Algebra Autumn 2013
Frank Sottile
17 October 2013

Seventh Homework

Hand in to Frank Tuesday 22 October:

38. Prove that the converse to Lagrange's Theorem holds for nilpotent groups. That is, if G is a finite nilpotent group and n divides the order of G, then G has a subgroup of order n.

    Hint: first prove it for p-groups.

Hand in to Frank Thursday 24 October:

39. Let G be a simple group. What are the **normal** subgroups of $G \times G$?

    Hint: Do number 40 first.

Hand in for the grader Thursday 24 October:

40. Let p be a prime number. How many simple subgroups does $\mathbb{Z}_p \times \mathbb{Z}_p$ have?

    Hint: Do the case of $p = 5$ explicitly. You might also consider this to be the Cartesian plane over the field with p elements.

41. Prove (without using the Feit-Thompson Theorem) that the following two statements are equivalent:

    (a) Every group of odd order is solvable.

    (b) The only simple groups of odd order are the abelian groups of prime order.

42. Let $H = \mathbb{Z}_3$ and $K = \mathbb{Z}_4$, and consider the homomorphism $\varphi\colon K \to \mathrm{Aut}(\mathbb{Z}_3)$ which sends the generator of $\mathbb{Z}_4$ to multiplication by -1. Show that $H \rtimes_\varphi K$ is a nonabelian group of order 12 that is not isomorphic to either A_4 or D_{12}.

43. Use semidirect products to classify all groups of order 28 up to isomorphism. (There are four isomorphism classes.)