Algebra II Winter 2021 — 22 February

Frank Sottile — Sixth Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 1 March.

1. For any homomorphism $f\colon A \to B$ of left R-modules, prove that the diagram

$$\begin{tikzcd} A \arrow[r, "\theta_A"] \arrow[d, "f"'] & A^{**} \arrow[d, "f^{**}"] \\ B \arrow[r, "\theta_B"] & B^{**} \end{tikzcd}$$

   is commutative. Here $\theta_A\colon A \to A^{**}$ is the natural map from the A-module to its second dual, and $f^{**}\colon A^{**} \to B^{**}$ is the double pullback map between second duals.

   Conclude that θ is a natural transformation from the identity functor on the category of left R-modules to the functor ** of taking second duals.

2. Let R be a ring and M a finitely-generated left R-module. Recall that $\mathrm{Hom}_R(M,R)$ is a right R-module. Show that if M is projective, then $\mathrm{Hom}_R(M,R)$ is projective.

   Show that if M is projective, then M is a reflexive R-module.

3. Let F be a free left R-module of infinite rank. Show that F is not reflexive.