Foundations of Mathematics — Math 300H Section 970

Eighth Homework: Due 1 November 2022

Definition: The *Fibonacci sequence* $\{f_n \mid n \geq 1\}$ is defined by $f_1 = f_2 = 1$ and for $n \geq 2$, $f_{n+1} = f_n + f_{n-1}$.

1. Let $a, r \in \mathbb{R}$ with $r \neq 1$. Prove that for every number $n \in \mathbb{N}$, $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$.

2. Explore the divisibility by 3 of positive powers of 4. (E.g. $4^n \mod 3$, for $n \in \mathbb{N}$.) Make a conjecture and prove it.

3. Write a proof in paragraph form of the inequality $3^n > 1 + 2^n$ for $n \geq 2$ using mathematical induction.

4. Consider the sequence $\{a_n \mid n \in \mathbb{N}\}$ defined by $a_1 = 1$, $a_2 = 3$ and for each $n \in \mathbb{N}$, $a_{n+2} = 3a_{n+1} - 2a_n$. Calculate the first eight terms of this sequence.

   Conjecture a formula for a_n and prove it using induction.

5. Consider the sequence $\{a_n \mid n \in \mathbb{N}\}$ defined by $a_1 = a_2 = 1$ and for each $n \in \mathbb{N}$, $a_{n+2} = \frac{1}{2}\left(a_{n+1} + \frac{2}{a_n}\right)$. Calculate the first six terms of this sequence.

   Prove, for all $n \in \mathbb{N}$, that $1 \leq a_n \leq 2$.

6. Compute the first 15 terms of the Fibonacci sequence (this will help for later problems). Note that the recursion $f_{n+1} = f_n + f_{n-1}$ may be rewritten $f_{n-1} = f_{n+1} - f_n$. Use this to extend the Fibonacci sequence to *negative* integers and compute the values of f_n for $-10 \leq n \leq 0$. Conjecture a formula for f_{-n} for $n \in \mathbb{N}$ and prove it by induction.

7. Explore sums of squares of the Fibonacci numbers and conjecture a formula for

$$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 .$$

   Prove your formula.

8. Evaluate the proposed proof of the following statement.

   Theorem. *For every positive integer n, we have* $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

   Proof. We proceed by induction. Note that the formula holds for $n = 1$. Assume that $1 + 3 + 5 + \cdots + (2k-1) = k^2$ for a positive integer k. We prove that $1 + 3 + 5 + \cdots + (2k+1) = (k+1)^2$. Observe that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k+1) &= (k+1)^2 \\ 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) &= (k+1)^2 \\ k^2 + (2k+1) &= (k+1)^2 \\ (k+1)^2 &= (k+1)^2 . \end{aligned}$$

$\square$