Algebra Autumn 2013
Frank Sottile
26 September 2013

Fifth Homework

Hand in to Frank Tuesday 1 October:

27. Let H be a subgroup of a group G and define the *core of H* to be

$$\text{core}(H) \ := \ \bigcap\{H^g \mid g \in G\}\,,$$

the intersection of all conjugates of H by elements of G.

Let $S := \{xH \mid x \in G\}$ be the set of left cosets of H in G. For each $g \in G$, define $g^* \colon S \to S$ by $g^*(xH) = gxH$.

(a) Show that g^* is an element of the symmetric group on the set S, $\text{Sym}(S)$.

(b) Show that the map $G \to \text{Sym}(S)$ given by $g \mapsto g^*$ is a group homomorphism whose kernel is the core of H.

Hand in to Frank Tuesday 8 October:

32. A Subset X of an abelian group F is *linearly independent* if $n_1x_1 + n_2x_2 + \cdots + n_kx_k = 0$ implies that $n_i = 0$ for all i, where $n_i \in \mathbb{Z}$ and $x_1, \ldots, x_k$ are distinct elements of X.

(a) Show that X is linearly independent if and only if every nonzero element of the subgroup $\langle X \rangle$ it generates may be written uniquely in the form $n_1x_1 + \cdots + n_kx_k$, where $n_i \in \mathbb{Z}$ and $x_1, \ldots, x_k$ are distinct elements of X.

(b) If F is free abelian of finite rank n, then it is *not* true that every linearly independent subset of n elements is a basis.

(c) If F is free abelian, then it is *not* true that every linearly independent subset of F may be extended to a basis of F.

(d) If F is free abelian, then it is *not* true that every generating set of F contains a basis fo F.

Hand in for the grader Tuesday 1 October:

28. A subgroup C of a group G is *characteristic* if, for any automorphism φ of G, we have $\varphi(C) = C$). Prove that any characteristic subgroup of a group G is normal.

29. Let F be a free group. Prove that the subgroup generated by all nth powers, $\{x^n \mid x \in F\}$, is a normal subgroup of F.

30. Let G be any group. Set *$M(G)$* to be the intersection of all subgroups of finite index in G. Prove that $M(G)$ is normal.

31. Show that free abelian group has a subgroup of index n, for any positive integer n.