Algebra II Winter 2021 12 April
Frank Sottile **Twelfth Homework**

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 19 April.

1. [15] Let $f := x^4 + ax^2 + b \in K[x]$ be an irreducible quartic over a field K of characteristic not 2. Let G be the Galois group of f. (Note that f is separable, due to the characteristic being not 2.) Show that
   (a) If b is a square in K, then $G = V$, the Klein four group.
   (b) If b is not a square in K, but $b(a^2 - 4b)$ is a square, then $G \simeq \mathbb{Z}/4\mathbb{Z}$ is a cyclic group.
   (c) If neither b nor $b(a^2 - 4b)$ are squares in K, then $G \simeq D_4$, the dihedral group.

2. [20] Determine the Galois groups of the following polynomials over $\mathbb{Q}$.
   (a) $x^4 + 4x + 4$
   (b) $x^4 + 3x + 3$
   (c) $x^4 - 7x^2 - 3x + 1$
   (d) $x^5 - 3x + 1$

3. Suppose that F/K is a Galois extension of degree p^n for p a prime and n a positive integer. Prove that for every degree d dividing p^n, (e.g. $d \in \{1, p, p^2, \ldots, p^n\}$) F/K has an intermediate field E with the degree of E/K equal to d and E/K is Galois.

4. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a Galois extension of $\mathbb{Q}$, determine its Galois group, and determine all intermediate fields.

Corrections:

In the first problem, suppose that a polynomial $f \in K[x]$ splits into linear factors in some extension field F of K. Then f has a multiple root in F if and only if f and its formal derivative f' have a common factor (this is done in Chapter III). But then they have a common factor in $K[x]$. Consequently, if f is irreducible in $K[x]$, it is not separable if and only if $f' = 0$. But then (necessarily) the characteristic of K, p, divides the degree of f, and a little more thought leads to the conclusion that there is a polynomial $g \in K[x]$ with $f(x) = g(x^p)$.

The reason for this discourse is to see that in the first problem, forbidding K to have characteristic 2, a polynomial of the given form is irreducible is then necessarily separable.

2(b) had a typo, the quintic is $x^5 - 3x + 1$, and not $x^5 - 2x + 1 = (x-1)(x^4 + x^3 + x^2 + x - 1)$.