Algebra II Winter 2021 5 April
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Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 12 April.

1. Let F/K be a field extension and $u_1, \dots, u_n \in F$. Show that $K(u_1, \dots, u_n)$ is the quotient field of the ring $K[u_1, \dots, u_n]$. (We have been assuming this. Please, do not give a circular argument.)
2. Let F/K be a finite Galois extension and L, M intermediate fields that are Galois over K.
   Prove that $[L \cap M\colon K] \cdot [LM\colon K] = [L\colon K] \cdot [M\colon K]$.
3. Suppose that F/K is a splitting field of a set S of polynomials in $K[x]$ and that E is an intermediate field. Show that F/E is a splitting field for S.
4. Show that no finite field K is algebraically closed; that is, there is a polynomial $f \in K[x]$ with no root in K.
5. Let E be an intermediate field of the field extension F/K.
   (a) Show that if $u \in F$ is separable over K, then it is separable over E.
   (b) Show that if F/K is separable, then so are F/E and E/K.
6. Let L, M be intermediate fields of a field extension F/K such that L/K is finite and Galois.
   Show that LM/M is a finite Galois extension and $\mathrm{Gal}(LM/M) \simeq \mathrm{Gal}(L/L \cap M)$.
7. Show that normality is not transitive. Hint: Let $r \in \mathbb{R}$ be a number such that $r^4 = 2$ and consider $\mathbb{Q}(r) \supset \mathbb{Q}(r^2) \supset \mathbb{Q}$.
8. Determine the Galois group of the rational polynomial $f = x^4 - 4x^2 + 1$.