Math 629, Homework #10 22 April 2018

11.2.0 Show that if p is a prime, then every binomial coefficient $\binom{p}{k}$ for $k \neq 0, p$ is divisible by p.

Use this to show that $2^p = (1+1)^p = 2 +$ terms divisible by p.

Recall the closed formula for a binomial coefficient,

$$\binom{p}{k} \;=\; \frac{p!}{k!(p-k)!}\,.$$

If $1 \leq k \leq p-1$, then every factor in the denominator is strictly less than p, so that the denominator is not divisible by p. Since p divides the numerator, and the fraction is an integer, it follows that p divides the binomial coefficient.

By the binomial formula we have

$$2^p \;=\; (1+1)^p \;=\; 1 + \tbinom{p}{1} + \cdots + \tbinom{p}{p-1} + 1\,.$$

That is, 2^p is the sum of all binomial coefficients $\binom{p}{k}$ for k from 0 to p. Since the first and last terms are 1, while the remaining terms are divisible by p, we see that $2^p - 2$ is divisible by p.

11.2.1 Using the result that $2^p = (1+1)^p = 2 +$ terms divisible by p to show that $3^p = (2+1)^p = 3 +$ terms divisible by p.

Again, let us use the boinomial theorem

$$3^p \;=\; (2+1)^p \;=\; 2^p + \sum_{k=1}^{p-1} \tbinom{p}{k} 1^k 2^{p-k} \quad + \; 1^p\,.$$

The same arguments as before lead us to conclude that $3^p = 2^p + 1 +$ terms divisible by $p = 2 + 1 +$ terms divisible by p.

11.2.2 Use these ideas to show that $n^p = n +$ terms divisible by p.

This is best to show by induction. Our induction hypothesis is that for every positive number n, there is a number N_n such that $n^p = n + p \cdot N_n$. We know this is true for $n = 1$, as in that case $N_1 = 0$.

Suppose that this is true for some number $n \geq 1$, so that $N_n := (n^p - n)/p$ is an integer. Then

$$(n+1)^p \;=\; n^p \;+\; \sum_{k=1}^{p-1} \tbinom{p}{k} 1^k n^{p-k} \quad + \; 1^p\,.$$

The middle sum is divisible by p, let $p{\cdot}M$ be this sum. Then $(n+1)^p = n^p + 1 + p{\cdot}M = n + 1 + p{\cdot}N_n + p{\cdot}M$. If we let $N_{n+1} := N_n + M$, we have established the assertion for $n+1$. This proves the statement for all positive integers $n \geq 1$.

11.2.3 Observe this divisibility property in the first few rows of Pascal's triangle.

Being a fan of the number 17 (can anyone guess why), let us look at the 17th row of Pascal's triangle:

1, 17, 136, 680, 2380, 6188, 12376, 19448, 24310, 24310, 19448, 12376, 6188, 2380, 680, 136, 17, 1

Dropping the initial 1 and using the symmetry, the next 8 terms are

$$1 \cdot 17\,,\; 8 \cdot 17\,,\; 40 \cdot 17\,,\; 140 \cdot 17\,,\; 364 \cdot 17\,,\; 728 \cdot 17\,,\; 1144 \cdot 17\,,\; 1430 \cdot 17\,.$$