Finance Data Science
Lecture 15: Chance Constraints and Risk Measures

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Notes

Outline

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Portfolio optimization problem

Data: (load `lect10data.mat`)

- Parameters t, c, β, integers n, k, p.
- Return vector $r \in \mathbf{R}^n$, initial position $x^0 \in \mathbf{R}^n$.
- Covariance matrix C, given by a factor model: $C = D + FF^T$, with D diagonal, F a $n \times k$ matrix.
- transaction cost function $TC(x) = c \cdot \|x - x^0\|_1$.
- Market impact function $MI(x) = \beta \cdot \sum_{i=1}^{n} |x_i - x_i^0|^{3/2}$.

Problem: Maximize portfolio return minus transaction costs and market impact, subject to the following constraints:

- An upper bound t on the portfolio variance.
- No shorting.
- The largest p positions do not represent more than 80% of the total position.

Provide a CVX code that solves this. Make sure the code prints out the final solution x.

Notes

Scenario uncertainty on returns

Assume return vectors and covariance matrices are uncertain, and scenarios are the pairs $(r^{(i)}, C^{(i)})$, $i = 1, \ldots, N$:

$$\max_x \ \alpha - \lambda t \ : \quad \begin{array}{l} \alpha \geq (r^{(i)})^T x, \quad i = 1, \ldots, N, \\ t \geq x^T C^{(i)} x, \quad i = 1, \ldots, N, \\ Ax \leq b, \quad Cx = d. \end{array}$$

- Above is a convex problem—which acronym would you use?
- Very versatile, but requires forming plausible scenarios.
- We may want to deal with a very large amount of such scenarios.

Notes

Interval uncertainty

A very common model derived from statistics is interval uncertainty:

$$\forall i \ : \ r_i \in [\hat{r}_i - \delta_i, \hat{r}_i + \delta_i],$$

where vector $\delta > 0$ contains the sizes of the intervals of confidence for r. This entails 2^n scenarios ... Previous approach won't scale!

Define *worst-case return:*

$$R(x) := \min_{r \,:\, |r - \hat{r}| \leq \delta} r^T x$$

Closed-form expression:

$$R(x) = \hat{r}^T x - \delta^T |x|$$

This is a *concave* function of x, so we can maximize it via CVX—without having to explicitly account for the 2^n scenarios!

Notes

Ellipsoidal uncertainty

We can even deal with an *infinite* amount of scnearios: Assume

$$r = \hat{r} + Ru \ : \ \|u\|_2 \leq \rho$$

This describes an ellipsoid with center $\hat{r}$ and "shape" determined by matrix R; $\rho > 0$ is an overall measure of uncertainty. This time, the number of scenarios is infinite ...

Define *worst-case return:*

$$R(x) := \min_{r,u} r^T x \ : \ r = \hat{r} + Ru, \ \ \|u\|_2 \leq \rho.$$

Closed-form expression:

$$R(x) = \hat{r}^T x - \rho \|R^T x\|_2.$$

Again a *concave* function of x, so we can maximize it via CVX.

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Chance constraints

Consider a single inequality $r^T x \geq t$ where x represent our portfolio position, r the return vector, and t a desired target.

If r is random, it makes sense to require that $\mathbf{Prob}\{r^T x \geq t\}$ is high. This is called a *chance constraint* .

Except in special cases, chance constraints are hard to deal with.

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Gaussian assumption

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Assume that the return vector follows a Gaussian distribution with mean $\hat{r}$ and covariance matrix C. That is, $r = \hat{r} + C^{1/2}u$, with $u \sim \mathcal{N}(0, I)$.

The chance constraint

$$\mathbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon$$

is equivalent to

$$\hat{r}^T x \geq t + \kappa(\epsilon)\|C^{1/2}x\|_2$$

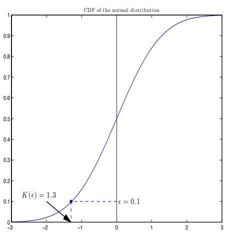
where $\kappa(\epsilon)$ is the negative of the inverse CDF of the normal distribution

$$\kappa(\epsilon) = -\Phi^{-1}(\epsilon), \;\; \Phi(\beta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-u^2/2} du.$$

Notes

Chance constraint

$$\mathbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon \iff \hat{r}^T x \geq t + \kappa(\epsilon)\|C^{1/2}x\|_2$$



- Above is an SOC constraint when $\epsilon < 1/2$.
- approach can be used for any linear inequality with Gaussian random coefficients.

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Robustness of chance constraints

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Previous chance constraint uses key Gaussian assumption . . . What if they are not met?

Assume returns are only known to have mean $\hat{r}$, covariance matrix C, but otherwise may follow any distribution.

Robust chance constraint: make sure that $\mathbf{Prob}\{r^T x \geq t\} \geq 1 - \epsilon$ no matter the *actual* distribution is (as long as it has mean $\hat{r}$ and covariance matrix C).

Notes

Chebyschev inequality

Chebyschev inequality allows to bound the chance constraints using the mean and covariance only.

Results in robust chance constraint:

$$\hat{r}^T x \geq t + \kappa(\epsilon) \|C^{1/2} x\|_2$$

where now

$$\kappa(\epsilon) = \frac{1-\epsilon}{\epsilon}.$$

- This is similar to the Gaussian case.
- The risk parameter is different (more conservative).

Notes

Measures of risk

There are many ways to measure the “risk” of a random variable. In our case the random variable is the portfolio’s return, $w^T \mathbf{r}$, with $\mathbf{r}$ the (random) return, and w the portfolio weight vector.

- *Variance:* expected squared deviation from the mean
- *Downside risk:* based on downside variance
- *Value-at-Risk* (VaR): looks at probability of loss being above a target
- *Conditional VaR:* expected loss given the loss is above a target
- *Worst-case variants:* when underlying probability distribution is partially known.
- Many more ...

Some are better suited to optimization ...

Notes

Value-at-Risk

In [1]:

> *"The Value at Risk measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval. Thus, if the VaR on an asset is \$ 100 million at a one-week, 95% confidence level, there is a only a 5% chance that the value of the asset will drop more than \$ 100 million over any given week."*

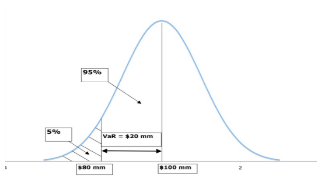
Notes

Value-at-Risk: mathematical expression

For $\beta \in [0, 1]$ (*e.g.*$\beta = 0.95$), The β-VaR of the portfolio is

$$\text{VaR}_\beta(w) := \inf \left\{ t \ : \ \text{Prob}(-w^T \mathbf{r} \le t) \right\} \ge \beta.$$

- The smallest value such that the probability of loss being lower than that value, has probability at least β.
- Captures the (negative) β-quantile.



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Properties of VaR

- VaR is hard to compute in general, even if distribution is perfectly known. Monte-Carlo methods have trouble computing the probability associated with a rare event!
- When probability is Gaussian, $\mathcal{N}(\hat{r}, C)$, and $\beta \geq 0.5$, there is a closed-form expression:

$$\text{VaR}_\beta(w) = \kappa^{\text{G}}(\beta)\sqrt{x^T C x} - \hat{r}^T w, \ \ \kappa^{\text{G}}(\beta) := \sqrt{2}\text{erf}^{-1}(2\beta - 1).$$

- In practice we do not even know the distribution!
- Worst-case VaR is the largest VaR among all distributions with given mean $\hat{r}$ and covariance matrix C:

$$\text{VaR}_\beta^{\text{WC}}(w) = \kappa^{\text{WC}}(\beta)\sqrt{x^T C x} - \hat{r}^T w, \ \ \kappa^{\text{WC}}(\beta) := \sqrt{\frac{\beta}{1-\beta}}.$$

Notes

Conditional VaR

The classical VaR examines at a (hopefully) rare event, but does not capture the amount of losses if that event happens.

Conditional Value-at-Risk: expected loss when loss higher than VaR:

$$\text{CVaR}_\beta(w) := \frac{1}{1-\beta}\,\mathbf{E}\left(-\mathbf{r}^T w \ : \ -w^T\mathbf{r} \geq \text{VaR}_\beta(w)\right).$$

Computing CVaR is difficult in general.

We have the alternate expression:

$$\text{CVaR}_\beta(w) := \min_t \ t + \frac{1}{1-\beta}\mathbf{E}(-\hat{r}^T w - t)_+.$$

Shows that CVaR, unlike VaR, is *convex* in w.

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Optimizing VaR

- In general, it is very difficult to compute, let alone optimize over w, the VaR of a portfolio with weight vector w.
- When $\beta \geq 0.5$, two exceptions: If we assume that the returns are Gaussian, or that the return distribution is only known up to its mean $\hat{r}$ and covariance matrix C, and we work with worst-case VaR.
- In both cases, a constraint such as $\mathrm{VaR}(w) \leq t$ can be written as a second-order cone one:

$$\kappa(\beta)\sqrt{x^T C x} \leq \hat{r}^T w + t,$$

for appropriate $\kappa(\beta)$.

Notes

CVaR

Optimizing the convex measure CVaR can be done several ways.

If we model the return distributions via scenarios, we may approximate the CVaR via a convex discrete sum. A sample set of returns $r^{(i)}$, $i = 1, \ldots, L$ yields the convex expression

$$\mathrm{CVaR}(w) \approx \min_t \; t + \frac{1}{L(1-\beta)} \sum_{i=1}^{L} (-w^T r^{(i)} - t)_+.$$

Alternatively, assuming the returns are Gaussian, or working with worst-case CVaR, leads to an expression of the form

$$\kappa(\beta)\sqrt{x^T C x} - \hat{r}^T w,$$

for some appropriate $\kappa(\beta)$.

Notes

Summary

- VaR, CVaR are defined only if distribution is full known.
- Except for simple cases (scenarios, Gaussian), the computation—let alone optimization—is very difficult and often unreliable.
- Optimizing worst-case / Gaussian VaR or worst-case / Gaussian / scenarii CVaR is easy, and involves LPs or SOCPs.

Notes

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