Algebra II Winter 2021 — 1 March
Frank Sottile — Answers to First Test

1. Let R be a commutative ring. Let $S \subset R$ be a multiplicative subset. Show that

$$J := \{a \in R \mid \exists s \in S \text{ with } as = 0\}$$

is an ideal of R, and its image in $R[S^{-1}]$ is $\{0\}$. Is this ideal the kernel of the natural map from R to $R[S^{-1}]$?

The easiest proof is to show that J is the kernel of the natural map $\iota\colon R \to R[S^{-1}]$ (as the kernel of a homomorphism is an ideal).

First, let $a \in J$ and $s \in S$ be such that $as = 0$. Since $\iota(a)$ is the (equivalence class) $\frac{as}{s} = \frac{0}{s}$, which is $\iota(0)$, we have $J \subset \ker(\iota)$.

Now, let $a \in \ker(\iota)$. Then, for any $s \in S$, $\iota(a) = 0 = \iota(0)$. Thus, in $R[S^{-1}]$, we have $\frac{as}{s} = \frac{0}{s}$. Recalling the definition of equality in $R[S^{-1}]$, this means that there exists $\sigma \in S$ such that $(as^2 - 0)\sigma = 0$, so that $as^2\sigma = 0$. As S is multiplicatively closed, we have that $a \in J$. $\square$

2. Let P be a left R module. State the definition that P is projective.

Let P be a projective R-module, and suppose that M and N are R-submodules of P such that $P = M \oplus N$. Prove that M (and hence also N) is projective, using the definition of projective, and not just quoting theorems from the class.

A left R-module P is *projective* if, for any exact sequence $A \to B \to 0$ of left R-modules and any R-module map $f\colon P \to B$, there is a map $g\colon P \to A$ so that $f = \alpha \circ g$. (Here, $\alpha\colon A \to B$ is the map in the exact sequence.)

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "f"] & \\ A \arrow[r, "\alpha"] & B \arrow[r] & 0 \end{tikzcd}$$

Suppose that P is a projective left R-module, and it is a direct sum of R-modules, $P = M \oplus N$. Let $A \to B \to 0$ be an exact sequence of left R-modules and $f\colon M \to B$ an R-module map.

Let $\iota\colon M \to P$ be the inclusion and $\pi\colon P \to M$ the projection induced by the decomposition $P = M \oplus N$. Observe that $\pi \circ \iota = 1_M$, the identity map on M. Consider the composition $f \circ \pi\colon P \to B$. As P is projective, this factors through the map α; there is a map $g\colon P \to A$ such that $\alpha \circ g = f \circ \pi$.

$$\begin{tikzcd} P \arrow[r, shift left, "\pi"] \arrow[d, "g"'] & M \arrow[l, shift left, "\iota"] \arrow[d, "f"] & \\ A \arrow[r, "\alpha"] & B \arrow[r] & 0 \end{tikzcd}$$

Precomposing with $\iota\colon M \to P$ gives $\alpha \circ (g \circ \iota) = f \circ (\pi \circ \iota) = f$. Thus f factors through the map α via the map $g \circ \iota$. This completes the proof that M is projective.

3. Let R be a ring and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a sequence of left R-modules and R-module homomorphisms. Suppose that for all left R-modules M the induced sequence,

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{\ \alpha_*\ } \mathrm{Hom}_R(M, B) \xrightarrow{\ \beta_*\ } \mathrm{Hom}_R(M, C)$$

is an exact sequence of abelian groups.

Show that this implies the original sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact.

As we know that $\mathrm{Hom}_R(R, A) \simeq A$ as abelian groups, when $M = R$ we obtain that the sequence is an exact sequence of abelian groups. Since the maps are maps of R-modules, this implies the desired result.

(For the forward map, left $f \in \mathrm{Hom}_R(R, A)$ be sent to $f(1_R) \in A$, which identifies the two as abelian groups. This also shows that $\alpha_* = \alpha$, and the same for the other maps.)