Appendix

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A Linear Algebra

This appendix explains the linear algebra tools used in the project. We implemented two data structures:

- $\mathbf{v} \in \mathbb{R}^3$: three–dimensional vectors (`Vec3`)
- $\mathbf{M} \in \mathbb{R}^{4\times4}$: 4×4 column–major matrices (`Mat4`)

A.1 `Vec3`: Three–Dimensional Vectors

A vector $\mathbf{v} = (x, y, z)$ is represented as a 3-component array. The `Vec3` module provides the following operations:

A.1.1 Normalization

Given $\mathbf{v}$, compute the normalized vector

$$\text{normalize}(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$$

If $\mathbf{v} = \mathbf{0}$, the zero vector $\mathbf{0}$ is returned.

A.1.2 Addition

$$\mathbf{a} + \mathbf{b} = (a_x + b_x,\ a_y + b_y,\ a_z + b_z)$$

A.1.3 Subtraction

$$\mathbf{a} - \mathbf{b} = (a_x - b_x,\ a_y - b_y,\ a_z - b_z)$$

A.1.4 Scalar Multiplication

For scalar s,

$$s\mathbf{a} = (s\,a_x,\ s\,a_y,\ s\,a_z)$$

A.1.5 Cross Product

The cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = \big(a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x\big)$$

A.2 `Mat4`: 4×4 Matrices and Transformations

Matrices are 4×4 arrays stored in column-major order. The `Mat4` module provides basic math tools for transformations used in computer graphics.

A.2.1 Identity Matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A.2.2 Perspective Projection

Let f_{ov} be the vertical field of view (camera's vertical viewing angle), a be the aspect ratio (screen width divided by height), and n, f be the the closest and farthest visible distances, we can calculate the perspective projection matrix:

$$\mathbf{M}_{\mathrm{persp}} = \begin{pmatrix} \dfrac{1}{a \tan\left(\dfrac{f_{\mathrm{ov}}}{2}\right)} & 0 & 0 & 0 \\ 0 & \dfrac{1}{\tan\left(\dfrac{f_{\mathrm{ov}}}{2}\right)} & 0 & 0 \\ 0 & 0 & \dfrac{f+n}{n-f} & -1 \\ 0 & 0 & \dfrac{2fn}{n-f} & 0 \end{pmatrix}$$

A.2.3 Look-At View Matrix

Let $\mathbf{e}$ be the camera position, $\mathbf{c}$ be the target point, and $\mathbf{u}$ be the world up vector. Firstly, we derive a new set of basis vectors:

$$\mathbf{z} = \frac{\mathbf{e} - \mathbf{c}}{\|\mathbf{e} - \mathbf{c}\|}, \qquad \mathbf{x} = \frac{\mathbf{u} \times \mathbf{z}}{\|\mathbf{u} \times \mathbf{z}\|}, \qquad \mathbf{y} = \mathbf{z} \times \mathbf{x}$$

Then the resulting view matrix is

$$\mathbf{M}_{\mathrm{lookAt}} = \begin{pmatrix} x_x & y_x & z_x & 0 \\ x_y & y_y & z_y & 0 \\ x_z & y_z & z_z & 0 \\ -\mathbf{x}\cdot\mathbf{e} & -\mathbf{y}\cdot\mathbf{e} & -\mathbf{z}\cdot\mathbf{e} & 1 \end{pmatrix}$$

A.2.4 Matrix Multiplication

For $\mathbf{C} = \mathbf{AB}$,

$$C_{ij} = \sum_{k=1}^{4} A_{ik} B_{kj}.$$

A.2.5 Translation (Post-Multiply)

For translation vector $\mathbf{t} = (t_x, t_y, t_z)$,

$$\mathbf{M}' = \mathbf{M} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{pmatrix}$$

A.2.6 Scaling (Post-Multiply)

For scale vector $\mathbf{s} = (s_x, s_y, s_z)$,

$$\mathbf{M}' = \mathbf{M} \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A.2.7 Rotation About the x-Axis (Post-Multiply)

For rotation angle θ,

$$\mathbf{M}' = \mathbf{M} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A.2.8 Rotation About the y-Axis (Post-Multiply)

For rotation angle θ,

$$\mathbf{M}' = \mathbf{M} \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A.2.9 Rotation About the z-Axis (Post-Multiply)

For rotation angle θ,

$$\mathbf{M}' = \mathbf{M} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A.2.10 Transformation of a Homogeneous Vector

For $\mathbf{v} = (x, y, z, w)$,

$$\mathbf{v}' = \mathbf{M}\mathbf{v}$$

B Trajectory Math

B.1 Linear Interpolation

For scalars a and b with interpolation parameter $p \in [0, 1]$,

$$\mathrm{lerp}(a, b, p) = a + (b - a)p.$$

B.2 3D Linear Interpolation

For vectors $\mathbf{a}, \mathbf{b}$, the interpolated vector is

$$\mathrm{lerp3}(\mathbf{a}, \mathbf{b}, p) = \big(\mathrm{lerp}(a_x, b_x, p),\ \mathrm{lerp}(a_y, b_y, p),\ \mathrm{lerp}(a_z, b_z, p)\big).$$

B.3 Yaw Angle

For a horizontal displacement $(\Delta x, \Delta z)$, the yaw angle with respect to the (x, z)-plane is

$$\psi = \mathrm{atan2}(\Delta x,\ \Delta z).$$