Foundations of Mathematics — Math 300H Section 970

YOUR NAME

Fifth Homework: Due 6 October 2022

Use English when possible. Answers should not just be symbols.

Definition: Let a, b, and m be integers. We say that a and b are *congruent modulo m* if m divides their difference, $a - b$. That is, when $m|(a-b)$. We write $a \equiv b \mod m$ when this occurs.

Definition: A real number x is a *rational number* when there exists integers n, d with $d \neq 0$ such that $x = n/d$. A real number that is not rational is an *irrational number*.

Definition: Suppose that a is a nonnegative real number. The *square root of a* is the unique nonnegative real number r such that $r^2 = a$. We write $\sqrt{a}$ for the square root of a.

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1. In our text (but not lectures), when we discuss congruence modulo a natural number m, we require that $m > 1$. Let us explore what happens in some extreme cases. Your answers need not be more than a few sentences long, and do not need a proof, but should show understanding.

   (a) Discuss what happens when $m = 1$ in the above definition for congruence modulo m. That is: what are its consequences, which numbers are congruent to others, and etc.

   (b) How about when $m = 0$? What are its consequences, etc.?

   (c) If we replace m by $-m$ in the definition, what changes?

2. Let a be an integer. Prove or find a counterexample to the statement that if a is odd, then $a^2 \equiv 1 \mod 8$. (Recall that we proved in class that $4|(a^2-1)$.)

3. Write up a nice, clean proof of the arithmetic-geometric mean: "For all positive real numbers x and y, we have $\sqrt{xy} \leq \frac{x+y}{2}$, and we have equality if and only if $x = y$."

4. Let x be a positive real number. Prove that $x + \frac{1}{x} \geq 2$.

   Hint: There is an easy way to do this and a hard way to do this.

5. Prove for every three real numbers x, y and z that $|x-z| \leq |x-y| + |y-z|$.

6. Prove that for all real numbers a, b, c, d, we have $(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$.

7. Is the following statement true or false? (If true, give a proof, if false, give a counterexample.) "For each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational."

8. Let A and B be sets. Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

9. In the context of the previous problem, under what conditions do those three sets form a partition (See Section 1.5 of our text) of $A \cup B$? Prove your assertion.

10. Prove that if A and B are sets such that $A \cup B \neq \emptyset$, then $A \neq \emptyset$ or $B \neq \emptyset$.