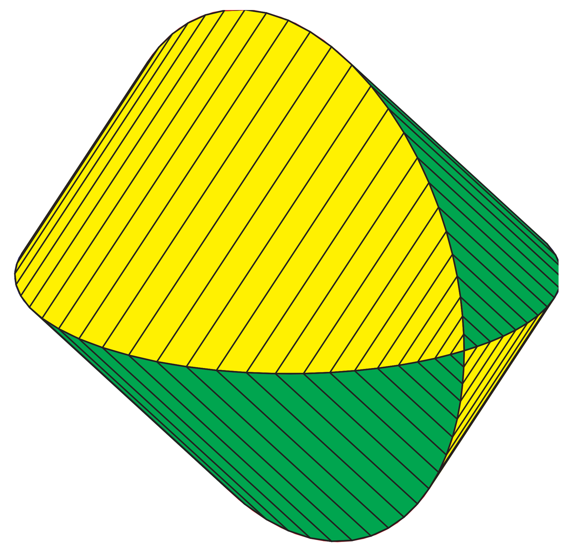
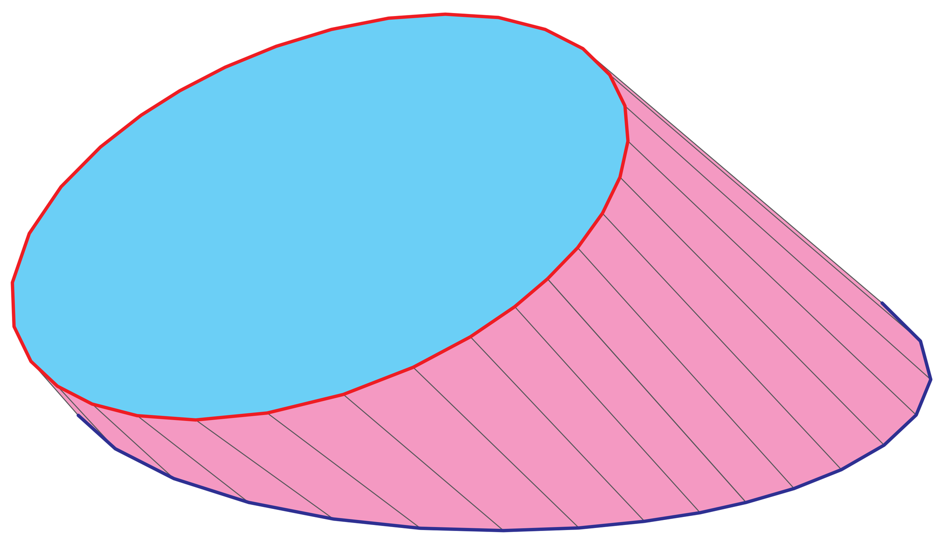


1 Introduction

Consider the convex hull of two circles in $\mathbb{RP}^3$. Its boundary has possibly two types of components: the disks of the circles and the ruled edge surface.



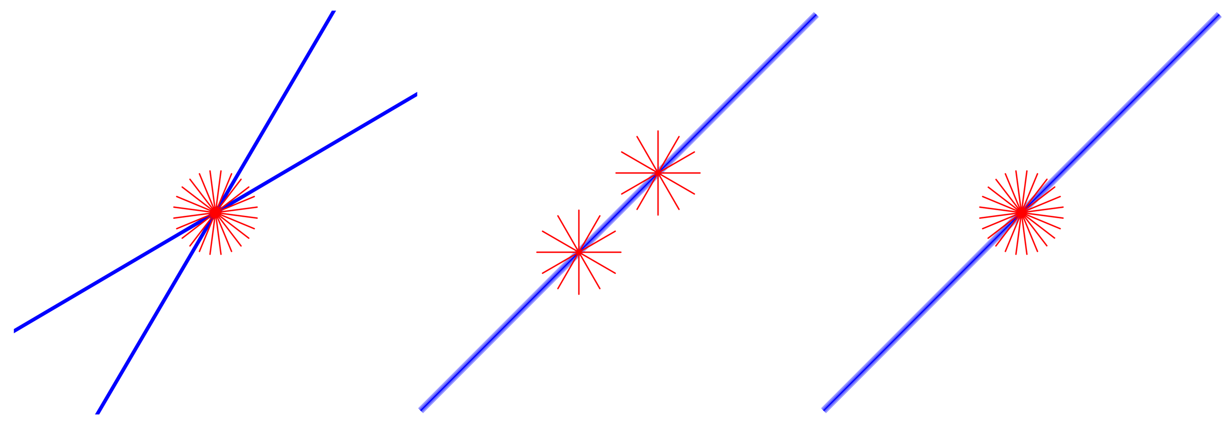
The edge surface is ruled by line segments (called stationary bisecants [1]) joining points whose tangents to the circles intersect. These stationary bisecants define a curve of bidegree (2, 2) in the product of the circles $\mathbb{RP}^1 \times \mathbb{RP}^1$. We classify the (2, 2)-curves which can occur.

2 Algebraic relaxation

Replace real circles by complex conics and study the curves of stationary bisecants.

Definition. A (2, 2)-curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is a curve defined by a form of bihomogeneous degree (2, 2).

A complete conic is either a smooth conic or a conic in one of three forms below:



Lemma. *The stationary bisecants to a pair of complete conics form a (2, 2)-curve.*

Each (2, 2)-curve C has four branch points (counting multiplicities), denoted by x_1, x_2, x_3, x_4 in one of the $\mathbb{CP}^1$ factors. If distinct, then the j-invariant of the curve C is defined by

$$j(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

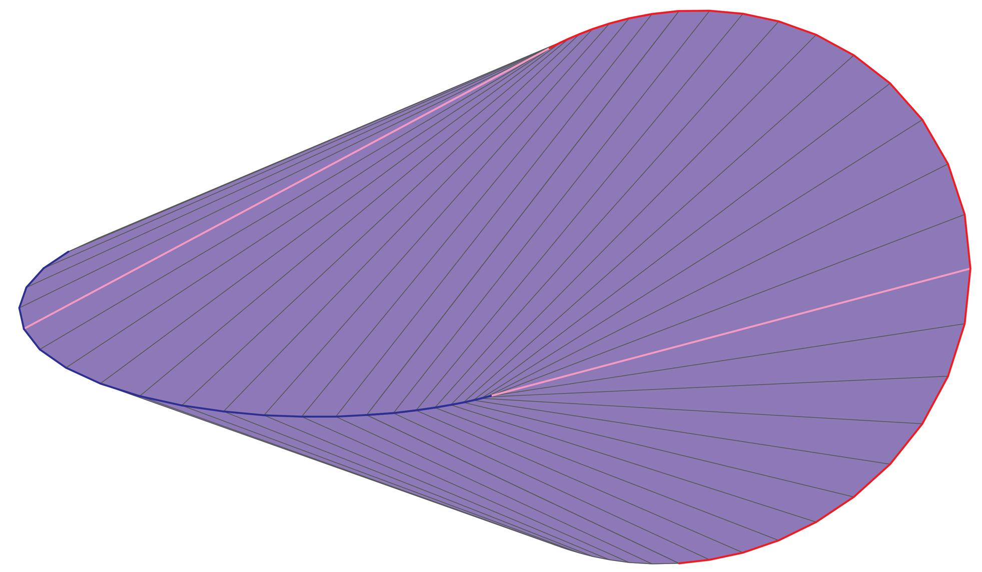
where λ is the cross-ratio

$$\lambda := \frac{(x_4 - x_1)(x_2 - x_3)}{(x_4 - x_3)(x_2 - x_1)}.$$

Thus, the j-invariant of a (2, 2)-curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is encoded in its four branch points.

3 Results

Geometry of branch points. *Let C_1, C_2 be two smooth conics in $\mathbb{CP}^3$ which do not lie on the same plane. A point x on the conic C_1 is a branch point if and only if the tangent line of the conic C_1 at the point x meets the conic C_2.*

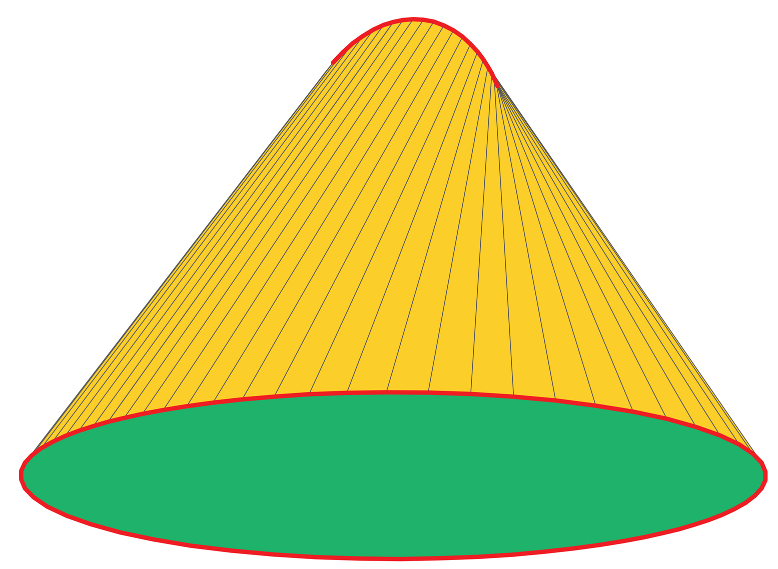


Theorem. *All (2, 2)-curves arise from pairs of complete conics.*

Theorem. *All smooth real (2, 2)-curves arise from pairs of circles.*

Moreover, we classify the real (2, 2)-curves that arise by the geometric positions of the pairs of circles. We do not know yet which pairs of circles can give us the cuspidal (2, 2)-curve.

4 Eye candy



5 Reference

[1] K. Ranestad, B. Sturmfels: On the convex hull of a space curve, Advances in Geometry, DOI 10.1515 / ADVGEOM.2011.021.