Finance Data Science
Lecture 6: Regression

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Notes

Outline

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What is regression?

In regression we are given a training set in the form of a matrix-vector pair:

$$X = [x_1, \dots, x_m] \in \mathbf{R}^{n \times m}, \quad y \in \mathbf{R}^m$$

where

- $x_i \in \mathbf{R}^n$ are m data points in n-dimensional "feature space";
- $y = (y_1, \dots, y_m)$ are corresponding "outputs" or "responses".

The goal of regression is to come up with a "prediction rule" $\hat{y}(x)$ that predicts the output for an unseen point $x \in \mathbf{R}^n$.

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Linear and non-linear prediction

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In linear prediction, we look for prediction rules of the form

$$\hat{y}(x) = w^T x + b$$

where $w \in \mathbf{R}^n$ and $b \in \mathbf{R}$ are the model parameters.

Most methods presented today are directly extended to "non-linear prediction rules", provided we work with non-linear features $\phi(x)$ instead of x, via

$$\hat{y}(x) = w^T \phi(x) + b.$$

Example:

$$\hat{y}(x) = w_1 x_1 + w_2 x_2 + w_3 x_1 x_2.$$

In a lecture 7 we explore these ideas in more detail; here we will focus on linear prediction rules.

Notes

Model fitting

To fit the model we usually solve a problem such as

$$\min_w \mathcal{L}(X^T w + b\mathbf{1}, y) + \lambda p(w),$$

where

- $\mathcal{L}$ is a convex loss function that encodes the error between the observed value and the predicted value;
- (w, b) are the model parameters;
- p is a penalty on the regression parameters;
- $\lambda > 0$ is a penalty parameter, obtained via cross-validation.

Most popular models are implemented in open-source packages such as scikit-learn [2].

Notes

Validation and testing

The cross-validation (over the penalty parameter λ) involves randomly selecting a subset of the data (representing say 70% of the data points), fitting the model, and testing on the remaining part via the prediction rule.

A new point is then given a predicted output via

$$\hat{y}(x) = w^T x + b.$$

Once that phase is done, we select the best value of the penalty parameter, and provide the final test results on an unseen test set.

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Ordinary least-squares
Definition

Given $X \in \mathbf{R}^{n \times m}$, $y \in \mathbf{R}^m$, the *Ordinary Least-Squares* (OLS) problem is

$$\min_w \|X^T w - y\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm, and $w \in \mathbf{R}^n$ is the variable.

- Problem is ubiquituous ones in engineering, sciences, economics and finance.
- Solved by Legendre, Gauss ($\sim$ 1850).
- Very mature solution technology via linear algebra (*e.g.*, SVD) techniques.
- One of the most basic convex problems, used inside many convex optimization algorithms.

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Least-squares
Applications

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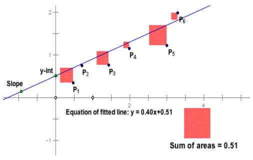
- Fitting auto-regressive models for log-return predictions.
- Various predictions in marketing, consumer credit, econometrics, etc.
- Solving simple portfolio optimization; index tracking.
- Generally, fitting models to data.

Notes

Interpretation

Smallest distance to consistency



OLS can be interpreted as finding the closest perturbation to "measurement" y to make equation $X^T w = y$ consistent (meaning, it has a solution w):

$$\min_{w,e} \|e\|_2 \;:\; X^T w = y + e.$$

e is noise that corrupted the measurement and made the model inconsistent.

Notes

Prediction

Previous interpretation useful in the context of prediction.

- In many cases, each column x_t of data matrix X corresponds to a measurement. (We use t to denote the column index.)
- The underlying model is

$$y_t = x_t^T w + e_t, \quad t = 1, \ldots, n,$$

  where $e \in \mathbf{R}^T$ is a noise vector. Assume e is random, with $\mathbf{E}\, e = 0$.

- *Question:* if we add one measurement (row x_{n+1}^T of X^T), what will be the corresponding output?
- *Answer:* since $\mathbf{E}\, e = 0$, the expected value of the new output y_{n+1} is

$$\hat{y}_{n+1} = x_{n+1}^T x.$$

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Example

Prediction via auto-regressive models

Auto-regressive (AR) model for time-series y_t:

$$y_t = w_1 y_{t-1} + \ldots + w_p y_{t-p} + e_t, \quad t = 1, 2, 3, \ldots$$

where vector $w \in \mathbf{R}^p$ determines the model parameters.

Find x by fitting based on $n + p$ observations of past data $(y_t)_{t=1}^{t=n+p}$

$$\min_w \|X^T w - y\|_2,$$

where $y = (y_{n+p}, \ldots, y_{p+1})$, and $p \times n$ X has t-th column equal to $(y_{n+p-t}, \ldots, y_{n+1-t})$.

(Each column of X corresponds to a new time point.)

Notes

Example

Prediction via auto-regressive models

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Once we've solved for w, we can make a prediction based on a new data value y_{n+p+1}:

$$\hat{y}_{n+p+1} = w_1 y_{n+p} + \ldots + w_n y_{n+1}.$$

Allows to form an average prediction error when we run the algorithm in a "sliding window" fashion.

Notes

Solution to OLS

If $p \times n$ matrix X is full row rank (XX^T is invertible), solution is unique:

$$w_{\mathrm{OLS}} = (XX^T)^{-1}Xy.$$

- Closed-form expression is rarely used. Algorithms such as QR decomposition or SVD are.
- Computational complexity grows as $\sim (pn^2 + n^3)$.
- Expression fails when X is not full rank. Then, nullspace of X^T describes ambiguity in solution. SVD methods can provide the whole subspace of solutions.

Notes

Regularized least-squares

Definition

In practice, OLS may provide solutions that are very sensitive to changes in input data (A, y).

Regularized LS:

$$\min_w \|X^T w - y\|_2^2 + \lambda \|w\|_2^2$$

where $\lambda > 0$ is the *regularization* parameter.

Stochastic interpretation:

$$\min_w \mathbf{E} \, \|(X + N)^T w - y\|_2^2$$

where N is random noise matrix, with $\mathbf{E}\, N = 0$ and $\mathbf{E}\, N^T N = \lambda I$.

Notes

Regularized least-squares
Solution

Solution: always unique, and given by

$$w_{\mathrm{RLS}} = (\lambda I + XX^T)^{-1} Xy.$$

- Parameter $\lambda > 0$ enforces invertibility.
- This parameter is usually chosen via cross-validation.
- Again, closed-form expression rarely used; linear algebra techniques use OLS method for the equivalent (OLS) problem

$$\min_w \left\| \begin{pmatrix} X^T \\ \sqrt{\lambda} I \end{pmatrix} w - \begin{pmatrix} y \\ 0 \end{pmatrix} \right\|_2 .$$

(Note that matrix involved is always full rank, not matter what data X is.)

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Example
AR model for prediction

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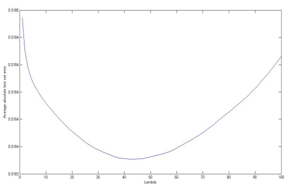

AR model for prediction via regularized LS: average prediction error vs. regularization parameter.

- *Data:* APPL log-returns.
- *Method:* AR model fitted via regularized LS.
- Curve shows average prediction error, with algorithm run in “sliding window” mode.

Notes

Motivation

We will examine different models based on a linear assumption: that, for a new data point $x \in \mathbf{R}^n$, the predicted value is an *affine* function of the input x:

$$\hat{y}(x) = x^T w + b.$$

where $w \in \mathbf{R}^n$ contains the *regression coefficients* and $b \in \mathbf{R}$ is an offset. (In lecture 7, we explore non-linear alternatives.)

Together, w, b are the parameters of the model, which we wish to "learn" from training data samples (x_i, y_i), $i = 1, \ldots, m$.

Notes

Generalized regression

We consider the problem

$$\min_w \mathcal{L}(X^T w + b\mathbf{1}, y) + \lambda p(w),$$

where

- $\mathcal{L}$ is a convex loss function that encodes the error between the observed value and the predicted value;
- (w, b) are the model parameters;
- p is a penalty on the regression parameters;
- $\lambda > 0$ is a penalty parameter.

When $\mathcal{L}(z, y) = \|z - y\|_2^2$, $p(w) = \|w\|_2^2$, we recover regularized least-squares.

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Playing with loss functions and penalties

Changing loss functions allos to cover these types of regression methods:

- Least-absolute deviation: to be less senstive to outliers than LS;
- Quantile regression: to predict intervals of confidence;
- Chebyschev regression: to work with largest errors only;
- KL divergence: to fit probability models

Typical penalties allow to

- l_1-norm: to enforce sparsity;
- l_2-norm (often, squared): to control statistical noise and improve prediction error;
- sum-block norms enable to enforce whole blocks of w to be zero.

Notes

LASSO

In LASSO, we solve the problem

$$\min_w \|X^T w - y\|_2^2 + \lambda \|w\|_1.$$

- Here the model encourages sparsity of the result, due to the term $\|w\|_1$ in the penalty.
- The motivation is to be able to *interpret* the results, by finding the features that are most “predictive”.
- In practice, we cross-validate the choices of λ. Alternatively: select features first by (pure) LASSO, then run regularized LS. Another alternative is seen next.

LASSO can be unstable (non-unicity of the result), esp. with correlated features.

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Elastic net

In Elastic net, we solve the problem

$$\min_w \|X^T w - y\|_2^2 + \lambda\|w\|_1 + \mu\|w\|_2^2,$$

with $\mu > 0$ an extra regularization parameter.

- Here the model still encourages sparsity of the result, due to the term $\|w\|_1$ in the penalty.
- But it balances the sparsity against some stability.
- And allows for a better control of sparsity.

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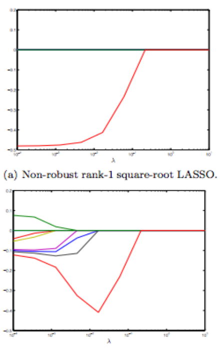
Controlling for sparsity



(a) Non-robust rank-1 square-root LASSO.

(b) Robust rank-1 square-root LASSO.

When data is low-rank, controlling sparsity is hard.

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Least-absolute deviation

In least-absolute deviation, we solve the problem

$$\min_w \ \|X^T w - y\|_1 + \lambda p(w)$$

with (for example) $p(w) = \|w\|_2^2$.

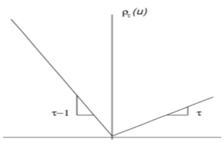
- Since the l_1 allows some elements of the vector $X^T w - y$ to be large, it can tolerate outliers better than l_2-norm loss.
- This method is robust, but unstable (it may change much in result to changes in the data).
- Adding a (squared) regularization term $p(w) = w^T w$ allows to control unstability.

Notes

Sample quantile

Given values $z_1, \ldots, z_m$, the *median* is given by

$$\min_q \sum_{i=1}^m |z_i - q|.$$



More generally, the minimizer for the problem

$$\min_q \ (1-\tau)\sum_{z_i<q}(q - z_i) + \tau \sum_{z_i \geq q}(z_i - q) = \sum_{i=1}^n \rho_\tau(z_i - q),$$

gives the τ% quantile, with

$$\rho_\tau(u) := \max(\tau u, (\tau - 1)u).$$

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Quantile regression

In quantile regression, we solve the problem

$$\min_{w} \sum_{i=1}^{m} \rho_\tau(x_i^T w - y_i) + \lambda p(w)$$

with (for example) $p(w) = \|w\|_2^2$.

- A linear or quadratic programming problem.
- Included in the StatsModel package `http://www.statsmodels.org/stable/index.html`.

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References

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