Foundations of Mathematics Math 300 Sections 902, 905

YOUR NAME

Third Homework: Due 28 September 2020

Definition: An integer a is *even* if there is an integer k such that $n = 2k$. An integer a is *odd* if there is an integer k such that $n = 2k+1$.
Definition: Let a and b be integers. We say that a *divides* b and write $a|b$ if there is an integer c such that $ac = b$.
Definition: A real number x is a *rational number* when there exists integers n, d with $d \neq 0$ such that $x = n/d$. A real number that is not rational is an *irrational number*.
Definition: Suppose that a is a nonnegative real number. The *square root of a* is the unique nonnegative real number r such that $r^2 = a$. We write $\sqrt{a}$ for the square root of a.
Definition: Suppose that a, b are positive real numbers. The *logarithm of a in base b*, written $\log_b(a)$, is the unique real number r such that $b^r = a$.

1. Consider the following statement:
   "Let $n \in \mathbb{Z}$. If $5 \not| (n^2 + 4)$, then $5 \not| (n-1)$ and $5 \not| (n+1)$."
   (a) Write its contrapositive
   (b) Construct a "know-show" table for a proof of this statement, in the form of a direct proof of the contrapositive. (You may find it useful to recycle code from previous homeworks)
   (c) Write your proof in paragraph form.
2. Write a proof in paragraph form of the following statement: "If n^2 is even, then n is even."
3. Write a proof in paragraph form of the following statement: "If nm is even, then m is even or n is even."
4. Is the following statement true or false? (If true, give a proof, if false, give a counterexample.)
   "For each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational."
5. Consider the definitions given on page 55 in the Sundstrom text on set equality and subsets.
   (a) Write each definition more mathematically in terms of elements of the sets, quantifiers and implications.
   (b) Write a proof in paragraph form of the statement: Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$. Both $\subset$ and $\subseteq$ denote 'subset'.
6. Write a proof in paragraph form of the following statement.
   "For all real numbers x and y, $x^2 = y^2$ if and only if $x = y$ or $x = -y$."
   (You may use that $\forall a, b \in \mathbb{R}, \quad ab = 0 \rightarrow a = 0$ or $b = 0$, but do not use anything about square roots, which could be a recipe for a misstep.)
7. Suppose that a, b, c are real numbers and that $ax^2 + bx + c = 0$ has two different solutions. Prove that the sum of the two solutions equals $-b/a$.
8. Using the definitions, prove by cases that for every integer n, $n^2 - n + 41$ is odd.

9. Prove that if m is odd, then $m^2 \equiv 1 \mod 8$.

10. For all integers a, b, c with $a \neq 0$, if $a \nmid (bc)$ then $a \nmid b$ and $a \nmid c$.

11. Write a proof in paragraph form of the following statement: "For all positive real numbers x, y we have $\sqrt{xy} \leq \frac{x+y}{2}$, and we have equality if and only if $x = y$."

12. Prove by *reductio ad absurdum* that an integer cannot be both even and odd.

13. Prove the following by contradiction (*reductio ad absurdum*):
    For all integers n, if n^2 is odd, then n is odd.

14. Prove that $\log_2 5$ is an irrational number.
    Can you find a (true) generalization of this statement, replacing 2 and/or 5 by other, nearly arbitrary positive integers?

15. Prove the following by contradiction (*reductio ad absurdum*):
    For all real numbers a and b with $b \geq 0$, if $a^2 \geq b$, then either $a \geq \sqrt{b}$ or $a \leq -\sqrt{b}$.

16. Is the following proposition true or false? (Justify your conclusion with a proof or counterexample).
    "For all nonnegative real numbers x and y, $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.