THE EQUIVARIANT CHOW RINGS OF QUOT SCHEMES

TOM BRADEN, LINDA CHEN, AND FRANK SOTTILE

ABSTRACT. We give a presentation for the (integral) torus-equivariant Chow ring of the quot scheme, a smooth compactification of the space of rational curves of degree d in the Grassmannian. For this presentation, we refine Evain's extension of the method of Goresky, Kottwitz, and MacPherson to express the torus-equivariant Chow ring in terms of the torus-fixed points and explicit relations coming from the geometry of families of torus-invariant curves. As part of this calculation, we give a complete description of the torus-invariant curves on the quot scheme and show that each family is a product of projective spaces.

INTRODUCTION

Let k be an algebraically closed field and let d, n, r be non-negative integers with $r < n$. We study the quot scheme $\mathcal{Q}_d := \mathcal{Q}_d(r, n)$ parametrizing quotient sheaves on $\mathbb{P}^1$ of the trivial vector bundle $\mathcal{O}_{\mathbb{P}^1}^n$ which have rank r and degree d. When $r > 0$, this is a compactification of the space $\mathcal{M}_d$ of parametrized rational curves of degree d on the Grassmannian $G(r, n)$ of r-dimensional quotients of k^n. Indeed, a morphism from $\mathbb{P}^1$ to $G(r, n)$ of degree d is equivalent to a quotient bundle $\mathcal{O}_{\mathbb{P}^1}^n \to \mathcal{T}$ of rank r and degree d.

Strømme [16] showed that $\mathcal{Q}_d(r, n)$ is a smooth, projective, rational variety of dimension $r(n - r) + nd$. He described the decomposition into Białynicki-Birula cells induced by an action of a one-dimensional torus T on $\mathcal{Q}_d$, thereby determining its Betti numbers. He also gave a presentation of its integral Chow ring (Theorem 5.3, *loc. cit.*) in terms of generators and relations. However, the set of generators is far from minimal, and the relations are given by the annihilator of a certain class, and are therefore non-explicit. He also gave a more elementary set of generators for its rational Chow ring. We compute the corresponding equivariant classes in Sections 5 and 6.

Later, Bertram used the geometry of $\mathcal{Q}_d$ to determine the (small) quantum cohomology ring of the Grassmannian [1]. He used a recursive description of the boundary $\mathcal{Q}_d \setminus \mathcal{M}_d$ to show that the 3-point genus zero Gromov-Witten invariants of the Grassmannian are equal to particular intersection numbers on $\mathcal{Q}_d$. By studying certain types of intersections on quot schemes, he obtained a quantum Schubert calculus for the Grassmannian. However, he did not need to compute the full cohomology ring of $\mathcal{Q}_d$.

Our main result is a presentation of the T-equivariant Chow ring $A_T^*(\mathcal{Q}_d)$ for an action of a torus T on $\mathcal{Q}_d$. This determines the ordinary Chow ring $A^*(\mathcal{Q}_d)$. Our presentation

gives $A_T^*(\mathcal{Q}_d)$ as an explicit subring of a direct sum of polynomial rings. When $k = \mathbb{C}$, the cycle class map induces an isomorphism between Chow and cohomology rings, so our result also determines the T-equivariant and ordinary cohomology rings of $\mathcal{Q}_d$.

Our presentation arises from an analysis of the localization map. When an algebraic torus T acts on a smooth variety X with finitely many fixed points X^T, the inclusion $i \colon X^T \hookrightarrow X$ induces the localization map of (integral) equivariant Chow rings

$$(1) \qquad\qquad i^* \colon A_T^*(X) \to A_T^*(X^T) \; = \; \bigoplus_{p \in X^T} A_T^*(p) \,.$$

Each summand $A_T^*(p)$ is canonically isomorphic to the symmetric algebra S of the character group of T. When $k = \mathbb{C}$, there is a similar localization for rational equivariant cohomology.

When $k = \mathbb{C}$, Chang and Skjelbred [4] showed that the image of the localization map in cohomology is cut out by the images of the T-equivariant cohomology of components of the one-skeleton of X (the points fixed by some codimension one subtorus of T). These components are closures of families of T-invariant curves. In particular, when X has finitely many one-dimensional T-orbits (whose closures are T-invariant curves), Goresky, Kottwitz, and MacPherson [11] used this to describe the image of the localization map for equivariant cohomology. Each T-invariant curve gives a relation, and these GKM relations cut out the image. Brion [3] showed that this remains true for rational equivariant Chow rings of varieties over any algebraically closed field.

The torus $T = T_{k^n} \times T_{\mathbb{P}^1}$ acts on $\mathcal{Q}_d$, where $T_{k^n} = (\mathbb{G}_m)^n$ acts diagonally on k^n and $T_{\mathbb{P}^1} = \mathbb{G}_m$ acts primitively on $\mathbb{P}^1$. This action has finitely many fixed points, but there are infinitely many T-invariant curves. The GKM relations remain valid, but are now insufficient to cut out the image of i^*. There will be extra relations coming from connected components of the one-skeleton of $\mathcal{Q}_d$. Brion [3] adapted the result of Chang and Skjelbred to Chow groups, showing that the relations given by these families are sufficient to cut out $A_T^*(X)$, rationally. Determining these relations explicitly is more difficult than for the GKM relations, however, and few cases have been worked out in detail.

One case for which the relations are known is when X is a Hilbert scheme of points on a toric surface, which has families of T-invariant curves. Following a sugestion of Brion, Evain [8] used Edidin and Graham's [6] version of the Atiyah-Bott-Berline-Vergne localization formula for equivariant Chow groups to give relations in terms of ideal-membership. The relations come from elements of the T-equivariant Chow rings of the families of T-invariant curves.

We discuss this in Section 4, and give a more explicit formula for Evain's relations when each component Y of $X^{T'}$ has smooth T-invariant subvarieties Z whose classes $[Z]$ generate $A_T^*(Y)$. We derive necessary and sufficient linear relations over $\mathbb{Q}$ from Evain's ideal-membership relations. Lastly, we show that if the T-weights of the tangent space at each fixed point are not too dependent (see Theorem 12), then Evain's relations also determine the integral Chow ring.

All of these additional hypotheses hold for $\mathcal{Q}_d$. In fact, the components Y for $\mathcal{Q}_d$ are relatively simple: all are products of projective spaces. As a result, we obtain explicit descriptions of the equivariant Chow ring of $\mathcal{Q}_d$, both rationally and integrally.

To describe the combinatorics of fixed points in $\mathcal{Q}_d^T$, we use the following notations. For an element $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we define $|\mathbf{a}| = \sum_i a_i$. We use addition and subtraction on $\mathbb{Z}^n$ considered as an abelian group, and denote the identity element by $\mathbf{0} = (0, \ldots, 0)$. If $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$, we set $\mathbf{a}! = \prod_i a_i!$, where $0! = 1$. Finally we use the partial order $\mathbf{a} \leq \mathbf{b}$ to mean $a_i \leq b_i$ for all i.

In Section 1, we give an explicit parametrization of the fixed point set $\mathcal{Q}_d^T$ by a set $\mathcal{F}$ of triples $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, where

- $\boldsymbol{\delta} \in \{0, 1\}^n$ takes the value 1 exactly $n - r$ times, so that $|\boldsymbol{\delta}| = n - r$, and
- $\mathbf{a}, \mathbf{b}$ are elements of $(\mathbb{Z}_{\geq 0})^n$ which satisfy $|\mathbf{a}| + |\mathbf{b}| = d$ and for which $\delta_i = 0$ implies $a_i = 0$ and $b_i = 0$.

The fixed points are maximally degenerate quotient sheaves supported at 0 and ∞; the data $\mathbf{a}$ and $\mathbf{b}$ describe the structure of the stalks as modules over $\mathcal{O}_{\mathbb{P}^1}$ and as representations of T.

Recall that $A_T^*(p) = S$, the symmetric algebra of the character group of T. We have $S = \mathbb{Z}[\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}]$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$, and $\mathbf{f}$ are dual to the obvious basis coming from the decomposition $T = T_{k^n} \times T_{\mathbb{P}^1}$. In particular, $\mathbf{f}$ restricts to the identity character on $T_{\mathbb{P}^1}$ and to the trivial character of $T_{\mathbb{C}^n}$. We write $S_{\mathbb{Q}}$ for $S \otimes_{\mathbb{Z}} \mathbb{Q}$ and sometimes $S_{\mathbb{Z}}$ for S, when we wish to emphasize our ring of scalars. We write $S^{\mathcal{F}}$ for the set of tuples of polynomials $(f_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \in S \mid (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F})$. Then $S^{\mathcal{F}} = A_T^*(Q_d^T)$, under the identification of $\mathcal{F}$ with Q_d^T. We exhibit the image of the localization map as a subring of $S^{\mathcal{F}}$.

In Section 2 we describe a finite set of T-invariant curves which span the tangent space at each fixed point. In Section 3, we describe the families of T-invariant curves and their closures. Based on this description, our relations for the image of $i_{\mathcal{Q}_d}^*$ are the following:

I. For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}') \in \mathcal{F}$ with $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$ except in positions i and j, and $\delta_i = \delta'_j = 1$ and $\delta_j = \delta'_i = 0$, we have

$$f_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \equiv f_{(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')} \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f}.$$

(Note that $(a'_j - a_i) = -(b'_j - b_i)$, since $a_i + b_i = a'_j + b'_j$.)

II. (a) For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, $(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}) \in \mathcal{F}$ with $\mathbf{a}$, $\mathbf{a}'$ agreeing except in positions i and j, we have

$$f_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \equiv f_{(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b})} \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f}.$$

(b) For any pair $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}') \in \mathcal{F}$ with $\mathbf{b}$, $\mathbf{b}'$ agreeing except in positions i and j, we have

$$f_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \equiv f_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}')} \mod \mathbf{e}_j - \mathbf{e}_i + (b_i - b'_j)\mathbf{f}.$$

(c) If we have $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$, $(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b})$, $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}') \in \mathcal{F}$ satisfying both of the previous conditions (with the same i and j), and in addition $a'_j - a_i = b_i - b'_j$, then

$$Df_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} - Df_{(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b})} - Df_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}')} + Df_{(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}')} \equiv 0 \mod \mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f}$$

where D is differentiation in the direction of $\mathbf{e}_j^{\vee}$, the dual basis vector to $\mathbf{e}_j$.

(c)$'$ Under the hypotheses of II(c),

$$f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} - f_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})} - f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')} + f_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')} \ \equiv \ 0 \mod (\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i)\mathbf{f})^2 \,.$$

III. For every $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F}$ and every $\mathbf{0} \le \mathbf{b}' < \mathbf{b}$,

$$\sum_{\mathbf{b}' \le \mathbf{c} \le \mathbf{b}} \frac{(-1)^{|\mathbf{c}|}}{(\mathbf{b} - \mathbf{c})!(\mathbf{c} - \mathbf{b}')!} D^{|\mathbf{b}| - |\mathbf{b}'| - 1} f_{(\boldsymbol{\delta},\mathbf{a}+\mathbf{c},\mathbf{b}-\mathbf{c})} \equiv 0 \mod \mathbf{f} \,,$$

where D is differentiation in the direction of $\mathbf{f}^\vee$, the dual basis vector to $\mathbf{f}$.

III$'$. For every $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}) \in \mathcal{F}$, $\mathbf{b} \ne \mathbf{0}$,

$$\sum_{0 \le \mathbf{c} \le \mathbf{b}} \frac{(-1)^{|\mathbf{c}|}}{\mathbf{c}!(\mathbf{b} - \mathbf{c})!} f_{(\boldsymbol{\delta},\mathbf{a}+\mathbf{c},\mathbf{b}-\mathbf{c})} \equiv 0 \mod \mathbf{f}^{|\mathbf{b}|}.$$

Relations I, II(a), and II(b) are standard GKM relations, while the rest come from families of T-invariant curves. In particular, relations II(c)/II(c)$'$ (respectively III/III$'$) come from certain families whose closures are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (respectively arbitrary products of projective spaces).

Theorem 1. *The rational equivariant Chow ring $A_T^*(\mathcal{Q}_d)_\mathbb{Q}$ is isomorphic to the set of tuples $f = (f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}) \in S_\mathbb{Q}^\mathcal{F}$ subject to the relations I, II(a)(b)(c), and III.*

The integral equivariant Chow ring $A_T^*(\mathcal{Q}_d)$ is isomorphic to the set of tuples $f = (f_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}) \in S_\mathbb{Z}^\mathcal{F}$ subject to the relations I, II(a)(b)(c)$'$, and III$'$.

We prove Theorem 1 in Section 4.5. Since the equivariant Chow ring determines the ordinary Chow ring for smooth spaces, this gives in principle a complete description of the Chow ring of $\mathcal{Q}_d$. The resulting computation of Betti numbers is the same as Stroome's computation.

The quot scheme $\mathcal{Q}_d$ represents the functor which associates to a scheme X the set of flat families over X of sheaves on $\mathbb{P}^1$ which are quotients of $\mathcal{O}_{\mathbb{P}^1}^n$ of rank r and degree d. Thus, there is a universal exact sequence

$$(2) \qquad\qquad 0 \ \to \ \mathcal{S} \ \to \ \mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}_d}^n \ \to \ \mathcal{T} \ \to 0$$

of sheaves on $\mathbb{P}^1 \times \mathcal{Q}_d$ where $\mathcal{T}$ has rank r and degree d. This sequence of sheaves is flat, and flatness implies that $\mathcal{S}$ is a vector bundle of degree $-d$ and rank $n - r$. In particular, points of $\mathcal{Q}_d$ are exact sequences of such sheaves on $\mathbb{P}^1$.

Strømme's generators of the rational cohomology ring of $\mathcal{Q}_d$ were Künneth components of the Chern classes of the tautological vector bundle $\mathcal{S}$ on $\mathbb{P}^1 \times \mathcal{Q}_d$. In Section 5, we describe the equivariant Chern classes of $\mathcal{S}$ in $A_T^*(\mathcal{Q}_d)$ and thus lifts of Strømme's generators to the equivariant Chow groups. In Section 6 we work this out explicitly for $\mathcal{Q}_2(0, 2)$, using Theorem 1 to describe the equivariant and ordinary Chow rings and giving explicit lifts of Strømme's generators as localized classes.

1. TORUS-FIXED POINTS OF $\mathcal{Q}_d$

The free module $\mathcal{O}_{\mathbb{P}^1}^n = k^n \otimes \mathcal{O}_{\mathbb{P}^1}$ has basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Write T_{k^n} for the group of diagonal matrices in this basis. Let $[x, y]$ be coordinates on $\mathbb{P}^1$ with x vanishing at 0 and

y at ∞. For $T_{\mathbb{P}^1}$ acting on $\mathbb{P}^1$ with fixed points 0 and ∞, the torus $T := T_{k^n} \times T_{\mathbb{P}^1}$ acts on $\mathcal{Q}_d$ naturally as indicated by the given splitting.

The T-fixed points are indexed by triples $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ in the set $\mathcal{F}$ of the Introduction. The fixed point corresponding to $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ is the sequence of sheaves on $\mathbb{P}^1$

$$\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \longrightarrow \mathcal{O}^n_{\mathbb{P}^1} \longrightarrow \mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$$

where $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is the image of the map

$$\bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(-a_i - b_i) \xrightarrow{\text{diag}(\delta_i x^{a_i} y^{b_i})} \mathcal{O}^n_{\mathbb{P}^1} .$$

We identify this fixed point with the subsheaf $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ of $\mathcal{O}^n_{\mathbb{P}^1}$.

We introduce the following notation. For natural numbers a, b, let $\mathcal{S}_{a,b}$ be the subsheaf of $\mathcal{O}_{\mathbb{P}^1}$ which is the image of the map

$$\mathcal{O}_{\mathbb{P}^1}(-a - b) \xrightarrow{x^a y^b} \mathcal{O}_{\mathbb{P}^1} .$$

Under the identification of modules over $\mathcal{O}_{\mathbb{P}^1}$ with saturated graded modules of the homogeneous coordinate ring $k[x, y]$, $\mathcal{S}_{a,b}$ is the ideal of $k[x, y]$ generated by $x^a y^b$. The quotient $\mathcal{O}_{\mathbb{P}^1}/\mathcal{S}_{a,b}$ is the skyscraper sheaf

$$\mathcal{T}_{a,b} = \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_0^a \oplus \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_\infty^b$$

on $\mathbb{P}^1$ supported at 0 and at ∞. Here, $\mathfrak{m}_p$ is the sheaf of ideals cutting out the point $p \in \mathbb{P}^1$. Then we have

$$\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = \bigoplus_{\delta_i=1} \mathcal{S}_{a_i,b_i} \cdot \mathbf{e}_i , \qquad \text{and}$$

$$\mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} = \bigoplus_{\delta_j=1} \mathcal{T}_{a_j,b_j} \cdot \mathbf{e}_j \oplus \bigoplus_{\delta_j=0} \mathcal{O}_{\mathbb{P}^1} \cdot \mathbf{e}_j .$$

In the sum, $\delta_j = 0$ means those j in $\{1, \ldots, n\}$ with $\delta_j = 0$, and the same for $\delta_j = 1$.

The tangent space to $\mathcal{Q}_d$ at this fixed point is $\text{Hom}(\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}, \mathcal{T}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})})$. Let $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_n$ be the basis dual to $\mathbf{e}_1, \ldots, \mathbf{e}_n$. For each i, j, set $\mathbf{E}_{ij} := \hat{\mathbf{e}}_i \otimes \mathbf{e}_j \in \text{Hom}(k^n, k^n)$. These $\mathbf{E}_{ij}$ form a basis for $\text{Hom}(k^n, k^n)$.

Theorem 2. *The tangent space $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ is canonically identified with*

$$(3) \qquad \bigoplus_{\delta_i=1}\bigoplus_{\delta_j=0} \text{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{O}_{\mathbb{P}^1}) \cdot \mathbf{E}_{ij} \quad \oplus \quad \bigoplus_{\delta_i=1}\bigoplus_{\delta_j=1} \text{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{T}_{a_j,b_j}) \cdot \mathbf{E}_{ij} .$$

We now give T-bases (bases of T-eigenvectors) for these summands and determine the corresponding weights. Fix a basis for the character group of T as follows. Extend the action of T_{k^n} on $\mathcal{O}^n_{\mathbb{P}^1}$ to T by letting the factor $T_{\mathbb{P}^1}$ act trivially. Then we abuse notation and denote the character of T acting on the ith basis vector $\mathbf{e}_i$ by the same symbol $\mathbf{e}_i$. Thus the dual basis element $\hat{\mathbf{e}}_i$ has T-weight $-\mathbf{e}_i$.

Similarly, extend the action of $T_{\mathbb{P}^1}$ on $\mathbb{P}^1$ to an action of T by letting T_{k^n} act trivially, and denote by $\mathbf{f}$ the character of T corresponding to the action on the dense orbit $\mathbb{P}^1 \setminus \{0, \infty\}$. More precisely, we can let $T_{\mathbb{P}^1} \cong \mathbb{C}^*$ act on the homogeneous coordinates

$k[x, y]$ of $\mathbb{P}^1$ by $q \cdot x = qx$ and $q \cdot y = y$. Thus T acts on the rational function $z := x/y$ with weight $\mathbf{f}$, and on the monomial $z^a = x^a y^{-a}$ with weight $a \mathbf{f}$.

The first sum of (3) involves spaces of the form $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{O}_{\mathbb{P}^1}) = H^0(\mathcal{S}_{a,b}^*)$. This space of sections has a monomial T-basis

$$\{z^c \mid -a \leq c \leq b\}.$$

For example, $H^0(\mathcal{S}_{1,2}^*) = k \cdot \{z^{-1}, z^0 = 1, z, z^2\}$. Thus if $\delta_i = 1$ and $\delta_j = 0$, then the piece $\mathrm{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{O}_{\mathbb{P}^1}) \cdot \mathbf{E}_{ij}$ of the tangent space has a monomial T-basis $z^c \cdot \mathbf{E}_{ij}$ for all $-a_i \leq c \leq b_i$. The basis element $z^c \cdot \mathbf{E}_{ij}$ has T-weight

(4) $\mathbf{e}_j - \mathbf{e}_i + c \mathbf{f} \,.$

The second sum of (3) involves spaces of the form $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$. Since $\mathcal{T}_{\alpha,\beta}$ is a skyscraper sheaf supported at 0 and ∞, a map $\phi \in \mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$ is determined by its actions at 0 and at ∞. At 0, $z = x/y$ is a local parameter, so the map ϕ becomes

$$\phi : \ z^a \mathbb{C}[z] \to \mathbb{C}[z]/\langle z^\alpha \rangle \,,$$

and thus has the form $z^{-a} f(z)$ where $f(z)$ has degree less than α. At ∞, z^{-1} is a local parameter, and the map ϕ has the form $z^b g(z^{-1})$ where $g(z)$ is a polynomial of degree less than β. Thus $\mathrm{Hom}(\mathcal{S}_{a,b}, \mathcal{T}_{\alpha,\beta})$ has the monomial T-basis

$$k \cdot \{z^{(\alpha - c) - a} \mid 1 \leq c \leq \alpha\} \ \oplus \ k \cdot \{z^{b - (\beta - c)} \mid 1 \leq c \leq \beta\} \,,$$

where elements in the first summand act by zero on the stalk at ∞, and elements of the second summand act by zero at 0.

Thus, if $\delta_i = \delta_j = 1$, then the summand $\mathrm{Hom}(\mathcal{S}_{a_i,b_i}, \mathcal{T}_{a_j,b_j}) \cdot \mathbf{E}_{ij}$ of (3) has a monomial T-basis

$$\{z^{(a_j - c) - a_i} \cdot \mathbf{E}_{ij} \mid 1 \leq c \leq a_j\} \cup \{z^{(b_j - c) - b_i} \cdot \mathbf{E}_{ij} \mid 1 \leq c \leq b_j\}$$

with corresponding T-weights

(5) $\mathbf{e}_j - \mathbf{e}_i + ((a_j - c) - a_i)\mathbf{f} \quad \text{and} \quad \mathbf{e}_j - \mathbf{e}_i + (b_i - (b_j - c))\mathbf{f} \,.$

We note that this discussion gives a basis for $T_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \mathcal{Q}_d$ consisting of

$$\sum_{\delta_j = 0} \sum_{\delta_i = 1} (a_i + b_i + 1) \quad + \quad \sum_{\delta_i = 1} \sum_{\delta_j = 1} (a_j + b_j) \ = \ (n - r) \cdot (d + r) + r \cdot d$$

$$= \ r \cdot (n - r) + nd \ = \ \dim \mathcal{Q}_d$$

elements, which shows that $\mathcal{Q}_d$ is smooth at the T-fixed point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$, and hence everywhere. In the next section, we will describe T-invariant curves in $\mathcal{Q}_d$ incident on $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$ whose tangent directions at $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$ coincide with this given T-basis of $T_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \mathcal{Q}_d$.

Example 3. The quot scheme $\mathcal{Q} := \mathcal{Q}_2(0, 2)$ of rank 0 and degree 2 quotients of $\mathcal{O}_{\mathbb{P}^1}^2$ has dimension $r(n-r) + dn = 2 \cdot 0 + 2 \cdot 2 = 4$. Note that the associated Grassmannian is a point. Since $r = 0$, the index $\boldsymbol{\delta}$ is the same for each fixed point, $\delta_1 = \delta_2 = 1$, and so the fixed points are indexed by quadruples (a_1, a_2, b_1, b_2) of non-negative integers whose sum is 2. Thus there are ten fixed points. We represent the fixed point (a_1, a_2, b_1, b_2)

by two columns of boxes superimposed on a horizontal line where the ith column has a_i boxes above the horizontal line and b_i boxes below it. For example,

$$\boxed{} \leftrightarrow (2,0,0,0), \qquad \boxed{} \leftrightarrow (1,1,0,0), \qquad \text{and} \qquad \boxed{} \leftrightarrow (1,0,0,1).$$

The fixed point corresponding to $\boxed{}$ is the exact sequence of sheaves on $\mathbb{P}^1$

$$0 \longrightarrow \begin{array}{c} \mathcal{O}(-2) \cdot \mathbf{e}_1 \\ \oplus \\ \mathcal{O} \cdot \mathbf{e}_2 \end{array} \begin{array}{c} \xrightarrow{x^2} \\ \\ \xrightarrow{1} \end{array} \begin{array}{c} \mathcal{O} \cdot \mathbf{e}_1 \\ \oplus \\ \mathcal{O} \cdot \mathbf{e}_2 \end{array} \longrightarrow \begin{array}{c} \mathcal{O}/\mathfrak{m}_0^2 \cdot \mathbf{e}_1 \\ \oplus \\ 0 \end{array} \longrightarrow 0,$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$. The tangent space at this fixed point is the sum of the two 2-dimensional T-invariant spaces of homomorphisms having the indicated T-bases.

$$\begin{aligned} \text{Hom}(\mathcal{O}, \mathcal{O}/\mathfrak{m}_0^2) \cdot \mathbf{E}_{21} &= k\{1, z\} \cdot \mathbf{E}_{21} \\ \text{Hom}(x^2\mathcal{O}, \mathcal{O}/\mathfrak{m}_0^2) \cdot \mathbf{E}_{11} &= k\{z^{-2}, z^{-1}\} \cdot \mathbf{E}_{11} \end{aligned}$$

As before, $z := x/y$ is a local parameter at 0 and z^{-1} is a local parameter at ∞. These basis elements have four distinct T-weights

$$\mathbf{e}_1 - \mathbf{e}_2, \ \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{f}, \ \text{and} \ -\mathbf{f}, \ -2\mathbf{f}.$$

The fixed point corresponding to $\boxed{}$ is the exact sequence of sheaves on $\mathbb{P}^1$,

$$0 \longrightarrow \begin{array}{c} \mathcal{O}(-1) \cdot \mathbf{e}_1 \\ \oplus \\ \mathcal{O}(-1) \cdot \mathbf{e}_2 \end{array} \begin{array}{c} \xrightarrow{x} \\ \\ \xrightarrow{y} \end{array} \begin{array}{c} \mathcal{O} \cdot \mathbf{e}_1 \\ \oplus \\ \mathcal{O} \cdot \mathbf{e}_2 \end{array} \longrightarrow \begin{array}{c} \mathcal{O}/\mathfrak{m}_0 \cdot \mathbf{e}_1 \\ \oplus \\ \mathcal{O}/\mathfrak{m}_\infty \cdot \mathbf{e}_2 \end{array} \longrightarrow 0.$$

The tangent space at this fixed point is the sum of four 1-dimensional T-invariant spaces of homomorphisms having bases and weights as indicated.

T-eigenspace	basis	T-weight
$\text{Hom}(x\mathcal{O}, \mathcal{O}/\mathfrak{m}_0) \cdot \mathbf{E}_{11}$	$z^{-1} \cdot \mathbf{E}_{11}$	$-\mathbf{f}$
$\text{Hom}(x\mathcal{O}, \mathcal{O}/\mathfrak{m}_\infty) \cdot \mathbf{E}_{12}$	$1 \cdot \mathbf{E}_{12}$	$\mathbf{e}_2 - \mathbf{e}_1$
$\text{Hom}(y\mathcal{O}, \mathcal{O}/\mathfrak{m}_0) \cdot \mathbf{E}_{21}$	$1 \cdot \mathbf{E}_{21}$	$\mathbf{e}_1 - \mathbf{e}_2$
$\text{Hom}(y\mathcal{O}, \mathcal{O}/\mathfrak{m}_\infty) \cdot \mathbf{E}_{22}$	$z \cdot \mathbf{E}_{22}$	$\mathbf{f}$

2. T-INVARIANT CURVES

We describe a collection of T-invariant curves whose tangent directions at a T-fixed point $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ form a basis for $T_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})} \mathcal{Q}_d$. A T-invariant curve on $\mathcal{Q}_d$ is, by definition, a flat family $\mathcal{S} \to \mathbb{P}^1$ with a T-action whose fibre $\mathcal{S}(s,t)$ over a point $[s,t] \in \mathbb{P}^1$ is a free subsheaf of $\mathcal{O}_{\mathbb{P}^1}^n$ of rank $n-r$ and degree $-d$. We exhibit each curve as a subsheaf $\mathcal{S}$ of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^n$ with a T-action that has rank $n-r$ and degree $(-d, -1)$ with $\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$ and $\mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')}$, where $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ is some other T-fixed point on $\mathcal{Q}_d$. Then $\mathcal{S}$ defines a T-invariant curve on $\mathcal{Q}_d$ connecting the two fixed points.

Each sheaf $\mathcal{S}$ has one of three types: I, II, or III. We describe them below and then argue that they have the desired properties. We write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$. We use the correspondence between sheaves over $\mathcal{O}$ and saturated modules over the bihomogeneous

coordinate ring $k[x,y][s,t]$ of $\mathbb{P}^1_{[x,y]} \times \mathbb{P}^1_{[s,t]}$. Then $\mathcal{O}^n$ is the rank n free module with basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

Type I: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another fixed point where the data $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ and $(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}')$ agree except in positions i and j, with $\delta_i = \delta_j' = 1$ and $\delta_j = \delta_i' = 0$. Note that $a_i + b_i = a_j' + b_j'$. Let $\mathcal{S}$ be the subsheaf of $\mathcal{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ except for its component in $\mathcal{O} \cdot \mathbf{e}_i + \mathcal{O} \cdot \mathbf{e}_j$, where it is the rank 1 and degree $-(a_i + b_i)$ subsheaf generated by the single element

$$(6) \qquad\qquad sx^{a_i}y^{b_i} \cdot \mathbf{e}_i + tx^{a_j'}y^{b_j'} \cdot \mathbf{e}_j \,.$$

Type II: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another T-fixed point where $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{b} = \mathbf{b}'$, and the data $\mathbf{a}$ and $\mathbf{a}'$ agree except in positions i and j with $i \neq j$. We suppose that i and j have been chosen so that $a_i < a_i'$. Then $a_j > a_j'$ and $c := a_i' - a_i = a_j - a_j' > 0$. Set $\gamma := a_i + b_i + c - a_j - b_j$.

If $\gamma \geq 0$, let $\mathcal{S}$ be the subsheaf of $\mathcal{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ except for its components in $\mathcal{O} \cdot \mathbf{e}_i + \mathcal{O} \cdot \mathbf{e}_j$, where it is the subsheaf generated by

$$(7) \qquad x^{a_i+c}y^{b_i} \cdot \mathbf{e}_i \,, \quad x^{a_j}y^{b_j} \cdot \mathbf{e}_j \,, \quad \text{and} \quad sx^{a_i}y^{b_i} \cdot \mathbf{e}_i + tx^{a_j-c}y^{b_j+\gamma} \cdot \mathbf{e}_j \,.$$

When $\gamma < 0$, replace the third generator by $sx^{a_i}y^{b_i-\gamma} \cdot \mathbf{e}_i + tx^{a_j-c}y^{b_j} \cdot \mathbf{e}_j$.

The remaining sheaves of type II are obtained by interchanging the roles of $\mathbf{a}$ and $\mathbf{b}$. That is, $\boldsymbol{\delta} = \boldsymbol{\delta}'$, $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b}, \mathbf{b}'$ agree except in positions $i \neq j$, and we further have that $c := b_i' - b_i = b_j - b_j' > 0$. Set $\gamma := a_i + b_i + c - a_j - b_j$. If $\gamma \geq 0$, let $\mathcal{S}$ be as before, except with the generators (7) replaced by

$$(8) \qquad x^{a_i}y^{b_i+c} \cdot \mathbf{e}_i \,, \quad x^{a_j}y^{b_j} \cdot \mathbf{e}_j \,, \quad \text{and} \quad sx^{a_i}y^{b_i} \cdot \mathbf{e}_i + tx^{a_j+\gamma}y^{b_j-c} \cdot \mathbf{e}_j \,.$$

If $\gamma < 0$, then the third generator will be $sx^{a_i-\gamma}y^{b_i} \cdot \mathbf{e}_i + tx^{a_j}y^{b_j-c}\mathbf{e}_j$.

Type III: Let $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ be another T-fixed point where $\boldsymbol{\delta} = \boldsymbol{\delta}'$ and the data $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{a}', \mathbf{b}')$ agree except in position i. Thus $\delta_i \neq 0$ and $a_i + b_i = a_i' + b_i'$. Let $\mathcal{S}$ be the subsheaf of $\mathcal{O}^n$ which agrees with both $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$, except for its component in $\mathcal{O} \cdot \mathbf{e}_i$, where it is the rank 1 and degree $-(a_i + b_i)$ subsheaf generated by

$$(9) \qquad\qquad sx^{a_i}y^{b_i} \cdot \mathbf{e}_i + tx^{a_i'}y^{b_i'} \cdot \mathbf{e}_i \,.$$

Theorem 4. *The subsheaves $\mathcal{S}$ of $\mathcal{O}^n_{\mathbb{P}^1 \times \mathbb{P}^1}$ of types I, II, and III are T-invariant and free of rank $n-r$ and degree $(-d, -1)$. They satisfy $\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$, and hence define T-invariant curves on $\mathcal{Q}_d$.*

Proof. The generators of $\mathcal{S}$ are T-invariant, except for those described by (6), (7), (8), and (9). But T acts transitively on those generators for $s \cdot t \neq 0$. Therefore, each sheaf $\mathcal{S}$ is T-invariant. In all cases, $\mathcal{S}$ has degree -1 with respect to $\mathbb{P}^1_{[s,t]}$.

The theorem is clear for the sheaves of types I and III, as they are constant on $\mathbb{P}^1_{[s,t]}$, except for the rank 1 components (6) and (9), each of which has degree $(-(a_i + b_i), -1)$. Specializing these generators at $[s,t] = [1,0]$ and $[0,1]$ shows that $\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ and $\mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$.

We use a Gröbner basis argument for the sheaves of type II. The Hilbert function for a submodule M of $\mathcal{O}^2$ equals the Hilbert function for the module of leading terms of

any Gröbner basis of M. As explained in [7, Ch. 15], a weight ω selecting these leading terms induces a $\mathbb{G}_m$-action on $\mathcal{O}^2$ whose restriction to the Gröbner basis of M generates a flat family over $\mathbb{A}^1$ whose special fibre is the module of leading terms.

For now, set $s = t = 1$. If $\mathbf{e}_i > \mathbf{e}_j$, then the generators (7) form a Gröbner basis for any position-over-monomial ordering, and the third generator has leading term $x^{a_i} y^{b_i} \cdot \mathbf{e}_i$. As $c > 0$, the module of leading terms is generated by $x^{a_i} y^{b_i} \cdot \mathbf{e}_i$ and $x^{a_j} y^{b_j} \cdot \mathbf{e}_j$, and so it has rank 2 and degree $-(a_i + b_i + a_j + b_j)$. The weight ω with $\omega(\mathbf{e}_i) = 0$ and $\omega(\mathbf{e}_j) = -1$ induces the leading terms and has corresponding $\mathbb{G}_m$-action $t.(\mathbf{e}_i, \mathbf{e}_j) = (\mathbf{e}_i, t\mathbf{e}_j)$, for $t \in \mathbb{G}_m$. This action on $\mathcal{S}(1,1)$ is the flat family of modules over $\mathbb{A}^1$ generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i , \quad x^{a_j} y^{b_j} \cdot \mathbf{e}_j , \quad \text{and} \quad x^{a_i} y^{b_i} \cdot \mathbf{e}_i + t x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j ,$$

which is just the part of $\mathcal{S}$ in $\mathcal{O} \cdot \mathbf{e}_i + \mathcal{O} \cdot \mathbf{e}_j$ restricted to the affine subset U of $\mathbb{P}^1_{[s,t]}$ where $s \neq 0$. Thus $\mathcal{S}|_U$ is a flat family over U of free subsheaves of $\mathcal{O}^n_{\mathbb{P}^1}$ of rank $n-r$ and degree $-d$, and $\mathcal{S}(1,0) = \mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$.

When $s = t = 1$ the generators (7) form a Gröbner basis when $\mathbf{e}_i < \mathbf{e}_j$, where the third generator has leading term $x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j$. The module of leading terms is generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i , \quad x^{a_j} y^{b_j} \cdot \mathbf{e}_j , \quad \text{and} \quad x^{a_j-c} y^{b_j+\gamma} \cdot \mathbf{e}_j .$$

Since $\gamma \geq 0$ and $c > 0$, saturating the ideal of $k[x,y]$ generated by $x^{a_j} y^{b_j}$ and $x^{a_j-c} y^{b_j+\gamma}$ by the irrelevant maximal ideal generated by x and y gives the ideal generated by $x^{a_j-c} y^{b_j}$. Thus the module of leading terms is generated by

$$x^{a_i+c} y^{b_i} \cdot \mathbf{e}_i , \quad \text{and} \quad x^{a_j-c} y^{b_j} \cdot \mathbf{e}_j .$$

As before, restricting $\mathcal{S}$ to the affine set of points $[s,t]$ of $\mathbb{P}^1$ where $t \neq 0$ gives a flat family of subsheaves of $\mathcal{O}^n_{\mathbb{P}^1}$ of rank $n-r$ and degree $-d$ with special fibre $\mathcal{S}(0,1) = \mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$. The same arguments suffice for the module generated by (8). $\qquad\square$

Theorem 5. *For any T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ in $\mathcal{Q}_d$, the set of tangent directions to the T-invariant curves induced by the sheaves $\mathcal{S}$ of types* I, II, *and* III *corresponds to the T-basis of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})} \mathcal{Q}_d$ defined in Section 1, and this correspondence is T-equivariant, respecting the weights. More specifically, at the T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$,*

    I. *The weight of the type* I *curve* (6) *is*

$$\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i) ,$$

    and such curves correspond to the first summand of (3).

    II. *The weight of the type* II *curve* (7) *is*

$$\mathbf{e}_j - \mathbf{e}_i + (a'_j - a_i) ,$$

    the weight of the type II *curve* (8) *is*

$$\mathbf{e}_j - \mathbf{e}_i + (b_i - b'_j) ,$$

    and such curves correspond to the second summand of (3) *when $i \neq j$.*

    III. *The weight of the type* III *curve* (9) *is*

$$(a'_j - a_i) ,$$

    and such curves correspond to the second summand of (3) *when $i = j$.*

Proof. In each of these curves, the T-fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is the point where $s = 1$ and $t = 0$. In what follows, we work locally, setting $s = 1$.

Note that the generator (6) of a type I sheaf may be rewritten

$$x^{a_i}y^{b_i} \cdot \left(\mathbf{e}_i + t \cdot x^{a'_j - a_i}y^{b'_j - b_i}\mathbf{E}_{ij}(\mathbf{e}_i)\right).$$

This shows that the tangent space to the curve when $t = 0$ is the T-basis element $x^{a'_j - a_i}y^{b'_j - b_i}\mathbf{E}_{ij}$ of $\mathrm{Hom}(\mathcal{S}_{a_i,b_i} \cdot \mathbf{e}_i, \mathcal{O}_{\mathbb{P}^1} \cdot \mathbf{e}_j)$. Thus the tangent spaces of type I curves at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ given by the first summand of (3). (Recall that in type I, we have $\delta_i = \delta'_j = 1$ and $\delta_j = \delta'_i = 0$.)

A similar analysis shows that the tangent space at $t = 0$ of the type III curve defined by (9) is spanned by $x^{a'_i - a_i}y^{b'_i - b_i}\mathbf{E}_{ii}$, and so the tangent spaces at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ of type III curves span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ given by the second summand of (3) when $i = j$.

For a curve of type II, note that the family of sheaves described by (7) is constant in a neighborhood of ∞. In a neighborhood of 0, it is given by

$$x^{a_i + c}\mathbf{e}_i, \quad x^{a_j}\mathbf{e}_j, \quad \text{and} \quad x^{a_i} \cdot \left(\mathbf{e}_i + tx^{(a_j - c) - a_i}\mathbf{E}_{ij}(\mathbf{e}_i)\right).$$

Thus $x^{(a_j - c) - a_i}\mathbf{E}_{ij}$ spans the tangent space at $t = 0$. A similar argument near ∞ for the sheaves described by (8) shows that the tangent spaces of type II curves at $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ span the component of $T_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}\mathcal{Q}_d$ given by the second summand of (3) when $i \neq j$. $\qquad\square$

A *moment graph* of a T-variety is a graph whose vertices correspond to T-fixed points and whose edges correspond to T-invariant curves, embedded into $\mathbb{R} \otimes \mathrm{Hom}(T, \mathbb{Z})$ so that the edge corresponding to a T-invariant curve is parallel to the weight of the action of T on the curve. More specifically, if C is a T-invariant curve joining fixed points p and q, then the edge from p to q in the moment graph is a positive multiple of the T-weight of T_pC. When $k = \mathbb{C}$ and we fix a Kähler form, there is a moment map $\mu \colon \mathcal{Q}_d \to \mathfrak{t}^*$ and the image of the T-fixed points and T-invariant curves is a moment graph.

When there are finitely many T-invariant curves, the Goresky-Kottwitz-MacPherson method to compute equivariant cohomology is conveniently expressed in terms of a moment graph, with one relation for each edge. When there are infinitely many T-invariant curves, there are additional relations coming from families of T-invariant curves, so it is better to work with the moment *multigraph*, where each family of T-invariant curves (which will appear as a connected component of parallel edges in the moment graph) is considered to form a single multiedge with more than 2 vertices, given by the fixed points in the closure of the family. To have a structure which determines the equivariant cohomology or Chow groups, we should label each multiedge with the topological type of the corresponding family.

Guillemin and Zara [12, 13, 14] have explored the combinatorial properties of moment graphs.

Example 6. Figure 1 represents a moment multigraph of $\mathcal{Q}_2(0, 2)$. Since the T-fixed points ⬜ and ⬜ have the same image in this and in any moment multigraph, we displace their images from their true positions for clarity. Similarly, some images of T-invariant curves are displaced or drawn as arcs.
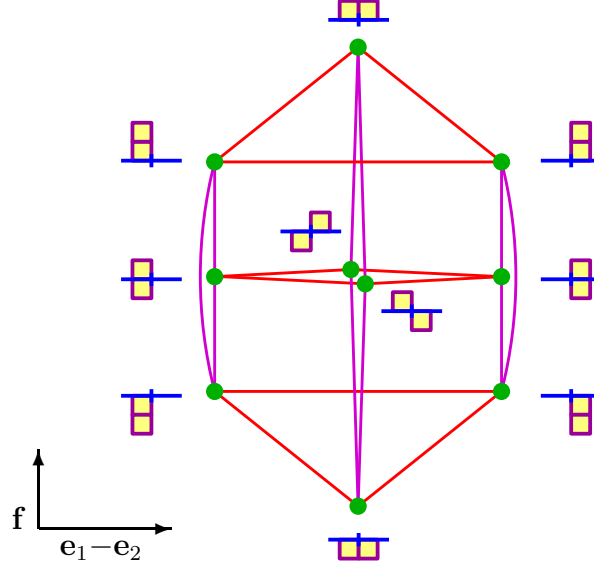
FIGURE 1. A moment multigraph of $\mathcal{Q}_2(0,2)$

The T-basis of the tangent space at ▢ has weights $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{f}, -\mathbf{f}, -2\mathbf{f}\}$. These correspond to the following four T-invariant curves:

Submodules of $\mathbf{e}_1\mathcal{O} \oplus \mathbf{e}_2\mathcal{O}$	$t = 0$	$s = 0$	Weight	Type
$x^2\mathbf{e}_1,\ x^2\mathbf{e}_2,\ s\mathbf{e}_1 + t\mathbf{e}_2$	$x^2\mathbf{e}_1,\ \mathbf{e}_2$	$\mathbf{e}_1,\ x^2\mathbf{e}_2$	$\mathbf{e}_1 - \mathbf{e}_2$	II
$x^2\mathbf{e}_1,\ x^2\mathbf{e}_2,\ sx\mathbf{e}_1 + ty\mathbf{e}_2$	$x^2\mathbf{e}_1,\ \mathbf{e}_2$	$x\mathbf{e}_1,\ x\mathbf{e}_2$	$\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{f}$	II
$(sx^2 + txy)\mathbf{e}_1,\ \mathbf{e}_2$	$x^2\mathbf{e}_1,\ \mathbf{e}_2$	$xy\mathbf{e}_1,\ \mathbf{e}_2$	$-\mathbf{f}$	III
$(sx^2 + ty^2)\mathbf{e}_1,\ \mathbf{e}_2$	$x^2\mathbf{e}_1,\ \mathbf{e}_2$	$y^2\mathbf{e}_1,\ \mathbf{e}_2$	$-2\mathbf{f}$	III

The T-basis to the tangent space at ▢ has weights $\{\pm\mathbf{f}, \pm(\mathbf{e}_2 - \mathbf{e}_1)\}$. These correspond to the following four T-invariant curves:

Submodules of $\mathbf{e}_1\mathcal{O} \oplus \mathbf{e}_2\mathcal{O}$	$t = 0$	$s = 0$	Weight	Type
$(sx + ty)\mathbf{e}_1,\ y\mathbf{e}_2$	$x\mathbf{e}_1,\ y\mathbf{e}_2$	$y\mathbf{e}_1,\ y\mathbf{e}_2$	$-\mathbf{f}$	III
$xy\mathbf{e}_1,\ y\mathbf{e}_2,\ sx\mathbf{e}_1 + tx\mathbf{e}_2$	$x\mathbf{e}_1,\ y\mathbf{e}_2$	$xy\mathbf{e}_1,\ \mathbf{e}_2$	$\mathbf{e}_2 - \mathbf{e}_1$	II
$x\mathbf{e}_1,\ xy\mathbf{e}_2,\ ty\mathbf{e}_1 + sy\mathbf{e}_2$	$x\mathbf{e}_1,\ y\mathbf{e}_2$	$\mathbf{e}_1,\ xy\mathbf{e}_2$	$\mathbf{e}_1 - \mathbf{e}_2$	II
$x\mathbf{e}_1,\ (sy + tx)\mathbf{e}_2$	$x\mathbf{e}_1,\ y\mathbf{e}_2$	$x\mathbf{e}_1,\ x\mathbf{e}_2$	$\mathbf{f}$	III

3. Families of T-invariant curves

Suppose that Z is a T-invariant curve on $\mathcal{Q}_d$. Let T' be the identity component of the stabilizer of Z in T, a codimension one subtorus of T. Then the action of T on Z factors through the quotient

$$T \longrightarrow T/T' \simeq \mathbb{G}_m.$$

This composition $\chi \colon T \to \mathbb{G}_m$ is a primitive weight parallel to the weight of the action of T on Z. Let Y be the component of the T'-fixed point locus $\mathcal{Q}_d^{T'}$ which contains Z. Then

$Y \setminus Y^T$ is foliated by one-dimensional orbits of T whose closures are T-invariant curves. We call Y the family of T-invariant curves on $\mathcal{Q}_d$ which contains Z. If $p \in Y^T \subset \mathcal{Q}_d^T$ is a T-fixed point of Y, then T_pY is a T-invariant linear subspace of $T_p\mathcal{Q}_d$ which is fixed pointwise by T'. In particular, all weights of the T-action on T_pY are multiples of χ.

To classify families of T-invariant curves, we first determine which T-weights of $T_p\mathcal{Q}_d$ are parallel. If the weight of a curve is not parallel to the weight of any other curve, it is isolated. By Theorem 5, it suffices to identify those T-invariant curves incident on a fixed point with parallel T-weights.

Theorem 7. *Two T-invariant curves $\mathcal{S}$, $\mathcal{S}'$ of types* I, II, *or* III *containing the fixed point $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ have parallel T-weights if and only if either*

    (1) *both $\mathcal{S}$ and $\mathcal{S}'$ have type* III *or*

    (2) *both $\mathcal{S}$ and $\mathcal{S}'$ have type* II, *and*

        (a) *$\mathcal{S}$ connects $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ to $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b})}$,*

        (b) *$\mathcal{S}'$ connects $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ to $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b}')}$,*

        (c) *$\mathbf{a}$ and $\mathbf{a}'$ agree except in positions i and j with $i \neq j$; $\mathbf{b}$ and $\mathbf{b}'$ agree except in positions i and j (same i,j); and $a_i + b_i = a'_j + b'_j$.*

Proof. For (1), note that the weight of T on a type III curve is parallel to $\mathbf{f}$.

If a curve does not have type III, then its weight has the form $\mathbf{e}_j - \mathbf{e}_i + c\mathbf{f}$, where $\delta_i = 1$ and either $\delta_j = 0$ if it has type I or $\delta_j = 1$ if it has type II. Thus $\mathcal{S}$ and $\mathcal{S}'$ have the same weight and type, $\boldsymbol{\delta} = \boldsymbol{\delta}'$, and the indices i and j in their definitions coincide.

Weights of curves of types I and II correspond to (4) and (5), respectively. Inspecting (4), shows that no two curves of type I can have the same weight. Inspecting (5) reveals that either a given curve $\mathcal{S}$ of type II has a unique weight, or else there is exactly one other type II curve $\mathcal{S}'$ with the same weight, and the two curves are as described in the statement of the theorem. $\qquad\square$

We show that all type III curves at a fixed point lie in a single family of T-invariant curves, and if two type II curves have the same weight then they lie in a 2-dimensional family. Together with the isolated T-invariant curves, this shows that the tangent spaces at a given fixed point T to families of T-invariant curves are the subspaces of $T_p\mathcal{Q}_d$ which are stabilized by codimension 1 subtori of T. It follows that these families contain all T-invariant curves in $\mathcal{Q}_d$.

Vertical families. A *vertical family* containing $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is parametrized by the product of projective spaces

$$\prod_{\delta_i=1} \mathbb{P}H^0(\mathcal{O}(a_i + b_i)) \;\simeq\; \prod_{\delta_i=1} \mathbb{P}^{a_i+b_i}.$$

It contains exactly the fixed points $\mathcal{S}_{(\boldsymbol{\delta}',\mathbf{a}',\mathbf{b}')}$ where $\boldsymbol{\delta} = \boldsymbol{\delta}'$, and $\mathbf{a}' + \mathbf{b}' = \mathbf{a} + \mathbf{b}$, along with all type III curves which connect them. These include all type III curves at each of these fixed points.

Consider the family $\mathcal{S}$ of submodules of $\mathcal{O}_{\mathbb{P}^1}^n$ generated by

$$\{\mathbf{e}_i s_i \mid \delta_i = 1, \quad s_i \in H^0(\mathcal{O}(a_i + b_i))\}.$$

The base of this family is $\prod_{\delta_i=1} \mathbb{P}H^0(\mathcal{O}(a_i + b_i))$, all subsheaves have rank $n - r$ and degree $-d$, and the foliation by T-invariant curves is given by the T-action on the base.

Horizontal families. If there exist i, j, c, c' such that $1 \leq c \leq a_j$, $1 \leq c' \leq b_j$, and $a_i + b_i + c + c' = a_j + b_j$, then the point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$ lies in a *horizontal family* parametrized by the product of two projective lines. Let $a_i' = a_i + c$, $a_j' = a_j - c$, $b_i' = b_i + c'$, $b_j' = b_j - c'$. Let $([s, t], [\sigma, \tau])$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathcal{S}$ be the submodule of $\mathcal{O}_{\mathbb{P}^1}^n$ which, except for its components in $\mathcal{O} \cdot \mathbf{e}_i + \mathcal{O} \cdot \mathbf{e}_j$, agrees with $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$. The component of $\mathcal{S}$ in $\mathcal{O} \cdot \mathbf{e}_i + \mathcal{O} \cdot \mathbf{e}_j$ is the subsheaf generated by

$$\mathbf{e}_j x^{a_j} y^{b_j}, \quad s\mathbf{e}_i x^{a_i'} y^{b_i} + t\mathbf{e}_j x^{a_j} y^{b_j'}, \quad \sigma\mathbf{e}_i x^{a_i} y^{b_i'} + \tau\mathbf{e}_j x^{a_j'} y^{b_j}, \quad \mathbf{e}_i x^{a_i'} y^{b_i'}.$$

Similar reasoning as for Theorem 4 shows that this defines a family of T-invariant curves over the base $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $([s, t], [\sigma, \tau])$. It contains four T-fixed points: setting $t = \tau = 0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})}$, setting $t = \sigma = 0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b})}$, setting $s = \tau = 0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b}')}$, and setting $s = \sigma = 0$ gives the fixed point $\mathcal{S}_{(\boldsymbol{\delta}, \mathbf{a}', \mathbf{b}')}$. This family also contains the four Type II curves connecting these four fixed points, given by setting exactly one of s, t, σ, or τ equal to zero. Furthermore, the data $(\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$ and $(\boldsymbol{\delta}', \mathbf{a}', \mathbf{b}')$ satisfy Theorem 7(2)(c), and any two T-invariant curves $\mathcal{S}$ and $\mathcal{S}'$ as in Theorem 7(2) lie in a unique horizontal family.

4. Algebraic extension of GKM theory

We discuss equivariant localization and an extension of the Goresky-Kottwitz-MacPherson relations when there are finitely many fixed points but infinitely many T-invariant curves. We work with equivariant Chow rings; similar results hold for equivariant cohomology. In fact, when $k = \mathbb{C}$, X is smooth and projective, and X^T is finite, the two theories coincide.

We first recall some properties of T-equivariant Chow rings as developed by Edidin and Graham [5] and Brion [3]. Next, we outline Evain's [8] development of ideas of Brion which extends the GKM relations to describe the T-equivariant Chow ring of a smooth variety with finitely many T-fixed points when there are infinitely many T-invariant curves. This description involves ideal-membership relations, one for each generator of the equivariant Chow ring of each family of T-invariant curves. When the generators are given by smooth subvarieties, these relations may be expressed in terms of tangent weights. This gives one form of our presentation for $A_T^*(\mathcal{Q}_d)$ in Theorem 1. We next give a variant of these relations using differential operators, which gives the other form of our presentation for $A_T^*(\mathcal{Q}_d)$. We then compute these relations for products of projective spaces, and finally deduce Theorem 1.

4.1. **Torus equivariant Chow rings.** When a linear algebraic group G acts on a smooth scheme X, Edidin and Graham [5] defined the equivariant Chow ring $A_G^*(X)$, using Totaro's algebraic approximation to the classifying space of G. It satisfies functorial properties under equivariant maps analogous to those for ordinary Chow rings [9], including proper pushforwards and pullbacks by local complete intersection morphisms.

When the group is a torus T, Brion [3] gave an alternative development of this theory which includes versions of the localization theorems that hold for equivariant cohomology.

He gave the following presentation for the equivariant Chow ring, analogous to the usual presentation of Chow groups. The equivariant Chow ring $A_T^*(p)$ of a point p is the integral symmetric algebra S of the character group $\hat{T}$ of T. Equivariant pullback makes $A_T^*(X)$ into an S-module.

Proposition 8 (Theorem 2.1 [3]). *The S-module $A_T^*(X)$ is defined by generators $[Y]$, for each T-invariant subvariety Y, and by relations $[div_Y(f)] - \chi[Y]$ for each rational function f on Y which is a T-eigenvector of weight χ; here χ is considered as an element of S in degree 1.*

It follows immediately that the usual Chow ring may be recovered from the S-module $A_T^*(X)$, as the quotient by the ideal S^+ of S generated by the character group $\hat{T}$.

Proposition 9 (Corollary 2.3.1 [3]). *We have*
$$A^*(X) \;=\; A_T^*(X) \otimes_S \mathbb{Z} \;=\; A_T^*(X)/S^+ A_T^*(X)\,.$$

The analogous statement in equivariant cohomology requires stronger hypotheses.

When $k = \mathbb{C}$, the connection between Chow groups and cohomology is given by the cycle map
$$A_T^*(X) \;\to\; H_{T,c}^*(X, \mathbb{Z})$$
to compactly supported (Borel-Moore) equivariant cohomology. If X is projective and the fixed point set X^T is finite, then the cycle map is an isomorphism.

Some statements below hold only for the rational equivariant Chow ring $A_T^*(X)_{\mathbb{Q}} := A_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. This is a module over the rational equivariant Chow ring $A_T^*(p)_{\mathbb{Q}}$ of a point p, which is the symmetric algebra $S_{\mathbb{Q}}$ of $\hat{T}_{\mathbb{Q}} := \hat{T} \otimes_{\mathbb{Z}} \mathbb{Q}$.

4.2. **Localization.** We now assume that X^T is finite, and that X has a decomposition into T-invariant affine cells $C_1, \ldots, C_m$ which can be ordered so that for $i = 1, \ldots, m$, the union $C_1 \cup \cdots \cup C_i$ is Zariski open. We will call such varieties *filtrable*; this is close to the terminology Brion used in [3], but he did not require that X^T be finite, and his cells were allowed to be vector bundles over components of X^T. If X is smooth and projective and X^T is finite, then Białynicki-Birula [2] showed that it is filtrable.

Let $i \colon X^T \to X$ be the inclusion of the subscheme of T-fixed points of X.

Proposition 10 (Corollary 3.2.1 [3]). *The S-module $A_T^*(X)$ is free. The map*
$$i^* \colon A_T^*(X) \to A_T^*(X^T)$$
is an injection.

Brion also established Chow ring versions of results of Chang and Skjelbred and of Goresky, Kottwitz, and MacPherson concerning the image of the localization map.

Proposition 11 (Sections 3.3 and 3.4 [3]).
  (a) *The image of the localization map $i^* A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is the intersection of the images of the localization maps*
$$i_{T'}^* \;\colon\; A_T^*(X^{T'})_{\mathbb{Q}} \;\to\; A_T^*(X^T)_{\mathbb{Q}}$$
  where T' runs over all codimension one subtori of T.

(b) *When T acts with finitely many fixed points and has finitely many invariant curves, then the image of the localization map $i^*\colon A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}} \simeq (S_{\mathbb{Q}})^{X^T}$ is the set of all tuples $(f_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ such that whenever p and q belong to the same irreducible T-invariant curve C, we have $f_p \equiv f_q$ modulo χ, where χ is the weight of the action of T on T_pC.*

Statement (a) is analogous to a theorem of Chang and Skjelbred [4] for equivariant cohomology. This result, together with the easy calculation of the equivariant Chow groups of $\mathbb{P}^1$, immediately gives (b), which is the Chow analog of the GKM relations for equivariant cohomology.

In general this result does not hold with $\mathbb{Z}$ coefficients. For instance, suppose that $\dim X = 2$, $x \in X^T$, and the weights of T on the tangent space T_xX are $a\chi$ and $a'\chi'$, where $\chi, \chi' \in \hat{T}$ are linearly independent primitive characters. Then condition (b) would say that if $(f_p) \in A_T^*(X^T)$ has $f_p = 0$ for $p \neq x$, then it is in the image of i^* if f_x is a multiple of $\operatorname{lcm}(a, a')\chi\chi'$. In fact, f_x must be a multiple of $aa'\chi\chi'$.

This is essentially the only obstruction to working with $\mathbb{Z}$ coefficients, at least if the fixed point set is finite. We say that the tangent weights of a T-variety X are *almost coprime* if whenever two T-weights of T_pX for $p \in X^T$ are divisible by the same integer $a > 1$, then they are parallel. With this added hypothesis, Brion's proof of Proposition 11 given by Brion works over $\mathbb{Z}$.

Theorem 12. *Let X be a smooth filtrable T-variety whose tangent weights are almost coprime. Then Proposition* 11 *holds with rational Chow groups replaced by integral Chow groups.*

4.3. **Evain's relations.** When T does not have finitely many invariant curves on X, then statement (b) of Proposition 11 fails, but by (a) we can still compute $A_T^*(X)$ if we know the images of $i_{T'}^*$ for all codimension one subtori T' of T. A finite set of such T' suffices, namely those which fix at least one T-invariant curve pointwise. Brion [3] and Goldin and Holm [10] have computed cases where the components of $X^{T'}$ are low-dimensional. Evain [8] recently described relations in the general case: we recall his results.

Let $Y = X^{T'}$ for T' a codimension one subtorus of T. By [15], Y is smooth. For $p \in Y^T = X^T$, let $e_p^T(Y) = e^T(TY)|_p$ be the localization of the equivariant Euler class of TY at p. Under the identification $A_T^*(p) = S$, this is the product of the T-weights on the tangent space T_pY.

Proposition 13 (Corollary 27 [8]). *A class $\alpha = (\alpha_p)_{p \in Y^T}$ in S^{Y^T} lies in $i_{T'}^* A_T^*(Y)$ if and only if*

(10) $$\sum_{p \in Y^T} \frac{\alpha_p \beta_p}{e_p^T(Y)} \quad \in \quad S$$

for every $\beta \in i_{T'}^* A_T^*(Y)$.

Remark on Evain's proof. The condition (10) is necessary, since if π is the projection of Y to a point, then the sum is simply $\pi_*(\alpha \cdot \beta)$, by the integration formula of Edidin

and Graham [6]. Note that since π_* is S-linear, it is enough to take β in a generating set of the S-module $i^*A_T^*(Y)$.

By Białynicki-Birula [2], there are two T-invariant cell decompositions C_p^+ and C_p^- for $p \in Y^T$ of X and an ordering of the fixed points Y^T such that the matrix with entries in S whose (p, q)-entry is

$$\pi_*([C_p^+] \cdot [C_q^-])$$

is unitriangular. Either set of classes $[C_p^+]$ or $[C_p^-]$ forms a basis for the S-module $A_T^*(X)$, and expressing the elements α and β in these two bases proves sufficiency.

Combining Proposition 13 with Proposition 11 and Theorem 12 gives the following criterion for membership in $i^*A_T^*(X)$.

Theorem 14. *A class* $\alpha = (\alpha_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ *lies in the image* $i^*A_T^*(X)_{\mathbb{Q}}$ *of the localization map if and only if for all* $Y = X^{T'}$ *for* T' *a codimension one subtorus of* T *we have*

$$\sum_{p \in Y^T} \frac{\alpha_p \beta_p}{e_p^T(Y)} \;\in\; S_{\mathbb{Q}}$$

for all β *in a set of* $S_{\mathbb{Q}}$-*module generators for* $i_{T'}^* A_T^*(Y)_{\mathbb{Q}}$.
If the tangent weights of X *are almost coprime, the same statement holds over* $\mathbb{Z}$.

Remark. When X is smooth, the relations in Theorem 14 can also be taken for Y running over all irreducible components of the union of the fixed points and the T-invariant curves, since such Y are just the connected components of the T'-fixed loci $X^{T'}$ for some codimension one subtorus T' of T. We call this union of fixed points and T-invariant curves the *one-skeleton of* X.

To apply Theorem 14, we need to know explicit generators of $A_T^*(Y)$, or more precisely their localizations to Y^T. By Proposition 8, one class of generators are the equivariant fundamental cycles $[Z]$ of T-invariant subvarieties Z of the components Y. These are easy to compute when Z is smooth, since if $p \in Z^T$ we have $[Z]_p = e_p^T(N_Z Y)$, the equivariant Euler class of the normal bundle to Z in Y, while if $p \in Y^T \setminus Z^T$, then $[Z]_p = 0$. It follows that $\frac{[Z]_p}{e_p^T(Y)} = \frac{1}{e_p^T(Z)}$ if $p \in Z^T$.

To see this, note that $[Z]_p \in A_T^*(p)$ is the pullback of $[Z]$ along the regular embedding $i_{p,Y} : p \to Y$. We factor $i_{p,Y}$ as the composition

$$p \xrightarrow{\; i_{p,Z} \;} Z \xrightarrow{\; i_Z \;} Y \,.$$

The class $[Z] \in A_T^*(Y)$ is the pushforward along i_Z of the unit class $1 = [Z] \in A_T^*(Z)$, and so we have

$$[Z]_p \;=\; i_{p,Z}^* i_Z^* i_{Z,*} 1 \;=\; i_{p,Z}^* e^T(N_Z Y) \;=\; e_p^T(N_Z Y) \,,$$

by the self-intersection formula for Chow rings.

Thus if we can find for each Y a collection $\mathcal{Z}_Y$ of smooth T-invariant subvarieties of Y so that the classes $[Z]$ for $Z \in \mathcal{Z}_Y$ generate $A_T^*(Y)$ as an S-module, we get the following more explicit version of Theorem 14.

Theorem 15. *A class $\alpha = (\alpha_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ lies in $i^* A_T^*(X)_{\mathbb{Q}}$ if and only if*

$$(11) \qquad \sum_{p \in Z^T} \frac{\alpha_p}{e_p^T(Z)} \; \in \; S_{\mathbb{Q}}$$

for all $Z \in \mathcal{Z}_Y$ and all components Y of the one-skeleton of X. If the tangent weights of X are almost coprime, the same statement holds over $\mathbb{Z}$.

The necessity of (11) does not require the argument above, since if $(\alpha_p) = i_* \alpha$ for $\alpha \in A_T^*(X)$, then the sum is just $\pi_*(\alpha|_Z)$, where π is the projection of Z to a point.

Obvious candidates for the subvarieties $[Z]$ are the closures of the Białynicki-Birula cells, since their classes form an S-basis for $A_T^*(Y)$. Unfortunately, they are not in general smooth—this was the case for Evain. However, for the quot schemes we study, they are smooth, as the connected components Y are products of projective spaces. More generally we can ask that for each component Y of $X^{T'}$ there is a torus T_Y containing T which acts on Y with finitely many orbits, so that Y is a smooth toric variety. The closures of the cells will be T_Y-orbit closures, and therefore smooth.

The relations of Theorem 15 are the same as those found by Goldin and Holm [10] for equivariant cohomology of Hamiltonian T-spaces. Their result applied where the spaces $X^{T'}$ are at most four-dimensional (over $\mathbb{R}$). Since this implies that the components of $X^{T'}$ are toric manifolds, Theorem 15 recovers their result.

4.4. Evain's relations as differential operators.

We rewrite this algebraic criterion in a different form. Suppose that Y is a smooth component of $X^{T'}$ and $Z \subset Y$ is a smooth T-invariant subvariety. The action of T on Y factors through a character $\eta \colon T \to \mathbb{C}^*$, so the weights of T on $T_p Z$ for $p \in Z^T$ are non-zero scalar multiples of η. Thus there exist numbers $d_p = d_p(Z)$ so that

$$e_p^T(Z) = d_p(Z) \cdot \eta^{\dim Z} .$$

The terms in (11) have a common denominator $\eta^{\dim Z}$, and so we may rewrite it as

$$\sum_{p \in Z^T} \frac{\alpha_p}{d_p(Z)} \; \in \; \eta^{\dim Z} S_{\mathbb{Q}} .$$

We can rewrite this condition using a linear differential operator. The ring $S_{\mathbb{Q}}$ is the symmetric algebra of $\hat{T}_{\mathbb{Q}}$, or dually the ring of polynomial functions on $\hat{T}_{\mathbb{Q}}^*$. Choose $\zeta \in \hat{T}_{\mathbb{Q}}^*$ for which $\zeta(\eta) \neq 0$. Then the operator $D = D_\zeta$ of differentiation in the direction of ζ acts on $S_{\mathbb{Q}}$. If $f \in S_{\mathbb{Q}}$ is divisible by η, then η^k divides f if and only if η^{k-1} divides Df, so the relation (11) is equivalent to

$$\sum_{p \in Z^T} d_p(Z)^{-1} D^j \alpha_p \equiv 0 \mod \eta, \quad \text{for all } 0 \leq j < \dim Z.$$

We give a variant of Theorem 15 which uses the last relation, but only with the maximum order derivative $j = \dim Z - 1$. In exchange, we must apply it using more subvarieties Z.

Let Y be a component of the one-skeleton of X, let η be the associated character of T, and consider the two Białynicki-Birula cell decompositions $\{C_p^- \mid p \in Y^T\}$ and

$\{C_p^+ \mid p \in Y^T\}$ induced by the T-action. Each cell $C = C_p^+, C_p^-$ is isomorphic to the T-vector space $T_pC \subset T_pY$. Suppose that within each cell $C = C_p^+$ we can find T-invariant affine subspaces $C_{p,1}, \ldots C_{p,\dim C}$ with $\dim C_{p,i} = i$ and which have smooth closures $Z_{p,i} = \overline{C_{p,i}}$. As before, this will be true if each Y is a toric variety for a larger torus T_Y containing T, since we can take each $Z_{p,i}$ to be the closure of a T_Y-orbit.

Theorem 16. *With these assumptions, a class $\alpha = (\alpha_p)_{p \in X^T} \in (S_{\mathbb{Q}})^{X^T}$ lies in $i^* A_T^*(X)_{\mathbb{Q}}$ if and only if*

$$(12) \qquad \sum_{q \in Z^T} d_q(Z)^{-1} D^{\dim Z - 1} \alpha_q \equiv 0 \mod \eta.$$

for all $Z = Z_{p,i}$ and for all components Y of the one-skeleton of X.

Proof. The necessity of the conditions (12) follows from the previous discussion.

To show they are sufficient, let U be an open union of the cells C_p^- and note that $Z_{p,i} \subset U$ if and only if $p \in U$. We use induction on the number of cells in U to show that the image of $i_U^* : A_T^*(U)_{\mathbb{Q}} \to A_T^*(U^T)_{\mathbb{Q}}$ is the set of $(\alpha_x)|_{x \in U^T}$ satisfying (12) for all $Z_{p,i} \subset U$.

When U is a single cell, this is immediate, as i_U^* is an isomorphism and there are no $Z_{p,i}$'s contained in U. Otherwise, suppose $\alpha = (\alpha_x)|_{x \in U^T}$ satisfies (12) for all $Z_{p,i} \subset U$. Let $C_p^- \subset U$ be a closed cell, and put $U' = U \setminus C_p^-$. There is an exact sequence[3, Proposition 3.2]

$$(13) \qquad 0 \to A_T^*(U)_{\mathbb{Q}} \xrightarrow{\rho} A_T^*(U')_{\mathbb{Q}} \times A_T^*(C_p^-)_{\mathbb{Q}} \to A_T^*(C_p^-)_{\mathbb{Q}}/(e^T(N)) \to 0 \,,$$

where $e^T(N)$ is the equivariant Euler class of the normal bundle N to C_p^- in X. Under the isomorphism $A_T^*(C_p^-) \cong S$, this is just the product of all the T-weights of N. The components of ρ are the restriction maps, while the map $A_T^*(C_p^-)_{\mathbb{Q}} \to A_T^*(C_p^-)_{\mathbb{Q}}/(e^T(N))$ is the natural quotient.

By the inductive hypothesis, $\alpha|_{(U')^T}$ lies in the image of $i_{U'}^*$. The exact sequence (13) implies that $A_T^*(U) \to A_T^*(U')$ is surjective and so we can write $\alpha = i_U^*\beta + \gamma$, with $\beta \in A_T^*(U)$ and $\gamma|_{(U')^T} = 0$. Since $i_U^*\beta$ satisfies the relations (12), so does γ. It will be enough to show that γ is in the image of i_U^*. But using the exact sequence (13), we see that this holds if and only if γ_p is a multiple of $c_d^T(N)$, which is a non-zero multiple of η^d, where $d = \operatorname{codim} C_p^- = \dim C_p^+$. But the relation (12) implies that $D^k\gamma_p \equiv 0 \pmod \eta$ for $0 \le k < d$. The result follows. $\qquad \square$

Example 17. Let $T = \mathbb{G}_m$ act on $X = \mathbb{P}^r$ by

$$t \cdot [x_0 : x_1 : \cdots : x_r] = [x_0 : tx_1 : \cdots : t^r x_r]$$

in homogeneous coordinates, where $t \in T$. For each $0 \le j \le r$, let $p_j \in X^T$ denote the T-fixed point corresponding to the jth standard basis vector $\mathbf{e}_j$. The tangent space $T_{p_j}X$ is $\mathbb{C}^n/\mathbb{C} \cdot \mathbf{e}_j$, with the action of T given by $t \cdot \overline{\mathbf{e}}_k = t^{k-j}\overline{\mathbf{e}}_k$. Thus $e_{p_j}(X) = (-1)^j j! (r-j)! \eta^r$, where η is the identity character. More generally, if $0 \le l \le n \le r$, let $Z_{l,m} = \mathbb{P}\operatorname{Span}\{\mathbf{e}_l, \mathbf{e}_{l+1}, \ldots, \mathbf{e}_m\}$. The same calculation gives

$$e_{p_j}(Z_{l,m}) = (-1)^{j-l}(j-l)!(m-j)!\eta^{m-l} \,.$$

We can apply Theorem 15 using the smooth subvarieties $Z_{0,l}$, for $1 \leq l \leq r$. Then $i^* A_T^*(X) \subset A_T^*(X^T)$ is the set of tuples $\alpha = (\alpha_0, \ldots, \alpha_r)$ where

$$(14) \qquad \sum_{0 \leq j \leq l} \frac{(-1)^j \alpha_j}{j!(l-j)!} \ \in \ \eta^l S$$

for all $1 \leq l \leq r$.

On the other hand, we can apply Theorem 16 using all the subvarieties $Z_{l,m}$. If D is differentiation on S in the direction of $\eta^\vee$, then α lies in the image of i^* if and only if

$$(15) \qquad \sum_{l \leq j \leq m} \frac{(-1)^{j-l} D^{m-l-1} \alpha_j}{(j-l)!(m-j)!} \ \in \ \eta S$$

for all $0 \leq l < m \leq r$. We could take one more derivative and ask that the resulting sums vanish, but this would not generalize to actions of higher dimensional tori.

When $X = \mathbb{P}^1$ we get exactly the GKM relation for a primitive action.

Example 18. These same arguments apply to products of projective spaces. Let $\mathbf{r} = (r_1, \ldots, r_n)$, and let $X = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$, where the action of $t \in T = \mathbb{G}_m$ on a point $([x_0^1 : x_1^1 : \cdots : x_{r_1}^1], \ldots, [x_0^n : x_1^n : \cdots : x_{r_n}^n])$ is given by multiplying x_j^i by t^j. The fixed points have the form $p_{\mathbf{j}} = (p_{j_1}, \ldots, p_{j_n})$, where p_{j_i} is the j_ith fixed point in $\mathbb{P}^{r_i}$, in the notation of Example 17, and $\mathbf{j} = (j_1, \ldots, j_n)$ satisfies $0 \leq \mathbf{j} \leq \mathbf{r}$, meaning that $0 \leq j_i \leq r_i$ for all $1 \leq i \leq n$.

For $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$ with $0 \leq l_i \leq m_i \leq r_i$ for all i, set $Z_{\mathbf{l},\mathbf{m}} = Z_{l_1,m_1} \times \cdots \times Z_{l_n,m_n}$. For each $\mathbf{l} \leq \mathbf{j} \leq \mathbf{m}$, the tangent space to $Z_{\mathbf{l},\mathbf{m}}$ at $p_{\mathbf{j}}$ is $\bigoplus_{i=1}^n T_{p_{j_i}} Z_{l_i,m_i}$. Using the computation from Example 17, we see that

$$e_{p_{\mathbf{j}}}^T(Z_{\mathbf{l},\mathbf{m}}) \ = \ (-1)^{|\mathbf{j}|}(\mathbf{j}-\mathbf{l})!(\mathbf{m}-\mathbf{j})!\eta^{|\mathbf{m}|-|\mathbf{l}|} \, .$$

Recall that for an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$, we put $\mathbf{a}! = a_1! \cdots a_n!$.

As in Example 17, we can either apply Theorem 15 using the subvarieties $Z_{\mathbf{0},\mathbf{l}}$, or Theorem 16 using all the $Z_{\mathbf{l},\mathbf{m}}$. The resulting conditions for a tuple $\alpha = (\alpha_{\mathbf{j}})_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{r}}$ to be in the image of the localization map are just (14) and (15), where the variables now represent elements of $\mathbb{Z}^n$ rather than scalars.

4.5. **Proof of Theorem 1.** We combine these localization results with the geometry of the quot scheme from Sections 1, 2, and 3 to produce a proof of Theorem 1.

The only non-primitive tangent weights are those with $i = j$ in (5), which are multiples of $\mathbf{f}$; These correspond to Type III curves. Thus the tangent weights are almost coprime, so Theorem 15 gives a correct description of the integral equivariant Chow ring.

The relations I, II(a) and II(b) are the GKM relations for the T-invariant curves of types I and II, as described in Section 2 using the identification of the T-weights of these curves with the tangent weights given by Theorem 5.

The relations II(c)/II(c)$'$ come from the horizontal families of T-invariant curves of type II. As described in Section 3, these are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, where the action on each factor is by the same primitive character. As in Example 18, we can apply the relation in Theorem 15 with $Y = \mathbb{P}^1 \times \mathbb{P}^1$ to get the relations II(c)/II(c)$'$. We get one new relation using $Z = Y$; smaller T-invariant subvarieties contained in Y are either

T-invariant curves, whose relations are already covered by II(a) and II(b), or points, which give no relation.

Finally, the relations III and III$'$ come from the vertical families. As described in Section 3, the vertical family containing $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ is isomorphic to

$$\prod_{\delta_i=1} \mathbb{P}H^0(\mathcal{O}(a_i+b_i)) \;\simeq\; \prod_{\delta_i=1} \mathbb{P}^{a_i+b_i}\,,$$

and the fixed points in the family are those $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a}',\mathbf{b}')}$ with $\mathbf{a}'+\mathbf{b}' = \mathbf{a}+\mathbf{b}$. The codimension one subtorus $T_{k^n} \subset T$ acts trivially on the family, and the remaining action of $T_{\mathbb{P}^1}$ is the one described in Example 18, using the monomial basis of $H^0(\mathcal{O}(a_i+b_i))$. Example 18 then gives exactly the relations III and III$'$. This proves Theorem 1. $\square$

5. Equivariant Chern classes on $\mathcal{Q}_d$

Recall that $\mathbb{P}^1 \times \mathcal{Q}_d$ has a universal exact sequence of sheaves

$$0 \;\rightarrow\; \mathcal{S} \;\rightarrow\; \mathcal{O}^n_{\mathbb{P}^1 \times \mathcal{Q}_d} \;\rightarrow\; \mathcal{T} \;\rightarrow 0$$

with $\mathcal{S}$ the tautological vector bundle of rank $n-r$.

Since both $\mathcal{Q}_d$ and $\mathbb{P}^1$ have cell decompositions, we have a Künneth decomposition of Chow rings, $A^*(\mathbb{P}^1 \times \mathcal{Q}_d) \simeq A^*(\mathbb{P}^1) \otimes_{\mathbb{Z}} A^*(\mathcal{Q}_d)$. Let $\pi \colon \mathbb{P}^1 \times \mathcal{Q}_d \to \mathcal{Q}_d$ be the projection. For each $1 \le i \le n-r$ we may decompose the Chern class $c_i(\mathcal{S})$

$$(16) \qquad\qquad c_i(\mathcal{S}) \;=\; \pi^* t_i + h\pi^* u_{i-1}\,,$$

where $t_i, u_i \in A^i(\mathcal{Q}_d)$ and h is the class of a point in $A^1(\mathbb{P}^1)$ pulled back to $\mathbb{P}^1 \times \mathcal{Q}_d$. Note that $u_0 = -d$. Strømme [16] proved that $A^*(\mathcal{Q}_d)_{\mathbb{Q}}$ is generated by the classes

$$\{t_1,\ldots,t_k,u_1,\ldots,u_{k-1}\}\,.$$

For each $1 \le i \le n-r$, the equivariant Chern class $c_i^T(\mathcal{S})$ localizes at a fixed point p to the ith elementary symmetric polynomial e_i in the T-weights of the fibre of $\mathcal{S}$ at p. The fixed points of $\mathbb{P}^1 \times \mathcal{Q}_d$ correspond to $\{0,\infty\} \times \mathcal{F}$. For $(\boldsymbol{\delta},\mathbf{a},\mathbf{b}) \in \mathcal{F}$, the bundle $\mathcal{S}_{(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}$ on $\mathbb{P}^1$ is a sum of line bundles $\mathcal{S}_{a_j,b_j}\mathbf{e}_j$ for $\delta_j = 1$. Since $\mathcal{S}_{a,b}$ has weight $a\mathbf{f}$ at 0 and $-b\mathbf{f}$ at ∞, the localizations of $c_i^T(\mathcal{S})$ are

$$c_{i,0,(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}^T \;=\; e_i(\{\mathbf{e}_j + a_j\mathbf{f} \mid \delta_j = 1\}) \quad \text{at } (0,(\boldsymbol{\delta},\mathbf{a},\mathbf{b}))$$
$$c_{i,\infty,(\boldsymbol{\delta},\mathbf{a},\mathbf{b})}^T \;=\; e_i(\{\mathbf{e}_j - b_j\mathbf{f} \mid \delta_j = 1\}) \quad \text{at } (\infty,(\boldsymbol{\delta},\mathbf{a},\mathbf{b}))\,.$$

We also have a Künneth decomposition in equivariant Chow cohomology:

$$A_T^*(\mathbb{P}^1 \times \mathcal{Q}_d) \;\overset{\sim}{\longleftarrow}\; A_T^*(\mathbb{P}^1) \otimes_S A_T^*(\mathcal{Q}_d)\,.$$

To see this, just imitate the argument for ordinary Chow cohomology; the equivariant Chow cohomology of a variety with an algebraic cell decomposition will be a free S-module, with a module basis given by the closures of the cells.

An equivariant Künneth decomposition of $c_i^T(\mathcal{S})$ analogous to (16) requires the choice of a lift of the class of a point to $A_T^1(\mathbb{P}^1)$. Localizing a class $x \in A_T^1(\mathbb{P}^1)$ gives an ordered pair $(x_0, x_\infty) \in \mathbb{Z}\mathbf{f} \oplus \mathbb{Z}\mathbf{f}$. Lifts of classes from $A^1(\mathbb{P}^1)$ are only well-defined modulo the span of $(\mathbf{f},\mathbf{f})$. Three possible choices for lifting the class of a point are:

$$(17) \qquad\qquad (i)\ (-\mathbf{f},0)\,, \qquad (ii)\ (0,\mathbf{f})\,, \quad \text{and} \quad (iii)\ \tfrac{1}{2}(-\mathbf{f},\mathbf{f})\,.$$

The symmetric lift (iii) requires rational coefficients. We will use this lift to express our formulas.

Given a lift $h \in A_T^1(\mathbb{P}^1)$ of the class of a point, the formula (16) defines equivariant lifts of the classes t_i, u_{i-1}. Strømme's result together with Proposition 9 implies that these classes generate $A_T^*(X)_{\mathbb{Q}}$ as an $S_{\mathbb{Q}}$-algebra.

Proposition 19. *The symmetric choice (iii) of lift $h \in A_T^1(\mathbb{P}^1)$ of the class of a point give the formula for the Künneth components t_i, u_{i-1} of the equivariant Chern class $c_i^T(\mathcal{S})$ of the bundle $\mathcal{S}$. We express this in terms of its localization at the fixed point of $\mathcal{Q}_d$ indexed by $p := (\boldsymbol{\delta}, \mathbf{a}, \mathbf{b})$.*

$$t_{i,p} \;=\; \frac{1}{2}(c_{i,0,p}^T + c_{i,\infty,p}^T) \qquad u_{i-1,p} \;=\; \frac{1}{\mathbf{f}}(c_{i,\infty,p}^T - c_{i,0,p}^T)\,.$$

6. Equivariant Chow ring of the quot scheme $\mathcal{Q}_2(0,2)$

We use Theorem 1 to describe the equivariant Chow ring of $\mathcal{Q}_2 := \mathcal{Q}_2(0,2)$. We first give a basis for $A_T^*(\mathcal{Q}_2)$ as a module over $S = \mathbb{Z}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}]$.

The equivariant Chow ring is the collection of tuples $(f_p \mid p \in \mathcal{Q}_2^T) \in S^{\mathcal{Q}_2^T}$ which satisfy the relations of Theorem 1. If p and q are connected by an edge in the moment multigraph (see Figure 1) with weight χ, then $f_p - f_q$ lies in the ideal generated by χ. These are the standard GKM relations.

There are two multiedges with four vertices in the multigraph, namely the vertical and horizontal lines of symmetry. They should be seen as flattened quadrangles, since they are images of subvarieties isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Each gives rise to an additional relation, as follows. Suppose that the quadrangle has four vertices a, b, c, d:

$$\xrightarrow{\;\chi\;} \qquad a \underset{b}{\overset{c}{\diagdown\diagup}} d$$

(Here, the edges are parallel with direction χ.) Then the tuples (f_p) must satisfy

$$f_a - f_b - f_c + f_d \;\in\; \chi^2 S\,.$$

These are relations of types II(c)$'$ (horizontal) and III$'$ (vertical) of Theorem 1.

The remaining multiedges with more than two vertices are the left and right vertical edges, both with three vertices. They should be seen as flattened triangles, since they come from subvarieties isomorphic to $\mathbb{P}^2$. The additional relations they induce are described as follows. Let the three fixed points on the multiedge be a, b, c, with b between a and c. Then

$$f_a - 2f_b + f_c \;\in\; \mathbf{f}^2 S\,.$$

This is relation of type III$'$ of Theorem 1.

When $k = \mathbb{C}$, we can construct an S-module basis for equivariant cohomology using equivariant Morse theory; as is well-known, a generic projection of the moment map to a line will give a Morse function which is perfect for equivariant cohomology. This results in an inductive algorithm to produce a basis, which is nicely expounded in [17] (see also [12]). Pick a direction vector $\mathbf{v}$ (corresponding to an element of the Lie algebra $\mathbf{t}_{\mathbb{R}}$) which does not annihilate the direction vector of any edge. Orient each edge to have positive pairing with $\mathbf{v}$, this is the Hasse diagram of a partial order on the fixed points

induced by $\mathbf{v}$. Then, using the relations described above, we can inductively construct a triangular basis with respect to this ordering. That is, if $f = f(p)$ corresponds to the fixed point p, it vanishes at q ($f_q = 0$) unless $p < q$, and f_p is the product of weights of edges pointing down from p. While this algorithm was motivated by Morse theory, it makes sense over any field k, if $\mathbf{v}$ is a linear function on the character group of T which does not annihilate any edge of the moment graph.

Set $\mathbf{e} := \mathbf{e}_1 - \mathbf{e}_2$ and pick the vector $\mathbf{v} = \mathbf{f} + \epsilon\mathbf{e}$, where $\epsilon > 0$ is small. One basis element is the identity $f(\text{▭})$, which localizes to 1 at each fixed point. We display each of the remaining nine in Figure 2 as a localization diagram, writing its localizations on a copy of a moment multigraph.
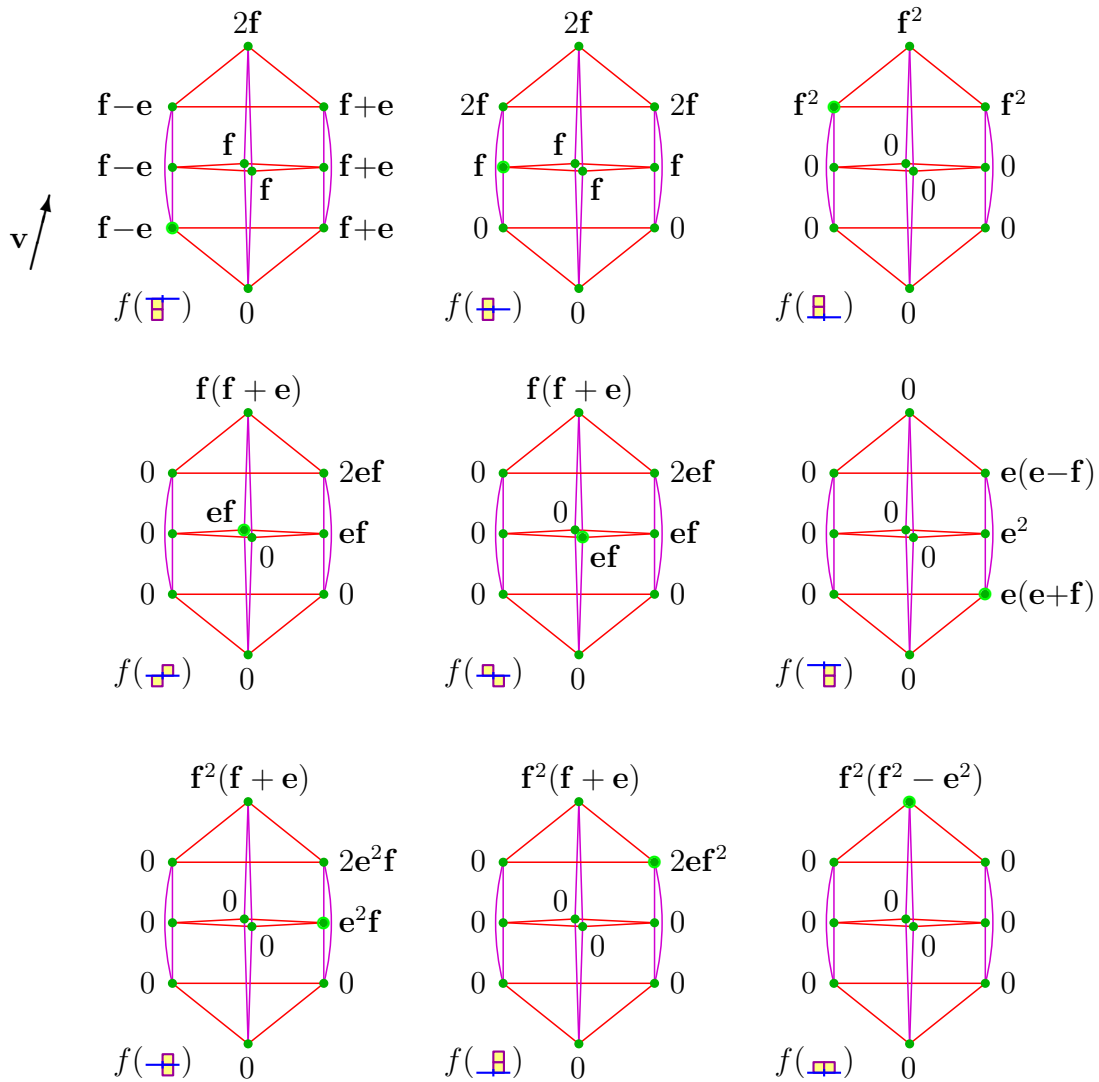
FIGURE 2. An S-module basis for $A_T^*(\mathcal{Q}_2(0,2))$.

Set $x := f(\boxplus) - \mathbf{f}$, $y := f(\boxplus) - \mathbf{f}$, and $z := f(\boxplus) - f(\boxplus)$. Figure 3 shows their localization diagrams. We show that they generate $A_T^*(\mathcal{Q}_2)_{\mathbb{Q}}$ as an $S_{\mathbb{Q}}$-algebra by showing
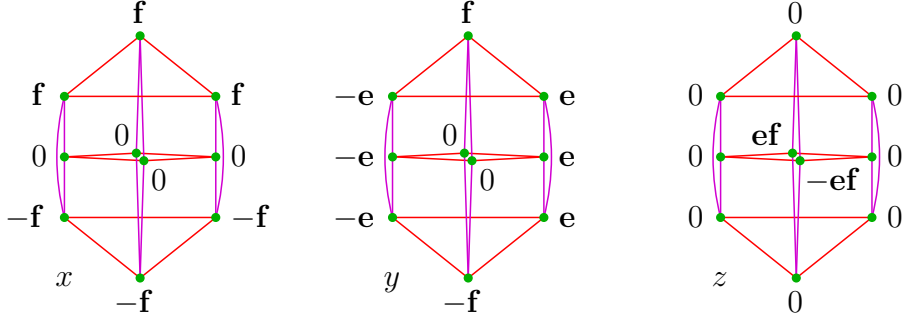

FIGURE 3. Generators for the $S_{\mathbb{Q}}$-algebra $A_T^*(\mathcal{Q}_2(0,2))_{\mathbb{Q}}$.

that each basis element $f(p)$ of degree greater than 2 lies in $S_{\mathbb{Q}}[x, y, z]$. Since

(18)
$$
\begin{aligned}
f(\boxplus) &= \tfrac{1}{2}x(x+\mathbf{f})\,, \\
f(\boxplus) &= \tfrac{1}{2}(y+\mathbf{e})(y-x)\,, \quad \text{and} \\
f(\boxplus) + f(\boxplus), &= (y+\mathbf{e})(x+\mathbf{f})\,,
\end{aligned}
$$

the four degree 2 basis elements lie in $S_{\mathbb{Q}}[x, y, z]$. The remaining three basis elements also lie in $S_{\mathbb{Q}}[x, y, z]$,

$$
yf(\boxplus) = f(\boxplus)\,, \qquad xf(\boxplus) = f(\boxplus)\,, \quad \text{and} \quad (y - \mathbf{e})f(\boxplus) = f(\boxplus)\,.
$$

Inspecting the localization diagrams of x, y, and z, shows that the following 5 expressions vanish in $A_T^*(\mathcal{Q}_2)$

(19) $\quad xz, \quad yz, \quad x(x^2 - \mathbf{f}^2), \quad (y^2 - \mathbf{e}^2)(y - x), \quad$ and $\quad z^2 - (y^2 - \mathbf{e}^2)(x^2 - \mathbf{f}^2)\,.$

In the lexicographic term order where $z > y > x > e > f$, these five polynomials form a Gröbner basis for the ideal $\mathcal{I}$ of $\mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, x, y, z]$ they generate with leading terms xz, yz, x^3, y^3, and z^2. There are ten standard monomials

$$
1, \ x, \ y, \ x^2, \ xy, \ y^2, \ z, \ x^2y, \ xy^2, \ \text{and} \ x^2y^2\,.
$$

Since $A_T^*(\mathcal{Q}_2)_{\mathbb{Q}}$ is free over $S_{\mathbb{Q}} = \mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}]$ of rank 10, we conclude that

$$
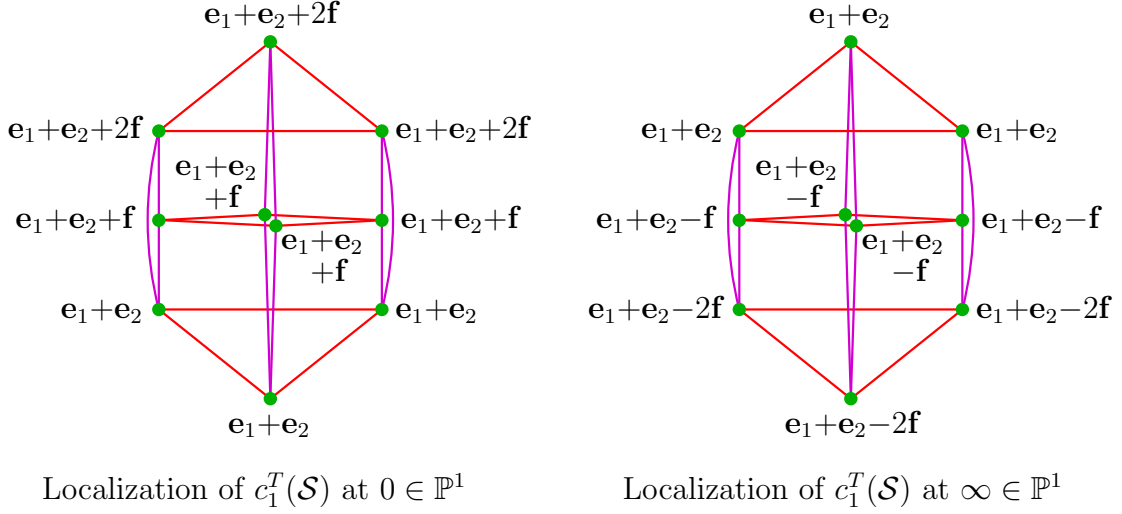A_T^*(\mathcal{Q}_2)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}, x, y, z]/\mathcal{I}\,.
$$

Using Proposition 9, we obtain the presentation of the rational Chow ring

$$
A^*(\mathcal{Q}_2)_{\mathbb{Q}} \simeq \mathbb{Q}[x, y, z]/\langle xz, \ yz, \ x^3, \ xy^2 - y^3, \ z^2 - x^2y^2\rangle\,.
$$

By (18), the integral Chow ring has a more complicated presentation.

We now consider Strømme's generators. Figure 4 shows the localization diagram of the first Chern class of $\mathcal{S}$. By the formula of Proposition 19, we have
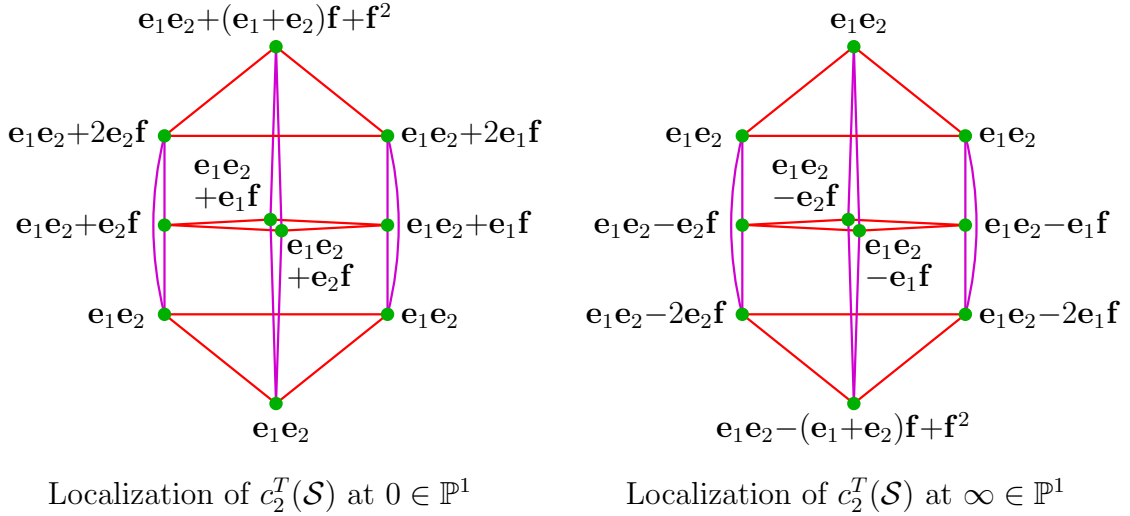
$$
c_1^T(\mathcal{S}) = \mathbf{e}_1 + \mathbf{e}_2 + x - 2h\,,
$$

FIGURE 4. First Chern class of $\mathcal{S}$.

so that
$$t_1 \;=\; \mathbf{e}_1 + \mathbf{e}_2 + x \qquad \text{and} \qquad u_0 \;=\; -2\,.$$

Figure 5 shows the localization diagram of the second Chern class of $\mathcal{S}$. By the formula

FIGURE 5. Second Chern class of $\mathcal{S}$.

of Proposition 19, we have
$$c_2^T(\mathcal{S}) \;=\; \mathbf{e}_1\mathbf{e}_2 + \tfrac{1}{2}x(y + \mathbf{e}_1 + \mathbf{e}_2) + \tfrac{1}{2}z \,-\, h(y + \mathbf{e}_1 + \mathbf{e}_2)\,,$$

so that
$$t_2 \;=\; \mathbf{e}_1\mathbf{e}_2 + \tfrac{1}{2}x(y + \mathbf{e}_1 + \mathbf{e}_2) + \tfrac{1}{2}z \qquad \text{and} \qquad u_1 \;=\; -(y + \mathbf{e}_1 + \mathbf{e}_2)\,.$$

The corresponding classes in $A^*(\mathcal{Q}_2)$ are

$$t_1 \;=\; x\,, \quad u_1 \;=\; -y\,, \quad \text{and} \quad t_2 \;=\; \tfrac{1}{2}(z + xy)\,.$$

Note that this shows that the claim in [16, Theorem (5.3)] that the classes t_i, u_{i-1} generate the integral Chow ring is false.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA 01003, USA
E-mail address: braden@math.umass.edu

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA
E-mail address: lchen@math.ohio-state.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA
E-mail address: sottile@math.tamu.edu