Algebra II Winter 2021 — 15 March

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Ninth Homework: Canonical Forms

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated.

Due Monday 22 March.

Let $\mathbb{F}$ be a field and V a finite-dimensional vector space over $\mathbb{F}$. Fix a linear transformation $T \in \mathrm{End}_{\mathbb{F}}(V)$ of the vector space V. Let t be an indeterminate and consider the map $\varphi_T \colon \mathbb{F}[t] \to \mathrm{End}_{\mathbb{F}}(V)$ given by $t \mapsto T$ (and $\mathbb{F} \ni 1 \mapsto Id_V$, where Id_V is the identity linear transformation on V).

Exercise 1: Show that φ_T is a ring homomorphism.

Definition. The *minimal polynomial* of T is the monic generator of the kernel of φ_T. (Recall that $\mathbb{F}[t]$ is a principal ideal domain.)

Lemma. *Under the map φ_T, V is a finitely generated torsion $\mathbb{F}[t]$-module.*

Exercise 2: Provide a proof of this statement.

Example. Let $f = a_0 + a_1 t + \cdots + a_{d-1}t^{d-1} + t^d$ be monic polynomial in $\mathbb{F}[t]$. Consider the quotient $\mathbb{F}[t]/\langle f\rangle$ as an $\mathbb{F}$-vector space and an $\mathbb{F}[t]$-module. Elements of $\mathbb{F}[t]$ act on this quotient by mutiplication.

Exercise 3: Show that $\{1, t, t^2, \ldots, t^{d-1}\}$ forms a basis for the $\mathbb{F}$-vector space $\mathbb{F}[t]/\langle f\rangle$.

What is the matrix R for the action of t on the vector space $\mathbb{F}[t]/\langle f\rangle$ with respect to this ordered basis?

(Treat elements of $\mathbb{F}[t]/\langle f\rangle$ as column vectors.)

Example. Let $\alpha \in \mathbb{F}$, $m \in \mathbb{N}$ positive, and consider the $\mathbb{F}[t]$-module $\mathbb{F}[t]/\langle (t-\alpha)^m\rangle$.

Exercise 4: Show that $\{1, (t-\alpha), (t-\alpha)^2, \ldots, (t-\alpha)^{m-1}\}$ is a basis for the $\mathbb{F}$-vector space $\mathbb{F}[t]/\langle (t-\alpha)^m\rangle$.

What is the matrix J for the action of t on $\mathbb{F}[t]/\langle (t-\alpha)^m\rangle$ with respect to this basis?

Definition. Let T, V be as above. An *eigenvector* of T is a nonzero vector $0 \neq v \in VB$ such that there exists $\lambda \in \mathbb{F}$ with $Tv = \lambda v$. The scalar λ is the *eigenvalue* of T corresponding to the eigenvector v.

Exercise 5: Let $\alpha \in \mathbb{F}$ and $m \in \mathbb{N}$ be positive.

(a) Show that $(t-\alpha)^{m-1}$ is the unique eigenvector for the action of t on the vector space $\mathbb{F}[t]/\langle (t-\alpha)^m\rangle$.

(b) Let $h \in \mathbb{F}[t]$. What are its eigenvectors and eigenvalues on the vector space $\mathbb{F}[t]/\langle (t-\alpha)^m\rangle$?

Theorem. (Rational Canonical Form) *Let $\mathbb{F}, T, V$ be as above. There is an ordered basus for V such that, in this basis, the linear transformation T has the block diagonal form*

$$\begin{pmatrix} R_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R_t \end{pmatrix},$$

where each diagonal block R_i has the form of the matrix in Exercise 3, for a polynomial $q_i(t)$ with $q_1|q_2|\cdots|q_t$. This diagonal form is unique.

Exercise 6: Prove this, using the decomposition of torsion modules over $\mathbb{F}[t]$ via invariant factors.

Theorem. (Jordan Canonical Form) *Let $\mathbb{F}, T, V$ be as above, and suppose that the minimal polyomial of T factors into liner factors (or that $\mathbb{F}$ is algebraically closed). There is an ordered basus for V such that, in this basis, the linear transformation T has the block diagonal form*

$$\begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_t \end{pmatrix} ,$$

where each diagonal block J_i has the form of the matrix in Exercise 4. This diagonal form is unique. The J_i are called Jordan blocks for T.

Exercise 7: Prove this, using the decomposition of torsion modules over $\mathbb{F}[t]$ via elementary divisors. What are the eigenvectors of T in this basis?

Example. A submodule M of $\mathbb{Z}^n$ is a free abelian group of rank $m \leq n$. Let

$$\varphi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^n$$

be any $\mathbb{Z}$-linear map with image M (with our assumptions, it is an injection).

Exercise 8: Show that there are bases for $\mathbb{Z}^m$ and $\mathbb{Z}^n$ such that

$$\varphi = \begin{pmatrix} \delta_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \delta_m & 0 & \cdots & 0 \end{pmatrix} ,$$

with $\delta_1 | \delta_2 | \cdots | \delta_m$ (and $\delta_m \neq 0$).

This is called the *Smith normal form of φ* More generally, (and classically) we may take φ to be a map between free modules $(\mathbb{F}[t])^m \to (\mathbb{F}[t])^n$ (with no assumptions on m, n, or φ (or much less classically, between free modules over a PID). Then there are bases so that φ is a diagonal matrix whose entries δ_i satisfy $\delta_1 | \delta_2 | \cdots | \delta_{\min\{m,n\}}$, with possibly $\delta_{\min\{m,n\}} = 0$. (Note that for all elements r, $r|0$.)

Bonus : Show that the diagonal entry δ_i is the greatest common divisor of the determinants of all $i \times i$ submatrices in the matrix representation of φ.