Finance Data Science
Lecture 12: Portfolio Optimization, II

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Outline

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Risk parity and budgeting

Motivations

Risk parity is an alternative to the classical mean-variance approach, where, instead of maximizing a risk-adjusted expected return, we focus on balancing ("budgeting") to so-called partial risks.

Partial risks correspond to the contribution of a given asset to the overall risk of the portfolio.

This approach is motivated by the fact that returns are very hard (impossible?) to estimate reliably, while estimating covariances is usually more reliable.

Partial risks

Let $x \in \mathbf{R}^n_+$ contain weights of a long-only portfolio. The (variance) risk is defined as

$$\sigma(x) := x^T C x = \sum_{i=1}^{n} x_i (Cx)_i = \frac{1}{2} \sum_{i=1}^{n} x_i \frac{\partial \sigma(x)}{\partial x_i}.$$

The *risk contributions* (partial risks) of asset i are defined to be

$$\sigma_i(x) = x_i (Cx)_i.$$

By construction:

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x).$$

Risk parity and budgeting

Risk parity is an alternative to the classical mean-variance approach, where, instead of maximizing a risk-adjusted expected return, we focus on balancing ("budgeting") partial risks.

This is motivated by the fact that returns are very hard (impossible?) to estimate reliably, while estimating covariances is usually more reliable.

Typically, we work with non-linear equality constraints of the form

$$\sigma_i(x) = \theta_i \sigma(x), \quad i = 1, \ldots, n,$$

with θ_i, $i = 1, \ldots, n$ a given set of parameters that sum to one, which defines our risk budgets.

A portfolio that satisfies the above with $\theta_i = 1/n$ is said to achieve *risk parity* , or *equal risk contributions* .

Risk budgeting and convexity

Risk budgeting problem: (for example)

$$\min_{x \in \mathcal{X}} \sigma(x) \ : \ \sigma_i(x) = \theta_i \sigma(x), \quad i = 1, \ldots, n,$$

where $\mathcal{X}$ is a set that corresponds to (usually) convex constraints, such as

- bounds on portfolio weights;
- bounds on transaction costs;
- constraints on diversification;
- etc.

The risk budgeting constraints are *non-linear equalities*, hence they are not convex.

Approaches:

- Using a non-linear solver such as `fmincon` in matlab might run into difficulties when there are many other constraints.
- Alternatively we can use convex models, as described next.

Unconstrained case

Fact: at the (unique) optimum of the convex problem

$$\min_x \sum_{i=1}^{n} x^T C x \ : \ \sum_{i=1}^{n} \theta_i \log x_i \geq 1, \ \ x \geq 0$$

we have

$$x_i(Cx)_i = \theta_i(x^T Cx), \ \ i = 1, \ldots, n.$$

Proof: uses optimality conditions for convex problems.

- Proves that we can always find a portfolio that satisfies given risk budgeting constraints.
- Approach does not generalize to the case when x is constraint (*e.g.*, upper bounds on weights, transaction costs, etc.)

Constrained case

A more flexible approach is to note that, for any given $x \geq 0$, the conditions

$$x_i(Cx)_i \geq \theta_i x^T Cx, \quad i = 1, \ldots, n,$$

are equivalent to the risk budfgeting constraints:

$$x_i(Cx)_i = \theta_i x^T Cx, \quad i = 1, \ldots, n,$$

Proof: by contradiction, assume one of the inequalities is strict ($>$), and sum them to get

$$x^T Cx = \sum_{i=1}^{n} x_i(Cx)_i > (x^T Cx) \sum_{i=1}^{n} \theta_i,$$

which, since $x \neq 0$, C is positive-definite and $\mathbf{1}^T \theta = 1$, leads to a contradiction.

Rotated second order cone

We proceed by noting that the conditions on a triple (z, u, v) with z a vector and u, v scalars:

$$uv \geq z^T z, \quad u \geq 0, \quad v \geq 0,$$

are convex (in (z, u, v)), and in fact, equivalent to the second-order cone constraints

$$\left\| \begin{pmatrix} 2z \\ u - v \end{pmatrix} \right\|_2 \leq u + v.$$

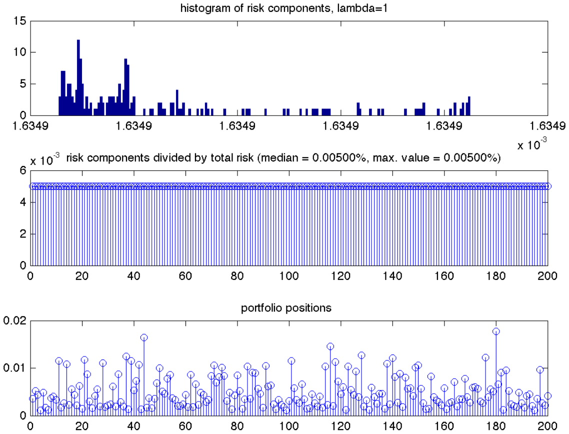
Proof: square the above; be careful in proving the correct signs!

Our inequality-based risk budget constraints then read

$$\left\| \begin{array}{c} 2\sqrt{\theta_i} Rx \\ x_i - (Cx)_i \end{array} \right\|_2 \leq x_i + (Cx)_i, \quad i = 1, \ldots, n,$$

where R is any matrix such that $C = RR^T$. These convex constraints allow us to solve any portfolio optimization problem with risk budget and other convex constraints.

Example



Risk parity achieved on a portfolio of 200 assets.

Outline

Index tracking problem

Goal: track a given index by investing in a universe of n assets. The problem can be expressed as

$$\min_{w \in \mathcal{W}} \|Rw - y\|_2^2,$$

where

- $R \in \mathbf{R}^{T \times n}$ contains (log-) return historical data, where each column contains the historical returns of a specific asset;
- $y \in reals^T$ contains the historical returns of the index to be tracked.
- Set $\mathcal{W}$ allows to constrain the replicating portfolio weights w (*e.g.*, no shorting, budget constraint).

The above can be extended to a forward-looking (as opposed to historical) setup, via a stochastic model ...

Stochastic model

Model the asset return vector $r \in \mathbf{R}^n$ as a random variable. Do the same with the index return, $\rho \in \mathbf{R}$. We assume that the mean and covariance matrix of the random vector $(r, \rho) \in \mathbf{R}^{n+1}$ are known:

$$\mathbf{E} \begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix}, \quad \mathrm{Cov}(r, \rho) = \begin{pmatrix} C & c \\ c^T & \gamma \end{pmatrix}.$$

Goal: minimize the variance of the tracking error $w^T r - \rho$, which is given by

$$\mathbf{E}(w^T r - \rho)^2 = w^T \tilde{C} w - 2\tilde{c}^T w + \tilde{\gamma},$$

where

$$\begin{pmatrix} \tilde{C} & \tilde{c} \\ \tilde{c}^T & \tilde{\gamma} \end{pmatrix} = \mathbf{E} \begin{pmatrix} r \\ \rho \end{pmatrix} \begin{pmatrix} r \\ \rho \end{pmatrix}^T = \begin{pmatrix} C & c \\ c^T & \gamma \end{pmatrix} + \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\rho} \end{pmatrix}^T.$$

We obtain a problem similar to the one obtained previously:

$$\min_w \; w^T \tilde{C} w - 2 w^T \tilde{c}$$

(Previously we had $C = R^T R$, $c = R^T y$.)

Constraints

In practice we work with many constraints:

- bounds on potfolio weights;
- no shorting;
- budget constraints;
- transaction costs;
- *Sparsity constraints:* we'd like to make sure we invest in just a few assets, so as to minimize the hassle of trading a large numbre of assets.

Next, we focus on the problem with no shorting, budget and cardinality (number of non-zeros) constraints:

$$\min_w \|Rw - y\|_2^2 \;:\; w \geq 0, \;\; w^T \mathbf{1} = 1, \;\; \|w\|_0 \leq k,$$

where $\|\cdot\|_0$ stands for cardinality and $k < n$ is given.

Where l_1-norm fails

The classical approach to deal with cardinality constraints is to use the l_1 norm. Unfortunately, this approach fails here, since the l_1 norm of w is . . . the constant 1!

Alternative approach: use the fact that

$$\|w\|_0 \geq \frac{\|w|_1}{\|w\|_\infty}$$

We obtain the non-convex relaxation:

$$\min_w \|Rw - y\|_2^2 \ : \ w \geq 0, \ \ w^T\mathbf{1} = 1, \ \ \frac{1}{k} \leq \max_{1 \leq i \leq n} w_i.$$

We need to solve n convex problems, each one corresponding to one variable fixed at the maximal value $1/k$:

$$(P_i) \ : \ \min_w \|Rw - y\|_2^2 \ : \ w \geq 0, \ \ w^T\mathbf{1} = 1, \ \ \frac{1}{k} \leq w_i.$$

Non-linear programming approach

We can refine the previous lower bound on cardinality, using the *fact* : for any distribution p,

$$H(p) := -\sum_{i=1}^{n} p_i \log p_i \leq \log \|p\|_0.$$

Proof: maximum entropy is attained for uniform distribution.

Note that the entropy function is concave (each term is). We obtain the non-convex relaxation

$$\min_w \|Rw - y\|_2^2 \;:\; w \geq 0, \;\; w^T \mathbf{1} = 1, \;\; -\sum_{i=1}^{n} p_i \log p_i \leq \log k.$$

To solve this, we can employ linearization techniques. The quality of the result depends highly on the initial point.

Outline

References

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