

1. Let  $R$  be a commutative ring. Let  $S \subset R$  be a multiplicative subset. Show that

$$J := \{a \in R \mid \exists s \in S \text{ with } as = 0\}$$

is an ideal of  $R$ , and its image in  $R[S^{-1}]$  is  $\{0\}$ . Is this ideal the kernel of the natural map from  $R$  to  $R[S^{-1}]$ ?

The easiest proof is to show that  $J$  is the kernel of the natural map  $\iota: R \rightarrow R[S^{-1}]$  (as the kernel of a homomorphism is an ideal).

First, let  $a \in J$  and  $s \in S$  be such that  $as = 0$ . Since  $\iota(a)$  is the (equivalence class)  $\frac{as}{s} = \frac{0}{s}$ , which is  $\iota(0)$ , we have  $J \subset \ker(\iota)$ .

Now, let  $a \in \ker(\iota)$ . Then, for any  $s \in S$ ,  $\iota(a) = 0 = \iota(0)$ . Thus, in  $R[S^{-1}]$ , we have  $\frac{as}{s} = \frac{0}{s}$ . Recalling the definition of equality in  $R[S^{-1}]$ , this means that there exists  $\sigma \in S$  such that  $(as^2 - 0)\sigma = 0$ , so that  $as^2\sigma = 0$ . As  $S$  is multiplicatively closed, we have that  $a \in J$ .  $\square$

2. Let  $P$  be a left  $R$  module. State the definition that  $P$  is projective.

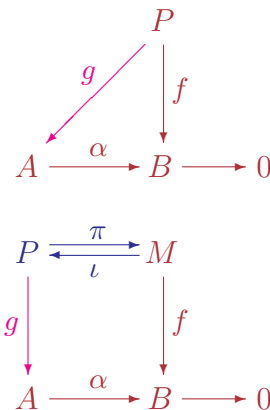
Let  $P$  be a projective  $R$ -module, and suppose that  $M$  and  $N$  are  $R$ -submodules of  $P$  such that  $P = M \oplus N$ . Prove that  $M$  (and hence also  $N$ ) is projective, using the definition of projective, and not just quoting theorems from the class.

A left  $R$ -module  $P$  is **projective** if, for any exact sequence  $A \rightarrow B \rightarrow 0$  of left  $R$ -modules and any  $R$ -module map  $f: P \rightarrow B$ , there is a map  $g: P \rightarrow A$  so that  $f = \alpha \circ g$ . (Here,  $\alpha: A \rightarrow B$  is the map in the exact sequence.)

Suppose that  $P$  is a projective left  $R$ -module, and it is a direct sum of  $R$ -modules,  $P = M \oplus N$ . Let  $A \rightarrow B \rightarrow 0$  be an exact sequence of left  $R$ -modules and  $f: M \rightarrow B$  an  $R$ -module map.

Let  $\iota: M \rightarrow P$  be the inclusion and  $\pi: P \rightarrow M$  the projection induced by the decomposition  $P = M \oplus N$ . Observe that  $\pi \circ \iota = 1_M$ , the identity map on  $M$ . Consider the composition  $f \circ \pi: P \rightarrow B$ . As  $P$  is projective, this factors through the map  $\alpha$ ; there is a map  $g: P \rightarrow A$  such that  $\alpha \circ g = f \circ \pi$ .

Precomposing with  $\iota: M \rightarrow P$  gives  $\alpha \circ (g \circ \iota) = f \circ (\pi \circ \iota) = f$ . Thus  $f$  factors through the map  $\alpha$  via the map  $g \circ \iota$ . This completes the proof that  $M$  is projective.



3. Let  $R$  be a ring and let  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  be a sequence of left  $R$ -modules and  $R$ -module homomorphisms. Suppose that for all left  $R$ -modules  $M$  the induced sequence,

$$0 \rightarrow \text{Hom}_R(M, A) \xrightarrow{\alpha_*} \text{Hom}_R(M, B) \xrightarrow{\beta_*} \text{Hom}_R(M, C)$$

is an exact sequence of abelian groups.

Show that this implies the original sequence  $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$  is exact.

As we know that  $\text{Hom}_R(R, A) \simeq A$  as abelian groups, when  $M = R$  we obtain that the sequence is an exact sequence of abelian groups. Since the maps are maps of  $R$ -modules, this implies the desired result.

(For the forward map, let  $f \in \text{Hom}_R(R, A)$  be sent to  $f(1_R) \in A$ , which identifies the two as abelian groups. This also shows that  $\alpha_* = \alpha$ , and the same for the other maps.)