Algebra Autumn 2023
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Seventh Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in for the grader Monday 16 October:

27. Let H be a subgroup of a group G and define the *core of H* to be
$$\text{core}(H) \;:=\; \bigcap\{H^g \mid g \in G\}\,,$$
the intersection of all conjugates of H by elements of G.

    Let $S := \{xH \mid x \in G\}$ be the set of left cosets of H in G. For each $g \in G$, define $g^*\colon S \to S$ by $g^*(xH) = gxH$.

    (a) Show that g^* is an element of the symmetric group on the set S, $\text{Sym}(S)$.

    (b) Show that the map $G \to \text{Sym}(S)$ given by $g \mapsto g^*$ is a group homomorphism whose kernel is the core of H.

28. Suppose that a group G has an element x with exactly *three* distinct conjugates. Show that G is not simple.

    Prove the same result if G has an element x with exactly *four* distinct conjugates.

    Bonus: This result remains true if G has an element with exactly five conjugates.

29. Let G be a finite group acting faithfully on a set S. Prove that if G is 2-transitive, then G is a primitive permutation group.

30. Let p be a prime number and C a subgroup of the symmetric group S_p of order p. Use the orbit-stabilizer theorem to determine the cardinality of the normalizer in S_p of C.

31. Suppose that G is a finite p-group. Show that G contains a normal subgroup of order q for every positive integer q dividing the order of G.