Finance Data Science
Lecture 12: Portfolio Optimization, II

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Outline

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Risk parity and budgeting

Motivations

Risk parity is an alternative to the classical mean-variance approach, where, instead of maximizing a risk-adjusted expected return, we focus on balancing ("budgeting") to so-called partial risks.

Partial risks correspond to the contribution of a given asset to the overall risk of the portfolio.

This approach is motivated by the fact that returns are very hard (impossible?) to estimate reliably, while estimating covariances is usually more reliable.

Partial risks

Let $x \in \mathbf{R}^n_+$ contain positions of a long-only portfolio. The (variance) risk is defined as

$$\sigma(x) := x^T C x = \sum_{i=1}^{n} x_i (Cx)_i = \frac{1}{2} \sum_{i=1}^{n} x_i \frac{\partial \sigma(x)}{\partial x_i}.$$

The *risk contributions* (partial risks) of asset i are defined to be

$$\sigma_i(x) = x_i (Cx)_i.$$

By construction:

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x).$$

Risk parity and budgeting

Risk parity is an alternative to the classical mean-variance approach, where, instead of maximizing a risk-adjusted expected return, we focus on balancing ("budgeting") partial risks.

This is motivated by the fact that returns are very hard (impossible?) to estimate reliably, while estimating covariances is usually more reliable.

Typically, we work with non-linear equality constraints of the form

$$\sigma_i(x) = \theta_i \sigma(x), \quad i = 1, \ldots, n,$$

with θ_i, $i = 1, \ldots, n$ a given set of parameters that sum to one, which defines our risk budgets.

A portfolio that satisfies the above with $\theta_i = 1/n$ is said to achieve *risk parity*, or *equal risk contributions*.

Risk budgeting and convexity

Risk budgeting problem: (for example)

$$\min_{x \in \mathcal{X}} \ \sigma(x) \ : \ \sigma_i(x) = \theta_i \sigma(x), \quad i = 1, \ldots, n,$$

where $\mathcal{X}$ is a set that corresponds to (usually) convex constraints, such as

- bounds on portfolio positions;
- bounds on transaction costs;
- constraints on diversification;
- etc.

The risk budgeting constraints are *non-linear equalities* , hence they are not convex.

Approaches:

- Using a non-linear solver such as `fmincon` in matlab might run into difficulties when there are many other constraints.
- Alternatively we can use convex models, as described next.

Unconstrained case

Fact: at the (unique) optimum of the convex problem

$$\min_x \sum_{i=1}^n x^T C x \; : \; \sum_{i=1}^n \theta_i \log x_i \geq 1, \;\; x \geq 0$$

we have

$$x_i (Cx)_i = \theta_i (x^T C x), \;\; i = 1, \ldots, n.$$

Proof: uses optimality conditions for convex problems.

- Proves that we can always find a portfolio that satisfies given risk budgeting constraints.
- Approach does not generalize to the case when x is constraint (*e.g.*, upper bounds on positions, transaction costs, etc.)

Constrained case

A more flexible approach is to note that, for any given $x \geq 0$, the conditions

$$x_i(Cx)_i \geq \theta_i x^T Cx, \quad i = 1, \ldots, n,$$

are equivalent to the risk budfgeting constraints:

$$x_i(Cx)_i = \theta_i x^T Cx, \quad i = 1, \ldots, n,$$

Proof: by contradiction, assume one of the inequalities is strict, and sum them to get

$$x^T Cx = \sum_{i=1}^{n} x_i(Cx)_i < (x^T Cx) \sum_{i=1}^{n} \theta_i,$$

which, since $x \neq 0$, C is positive-definite and $\mathbf{1}^T \theta = 1$, leads to a contradiction.

Rotated second order cone

We proceed by noting that the conditions on a triple (z, u, v) with z a vector and u, v scalars:

$$uv \geq z^T z, \quad u \geq 0, \quad v \geq 0,$$

are convex (in (z, u, v)), and in fact, equivalent to the second-order cone constraints

$$\left\| \begin{pmatrix} 2z \\ u - v \end{pmatrix} \right\|_2 \leq u + v.$$

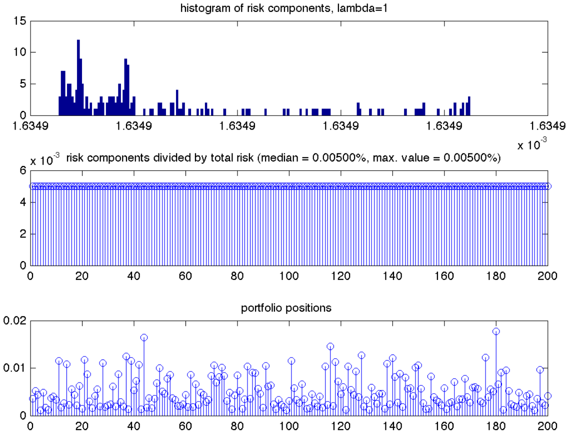
Proof: square the above; be careful in proving the correct signs!

Our inequality-based risk budget constraints then read

$$\left\| \begin{matrix} 2\sqrt{\theta_i} Rx \\ x_i - (Cx)_i \end{matrix} \right\|_2 \leq x_i + (Cx)_i, \quad i = 1, \ldots, n,$$

where R is any matrix such that $C = RR^T$. These convex constraints allow us to solve any portfolio optimization problem with risk budget and other convex constraints.

Example



Risk parity achieved on a portfolio of 200 assets.

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References

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