Finance Data Science
Lecture 15: Further Topics

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Outline

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Portfolio optimization problem

Data: (load `lect10data.mat`)

- Parameters t, c, β, integers n, k, p.
- Return vector $r \in \mathbf{R}^n$, initial position $x^0 \in \mathbf{R}^n$.
- Covariance matrix C, given by a factor model: $C = D + FF^T$, with D diagonal, F a $n \times k$ matrix.
- transaction cost function $TC(x) = c \cdot \|x - x^0\|_1$.
- Market impact function $MI(x) = \beta \cdot \sum_{i=1}^n |x_i - x_i^0|^{3/2}$.

Problem: Maximize portfolio return minus transaction costs and market impact, subject to the following constraints:

- An upper bound t on the portfolio variance.
- No shorting.
- The largest p positions do not represent more than 80% of the total position.

Provide a CVX code that solves this. Make sure the code prints out the final solution x.

CVX code

```
cvx_begin
variables x(n,1) z(k,1);
maximize( r'*x - c*norm(x-x0,1)-beta*sum(pow_abs(x-x0,1.5)) )
subject to
x >= 0;
sum_largest(x,p) <= .8*sum(x);
x'*D*x + z'*z <= t;
z == F'*x;
cvx_end
```

- Introducing variable $z \in \mathbf{R}^k$ allows to exploit factor structure and speed up computation (note: we never form the covariance matrix!).
- Use `pow_abs` to model market impact.

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Measures of risk

There are many ways to measure the “risk” of a random variable. In our case the random variable is the portfolio’s return, $w^T\mathbf{r}$, with $\mathbf{r}$ the (random) return, and w the portfolio weight vector.

- *Variance:* expected squared deviation from the mean
- *Downside risk:* based on downside variance
- *Value-at-Risk* (VaR): looks at probability of loss being above a target
- *Conditional VaR:* expected loss given the loss is above a target
- *Worst-case variants:* when underlying probability distribution is partially known
- Many more . . .

Some are better suited to optimization . . .

Value-at-Risk

For $\beta \in [0, 1]$, The β-VaR of the portfolio is

$$\text{VaR}_\beta(w) := \inf \left\{ t \ : \ \text{Prob}(-w^T \mathbf{r} \leq t) \right\} \geq \beta.$$

- The smallest value such that the probability of portfolio return being less than that value, is highly likely.
- Captures the β-quantile.
-

Easy to compute when probability is Gaussian, $\mathcal{N}(\hat{r}, \Sigma)$:

$$\text{VaR}_\beta(w) =$$

In practice we do not even know the distribution!

Using another measure of risk

If C is the empirical covariance matrix, the portfolio variance is

$$\sum_{t=1}^{T} [(r_t - \hat{r})^T x]^2$$

where $r_t \in \mathbf{R}^n$ is the return vector at time t (in the past), and $\hat{r} \in \mathbf{R}^n$ is the time-average of the returns.

A related measure of risk would be

$$\sum_{t=1}^{T} |(r_t - \hat{r})^T x|$$

Questions:

- Can you use this measure instead of the classical variance?
- What could be advantages for this?

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Scenario uncertainty on returns

Assume return vectors and covariance matrices are uncertain, and scenarios are the pairs $(r^{(i)}, C^{(i)})$, $i = 1, \ldots, N$:

$$\max_x \ \alpha - \lambda t \ : \quad \begin{array}{l} \alpha \geq (r^{(i)})^T x, \quad i = 1, \ldots, N, \\ t \geq x^T C^{(i)} x, \quad i = 1, \ldots, N, \\ Ax \leq b, \quad Cx = d. \end{array}$$

Interval uncertainty

A very common model derived from statistics is interval uncertainty:

$$\forall\, i \;:\; r_i \in [\hat{r}_i - \delta_i, \hat{r}_i + \delta_i],$$

where vector $\delta > 0$ contains the sizes of the intervals of confidence for r.

This entails 2^n scenarios . . . Previous approach won't scale!

Interval uncertainty

Scalable approach

Worst-case return:

$$R(x) := \min_{r \,:\, |r-\hat{r}|\leq\delta} r^T x$$

Solution:

$$R(x) = \hat{r}^T x - \delta^T |x|$$

This is a *concave* function of x, so we can maximize it via CVX.

Ellipsoidal uncertainty

Assume

$$r = \hat{r} + Ru \; : \; \|u\|_2 \leq 1$$

This describes an ellipsoid with center $\hat{r}$ and "shape" determined by matrix R.

This time, the number of scenarios is infinite . . .

Ellipsoidal uncertainty

Scalable approach

Worst-case return:

$$R(x) := \min_{r=\hat{r}+Ru \,:\, \|u\|_2 \le 1} r^T x$$

Solution:

$$R(x) = \hat{r}^T x - \|R^T x\|_2$$

This is a *concave* function of x, so we can maximize it via CVX.

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Chance constraints

Consider a single inequality $r^T x \geq t$ where x represent our portfolio position, r the return vector, and t a desired target.

If r is random, it makes sense to require that $\textbf{Prob}\{r^T x \geq t\}$ is high. This is called a *chance constraint* .

Except in special cases, chance constraints are hard to deal with.

Gaussian assumption

Assume that the return vector follows a Gaussian distribution with mean $\hat{r}$ and covariance matrix C. That is, $r = \hat{r} + C^{1/2}u$, with $u \sim \mathcal{N}(0, I)$.

The chance constraint

$$\textbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon$$

is equivalent to

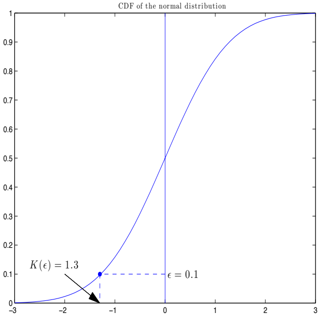
$$\hat{r}^T x \geq t + \kappa(\epsilon)\|C^{1/2}x\|_2$$

where $\kappa(\epsilon)$ is the negative of the inverse CDF of the normal distribution

$$\kappa(\epsilon) = -\Phi^{-1}(\epsilon), \quad \Phi(\beta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-u^2/2} du.$$

Chance constraint

$$\textbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon \Longleftrightarrow \hat{r}^T x \geq t + \kappa(\epsilon)\|C^{1/2}x\|_2$$



- Above is an SOC constraint when $\epsilon < 1/2$.
- approach can be used for any linear inequality with Gaussian random coefficients.

Value-at-Risk

Clearly, when the target return t grows, the portfolio return condition $\textbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon$ becomes harder to satisfy.

The Value-at-Risk at level ϵ is the largest target t attainable until the chance constraint becomes infeasible. Under the Gaussian assumption the VaR becomes

$$\hat{r}^T x - \kappa(\epsilon) \|C^{1/2} x\|_2.$$

- This is similar to the risk-adjusted return of the basic mean-variance model.
- The risk parameter is $\kappa(\epsilon)$.
- The risk measure is based on standard deviation, not variance.

Robustness of chance constraints

Previous chance constraint uses key Gaussian assumption . . . What if they are not met?

Assume returns are only known to have mean $\hat{r}$, covariance matrix C, but otherwise may follow any distribution.

Robust chance constraint: make sure that $\textbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon$ no matter the *actual* distribution is (as long as it has mean $\hat{r}$ and covariance matrix C).

Chebyschev inequality

Chebyschev inequality allows to bound the chance constraints using the mean and covariance only.

Results in robust chance constraint:

$$\hat{r}^T x \geq t + \kappa(\epsilon) \| C^{1/2} x \|_2$$

where now

$$\kappa(\epsilon) = \frac{1 - \epsilon}{\epsilon}.$$

- This is similar to the Gaussian case.
- The risk parameter is different (more conservative).

Outline

References

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