THE MODERN SCHUBERT CALCULUS

FRANK SOTTILE

ABSTRACT. We describe the classical Schubert calculus and some current developments in the Schubert Calculus of Enumerative Geometry. **More Later.**

To the memory of my teacher and friend, D. M. Carr, 1940–2000.

INTRODUCTION

In 1886, Herman Cäser Hannibal Schubert [36, 34] gave a formula for the number $d_{m,p}$ of p-dimensional subspaces (p-planes) in $\mathbb{C}^{m+p}$ which have non-trivial intersection with each of mp general m-planes. (The number mp ensures there will be finitely many such p-planes.)

$$d_{m,p} \ := \ \frac{1!\,2!\cdots(p-2)!\,(p-1)!\cdot(mp)!}{(m)!\,(m+1)!\cdots(m+p-1)!}\ . \tag{0.1}$$

Schubert's contemporaries Mario Pieri [32] and Giovanni Zeno Giambelli [15] gave formulas which enable the computation of the numbers of p-planes satisfying more general incidence conditions imposed by flags of linear subspaces.

By the middle of the 20th century, Van der Waerden [50], Ehresmann [7], Hodge [17, 18], and others had justified these formulas as rules for computing in the cohomology ring of the Grassmannian of p-planes in $\mathbb{C}^{m+p}$ with respect to its basis of Schubert cycles. This Schubert basis of cohomology is important in many areas of mathematics—most notably in algebraic combinatorics through Young tableaux and Schur polynomials [44, Ch. 7] and in representation theory through characters of irreducible representations of general linear and symmetric groups [12]. A high point of this interaction is the Littlewood-Richardson rule for expressing the product of two Schubert cycles in the Schubert basis [31, 37, 47]. The Littlewood-Richardson rule has further applications to diverse areas including eigenvalues of sums of hermitian matrices and invariant factors of products of matrices [14].

This field of Schubert calculus, which studies the geometry and combinatorics of Grassmannians, continues to be a vital area of mathematics. Currently, the subject is in a golden age, over 115 years after its inception. For example, Ravi Vakil [49] has recently given a geometric proof of the Littlewood-Richardson rule, a long-standing open problem. His proof also solves two other long-standing problems in the Schubert calculus, concerning real solutions and transversality in positive characteristic.

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Other recent highlights include Buch's Littlewood-Richardson rule for K-theory of a Grassmannian [2], and Belkale's geometric proof of the Horn and saturation conjecture [1]. This survey will discuss these and many other recent advances in this subject.

We begin in the first section discussing classical achievements of the Schubert calculus and the geometry of Grassmannians. There we also describe the relation of the Schubert calculus to representation theory and give some open problems in this classical Schubert calculus. The second section is the heart of this survey—we discuss aspects of the modern Schubert calculus that involve the Grassmannian, including the advances mentioned above. We will also discuss an application of the classical and quantum Schubert calculus to a problem from systems theory. Finally in Section 3, we discuss extensions to more general flag manifolds.

In his treatise, "Kalkül der abzählenden Geometrie"[†] [33], Schubert declared enumerative geometry to be concerned with all questions of the following form: How many geometric figures of a fixed type satisfy certain given conditions? He proposed answers to many such problems using a formal calculus of symbols representing geometric conditions. Hilbert's 15th problem [16] asked for (i) a rigorous foundation of Schubert's enumerative calculus and (ii) a verification of the numbers he obtained.

A consequence of Hilbert's 15th problem is that the term "Schubert calculus" often refers both to Intersection Theory and to Enumerative Geometry, which correspond roughly to the first and second parts of Hilbert's 15th problem. We take a narrower interpretation that the Schubert calculus is concerned with the Grassmannian and related spaces, that is, when the geometric figures we are counting are linear subspaces of a vector space satisfying incidence conditions imposed by other linear subspaces.

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1. The Classical Schubert Calculus

We give two approaches to the enumerative problems of the Introduction. The first involves the cohomology ring of Grassmannians, and the second the equations for Grassmannians in a natural projective embedding. We begin with the classical problem of four lines in 3-space, then describe the basics of the geometry and cohomology of Grassmannians, continue with its relation to other parts of mathematics, and conclude with a discussion of equations for the Grassmannian. A nice discussion of much of this material from a different perspective is presented in the article of Kleiman and Laksov [28].

1.1. The problem of four lines. T there are two lines in complex 3-space that meet each of 4 (general) lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4. Up to a projective change of coordinates, we may assume that ℓ_1, ℓ_2, and ℓ_3 are given parametrically as follows

$$\ell_1 \ :\ (t,0,0) \qquad \ell_2 \ :\ (t,1,t) \qquad \ell_3 \ :\ (t,-1,-t)\ .$$

[†] "Calculus of Enumerative Geometry"

These lines lie on the quadric surface Q of Figure 1 defined by the equation $z = xy$.

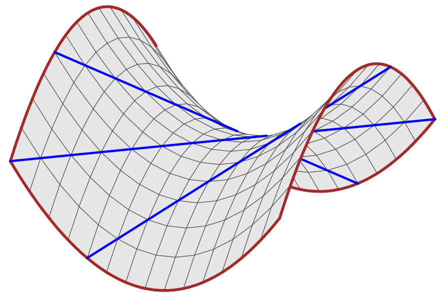


FIGURE 1. Three lines on the quadric surface $z = xy$

The set of lines meeting each of ℓ_1, ℓ_2, and ℓ_3 are those having parameterization (x_0, t, tx_0) for x_0 some constant.

Consider now the last line, ℓ_4. Since Q is defined by a quadratic equation, this line meets Q in 2 points, say (x_1, y_1, x_1y_1) and (x_2, y_2, x_2y_2). Thus the two lines

$$(x_1, t, tx_1) \quad \text{and} \quad (x_2, t, tx_2)$$

are the lines that meet each of the given lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4.

The problem of the Introduction, and more generally the Schubert calculus, are vast generalizations of this problem.

1.2. GEOMETRY OF THE GRASSMANNIAN. Let m, p be positive integers, set $n := m+p$, and let $\mathrm{Gr}_{p,n}$ be the collection of all p-dimensional subspaces (p-planes) in $\mathbb{C}^n$. This **Grassmannian** is a compact complex manifold of dimension mp. Indeed, in §1.6 below, we give equations that describe $\mathrm{Gr}_{p,n}$ as a closed subset of the complex projective space $\mathbb{P}(\bigwedge^p \mathbb{C}^n)$. The general linear group $GL(n, \mathbb{C})$ acts transitively on the Grassmannian, so it is a manifold. Lastly, the association

$$\mathrm{Mat}_{p\times m}\mathbb{C} \ni X \longmapsto \text{row space}\,[\,X \;:\; I_p\,] \tag{1.1}$$

where I_p is the p by p identity matrix, defines a holomorphic bijection between $\mathrm{Mat}_{p\times m}\mathbb{C}$ and an open subset of $\mathrm{Gr}_{p,n}$, this shows that $\mathrm{Gr}_{p,n}$ has complex dimension mp. When $p = 2$, these Grassmannians appeared in the enumerative problems of Section 1.1: A line in projective space $\mathbb{P}^{n-1}$ corresponds to a plane in the underlying vector space $\mathbb{C}^n$. In particular, the problem of four lines concerns the Grassmannian $\mathrm{Gr}_{2,4}$ of 2-planes in $\mathbb{C}^4$, which Schubert studied in [33].

Suppose we represent a p-plane $H \in \mathrm{Gr}_{p,n}$ as the row space of a p by n matrix. While there are many matrices representing H, applying Gaussian elimination to any representative gives the unique echelon matrix representative of H.

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{1.2}$$

Here, an asterix $(*)$ represents an arbitrary complex number.

Write $\mathbb{Y}_{m,p}$ for the collection of all decreasing sequences $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$ of integers, called **partitions**, which have length p and satisfy $m \geq \lambda_1$. For $\lambda \in$

$\mathbb{Y}_{m,p}$, the **Schubert cell** X_λ° is the set of all p-planes in $\mathbb{C}^n$ whose echelon matrix representative has last non-zero entry (pivot) of row b in column $m+b-\lambda_b$. If $\emptyset$ is the partition $(0,\ldots,0)$, then $X_\emptyset^\circ$ is the set of p-planes parameterized by the local coordinates (1.1). Since the echelon matrix representative of a p-plane in X_λ° has λ_b *fewer* undetermined entries in row b than does a p-plane in $X_\emptyset^\circ$, we see that $X_\lambda^\circ \simeq \mathbb{C}^{mp-|\lambda|}$, where $|\lambda| := \lambda_1+\cdots+\lambda_p$.

Let $\{e_1,\ldots,e_n\}$ be the basis of $\mathbb{C}^n$ corresponding to the columns of our p by n matrices. For each b, set $F_b := \langle e_1,\ldots,e_b\rangle$, the linear span of the vectors $\{e_1,\ldots,e_b\}$. Let $F_\bullet$ be the collection of subspaces $F_1, F_2, \ldots, F_n$. Then the Schubert cell X_λ° equals

$$X_\lambda^\circ F_\bullet \;:=\; \{H \in \mathrm{Gr}_{p,n} \mid \dim H\cap F_a = b \text{ for } i_b \le a < i_{b+1}\}\,, \tag{1.3}$$

where $i_b := m+b-\lambda_b$. Denote the resulting sequence $i_1 < i_2 < \ldots < i_p$ by $I(\lambda)$.

More generally, given a **complete flag** $F_\bullet$ ($F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$, with $\dim F_b = b$) and a partition $\lambda \in \mathbb{Y}_{m,p}$, (1.3) defines the Schubert cell $X_\lambda^\circ F_\bullet$. Since each p-plane has a unique echelon representative, for any flag $F_\bullet$, the Grassmannian is the disjoint union of these Schubert cells

$$\mathrm{Gr}_{p,n} \;=\; \coprod_{\lambda\in\mathbb{Y}_{m,p}} X_\lambda^\circ F_\bullet\,.$$

In fact, this is a CW-decomposition of $\mathrm{Gr}_{p,n}$ into even (real) dimensional cells. This implies that the (integral) cohomology groups $H^*\mathrm{Gr}_{p,n}$ of the Grassmannian have a basis given by classes Poincaré dual to the fundamental homology cycles of the closures of the Schubert cells [7].

These closures are the **Schubert varieties** (introduced by Schubert [35]), which are defined by replacing the equalities in (1.3) by inequalities

$$X_\lambda F_\bullet \;=\; \{H \in \mathrm{Gr}_{p,n} \mid \dim H\cap F_{m+b-\lambda_b} \ge b \text{ for } b=1,2,\ldots,p\}\,.$$

For $H \in \mathrm{Gr}_{p,n}$, $\dim H \cap F_{m+b} \ge b$. Thus the condition imposed by λ_b is that H has this same intersection with a smaller (by dimension λ_b) subspace of the flag $F_\bullet$.

Partitions are partially ordered by componentwise comparison: $\mu \le \lambda$ when $\mu_b \le \lambda_b$ for all b. Similarly, Schubert varieties are partially ordered by inclusion. These orderings are related in the following way. For $\lambda, \mu \in \mathbb{Y}_{p,n}$,

$$\lambda \le \mu \quad \text{if and only if} \quad X_\lambda F_\bullet \supset X_\mu F_\bullet\,.$$

The Schubert calculus of enumerative geometry counts subspaces of a vector space satisfying incidence conditions imposed by other linear subspaces. In the geometric language we have developed, partitions $\lambda \in \mathbb{Y}_{p,n}$ encode the incidence conditions that may be imposed on p-planes in $\mathbb{C}^n$ by flags of linear subspaces, and the Schubert variety $X_\lambda F_\bullet$ is the locus of p-planes satisfying the **Schubert condition** λ imposed by $F_\bullet$. The general enumerative problem we study is the following. Given partitions $\lambda^1, \lambda^2, \ldots, \lambda^s \in \mathbb{Y}_{p,n}$ and flags $F_\bullet^1, F_\bullet^2, \ldots, F_\bullet^s$ in general position, what is the number of points in the intersection

$$X_{\lambda^1}F_\bullet^1 \cap X_{\lambda^2}F_\bullet^2 \cap \cdots \cap X_{\lambda^s}F_\bullet^s\ ? \tag{1.4}$$

When $m=p=2$, $s=4$ and $\lambda^i = (1,0)$, this is the problem of four lines in Section 1.1.

1.3. Cohomology and Intersection Theory. One tool for determining the expected numbers of points in an intersection (1.4) is the cohomology ring of the Grassmannian. For us, the most important properties of cohomology groups of smooth complex algebraic varieties X of dimension d are the following.

(1) Each irreducible k-dimensional subvariety Y of X determines a fundamental cycle $[Y]$ in $H_{2k}X$, and therefore by Poincaré duality, a cohomology class, also denoted $[Y]$, in $H^{2d-2k}X = H^{2c}X$, where c is the codimension of Y in X.

(2) This construction extends in a natural way to reducible subvarieties Y, Z of X so that if $Y \cap Z$ shares no components with either Y or Z, then

$$[Y \cup Z] \;=\; [Y] + [Z]\,.$$

(3) Suppose Y, Z are irreducible subvarieties of X of codimensions b and c respectively, and each component V_i of $Y \cap Z$ has codimension $b + c$ in X. Then $Y \cap Z$ is a **proper** intersection, and there exist positive integral intersection multiplicities a_i associated to each component V_i such that

$$[Y] \cdot [Z] \;=\; \sum a_i\,[V_i]\,.$$

When Y meets Z transversally along an open subset of V_i, then the multiplicity a_i is 1 and $Y \cap Z$ is **generically transverse** along V_i.

(4) For any point $pt \in X$, $H^{2d}X = \mathbb{Z} \cdot [pt]$, and so there is a degree map

$$\deg \;:\; H^*X \;\twoheadrightarrow\; H^{2n}X \;\longrightarrow\; \mathbb{Z}$$

given by associating a cohomology class y to the coefficient of $[pt]$ in that class.

Schubert's enumerative calculus [33] involved treating geometric conditions ρ, σ, τ on figures of some type as formal symbols, with $\rho + \sigma$ interpreted as the condition that either ρ or σ holds, and $\rho \cdot \sigma$ as the condition that both ρ and σ hold when ρ and σ are independent. Conditions ρ and σ are **numerically equivalent** if, whenever τ is a condition such that only finitely many figures satisfy both ρ and τ and also both σ and τ, then these numbers are equal. While these ideas originated in work of Chasles [6] on conics, they were systematized and used to great effect by Schubert.

When there is a smooth complex algebraic parameter space X of figures, then a geometric condition defines a subvariety of X, and thus a class in the cohomology ring of X. By properties 1–4 above, these cohomology classes have the properties Schubert used, *when used for enumeration*.† We refer to Kleiman's excellent survey [26] for a discussion of this justification of Schubert's formal enumerative calculus.

The Schubert calculus is meaningful and has an identical description for any field. In this setting, the justification of Schubert's enumerative calculus is accomplished by Intersection Theory, which replaces the cohomology ring by the Chow ring. (See also the survey of Kleiman [27] or the books of Fulton [11, 13].) The Chow ring has a map to cohomology (singular cohomology over the complex numbers, and étale in general) which is neither injective nor surjective in general. In contrast, for the varieties we study, the Chow ring does not depend upon the field, and is isomorphic to the cohomology ring of the complex variety. This is because the geometric arguments

†For general algebraic varieties (but not the ones considered here), it is an open problem whether numerical equivalence coincides with homological equivalence.

given below for the complex Grassmannian do not depend upon the field, and thus establish the same results for the Chow ring.

1.4. Cohomology of the Grassmannian. The **Schubert class** $\sigma_\lambda := [X_\lambda F_\bullet]$ is the cohomology class Poincaré dual to the fundamental homology cycle of the Schubert variety $X_\lambda F_\bullet$. Define the dual $\lambda^\vee$ of a partition λ by

$$\lambda_b + \lambda^\vee_{p+1-b} \;=\; m \quad \text{for} \quad b = 1, 2, \ldots, p\,.$$

The following Basis Theorem of Schubert [36] was rigorously established by Ehresmann [7] for cohomology and by Hodge [17] for the Chow ring.

Theorem 1.5. *The Schubert cycles $\{\sigma_\lambda \mid \lambda \in \mathbb{Y}_{p,n}\}$ form an integral basis of the cohomology groups $H^*\mathrm{Gr}_{p,n}$ for the Grassmannian with $\sigma_\lambda \in H^{2|\lambda|}\mathrm{Gr}_{p,n}$. Furthermore, this Schubert basis is self-dual under the intersection pairing*

$$\begin{array}{rcl} H^*\mathrm{Gr}_{p,n} \times H^*\mathrm{Gr}_{p,n} & \longrightarrow & H^{2mp}\mathrm{Gr}_{p,n} \;\simeq\; \mathbb{Z}\cdot[pt]\,,\\ (x,y) & \longmapsto & \deg(x\cdot y) \end{array}$$

with σ_λ dual to $\sigma_{\lambda^\vee}$.

Sketch of proof. We previously observed that the Schubert cycles form a basis for the cohomology groups. To show this basis is self-dual, let $F_\bullet$ and $F'_\bullet$ be flags in linear general position and λ, μ be partitions in $\mathbb{Y}_{p,n}$. Then the intersection of Schubert varieties $X_\mu F_\bullet \cap X_{\lambda^\vee} F'_\bullet$ is empty unless $\mu \leq \lambda$. This is because any p-plane H in the intersection must meet $F_{m+b-\mu_b} \cap F'_{p+1-b+\lambda_b}$ non-trivially, and this is non-empty only if $\mu_b \leq \lambda_b$. In fact, H is spanned by its intersections with the subspaces $V_b := F_{m+b-\mu_b} \cap F'_{p+1-b+\lambda_b}$, for $b = 1, 2, \ldots, p$.

When $\lambda = \mu$, each subspace V_b is 1-dimensional. Thus $X_\mu F_\bullet \cap X_{\lambda^\vee} F'_\bullet$ is a single point as $H \in X_\mu F_\bullet \cap X_{\lambda^\vee} F'_\bullet$ is necessarily spanned by the V_b. A calculation in local coordinates shows that intersection is transverse, which completes the proof. ∎

This self-duality of the Schubert basis shows that the *integral* cohomology ring of the Grassmannian satisfies Poincaré duality (there is no torsion and the intersection pairing is unimodular!). The Basis Theorem has an important corollary.

Corollary 1.6. *If $Y \subset \mathrm{Gr}_{p,n}$, then $[Y] = \sum_\lambda c^\lambda_{[Y]}\,\sigma_\lambda$, where $c^\lambda_{[Y]} = \#(Y \cap X_{\lambda^\vee} F_\bullet)$, when this intersection is transverse.*

Schubert [36] gave the first non-trivial product formula.

Theorem 1.7. *Let $p, n > 1$. For any $\mu \in \mathbb{Y}_{p,n}$, we have*

$$\sigma_\mu \cdot \sigma_1 \;=\; \sum_{\mu \lessdot \lambda} \sigma_\lambda\,.$$

Here, $\mu \lessdot \lambda$ (μ covers λ) means that $\mu < \lambda$ and there is no ν with $\mu < \nu < \lambda$. In practice is means that there is a unique index b with $\mu_b \neq \lambda_b$, and $\mu_b + 1 = \lambda_b$.

Proof. Since the Schubert variety $X_1 F_\bullet$ depends only on the m-plane F_m in $F_\bullet$,

$$X_1 F_\bullet \;=\; \{H \in \mathrm{Gr}_{p,n} \mid H \cap F_m \neq \{0\}\}\,,$$

we write $X_1 F_m$ or $X_1 K$ with $\dim K = m$ for this variety.

Let $F_\bullet$ be the standard flag where $F_b = \langle e_1, \ldots, e_b\rangle$ for $e_1, \ldots, e_n$ a basis for $\mathbb{C}^n = \mathbb{C}^{m+p}$. Set $K := \langle e_j \mid j \neq m + b - \mu_b \text{ for } b = 1, 2, \ldots, p\rangle$. The theorem follows by showing

$$X_\mu F_\bullet \cap X_1 K \;=\; \sum_{\mu \lessdot \lambda} X_\lambda F_\bullet\,, \tag{1.8}$$

and this intersection is generically transverse.

The Schubert variety $X_\mu F_\bullet$ is parameterized by the set M_μ of p by n matrices of full rank whose entries satisfy $x_{b,a} = 0$ for $a > i_b$ where $I(\mu) = i_1 < \ldots i_p$ with $i_b := m + b - \mu_b$

$$M_\mu \;=\; \left[\begin{array}{ccccccccc} x_{1,1} & \cdots & x_{1,i_1} & 0 & & & \cdots & & 0 \\ x_{2,1} & & \cdots & & x_{2,i_2} & 0 & & \cdots & 0 \\ \vdots & & & & & \ddots & \ddots & & \vdots \\ x_{p,1} & & & \cdots & & & x_{p,i_p} & 0 \;\cdots & 0 \end{array}\right].$$

The m-plane K is the row space of the matrix obtained by deleting rows $i_1, \ldots, i_p$ from the identity matrix I_n. The condition for the row space of a matrix $M \in M_\mu$ (a p-plane in $X_\mu F_\bullet$) to meet K non-trivially (and hence lie in the intersection (1.8)) is

$$0 \;=\; \det\left[\begin{array}{c} K \\ M \end{array}\right] \;=\; \pm\, x_{1,i_1} \cdot x_{2,i_2} \cdots x_{p,i_p}\,.$$

If $\mu < \lambda$, then $\mu_b < \lambda_b$ for some b and so $x_{b,i_b} = 0$ on M_λ. Conversely, on M_μ the zero scheme of the polynomial $x_{1,i_1} \cdot x_{2,i_2} \cdots x_{p,i_p}$ is the union of the sets M_λ for $\mu < \lambda$, and there are generically no multiplicities: When $\mu \lessdot \lambda$ with $\mu_b + 1 = \lambda_b$, but $\mu_j = \lambda_j$ for $j \neq b$, then $M_\lambda \subset M_\mu$ is defined by the vanishing of x_{b,i_b}. This proves (1.8), and shows that the intersection is transverse, proving the theorem. $\blacksquare$

When $m = p = 2$ and $\mu = 1$ (and we work with lines in projective 3-space), the Schubert variety $X_1\ell$ is the set of lines which meet ℓ. In this case, the special position in the proof above (which goes back to Schubert [33]) is that ℓ_1 and ℓ_2 are lines that meet in a point p and span a plane H. If $Y(p)$ is the Schubert variety of lines containing p and $Z(H)$ the Schubert variety of lines contained in H, then

$$X_1\ell_1 \cap X_2\ell_2 \;=\; Y(p) \cup Z(H)\,,$$

as a line meeting both ℓ_1 and ℓ_2 either meets p or lies in H.

Such specializations solve the problem of four lines from Section 1.1 as follows. Suppose the four lines $\ell_1, \ldots, \ell_4$ lie along the edges of a tetrahedron with no three forming a triangle (the solid lines in Figure 2). The dashed lines spanned by the remaining 2 edges are the lines that meet each of $\ell_1, \ldots, \ell_4$. Choosing the coordinate lines in $\mathbb{P}^3$, this tetrahedron may be realized over any field, showing again how this problem of 4 lines may be solved over any field.

If we iterate Theorem 1.7, then we obtain

$$\sigma_\mu \cdot (\sigma_1)^a \;=\; \sum_{\mu \lessdot \mu^1 \lessdot \cdots \lessdot \mu^a = \lambda} \sigma_\lambda \;=\; \sum_{\lambda\,:\,|\lambda| = |\mu| + a} f^\lambda_\mu\, \sigma_\lambda\,,$$

where f^λ_μ is the number of (saturated) chains in Young's lattice from μ to λ.

The partition $(m^p) = (m, \ldots, m)$ is the maximal element of $\mathbb{Y}_{p,n}$, and $X_{(m^p)}F_\bullet = \{F_p\}$, so $\sigma_{(m^p)}$ is the class of a point. Thus $f^{(m^p)}_\mu = \deg(\sigma_\mu \cdot \sigma_1^{mp-|\mu|})$, which counts the

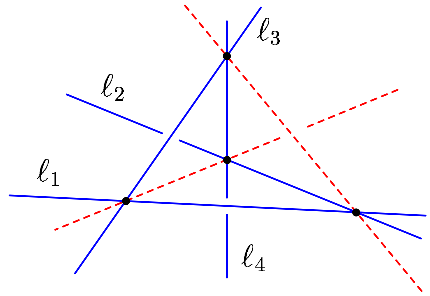


FIGURE 2. Degenerate configuration of 4 lines.

p-planes in $X_\mu F_\bullet$ that have non-trivial intersection with $mp-|\mu|$ different m-planes, when the corresponding Schubert varieties meet transversally. While these numbers are easily computed recursively, Schubert [34] gave the following formula

$$f_{\mu^\vee}^{(m^p)} \;=\; f_\emptyset^\mu \;=\; \frac{|\mu|!\prod_{i<j}(\mu_i-\mu_j+j-i)}{(\mu_1+p-1)!\cdots(\mu_j+p-j)!\cdots\mu_p!}\,. \tag{1.9}$$

When $\mu=(m^p)$, we recover the formula (0.1) for $d_{m,p}$.

Pieri's rule determines the multiplicative structure the cohomology ring of the Grassmannian. For $1\leq a\leq m$, let $\sigma_a=\sigma_{(a,0,\ldots,0)}$, a **special Schubert class**.

Theorem 1.10. *For any $\mu\in\mathbb{Y}_{m,p}$ and $1\leq a\leq m$, we have*

$$\sigma_\mu\cdot\sigma_a \;=\; \sum\sigma_\lambda\,,$$

the sum over all $\lambda\in\mathbb{Y}_{m,p}$ with $|\lambda|=|\mu|+a$, and

$$m\geq\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\cdots\geq\lambda_p\geq\mu_p\geq 0\,. \tag{1.11}$$

This was proven in the case of $p=2$ by Schubert [35], the case when $a=m-1$ by Castelnuovo [4], and Pieri [32] established the general case.

Sketch of Proof. By Corollary 1.6, it suffices to show that if $|\lambda|=|\mu|+a$ and $F_\bullet$, $F'_\bullet$, and $F''_\bullet$, are in general position, then

$$\#(X_\mu F_\bullet\cap X_{\lambda^\vee}F'_\bullet\cap X_aF''_\bullet) \;=\; \begin{cases}1 & \text{if (1.11) holds,}\\ 0 & \text{otherwise.}\end{cases} \tag{1.12}$$

Any p-plane H in $X_\mu F_\bullet\cap X_{\lambda^\vee}F'_\bullet$ lies in the linear span V of the subspaces $V_b:=F_{m+b-\mu_b}\cap F'_{p+1-b+\lambda_b}$. One may check that V has dimension at most $p+a$, with equality only when (1.11) holds. As $X_aF''_\bullet$ is the collection of all p-planes H which meet F''_{m+1-a} non-trivially, the triple intersection will be empty unless F''_{m+1-a} meets V. This implies that $\dim V\geq p+a$ and so (1.11) holds.

When condition (1.11) holds, $V\cap F''_{m+1-a}$ is 1-dimensional and spanned by the single vector $v=\oplus v_b$ where $0\neq v_b\in V_b$. Thus $H=\langle v_1,v_2,\ldots,v_p\rangle$ is the unique p-plane in the triple intersection. A calculation in local coordinates shows this intersection is transverse, which completes proof. ∎

Corollary 1.13. *The cohomology ring $H^*\mathrm{Gr}_{m,p}$ is generated by the special Schubert cycles $\sigma_1,\sigma_2,\ldots,\sigma_m$.*

Proof. Let $R \subset H^*\mathrm{Gr}_{m,p}$ be the subring generated by the special Schubert cycles. For $\lambda, \mu \in \mathbb{Y}_{m,p}$, write $\mu \prec \lambda$ if the last non-zero component of $\lambda - \mu$ is positive. We induct on the total order $\prec$ on $\mathbb{Y}_{m,p}$ to show that $\sigma_\mu \in R$ for each $\mu \in \mathbb{Y}_{m,p}$.

Let $\mu \in \mathbb{Y}_{m,p}$ and suppose that for $\nu \prec \mu$, we have $\sigma_\nu \in R$. Let k be the index with $\mu_k > 0$ but $\mu_{k+1} = 0$, and set $\nu = (\mu_1, \ldots, \mu_{k-1}) \prec \mu$. Then by Pieri's rule, $\sigma_\nu \cdot \sigma_{\mu_k}$ has leading term σ_μ, and all other terms σ_λ have $\lambda \prec \mu$. Thus

$$\sigma_\mu \;=\; \sigma_\nu \cdot \sigma_{\mu_k} - \text{ other terms } \in R. \qquad \blacksquare$$

This proof suggests a recursive method to write σ_μ as a polynomial in the special Schubert cycles. For example, in $H^*\mathrm{Gr}_{2,5}$, let $\mu = (3,2)$. Then we have

$$\begin{aligned} \sigma_3 \cdot \sigma_2 &= \sigma_{3,2} + \sigma_{4,1} + \sigma_5 \\ \sigma_4 \cdot \sigma_1 &= \phantom{\sigma_{3,2} + {}} \sigma_{4,1} + \sigma_5\,, \end{aligned}$$

and so

$$\sigma_{3,2} \;=\; \sigma_3 \cdot \sigma_2 - \sigma_4 \cdot \sigma_1 \;=\; \det \begin{bmatrix} \sigma_3 & \sigma_4 \\ \sigma_1 & \sigma_2 \end{bmatrix}.$$

The generalization of this determinantal formula is due to Giambelli [15].

Theorem 1.14. *Set $\sigma_0 = 1$ and $\sigma_a = 0$ if $a < 0$ or $a > m$. Then, for any $\lambda \in \mathbb{Y}_{m,p}$ with $\lambda_{k+1} = 0$, we have*

$$\sigma_\lambda \;=\; \det \begin{bmatrix} \sigma_{\lambda_1} & \sigma_{1+\lambda_1} & \cdots & \sigma_{k-1+\lambda_1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \cdots & \sigma_{k-2+\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\lambda_k-k+1} & \sigma_{\lambda_k-k+2} & \cdots & \sigma_{\lambda_k} \end{bmatrix} \;=\; \det[\sigma_{\lambda_i+j-i}]\,.$$

This formula of Giambelli may be proven using elementary algebra (see [18]): Induct on k and expand the determinant along its last column, expressing it as an alternating sum of terms of the form $\sigma_\mu \cdot \sigma_{k-i+\lambda_i}$. Expanding these using Pieri's rule, all terms cancel in pairs, except the term σ_λ.

These results, Schubert's Basis and Duality Theorem, Pieri's rule, and Giambelli's formula, form the core of the classical Schubert calculus. Their importance is reflected by common terminology. A **Pieri formula** is any formula for the product of a geometric class with a generator of the cohomology ring. A **Giambelli formula** is any formula expressing these geometric classes in terms of the ring generators.

These fundamental results also hold for the Chow ring of the Grassmannian of p-planes in $\mathbb{K}^n$, for any field $\mathbb{K}$. They enable the computation of the expected number $\deg(\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^s})$ of points in the intersection $X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^s}F^s_\bullet$.

The question remains whether this expected number has enumerative significance. That is, whether or not this intersection is transverse for general flags, so that all solutions appear with multiplicity 1. This basic principle of the classical Schubert calculus is not known to hold in general. Kleiman's Transversality Theorem [25] establishes this principle for fields of characteristic zero.

Theorem 1.15. *Let X be an algebraic variety equipped with a transitive action of a connected algebraic group G. Let $Y, Z \subset X$ be subvarieties. Then there exists a dense open subset U of G such that for $g \in U$, the translate gY of Y meets Z properly, and in characteristic zero, $gY \cap X$ is generically transverse.*

Kleiman [25] also showed that this restriction to characteristic zero is necessary. He exhibits a subvariety Z of $\mathrm{Gr}_{2,2}$ over a field of characteristic $q > 0$ having the property that Z does not meet any translate of a particular Schubert variety transversally. However, Z is not a Schubert variety, and so the question remained if general translates of Schubert varieties meet generically transitively in positive characteristic.

This was recently settled in the affirmative. Partial answers were given in [38] for all Schubert varieties in $\mathrm{Gr}_{2,p}$ and $\mathrm{Gr}_{m,2}$, and for codimension 1 Schubert varieties [42]. Vakil has just proven this in full generality [49].

Theorem 1.16. *For any partitions $\lambda^1, \lambda^2, \ldots, \lambda^s \in \mathbb{Y}_{m,p}$, there is an open subset U of $(\mathbb{F}\ell_n)^s$ consisting of flags $(F^1_\bullet, F^2_\bullet, \ldots, F^s_\bullet)$ such that*

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^s}F^s_\bullet$$

is generically transverse.

1.5. The Littlewood-Richardson rule. These fundamental results of Schubert, Pieri, and Giambelli do not give a satisfactory procedure for writing a product of Schubert cycles $\sigma_\mu \cdot \sigma_\nu$ in terms of the basis of Schubert cycles. There are **Littlewood-Richardson** constants $c^\lambda_{\mu,\nu}$ for $\lambda, \mu, \nu \in \mathbb{Y}_{m,p}$ defined by the identity

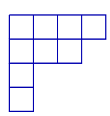
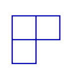
$$\sigma_\mu \cdot \sigma_\nu \;=\; \sum_\lambda c^\lambda_{\mu,\nu}\, \sigma_\lambda\,.$$

These constants have the following geometric interpretation

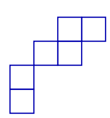
$$c^\lambda_{\mu,\nu} \;=\; \#(X_\mu F_\bullet \cap X_\nu F'_\bullet \cap X_{\lambda^\vee} F''_\bullet)\,,$$

when $F_\bullet, F'_\bullet, F''_\bullet$ are general flags in $\mathbb{C}^n$. In particular, the $c^\lambda_{\mu,\nu}$ are nonnegative, a fact that is not evident from the formulas of Section 1.4.

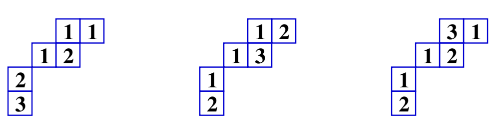
Littlewood and Richardson [31] gave a combinatorial formula for these constants which was proven by Schützenberger [37] and Thomas [47]. The **Young diagram** of a partition λ is a left-justified array of boxes with λ_i boxes in the ith row. Thus

 and 

are the Young diagrams of $(4,3,1,1)$ and $(2,1)$, respectively. Given $\mu \leq \lambda$, there is a corresponding inclusion of Young diagrams, with the **skew Young diagram** λ/μ their set-theoretic difference. Thus



is the skew diagram $(4,3,1,1)/(2,1)$. A (skew) **Young tableau** is a filling of the boxes in a (skew) Young diagram with positive integers that weakly increase across each row and strictly increase down each column. For example, the first two fillings of $(4,3,1,1)/(2,1)$ below are Young tableaux, while the third is not.



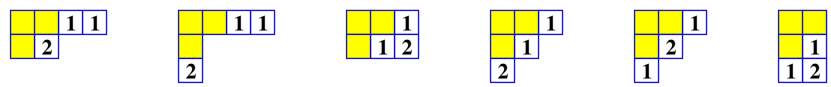
A **Littlewood-Richardson tableau** T is a Young tableau satisfying the following condition: If we list the integers in the boxes of T from right to left starting from the top row and working down, then among the first t elements of this list, the number j occurs at least as often as does $j+1$, for each t and j. The **weight** of a tableau T is the sequence whose ith entry is the number of occurrences of the integer i in T. The two tableaux above each have weight $(3,2,1)$, but only the first is a Littlewood-Richardson tableau. We state the Littlewood-Richardson rule.

Theorem 1.17. *The coefficient $c^{\lambda}_{\mu,\nu}$ is the number of Littlewood-Richardson tableaux of shape λ/μ and weight ν.*

For example, in $H^*\mathrm{Gr}_{4,3}$, we have

$$\sigma_{(2,1)} \cdot \sigma_{(2,1)} \;=\; \sigma_{(4,2)} + \sigma_{(4,1,1)} + \sigma_{(3,3)} + 2\cdot\sigma_{(3,2,1)} + \sigma_{(2,2,2)}\,,$$

as we can see by listing all Littlewood-Richardson tableaux of shape $\lambda/(2,1)$ and weight $(2,1)$, where $\lambda_1 \leq 4$ and λ has length at most 3.



Littlewood and Richardson proposed their rule in the context of representations of $GL(n,\mathbb{C})$ and the proofs by Schützenberger and Thomas were purely combinatorial. The connection between these fields and the Schubert calculus is through the Schur polynomial basis of the ring of symmetric polynomials. One of the most spectacular and recent advances in the Schubert calculus is a geometric proof of the Littlewood-Richardson rule by Vakil [49]. His result has many implications, including Theorem 1.16. We will describe this development in Section 2.

Let Λ be the ring of symmetric polynomials in the indeterminates $x_1,\dots,x_p$. For any partition λ of length p, Cauchy [5] defined the polynomial

$$s_\lambda \;=\; s_\lambda(x_1,\dots,x_n) \;:=\; \frac{\det[x_i^{\lambda_j+p-j}]}{\det[x_i^{p-j}]}\,, \tag{1.18}$$

now called a **Schur polynomial**. Since the numerator is alternating, it is divisible by the Van der Monde determinant, which is the denominator, and so the quotient is a polynomial that is symmetric in $(x_1,\dots,x_p)$.

Multiplication by the Van der Monde gives an isomorphism between Λ and its module of alternating polynomials. The numerator in (1.18) has lexicographic leading term $x_1^{\lambda_1+p-1}\cdots x_p^{\lambda_p}$, so these numerators form a basis for the module of alternating polynomials. Thus the Schur polynomials form a basis for Λ.

Among the Schur polynomials are the complete symmetric polynomials

$$h_j \;=\; h_j(x_1,\dots,x_p) \;:=\; \sum_{i_1\leq i_2\leq\cdots\leq i_j} x_{i_1}x_{i_2}\cdots x_{i_j} \;=\; s_{(j)}\,,$$

for $1\leq j$. These generate Λ as a ring, $\Lambda = \mathbb{Z}[h_1,\dots,h_p]$.

Jacobi [23] discovered and Trudi [48] proved the following formula

$$s_\lambda(x_1,\dots,x_p) \;=\; \det[h_{\lambda_i+j-i}(x_1,\dots,x_p)]_{i,j=1,2,\dots,k}\,,$$

where $\lambda_{k+1}=0$, $h_0=1$, and $h_a=0$ for $a<0$. These facts, that the Schur polynomials are a basis for Λ, that Λ is freely generated by the h_j, and this formula of Jacobi-Trudi, are formally equivalent to the Basis Theorem, Corollary 1.13, and to the Giambelli formula. This was observed by Lesieur [30]. We state that precisely.

Theorem 1.19. *The association $h_j \mapsto \sigma_j$ defines a surjective ring homomorphism $\Lambda \twoheadrightarrow H^*\mathrm{Gr}_{m,p}$ with*

$$s_\lambda \longmapsto \begin{cases} \sigma_\lambda & \text{if } \lambda_1 \leq m \\ 0 & \text{if } \lambda_1 > m \end{cases},$$

and whose kernel has a basis $\{s_\lambda \mid \lambda_1 > m\}$.

Thus any formula involving Schur polynomials is equivalent to a formula involving Schubert cycles—if we set $\sigma_\lambda = 0$ when $\lambda_1 > m$. For example, there is a Pieri formula for the product $s_\mu \cdot h_a$ as in Theorem 1.10, expressed as the same sum, but with the restriction $\lambda_1 \leq m$ removed. The terminology 'Pieri formula' was introduced into algebraic combinatorics by Alain Lascoux.

Weyl modules V_λ are the finite dimensional irreducible representations of the Lie algebra $\mathfrak{gl}_p$ of $p \times p$ complex matrices. Here λ is a partition with $\lambda_{p+1} = 0$. Every finite dimensional representation of $\mathfrak{gl}_p$ is isomorphic to a direct sum of Weyl modules in an essentially unique way. Thus the representation ring R of $\mathfrak{gl}_p$ has an integral basis consisting of the Weyl modules. It was in this context that Littlewood and Richardson formulated their rule for decomposing a tensor product $V_\mu \otimes V_\nu$ of two Weyl modules in as a direct sum of other Weyl modules,

$$V_\mu \otimes V_\nu \;=\; \bigoplus_\lambda V_\lambda^{\oplus c^\lambda_{\mu,\nu}}\,.$$

The character $\chi(V)$ of a finite-dimensional $\mathfrak{gl}_p$-module V evaluated at $x \in \mathfrak{gl}_p$ is a symmetric polynomial in the (generalized) eigenvalues of x. This is because $\chi(V)$ takes the same value at all $GL(p,\mathbb{C})$-conjugates of x and the $\mathfrak{gl}_p$-action on V is given by polynomials. It is a wonderful fact that the character $\chi(V_\lambda)$ of a Weyl module evaluated $x \in \mathfrak{gl}_p$ is Schur polynomial s_λ evaluated at the eigenvalues of x. Thus the representation ring of $\mathfrak{gl}_p$ with its basis of Weyl modules is isomorphic to the ring of symmetric polynomials with its basis of Schur polynomials.

The formal relation between the algebra Λ of symmetric polynomials and the cohomology ring $H^*\mathrm{Gr}_{m,p}$ of the Grassmannian of Theorem 1.19 was made conceptual by Horrocks [20]. The map $\Lambda \twoheadrightarrow H^*\mathrm{Gr}_{m,p}$ is given by evaluating symmetric polynomials at the Chern roots of the tautological p-plane bundle over $\mathrm{Gr}_{m,p}$. Horrocks used the representation theory of $\mathfrak{gl}_p$ to realize invariant forms representing Schubert cycles in terms of the Chern roots. Another conceptualization is due to Carrell [3]. Tamvakis recently gave a more direct relation using the Chern-Weil theory of characteristic classes [46].

An even more direct link should exist which has the following form. A $\mathfrak{gl}_p$-module V is homogeneous of degree d if the scalar matrix xI_p acts on V by multiplication by x^d. Any module is a direct sum of homogeneous submodules. Given a homogeneous $\mathfrak{gl}_p$-module V of degree d, we ask for a natural construction of a subvariety $Y(V) \subset \mathrm{Gr}_{m,p}$ of codimension d whose cycle class $[Y(V)]$ is the image of the character $\chi(V)$ in $H^*\mathrm{Gr}_{m,p}$. This association $V \mapsto Y(V)$ would ideally associate an intersection of subvarieties to a tensor product. Such an intersection may not be proper, so we should only expect that

$$Y(V \otimes W) \;\subset\; Y(V) \cap Y(W).$$

While the Littlewood-Richardson rule gives the number $c^{\lambda}_{\mu,\nu}$ of points in a triple intersection

$$X_\mu F_\bullet \cap X_\nu F'_\bullet \cap X_{\lambda^\vee} F''_\bullet\,, \tag{1.20}$$

for general flags $F_\bullet, F'_\bullet, F''_\bullet$, it does not give a satisfactory answer to the following fundamental problem.

Decide when (1.20) is non-empty for general (and hence all) $F_\bullet, F'_\bullet, F''_\bullet$.

That is, when is $c^{\lambda}_{\mu,\nu} \neq 0$?

The resolution of this problem is one of the most exciting recent stories in mathematics, with the final step due to Knutson and Tao [29]. The impetus for that work was a question about linear inequalities among the eigenvalues of three hermitian matrices whose sum is zero. This problem has connections to many other parts of mathematics, and its solution involved the representation theory of $\mathfrak{gl}_p$, geometric invariant theory, symplectic geometry, and novel combinatorics. For the full story, see Fulton's survey [14].

Briefly, let $T_p := \{(\lambda,\mu,\nu) \mid c^{\lambda}_{\mu,\nu} \neq 0\}$, a subset of $\mathbb{N}^{3p}$. Then T_p is a saturated sub-semigroup of $\mathbb{N}^{3p}$ defined by the **Horn inequalities**

$$\sum_{i=1}^{r} \lambda_{i+\alpha_{r+1-i}} \;\leq\; \sum_{i=1}^{r} \mu_{i+\beta_{r+1-i}} \;+\; \sum_{i=1}^{r} \nu_{i+\gamma_{r+1-i}}\;, \tag{$*\alpha\beta\gamma$}$$

for all $(\alpha,\beta,\gamma) \in \mathbb{Y}_{r,p-r}$ with $c^{\alpha}_{\beta,\gamma} \neq 0$. Thus, for partitions (λ,μ,ν) of length p with $|\lambda| = |\mu| + |\nu|$, the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is non-zero if and only if ($*\alpha\beta\gamma$) holds for all $r = 1,2,\ldots,p$ and all $\alpha,\beta,\gamma \in \mathbb{Y}_{r,p-r}$ with $c^{\alpha}_{\beta,\gamma} \neq 0$, and these in turn are determined by similar inequalities.

The work of Knutson and Tao, like the original proof of the Littlewood-Richardson rule, was purely combinatorial, and did not involve the geometry of the Grassmannian. It is not hard to see that the Horn inequalities ($*\alpha\beta\gamma$) are necessary for for a triple intersection (1.20) to be non-empty. Briefly, if H is a p-plane in such a triple intersection, then we may restrict the three flags $F_\bullet, F'_\bullet$, and $F''_\bullet$ to H and obtain three general complete flags $E_\bullet, E'_\bullet$, and $E''_\bullet$ in $H \simeq \mathbb{C}^p$.

Now, suppose that we have a triple $(\alpha,\beta,\gamma) \in \mathbb{Y}_{r,p-r}$ with $c^{\alpha}_{\beta,\gamma} \neq 0$. Then the triple intersection of Schubert varieties in the Grassmannian of r-planes in H

$$X_\beta E_\bullet \cap X_\gamma E'_\bullet \cap X_{\alpha^\vee} E''_\bullet$$

is non-empty. An r-plane K in this triple intersection satisfies three Schubert conditions in the Grassmannian of r-planes in the original $\mathbb{C}^n$, and the *corresponding* triple intersection in $\mathrm{Gr}_{r,n}$ is non-empty, which implies that the sum of the codimension is at most $\dim(\mathrm{Gr}_{r,n})$. This condition turns out to be equivalent to the inequality ($*\alpha\beta\gamma$).

Recently, Prakash Belkale [1] proved the other direction, that if $(\lambda,\mu,\nu) \in \mathbb{Y}_{p,n}$ satisfy ($*\alpha\beta\gamma$) for all $(\alpha,\beta,\gamma) \in \mathbb{Y}_{r,p-r}$ with $c^{\alpha}_{\beta,\gamma} \neq 0$, then the triple intersection (1.20) is non-empty. The key point was to fix $H \in \mathrm{Gr}_{p,n}$† and consider triples $F_\bullet, F'_\bullet, F''_\bullet$ of flags with H a point in the intersection (1.20). He then showed that the Horn inequalities implied that, for general such triples, the intersection (1.20) is transverse at H. A dimension count implies that for general $F_\bullet, F'_\bullet, F''_\bullet$ (no conditions regarding H), the intersection (1.20) is nonempty.

†Fix the indexing. Need $\mathrm{Gr}_{p,n}$ and not $\mathrm{Gr}_{m,p}$

1.6. Equations for Grassmann manifolds. A different understanding of the intersection number $d_{m,p}$ is given by the equations of the Grassmannian in a natural projective embedding. The pth exterior power of the inclusion of a p-planes H into $\mathbb{C}^n = \mathbb{C}^{m+p}$ is

$$\mathbb{C} \;\simeq\; \wedge^p H \;\longrightarrow\; \wedge^p\mathbb{C}^n \;\simeq\; \mathbb{C}^{\binom{n}{p}}\,.$$

the association $H \mapsto \wedge^p H \in \mathbb{P}^{\binom{n}{p}-1}$ defines the **Plücker embedding** of the Grassmannian $\mathrm{Gr}_{m,p}$ into $\mathbb{P}^{\binom{n}{p}-1}$.

This Plücker embedding is given explicitly as follows. Represent a p-plane H as the row space of a p by n matrix X with entries $x_{i,j}$. The columns of X correspond to a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{C}^n$, and this induces a basis for $\wedge^p\mathbb{C}^n$:

$$e_I \;:=\; e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_p} \qquad \text{for} \quad I\colon i_1 < i_2 < \cdots < i_p\,.$$

We write this set of sequences as $\binom{[n]}{p}$, the subsets of $\{1,2,\ldots,n\}$ with cardinality p. This basis gives Plücker coordinates p_I for $\mathbb{P}^{\binom{n}{p}-1}$, and the p_Ith coordinate of $\wedge^p H$ is the Ith maximal minor of the matrix X,

$$\begin{pmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_p}\\ x_{2,i_1} & x_{2,i_2} & \cdots & x_{2,i_p}\\ \vdots & \vdots & \ddots & \vdots\\ x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_p}\end{pmatrix}.$$

Associating each variable p_I to the Ith maximal minor of the matrix X defines a map $\varphi\colon \mathbb{C}[p_I] \to \mathbb{C}[x_{i,j}]$, which is the algebraic counterpart of the composite

$$\mathrm{Mat}_{p\times n} \xrightarrow{\text{row space}} \mathrm{Gr}_{m,p} \xrightarrow{\text{Plücker}} \mathbb{P}^{\binom{n}{p}-1}\,.$$

The first map is a surjection and the second an injection.

Thus the kernel of the map φ is the homogeneous ideal $\mathcal{I}_{m,p}$ of the Grassmannian, and so its image in $\mathbb{C}[x_{i,j}]$ is a subring isomorphism to the homogeneous coordinate ring of the Grassmannian. This fact has a consequence for linear algebra.

Proposition 1.21. *The ideal $\mathcal{I}_{m,p}$ is the ideal of algebraic relations among the maximal minors of a generic p by m matrix.*

Numerical invariants of the Grassmannian such as its dimension and degree may be determined by studying the kernel and image of the map φ. This has significance for our enumerative problem. Let Y be a fixed m by n matrix and X a p by n matrix, both of full rank. Then their row spaces K and H meet non-trivially ($H \in X_1K$) if and only if

$$0 \;=\; \begin{bmatrix} X\\ Y\end{bmatrix}.$$

Laplace expansion of the determinant along the rows of X gives an equation which is *linear* in the Plücker coordinates $p_I(H)$ of H.

$$0 \;=\; \sum_{I\in\binom{[n]}{p}} k_I\cdot p_I(H)\,, \tag{1.22}$$

where k_I is the appropriately signed maximal minor of Y complementary to the columns indexed by I, which is a Plücker coordinate of K.

Thus our original enumerative problem from the Introduction of determining those p-planes H that meet mp general m-planes $K_1, K_2, \ldots, K_{mp}$ non-trivially corresponds to solving mp *linear* equations of the form (1.22) on $\mathrm{Gr}_{p,m}$. Geometrically, this asks for the points common to the Grassmannian and mp hyperplanes, in its Plücker embedding. The expected number of solutions to this is just the degree of the Grassmannian in its Plücker embedding.

For our descriptions of the Plücker ideal of the Grassmannian, we extend our indexing of the Plücker variables p_I, allowing arbitrary sequences I of length p, with the following identification

$$p_{i_1,i_2,\ldots,i_p} \;=\; -p_{i_1,\ldots,i_{j-1},\, i_{j+1},i_j,\,\ldots,i_p}\,.$$

Suppose A, B, and C are sequences of numbers in $\{1,\ldots,n\}$ of respective lengths $t-1$, $p+1$, and $p-t$. For a subset $J \subset \{1,\ldots,p+1\}$, the subsequence $B|_J$ of B consists of the elements of B indexed by J, and $B|_{J^c}$ is the complementary subsequence. Define the polynomial $[A\dot{B}C]$ to be

$$\sum_{J\in\binom{[p+1]}{t}} (-1)^{\sum j_k - k} p_{A,B|_{J^c}} \cdot p_{B|_J,c}\,,$$

where $A, B|_{J^c}$ is the concatenation of the sequences A and $B|_{J^c}$. These are now called the **Van der Waerden syzygies**. Invariant theory shows they lie in the Plücker ideal as they vanish when evaluated at the maximal minors of a $p \times n$ matrix: The polynomial $[A\dot{B}C]$ is an alternating $p+1$-form in the (length p) columns of the matrix indexed by B.

The Van der Waerden syzygies generate the Plücker ideal, the subset of straightening syzygies constitute a minimal Gröbner basis for the ideal in a natural term order. We describe the straightening syzygies. Indices $J \in \binom{[n]}{p}$ are partially ordered by componentwise comparison

$$I \;\le\; J \qquad \text{if } i_k \;\ge\; j_k \text{ for } k = 1,\ldots,p\,.$$

We call this the Bruhat order, as it is isomorphic to the Bruhat order on Young's lattice of partitions under the association $I \leftrightarrow \lambda(I)$, where, if $I = i_1 \;<\; \cdots \;<\; i_p$, then the partition $\lambda(I)$ is $m+1-i_1 \;\ge\; m+2-i_2 \;\ge\; \cdots \;\ge\; m+p-i_p$. This is the same association of sequences to partitions that we saw before. If we set $F_\bullet$ to be the standard flag, then

$$X_\lambda F_\bullet \;=\; \{H \in \mathrm{Gr}_{p,n} \mid p_I(H) = 0 \text{ for } I \not\le I(\lambda)\}\,.$$

Suppose $I, J \in \binom{[n]}{p}$ are incomparable ($I \not\le J$, $J \not\le J$) with $i_1 < j_1$. Then there is a smallest index t with $i_t > j_t$. Set $A := i_1 < \cdots < i_{t-1}$, $B := j_1 < \cdots < j_t < i_t < \cdots < i_p$, and $C := j_{t+1} < \cdots < j+p$. Then the straightening syzygy $S(I,J)$ is the Van der Waerden syzygy $[A\dot{B}C]$. Let $\prec_{drl}$ be the degree reverse lexicographic term order on $\mathbb{C}[p_I]$ induced by the linear ordering of the variables p_I where $p_I \prec_{drl} p_J$ if I precedes J lexicographically. Sturmfels and White [45] reinterpreted the classical work of Young [53] and Hodge-Pedoe [19] as follows.

Theorem 1.23. *The set of straightening syzygies $S(I,J)$ for incomparable pairs $I, J \in \binom{[n]}{p}$ forms a minimal Gröbner basis in the term order $\prec_{drl}$. The initial term of $S(I,J)$ is $p_I p_J$.*

Since the straightening syzygies $S(I,J)$ have distinct initial terms, they are linearly independent. This result has several immediate corollaries. Let $C_{m,p}$ be the set of all maximal chains in this Bruhat order.

Corollary 1.24. *The initial ideal $in_{\prec_{drl}}\mathcal{I}$ of the Plücker ideal is generated by square-free quadratic monomials, and it factors as an intersection of ideals, one for each maximal chain in $C_{m,p}$.*

$$in_{\prec_{drl}}\mathcal{I} \;=\; \langle p_I p_J \mid I,J \in \tbinom{[n]}{p} \text{ are incomparable}\rangle \tag{1.25}$$

$$\;=\; \bigcap_{c\in C_{m,p}} \langle p_I \mid I \notin c\rangle\,. \tag{1.26}$$

Proof. The description (1.25) is an immediate consequence of Theorem 1.23, the initial ideal is generated by the initial terms of the elements $S(I,J)$ which form the Gröbner basis.

The inclusion of the initial ideal in the intersection follows as no chain can contain both I and J, when I and J are incomparable. For the other inclusion, consider a monomial $p := p_{I_1}p_{I_2}\cdots p_{I_N}$ that is not in the initial ideal. Then the indices $I_1, I_2, \ldots, I_N$ must be comparable, that is, there is a chain $c \in C_{m,p}$ containing $I_1, I_2, \ldots, I_N$. But then p does not lie in $\langle p_I \mid I \notin c\rangle$. ∎

Initial ideals have an important geometric consequence.

Proposition 1.27. *Let X be a subvariety of a projective space $\mathbb{P}^n$ over a field $\mathbb{K}$ with initial ideal $\mathcal{J}$ with respect to some term order. Then there is a $\mathbb{K}^\times$-action on $\mathbb{P}^n$ inducing a 1-parameter (flat) deformation of X into the algebraic scheme of $\mathcal{J}$.*

By the expression (1.26) for the initial ideal $in_{\prec_{drl}}\mathcal{I}$ of the Grassmannian, its scheme is simply the union of $\#C_{m,p}$ coordinate planes in Plücker space.

$$\bigcup_{c\in C_{m,p}} \operatorname{Span}\langle p_I \mid I \in c\rangle\,,$$

and so the Grassmannian has a flat deformation to this union. Since flatness preserves numerical invariants of algebraic varieties, such as degree, we can compute the degree of the Grassmannian.

Theorem 1.28. *The degree of the Grassmannian $\mathrm{Gr}_{p,n}$ in its Plücker embedding is $\#C_{m,p}$.*

This gives yet another derivation that the intersection number for the problem of the introduction is the number of maximal chains in the Bruhat order.

All of these ideals, the Plücker ideal and its initial ideal, and even the ideal of a Schubert variety,

$$\mathcal{I}(X_\lambda F_\bullet) \;=\; \mathcal{I}_{m,p} + \langle p_J \mid J \not\leq I(\lambda)\rangle\,,$$

are all finely tuned to the Bruhat order. This has the following consequence. The same flat deformation of the Grassmannian that we just used transforms a Schubert variety into a union of coordinate planes, one for each chain in the interval $[\mu, top]$ in the Bruhat order. Further details may be found in ?Sturmfels?, ?Sturmfels-White?.

2. New developments in Schubert calculus for Grassmannians

Recently, this classical Schubert calculus has been extended and deepened through the work of many outstanding young mathematicians. Here, we keep to our main theme of the Grassmannian. We begin with some deepening of the classical results of Section 1 and then move on to extensions in K-theory, equivariant cohomology, and quantum cohomology.

2.1. **Reality.**

The observation in Section 1 that the problems of four lines can have both of its solution lines be real is clasical. Strangely, questions of reality were not pursued classically. Fulton, in his 1984 booklet on Intersection Theory [11, p. ??], may have been the first to ask whether enumerative geometric problems can have all their solutions be real. We call such an enumerative geometric problem **fully real**. Subsequent results, primarily in the Schubert calculus, suggest that fully real enumerative geometic problems may be the norm. For a survey of this story and its relation to solving equations, see [43].

The first substantial evidence for this ubiquity of full reality was given in 1997 [38]. There, it was shown that the Schubert calculus on the Grassmannian $\mathrm{Gr}_{2,n}$ of 2-dimensional linear subspaces of a vector space is fully real. The main technique was the construciton of delicate geometric deformations that transform a general intersection of Schubert varieties

$$X_{\lambda^1}F_\bullet{}^1 \cap X_{\lambda^2}F_\bullet{}^2 \cap \cdots \cap X_{\lambda^s}F_\bullet{}^s \tag{2.1}$$

on $\mathrm{Gr}_{2,n}$ into a union of Schubert varieties that is free free of multiplicities. Furthermore, the Schubert in that union could be spoecified to be real. When (2.1) is 0-dimensional (the situation in an enumerative problem) this implied that there was a choice of real flags $F_\bullet{}^1, F_\bullet{}^2, \ldots, F_\bullet{}^s$ such that every point in the intersection (2.1) is real.

That argument had two byproducts. First, it was necessary to argue from first principles that the intersection (2.1) was transverse, and those arguments were independent of the field. This proved that a general collection of Schubert varieties in $\mathrm{Gr}_{2,n}$ meet transversally in any characteristic. A second byproduct was a geometric proof for the intersection numbers in $\mathrm{Gr}_{2,n}$.

(1) Shapiro
(2) Evidence, ERA AMS article with proof
(3) Numerical algorithms
(4) Eremenko-Gabrielov rational functions
(5) Eremenko-Gabrielov real-degree
(6) Vakil's work.

2.2. K-**theory.**

2.3. **Equivariant cohomology.**

2.4. **Quantum Cohomology and systems theory.**

Giambelli-Pieri-LR rule

Equations.

Reality

Transversality

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Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA

E-mail address: `sottile@math.umass.edu`

URL: `http://www.math.umass.edu/~sottile`