Finance Data Science
Lecture 13: Robust Portfolio Optimization

Laurent El Ghaoui

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Haas School of Business
UC Berkeley

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Recap: SOCP

Robust Optimization
Uncertainties
Robust LP
Chance Constraints

Covariance Robustness
Worst-case risk
Scenarios models

Summary and References

Outline

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Second-Order Cone Program

A problem of the form

$$\min_x \ c^T x \ : \ \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \ \ i = 1, \ldots, m,$$

is a *second-order cone program*. It can be solved via CVX, MOSEK, CPLEX, via efficiency close to that for LPs.

SOCP's include as special cases:

- LP's and QP's.
- Problems with convex quadratic constraints and objective (QCQP).

SOCPs arise in many applications, including finance [5].

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Handling quadratic constraints

The convex quadratic constraint on x

$$x^T Qx \leq t$$

where Q is positive semi-definite, is equivalent to

$$\left\| \begin{pmatrix} 2Rx \\ (t-1) \end{pmatrix} \right\|_2 \leq (t+1).$$

(Here, R is any matrix with $R^T R = Q$.)

This allows to model QP's and QCQP's as SOCP's—we just have to find a factor R for the matrix Q (guaranteed to exist since Q is PSD).

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QCQP

Quadratically constrained quadratic program (QCQP):

$$\min_x \; c^T x + x^T Q x \; : \; c_i^T x + x^T Q_i x \le d_i, \quad i = 1, \ldots, m$$

- A convex program, provided $Q_0, \ldots, Q_m$ are all positive semi-definite (PSD).
- Can be reduced to an SOCP, with same number of variables and constraints.
- Most commercial software requires the transformation to SOCP standard form to be done by the user; CVX does not.

Modeling powers of variables

Market impact models often involve power functions, such as

$$f(x) = \sum_{i=1}^{n} |x_i - x_i^0|^p,$$

where x^0 is the given initial position, and p is a non-negative scalar. These functions are convex for $p \geq 1$.

SOCP framework can also handle powers (for $p \geq 1$). For example, the constraint on $x \in \mathbf{R}$: $|x|^{3/2} \leq 1$ is equivalent to the existence of z, w, v such that

$$wt \geq z^2, \quad z \geq w^2, \quad z \geq |x|.$$

The first two constraints are (rotated) second-order cone constraints; the last one is equivalent to two linear inequalities.

In CVX, no need to do this! Use the function `pow_abs`.

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Convex optimization models

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"Nominal" optimization problem:

$$\min_x \; f_0(x) \;:\; f_i(x) \le 0, \quad i = 1, \ldots, m$$

f_0, f_i's are convex.

- Includes many problems arising in decision making, statistics.
- Efficient (polynomial-time) algorithms.
- Convex relaxations for non-convex problems.

Uncertainties are a pain!!

In practice, problem data is uncertain:

- *Estimation* errors affect problem parameters.
- *Implementation* errors affect the decision taken.

Uncertainties often lead to highly unstable solutions, or much degraded realized performance.

These problems are compounded in problems with multiple decision periods.

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Robust counterpart

"Nominal" optimization problem:

$$\min_x f_0(x) \ : \ f_i(x) \leq 0, \quad i = 1, \ldots, m.$$

Robust counterpart:

$$\min_x \max_{u \in \mathcal{U}} f_0(x, u) \ : \ \forall \, u \in \mathcal{U}, \quad f_i(x, u) \leq 0, \quad i = 1, \ldots, m$$

- functions f_i now depend on a second variable u, the "uncertainty", which is constrained to lie in given set $\mathcal{U}$.
- Inherits convexity from nominal. Very tractable in some practically relevant cases.
- Complexity is high in general, but there are systematic ways to get relaxations.

Robust chance counterpart

(Assume for simplicity there are no constraints)

$$\min_x \max_{p \in \mathcal{P}} \mathbf{E}_p f_0(x, u).$$

- Uncertainty is now random, obeys distribution p.
- Distribution p is only known to belong to a class $\mathcal{P}$ (*e.g.*, unimodal, given first and second moments).
- Complexity is high in general, but there are systematic ways to get relaxations.
- Rich variety of related models, including Value-at-Risk constraints.

In this lecture: our main goal is to introduce some important concepts in robust optimization, *e.g.* robust counterparts, distributional robustness.

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Uncertainty models

Nominal problem:

$$\min_x \; c^T x \; : \; a_i^T x \leq b_i, \quad i = 1, \ldots, m.$$

We assume that $a_i = \hat{a}_i + \rho u_i$, where

- $\hat{a}_i$'s are the nominal coefficients.
- u_i's are the uncertain vectors, with $u_i \in \mathcal{U}_i$ but otherwise unknown.
- $\rho \geq 0$ is a measure of uncertainty.

Assumption that uncertainties affect each constraint independently is done without loss of generality.

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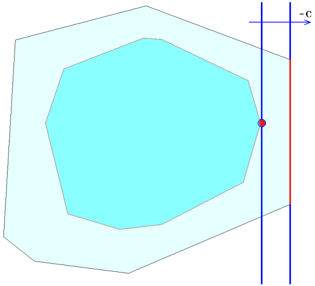
Robust counterpart

Robust counterpart:

$$\min_x \; c^T x \;:\; \forall u_i \in \mathcal{U}_i, \;\; (\hat{a}_i + \rho u_i)^T x \le b_i, \;\; i = 1, \ldots, m.$$

Solution may be hard, but becomes easy when:

- $\mathcal{U}_i$ are polytopic, given by their vertices (“scenarios”);
- $\mathcal{U}_i$’s are “simple” sets such as ellipsoids, boxes, etc.
- Complexity governed by the support functions of sets $\mathcal{U}_i$.



Robust LP with ellipsoidal uncertainty.

Chance constraints

Simple case

Consider an LP, and assume one of the constraints is $a^T x \leq b$, where $x \in \mathbf{R}^n$ is the decision variable.

If a is random, we can often deal with the chance constraint

$$\text{Prob}\left\{a^T x \leq b\right\} \geq 1 - \epsilon$$

easily. For example, if a is Gaussian with mean $\hat{a}$ and covariance matrix Γ, above is equivalent to

$$\hat{a}^T x + \kappa(\epsilon)\|\Gamma^{1/2} x\|_2 \leq b,$$

where $\kappa(\cdot)$ is a known function that is positive when $\epsilon < 0.5$.

Distributional robustness

Consider instead

$$\sup_{p \in \mathcal{P}} \mathbf{Prob}_p \left\{ (u,1)^T W (u,1) > 0 \right\} \leq \epsilon$$

where the sup is taken with respect to all distributions p in a specific class $\mathcal{P}$, specifying *e.g.*:

- Moments.
- Symmetry, unimodality.

Example

Transaction costs In many financial decision problems, the transaction costs can be modeled with

$$T(x,u) = \|A(x)u + b(x)\|_1,$$

for appropriate affine $A(\cdot)$, $b(\cdot)$.

Example:

$$\sum_{t=1}^{T} |x_{t+1} - x_t|$$

with decision variable x_t an affine function of u.

This leads to consider quantities such as

$$\max_{u \sim (0,I)} \mathbf{E}\, T(x,u)$$

where $u \sim (0, I)$ refers to distributions with zero mean and unit covariance matrices.

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A useful result

For given $m \times d$ matrix A and d-vector b, define

$$\phi := \max_{u \sim (0,I)} \mathbf{E} \, \|Au + b\|_1$$

Let a_i denote the i-th row of A ($1 \le i \le m$). Then

$$\frac{2}{\pi}\psi \le \phi \le \psi,$$

where

$$\psi := \sum_{i=1}^{m} \left\| \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\|_2 .$$

Note: ψ is convex in A, b, which allows to minimize it if A, b are affine in the decision variables. Leads to second-order cone constraint.

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Covariance uncertainty: robust approach

Assume that covariance matrix is unknown but bounded in a given set $\mathcal{U}$. Key assumption: every matrix in $\mathcal{U}$ is positive semi-definite (PSD). (We write: $C \succeq 0$.)

Robust counterpart:

$$\max_{x} \min_{C \in \mathcal{U}} r^T x - \lambda \cdot x^T C x \;:\; Ax \leq b, \;\; Cx = d,$$

- Decision-maker is in a “game” against the uncertainty.
- The shape and size of the uncertainty set $\mathcal{U}$ guides its decision.

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Worst-case risk function

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Robust portfolio problem:

$$\max_x \; r^T x - \lambda \cdot R(x) \;:\; Ax \leq b, \;\; Cx = d,$$

where

$$R(x) := \max_{C \in \mathcal{U}} \; x^T C x$$

is the *wort-case risk* of a given portfolio x.

- By construction (as the maximum of convex functions of x), the worst-case risk is a *convex* function of x.
- Here, we use our key assumption to say that $x \to x^T C x$ is convex, since C is PSD whenever $C \in \mathcal{U}$.

Scenario model

Assume that $\mathcal{U} = \{C^{(1)}, \ldots, C^{(N)}\}$, where each $C^{(i)}$, $i = 1, \ldots, N$ is a particular "view" on the market. Our key assumption becomes $C^{(i)}$ is PSD for every i.

Robust counterpart:

$$\max_x \; r^T x - \lambda \max_{1 \le i \le N} \; x^T C^{(i)} x \;:\; Ax \le b, \;\; Cx = d,$$

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Robust counterpart as a QCQP

Robust counterpart is a QCQP (hence an SOCP):

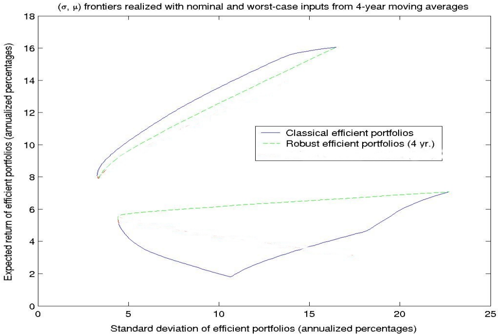
$$\max_x \; r^T x - \lambda t \; : \quad \begin{array}{l} t \geq x^T C^{(i)} x, \quad i = 1, \ldots, N, \\ Ax \leq b, \quad Cx = d, \end{array}$$

Can generalize this to return vector also uncertain, and scenarios are the pairs $(r^{(i)}, C^{(i)})$, $i = 1, \ldots, N$:

$$\max_x \; \alpha - \lambda t \; : \quad \begin{array}{l} \alpha \geq (r^{(i)})^T x, \quad i = 1, \ldots, N, \\ t \geq x^T C^{(i)} x, \quad i = 1, \ldots, N, \\ Ax \leq b, \quad Cx = d. \end{array}$$

Example

From [1]:



- Under *nominal* conditions, robust portfolio is slightly not as efficient as classical;
- Robust portfolio withstands deviations from nominal return conditions much better.

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Summary

- Can handle noise in both return *and* covariance matrix.
- Robustness constraints can be handled with simple uncertainty set (scenarios, boxes, ellipsoids).
- Chance constraints are hard in general, but robust chance constraints are usually easier.
- Most of these problems reduce to SOCPs.
- Framework very flexible and allows more complicated uncertainty sets.

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