Algebra Autumn 2023
Frank Sottile
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Second Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in to Frank Wednesday 6 September: (Have this on a separate sheet of paper.)

6. Show that a group G which is isomorphic to each of its proper subgroups is cyclic.

   What are the possibilities for the group G?

Hand in for the grader Wednesday 6 September: (Have this separate from #6.)

7. Show that $\mathbb{Q}$ has a chain of cyclic subgroups $\langle a_1\rangle \subsetneq \langle a_2\rangle \subsetneq \langle a_3\rangle \subsetneq \cdots$ with $\mathbb{Q} = \bigcup\{\langle a_i\rangle \mid i \in \mathbb{Z}\}$. Deduce that $\mathbb{Q}$ has no maximal cyclic subgroup.

8. Suppose that $\varphi\colon G \to H$ is a group homomorphism. Fill in the details of the assertion in class: If φ is a bijection, then the inverse function $\varphi^{-1}\colon H \to G$ is also a homomorphism. Do this by checking that it preserves the identity, sends products to products, and sends inverses to inverses.

9. Let D_4 be the group under matrix multiplication generated by the real 2×2 matrices $S := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ and $R := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. Show that D_4 is a nonabelian group of order 8.

   Let $\square$ be the square with vertices $(\pm 1, \pm 1)$ in $\mathbb{R}^2$. Show that D_4 acts on $\square$, and is its group of symmetries, called the *dihedral group* of order 8.

10. Let Q_8 be the group generated by the complex 2×2 matrices $\mathbf{i} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ and $\mathbf{j} := \left(\begin{smallmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{smallmatrix}\right)$. Show that Q_8 is a nonabelian group of order 8. Hint: Observe that $\mathbf{ij} = \mathbf{ji}^3$, so that every element of Q_8 has the form $\mathbf{i}^a\mathbf{j}^b$. Note further that $\mathbf{i}^4 = \mathbf{j}^4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$, the identity. This is the *quaternion* group.

    Comparing subgroups, or the number of elements of different orders, show that Q_8 is not isomorphic to D_4.

11. The *center* of a group G is the set $C(G) := \{a \in G \mid ag = ga \text{ for all } g \in G\}$.

    For $g \in G$, the *centralizer of g* is the set $C_G(g) := \{a \in G \mid ag = ga\}$. Prove that $C(G)$ and $C_G(g)$ are subgroups of G.