Algebra II Winter 2021 25 January
Frank Sottile Second Homework

Write your answers neatly, in complete sentences, and prove all assertions. Start each problem on a new page (this makes it easier in Gradescope). Revise your work before handing it in, and submit a .pdf created from a LaTeX source to Gradescope. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Due Monday 1 February.

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1. Let $S := \{1, 2, 2^2 = 4\}$, a multiplicatively closed subset of $R := \mathbb{Z}/6\mathbb{Z}$.

   Determine (compute and identify) the ring $R[S^{-1}]$ of fractions of R by S.

2. Let R be a commutative ring and suppose that $S \subset R$ is a multiplicatively closed subset (multiplicative sub-semigroup of R). Identify the kernel of the canonical map $\iota\colon R \to R[S^{-1}]$.

3. Show that for any ring R and R-module M, $\text{Hom}_R(R, M) \simeq (M, +, 0)$, as abelian groups.

4. Let R be a ring and A be an abelian group. For $r \in R$ and $f \in \text{Hom}_{\mathbb{Z}}(R, A)$, define $r.f\colon R \to A$ by $(r.f)(x) = f(xr)$ for $x \in R$. Show that this gives $\text{Hom}_{\mathbb{Z}}(R, A)$ the structure of an R-module. (Part of this problem is showing that $r.f \in \text{Hom}_{\mathbb{Z}}(R, A)$.)

5. Let R be a ring and A, B, M, and N be R-modules. Let $f \in \text{Hom}_R(A, M)$ and $g \in \text{Hom}_R(N, B)$. For $\varphi \in \text{Hom}_R(M, N)$, define $f^*(\varphi) := \varphi \circ f$ and $g_*(\varphi) := g \circ \varphi$. Show that these give homomorphisms of abelian groups,
$$f^*\colon \text{Hom}_R(M, N) \to \text{Hom}_R(A, N) \quad \text{and} \quad g_*\colon \text{Hom}_R(M, N) \to \text{Hom}_R(M, B).$$
   Show that $f \mapsto f^*$ is a homomorphism of abelian groups $\text{Hom}_R(A, M) \to \text{Hom}_Z(\text{Hom}_R(M, N), \text{Hom}_R(A, N))$.

6. Let M be an R-module. Show that $\text{Hom}_R(M, M)$ is a ring whose product is the composition of functions. It is called the *endomorphism ring* of M, written $\text{End}(M)$.

   Show that M is a left $\text{End}(M)$-module under the action by elements $f \in \text{End}(M)$ defined by $f.m = f(m)$, for $m \in M$.

7. An R-module M is *simple* if its only submodules are 0 and M. Prove that every simple R-module is cyclic.

   Prove *Schur's Lemma*, that when M is simple, $\text{End}(M)$ is a division ring.

8. (Five Lemma). Consider the following commutative diagram of R-modules, with exact rows:
$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$
   (a) Prove that if f_1 is a surjection and f_2, f_4 are injections, then f_3 is an injection.
   (b) Prove that if f_5 is an injection and f_2, f_4 are surjections, then f_3 is a surjection.

9. (Splicing short exact sequences). If $0 \to A \to B \xrightarrow{f} C \to 0$ and $0 \to C \xrightarrow{g} D \to E \to 0$ are short exact sequences of R-modules, then the sequence $0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is exact.
   Show that every exact sequence may be obtained by splicing together suitable short exact sequences in this manner.