

# Graphs Whose Spectral Band Functions are Perfect Morse Functions

Minisymposium on Spectral Theory and Applications  
of Schrödinger Operators

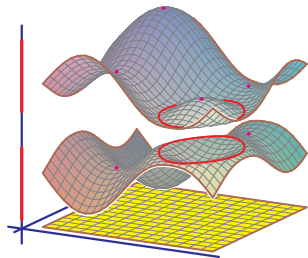
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Frank Sottile

Texas A&M University

sottile@tamu.edu



Work with Matthew Faust of Michigan State.

Not Morse functions



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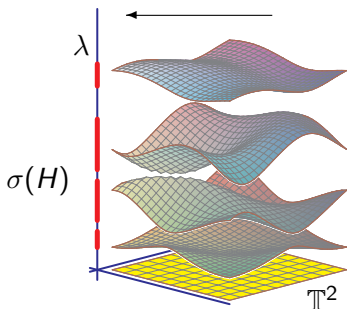
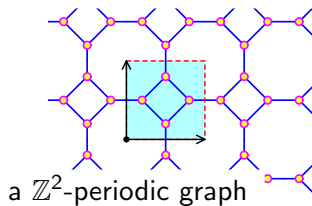
# Periodic Graph Operators

A  $\mathbb{Z}^d$ -periodic graph  $\Gamma = (\mathcal{V}, \mathcal{E})$  with  
periodic functions  $V: \mathcal{V} \rightarrow \mathbb{R}$  and  
 $E: \mathcal{E} \rightarrow \mathbb{R}$  (a *labeling*)

$\rightsquigarrow$  a discrete periodic operator

$H = V + \Delta$  on  $\ell_2(\mathcal{V})$ .

$\Delta$  is a weighted graph Laplacian.



As  $H$  is self-adjoint, its spectrum  $\sigma(H)$   
is a closed subset of  $\mathbb{R}$ .

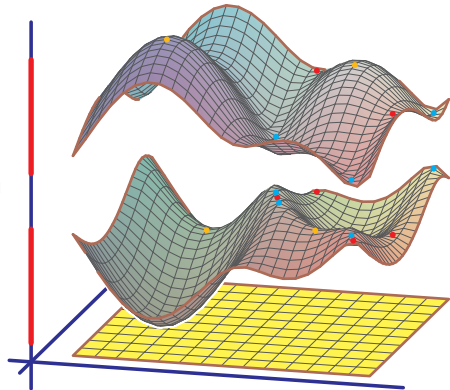
Viewed through the lens of unitary  
characters  $\mathbb{T}^d$  of  $\mathbb{Z}^d$ , the spectrum is the  
image of  $|\mathcal{V}/\mathbb{Z}^d|$  *spectral band  
functions*  $\lambda_i: \mathbb{T}^d \rightarrow \mathbb{R}$ .

This also gives a *Floquet matrix*  
 $H(z)$  of Laurent polynomials such that  
 $\det(H(z) - \lambda_i(z)) = 0$ .

# Spectral Edges Conjecture

**Kuchment:** Physicists typically assume that all extrema of spectral band functions are nondegenerate critical points, which implies many important physical properties.

The *spectral edges conjecture* for a periodic graph  $\Gamma$  posits that this occurs for a general labeling of  $\Gamma$ .



**Easy fact:** If  $z \in \mathbb{T}^d$  satisfies that  $z^2 = 1$  ( $z$  is a *corner point*), then it is a critical point of every spectral band function.

By the Morse inequalities, if every critical point of a spectral band function is nondegenerate, then it has at least  $2^d$  critical points.

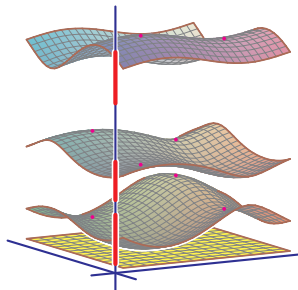
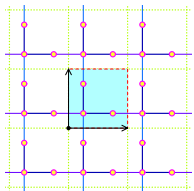
(This is because  $2^d$  is the total Betti number of  $\mathbb{T}^d$ .)

# Perfect Morse Functions

A spectral band function has at least  $2^d$  critical points.

When it has exactly  $2^d$  critical points, all nondegenerate, then the spectral band function is a *perfect Morse function*.

The Lieb lattice is a graph whose spectral band functions can be perfect Morse functions:



We seek conditions which imply that all spectral band functions are perfect Morse functions.

This very strong condition implies the spectral edges conjecture.

# Minimally Supported Periodic Graphs

The general Floquet matrix  $H(z)$  of a connected periodic graph  $\Gamma$  has a monomial  $z^{\pm\alpha} \iff \Gamma$  has an edge between a fundamental domain and a translate by  $\alpha$ .

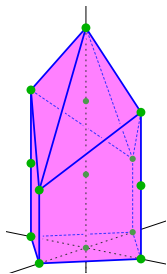
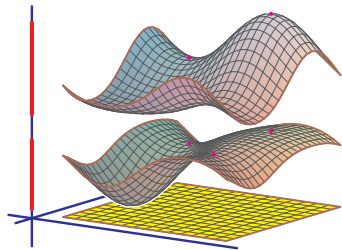
Such edges span  $\mathbb{Z}^d$ .

The Floquet matrix  $H(z)$  is *minimally supported* if the only monomials of  $z$  in  $\det H(z)$  are  $z^0$ ,  $z_i$ ,  $z_i^{-1}$ , for  $i \in [d]$ .

A graph is *minimally supported* if its general Floquet matrix is minimally supported.

This is a property of the Newton polytope of  $\det(H(z) - \lambda)$ .

**Faust-Robinson-S.:** *If  $\Gamma$  is minimally supported, each spectral band has at most  $2^d$  isolated critical points.*



# Flat Band Trichotomy

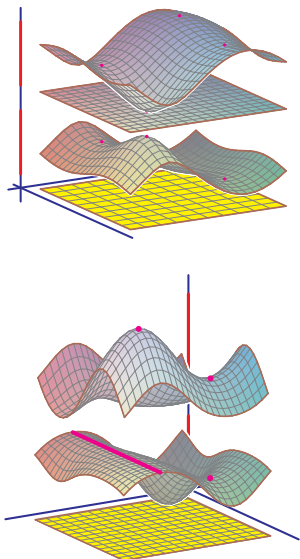
A labeled graph  $\Gamma$  has a *flat band* if  $D(z, \lambda) := \det(H(z) - \lambda)$  has a factor  $\lambda - \lambda_0$ .

$\Gamma$  has a *flat sub-band* if, after substituting  $\pm 1$  for  $z_i$  in  $D(z, \lambda)$  for some *but not all* coordinates, there is a factor  $\lambda - \lambda_0$ . This implies a  $\mathbb{T}^r$  of critical points for some  $r < d$ .

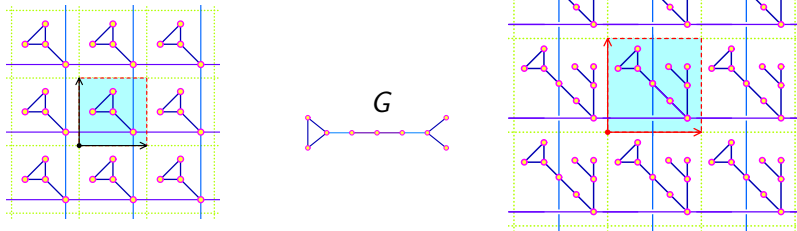
**Trichotomy (Faust-S.):** *If a labeled  $\mathbb{Z}^d$  periodic graph  $\Gamma$  is minimally supported, then either*

- (1)  $\Gamma$  has a flat band.
- (2)  $\Gamma$  has a flat sub-band.
- (3) Every spectral band function is a perfect Morse function.

These can be wild. [html](#)



# Isthmus connected graphs



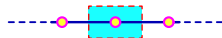
A periodic graph  $\Gamma$  is *isthmus-connected* if it has a fundamental domain  $G$  with an induced path such that each adjacent fundamental domain is connected through a unique vertex in the interior of the path.

**Isthmus (Faust-S.):** Let  $\Gamma$  be an isthmus connected graph. For any choice of non-zero edge parameters, there is an *explicitly described* dense open subset  $U$  of potentials with the property:

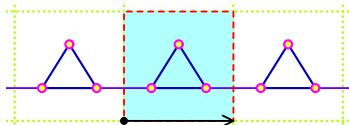
*For any labeling of  $\Gamma$  with these edge weights and potential  $V \in U$ , every spectral band function is a perfect Morse function.*

# Parallel Extensions

Let  $\mathbb{Z}_*$  be the 1-dimensional square lattice.



The parallel extension of a  $\mathbb{Z}^d$ -periodic labeled graph  $\Gamma$  is the  $\mathbb{Z}^{d+1}$ -periodic graph  $\mathbb{Z}_* \times \Gamma$ .



minimally connected

**Parallel Extension (Faust-S.):**

*If every spectral band function of a labeled periodic graph  $\Gamma$  is perfect Morse function, then every spectral band function of its parallel extension is a perfect Morse function.*

