Algebra II Winter 2021 — 1 February
Frank Sottile — Third Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 8 February.

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1. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. The set $\text{End}_{\mathbb{F}}(V)$ of linear transformations $T\colon V \to V$ of V forms a ring with multiplication the composition of maps. (If $n = \dim_{\mathbb{F}}(V)$, then $\text{End}_{\mathbb{F}}(V) \simeq \text{Mat}_{n\times n}(\mathbb{F})$, given by any ordered basis of V.) Verify that V is naturally an $\text{End}_{\mathbb{F}}(V)$-module and identify its submodules.

2. Let $\mathbb{F}$ be a field, V a finite-dimensional vector space over $\mathbb{F}$, and $T\colon V \to V$ a linear transformation.

   Show that the ring homomorphism induced by $x \mapsto T$ equips V with the structure of a module over the polynomial ring $\mathbb{F}[x]$.

   What are the $\mathbb{F}[x]$-submodules of V under this action ?

3. Suppose that $\phi\colon M \to N$ and $\psi\colon N \to M$ are R-module homomorphisms such that $\psi \circ \phi = 1_M$ (the identity map on M). Prove that $N = \text{image}(\phi) \oplus \text{kernel}(\psi)$.

4. Suppose that R is a principal ideal domain, A a left R-module, and $p \in R$ a prime (and hence also irreducible). Recall that R/pR is a field.

   Show that both $pA := \{pa \mid a \in A\}$ and $A[p] := \{a \in A \mid pa = 0\}$ are R-submodules of A.

   Show that A/pA and $A[p]$ are both naturally vector spaces over R/pR. (Part of this is interpreting 'naturally', but there is only one sensible choice of the action.)

5. Let V be a vector space over a division ring D and S the set of all subspaces of V, partially ordered by inclusion.

   Show that S is a *complete lattice* (defined in Exercise 7.2 of the Introdution to Hungerford) with least upper bound of U, W equal to $U + W$ and greatest lower bound $U \cap W$.

   Show that S is *complemented*; for all $W \in S$, there is a $U \in S$ such that $W + U = V$ and $W \cap U = \{0\}$.

   Show that S is *modular*; for $A, B, C \in S$ with $C \subset A$, we have $A \cap (B + C) = (A \cap B) + C$.

6. If F and G are free modules over a ring with the invariant dimension property, show that $\text{rank}(F \oplus G) = \text{rank}(F) + \text{rank}(G)$.

7. Let R be a ring with no zero divisors such that for all $r, s \in R$, there are $a, b \in R$, not both zero, such that $ar + bs = 0$. Show that if $R = M \oplus N$ as R-modules, then one of M or N is the 0-module, $\{0\}$. Use this to show that R has the invariant dimension property.

8. Show that if F is a free module over a ring R such that F has a basis of cardinality an integer $n \geq 1$, and *another* basis with cardinality $n+1$, then F has a basis of every cardinality $m \in \mathbb{N}$ with $m \geq n$.

   Note that the ring R is necessarily noncommutative.