Foundations of Mathematics — Math 300H Section 970

YOUR NAME

Third Homework: 20 September 2021

Use English when possible. Answers should not just be symbols.

1. Determine whether each of the following sentences is a statement, a predicate (open statement), or neither.
   (a) The Boston Celtics have won 16 NBA championships.
   (b) The plane is leaving in four minutes.
   (c) Get a note from your doctor.
   (d) Is this the best that you can do?
   (e) Excessive exposure to the sun may cause melanoma.
   (f) $5 \cdot 2 = 9$.
   (g) Someone in the room is a murderer.
   (h) $x^2 + 1 \neq 0$.
   (i) For every real number x, $x^2 + 1 \neq 0$.
   (j) The equation for a circle of radius 1 centered at the origin is $x^2 + y^2 = 1$.
   (k) If m and n are even integers, then mn is odd.

2. For each of the following statements, determine if it has any universal or existential quantifiers. If it has universal quantifiers, rewrite it in the form "for all...". If it has existential quantifiers, rewrite it in the form "there exists ... such that ...". Introduce variables where appropriate.
   (a) The area of a rectangle is its length times its width.
   (b) A triangle may be equilateral.
   (c) $8 - 8 = 0$.
   (d) The sum of an even integer and an odd integer is even.
   (e) For every even integer, there is an odd integer such that the sum of the two is odd.
   (f) A function that is continuous on the closed interval $[a, b]$ is integrable on $[a, b]$.
   (g) A function is continuous on $[a, b]$ whenever it is differentiable on $[a, b]$.
   (h) A real-valued function that is continuous at 0 is not necessarily differentiable at 0.
   (i) All positive real numbers have a square root.
   (j) The smallest positive integer is 1.

3. Write a useful negation of each statement in Exercise 2.

4. Negate each of the following statements (which are important definitions in mathematics). Assume that the symbols f, K, a, and l are defined.

   (a) For every $x \in K$, if $x \neq 0$, then there is a $y \in K$ such that $xy = 1$.

   (b) For every real number $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in \mathbb{R}$ with $x \neq a$ and $|x - a| < \delta$, then $|f(x) - l| < \epsilon$.

   (c) For every real number $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in \mathbb{R}$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

5. For statements P and Q show that $\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$ is a tautology. Then state $\big(P \wedge (P \Rightarrow Q)\big) \Rightarrow Q$ in words. (This is an important argument form, called *modus ponens*.)

6. For statements P, Q and R show that $\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow (P \Rightarrow R)$ is a tautology. (This is another important argument form, called *syllogism*.)

   Give an example of a valid syllogism involving Socrates, and give an example of an invalid syllogism involving Socrates.

7. For statements P, Q and R show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent. Can you show this without a truth table?

8. Define an open sentence $R(x)$ over some domain S (not mathematical, but in words), and then state $\forall x \in S,\ R(X)$ and $\exists x \in S,\ R(x)$ in words.

9. State the negation of the following quantified statements:

   (a) For every rational number r, the number $1/r$ is rational,

   (b) There exists a rational number r such that $r^2 = 2$.

10. Determine the truth value of each of the following statements. (List which are true)

    (a) $\exists x \in \mathbb{R},\ x^2 - x = 0$.

    (b) $\forall n \in \mathbb{N},\ n + 1 \geq 2$.

    (c) $\forall x \in \mathbb{R},\ \sqrt{x^2} = x$.

    (d) $\exists x \in \mathbb{Q},\ 3x^2 - 27 = 0$.

    (e) $\exists x \in \mathbb{R},\ \exists y \in \mathbb{R},\ x + y + 3 = 8$.

    (f) $\forall x, y \in \mathbb{R},\ x + y + 3 = 8$.

    (g) $\exists x, y \in \mathbb{R},\ x^2 + y^2 = 9$.

    (h) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x^2 + y^2 = 9$.

11. Let a, b, and c be integers. Consider the following conditional statement:

$$\text{If } a \text{ divides } bc\text{, then } a \text{ divides } b \text{ or } a \text{ divides } c.$$

Which of the following statements have the same meaning as this conditional statement, and which are negations of this conditional statement:

(a) If a divides b or a divides c, then a divides bc.

(b) If a does not divide b or a does not divide c, then a does not divide bc.

(c) a divides bc, a does not divide b, and a does not divide c.

(d) If a does not divide b and a does not divide c, then a does not divide bc.

(e) a does not divide bc or a divides b or a divides c.

(f) If a divides bc and a does not divide c, then a divides b.

(g) If a divides bc or a does not divide b, then a divides c.

12. Give a definition of each of the following and then state a characterization of each.

(a) Two lines in the plane are perpendicular.

(b) A rational number.

13. Give a valid definition of an odd integer.

14. Prove that if a, b, c are odd integers such that $a+b+c=0$, then $abc<0$. (You may use any well-known properties of integers here.)

15. Prove that if a and c are odd integers, then $ab+bc$ is an even integer for every integer b.

Sketch it first (perhaps in a table form), and then write it in paragraph form.