Teaching Manual: Calculus I

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PREFACE

A Sound knowledge of engineering mathematics is a 'sine qua non' for the modern engineer to attain new heights in all aspects of engineering practice. Calculus I (with code GSU 07101) is the first among four courses in engineering mathematics offered at Dar es Salaam Institute of Technology (DIT) a for a bachelor's degree programme. The course aims at enabling students get knowledge and skills of limits and continuity, laplace transforms, fourier series and functions of several variables and use it to solve engineering related problems.

This manual is written in a lucid, easy to understand language. Each topic has been well coved in scope, content and several worked out examples, carefully selected to cover all aspects of the topic are presented. There are more than 80 worked examples. At the end of the manual a number of practice exercises are provided. Also selected reading texts for further references are provided.

This manual has put into practice since 2005 here at DIT. We are hopeful that the manual will be useful to both students and teachers not only at DIT but also to other engineering colleges.

In spite of our best efforts, some errors might have crept in the manual. Any such errors and all suggestions for improving future editions of the manual are welcome and will be greatly acknowledged.

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Chapter 1

Theory of Limits

1.1 Definitions of a Limit

1.1.1 An informal definition of limit

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit** L as x approaches a, and

$$\lim_{x \to a} f(x) = L$$

This is,

$$\lim_{x \to a} f(x) = L, \quad \text{for some number} \quad L, \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Example 1.1 *Evaluate*

$$\lim_{x \to -3} \frac{3x+9}{x^2-9}$$

Solution *We examine some function values for x near -3.*

x	-2.80	-2.84	-2.89	-2.93	-2.98	-3.02	-3.07	-3.11	-3.16	-3.20
f(x)	-0.52	-0.51	-0.51	-0.51	-0.50	-0.50	-0.49	-0.49	-0.49	-0.48

Based on this numerical evidence, it's reasonable to conjecture that

$$\lim_{x \to -3^-} \frac{3x+9}{x^2-9} = \lim_{x \to -3^+} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Further, note that

$$\lim_{x \to -3^-} \frac{3x+9}{x^2-9} = \lim_{x \to -3^-} \frac{3(x+3)}{(x+3)(x-3)} = \lim_{x \to -3^-} \frac{3}{x-3} = -\frac{1}{2},$$

since $(x-3) \to -6$ as $x \to -3$. Again, the cancellation of the factors of $(x+3)$ is valid since in the limit as $x \to -3$, x is close to -3, but $x \neq -3$, so that $x+3 \neq 0$.
Likewise,

$$\lim_{x \to -3^+} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Finally, since the function approaches the $same$ value as $x \to -3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x \to -3} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Example 1.2 *Determine whether*

$$\lim_{x \to 3} \frac{3x+9}{x^2-9} \quad \textit{exists}$$

Solution

We first compute some function values for x near to 3.

x	2.80	2.84	2.89	2.93	2.98	3.02	3.07	3.11	3.16	3.20
f(x)	-15.00	-19.29	-27.00	-45.00	-135.00	135.00	45.00	27.00	19.29	15.00

Based on this numerical evidence, it appears that, as $x \to 3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus

$$\lim_{x \to 3^+} \frac{3x+9}{x^2-9} \quad \textit{does not exist}$$

Similarly, from the table of values for $x < 3$, we can say that

$$\lim_{x \to 3^-} \frac{3x+9}{x^2-9} \quad \textit{does not exist}$$

Since neither one-side limit exist, we say

$$\lim_{x \to 3} \frac{3x+9}{x^2-9} \quad \textit{does not exist}$$

Here, we considered both one-side limit for the sake of completeness. Of course, you should keep in mind that if $either$ one-side limit fails to exist, then the limit does not exist.

Example 1.3 *Evaluate*

$$\lim_{x \to 0} \frac{x}{|x|}$$

Solution

$$\begin{aligned} \lim_{x \to 0^+} \frac{x}{|x|} &= \lim_{x \to 0^+} \frac{x}{x} \quad (\textit{since} \quad |x| = x, \quad \textit{when} \quad x > 0) \\ &= \lim_{x \to 0^+} 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to 0^-} \frac{x}{|x|} &= \lim_{x \to 0^-} \frac{x}{-x} \quad (\textit{since} \quad |x| = -x, \quad \textit{when} \quad x < 0) \\ &= \lim_{x \to 0^-} -1 \\ &= -1 \end{aligned}$$

It now follows that

$$\lim_{x \to 0} \frac{x}{|x|} \quad \textit{does not exist}$$

Example 1.4 *Evaluate $\lim_{x \to 0} f(x)$, where f is defined by*

$$f(x) = \begin{cases} x^2 + 2\cos x + 1, & \textit{for} \quad x < 0 \\ \sec x - 4, & \textit{for} \quad x \geq 0 \end{cases}$$

Solution *Since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one-sided limits. We have*

$$\lim_{x \to 0^-} = \lim_{x \to 0^-} (x^2 + 2\cos x + 1) = 2\cos 0 + 1 = 3.$$

Also, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\sec x - 4) = \sec 0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x \to 0} f(x)$ does not exist.

Limit Rules

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and k is a constant, then

1. Limit of a sum:
$$\lim_{x\to a}[f(x) + g(x)] = L + M$$
2. Limit of a difference:
$$\lim_{x\to a}[f(x) - g(x)] = L - M$$
3. Limit of a product:
$$\lim_{x\to a} f(x)g(x) = LM$$
4. Limit of a multiple:
$$\lim_{x\to a} kf(x) = kL$$
5. Limit of a quotient:
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{if} \quad M \neq 0.$$

   If m is an integer and n is a positive integer, then

6. Limit of a power:
$$\lim_{x\to a}[f(x)]^{m/n} = L^{m/n},$$

   provided $L > 0$ if n is even, and $L \neq 0$ if $m < 0$. If $f(x) \leq g(x)$ on an interval containing a in its interior, then

7. Order is preserved:
$$L \leq M$$

Rules 1-6 are also valid right limits and left limits. So is Rule 7, under the the assumption that $f(x) \leq g(x)$ on the open interval extending in the appropriate direction from a.

Example 1.5 *Evaluate*
$$\lim_{x\to 0}(x\cot x)$$

Solution

$$\begin{aligned}
\lim_{x\to 0}(x\cot x) &= \lim_{x\to 0}\left(x\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}\cos x\right) \\
&= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}\cos x\right) \\
&= \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1
\end{aligned}$$

Limits at infinity and negative infinity (informal definition)

If the function f is defined on an interval (a, ∞) and if we can ensure that $f(x)$ is as close as we want to the number L by taking x large enough, then we say that $f(x)$ *approaches the limit L as x approaches infinity*, and we write
$$\lim_{x\to\infty} f(x) = L.$$

If f is defined on an interval $(-\infty, b)$ and if we can ensure that $f(x)$ is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that $f(x)$ *approaches the limit M as x approaches negative infinity*, and we write
$$\lim_{x\to-\infty} f(x) = M.$$

Example 1.6 *Find*
$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right).$$

Solution *We are trying to find the limit of the difference of two functions, each of which becomes arbitrarily large as x increases to infinity. We rationalize the expression by multiplying the numerator and the denominator (which is 1) by the conjugate expression $\sqrt{x^2+x}+x$:*

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+x}-x) &= \lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x}\\
&= \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2(1+\frac{1}{x})}+x}\\
&= \lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}
\end{aligned}$$

(Here, $\sqrt{x^2}=x$ because $x>0$ as $x\to\infty$.)

1.1.2 The formal definition of a limit

When we say that $f(x)$ has limit L as x approaches a, we are really saying that we can ensure that the *error* $|f(x)-L|$ will be less than *any* allowed tolerance, no matter how small, by taking x *close enough* to a (but not equal to a). It is traditional to use ϵ, the Greek letter "epsilon", for the size of the allowable *error* and δ, the Greek letter "delta" for the *difference* $x-a$ that measures how close x must be to a to ensure that the error is within that tolerance. These are the letters that Cauchy and Weierstrass used in their pioneering work on limits and continuity in the nineteenth century.
If ϵ is any positive, *no matter how small*, we must be able to ensure that $|f(x)-L|<\epsilon$ by restricting x to be *close enough* to (but not equal to) a. How close is close enough? It is sufficient that the distance $|x-a|$ from x to a be less than a positive number δ that depends on ϵ. (See Figure 1.1.) If we can find such δ for any positive ϵ, we are entitled to conclude that $\lim_{x\to a}f(x)=L$.

A formal definition of limit

We say that $f(x)$ *approaches the limit* L as x *approaches* a, and we write

$$\lim_{x\to a}f(x)=L,$$

if the following condition is satisfied: for every number $\epsilon>0$ there exists a number $\delta>0$, possibly depending on ϵ, such that

$$\text{if}\quad 0<|x-a|<\delta,\quad\text{then}\quad |f(x)-L|<\epsilon.$$

Example 1.7 Testing the Definition
The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. This following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.

Example 1.8 *Show that*
$$\lim_{x\to 1}(5x-3)=2.$$

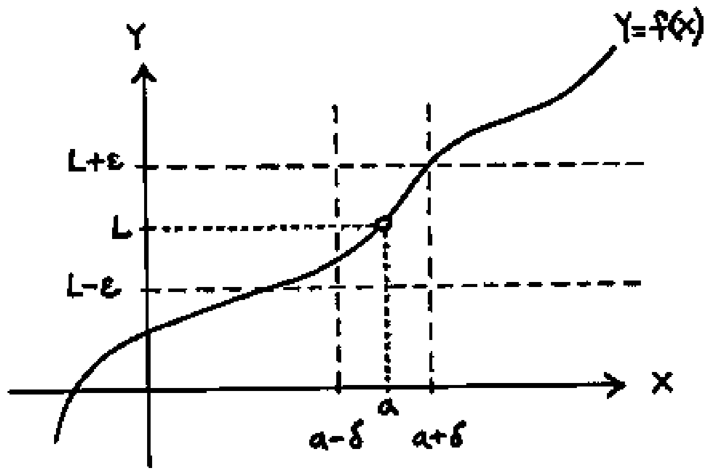


Fig. 1.1: If $x \neq a$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$

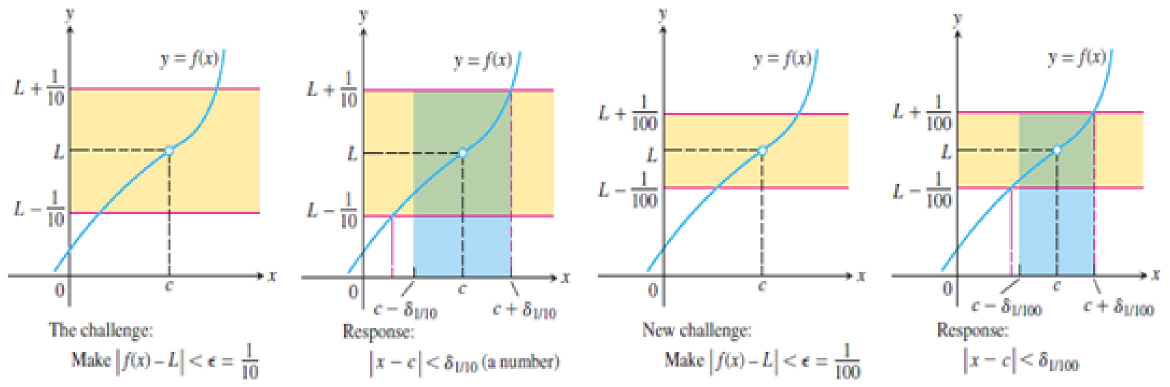


Fig. 1.2:

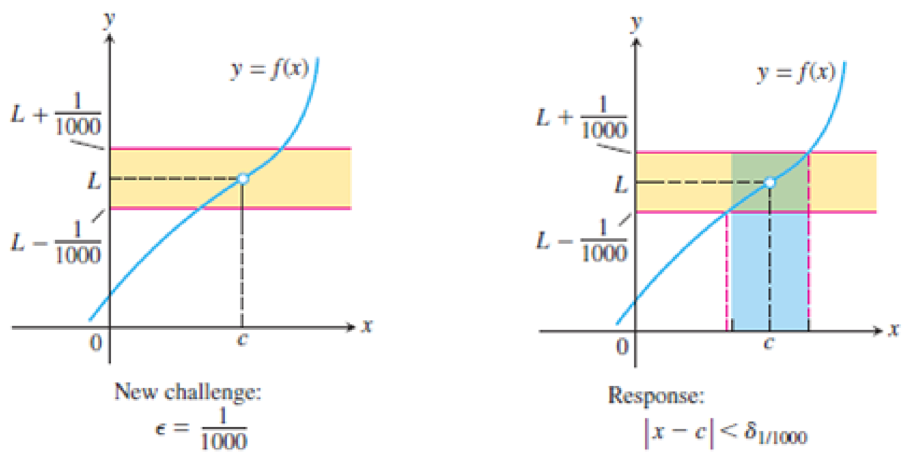


Fig. 1.3:

Solution *Set* $c = 1, f(x) = 5x - 3$, *and* $L = 2$ *in the definition of limit. For any given* $\epsilon > 0$, *we have to find a suitable* $\delta > 0$ *so that if* $x \neq 1$ *and* x *is within distance* δ *of* $c = 1$, *that is, whether*

$$0 < |x - 1| < \delta,$$

it is true that if $f(x)$ *is within distance* ϵ *of* $L = 2$, *so*

$$|f(x) - 2| < \epsilon.$$

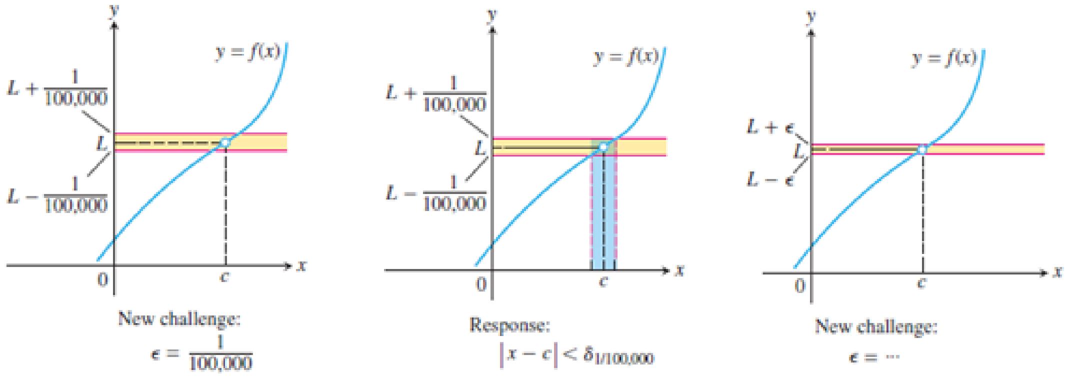


Fig. 1.4:

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x-3)-2| = &|5x-5| < \epsilon \\ &5|x-1| < \epsilon \\ &|x-1| < \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (see figure 1.5). If $0 < |x-1| < \delta = \epsilon/5$, then

$$|(5x-3)-2| = |5x-5| = 5|x-1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x\to 1}(5x-3) = 2$.
The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x-1| < \delta$ imply $|5x-5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work.

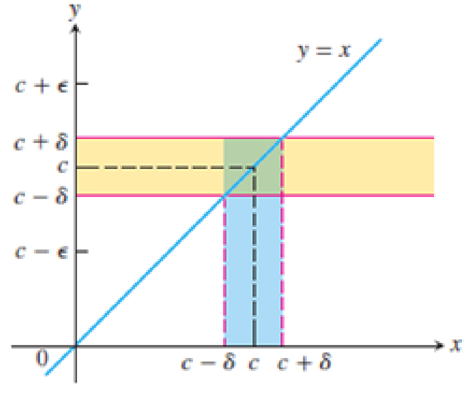


Fig. 1.5:

Example 1.9 *Prove the following results*

(a) $\lim_{x\to c} x = c$

(b) $\lim_{x\to c} k = k \quad$ *(k constant)*

Solution *(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x*

$$0 < |x-c| < \delta \quad \textit{implies} \quad |x-c| < \epsilon$$

The implication will hold if δ equals ϵ or any smaller positive number (see figure 1.6 (a)). This proves that $\lim_{x\to c} x = c$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \textit{implies} \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and implication will hold (see figure 1.6 (b)). This proves that $\lim_{x \to c} k = k$.

(a) (b)

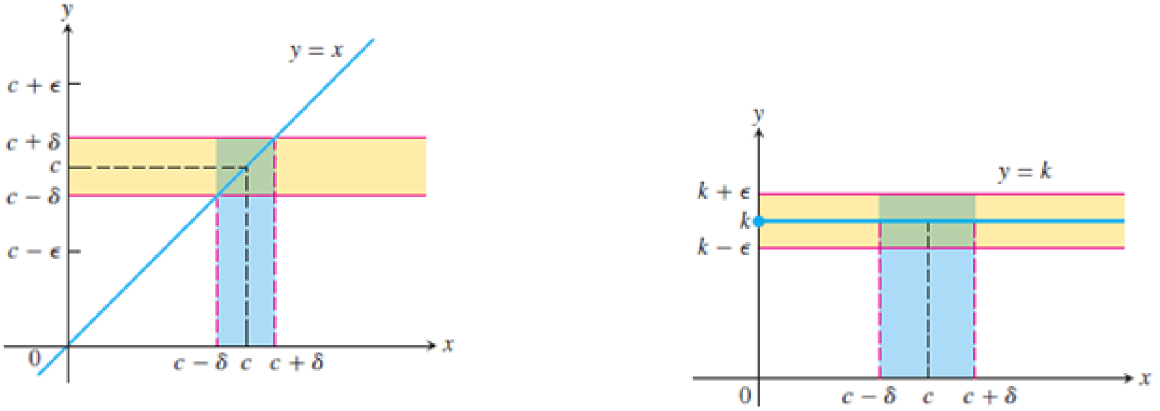


Fig. 1.6:

Finding Deltas Algebraically for Given Epsilons

In the above examples, the intervals for values about c for which $|f(x) - L|$ was less than ϵ was symmetric about c and we could take δ to be half the length of that interval. When such symmetry is absent, as it usually is, we take δ to be the distance from c to the interval's $nearer$ endpoint.

Example 1.10 *For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x*

$$0 < |x - 5| < \delta \Rightarrow |\sqrt{x-1} - 2| < 1.$$

Solution *We organize the search into two steps*

1. *Solving the inequality $|\sqrt{x-1} - 2| < 1$ to find an interval containing $x = 5$ on which the inequality holds for all $x \neq 5$.*

$$\begin{aligned}
&|\sqrt{x-1}| < 1 \\
&-1 < \sqrt{x-1} - 2 < 1 \\
&1 < \sqrt{x-1} < 3 \\
&1 < x - 1 < 9 \\
&2 < x < 10
\end{aligned}$$

   The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value $\delta > 0$ to place the centered interval $5 - \delta < x < 5 + \delta$ (centered at $x = 5$) inside the interval $(2, 10)$. The distance from 5 to the nearer endpoint of $(2, 10)$ is 3 (see figure 1.7 (a)). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1} - 2| < 1$ (see figure 1.7 (b)):*

(a) (b)

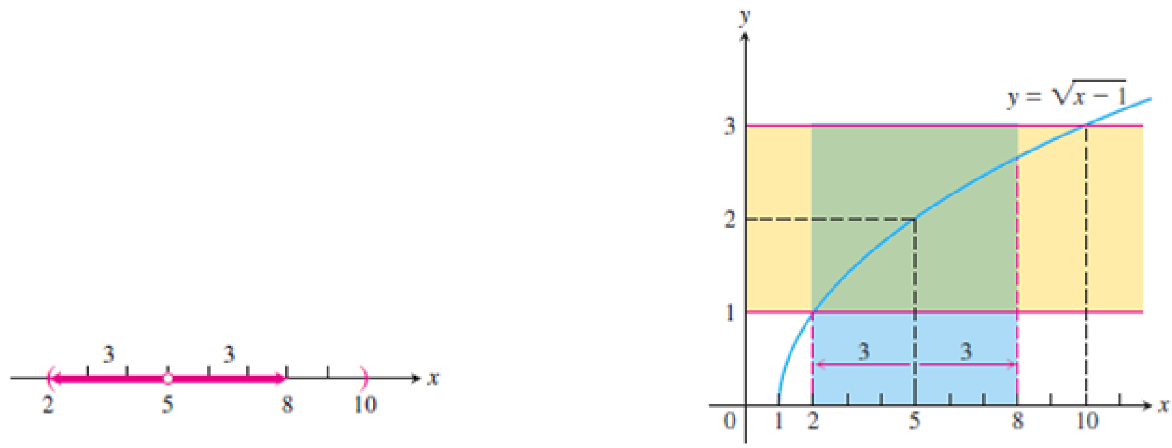


Fig. 1.7:

How to Find Algebraically a δ for a Given f, L, c, and $\epsilon > 0$

The process of find a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

can be accomplished in two steps

1. Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$.

2. Find a value of $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq c$ in this δ-interval.

Example 1.11 *Prove that* $\lim_{x \to 2} f(x) = 4$ *if*

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution *Our task to show that given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* x

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$

1. *Solve the inequality* $|f(x) - 4| < \epsilon$ *to find an open interval containing* $x = 2$ *on which the inequality holds for all* $x \neq 2$.
For $c \neq 2$, *we have* $f(x) = x^2$, *and the inequality to solve is* $|x^2 - 4| < \epsilon$:

$$\begin{aligned} |x^2 - 4| &< \epsilon \\ -\epsilon < x^2 - 4 &< \epsilon \\ 4 - \epsilon < x^2 &< 4 + \epsilon \\ \sqrt{4 - \epsilon} < |x| &< \sqrt{4 + \epsilon} \\ \sqrt{4 - \epsilon} < x &< \sqrt{4 + \epsilon} \end{aligned}$$

The inequality $|f(x) - 4| < \epsilon$ *holds for all* $x \neq 2$ *in the open interval* $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$ *(see figure below)*

2. *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.
Take δ *to be the distance from* $x = 2$ *to the nearer endpoint of* $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$. *In*

other worlds, take $\delta = min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$*, the minimum (the smaller) of the two numbers* $2 - \sqrt{4-\epsilon}$ *and* $\sqrt{4+\epsilon} - 2$*. If* δ *has this or any smaller positive value, the inequality* $0 < |x-2| < \delta$ *will automatically place* x *between* $\sqrt{4-\epsilon}$ *and* $\sqrt{4+\epsilon}$ *to make* $|f(x) - 4| < \epsilon$*. For all* x*,*

$$0 < |x-2| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$

This completes the proof for $\epsilon < 4$*.*
If $\epsilon \geq 4$*, then we take* δ *to the distance from* $x = 2$ *to the nearer endpoint of the interval* $(0, \sqrt{4+\epsilon})$*. In other words, take* $\delta = min\{2, \sqrt{4+\epsilon} - 2\}$*. (see figure 1.8)*

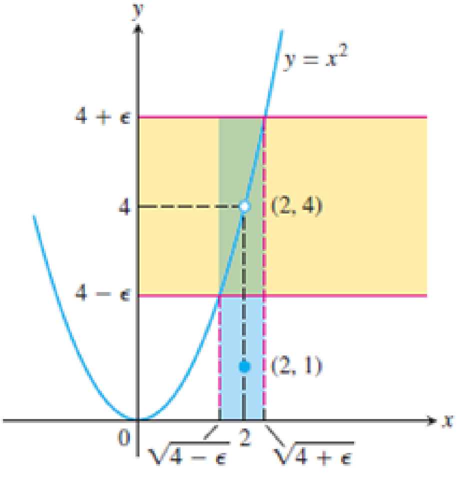


Fig. 1.8:

1.2 Continuity

In ordinary language, to say that a certain process is " continuous" is to say that it goes on without interruption and without abrupt changes. In mathematics the word "continuous" has much the same meaning.
The concept of continuity is so important in calculus and its applications that we discuss it with some care. First we treat *continuity at a point c* (a number c), and then we discuss *continuity on an interval.*

Continuity at a Point

The basic idea is a follows: We are given a function f and a number c. We calculate (if we can) both $\lim_{x\to c} f(x)$ and $f(c)$. If these two numbers are equal, we say that f is *continuous* at c. Here is the definition formally stated.

Definition 1.1 *Let* f *be a function defined at least on an open interval* $(c-p, c+p)$*. We say that* f *is continuous at* c *if*

$$\lim_{x\to c} f(x) = f(c)$$

If the domain of f contains an interval $(c-p, c+p)$, then f can fail to be continuous at c for only one of two reasons: either

(i) f has a limit as x tends to c, but $\lim_{x\to c} \neq f(c)$ or

(ii) f has no limit as x tends to c.

In case (i) the number c is called a*removable discontinuity*. The discontinuity can be removed by redefining f at c. If the limit is L, redefine f at c to be L.
In case (ii) the number c is called an *essential* discontinuity. You can change the value of f at a billion points in any way you like. The discontinuity will remain.
The function depicted in Figure 1.9 (a) has a removable discontinuity at c. The discontinuity can be removed by lowering the dot into place (i.e., by redefining f at c to be L).
The function depicted in Figures 1.9 (b), 1.10, and 1.11 have essential discontinuity at c. The discontinuity in Figure 1.9 (a) is, for obvious reasons, called *jump* discontinuity. The functions of Figure 1.10 have *infinite* discontinuities.
In Figure 1.11, have tried to portray the Dirichlet function

$$f(x) = \begin{cases} 1, & x \quad \text{rational} \\ -1 & x \quad \text{irrational} \end{cases}$$

(a) (b)

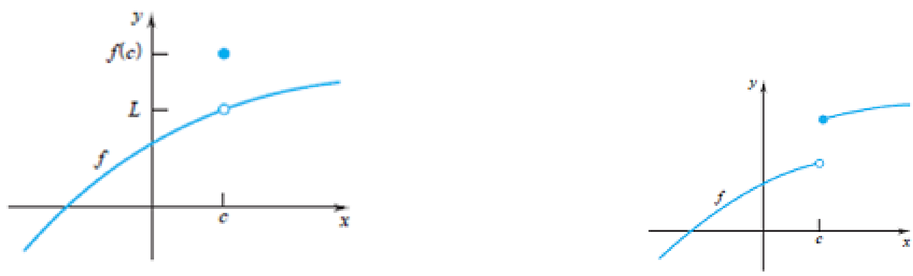


Fig. 1.9:

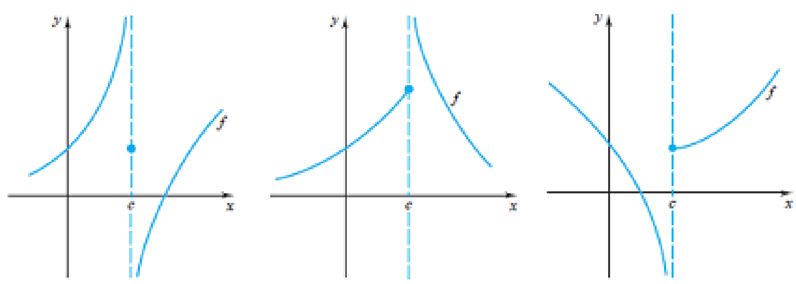


Fig. 1.10:

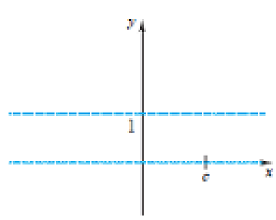


Fig. 1.11:

At no point c does f have a limit. Each point is an essential discontinuity. The function is everywhere discontinuous.
Most of the functions that you have encountered so far are continuous at each point of their domains. In particular, this is true for polynomials P,

$$\lim_{x \to c} P(x) = P(c), \tag{1.1}$$

for rational functions (quotients of polynomials) $R = P/Q$

$$\lim_{x\to c} R(x) = \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = R(c) \quad \text{provided} \quad Q(c) \neq 0, \tag{1.2}$$

and for the absolute value function,

$$\lim_{x\to c} |x| = |c|. \tag{1.3}$$

As you were asked to show earlier,

$$\lim_{x\to c} \sqrt{x} = \sqrt{c} \quad \text{for each} \quad c > 0.$$

Theorem 1.1 *If f and g are continuous at c, then*

(i) $f + g$ is continuous at c;

(ii) $f - g$ is continuous at c;

(iii) αf is continuous at c for each real α;

(iv) $f.g$ is continuous at c;

(v) f/g is continuous at c provided $g(c) \neq 0$

Example 1.12 *The function $F(x) = 3|x| + \frac{x^3-x}{x^2-5x+6} + 4$ is continuous at all real numbers other than 2 and 3. You can see this by noting that*

$$F = 3f + g/h + k$$

where

$$f(x) = |x|, \quad g(x) = x^3 - x, \quad h(x) = x^2 - 5x + 6, \quad k(x) = 4$$

Since f, g, h, k are everywhere continuous, F is continuous except at 2 and 3, the number at which h takes on the value 0. (At those numbers F is not defined)

Definition 1.2 One-side Continuity
A function f is called

$$\textit{continuous from the left at } c \quad \text{if} \quad \lim_{x\to c^{-1}} f(x) = f(c).$$

It is called

$$\textit{continuous from the right at } c \quad \text{if} \quad \lim_{x\to c^{+}} f(x) = f(c)$$

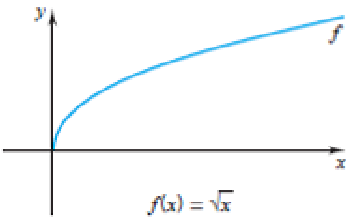
The function of Figure 1.12 (a) is continuous from the right at 0; the function of Figure 1.12 (b) is continuous from left at 1. It follows from definition 1.1 that a function is continuous at c iff it is continuous from both sides at c. Thus f is continuous at c iff $f(c)$, $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$ all exist and are equal.

Example 1.13 *Determine the discontinuities, if any, of the following function:*

$$f(x) = \begin{cases} 2x+1, & x \leq 0 \\ 1, & 0 < x \leq 1 \\ x^2+1, & x > 1 \end{cases}$$

Solution *Clearly f is continuous at each point in the open intervals $(-\infty, 0), (0, 1), (1, \infty)$ (On each of these intervals f is a polynomial, see Figure 1.13). Thus, we have to check*

(a) (b)



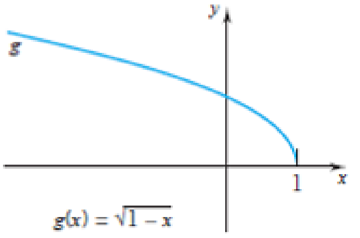


Fig. 1.12:

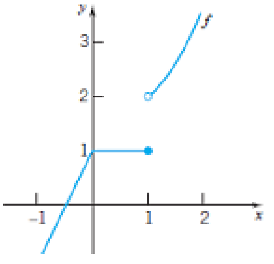


Fig. 1.13:

the behaviour of f at $x = 0$ and $x = 1$. The figure suggests that f is continuous at 0 and discontinuous at 1. Indeed, that is the case:

$$f(0) = 1, \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} f(2x+1) = 1, \quad \lim_{x\to x^+} = \lim_{x\to 0^+}(1) = 1$$

. This makes f continuous at 0. This situation is different at $x = 1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(1) = 1 \quad and \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+1) = 2$$

Thus f has an essential discontinuity at 1, a jump discontinuity.

Example 1.14 *Determine the discontinuities, if any, of the following function:*

$$f(x) = \begin{cases} x^3, & x \leq -1 \\ x^2 - 2, & -1 < x < 1 \\ 6 - x, & 1 \leq x < 4 \\ \frac{6}{7-x}, & 4 < x < 7 \\ 5x + 2, & x \geq 7. \end{cases}$$

Solution *It should be clear that f is continuous at each point of the open intervals $(-\infty, -1), (-1, 1), (1, 4), (4, 7), (7, \infty)$. All we have to check is the behaviour of f at $x = -1, 1, 4, 7$. To do so, we apply definition 1.1*

The function is continuous at $x = -1$ since $f(-1) = (-1)^3 = -1$, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(x^3) = -1$, and $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(x^2 - 2) = -1$. Our findings at the other three points are displayed in the following chart. Try to verify each entry.

c	*f(c)*	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	*Conclusion*
1	*5*	*-1*	*5*	*discontinuous*
4	*not defined*	*2*	*2*	*discontinuous*
7	*37*	*does not exist*	*37*	*discontinuous*

The discontinuity at $x = 4$ is removable: if we redefine f at 4 to be 2, then f becomes continuous at 4. The numbers 1 and 7 are essential discontinuities. The discontinuity at 1 is a jump discontinuity; the discontinuity at 7 is an infinite discontinuity: $f(x) \to \infty$ as $x \to 7^-$

Chapter 2

Laplace Transforms

In mathematics, the Laplace transform is a widely used integral transform. The Laplace transform has many important applications throughout the sciences. It is named for Pierre-Simon Laplace who introduced the transform in his work on probability theory. The Laplace transform is related to the Fourier transform, but whereas the Fourier transform resolves a function or signal into its modes of vibration, the Laplace transform resolves a function into its moments. Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In physics and engineering, it is used for analysis of linear systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In this analysis, the Laplace transform is often interpreted as a transformation from the time-domain, in which inputs and outputs are functions of time, to the frequency-domain, where the same inputs and outputs are functions of complex angular frequency, in radians per unit time. Also in engineering and physics; the output of a linear time invariant system can be calculated by convolving its unit impulse response with the input signal. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behaviour of the system, or in synthesizing a new system based on a set of specifications. Performing this calculation in Laplace space turns the convolution into a multiplication; the latter being easier to solve because of its algebraic form. The Laplace transform can also be used to solve differential equations and is used extensively in electrical engineering. The Laplace transform reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The original differential equation can then be solved by applying the inverse Laplace transform.

2.1 Definition

Let $f(t)$ be a real valued function defined for all $t \geq 0$. Then the Laplace transform of $f(t)$ denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{2.1}$$

where s is a real or a complex number.
If the integral on the right hand side (2.1) exists, it is a function of s and is usually denoted by $F(s)$. Here s is called the parameter.
Thus

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

The Laplace transform is a linear transform, by which is meant that:

1. The transform of a sum (or difference) of expression is the sum (or difference) of the individual transforms. That is

$$L\{f(t) \pm g(t)\} = L\{f(t)\} \pm L\{g(x)\}$$

2. The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is

$$L\{kf(t)\} = kL\{f(t)\}.$$

Proof: (1) we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Therefore,

$$\begin{aligned} L\{kf(t)\} &= \int_0^\infty e^{-st} kf(t)dt \\ &= k\int_0^\infty e^{-st} f(t)dt \\ &= kL\{f(t)\} \end{aligned}$$

(2) Consider

$$\begin{aligned} L\{f(t)+g(t)\} &= \int_0^\infty e^{-st}[f(t)+g(t)]dt \\ &= \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} g(t)dt \\ &= L\{f(t)\} + L\{g(t)\} \end{aligned}$$

2.2 Laplace Transforms of Some Standard Functions

2.2.1 Laplace transform of a constant

Let $f(x) = a$. Where a is a constant. Then from the definition of Laplace transform, we get

$$\begin{aligned} L(a) &= \int_0^\infty e^{-st} a dt \\ &= a\int_0^\infty e^{-st} dt \\ &= a\left[\frac{e^{-st}}{-s}\right]_0^\infty \\ &= \frac{-a}{s}[e^{-\infty} - e^0] \\ &= \frac{a}{s} \quad \text{since} \quad e^{-\infty} = 0 \quad \text{and} \quad e^0 = 1 \end{aligned}$$

Hence

$$L(a) = \frac{a}{s}. \tag{2.2}$$

In particular cases, $L(1) = \frac{1}{s}$.

2.2.2 Laplace transform of e^{at}

Substituting $f(t) = e^{at}$ in the definition of Laplace transform, we get

$$\begin{aligned}
L(e^{at}) &= \int_0^\infty e^{-st} e^{at} dt \\
&= \int_0^\infty e^{(-s+a)t} dt \\
&= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty \\
&= \frac{-1}{s-a} [e^{-\infty} - e^0] \\
&= \frac{1}{s-a} \quad \text{if} \quad s > a > 0
\end{aligned}$$

Therefore

$$L(e^{at}) = \frac{1}{s-a}, s > a > 0 \tag{2.3}$$

Replacing a by $-a$, we get

$$L(e^{-at}) = \frac{1}{s+a}, s > -a. \tag{2.4}$$

2.2.3 Laplace transform of $\sinh at$

We have

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

Substituting

$$f(t) = \sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\begin{aligned}
L(\sinh at) &= L\{\frac{e^{at} - e^{-at}}{2}\} = \frac{1}{2} \left[L(e^{at}) - L(e^{-at}) \right] \\
&= \frac{1}{2}\{\frac{1}{s-a} - \frac{1}{s+a}\} \\
&= \frac{a}{s^2 - a^2}, \quad \text{if} \quad s > a
\end{aligned}$$

Hence

$$L(\sinh at) = \frac{a}{s^2 - a^2}, s > a \tag{2.5}$$

2.2.4 Laplace transform of $\cosh at$

We have

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

Substituting

$$f(t) = \cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\begin{aligned}
L(\cosh at) &= L\{\frac{e^{at} + e^{-at}}{2}\} = \frac{1}{2} \left[L(e^{at}) + L(e^{-at}) \right] \\
&= \frac{1}{2}\{\frac{1}{s-a} + \frac{1}{s+a}\} \\
&= \frac{s}{s^2 - a^2}, \quad \text{if} \quad s > a
\end{aligned}$$

Hence

$$L(\cosh at) = \frac{s}{s^2 - a^2}, s > a \tag{2.6}$$

2.2.5 Laplace transform of $\sin at$ and $\cos at$

We know by Euler's formula that

$$e^{iat} = \cos at + i \sin at$$

Therefore

$$\begin{aligned} L(\cos at + i \sin at) &= L(e^{iat}) \\ &= \frac{1}{s - ia} \\ &= \frac{s + ia}{(s - ia)(s + ia)} \\ &= \frac{s + ia}{s^2 + a^2} \\ &= \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2} \end{aligned}$$

On equating the real and imaginary parts, we obtain

$$\begin{aligned} L(\cos at) &= \frac{s}{s^2 + a^2} \\ L(\sin at) &= \frac{a}{s^2 + a^2} \end{aligned} \tag{2.7}$$

2.2.6 Laplace transform of t^n

Let $f(t) = t^n$, where n is a non-negative real number or n is a negative non-integers. Then from the definition

$$L(t^n) = \int_0^\infty e^{-st} t^n dt$$

Substitute $st = x$, so that $dt = \frac{dx}{s}$ and $t = \frac{x}{s}$. When $t = 0$, $x = 0$ and when $t = \infty$, $x = \infty$. Therefore

$$\begin{aligned} L(t^n) &= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dt}{s} \\ &= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx \\ &= \frac{1}{s^{n+1} \Gamma(n+1)} \end{aligned}$$

Thus

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \tag{2.8}$$

In particular if n is a non-negative integers, we have

$$\Gamma(n+1) = n!$$

Hence,

$$L(t^n) = \frac{n!}{s^{n+1}} \tag{2.9}$$

where n is a non-negative integer.

Example 2.1

$$\begin{aligned} L(\sin^2 at) &= L\left(\frac{1-\cos 2at}{2}\right) \\ &= \frac{1}{2}L(1-\cos 2at) \\ &= \frac{1}{2}[L(1) - L(\cos 2at)] \\ &= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+(2a)^2}\right] \\ &= \frac{1}{2}\left[\frac{s^2+4a^2-s^2}{s(s^2+4a^2)}\right] \\ &= \frac{2a^2}{s(s^2+4a^2)} \end{aligned}$$

Example 2.2

$$\begin{aligned} L(\sin 5t \cos 3t) &= L\{\frac{1}{2}[\sin 8t + \sin 2t]\} \\ &= \frac{1}{2}\{L(\sin 8t) + L(\sin 2t)\} \\ &= \frac{1}{2}\left[\frac{8}{s^2+8^2} + \frac{2}{s^2+2^2}\right] \\ &= \frac{5(s^2+16)}{(s^2+64)(s^2+4)} \end{aligned}$$

Example 2.3

$$\textit{If} \quad f(t) = \begin{cases} 2, & 0 < t < 3 \\ t, & t > 3 \end{cases} \quad , \quad \textit{find} \quad L\{f(t)\}$$

Solution. *Now,*

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\ &= \int_0^3 e^{-st}.2dt + \int_3^\infty e^{-st}.tdt \\ &= 2\left[\frac{e^{-st}}{-s}\right]_0^3 + \left[t\frac{e^{-st}}{-s} - 1.\frac{e^{-st}}{(-s)^2}\right]_3^\infty \\ &= \frac{-2}{s}\left[e^{-3s} - 1\right] + 0 - \left[-\frac{3e^{-2s}}{s} - \frac{e^{-3s}}{s^2}\right] \\ &= \frac{2}{s} + \frac{s+1}{s^2}.e^{-3s} \end{aligned}$$

2.3 Laplace Transforms of the form $e^{at}f(t)$

If the Laplace transform of $f(t)$ is known, then the Laplace transform of $e^{at}f(t)$ where a is a constant can be determined by using the shifting property.

Shifting property:

If

$$L\{f(t)\} = F(s)$$

then

$$L\{e^{at}f(t)\} = F(s-a)$$

Proof we have

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t) = F(s)$$

Therefore

$$\begin{aligned} L\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a). \end{aligned}$$

Replacing a by $-a$, we get,

$$L\{e^{-at}f(t)\} = F(s+a)$$

Example 2.4 *Find*

$$L(e^{-t}\cos^2 3t)$$

Solution. *Consider*

$$\begin{aligned} L(cos^2 3t) &= L\left(\frac{1+\cos 6t}{2}\right) \\ &= \frac{1}{2}[L(1) + L(\cos 6t)] \\ &= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+6^2}\right] \\ &= \frac{s^2+18}{s(s^2+36)} \end{aligned}$$

Therefore

$$\begin{aligned} L(e^{-t}\cos^2 3t) &= \frac{(s+1)^2+18}{(s+1)((s+1)^2+36)} \qquad (s \to s+1) \\ &= \frac{s^2+2s+19}{(s+1)(s^2+2s+37)} \end{aligned}$$

Example 2.5 *Find*

$$e^{3t}\sin^3 2t$$

Solution. *We have*

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

Hence

$$\begin{aligned} \sin^3 2t &= \frac{1}{4}(3\sin 2t - \sin 6t) \\ L(\sin^3 2t) &= \frac{1}{4}[3L(\sin 2t - L(\sin 6t)] \\ &= \frac{1}{4}\left[3.\frac{2}{s^2+2^2} - \frac{6}{s^2+6^2}\right] \\ &= \frac{48}{(s^2+4)(s^2+36)} \end{aligned}$$

By using shifting Rule, we get, $s \to s-3$

$$\begin{aligned} L\{e^{3t}\sin^3 2t\} &= \frac{48}{[(s-3)^2+4][(s-3)^2+36]} \\ &= \frac{48}{(s^2-6s+13)(s^2-6s+45)} \end{aligned}$$

In view of the shifting property we can find the Laplace transform of the standard functions discussed in the preceding section multiplied by e^{at} or e^{-at}

1. $L(\sin bt) = \frac{b}{s^2+b^2}$ $\quad L(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$
2. $L(\cos bt) = \frac{s}{s^2+b^2}$ $\quad L(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$
3. $L(\sinh bt) = \frac{b}{s^2-b^2}$ $\quad L(e^{at}\sinh bt) = \frac{b}{(s-a)^2-b^2}$
4. $L(\cosh bt) = \frac{s}{s^2-b^2}$ $\quad L(e^{at}\cosh bt) = \frac{s-a}{(s-a)^2-b^2}$
5. $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ $\quad L(e^{at}t^n) = \frac{\Gamma(n+1)}{(s-a)^{n+1}},$ for $\quad n \neq 0$

2.4 Laplace Transforms of the form $t^n f(t)$ where n is positive integer

In this section we shall find the Laplace transforms of the function of the form $t^n f(t)$ where n is a positive integer if the Laplace transform of $f(t)$ is known.

Theorem 2.1 *If* $L\{f(t)\} = F(s)$ *then*

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

Proof: We shall prove the theorem for $n = 1$ i.e.

$$L\{tf(t)\} = -\frac{d}{ds}\{F(s)\}$$

We have,

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Differentiating w.r.t. 's', we have

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{d}{ds}\{e^{-st} f(t)\}dt$$

In the R.H.S. we shall apply Leibnitz rule for differentiation under integral sign,

$$\begin{aligned}
&= \int_0^\infty e^{-st}(-t)f(t)dt \\
&= -\int_0^\infty e^{-st}\{tf(t)\}dt \\
&= -L\{tf(t)\} \\
\therefore \quad L\{tf(t)\} &= \frac{-d}{ds}\{F(s)\} = \frac{-d}{ds}[L\{f(t)\}] \\
\text{Further,} \qquad L\{t^2 f(t)\} &= L\{t[tf(t)]\} = \frac{-d}{ds}L\{tf(t)\} \\
&= \frac{-d}{ds}\left[\frac{-d}{ds}L\{f(t)\}\right] \\
&= (-)^2 \frac{d^2}{ds^2}L\{f(t)\} \\
&= (-1)^2 \frac{d^2}{ds^2}\{F(s)\}
\end{aligned}$$

By repeating this process of the above theorem, we get

$$L\{t^n f(t)\} = (-1)^n . \frac{d^n}{ds^n}\{F(s)\}.$$

2.5 Laplace Transforms of $\frac{f(t)}{t}$

If $Lf(t)$ is known then we can find the Laplace transform of $\frac{f(t)}{t}$ by using the following.

Theorem 2.2 *If* $L\{f(t)\} = F(s)$ *and* $\lim_{t\to 0}\frac{f(t)}{t}$ *exists then*

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds \tag{2.10}$$

Proof: We have, $F(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ On integrating on both sides w.r.t. s from s to ∞, we get

$$\begin{aligned}\int_s^\infty F(s)ds &= \int_s^\infty \left[\int_0^\infty e^{-st}f(t)dt\right]ds &(2.11)\\ &= \int_0^\infty \left[\int_s^\infty e^{-st}ds\right]f(t)dt, &(2.12)\end{aligned}$$

(By changing the order of integration.)
Now

$$\int_s^\infty e^{-st}ds = \left[\frac{-e^{-st}}{-t}\right]_s^\infty = \frac{-1}{t}[e^{-\infty} - e^{-st}] = \frac{+e^{-st}}{t} \qquad (e^{-\infty} = 0)$$

$\therefore$ Eqn (2.12) gives,

$$\begin{aligned}\int_s^\infty F(s)ds &= \int_0^\infty \frac{e^{-st}}{t}f(t)dt = L\left\{\frac{f(t)}{t}\right\}\\ \therefore \qquad \int_s^\infty F(s) &= L\left\{\frac{f(t)}{t}\right\}\end{aligned}$$

This complete the proof.

Example 2.6

$$\begin{aligned}L\{t\sin at\} &= \frac{-d}{ds}L(\sin at)\\ &= \frac{-d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\\ &= \frac{2as}{(s^s+a^2)^2}\end{aligned}$$

Example 2.7 $L(te^{-2t}\cos 2t)$. *From theorem (2.1) above we have*

$$L(t\cos 2t) = \frac{s^2-4}{(s^2+4)^2}$$

Hence by using shifting Rule, we get $s \to s+2$

$$\begin{aligned}L\{e^{-2t}t\cos 2t\} &= \frac{(s+2)^2-4}{[(s+2)^2+4]^2}\\ &= \frac{s(s+4)}{(s^2+4s+8)^2}\end{aligned}$$

Example 2.8 $L(t^3\sin t)$. *We have* $L(\sin t) = \frac{1}{s^2+1}$

$$\begin{aligned}\therefore \qquad L(t^3\sin t) &= (-1)^3.\frac{d^3}{ds^3}\left\{\frac{1}{s^2+1}\right\}\\ &= \frac{24s(s^2-1)}{(s^2+1)^4}\end{aligned}$$

Example 2.9 $L\left\{\frac{1-e^{at}}{t}\right\}$. *Now* $\lim_{t\to 0}\frac{1-e^{at}}{t} = \lim_{t\to 0}\frac{-ae^{at}}{1} = -a$ *(By using L'Hospital Rule).*

$$\begin{aligned}
\textit{Also} \qquad L(1-e^{at}) &= L(1) - L(e^{at}) \\
&= \frac{1}{s} - \frac{1}{s-a} \\
L\left\{\frac{1-e^{at}}{t}\right\} &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-a}\right) ds \\
&= [\ln s - \ln(s-a)]_s^\infty \\
&= \left[\ln\left(\frac{1}{1-\frac{a}{s}}\right)\right]_s^\infty \\
&= -\ln\frac{s}{s-a} \\
&= \ln\left(\frac{s-a}{s}\right)
\end{aligned}$$

Example 2.10 $L\left\{\frac{1-\cos at}{t}\right\}$. *Consider* $\lim_{t\to 0}\frac{1-\cos at}{t} = \lim_{t\to 0}\frac{a\sin at}{1} = 0$ *(By using L'Hospital Rule).*

$$\begin{aligned}
\textit{We have} \qquad L(1-\cos at) &= L(1) - L(\cos at) \\
&= \frac{1}{s} - \frac{s}{s^2+a^2} \\
L\left\{\frac{1-\cos at}{t}\right\} &= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds \\
&= \left.\ln s - \frac{1}{2}\ln(s^2+a^2)\right]_s^\infty \\
&= \left.\frac{1}{2}\ln\frac{s^2}{s^2+a^2}\right]_s^\infty \\
&= \left.\frac{1}{2}\ln\frac{1}{1+\frac{a^2}{s^2}}\right]_s^\infty \\
&= \frac{1}{2}\left\{\ln 1 + \ln\left(1+\frac{a^2}{s^2}\right)\right\} \\
&= \frac{1}{2}\ln\frac{s^2+a^2}{s^2}
\end{aligned}$$

2.6 Laplace Transforms of Unit Step Function and Unit Impulse Function

Unit Step Function (Heaviside function)

The unit step function or Heaviside function $u(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0, & \text{when} \quad t \le a \\ 1, & \text{when} \quad t > a \end{cases} \tag{2.13}$$

where $a \ge 0$. The graph of this function is as shown in Figure 2.1.

Table 2.1: Laplace Transform Pairs

$f(t)$	$F(s) = L\{f(t)\}$	Conditions on s
1. 1	$1/s$	$s > 0$
2. t	$1/s^2$	$s > 0$
3. $t^n(n = 1, 2, \dots$	$n!/s^{n+1}$	$s > 0$
4. $t^a(a > -1)$	$\Gamma(a+1)/s^{a+1}$	$s > a$
5. e^{at}	$1/(s-a)$	$s > a$
6. $t^n e^{at}(n = 1, 2, \dots)$	$n!/(s-a)^{n+1}$	$s > a$
7. $H(t-a)$	e^{-as}/s	$s \geq a$
8. $\delta(t-a)$	e^{-as}	$s > 0, a > 0$
9. $\sin at$	$a/(s^2+a^2)$	$s > 0$
10. $\cos at$	$s/(s^2+a^2)$	$s > 0$
11. $t \sin at$	$2as/(s^2+a^2)^2$	$s > 0$
12. $t \cos at$	$(s^2-a^2)/(s^2+a^2)^2$	$s > 0$
13. $e^{at} \sin at$	$b/[(s-a)^2+b^2]$	$s > a$
14. $e^{at} \cos at$	$(s-a)/[(s-a)^2+b^2]$	$s > a$
15. $(1/2a^3) \sin at - (1/2a^2) t \cos at$	$1/(a^2+a^2)^2$	$s > 0$
16. $(1/2a) \sin at + (1/2a) t \cos at$	$s^2/(s^2+a^2)^2$	$s > 0$
17. $1 - \cos at$	$a^2/[s(s^2+a^2)]$	$s > 0$
18. $at - \sin at$	$a^2/[s^2(s^2+a^2)]$	$s > 0$
19. $\sinh at$	$a/(s^2-a^2)$	$s > \lvert a\rvert$
20. $\cosh at$	$s/(s^2-a^2)$	$s > \lvert a\rvert$
21. $(1/2a^3) \sin at + (1/2a^2) t \cosh at$	$1/(s^2-a^2)^2$	$s > \lvert a\rvert$
22. $(1/2a) t \sinh at$	$s/(s^2-a^2)^2$	$s > \lvert a\rvert$
23. $(1/2a) \sinh at + (1/2) t \cosh at$	$s^/(s^2-a^2)^2$	$s > \lvert a\rvert$
24. $\sinh at - \sin at$	$2a^3/(s^4-a^4)$	$s > \lvert a\rvert$
25. $\cosh at - \cos at$	$2a^2 s/(s^4-a^4)$	$s > \lvert a\rvert$

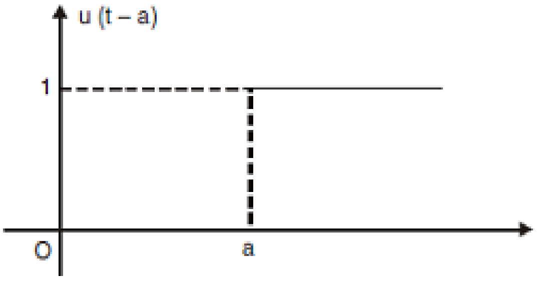


Fig. 2.1: The unit step function

2.6.1 Properties Associated with the Unit Step Function

(i) $L\{u(t-a)\} = \frac{e^{-as}}{s}$

(ii) $L\{f(t-a)u(t-a)\} = e^{-as}F(s) = e^{-as}L\{f(t)\}$.

Proof: (i) Using the definition of Laplace transform, we have

$$\begin{aligned} L\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)dt \\ &= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt \\ &= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt \\ &= 0 + \left.\frac{e^{-st}}{-s}\right]_a^\infty \\ &= \frac{-1}{s}[e^{-\infty} - e^{-sa}] = \frac{e^{-sa}}{s} \end{aligned}$$

Thus, $$L\{u(t-a)\} = \frac{e^{-sa}}{s}$$

(ii) By definition, we have

$$\begin{aligned} L\{f(t-a)u(t-a)\} &= \int_0^\infty e^{-st}f(t-a)u(t-a)dt \\ &= \int_0^a e^{-st}f(t-a)u(t-a)dt + \int_a^\infty e^{-st}f(t-a)u(t-a)dt \\ &= \int_0^a e^{-st}f(t-a)(0)dt + \int_a^\infty e^{-st}f(t-a).1dt \\ &= \int_a^\infty e^{-st}f(t-a)dt \end{aligned}$$

Substituting $t-a=x$ so that $dt=dx$, when $t=a$, $x=0$, when $t=\infty$, $x=\infty$, $t=a+x$. Hence

$$\begin{aligned} L\{f(t-a)u(t-a)\} &= \int_0^\infty e^{-s(a+x)}f(x)dx \\ &= e^{-sa}\int^{-sx} f(x)dx \\ &= e^{-as}\int_0^\infty e^{-st}f(t)dt \qquad \text{change x to t} \\ &= e^{-as}L\{f(t)\} \\ &= e^{-as}F(s). \qquad \textbf{Hence proved.} \end{aligned}$$

Example 2.11 *Find the Laplace transform of* $(2t-1)u(t-2)$
Solution. *Now* $2t-1 = 2(t-2)+3$. *Therefore, using Heaviside shift theorem, we get*

$$\begin{aligned}
L\{(2t-1)u(t-2)\} &= L\{[2(t-2)+3]u(t-2)\} \\
&= e^{-2s}L(2t+3) \qquad \textit{Replacing t-2 by t} \\
&= e^{-2s}\{2L(t)+L(3)\} \\
&= e^{-2s}\left\{\frac{2}{s^2}+\frac{3}{s}\right\}
\end{aligned}$$

Example 2.12 *Find the Laplace transform of* $t^2u(t-3)$
Solution. *Now* $t^2 = [(t-3)+3]^2 = (t-3)^2+6(t-3)+9$
Then

$$\begin{aligned}
L\{t^2u(t-3)\} &= L\{[(t-3)^2+6(t-3)+9]u(t-3)\} \\
&= e^{-3s}L(t^2+6t+9) \qquad \textit{Replacing t-3 by t} \\
&= e^{-3s}[L(t^2)+6L(t)+9] \\
&= e^{-3s}\left\{\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right\} \qquad \textit{Using Heaviside shift theorem}
\end{aligned}$$

Example 2.13 *Find the Laplace transform of* $u(t-\pi/2).\cos 2(t-\pi/2)$
Solution.

$$\begin{aligned}
L\{u(t-\pi/2).\cos 2(t-\pi/2)\} &= e^{-\pi s/2}.F(s) \qquad \textit{where} \quad F(s) = L(\cos 2t) \\
&= e^{-\pi s/2}\left(\frac{s}{s^2+4}\right) = \frac{s.e^{-\pi s/2}}{s^2+4}
\end{aligned}$$

Example 2.14 *Express the following function in terms of the Heaviside's unit step function and hence find its Laplace transform.*

$$f(t) = \begin{cases} e^{-t}, & 0 < t < 3 \\ 0, & t > 3 \end{cases}$$

Solution.

$$\begin{aligned}
\textit{Now} \qquad f(t) &= e^{-t} + [0 - e^{-t}]u(t-3) \\
&= e^{-t} - e^{-t}u(t-3) \\
&= e^{-t} - e^{-(t-3)}u(t-3)e^{-3} \\
\therefore \qquad L\{f(t)\} &= L(e^{-t} - e^{-3}L\{e^{-(t-3)}u(t-3)\} \\
&= \frac{1}{s+1} - e^{-3}e^{-3s}L(e^{-t}) \\
&= \frac{1}{s+1} - e^{-3(s+1)}.\frac{1}{s+1} \\
&= \frac{1-e^{-3(s+1)}}{s+1}
\end{aligned}$$

2.6.2 Laplace Transform of the Unit Impulse Function

Graphically the Dirac delta or unit impulse $\delta(t-a)$ is represented by the horizontal axis with a vertical line of infinite length at $t = a$ (see figure 2.2.

Following are the results of Laplace Transform of Dirac Delta Functions:

$$L\{(\delta(t-a)\} = e^{-as}$$

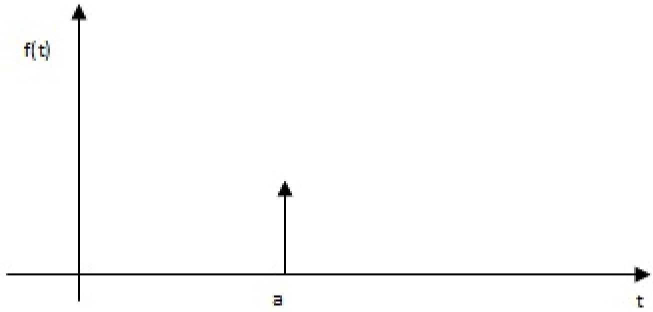


Fig. 2.2: Unit impulse function

$$L\{\delta(t)\} = 1$$

Let us deal with the more general case of $L\{f(t).\delta(t-a)\}$
We have

$$L\{f(t).\delta(t-a)\} = \int_0^\infty e^{-st}.f(t).\delta(t-a)dt.$$

Now the integrand $e^{-st}.f(t).\delta(t-a) = 0$ for all values of t except at $t = a$ at which point $e^{-st} = e^{-as}$, and $f(t) = f(a)$.

$$\therefore L\{f(t).\delta(t-a)\} = f(a).e^{-as}\int_0^\infty \delta(t-a)dt = f(a).e^{-as}(1)$$

$$\therefore L\{f(t).\delta(t-a)\} = f(a)e^{-as}$$

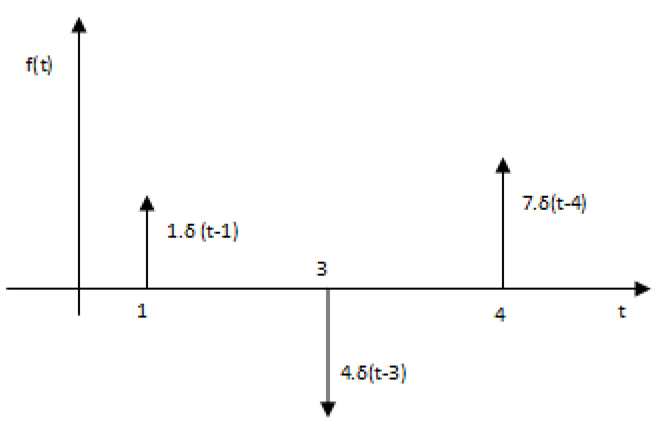
e.g.

$$L\{6.\delta(t-4)\}, \quad a = 4, \qquad \therefore L\{6.\delta(t-4)\} = 6e^{-4s}$$
$$L\{t^3.\delta(t-2)\} \quad a = 2, \qquad \therefore L\{t^3.\delta(t-2)\} = 8e^{-2s}$$

Example 2.15

Impulses of 1, 4, 7 units occurs at $t = 1$, $t = 3$ and $t = 4$ respectively, in directions shown in figure below. Write down an expression for $f(t)$ and determine its Laplace transform.
Solution We have $f(t) = 1.\delta(t-1) - 4.\delta(t-3) + 7.\delta(t-4)$. Then

$$L\{f(t)\} = e^{-s} - 4e{-}3s + 7e^{-4s}$$



2.7 Laplace Transforms of Periodic Functions

A function $f(t)$ is said to be periodic function with period $\alpha > 0$, if $f(t+\alpha = f(t)$.

Theorem 2.3 *If $f(t)$ is a periodic function of period $\alpha > 0$, then*

$$L\{f(t)\} = \frac{1}{1-e^{-s\alpha}} \int_0^\alpha e^{-st} f(t)dt \tag{2.14}$$

Example 2.16

If $f(x) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$ and $f(t) = f(t+4)$, find $L\{f(t)\}$.

Solution Since $f(t)$ is a periodic function with period $\alpha = 4$ from 2.14, we get

$$L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)dt$$

Now,

$$\begin{aligned}
\int_0^4 e^{-st} f(t)dt = &\int_0^2 e^{-st}.3tdt + \int_2^4 e^{-st} f(t)dt \\
=&3\left[t.\frac{e^{-st}}{-s} - 1.\frac{e^{-st}}{(-s)^2}\right]_0^2 + \left[6\frac{e^{-st}}{-s}\right]_2^4 \\
=&3\left[-2\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right] - 3\left[0 - \frac{1}{s^2}\right] - \frac{6}{s}\left(e^{-4s} - e^{-2s}\right) \\
=&\frac{3}{s^2} - \frac{3e^{-2s}}{s^2} - \frac{6e^{-4s}}{s} \\
=&\frac{3}{s^2}\left(1 - e^{-2s} - 2se^{-4s}\right)
\end{aligned}$$

$$\therefore L\{f(t)\} = \frac{3\left(1 - e^{-2s} - 2se^{-4s}\right)}{s^2\left(1-e^{-4s}\right)}.$$

Chapter 3

Inverse Laplace Transforms

3.1 Introduction

If $L\{f(t)\} = F(s)$, then $f(t)$ is called the Inverse Laplace Transform of $F(s)$ and symbolically, we write $f(t) = L^{-1}\{F(s)\}$. Here L^{-1} is called the inverse Laplace transform operator. For example

$$(i) \qquad L(e^{at}) = \frac{1}{s-a}, \quad s > a, \quad L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$(ii) \qquad L(t) = \frac{1}{s^2}, \quad L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

3.2 Inverse Laplace Transforms of Some Standard Functions

1. Since $L(1) = \frac{1}{s}$, $\quad L^{-1}\left\{\frac{1}{s}\right\} = 1$
2. $L(e^{at}) = \frac{1}{s-a}, \quad s > a, \quad L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

Replacing a by -a, we get

$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

3. $L(\sin at) = \frac{a}{s^2+a}$, $\quad L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at, \quad L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$
4. $L(\cos at) = \frac{s}{s^2+a^2}$, $\quad L^{-}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$
5. $L(\sinh at) = \frac{a}{s^2-a^2}$, $\quad L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at, \quad L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$
6. $L(\cosh at) = \frac{s}{s^2-a^2}$, $\quad L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
7. $L(t^n) = \frac{n!}{s^{n+1}}$ where n is a positive integer, we get

$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

Replacing n by $n-1$, we get

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$$

In particular

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t^2-1}{(2-1)!} = t$$

$$L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^{3-1}}{(3-1)!} = \frac{t^2}{2}$$

Since,

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}, \quad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}, \quad n > -1$$

Table 3.1: The following table gives list of the Inverse Laplace Transform of some standard functions.

S.No.	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	S.No.	$F(s)$	$f(t) = L^{-1}\{F(s)\}$
1.	$\frac{1}{s-a}$	e^{at}	6.	$\frac{s}{s^2-a^2}$	$\cos h\ at$
2.	$\frac{1}{s+a}$	e^{-at}	7.	$\frac{1}{s}$	1
3.	$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$	8.	$\frac{1}{s^2}$	t
4.	$\frac{s}{s^2+a^2}$	$\cos at$	9.	$\frac{1}{s^n}, n = 1, 2, \ldots$	$\frac{t^{n-1}}{(n-1)!}$
5.	$\frac{1}{s^2-a^2}$	$\frac{\sin h\ at}{a}$	10.	$\frac{1}{s^{n+1}}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$

3.3 Properties of inverse Laplace transform

1. **Linearity Property**
   If a and b are two constants then

$$L^{-1}\{aF(a) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$$

This result can be extended to more than two functions. This shows that like L, L^{-1} is also a linear operator.

Example 3.1

$$L^{-1}\left\{\frac{2}{s-3} - \frac{3s}{s^2+16} + \frac{4}{s^2-9}\right\} = 2L^{-1}\left\{\frac{1}{s-3}\right\} - 3L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{s^2-3^2}\right\}$$
$$= 2e^{3t} - 3\cos 4t + 4\frac{\sinh 3t}{3}$$

2. **Shifting Property**
   If

$$L^{-1}\{F(s)\} = f(t)$$

then

$$\{L^{-1}F(s-a)\} = e^{at}f(t) = e^{at}L^{-1}\{F(s)\}$$

This follows immediately from the result.
If

$$L\{f(t)\} = F(s)$$

then

$$L\{e^{at}f(t)\} = F(s-a)$$

Replacing a by $-a$, we get

$$L^{-1}\{F(s+a)\} = e^{-at}f(t) = e^{-at}L^{-1}\{F(s)\}$$

Example 3.2

$$\begin{aligned}
L^{-1}\left\{\frac{1}{s^2-2s+5}\right\} &= L^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\} \\
&= e^t L^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\
&= e^t \frac{\sin 2t}{2} \\
&= \frac{1}{2} e^t \sin 2t.
\end{aligned}$$

Example 3.3

$$\begin{aligned}
L^{-1}\left\{\frac{s-3}{s^2-6s+13}\right\} &= L^{-1}\left\{\frac{s-3}{(s-3)^2+2^2}\right\} \\
&= e^{3t} L^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\
&= e^{3t}\cos 2t.
\end{aligned}$$

3.4 Inverse Laplace Transforms using Partial Fractions

In this method we first resolve the given rational function of s into partial fractions and then find the inverse Laplace transform of each fraction.

Example 3.4 *Find the inverse Laplace transform of*

$$\frac{2s-1}{s^2-5s+6}$$

Solution

$$\text{Let} \qquad \frac{2s-1}{s^2-5s+6} = \frac{2s-1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

$$2s-1 = A(s-3) + B(s-2)$$

$$\text{Put} \qquad s = 3, \quad B = 5$$

$$\text{Put} \qquad s = 2, \quad A = -3$$

$$\therefore \qquad \frac{2s-1}{s^2-5s+6} = -\frac{3}{s-2} + \frac{5}{s-3}$$

$$\begin{aligned}
\text{Then} \qquad L^{-1}\left\{\frac{2s-1}{s^2-5s+6}\right\} &= -3L^{-1+}\left\{\frac{1}{s-1}\right\} + 5L^{-1}\left\{\frac{1}{s-3}\right\} \\
&= -3e^{2t} + 5e^{3t}
\end{aligned}$$

Example 3.5 *Find the inverse Laplace transform of*

$$\frac{s}{(2s-1)(3s-1)}$$

$$\text{Let} \qquad \frac{s}{(2s-1)(3s-1)} = \frac{A}{2s-1} + \frac{B}{3s-1}$$

$$S = A(3s-1) + B(2s-1)$$

$$\textit{Put} \qquad s = \frac{1}{2}, \quad A = 1, \quad \textit{and} \quad s = \frac{1}{3}, \quad B = -1$$

$$\frac{s}{(2s-1)(3s-1)} = \frac{1}{2s-1} - \frac{1}{3s-1}$$

$$\begin{aligned}
\therefore \qquad L^{-1}\left\{\frac{s}{(2s-1)(3s-1)}\right\} &= L^{-1}\left\{\frac{1}{2s-1}\right\} - L^{-1}\left\{\frac{1}{3s-1}\right\} \\
&= L^{-1}\left\{\frac{1}{2\left(s-\frac{1}{2}\right)}\right\} - L^{-1}\left\{\frac{1}{3\left(s-\frac{1}{3}\right)}\right\} \\
&= \frac{1}{2}L^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s-\frac{1}{3}}\right\} \\
&= \frac{1}{2}e^{\frac{1}{2}t} - \frac{1}{3}e^{\frac{1}{3}t}.
\end{aligned}$$

Example 3.6 *Find the inverse Laplace transform of*

$$\frac{2s-3}{(s-1)(s-2)(s-3)}$$

Solution.

$$\textit{Let} \qquad \frac{2s-3}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$2s-3 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

$$\textit{Put} \qquad s = 1, \quad \Rightarrow A = \frac{-1}{2}$$

$$s = 3, \quad \Rightarrow C = \frac{3}{2}$$

$$s = 2, \quad \Rightarrow B = -1$$

$$\textit{Thus,} \qquad \frac{2s-3}{(s-1)(s-2)(s-3)} = \frac{-\frac{1}{2}}{s-1} - \frac{1}{s-2} + \frac{\frac{3}{2}}{s-3}$$

$$\begin{aligned}
L^{-1}\left\{\frac{2s-3}{(s-1)(s-2)(s-3)}\right\} &= \frac{-1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\frac{1}{s-2} + \frac{3}{2}L^{-1}\frac{1}{s-3} \\
&= \frac{-1}{2}e^{t} - e^{2t} + \frac{3}{2}e^{3t}.
\end{aligned}$$

Example 3.7 *Find the inverse Laplace transform of*

$$\frac{4s+5}{(s-1)^2(s+2)}$$

Solution

$$\textit{Let} \qquad \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

$$\textit{Put} \qquad s = 1, \quad \Rightarrow B = 3$$

$$s = -2, \quad \Rightarrow C = \frac{-1}{3}$$

To find A, put $s = 0$,

$$\textit{Then} \qquad 5 = -2A + 2B + C$$

$$\textit{This gives} \qquad A = \frac{1}{3}$$

Thus the partial fraction is

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{\frac{1}{3}}{s-1} + \frac{3}{(s-1)^2} - \frac{\frac{1}{3}}{s+2}$$

$$\begin{aligned}
\therefore \quad L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\} &= \frac{1}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\} \\
&= \frac{1}{3}e^t + 3e^t\left[L^{-1}\left\{\frac{1}{s^2}\right\}\right] - \frac{1}{3}e^{-2t} \\
&= \frac{1}{3}e^t + 3e^t.t - \frac{1}{3}e^{-2t} \\
&= \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}.
\end{aligned}$$

3.5 Inverse Laplace Transforms of the Functions of the Form $\frac{F(s)}{s}$

We have proved that if

$$L\{f(t)\} = F(s)$$

$$\text{then} \qquad L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

$$\text{Hence} \qquad L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)dt \tag{3.1}$$

Example 3.8 *Evaluate*

$$L^{-1}\left\{\frac{1}{s(s+a)}\right\}$$

Solution

$$\textit{Consider} \qquad L^{-1}\left\{\frac{1}{(s+a)}\right\} = e^{-at}$$

Using equation (3.1), we get

$$L^{-1}\left\{\frac{1}{s(s+a)}\right\} = \int_0^t e^{-at}dt = \left[\frac{e^{-at}}{-a}\right]_0^t = \frac{1}{a}\left(1 - e^{-at}\right)$$

Example 3.9 *Evaluate*

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$$

$$\textit{We have} \qquad L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$\begin{aligned}
\therefore \quad L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} &= \int_0^t \frac{1}{a}\sin at dt \\
&= \frac{1}{a}\left[\frac{(-\cos at)}{a}\right]_0^t \\
&= \frac{1}{a^2}(1 - \cos at)
\end{aligned}$$

3.6 Inverse Functions of Step Functions

The main points are

(a)

$$u(t-c) = \begin{cases} 0, & 0 < t < c \\ 1, & t \geq c \end{cases}$$

(b)

$$L\{u(t-c)\} = \frac{e^{-cs}}{s}$$

$$L\{u(t)\} = \frac{1}{s}$$

(c)

$$L\{u(t-c).f(t-c)\} = e^{-cs}.F(s) \qquad \text{where} \quad F(s) = L\{f(t)\}$$

(d) If

$$F(s) = L\{f(t)\},$$

then

$$e^{-cs}.F(s) = L\{u(t-c).f(t-c)\}$$

Example 3.10 *Find the function whose transform is* $\frac{e^{-4s}}{s^2}$.
The numerator corresponds to e^{-cs} *where* $c = 4$ *and therefore indicates* $u(t-4)$.
Then

$$\frac{1}{s^2} = F(s) = L(t) \quad \therefore \quad f(t) = t$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} = u(t-4).(t-4)$$

Remember that in writing the final result, $f(t)$ *is replaced by* $f(t-c)$.

Example 3.11 *Determine*

$$L^{-1}\left\{\frac{s.e^{-s}}{s^2+9}\right\}$$

Solution
The numerator contains e^{-s} *which indicates* $u(t-1)$. *Also*

$$\frac{s}{s^2+9} = F(s) = L(\cos 3t)$$

$$\therefore \quad f(t) = \cos 3t \quad \therefore \quad f(t-1) = \cos 3(t-1).$$

$$\therefore \quad L^{-1}\left\{\frac{s.e^{-s}}{s^2+9}\right\} = u(t-1).\cos 3(t-1)$$

Remember that, having obtained $f(t)$, *the result contains* $f(t-c)$.

Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Chapter 4

Some Applications of Laplace Transforms

4.1 Laplace Transforms of the Derivatives

If the Laplace transform of $f(t)$ is known then by the following results we can find the Laplace transforms of the derivatives $f'(t), f''(t), \cdots, f^n(t)$ and $\int_0^t f(t)dt$.

Laplace transforms of the derivatives: *Functions of exponential order.*

A continuous function $f(t), t > 0$ is said to be exponential order if

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

Theorem 4.1 *If $f(t)$ is of exponential order and $f'(t)$ is continuous then*

$$L\{f'(t)\} = sL\{f(t)\} - f(0) \tag{4.1}$$

Proof: *By the definition of Laplace transform, we have*

$$L\{f'\}(t) = \int_0^\infty e^{-st} f'(t)dt$$

Applying integration by parts

$$\begin{aligned}
&= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)e^{-st}(-s)dt \\
&= 0 - f(0) + s\int_0^\infty e^{-st} f(t)dt \\
&= -f(0) + s\int_0^\infty e^{-st} f(t)dt \\
&= -f(0) + sL\{f(t)\} \\
L\{f'(t)\} &= sL\{f(t)\} - f(0)
\end{aligned}$$

Laplace transform of $f''(t)$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0) \tag{4.2}$$

Let

$$f'(t) = g(t) \quad \text{so that} \quad f''(t) = g'(t)$$

Consider

$$\begin{aligned}
Lf''(t) &= L\{g'(t)\} \\
&= sL\{g(t)\} - g(0), \quad \text{using (4.1)} \\
&= sL\{f'(t)\} - f'(0) \\
&= s[sL\{f(t)\} - f(0)] - f'(0) \\
L\{f''(t)\} &= s^2L\{f(t)\} - sf(0) - f'(0)
\end{aligned}$$

Similarly,

$$L\{f'''(t)\} = s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0) \tag{4.3}$$

$$L\{f^n(t)\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0)\cdots f^{n-1} \tag{4.4}$$

If $f(t) = y$ then (4.4) can be written in the form

$$L(y^n) = s^nL(y) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$$

where $y', y'', \cdots y^{(n)}$ denotes the successive derivatives.

4.2 Solution of Linear Differential Equations

One of the important application of Laplace transform is to solve linear differential equations with constant coefficients with initial conditions. For example, consider a second order linear differential equation

$$\frac{d^2y}{dx^2} + a_0\frac{dy}{dx} + a_1y = f(t)$$

$$\text{i.e.} \qquad y'' + a_0y' + a_1y = f(t)$$

where a_0, a_1 are constants with initial conditions $y(0) = A$ and $y'(0) = B$.
Taking Laplace transforms on both sides of the above equation and using the formulae on Laplace transforms of the derivatives y' and y''.
We recall the formulae for immediate reference.

$$\begin{aligned}
L(y') &= sL(y) - y(0) \\
L(y'') &= s^2L(y) - sy(0) - y'(0) \\
L(y''') &= s^3L(y) - s^2y(0) - sy'(0) - y''(0)
\end{aligned}$$

and so on.
To solve a differential equation by Laplace transforms, we go through four distinct stages

(a) Rewrite the equation in terms of Laplace transforms.

(b) Insert the given initial conditions.

(c) Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solutions.

The procedure is summarized in Figure (4.1).

Example 4.1 *Solve using Laplace transforms.*

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

$$\textit{given that} \qquad y(0) = 0 \quad \textit{and} \quad y'(0) = 0$$

Solution. *Given equation is* $y'' - 3y' + 2y = e^{3t}$.
Taking Laplace transforms on both sides, we get

$$L(y'') - 3L(y') + 2L(y) = L(e^{3t})$$

$$\textit{i.e.} \qquad s^2L(y) - sy(0) - y'(0) - 3[sL(y) - y(0)] + 2L(y) = \frac{1}{s-3}$$

where $y(0) = 0$ *and* $y'(0) = 0$

$$\textit{i.e.,} \qquad (s^2 - 3s + 2)L(y) = \frac{1}{s-3} \qquad \textit{using the initial conditions}$$

$$\begin{aligned} \textit{i.e.,} \qquad L(y) &= \frac{1}{(s^2 - 3s + 2)(s-3)} \\ &= \frac{1}{(s-1)(s-2)(s-3)} \end{aligned}$$

$$\begin{aligned} \therefore \qquad y &= L^{-1}\left[\frac{1}{(s-1)(s-2)(s-3)}\right] \\ &= L^{-1}\left[\frac{\frac{1}{2}}{s-1} - \frac{1}{s-2} + \frac{\frac{1}{2}}{s-3}\right] \qquad \textit{using partial fractions} \\ &= \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-3}\right\} \\ y &= \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t} \end{aligned}$$

This is the required solution.

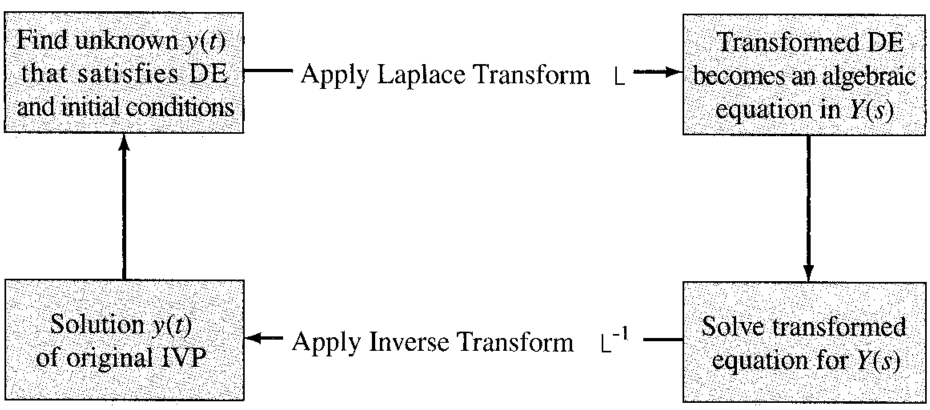


Fig. 4.1:

Example 4.2 *Solve using Laplace transforms* $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = \sin t$, *given* $y(0) = \frac{1}{10}$ *and* $y'(0) = \frac{21}{10}$.
Solution. *Given equation is*

$$y'' - 5y' + 6y = \sin t$$

Taking Laplace transforms on both sides, we get

$$L(y'') - 5L(y') + 6L(y) = L(\sin t)$$

$$\textit{i.e.,} \qquad s^2L(y) - sy(0) - y'(0) - 5[sL(y) - y(0)] + 6L(y) = \frac{1}{s^2+1}$$

where $y(0) = \frac{1}{10}$ *and* $y'(0) = \frac{21}{10}$ *i.e.,*

$$s^2 L(y) - \frac{s}{10} - \frac{21}{10} - 5\left[sL(y) - \frac{1}{10}\right] + 6L(y) = \frac{1}{s^2+1}$$

$$\begin{aligned}
(s^2 - 5s + 6)L(y) &= \frac{1}{s^2+1} + \frac{1}{10}(s+16) \\
L(y) &= \frac{1}{(s^2+1)(s^2-5s+6)} + \frac{1}{10}\frac{s+16}{(s^2-5s+6)} \\
\text{i.e.,} \qquad L(y) &= \frac{1}{(s^2+1)(s-3)(s-2)} + \frac{1}{10}\frac{s+16}{(s-2)(s-3)} \\
&= \frac{\frac{1}{10}s + \frac{1}{10}}{s^2+1} + \frac{\frac{-1}{5}}{s-2} + \frac{\frac{1}{10}}{s-3} + \frac{1}{10}\left[\frac{-18}{s-2} + \frac{19}{s-3}\right] \qquad \text{By using partial fractions} \\
&= \frac{-2}{s-2} + \frac{2}{s-3} + \frac{1}{10}\left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= L^{-1}\left\{\frac{-2}{s-2} + \frac{2}{s-3} + \frac{1}{10}\left[\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]\right\} \\
y &= -2L^{-1}\left\{\frac{1}{s-2}\right\} + 2L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{10}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}\right] \\
&= -2e^{2t} + 2e^{3t} + \frac{1}{10}(\cos t + \sin t)
\end{aligned}$$

Simultaneous differential equations

Convert the simultaneous differential equations into simultaneous algebraic equations by taking the Laplace transform of each equation in turn. Insert the initial values. Solve the simultaneous algebraic equations in the usual manner and take the inverse Laplace transform of the algebraic solutions to find the solutions to the simultaneous differential equations.

Example 4.3 *Solve the pair of simultaneous equations*

$$\begin{aligned}
y' - x &= e^t \\
x' + y &= e^{-t}
\end{aligned}$$

given that at $t = 0$*,* $x = 0$*, and* $y = 0$*.*

(a) We first express both equations in Laplace transforms.

$$\begin{aligned}
(sL(y) - y(0)) - L(x) &= \frac{1}{s-1} \\
(sL(x) - x(0)) - L(y) &= \frac{1}{s+1}
\end{aligned}$$

(b) Then we insert the initial conditions, $x(0) = 0$ *and* $y(0) = 0$

$$\therefore \qquad \left.\begin{aligned}
sL(y) - L(x) &= \frac{1}{s-1} \\
sL(x) - L(y) &= \frac{1}{s+1}
\end{aligned}\right\}$$

(c) *We now solve these for $L(y)$ and $L(x)$ by normal algebraic method. Eliminating $L(y)$ we have*

$$\begin{aligned}
sL(y) - L(x) &= \frac{1}{s-1}\\
sL(y) + s^2L(x) &= \frac{s}{s+1}\\
\therefore \quad (s^2+1)L(x) &= \frac{2}{s+1} - \frac{1}{s-1} = \frac{s^2-2s-1}{(s+1)(s-1)}\\
\therefore \quad L(x) &= \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)}
\end{aligned}$$

$$\begin{aligned}
L(x) = \frac{s^2-2s-1}{(s-1)(s+1)(s^2+1)} &\equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}\\
\therefore \quad s^2-2s-1 =& A(s+1)(s^2+1) + B(s-1)(s^2+1)\\
&+ (s-1)(s+1)(Cs+D)
\end{aligned}$$

Putting $s=1$ and $s=-1$ gives $A=-\frac{1}{2}$ and $B=-\frac{1}{2}$.
Comparing coefficients of s^3 and the constant terms gives $C=1$ and $D=1$.

$$\therefore \quad L(x) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{ss+1}{s^2+1}$$

$$x = -\frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + \cos t + \sin t$$

and

$$\left.\begin{aligned}
s^2L(y) - sL(x) &= \frac{s}{s-1}\\
L(y) + sL(x) &= \frac{1}{s+1}
\end{aligned}\right\}$$

$$\therefore \quad (s^2+1)L(y) = \frac{s}{s-1} + \frac{1}{s+1} = \frac{s^2+2s-1}{(s-1)(s+1)}$$

$$\begin{aligned}
\therefore \quad L(y) = \frac{s^2+2s-1}{(s-1)(s+1)(s^2+1)} &\equiv \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}\\
\therefore \quad s^2+2s-1 =& A(s+1)(s^2+1) + B(s-1)(s^2+1)\\
&+ (s-1)(s+1)(Cs+D)
\end{aligned}$$

Putting $s=1$ and $s=-1$ gives $A=\frac{1}{2}$ and $B=\frac{1}{2}$. Equating coefficients of s^3 and the constant terms gives $C=-$ and $D=1$.

$$\therefore \quad L(y) = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore \quad y = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - \cos t + \sin t$$

Example 4.4 *Solve the equations*

$$2y' - 6y + 3x = 0$$

$$3x' - 3x - 2y = 0$$

given that $x(0)=1$ and $y(0)=3$.
Expressing these in Laplace transforms, we have

$$2(sL(y) - y(0)) - 6L(y) + 3L(x) = 0$$

$$3(sL(x) - x(0)) - 3L(x) - 2L(y) = 0$$

Then we insert the initial conditions and simplify, obtaining

$$\begin{aligned} 3L(x) + (2s-6)L(y) &= 6 \\ (3s-3)L(x) - 2L(y) &= 3 \end{aligned}$$

(a) *To find* $L(x)$*, multiply the second equation by* $s-3$ *and add to the first equation i.e.*

$$\left.\begin{aligned} 3L(x) + (2s-6)L(y) &= 6 \\ (s-3)(3s-3)L(x) - (2s-6)L(y) &= 3(s-3) \end{aligned}\right\}$$

Add to obtain

$$\begin{aligned} [(s-3)(3s-3)+3]L(x) &= 3s-9+1 \\ \therefore \quad (3s^2-12s+12)L(x) &= 3s-3 \\ (s^2-4s+4)L(x) &= s-1 \end{aligned}$$

$$\therefore \quad L(x) = \frac{s-1}{(s-2)^2} \equiv \frac{A}{s-2} + \frac{B}{(s-2)^2} = \frac{A(s-2)+B}{(s-2)^2}$$

$$\therefore \quad s-1 = A(s-2)+B$$

giving $A = 1$ *and* $B = 1$

$$\therefore \quad L(x) = \frac{1}{s-2} + \frac{1}{(s-2)^2} \qquad \therefore \quad x = e^{2t} + te^{2t}$$

(b) *Similarly, we can find* $L(y)$

$$L(y) = \frac{6s-9}{2(s-2)^2} \equiv \frac{1}{2}\left\{\frac{A}{s-2} + \frac{B}{(s-2)^2}\right\} = \frac{1}{2}\left\{\frac{A(s-2)+B}{(s-2)^2}\right\}$$

$$\therefore \quad 6s-9 = A(s-2)+B \quad \therefore \quad A = 6; \quad B = 3$$

$$\therefore \quad L(y) = \frac{1}{2}\left\{\frac{6}{s-2} + \frac{3}{(s-2)^2}\right\} \quad \therefore \quad y = \frac{1}{2}\left\{6e^{2t} + 3te^{2t}\right\}$$

Chapter 5

Fourier Series

5.1 Introduction to Fourier Series

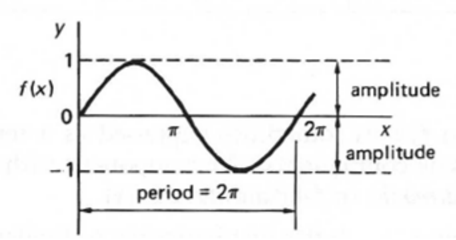
We have seen earlier that many functions can be expressed in the form of infinite series. Problems involving various forms of oscillations are common in fields of modern technology and *Fourier series*, with which we shall now be concerned, enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of a Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin's and Taylor's series require the function to be continuous throughout.

Fourier series provides a method of analyzing periodic functions into their constituent components. Alternating current and voltages, displacement, velocity and acceleration of slider-crank mechanisms and acoustic waves are typical practical examples in engineering and science where periodic functions are involved and often requiring analysis.

5.2 Periodic functions

A function $f(x)$ is said to be *periodic* if $f(x+T) = f(x)$ for all values of x, where T is some positive number. T is the interval between two successive repetitions and is called the *period* of the function $f(x)$. For example, $y = \sin x$ is periodic in x with period 2π since $\sin x = \sin(x+2\pi) = \sin(x+4\pi)$, and so on. In general, if $y = \sin \omega t$ then the period of the waveform is $2\pi/\omega$.

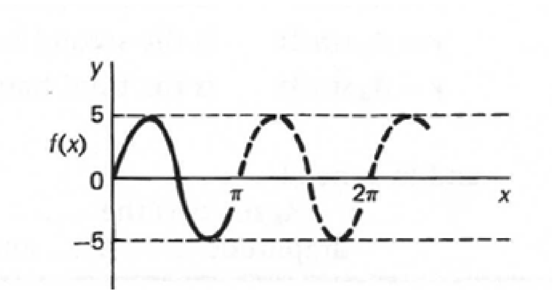
Example 5.1 *The function* $\sin x$ *has periods* $2\pi, 4\pi, 6\pi, \ldots,$ *since* $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ *all are equal* $\sin x$*. However,* 2π *is the* **least period** *or the period of* $\sin x$*.*

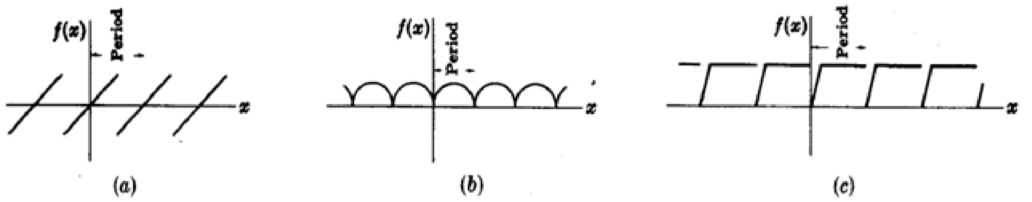


Example 5.2 *The period of* $\sin nx$ *or* $\cos nx$*, where* n *is a positive integer, is* $2\pi/n$*.*
e.g. $y = 5\sin 2x$*, the amplitude is* 5*. The period is* π *(or* 180^o*) and there are thus 2 complete cycles in* 360^o

Example 5.3 *The period of* $\tan x$ *is* π*.*

Other examples of periodic functions are





Integrals of periodic functions

1. $$\int_{-\pi}^{\pi} dx = [x]_{-\pi}^{\pi} = 2\pi$$

2. $$\int_{-\pi}^{\pi} \cos nx dx = 0$$

3. $$\int_{-\pi}^{\pi} \sin nx dx = 0$$

4. $$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \pi\delta_{mn}$$

   $$\text{where} \quad \delta_{mn} = \begin{cases} 1 & \text{if} m = n \\ 0 & \text{if} m \neq n \end{cases}$$

   δ_{mn} is called the Kronecker delta.

5. $$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \pi\delta_{mn}$$

6. $$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$$

Note that the same results are obtained no matter what the end points of the integrals are, *provided that the interval between them is one period*. So, for example

$$\begin{aligned} \int_{k}^{k+2\pi} \cos nx dx &= \left[\frac{\sin nx}{n}\right]_{k}^{k+2\pi} \qquad (n \neq 0) \\ &= \frac{\sin(nk + 2n\pi)}{n} - \frac{\sin nk}{n} \\ &= 0 \qquad \text{because} \quad \sin(x + 2n\pi) = \sin x \end{aligned}$$

5.3 Orthogonal functions

If two different functions $f(x)$ and $g(x)$ are defined on the interval $a \leq x \leq b$ and

$$\int_a^b f(x)g(x)dx = 0$$

then we say that the two functions are *orthogonal* to each other on the interval $a \leq x \leq b$. In the previous section we have seen that the trigonometric functions $\sin nx$ and $\cos nx$ where $n = 0, 1, 2, \ldots$ form an infinite collection of periodic functions that are mutually orthogonal on the interval $-\pi \leq x \leq \pi$, indeed on any interval of width 2π. That is

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \qquad \text{for} \quad m \neq n$$

and

$$\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0 \qquad \text{for} \quad m \neq n$$

5.4 Odd and Even functions

Before taking up our study of Fourier series, in the next chapter, we need to define even, odd and periodic functions (period functions already defined).
Let f be defined on an x interval, finite or infinite, that is centered at $x = 0$. We say that f is an *even* function if

$$f(-x) = f(x) \tag{5.1}$$

and an *odd* function if

$$f(-x) = -f(x) \tag{5.2}$$

for all x in that interval. That is, the graph of f is *symmetric* about $x = 0$ if f is even, and *antisymmetric* about $x = 0$ if f is odd. Examples are shown in 5.1. For example, $5, x^2, 3x^4, \cos x, \sin|x|$, and e^{-x^2} are even and $x, 3x^3, 2x^5, \sin x$ and $x \cos x$ are odd.

(a) (b)

Fig. 5.1: (a) Even and (b) odd function

There are several useful algebraic properties of even and odd functions, such as the following:

$$even + even = even \tag{5.3a}$$
$$even \times even = even \tag{5.3b}$$
$$odd + odd = odd \tag{5.3c}$$
$$odd \times odd = even \tag{5.3d}$$
$$even \times odd = odd \tag{5.3e}$$

To prove (5.3e), for example, let $F(x)$ be even and let $G(x)$ be odd. Then $F(-x)G(-x) = F(x)[-G(x)] = -F(x)G(x)$, in accord with 5.2. In addition, two useful integral properties are as follows. If f is even, then

$$\int_{-A}^{A} f(x) = 2\int_{0}^{A} f(x)dx, \quad (f \quad \text{even}) \text{ and if } f \text{ is odd, then} \tag{5.4a}$$
$$\int_{-A}^{A} f(x)dx = 0, \quad (f \quad \text{odd}) \tag{5.4b}$$

for if we interpret the integral in (5.4a) as areas (positive above the x axis, negative below it) then the area $\int_{-A}^{0} f(x)dx$ is equal to the $\int_{0}^{A} f(d)dx$ due the symmetry of the graph of f. And in the case of (5.4b)the area $\int_{-A}^{0} f(x)dx$ and $\int_{0}^{A} f(x)dx$ are negatives of each other, due to the antisymmetry of the graph of f, and hence cancel.
Alternatively, (5.3a) and (5.3b) follow directly from (5.1) and (5.2), respectively. For example, if f is odd, then

$$\begin{aligned}
\int_{-A}^{A} f(x)dx &= \int_{-A}^{0} f(x)dx + \int_{0}^{A} f(x)dx \\
&= \int_{0}^{A} f(-t)(-dt) + \int_{0}^{A} f(x)dx \quad (x = -t) \\
&= \int_{0}^{A} f(-t)(dt) + \int_{0}^{A} f(x)dx \\
&= \int_{0}^{A} -f(t)dt + \int_{0}^{A} f(x)dx \quad (\text{oddness of f}) \\
&= -\int_{0}^{A} f(x)(dt) + \int_{0}^{A} f(x)dx \quad (x = t) \\
&= 0
\end{aligned}$$

as stated in (5.4b)

Note carefully that a given function is not necessary even or odd: it may be both even and odd, or it may be either. Every function can be uniquely decomposed into sum of an even function, say f_e and an odd function, say f_o, as demonstrated by the simple identity

$$\begin{aligned}
f(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\
&\equiv f_e(x) + f_o(x)
\end{aligned} \tag{5.5}$$

for observe that

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x)$$

and, similarly, that $f_e(-x) = f_e(x)$.

Example 5.4 *Surely $f(x) = e^x$ is neither even nor odd. Since (Fig. 5.2) it is neither symmetry nor antisymmetry about $x = 0$. Putting $f(x) = e^x$ and $f(-x) = e^{-x}$ into (5.5) gives*

$$f_e(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad f_o = \frac{e^x - e^{-x}}{2}$$

as the even and odd part of e^x respectively. In fact, we recognize these functions as $\cosh x$ and $\sinh x$. So it is interesting that we can think of $\cosh x$ and $\sinh x$ as the even and odd parts of e^x respectively.

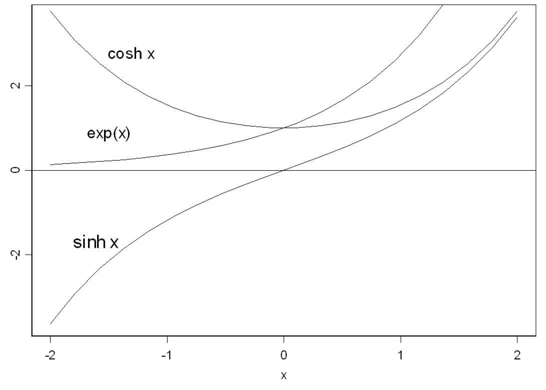


Fig. 5.2: Even and odd parts of e^x

5.5 The Fourier Series of a Function

Let $f(x)$ be defined on $[-L, L]$. We want to choose numbers $a_0, a_2, \dots$ and $b_1, b_2, \dots$ such that

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}[a_k \cos(k\pi x/L) + b_k \sin(k\pi x/L)] \tag{5.6}$$

This decomposition of the function into a sum of terms , each representing the influence of a different fundamental frequency on the behavior of the function.
To determine a_0, integrate equation (5.24) term by term to get

$$\begin{aligned}\int_{-L}^{L} f(x)dx =& \frac{1}{2}\int_{-L}^{L} a_0 dx \\ &+ \sum_{k=1}^{\infty}\left(a_k \int_{-L}^{L}\cos(k\pi x/L)dx + b_k \int_{-L}^{L}\sin(k\pi x/L)dx\right) \\ =& \frac{1}{2}a_0(2L) = \pi a_0\end{aligned}$$

because all of the integrals in the summation are zero. Then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{5.7}$$

To solve for the other coefficients in the proposed equation (5.24), we will use the following three facts, which follow by routine integrations. Let m and n be integers. Then

$$\int_{-L}^{L}\cos(n\pi x/L)\sin(m\pi x/L)dx = 0. \tag{5.8}$$

Furthermore, if $n \neq m$ then

$$\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)dx = \int_{-L}^{L} \sin(n\pi x/L)\sin(m\pi x/L)dx = 0. \tag{5.9}$$

And, if $n \neq 0$, then

$$\int_{-L}^{L} \cos^2(n\pi x/L)dx = \int_{-L}^{L} \sin^2(n\pi x/L)dx = L. \tag{5.10}$$

Now let n be any positive integer. To solve for a_n, multiply equation (5.24) by $\cos(n\pi x/L)$ and integrate the resulting equation to get

$$\int_{-L}^{L} f(x)\cos(n\pi x/L)dx = \frac{1}{2}a_0 \int_{-L}^{L} \cos(n\pi x/L)dx$$
$$+ \sum_{k=1}^{\infty}\left(a_k \int_{-L}^{L} \cos(k\pi x/L)\cos(n\pi x/L)dx + b_k \int_{-L}^{L} \sin(k\pi x/L)\cos(n\pi x/L)dx\right)$$

Because equations (5.26) and (5.27), all of the terms on the right are zero except the coefficient of a_n, which occurs in the summation when $k = n$. The last equation reduces to

$$\int_{L}^{L} f(x)\cos(n\pi x/L)dx = a_n \int_{-L}^{L} \cos^2(n\pi x/L)dx = a_n L$$

by equation (5.28). Therefore

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx. \tag{5.11}$$

This expression contains a_0 if we let $n = 0$.
Similarly, if we multiply equation (5.24) by $\sin(n\pi x/L)$ and integrate, we obtain

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx. \tag{5.12}$$

The numbers

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5.13}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5.14}$$

are called the *Fourier coefficients of f on [-L,L]*. When these numbers are used, the series in (5.24) is called the *Fourier series of f on [-L,L]*.

Example 5.5 *(a) Find the Fourier coefficients corresponding to the function*

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \quad \textit{Period} \quad = 10 \end{cases}$$

(b) Write the corresponding Fourier series

Solution

The graph of $f(x)$ is

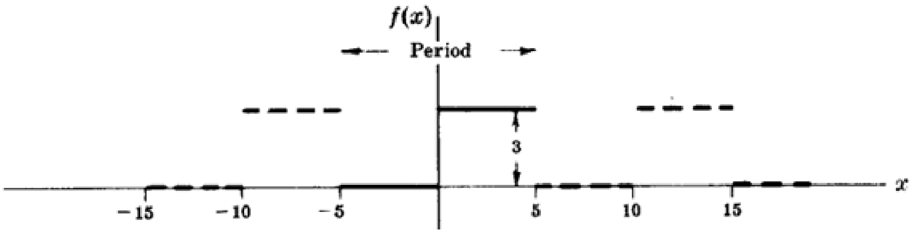


Fig. 5.3:

(a) *Period* $= 2L = 10$ *and* $L = 5$. *Choose the interval* c *to* $c + 2L$ *as* -5 *to* 5, *so that* $c = -5$. *Then*

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos(n\pi x/L)dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos(n\pi x/5)dx \\
&= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos(n\pi x/5)dx + \int_0^5 (3)\cos(n\pi x/5)dx\right\} = \frac{3}{5}\int_0^5 \cos(n\pi x/5) \\
&= \frac{3}{5}\left[\frac{5}{n\pi}\sin(n\pi x/5)\right]_0^5 = 0 \quad \textit{if} \quad n \neq 0
\end{aligned}
$$

If $n = 0$

$$a_0 = \frac{3}{5}\int_0^5 \cos(0\pi x/5)dx = \frac{3}{5}\int_0^5 dx = 3.$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin(n\pi x/L)dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin(n\pi x/5)dx \\
&= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin(n\pi x/5)dx + \int_0^5 (3)\sin(n\pi x/5)dx\right\} = \frac{3}{5}\int_0^5 \sin(n\pi x/5) \\
&= \frac{3}{5}\left[-\frac{5}{n\pi}\cos(n\pi x/5)\right]_0^5 = \frac{3(1-\cos n\pi)}{n\pi}
\end{aligned}
$$

(b) *The corresponding Fourier series is*

$$
\begin{aligned}
\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/L) + b_n\sin(n\pi/L)) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin(n\pi x/5) \\
&= \frac{3}{2} + \frac{6}{\pi}\left(\sin(x\pi/5) + \frac{1}{3}\sin(3x\pi/5) + \frac{1}{5}\sin(5\pi x/5) + \ldots\right.
\end{aligned}
$$

5.6 Dirichlet Conditions

Suppose that

1. $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$
2. $f(x)$ is periodic outside $(-L, L)$ with period $2L$
3. $f(x)$ and $f'(x)$ are piecewise continuous in $-L, L$.

Then the series (5.24) with Fourier coefficients converges to

(a) $f(x)$ if x is a point of continuity

(b) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right and left- hand limits of $f(x)$ at x and represent $\lim_{\epsilon\to+0} f(x+\epsilon)$ and $\lim_{\epsilon\to 0+} f(x-\epsilon)$, respectively.

Example 5.6 *How should* $f(x)\begin{cases} 0 & -5<x<0 \\ 3 & 0<x<5 \end{cases}$ *be defined at* $x=-5, x=0$*, and* $x=5$ *in order that the Fourier series will converge to* $f(x)$ *for* $-5\le x\le 5$*?*

Solution

Since $f(x)$ *satisfies the Dirichlet conditions, we can say that the series converges to* $f(x)$ *at all points of continuity and to* $\frac{f(x+0)+f(x-0)}{2}$ *at points of discontinuity. At* $x=-5, 0,$ *and* 5*, which are points of discontinuity, the series converge to* $(3+0)/2=3/2$ *as seen from the graph. If we redefine* $f(x)$ *as follows,*

$$f(x)=\begin{cases} 3/2 & x=-5 \\ 0 & -5<x<0 \\ 3/2 & x=0 \\ 3 & 0<x<5 \\ 3/2 & x=5 \end{cases} \quad \textit{Period} \quad =10$$

Then the series will converge to $f(x)$ *for* $-5\le x\le 5$.

5.7 Fourier series for periodic functions of period 2π

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic functions just discussed. That is, if $f(x)$ is defined on the interval $-\pi\le x\le\pi$ where $f(x+2\pi n)=f(x)$ then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

The coefficients of a_n

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\cos mx dx \\
=&\int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right)\cos mx dx \\
=&\frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx dx+\sum_{n=1}^{\infty}a_n\int_{-\pi}^{\pi}\cos nx\cos mx dx \\
&+\sum_{n=1}^{\infty}b_n\int_{-\pi}^{\pi}\sin nx\cos mx dx \\
=&\frac{a_0}{2}\times 0+\sum_{n=1}^{\infty}a_n\pi\delta_{nm}+\sum_{n=1}^{\infty}b_n\times 0 \\
=&a_m\pi
\end{aligned}$$

and so

$$a_m=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n=0,1,2,\ldots$$

The coefficient of b_n

$$
\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\sin mx dx \\
&= \int_{-\pi}^{\pi}\left(\frac{a_0}{2}\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right)\sin mx dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi}\sin mx dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin mx dx \\
&\quad + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin nx\sin mx dx \\
&= \frac{a_0}{2}\times 0 + \sum_{n=1}^{\infty} a_n\times 0 + \sum_{n=1}^{\infty} b_n\pi\delta_{nm} \\
&= b_m\pi \\
\therefore\quad b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 0,1,2,\ldots
\end{aligned}
$$

Example 5.7 *Expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series if (a) the period is 2π*
Solution
(a) The graph of $f(x)$ with period 2π is shown below

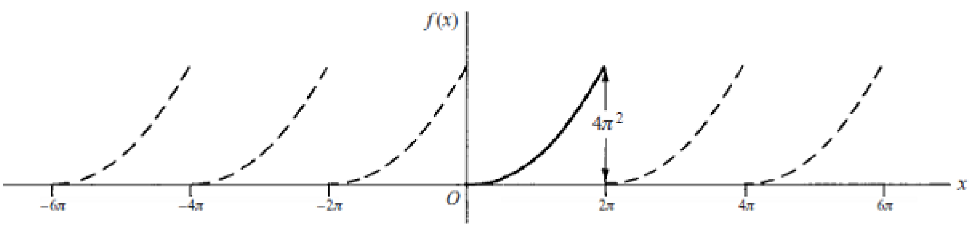


Fig. 5.4:

Period $= 2L = 2\pi$. Choosing $c = 0$, we have

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos(n\pi x/L)dx = 1\pi\int_0^{2\pi} x^2\cos nx dx \\
&= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{4}{n^2}, \quad n \neq 0
\end{aligned}
$$

If $n = 0$, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin(n\pi x/L)dx = 1\pi\int_0^{2\pi} x^2\sin nx dx \\
&= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = \frac{-4\pi}{n}
\end{aligned}
$$

Then $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.
This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

Example 5.8 *Find the Fourier series of $f(x) = x - x^2, \quad -\pi < x < \pi$*
Solution

Let $f(x) = x - x^2$ *for* $-\pi \leq x \leq \pi$. *Here* $L = \pi$. *Compute*

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = -\frac{2}{3}\pi^2, \\
a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nxdx \\
=& \frac{4\sin n\pi - 4n\pi\cos n\pi - 2n^2\pi^2\sin n\pi}{n^3\pi} \\
=& -\frac{4}{n^2}\cos n\pi = -\frac{4}{n^2}(-1)^n \\
=& \frac{4(-1)^{n+1}}{n^2}
\end{aligned}$$

and

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nxdx \\
=& 2\sin n\pi - 2n\pi\cos nx \\
=& -\frac{2}{n}\cos nx = -\frac{2}{n}(-1)^n \\
=& \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

We have used the facts that $\sin nx = 0$ *and* $\cos nx = (-1)^n$ *if* n *is an integer. The Fourier series of* $f(x) = x - x^2$ *on* $[-\pi, \pi]$ *is*

$$-\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right].$$

5.8 Products of odd and even functions

Theorem 5.1 *If* $f(x)$ *is defined over the interval* $-\pi < x < \pi$ *and* $f(x)$ *is* $even$, *then the Fourier series for* $f(x)$ *contains* $cosine\ terms$ *only. Included in this is* a_0 *which may be regarded as* $a_n\cos nx$ *with* $n = 0$.

Proof: Since $f(x)$ is even, $\int_{-\pi}^{0} f(x)dx = \int_0^{\pi} f(x)dx$
(a)$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)$ $\therefore$ $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$.
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$.
But $f(x)\cos nx$ is the product of two even functions and therefore itself even.
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$.
Because, since $f(x)\sin nxdx$ is the product of an even function and an odd function, it is itself odd.
$\therefore$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0$ $\therefore$ $b_n = 0$
Therefore, there are no sine terms in the Fourier series for $f(x)$.

Theorem 5.2 *If* $f(x)$ *is an* odd *function defined over the interval* $-\pi < x < \pi$, *then the Fourier series for* $f(x)$ *contains* $sine\ terms$ *only.*

Proof: Since $f(x)$ is an odd function, $\int_{-\pi}^{0} f(x)dx = -\int_0^{\pi} f(x)dx$.
(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$. But $f(x)$ is odd $\therefore$ $a_0 = 0$
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = 0$ because $f(x)\cos nx$ is odd function (ie product of odd and even)

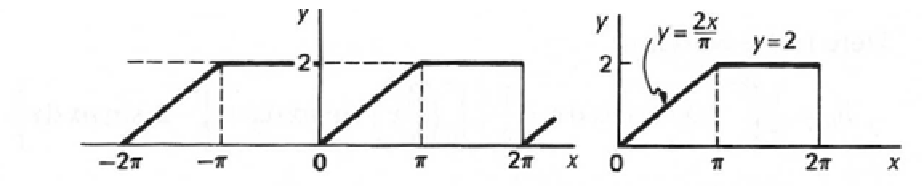


Fig. 5.5:

Example 5.9 *Determine the Fourier series for the function shown in figure 5.12nc*
Solution*: This is neither odd nor even. Therefore we must find a_0, a_n and b_n.*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

(a)

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left\{\int_0^{\pi}\frac{2}{\pi}xdx + \int_{\pi}^{2\pi} 2dx\right\} \\ =& \frac{1}{\pi}\left\{\left[x^2/\pi\right]_0^{\pi} + [2x]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}(\pi + 4\pi - 2\pi) = 3 \quad \therefore \quad a_0 - 3 \end{aligned}$$

(b)

$$\begin{aligned} a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\left\{\int_0^{\pi}(2x/\pi)\cos nxdx + \int_{\pi}^{2\pi} 2\cos nxdx\right\} \\ =& \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nxdx + \int_{\pi}^{2\pi}\cos nxdx\right\} \\ a_n =& \frac{2}{\pi}\left\{\frac{1}{\pi}(0-0) - \frac{1}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\} \\ =& \frac{2}{\pi}\left\{-\frac{1}{\pi n^2}(-(-1)^n + 1) + (0-0)\right\} \\ =& -\frac{2}{\pi^2 n^2}(1-(-1)^n) \end{aligned}$$

and so $a_0 = 0$ (n even) and $a_n = -\frac{4}{\pi^2 n^2}$ (n odd)

$$\begin{aligned} b_n =& \frac{1}{\pi}\int^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\left\{\int_0^{\pi}(2x/\pi)\sin nxdx + \int_{\pi}^{2\pi} 2\sin nxdx\right\} \\ =& \frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos nxdx + \int_{\pi}^{2x}\sin nxdx\right\} \\ =& \frac{2}{\pi}\left\{\frac{1}{\pi n}(-\pi\cos n\pi) + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\} \\ =& \frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}(\cos 2\pi n - \cos n\pi)\right\} \\ =& \frac{2}{\pi}\left\{-\frac{1}{n}\cos 2n\pi\right\} = -\frac{2}{\pi n}\cos 2n\pi \end{aligned}$$

But $\cos 2n\pi = 1 \;.\; \therefore \quad b_n = -\frac{2}{\pi n}$

$$\begin{aligned} f(x) =& \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\} \\ & -\frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right\} \end{aligned}$$

Theorem 5.3 *Fourier's theorem*
This theorem states that a periodic function that satisfies certain conditions can be expressed as the sum of a number of sine functions of different amplitudes, phases and periods. That is, if $f(t)$ is a periodic function with period T then

$$\begin{aligned} f(t) =& A_0 + A_1 \sin(\omega t + \phi_1) + A_2 \sin(2\omega t + \phi_2) + \dots \\ &+ A_n \sin(n\omega t + \phi_n) + \dots \end{aligned} \tag{5.15}$$

Where As and ϕs are constants and $\omega = 2\pi/T$ is the frequency of $f(t)$. The term $A_1 \sin(\omega t + \phi_1)$ is called the *first harmonic* or *fundamental mode*, and it has the same frequency ω as the parent function $f(t)$. The term $A_n \sin(n\omega t + \phi_n)$ is called the *n-th harmonic*, and it has frequency $n\omega$, which is n times that of the fundamental. A_n denotes the *amplitude* of the n-th harmonic and ϕ_n is its *phase angle*, measuring the lag or lead of the n-th harmonic with reference to a pure sine wave of the same frequency.
Since

$$\begin{aligned} A_n \sin(n\omega t + \phi_n) &\equiv (A_n \cos\phi_n) \sin n\omega t + (A_n \sin\phi_n) \cos n\omega t \\ &\equiv b_n \sin n\omega t + a_n \cos n\omega t \end{aligned}$$

where

$$b_n = A_n \cos\phi_n, \quad a_n = A_n \sin\phi_n \tag{5.16}$$

the expression 5.33 can be written as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{5.17}$$

where $a_0 = 2A_0$ (we shall see later that taking the first term as $\frac{1}{2}a_0$ rather than a_0 is a convenience that enables us to make a_0 fit a general result). The expression (5.35) is called the *Fourier series expansion* of the function $f(t)$, and as and bs are called respectively as the *in-phase* and *phase quadrature components* of the n-th harmonic, this terminology arising from the use of the phasor notation $e^{in\omega t} = \cos n\omega t + i \sin n\omega t$. Clearly, 5.33 is the alternative representation of the Fourier series with the amplitude and phase of the n-th harmonic being determined from (5.34) as

$$A_n = \sqrt{(a_n^2 + b_n^2)}, \quad \phi_n = \tan^{-1}(a_n/b_n)$$

with care being taken over choice of quadrant.

The Fourier coefficients are given by

$$a_n = \frac{2}{T} \int_d^{d+T} f(t) \cos n\omega t dt \quad (n = 0, 1, 2, \dots) \tag{5.18}$$

$$b_n = \frac{2}{T} \int_d^{d+T} f(t) \sin n\omega t dt \quad (n = 0, 1, 2, \dots) \tag{5.19}$$

Example 5.10 *A periodic function $f(t)$ of period 4 (that is , $f(t+4) = f(t)$) is defined in the range $-2 < t < 2$ by*

$$f(t) = \begin{cases} 0 & (-2 < t < 0) \\ 1 & (0 < t < 2) \end{cases}$$

Sketch a graph of $f(t)$ for $-6 \le t \le 6$ and obtain a Fourier series expansion for the function.
Solution *we have*

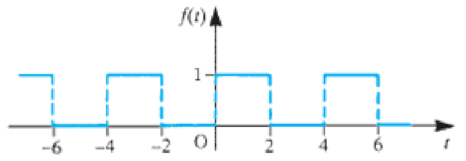


Fig. 5.6:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)dt = \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_{0}^{2} 1dt\right) = 1$$

$$\begin{aligned} a_n =& \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{1}{2}n\pi t dt \quad (n = 1, 2, 3, \dots) \\ =& \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_{0}^{2} \cos\frac{1}{2}n\pi t dt\right) = 0 \end{aligned}$$

and

$$\begin{aligned} b_n =& \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{1}{2}n\pi t dt \quad (n = 1, 2, 3, \dots) \\ =& \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_{0}^{2} \sin\frac{1}{2}n\pi t dt\right) = \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}[1 - (-1)^n] \\ =& \begin{cases} 0 & (\textit{even} \quad n) \\ 2/n\pi & (\textit{odd} \quad n) \end{cases} \end{aligned}$$

Thus, from (5.35), the Fourier series expression of $f(t)$ is

$$\begin{aligned} f(t) =& \frac{1}{2} + \frac{2}{\pi}(\sin\frac{1}{2}\pi t + \frac{1}{3}\sin\frac{3}{2}\pi t + \frac{1}{5}\sin\frac{5}{2}\pi t + \dots) \\ =& \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{1}{2}(2n-1)\pi t \end{aligned}$$

5.9 Half Range Fourier Sine and Cosine Series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx & \text{for half range sine series} \\ b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx & \text{for half range cosine series} \end{cases}$$

Example 5.11 *Expand $f(x) = x, 0 < x < 2$, in a half range (a) sine series, (b) cosine series*

Solution *(a) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 5.14. This is sometimes called the odd extension of $f(x)$. Then* $2L = 4, L = 2$

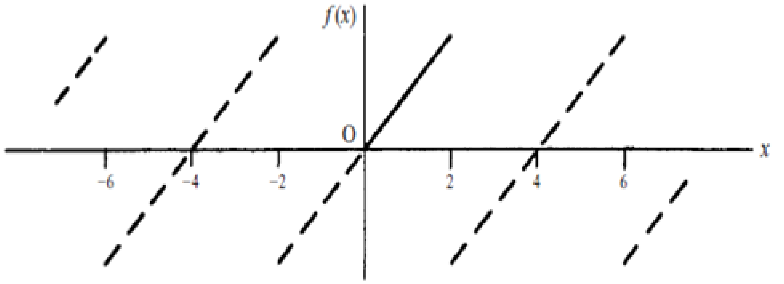


Fig. 5.7:

Thus, $a_n = 0$ *and*

$$\begin{aligned} b_n =& \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 (x)\sin\frac{n\pi x}{2}dx \\ =& \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi \end{aligned}$$

Then

$$\begin{aligned} f(x) =& \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2} \\ =& \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \dots\right) \end{aligned}$$

(b) Extend the function of $f(x)$ *to that of the even function of period 4 shown in figure 5.15. This is the even extension of* $f(x)$*. Then* $2L = 4, L = 2.$

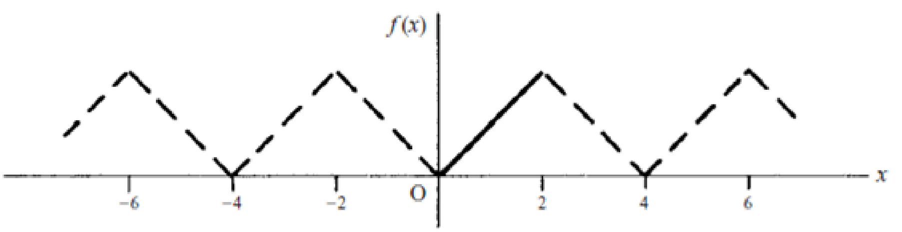


Fig. 5.8:

Thus, $b_n = 0$ *and*

$$\begin{aligned} a_n =& \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 (x)\cos\frac{n\pi x}{2}dx \\ =& \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 \\ =& \frac{4}{n^2\pi^2}(\cos n\pi - 1) \quad \textit{If} \quad n \neq 0 \end{aligned}$$

If $n = 0$, $a_0 = \int_0^2 x dx = 2.$
Then

$$\begin{aligned} f(x) =& 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2} \\ =& 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right) \end{aligned}$$

5.10 Complex Fourier Series

The Euler identities $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$ allow us to write

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

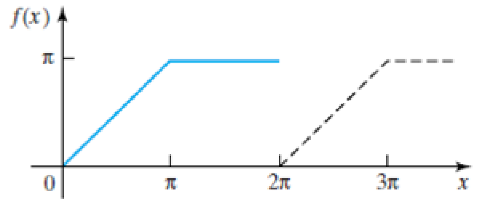


Fig. 5.9: The function $f(x)$ defined for $0 \leq x < 2\pi$

When these results are used in the real variable Fourier series representation of $f(x)$ over the interval $-L \leq x \leq L$, it becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \right) + b_n \left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) \right],$$

and after grouping terms we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} \right) e^{in\pi x/L} + \sum_{n=1}^{\infty} \left(\frac{a_n + ib_n}{2} \right) e^{-in\pi x/L} \tag{5.20}$$

If we now define

$$c_0 = a_0, c_n = \frac{a_n - ib_n}{2}, \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2} \quad \text{for} \quad n = 1, 2 \ldots, \tag{5.21}$$

the Fourier series presentation in (5.38) becomes

$$f(x) = \lim_{k \to \infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for} \quad -L \leq x \leq L. \tag{5.22}$$

This is the *complex* or *exponential* form of the Fourier series representation of $f(x)$.
If real functions $f(x)$ are considered, the Fourier coefficients a_n and b_n are real, and (5.39) then shows that c_n and c_{-n} are complex conjugates, because $c_{-n} = \bar{c_n}$. To proceed further we now make use of the fact that the functions $\exp(im\pi x/L)$ and $\exp(-in\pi x/L)$ are orthogonal over the interval $-L \leq x \leq L$, because integration shows that

$$\int_{-L}^{L} e^{im\pi x/L} e^{in\pi x/L} dx = \begin{cases} 0, & \text{for} \quad m \neq -n \\ 2\pi & \text{for} \quad m = -n \end{cases} \quad \text{for} \quad m, n \quad \text{positive integers}$$

Multiplication of (5.40) by $\exp(-im\pi x/L)$, following by integration over $-L \leq x \leq L$ and use of the above orthogonality condition gives

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.23}$$

Collecting these results we arrive at the following definition.

The complex form of a Fourier series

Let the real function $f(x)$ be defined on the interval $-L \leq x \leq L$. Then the complex Fourier series representation of $f(x)$ is

$$f(x) = \lim_{k\to\infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for} \quad -L \leq x \leq L,$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Example 5.12 *Find the complex Fourier series representation of*

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Solution *As the function of $f(x)$ is defined on the interval $-\pi < x < \pi$, we have $L = \pi$, so the coefficients c_n are given by*

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1 dx = \frac{1}{2}$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx} dx = \frac{1}{n\pi}\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right), \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

The coefficients c_n reduce to the real values

$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2} \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

so $c_n = c_{-n}$ because c_n is an even function of n. Consideration of the function $\sin(n\pi/2)$ *for integer values of n shows that*

$$c_{2n-1} = \frac{(-1)^{n-1}}{\pi(2n-1)} \quad \textit{and} \quad c_{2n} = 0 \quad \textit{for} \quad n = 1, 2, \ldots$$

Thus, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2} + \lim_{k\to\infty} \sum_{n=-k}^{k} c_n (e^{inx} + e^{-inx}).$$

with $e^{inx} + e^{-inx} = 2\cos nx$, the complex Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)}$$

Example 5.13 *Find the complex Fourier series representation of*

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 4 \end{cases}$$

Solution *The function* $f(x)$ *is defined on the interval* $0 \leq x \leq 2L$*, with* $2L = 4$*, so* $L = 2$*. Thus, the complex Fourier coefficients* c_n *are given by*

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{-in\pi/2}dx = \frac{1}{4}\int_1^4 e^{-in\pi/2}dx, \quad \textit{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Setting $n = 0$ *gives*

$$c_0 = \frac{3}{4},$$

whereas

$$c_n = \frac{i}{2\pi n}[1 - e^{in\pi/2}], \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

So the complex Fourier series representation of $f(x)$ *is*

$$f(x) = c_0 + \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/2},$$

with c_0 *and* c_n *defined as shown.*

5.11 The Fourier Series of a Function

Let $f(x)$ be defined on $[-L, L]$. We want to choose numbers $a_0, a_2, \ldots$ and $b_1, b_2, \ldots$ such that

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}[a_k \cos(k\pi x/L) + b_k \sin(k\pi x/L)] \tag{5.24}$$

This decomposition of the function into a sum of terms , each representing the influence of a different fundamental frequency on the behavior of the function.
To determine a_0, integrate equation (5.24) term by term to get

$$\begin{aligned}\int_{-L}^{L} f(x)dx =& \frac{1}{2}\int_{-L}^{L} a_0 dx \\ &+ \sum_{k=1}^{\infty}\left(a_k \int_{-L}^{L} \cos(k\pi x/L)dx + b_k \int_{-L}^{L} \sin(k\pi x/L)dx\right) \\ =& \frac{1}{2}a_0(2L) = \pi a_0\end{aligned}$$

because all of the integrals in the summation are zero. Then

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \tag{5.25}$$

To solve for the other coefficients in the proposed equation (5.24), we will use the following three facts, which follow by routine integrations. Let m and n be integers. Then

$$\int_{-L}^{L} \cos(n\pi x/L)\sin(m\pi x/L)dx = 0. \tag{5.26}$$

Furthermore, if $n \neq m$ then

$$\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)dx = \int_{-L}^{L} \sin(n\pi x/L)\sin(m\pi x/L)dx = 0. \tag{5.27}$$

And, if $n \neq 0$, then

$$\int_{-L}^{L} \cos^2(n\pi x/L)dx = \int_{-L}^{L} \sin^2(n\pi x/L)dx = L. \tag{5.28}$$

Now let n be any positive integer. To solve for a_n, multiply equation (5.24) by $\cos(n\pi x/L)$ and integrate the resulting equation to get

$$\begin{aligned}&\int_{-L}^{L} f(x)\cos(n\pi x/L)dx = \frac{1}{2}a_0 \int_{-L}^{L} \cos(n\pi x/L)dx \\ &+ \sum_{k=1}^{\infty}\left(a_k \int_{-L}^{L} \cos(k\pi x/L)\cos(n\pi x/L)dx + b_k \int_{-L}^{L} \sin(k\pi x/L)\cos(n\pi x/L)dx\right)\end{aligned}$$

Because equations (5.26) and (5.27), all of the terms on the right are zero except the coefficient of a_n, which occurs in the summation when $k = n$. The last equation reduces to

$$\int_{L}^{L} f(x)\cos(n\pi x/L)dx = a_n \int_{-L}^{L} \cos^2(n\pi x/L)dx = a_n L$$

by equation (5.28). Therefore

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx. \tag{5.29}$$

This expression contains a_0 if we let $n = 0$.
Similarly, if we multiply equation (5.24) by $\sin(n\pi x/L)$ and integrate, we obtain

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx. \tag{5.30}$$

The numbers

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5.31}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{5.32}$$

are called the *Fourier coefficients of f on [-L,L]*. When these numbers are used, the series in (5.24) is called the *Fourier series of f on [-L,L]*.

Example 5.14 *(a) Find the Fourier coefficients corresponding to the function*

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \quad \textit{Period} \quad = 10 \end{cases}$$

(b) Write the corresponding Fourier series
Solution
The graph of $f(x)$ is

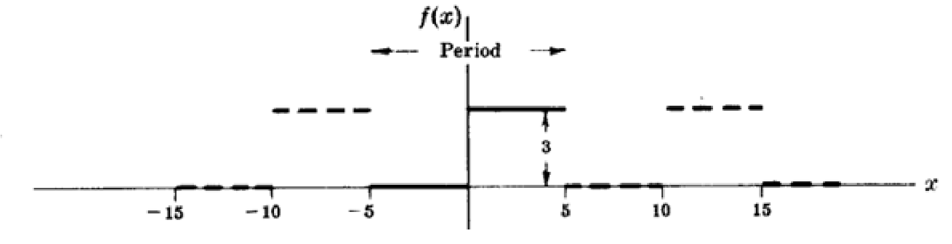


Fig. 5.10:

(a) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_{c}^{c+2L} f(x)\cos(n\pi x/L)dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos(n\pi x/5)dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos(n\pi x/5)dx + \int_{0}^{5}(3)\cos(n\pi x/5)dx\right\} = \frac{3}{5}\int_{0}^{5}\cos(n\pi x/5) \\ &= \frac{3}{5}\left[\frac{5}{n\pi}\sin(n\pi x/5)\right]_0^5 = 0 \quad \textit{if} \quad n \neq 0 \end{aligned}$$

If $n = 0$

$$a_0 = \frac{3}{5}\int_0^5 \cos(0\pi x/5)dx = \frac{3}{5}\int_0^5 dx = 3.$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin(n\pi x/L)dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin(n\pi x/5)dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin(n\pi x/5)dx + \int_0^5 (3)\sin(n\pi x/5)dx\right\} = \frac{3}{5}\int_0^5 \sin(n\pi x/5) \\ &= \frac{3}{5}\left[-\frac{5}{n\pi}\cos(n\pi x/5)\right]_0^5 = \frac{3(1-\cos n\pi)}{n\pi} \end{aligned}$$

(b) The corresponding Fourier series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/L) + b_n\sin(n\pi/L)) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin(n\pi x/5) \\ &= \frac{3}{2} + \frac{6}{\pi}\left(\sin(x\pi/5) + \frac{1}{3}\sin(3x\pi/5) + \frac{1}{5}\sin(5\pi x/5) + \ldots\right. \end{aligned}$$

5.12 Dirichlet Conditions

Suppose that

1. $f(x)$ is defined except possibly at a finite number of points in $(-L, L)$
2. $f(x)$ is periodic outside $(-L, L)$ with period $2L$
3. $f(x)$ and $f'(x)$ are piecewise continuous in $-L, L$.

Then the series (5.24) with Fourier coefficients converges to

(a) $f(x)$ if x is a point of continuity

(b) $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Here $f(x+0)$ and $f(x-0)$ are the right and left- hand limits of $f(x)$ at x and represent $\lim_{\epsilon\to+0} f(x+\epsilon)$ and $\lim_{\epsilon\to 0+} f(x-\epsilon)$, respectively.

Example 5.15 *How should* $f(x)\begin{cases} 0 & -5<x<0 \\ 3 & 0<x<5 \end{cases}$ *be defined at* $x=-5, x=0$, *and* $x=5$ *in order that the Fourier series will converge to* $f(x)$ *for* $-5\le x\le 5$*?*

Solution
Since $f(x)$ *satisfies the Dirichlet conditions, we can say that the series converges to* $f(x)$ *at all points of continuity and to* $\frac{f(x+0)+f(x-0)}{2}$ *at points of discontinuity. At* $x=-5, 0,$ *and* 5*, which are points of discontinuity, the series converge to* $(3+0)/2 = 3/2$ *as seen from the graph. If we redefine* $f(x)$ *as follows,*

$$f(x) = \begin{cases} 3/2 & x=-5 \\ 0 & -5<x<0 \\ 3/2 & x=0 \\ 3 & 0<x<5 \\ 3/2 & x=5 \end{cases} \quad \textit{Period} \quad = 10$$

Then the series will converge to $f(x)$ *for* $-5\le x\le 5$.

5.13 Fourier series for periodic functions of period 2π

Given that certain conditions are satisfied then it is possible to write a periodic function of period 2π as a series expansion of the orthogonal periodic functions just discussed. That is, if $f(x)$ is defined on the interval $-\pi \leq x \leq \pi$ where $f(x + 2\pi n) = f(x)$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

The coefficients of a_n

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \cos mx dx \\
=&\int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \right) \cos mx dx \\
=&\frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx \\
&+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx \\
=&\frac{a_0}{2} \times 0 + \sum_{n=1}^{\infty} a_n \pi \delta_{nm} + \sum_{n=1}^{\infty} b_n \times 0 \\
=&a_m \pi
\end{aligned}$$

and so

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \ldots$$

The coefficient of b_n

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \sin mx dx \\
=&\int_{-\pi}^{\pi} \left(\frac{a_0}{2} \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \right) \sin mx dx \\
=&\frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx dx \\
&+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx dx \\
=&\frac{a_0}{2} \times 0 + \sum_{n=1}^{\infty} a_n \times 0 + \sum_{n=1}^{\infty} b_n \pi \delta_{nm} \\
=&b_m \pi \\
\therefore \quad b_n =&\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 0, 1, 2, \ldots
\end{aligned}$$

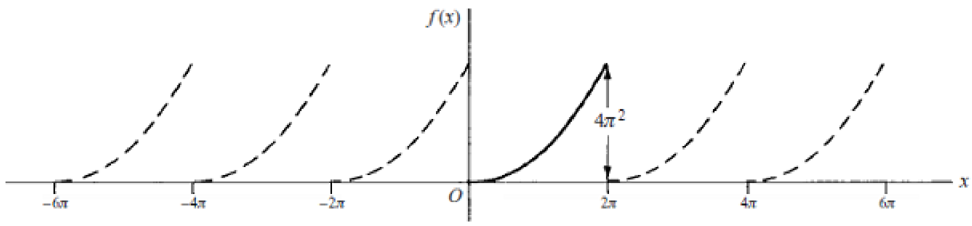


Fig. 5.11:

Example 5.16 *Expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series if (a) the period is 2π*
Solution
(a) The graph of $f(x)$ with period 2π is shown below
Period $= 2L = 2\pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos(n\pi x/L)dx = 1\pi\int_0^{2\pi} x^2\cos nxdx$$
$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{4}{n^2}, \quad n \neq 0$$

If $n = 0$, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{8\pi^2}{3}$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin(n\pi x/L)dx = 1\pi\int_0^{2\pi} x^2\sin nxdx$$
$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.
This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

Example 5.17 *Find the Fourier series of $f(x) = x - x^2, \quad -\pi < x < \pi$*
Solution
Let $f(x) = x - x^2 \quad$ for $\quad -\pi \leq x \leq \pi$. Here $L = \pi$. Compute

$$\begin{aligned}
a_0 =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = -\frac{2}{3}\pi^2,\\
a_n =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nxdx\\
=& \frac{4\sin n\pi - 4n\pi\cos n\pi - 2n^2\pi^2\sin n\pi}{n^3\pi}\\
=& -\frac{4}{n^2}\cos n\pi = -\frac{4}{n^2}(-1)^n\\
=& \frac{4(-1)^{n+1}}{n^2}
\end{aligned}$$

and

$$\begin{aligned}
b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nxdx\\
=& 2\sin n\pi - 2n\pi\cos nx\\
=& -\frac{2}{n}\cos nx = -\frac{2}{n}(-1)^n\\
=& \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

We have used the facts that $\sin nx = 0$ and $\cos nx = (-1)^n$ if n is an integer. The Fourier series of $f(x) = x - x^2$ on $[-\pi, \pi]$ is

$$-\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right].$$

5.14 Products of odd and even functions

Theorem 5.4 *If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is $even$, then the Fourier series for $f(x)$ contains $cosine\ terms$ only. Included in this is a_0 which may be regarded as $a_n \cos nx$ with $n = 0$.*

Proof: Since $f(x)$ is even, $\int_{-\pi}^{0} f(x)dx = \int_0^{\pi} f(x)dx$
(a)$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x) \quad \therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx.$
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx.$
But $f(x)\cos nx$ is the product of two even functions and therefore itself even.
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx.$
Because, since $f(x)\sin nxdx$ is the product of an even function and an odd function, it is itself odd.
$\therefore \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = 0 \quad \therefore \quad b_n = 0$
Therefore, there are no sine terms in the Fourier series for $f(x)$.

Theorem 5.5 *If $f(x)$ is an odd function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains $sine\ terms$ only.*

Proof: Since $f(x)$ is an odd function, $\int_{-\pi}^{0} f(x)dx = -\int_0^{\pi} f(x)dx.$
(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$ But $f(x)$ is odd $\therefore \quad a_0 = 0$
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = 0$ because $f(x)\cos nx$ is odd function (ie product of odd and even)

Example 5.18 *Determine the Fourier series for the function shown in figure 5.12nc*

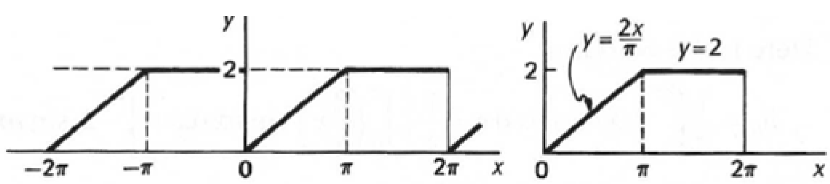


Fig. 5.12:

Solution: *This is neither odd nor even. Therefore we must find a_0, a_n and b_n.*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

(a)

$$\begin{aligned} a_0 =& \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\left\{\int_0^{\pi}\frac{2}{\pi}xdx + \int_{\pi}^{2\pi} 2dx\right\} \\ =& \frac{1}{\pi}\left\{\left[x^2/\pi\right]_0^{\pi} + \left[2x\right]_{\pi}^{2\pi}\right\} = \frac{1}{\pi}(\pi + 4\pi - 2\pi) = 3 \quad \therefore \quad a_0 - 3 \end{aligned}$$

(b)

$$
\begin{aligned}
a_n =& \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx = \frac{1}{\pi}\left\{\int_0^{\pi}(2x/\pi)\cos nx dx + \int_{\pi}^{2\pi} 2\cos nx dx\right\}\\
=&\frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\sin nx dx + \int_{\pi}^{2\pi}\cos nx dx\right\}\\
a_n =& \frac{2}{\pi}\left\{\frac{1}{\pi}(0-0) - \frac{1}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\}\\
=&\frac{2}{\pi}\left\{-\frac{1}{\pi n^2}(-(-1)^n+1) + (0-0)\right\}\\
=&-\frac{2}{\pi^2 n^2}(1-(-1)^n)
\end{aligned}
$$

and so $a_0 = 0$ *(n even) and* $a_n = -\frac{4}{\pi^2 n^2}$ *(n odd)*

$$
\begin{aligned}
b_n =& \frac{1}{\pi}\int^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\left\{\int_0^{\pi}(2x/\pi)\sin nx dx + \int_{\pi}^{2\pi} 2\sin nx dx\right\}\\
=&\frac{2}{\pi}\left\{\frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos nx dx + \int_{\pi}^{2x}\sin nx dx\right\}\\
=&\frac{2}{\pi}\left\{\frac{1}{\pi n}(-\pi\cos n\pi) + \frac{1}{\pi n}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right\}\\
=&\frac{2}{\pi}\left\{-\frac{1}{n}\cos n\pi + (0-0) - \frac{1}{n}(\cos 2\pi n - \cos n\pi)\right\}\\
=&\frac{2}{\pi}\left\{-\frac{1}{n}\cos 2n\pi\right\} = -\frac{2}{\pi n}\cos 2n\pi
\end{aligned}
$$

But $\cos 2n\pi = 1$. $\therefore \quad b_n = -\frac{2}{\pi n}$

$$
\begin{aligned}
f(x) =& \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right\}\\
&-\frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \dots\right\}
\end{aligned}
$$

Theorem 5.6 *Fourier's theorem*
This theorem states that a periodic function that satisfies certain conditions can be expressed as the sum of a number of sine functions of different amplitudes, phases and periods. That is, if $f(t)$ is a periodic function with period T then

$$
\begin{aligned}
f(t) =& A_0 + A_1\sin(\omega t + \phi_1) + A_2\sin(2\omega t + \phi_2) + \dots\\
&+ A_n\sin(n\omega t + \phi_n) + \dots
\end{aligned}
\tag{5.33}
$$

Where As and ϕs are constants and $\omega = 2\pi/T$ is the frequency of $f(t)$. The term $A_1\sin(\omega t + \phi_1)$ is called the *first harmonic* or *fundamental mode*, and it has the same frequency ω as the parent function $f(t)$. The term $A_n\sin(n\omega t + \phi_n)$ is called the *n-th harmonic*, and it has frequency $n\omega$, which is n times that of the fundamental. A_n denotes the *amplitude* of the n-th harmonic and ϕ_n is its *phase angle*, measuring the lag or lead of the n-th harmonic with reference to a pure sine wave of the same frequency.
Since

$$
\begin{aligned}
A_n\sin(n\omega t + \phi_n) &\equiv (A_n\cos\phi_n)\sin n\omega t + (A_n\sin\phi_n)\cos n\omega t\\
&\equiv b_n\sin n\omega t + a_n\cos n\omega t
\end{aligned}
$$

where

$$b_n = A_n \cos\phi_n, \quad a_n = A_n \sin\phi_n \tag{5.34}$$

the expression 5.33 can be written as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{5.35}$$

where $a_0 = 2A_0$ (we shall see later that taking the first term as $\frac{1}{2}a_0$ rather than a_0 is a convenience that enables us to make a_0 fit a general result). The expression (5.35) is called the *Fourier series expansion* of the function $f(t)$, and as and bs are called respectively as the *in-phase* and *phase quadrature components* of the n-th harmonic, this terminology arising from the use of the phasor notation $e^{in\omega t} = \cos n\omega t + i \sin n\omega t$. Clearly, 5.33 is the alternative representation of the Fourier series with the amplitude and phase of the n-th harmonic being determined from (5.34) as

$$A_n = \sqrt{(a_n^2 + b_n^2)}, \quad \phi_n = \tan^{-1}(a_n/b_n)$$

with care being taken over choice of quadrant.

The Fourier coefficients are given by

$$a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t dt \quad (n = 0, 1, 2, \dots) \tag{5.36}$$

$$b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin n\omega t dt \quad (n = 0, 1, 2, \dots) \tag{5.37}$$

Example 5.19 *A periodic function $f(t)$ of period 4 (that is , $f(t+4) = f(t)$) is defined in the range $-2 < t < 2$ by*

$$f(t) = \begin{cases} 0 & (-2 < t < 0) \\ 1 & (0 < t < 2) \end{cases}$$

Sketch a graph of $f(t)$ for $-6 \le t \le 6$ and obtain a Fourier series expansion for the function.
Solution *we have*

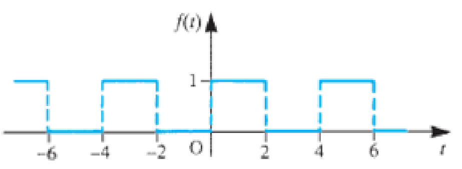


Fig. 5.13:

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)dt = \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_0^2 1dt\right) = 1$$

$$\begin{aligned} a_n =& \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{1}{2}n\pi t dt \quad (n = 1, 2, 3, \dots) \\ =& \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_0^2 \cos\frac{1}{2}n\pi t dt\right) = 0 \end{aligned}$$

and

$$\begin{aligned}
b_n =& \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{1}{2}n\pi t dt \quad (n = 1, 2, 3, \dots) \\
=& \frac{1}{2}\left(\int_{-2}^{0} 0dt + \int_{0}^{2} \sin\frac{1}{2}n\pi t dt\right) = \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}[1-(-1)^n] \\
=& \begin{cases} 0 & (\textit{even} \quad n) \\ 2/n\pi & (\textit{odd} \quad n) \end{cases}
\end{aligned}$$

Thus, from (5.35), the Fourier series expression of $f(t)$ is

$$\begin{aligned}
f(t) =& \frac{1}{2} + \frac{2}{\pi}(\sin\frac{1}{2}\pi t + \frac{1}{3}\sin\frac{3}{2}\pi t + \frac{1}{5}\sin\frac{5}{2}\pi t + \dots) \\
=& \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{1}{2}(2n-1)\pi t
\end{aligned}$$

5.15 Half Range Fourier Sine and Cosine Series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx & \text{for half range sine series} \\ b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx & \text{for half range cosine series} \end{cases}$$

Example 5.20 *Expand $f(x) = x, 0 < x < 2$, in a half range (a) sine series, (b) cosine series*

Solution *(a) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 5.14. This is sometimes called the odd extension of $f(x)$. Then* $2L = 4, L = 2$

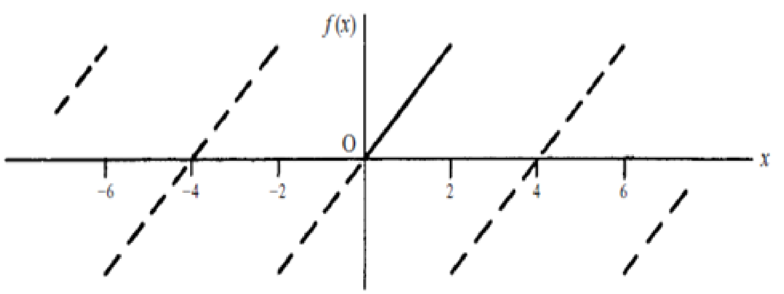


Fig. 5.14:

Thus, $a_n = 0$ and

$$\begin{aligned}
b_n =& \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 (x)\sin\frac{n\pi x}{2}dx \\
=& \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi
\end{aligned}$$

Then

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{-4}{n\pi} \cos n\pi \sin \frac{n\pi x}{2} \\
&= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \dots\right)
\end{aligned}
$$

(b) Extend the function of $f(x)$ to that of the even function of period 4 shown in figure 5.15. This is the even extension of $f(x)$. Then $2L = 4, L = 2$.

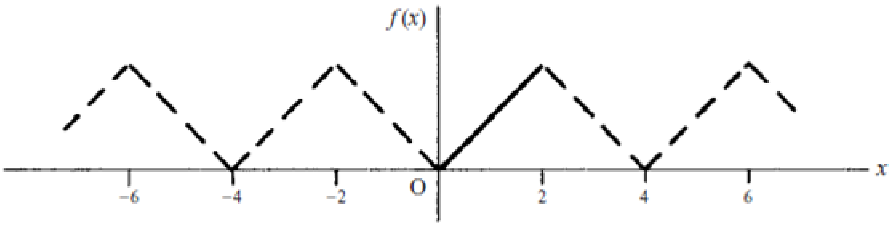


Fig. 5.15:

Thus, $b_n = 0$ and

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{2}\int_0^2 (x)\cos\frac{n\pi x}{2}dx \\
&= \left\{ (x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 \\
&= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \quad \textit{If} \quad n \neq 0
\end{aligned}
$$

If $n = 0$, $a_0 = \int_0^2 x dx = 2$.
Then

$$
\begin{aligned}
f(x) &= 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2} \\
&= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right)
\end{aligned}
$$

5.16 Complex Fourier Series

The Euler identities $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$ allow us to write

$$
\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}
$$

When these results are used in the real variable Fourier series representation of $f(x)$ over the interval $-L \leq x \leq L$, it becomes

$$
f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}\right) + b_n\left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}\right)\right],
$$

and after grouping terms we have

$$
f(x) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\pi x/L} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-in\pi x/L} \tag{5.38}
$$

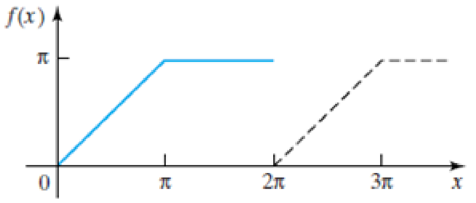


Fig. 5.16: The function $f(x)$ defined for $0 \le x < 2\pi$

If we now define

$$c_0 = a_0, c_n = \frac{a_n - ib_n}{2}, \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2} \quad \text{for} \quad n = 1, 2 \ldots, \tag{5.39}$$

the Fourier series presentation in (5.38) becomes

$$f(x) = \lim_{k \to \infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for} \quad -L \le x \le L. \tag{5.40}$$

This is the *complex* or *exponential* form of the Fourier series representation of $f(x)$.
If real functions $f(x)$ are considered, the Fourier coefficients a_n and b_n are real, and (5.39) then shows that c_n and c_{-n} are complex conjugates, because $c_{-n} = \bar{c_n}$. To proceed further we now make use of the fact that the functions $\exp(im\pi x/L)$ and $\exp(-in\pi x/L)$ are orthogonal over the interval $-L \le x \le L$, because integration shows that

$$\int_{-L}^{L} e^{im\pi x/L} e^{in\pi x/L} dx = \begin{cases} 0, & \text{for} \quad m \ne -n \\ 2\pi & \text{for} \quad m = -n \end{cases} \quad \text{for} \quad m, n \quad \text{positive integers}$$

Multiplication of (5.40) by $\exp(-im\pi x/L)$, following by integration over $-L \le x \le L$ and use of the above orthogonality condition gives

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.41}$$

Collecting these results we arrive at the following definition.

The complex form of a Fourier series

Let the real function $f(x)$ be defined on the interval $-L \le x \le L$. Then the complex Fourier series representation of $f(x)$ is

$$f(x) = \lim_{k \to \infty} \sum_{n=-k}^{k} c_n e^{in\pi x/L} \quad \text{for} \quad -L \le x \le L,$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Example 5.21 *Find the complex Fourier series representation of*

$$f(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

Solution *As the function of $f(x)$ is defined on the interval $-\pi < x < \pi$, we have $L = \pi$, so the coefficients c_n are given by*

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1dx = \frac{1}{2}$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-inx}dx = \frac{1}{n\pi}\left(\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}\right), \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

The coefficients c_n reduce to the real values

$$c_n = \frac{1}{n\pi}\sin\frac{n\pi}{2} \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

so $c_n = c_{-n}$ because c_n is an even function of n. Consideration of the function $\sin(n\pi/2)$ for integer values of n shows that

$$c_{2n-1} = \frac{(-1)^{n-1}}{\pi(2n-1)} \quad \textit{and} \quad c_{2n} = 0 \quad \textit{for} \quad n = 1, 2, \ldots$$

Thus, the complex Fourier series representation of $f(x)$ is

$$f(x) = \frac{1}{2} + \lim_{k\to\infty}\sum_{n=-k}^{k} c_n(e^{inx} + e^{-inx}).$$

with $e^{inx} + e^{-inx} = 2\cos nx$, the complex Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)}$$

Example 5.22 *Find the complex Fourier series representation of*

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 4 \end{cases}$$

Solution *The function $f(x)$ is defined on the interval $0 \le x \le 2L$, with $2L = 4$, so $L = 2$. Thus, the complex Fourier coefficients c_n are given by*

$$c_n = \frac{1}{4}\int_0^4 f(x)e^{-in\pi/2}dx = \frac{1}{4}\int_1^4 e^{-in\pi/2}dx, \quad \textit{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Setting $n = 0$ gives

$$c_0 = \frac{3}{4},$$

whereas

$$c_n = \frac{i}{2\pi n}[1 - e^{in\pi/2}], \quad \textit{for} \quad n = \pm 1, \pm 2, \ldots$$

So the complex Fourier series representation of $f(x)$ is

$$f(x) = c_0 + \lim_{k\to\infty}\sum_{n=-k}^{k} c_n e^{in\pi x/2},$$

with c_0 and c_n defined as shown.

Chapter 6

Functions of Several Variables

6.1 Functions of Two or More Variables

Definition 6.1 *Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number*

$$w = f(x_1, x_2, \ldots, x_n)$$

each element in D. The set D is the function $domain$. The set of w-values taken on by f is the function's range. The symbol w is the $dependent\ variable$ of f, and f is said to be a function of the $n\ independent\ variables\ x_1$ to x_n. We also call the x_j's the function's $input\ variables$ and call w the function's $output\ variables.$

If f is a function of two independent variables, we usually call the independent variables x and y and the dependent variable z, and we picture the domain of f as a region in the xy-plane (Figure 6.1). If f is a function of three independent variables, we call the independent variables x, y and z and the dependent variable w, and we picture the domain as a region in space.
In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r, h)$. To be more specific, we might replace the notation $f(r, h)$ by the formula that calculates the value of V from the values of r and h, and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variables of the function.

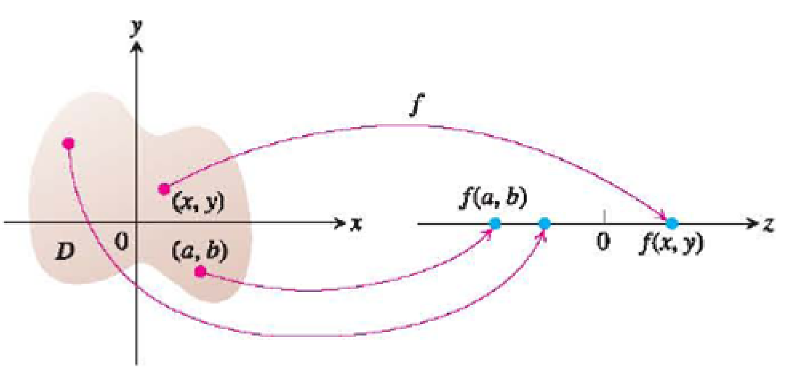


Fig. 6.1: An arrow diagram for the function $z = f(x, y)$

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variables. For example, the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

Domains and Ranges

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}$, y cannot be less than x^2. If $f(x, y) = \frac{1}{xy}$, xy cannot be zero. The domain of a function is assumed to the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly. The range consists of the set of output values for the dependent variable.

Example 6.1 *(a) These are functions of two variables(Table 6.1). Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z*
(b) These are functions of three variables with restrictions on some of the their domains

Table 6.1:

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0)U(0, \infty)$
$z = \sin xy$	Entire plane	[-1, 1]

(Table 6.2)

Table 6.2:

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \frac{1}{x^2+y^2+z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

6.2 Partial Derivatives

Functions of Two Variables

Let f be a function of x and y; take for example

$$f(x, y) = 3x^2 y - 5x \cos \pi y.$$

The partial derivatives of f with respect to x is the function f_x obtained by differentiating f with respect to x, keeping y fixed. In this case

$$f_x(x, y) = 6xy - 5 \cos \pi y.$$

The partial derivative of f with respect to y is the function f_y obtained by differentiating f with respect to y, keeping x fixed. In this case

$$f_y(x, y) = 3x^2 + 5\pi x \sin \pi y.$$

These partial derivatives are limits:

Definition 6.2 **Partial Derivatives (two variables)**
Let f be a function of two variables x, y. The partial derivatives of f with respect to x and with with respect to y are the functions f_x and f_y defined by setting

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided these limits exist.

Example 6.2 *For the function $f(x, y) = x \arctan xy$*

$$f_x(x, y) = x\frac{x}{1 + (xy)^2} + \arctan xy = \frac{xy}{1 + x^2y^2} + \arctan xy$$

$$f_y(x, y) = x\frac{x}{1 + (xy)^2} = \frac{x^2}{1 + x^2y^2}$$

In the one variable case, $f'(x_0)$ gives the rate of change of $f(x)$ with respect to x at $x = x_0$. In the two-variable case, $f_x(x_0, y_0)$ gives the rate of change of $f(x, y_0)$ with respect to x at $x = x_0$, and $f_y(x_0, y_0)$ gives the rate of change of $f(x_0, y)$ with respect to y at $y = y_0$.

Example 6.3 *For the function $f(x, y) = e^{xy} + \ln(x^2 + y)$,*

$$f_x(x, y) = ye^{xy} + \frac{2x}{x^2 + y} \quad \textit{and} \quad f_y(x, y) = xe^{xy} + \frac{1}{x^2 + y}$$

The number

$$f_x(2, 1) = e^2 + \frac{4}{4 + 1} = e^2 + \frac{4}{5}$$

gives the rate of change with respect to x of the function

$$f(x, 1) = e^x + \ln(x^2 + 1) \quad \textit{at} \quad x = 2;$$

the number

$$f_y(2, 1) = 2e^2 + \frac{1}{4 + 1} = 2e^2 + \frac{1}{5}$$

gives the rate of change with respect to y of the function

$$f(2, y) = e^{2y} + \ln(4 + y) \quad \textit{at} \quad y = 1$$

Example 6.4 *Let $f(x, y) = 4 - 2x^2 - y^2$. Find the slope of the tangent line at the point $(1, 1)$ on the curve formed by the intersection of the surface $z = f(x, y)$ and (a) the plane $y = 1$ (b) the plane $x = 1$*

Solution

(a) The slope of the tangent line at any point on the curve formed by the intersection of the plane $y = 1$ and the surface $z = 4 - 2x^2 - y^2$ is given by

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(4 - 2x^2 - y^2) = -4x$$

In particular, the slope of the required tangent line is

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = -4(1) = -4$$

(b) The slope of the tangent line at any point on the curve formed by the intersection of the plane $x = 1$ and the surface $z = 4 - 2x^2 - y^2$ is given by

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(4 - 2x^2 - y^2) = -2y$$

In particular, the slope of the required tangent line is

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -2(1) = -2$$

(See Figure 6.2)

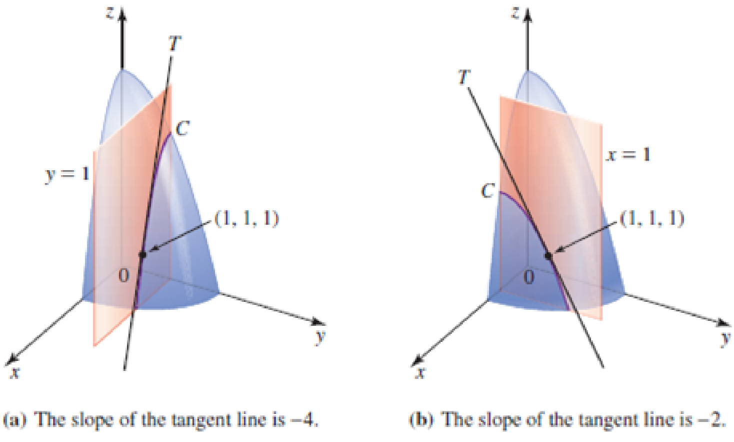


Fig. 6.2:

Example 6.5 Electrostatic Potential *Figure 6.3 shows a crescent-shaped region R that lies inside the disk $D_1 = \{(x, y)|(x - y)^2 + y^2 \leq 4\}$ and outside the disk $D_1 = \{(x, y)|(x - y)^2 + y^2 \leq 1\}$. Suppose that the electrostatic potential along the inner circle is kept at 50 volts and the electrostatic potential along the outer circle is kept at 100 volts. Then the electrostatic potential at any point (x, y) in R is given by*

$$U(x, y) = 150 - \frac{200x}{x^2 + y^2}$$

volts

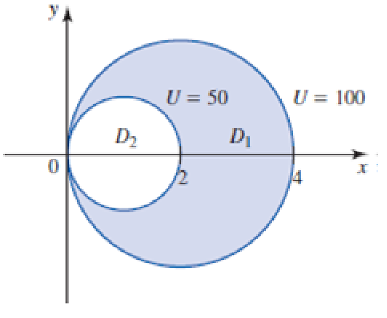


Fig. 6.3: The electrostatic potential inside the crescent-shaped region is $U(x, y)$

(a) Compute $U_x(x, y)$ and $U_y(x, y)$
(b) Compute $U_x(3, 1)$ and $U_y(3, 1)$ and interpret your results.

Solution

$$\textit{(a)}\quad U_x(x,y) = \frac{\partial}{\partial x}\left[100 - \frac{200x}{x^2+y^2}\right] = -\frac{\partial}{\partial x}\left(\frac{200x}{x^2+y^2}\right)$$
$$= -\frac{(x^2+y^2)\frac{\partial}{\partial x}(200x) - 200x\frac{\partial}{\partial x}(x^2+y2)}{(x^2+y^2)^2}$$
$$= -\frac{200(x^2+y^2) - 200x(2x)}{(x^2+y^2)^2} = \frac{200(x^2-y^2)}{(x^2+y^2)^2}$$
$$U_y(x,y) = \frac{\partial}{\partial y}\left[150 - \frac{200x}{x^2+y^2}\right] = -\frac{\partial}{\partial y}\left(\frac{200x}{x^2+y^2}\right)$$
$$= -200x\frac{\partial}{\partial y}(x^2+y^2)^{-1}$$
$$= -200x(-1)(x^2+y^2)^{-2}\frac{\partial}{\partial y}(x^2+y^2)$$
$$= 200x(x^2+y^2)^{-2}(2y) = \frac{400xy}{(x^2+y^2)^2}$$

$$\textit{(b)}\quad U_x(3,1) = \frac{200(9-1)}{(9+1)^2} = 16 \quad \textit{and} \quad U_y(3,1) = \frac{400(3)(1)}{(9+1)^2} = 12$$

This tell us that the rate of change of the electrostatic potential at the point (3,1) in the x-direction is 16 volts per unit change in x with y held fixed at 1, and the rate of change of the electrostatic potential at the point (3,1) in the y-direction is 12 volts per unit change in y with x held fixed at 3.

6.3 Higher-Oder Derivatives

Consider the function $z = f(x,y)$ of two variables. Each of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are functions of x and y. Therefore, we may take the partial derivatives of the these functions to obtain the four *second-order partial derivatives*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

(See Figure 6.4)

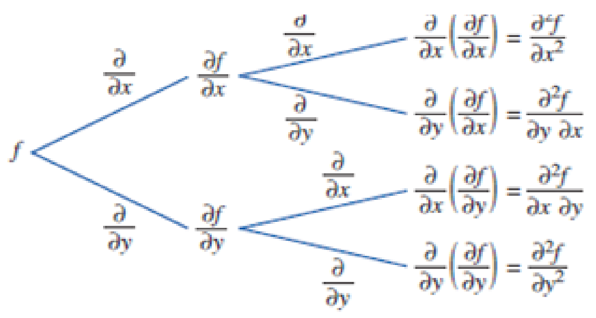


Fig. 6.4:

Notation

Partial derivatives of second and higher orders are calculated by taking partial derivatives of already calculated partial derivatives. The order in which the differentiations are performed is indicated in the notations used. If $z = f(x, y)$, we can calculate *four* partial derivatives of second order, namely, two **pure** second partial derivatives with respect to x or y,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = f_{11}(x, y) = f_{xx}(x, y),$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial z}{\partial y} = f_{22}(x, y) = f_{yy}(x, y),$$

and two **mixed** second partial derivatives with respect to x and y,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = f_{21}(x, y) = f_{yx}(x, y),$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = f_{12}(x, y) = f_{xy}(x, y).$$

Again, we remark that the notations f_{11}, f_{12}, f_{21} and f_{22} are usually preferable to f_{xx}, f_{xy}, f_{yx}, and f_{yy}, although the latter are often used in partial differential equations. Note that f_{12} indicates differentiation of f *first* with respect to its first variable and *then* with respect to its second variable; f_{21} indicates the opposite order of differentiation. The subscript closest to f indicates which differentiation occurs first.
Similarly, if $w = f(x, y, z)$, then

$$\frac{\partial^5 w}{\partial y \partial x \partial y^2 \partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial w}{\partial z} = f_{32212}(x, y, z) = f_{zyyxy}(x, y, z).$$

Example 6.6 *Find the four second partial derivatives of $f(x, y) = x^3y^4$*
Solution

$$\begin{array}{ll} f_1(x, y) = 3x^3y^4, & f_2(x, y) = 4x^3y^3 \\ f_{11}(x, y) = \frac{\partial}{\partial x}(3x^2y^4) = 6xy^4, & f_{21}(x, y) = \frac{\partial}{\partial x}(4x^3y^3) = 12x^2y^3, \\ f_{12}(x, y) = \frac{\partial}{\partial y}(3x^2y^4) = 12x^2y^3 & f_{22} = \frac{\partial}{\partial y}(4x^3y^3) = 12x^3y^2. \end{array}$$

Example 6.7 *Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ for the function $f(x, y, z) = e^{x-2y+3z}$*
Solution

$$\begin{aligned} f_{223}(x, y, z) =& \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial y}e^{x-2y+3z} \\ =& \frac{\partial}{\partial z}\frac{\partial}{\partial y}(-2e^{x-2y+3z}) \\ =& \frac{\partial}{\partial z}(4e^{x-2y+3z}) = 12e^{x-2y+3z}, \\ f_{232}(x, y, z) =& \frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial y}e^{x-2y+3z} \\ =& \frac{\partial}{\partial y}\frac{\partial}{\partial z}(-2e^{x-2y+3z}) \\ =& \frac{\partial}{\partial y}(-6e^{x-2y+3z}) = 12e^{x-2y+3z}, \\ f_{322}(x, y, z) =& \frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}e^{x-2y+3z} \\ =& \frac{\partial}{\partial y}\frac{\partial}{\partial y}(3e^{x-2y+3z}) \\ =& \frac{\partial}{\partial y}(-6e^{x-2y+3z}) = 12e^{x-2y+3z}. \end{aligned}$$

6.4 The Total Differential

Definition 6.3 *Differentials*
Let $z = f(x,y)$, and let $\triangle x$ and $\triangle y$ be increments of x and y, respectively. The **differentials** dx **and** dy *of the independent variables x and y are*

$$dx = \triangle x \quad \text{and} \quad dy = \triangle y$$

The **differential** z, *or* **total differential**, *of the dependent variable z is*

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy$$

Example 6.8 *Let $z = f(x,y) = 2x^2 - xy$*
(a) Find the differential dz
(b) Compute the value of dz if (x,y) changes from $(1,1)$ to $(0.98, 1.03)$, and compare your result with the value of $\triangle z$ obtained in Example 6.7
Solution
(a) $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (4x - y)dx - xdy$
(b) Here $x = 1, y = 1, dx = \triangle x = -0.02$, and $dy = \triangle y = 0.03$. Therefore, $dz = [4(1) - 1](-0.02) - 1(0.03) = -0.09$
The value of $\triangle z$ obtained in Example 6.7 was -0.0886, so dz is a good approximation of $\triangle z$ in this case. Observe that it is easier to compute dz than to compute $\triangle z$.

Example 6.9 *A storage tank has the shape of a right circular cylinder. Suppose that the radius and height of the tank are measured at $1.5m$ and $5m$, respectively, with a possible error of $0.05m$ and $0.1m$, respectively. Use differentials to estimate the maximum error in calculating the capacity of the tank.*
Solution *The capacity (volume) of the tank is $V = \pi r^2 h$. The error in calculating the capacity of the tank is given by*

$$\triangle \approx dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = 2\pi r h dr + \pi r^2 dh$$

Since the errors in the measurement of r and h are at most $0.05m$ and 0.1, respectively, we have $dr = 0.05$ and $dh = 0.1$. Therefore, taking $r = 1.5, h = 5, dr = 0.05$, and $dh = 0.1$, we obtain

$$\begin{aligned} dV =& 2\pi r h dr + \pi r^2 dh \\ \approx& 2\pi(1.5)(5)(0.05)^2(0.1) = 0.975\pi \end{aligned}$$

Thus, the maximum error in calculating the volume of the storage tank is approximately 0.975π, or $3.1m^3$.

Theorem 6.1 *Theorems on Differentials*
In the following we shall assume that all functions have continuous first partial derivatives in s region $\Re$, i.e., the functions are continuously differentiable in $\Re$.

1. *If $z = f(x_1, x_2, \ldots, x_n)$, then*

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n \tag{6.1}$$

   regardless of whether the variables $x_1, x_2, \ldots, x_n$ are independent or dependent on other variables. In (6.1) we often use z in place of f.

2. *If $f(x_1, x_2, \ldots, x_n) = c$, a constant, then $df = 0$. Note that in this case $x_1, x_2, \ldots, x_n$ cannot all be independent variables.*

3. *The expression $P(x,y)dx+Q(x,y)dy$ or briefly $Pdx+Qdy$ is the differential of $f(x,y)$ if and only if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ In such case $Pdx+Qdy$ is called an exact differential.*
   Note: Observe that $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ implies that $\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}$.

4. *The expression $P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$ or briefly $Pdx+Qdy+Rdz$ is the differential of $f(x,y,z)$ if and only if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$. In such case $Pdx+Qdy+Rdz$ is called an exact differential.*

Example 6.10 *(a) Let $U=x^2e^{y/x}$. Find dU. (b) Show that $(3x^2y-2y^2)dx+(x^3-4xy+6y^2)dy$ can be written as an exact differential of a function $\phi(x,y)$ and find this function.*
(a) Method 1:

$$\frac{\partial U}{\partial x}=x^2e^{y/x}(-y/x^2)+2xe^{y/x}, \quad \frac{\partial U}{\partial y}=x^2e^{y/x}(1/x)$$

Then

$$dU=\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy=(2xe^{y/x}-ye^{y/x})dx+xe^{y/x}dy$$

Method 2:

$$\begin{aligned} dU=&x^2d(e^{y/x})+e^{y/x}d(x^2)=x^2e^{y/x}d(y/x)+2xe^{y/x}dx\\ =&x^2e^{y/x}(\frac{xdy-ydx}{x^2})+2xe^{y/x}dx=(2xe^{y/x}-ye^{y/x})dx+xe^{y/x}dx \end{aligned}$$

(b) Method 1
Suppose that
$(3x^2y-2y^2)dx+(x^3-4xy+6y^2)dy=d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy.$
Then (1) $\frac{\partial \phi}{\partial x}=3x^2y-2y^2$, (2) $\frac{\partial \phi}{\partial y}=x^3-4xy+6y^2$
From (1), integrating with respect to x keeping y constant, we have

$$\phi=x^3y=2xy^2+F(y)$$

where $F(y)$ is the "constant" of integration. Substituting this into (2) yields
$x^3-4xy+F'(y)=x^3-4xy+6y^2$ *from which $F'(y)=6y^2$, i.e., $F(y)=2y^3+c$*
Hence, the required function is $\phi=x^3y-2xy^2+2y^3+c$, where c is an arbitrary constant. Note that the existence of such a function is guaranteed, since if $P=3x^2y-2y^2$ and $Q=x^3-4xy+6y^2$, then $\partial P/\partial y=3x^2-4y=\partial Q/\partial x$ identically. If $\partial P/\partial y \neq \partial Q/\partial x$ this function would not exist and the given expression would not be an exact differential.
Method 2

$$\begin{aligned} (3x^2y-2y^2)dx+(x^3-4xy+6y^2)dy=&(3x^2ydx+x^3dy)-(2y^2dx+4xydy)+6y^2dy\\ =&d(x^3y)-d(2xy^2)+d(2y^3)=d(x^3y-2xy^2+2y^3)\\ =&d(x^3y-2xy^2+2y^3+c) \end{aligned}$$

Then the required function is $x^3y-2xy^2+2y^3+c$.

6.5 Chain Rule

Case 1: $z=f(x,y), x=g(t), y=h(t)$ and compute $\frac{dz}{dt}$.
In this case we are going to compute an ordinary derivative since z really would be a function of t only if we were to substitute in for x and y.
The chain rule of this case is,

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$$

So, basically what we are doing here is differentiating f with respect to each variable in it and then multiplying each of these by the derivative of that variable with respect to t. The final step is to then add all this up.

Example 6.11 *Compute $\frac{dz}{dt}$ for each of the following*

(a) $z = xe^{xy}$, $x = t^2$, $y = t^{-1}$

(b) $z = x^2y^3 + y\cos x$, $x = \ln(t^2)$, $y = \sin(4t)$

Solution

(a) $z = xe^{xy}$, $x = t^2$, $y = t^{-1}$
There really isn't all that much to do here other than using the formula.

$$\begin{aligned}\frac{dz}{dt} =& \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ =&(e^{xy} + yxe^{xy})(2t) + x^2e^{xy}(-t^{-2})\\ =&2t(e^{xy} + yxe^{xy}) - t^{-}2x^2e^{xy}\end{aligned}$$

So, technically we have computed the derivative. However, we should probability go head and substitute in for x and y as well at this point since we have already got t's in the derivative. Doing this gives,

$$\frac{dz}{dt} = 2t(e^t + te^t) - t^{-2}t^4e^t = 2te^t + t^2e^t$$

Note that in this case it might actually have been easier to just substitute in for x and y in the original function and just compute the derivative as we normally would. For comparisons sake let's do that.

$$z = t^2e^t \Rightarrow \frac{dz}{dt} = 2te^t + t^2e^t$$

(b) $z = x^2y^3 + y\cos x, \quad x = \ln(t^3), \quad y = \sin(4t)$
Okay, in this case it would almost definitely be more work to do the substitution first so we'll use the chain rule first and then substitute

$$\begin{aligned}\frac{dz}{dt} =&(2xy^3 - y\sin x)(2/t) + (3x^2y^2 + \cos x)(4\cos(4t))\\ =&\frac{4\sin^3(4t)\ln t^2 - 2\sin(4t)\sin(\ln t^2)}{t} + 4\cos(4t)(3\sin^2(4t)[\ln t^2]^2 + \cos(\ln t^2))\end{aligned}$$

Note that sometimes, because of the significant mess of the final answer, we will only simplify the first step a little and leave the answer in terms of x, y, and t. This is dependent upon the situation, class and instructor however and for this class we will pretty much always be substituting in for x and y.

Let's suppose that we have the following situation,

$$z = f(x,y) \quad y = g(x)$$

In this case the chain rule for $\frac{dz}{dx}$ becomes,

$$\frac{dz}{dx} = \frac{\partial f}{dx}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

In the first term we are using the fact that, $\frac{dx}{dx} = \frac{d}{dx}(x) = 1$.

Example 6.12 *Compute $\frac{dz}{dx}$ for $z = x\ln(xy) + y^3, \quad y = \cos(x^2+1)$*
Solution
We'll just plug into the formula.

$$\begin{aligned}\frac{dz}{dx} &= \left(\ln(xy) + x\frac{y}{xy}\right) + \left(x\frac{x}{xy} + 3y^2\right)\left(-2x\sin\left(x^2+1\right)\right)\\ &= \ln\left(x\cos(x^2+1)\right) + 1 - 2x\sin(x^2+1)\left(\frac{x}{\cos(x^2+1)} + 3\cos^2(x^2+1)\right)\\ &= \ln(x\cos(x^2+1)) + 1 - 2x^2\tan(x^2+1) - 6x\sin(x^2+1)\cos^2(x^2+1)\end{aligned}$$

Case 2: $z = f(x,y), \quad x = g(s,t), \quad y = h(s,t)$ and compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$
In this case if we were to substitute in for x and y we would get that z is a function of s and t and so it makes sense that we would be computing partial derivatives here and that there would be two of them.
Here is the chain rule for both these cases.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Example 6.13 *Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = e^{2r}\sin(3\theta)$, $r = st - t^2$, $\theta = \sqrt{s^2+t^2}$*
Solution
Here is the chain rule for $\frac{\partial z}{\partial s}$.

$$\begin{aligned}\frac{\partial z}{\partial s} =& (2e^{2r}\sin(3\theta))(t) + (3e^{2r}\cos(3\theta))\frac{s}{\sqrt{s^2+t^2}}\\ =& t\left(2e^{2(st-t^2)}\sin(3\sqrt{s^2+t^2})\right) + \frac{3se^{2(st-t^2)}\cos(3\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\end{aligned}$$

Now the chain rule for $\frac{\partial z}{\partial t}$.

$$\begin{aligned}\frac{\partial z}{\partial t} =& (2e^{2r}\sin(3\theta))(s-2t) + (3e^{2r}\cos(3\theta))\frac{t}{\sqrt{s^2+t^2}}\\ =& (s-2t)\left(2e^{2(st-t^2)}\sin(3\sqrt{s^2+t^2})\right) + \frac{3te^{2(st-t^2)}\cos(3\sqrt{s^2+t^2})}{\sqrt{s^2+t^2}}\end{aligned}$$

Case 3: Suppose that z is a function of n variables, $x_1, x_2, \ldots, x_n$, and that each of these variables are in turn functions of m variables, $t_1, t_2, \ldots, t_m$. Then for any variable $t_i, i = 1, 2, \ldots, m$ we have the following,

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_m}\frac{\partial x_n}{\partial t_i}$$

Wow. That's a lot to remember. There is actually an easier way to construct all the chain rules that we have discussed in the section or will look at in later examples. We can build up a **tree diagram** that will give us the chain rule for any situation. To see how these works let's go back and take a look at the chain rule for $\frac{\partial z}{\partial x}$ given that $z = f(x,y), x = g(s,t), y = h(s,t)$. We already know what this is, but it may help to illustrate the tree diagram if we already know the answer. For reference here is the chain rule for this case,

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Here is the tree diagram for this case (see figure 6.5 (a)).
Or

$$w = f(x,y,z), \quad x = g_1(s,t,r), \quad y = g_2(s,t,r), \quad \text{and} \quad z = g_3(s,t,r)$$

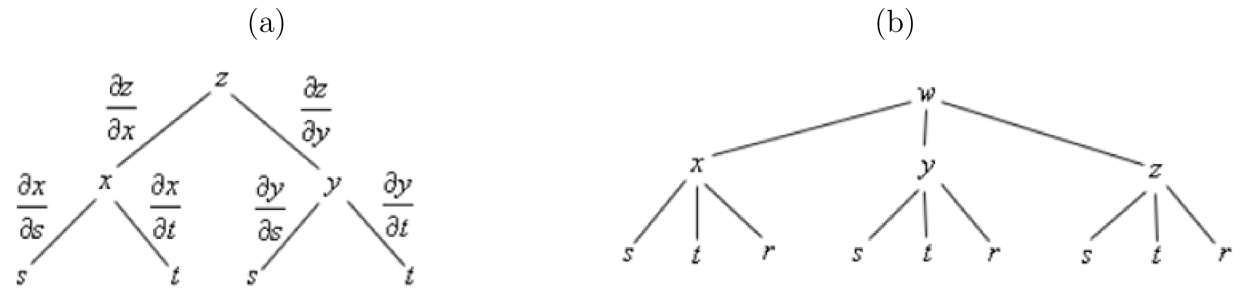


Fig. 6.5:

Here is the tree diagram for this situation (see figure 6.5 (b)).
From this it looks like the derivative will be

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

Example 6.14 *(Laplace's equation in polar coordinates) If $z = f(x, y)$ has continuous partial derivatives of second order, and if $x = r\cos\theta$ and $y = r\sin\theta$, show that*

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

First note that

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Thus,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}.$$

Now differentiate with respect to r again. Remember that r and θ are independent variables, so the factors $\cos\theta$ and $\sin\theta$ can be regarded as constants. However, $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ depend on x and y and, therefore, on r and θ.

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\\
&= \cos\left(\cos\theta\frac{\partial^2 z}{\partial x^2} + \sin\theta\frac{\partial^2 z}{\partial y\partial x}\right) + \sin\theta\left(\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \sin\theta\frac{\partial^2 z}{\partial y^2}\right).\\
&= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

we have used the equality of mixed partials in the last line. Similarly,

$$\frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}.$$

When we differentiate a second time with respect to θ, we can regard r as constant, but each term above is still a product of two functions that depend on θ. Thus,

$$\begin{aligned}
\frac{\partial^2 z}{\partial \theta^2} &= -r\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial}{\partial \theta}\frac{\partial z}{\partial x}\right) + r\left(-\sin\theta\frac{\partial z}{\partial y} + \cos\theta\frac{\partial}{\partial \theta}\frac{\partial z}{\partial y}\right)\\
&= -r\frac{\partial z}{\partial r} - r\sin\theta\left(-r\sin\theta\frac{\partial^2 z}{\partial x^2} + r\cos\theta\frac{\partial^2 z}{\partial y\partial x}\right)\\
&\quad + r\cos\theta\left(-r\sin\theta\frac{\partial^2}{\partial x\partial y} + r\cos\theta\frac{\partial^2 z}{\partial y^2}\right)\\
&= -r\frac{\partial z}{\partial r} + r^2\left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2}\right).
\end{aligned}$$

Combining these results, we obtain the desired formula:

$$\frac{\partial z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

6.6 Applications of Partial Derivatives

Local Extreme Values

Two Variables

We suppose for the moment that $f = f(x, y)$ is defined on an open connected set and is continuously differentiable there. The graph of f is surface

$$z = f(x, y).$$

Where f has a local maximum, the surface has a local high point. Where f has a local minimum, the surface has a local low point. Where f has either a local maximum or a local minimum, the gradient is 0 and therefore the tangent plane is horizontal. See Figure 6.6 (a).

A zero gradient signals the possibility of a local extreme value; it does not guarantee it. For example, in the case of the saddle-shaped surface of Figure 6.6 (b), there is a horizontal tangent plane at the origin and therefore the gradient is zero there, yet the origin gives neither a local maximum nor a local minimum.

Critical points at which the gradient is zero are called *stationary points*. The stationary points that do not give rise to local extreme values are called *saddle points*.

(a) (b)

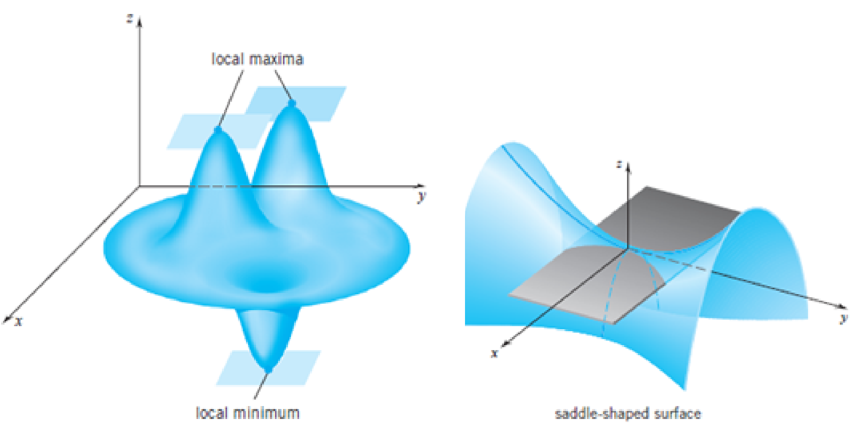


Fig. 6.6:

Definition 6.4

1. *A function $f(x, y)$ has a* **relative minimum** *at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in some region around (a, b).*

2. *A function $f(x, y)$ has a* **relative maximum** *at the point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some region around (a, b)*

Note that this definition does not say that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point (a, b) the function will always be larger than $f(a, b)$. Outside of that region it is completely possible for the function to be smaller. Likewise, a relative maximum only say that around (a, b) the function will always be smaller than $f(a,b)$. Again, outside of the region it is completely possible that the function will be larger.

Fact

If the point (a,b) is a relative extrema of the function $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$ and in fact we'll have $\nabla f(a,b) = \vec{0}$.

Fact

Suppose that (a,b) is a critical point of $f(x,y)$ and that the second order partial derivatives are continuous in some region that contains (a,b). Next define,

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

We then have the following classifications of the critical point.

1. If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a relative minimum.

2. If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is relative maximum.

3. If $D < 0$ then (a,b) is a saddle point.

4. If $D = 0$ then (a,b) may be a relative minimum, relative maximum or a saddle point. Other techniques would be need to be used to classify the critical point.

Example 6.15 *Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.*

Solution

We first need all the first order (to find the critical points) and second order (to classify the critical points) partial derivatives so, let's get those.

$$\begin{array}{lll} f_x = 3x^2 - 3y & f_y = 3y^2 - 3x & \\ f_{xx} = 6x & f_{yy} = 6y & f_{xy} = -3 \end{array}$$

Let's first find the critical points. Critical points will be solutions to the system of equations

$$\begin{aligned} f_x &= 3x^2 - 3y = 0 \\ f_y &= 3y^2 - 3x = 0 \end{aligned}$$

This is a non-linear system of equations and these can, on occasion, be difficult to solve. However, in this case it's not too bad. We can solve the first equation for y as follows,

$$3x^2 - 3y = 0 \quad \Rightarrow y = x^2$$

Plugging this into the second equation gives,

$$3(x^2)^2 - 3x = 3x(x^3 - 1) = 0$$

From this we can see that we must have $x = 0$ or $x = 1$. Now use the fact that $y = x^2$ to get the critical points

$$\begin{array}{lll} x = 0: & y = 0^2 = 0 & \Rightarrow \quad (0,0) \\ x = 1: & y = 1^1 = 1 & \Rightarrow \quad (1,1) \end{array}$$

So, we get two critical points. All we need to do now is classify them. To do this we need D. Here is the general formula for D.

$$\begin{aligned} D(x,y) =& f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 \\ =& (6x)(6y) - (-3)^2 \\ =& 36xy - 9 \end{aligned}$$

To classify the critical points all that we need to do is plug in the critical points and use the fact above to classify them.

$$(0,0): \quad D = D(0,0) = -9 < 0$$

So, for $(0,0)$ D is negative and so this must a saddle point.

$$(11): \quad D = D(1,1) = 36 - 9 = 27 > 0 \quad f_{xx}(1,1) = 6 > 0$$

For $(1,1)$ D is positive and f_{xx} is positive and so we must have a relative minimum.

Example 6.16 *Find and classify all the critical points for $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$*
Solution
As with the first example we will first need to get all the first and second order derivatives.

$$\begin{array}{lll} f_x = 6xy - 6x & f_y = 3x^2 + 3y^2 - 6y & \\ f_{xx} = 6y - 6 & f_{yy} = 6y - 6 & f_{xy} = 6x \end{array}$$

we will first need the critical points. The equations that we will need to solve this time are,

$$\begin{aligned} 6xy - 6x &= 0 \\ 3x^2 + 3y^2 - 6y &= 0 \end{aligned}$$

These equation are a little trickier to solve than the first set, but once you see what to do they really are'nt terribly bad.
First, let's notice that we can factor out a $6x$ for the first equation to get

$$6x(y-1) = 0$$

So, we can see that the first equation will be zero if $x = 0$ or $y = 1$. Be careful to not just cancel the x from both sides. If we had done that we would have missed $x = 0$.
To find the critical points we can plug these (individually) into the second equation and solve for the remaining variable.

$$\begin{array}{llll} x = 0: & 3y^2 - 6y = 3y(y-2) = 0 & \Rightarrow & y = 0, y = 2 \\ Y = 1: & 3x^2 - 3 = 3(x^2 - 1) = 0 & \Rightarrow & x = -1, x = 1 \end{array}$$

So, if $x = 0$ we have the following critical points,

$$(0,0), \quad (0,2)$$

and if $y = 1$ the critical points are,

$$(1,1) \quad (-1,1)$$

Now all we need to do is classify the critical points. To do this we'll need the general formula for D.

$$D(x,y) = (6y-6)(6y-6) - (6x)^2 = (6y-6)^2 - 36x^2$$

$$\begin{array}{rll} (0,0): & D = D(0,0) = 36 > 0 & f_{xx}(0,0) = -6 < 0 \\ (0,2): & D = D(0,2) = 36 > 0 & f_{xx}(0,2) = 6 > 0 \\ (1,1): & D = D(1,1) = -36 < 0 & \\ (-1,1): & D = D(-1,1) = -36 < 0 & \end{array}$$

So, it looks like we have the following classifications of each of these critical points

$$\begin{array}{rl} (0,0): & \textit{Relative Maximum} \\ (0,2): & \textit{Relative Minimum} \\ (1,1): & \textit{Saddle point} \\ (-1,1): & \textit{Saddle point} \end{array}$$

Exercise One

1. Use $\epsilon - \delta$ definition of limits to prove that:

(i)	$\lim_{x\to 1}(3x+1)=4$	(ii)	$\lim_{x\to 2}(5-2x)=1$	(iii)	$\lim_{x\to 0} x^2=0$
(iv)	$\lim_{x\to 2}\frac{x-2}{1+x^2}=0$	(v)	$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$	(vi)	$\lim_{x\to 2} x^3=8$
(vii)	$\lim_{x\to -1}\frac{x+1}{x^2-1}=-\frac{1}{2}$	(viii)	$\lim_{x\to 3}\frac{x-2}{x+1}=\frac{1}{4}$	(ix)	$\lim_{x\to 2} 7x+4=18$

2. State with reason(s) whether the function is continuous or discontinuous.

(i) $f(x)=\begin{cases} x & x<0 \\ x^2 & x\geq 0\end{cases}$ (ii) $f(x)=\begin{cases} \frac{1}{x^2} & x\neq 0 \\ 0 & x=0\end{cases}$

(iii) $f(x)=\begin{cases} x^2 & x\leq 1 \\ 0.987 & x>1\end{cases}$

3. (i) Find k so that $f(x)=\begin{cases} x^2 & x\leq 2 \\ k-x^2 & x>2\end{cases}$ is a continuous function.

   (ii) Find m so that $f(x)=\begin{cases} x-m & x\leq \\ 1-mx & x>2\end{cases}$ is continuous for all x.

4. Evaluate the following limits if exist

   (i) $\lim_{x\to 4}\frac{x^2-7x+12}{x-4}$

   (ii) $\lim_{x\to 3} f(x)$ where $f(x)\begin{cases} x^2+5 & \text{if } x\neq 3 \\ 7 & \text{if } x=3\end{cases}$

   (iii) $\lim_{x\to 2} g(x)$ where $g(x)\begin{cases} x^2+9 & \text{if } x<2 \\ x^2-3 & \text{if } x\geq 2\end{cases}$

   (iv) $\lim_{x\to 5}\frac{3x-15}{\sqrt{x^2-10x+25}}$

Exercise Two

1. Find the Laplace transform of the following functions

(i)	$f(x) = 3 + 2x^2$	(ii)	$f(x) = 5\sin 3x - 17e^{-2x}$
(iii)	$f(x) = 2\sin x + 3\cos 2x$	(iv)	$f(x) = xe^{4x}$
(v)	$f(x) = e^{-2x}\sin 5x$	(vi)	$f(x) = \sinh x \cos x$
(vii)	$f(x) = e^{-x}x\cos 2x$	$(viii)$	$f(x) = x^{\frac{7}{2}}$
(ix)	$f(x) = \frac{\sin 3x}{x}$	(x)	$f(x) = \begin{cases} e^x & x \le 2 \\ 3 & x > 2 \end{cases}$

2. Find the Laplace transforms of the following functions
(i) $f(t) = (t^2+t+1)u(t-1)$ (ii) $f(t) = e^{3t}u(t-2)$ (iii) $(\sin t + \cos t)u(t - \pi/2)$

3. Express $f(x)$ in terms of Heavisides unit step function and find its Laplace transform
(i) $f(t) = \begin{cases} t^2 & o < t < 2 \\ 4t & 2 < t < 4 \\ 8 & t > 4 \end{cases}$ (ii) $f(t) = \begin{cases} e^{-t} & 0 < t < 3 \\ 0 & t > 3 \end{cases}$

4. Find (i) $\mathcal{L}\{2\delta(t-1) + 3\delta(t-2) + 4\delta(t-3)\}$ (ii) $\mathcal{L}\{\cosh 3t\delta(t-2)\}$

5. Find (i) $\mathcal{L}^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\}$ (ii)$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\}$ (iii) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\}$ (iv) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$
(v) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+9}\right\}$ (vi) $\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\}$ (vii) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)+(s^2+4s+8)}\right\}$

6. Use Laplace transforms to solve
(i) $y' - 5y = 0;\quad y(0) = 2$ (ii) $y' + y = \sin x;\quad y(0) = 1$
(iii) $y'' - 3y' + 4y = 0;\quad y(0) = 1,\quad y'(0) = 5$ (iv) $y'' - y' - 2y = 4x^2;\quad y(0) = 1, y'(0) = 4$
(v) $y''' + y' = e^x; y(0) = y'(0) = y''(0) = 0$

7. A resistance R in series with inductance L is connected e.m.f $E(t)$. The current is given by $L\frac{di}{dt} + Ri = E(t)$. If the switch is connected at $t = 0$ and disconnected at $t = a$, find the current i in terms of t.

8. Use Laplace transforms to solve
(i) $\left.\begin{aligned} v' - u + v &= 0 \\ u' + u - v &= 0 \end{aligned}\right\}$ with $u(0) = 1,\quad v(0) = 2$
(ii) $\left.\begin{aligned} y' + z &= x \\ z' + 4y &= 0 \end{aligned}\right\}$ with $y(0) = 1,\quad z(0) = -1$
(iii) $\left.\begin{aligned} w' - y + 2z &= 3e^{-x} \\ -2w' + 2y' + z &= 0 \\ 2w' - 2y + z' + 2z'' &= 0 \end{aligned}\right\}$ with $w(0) = 1$, $w'(0) = 1,\quad y(0) = 2,\quad z(0) = 2,$
$z'(0) = -2$

Exercise Three

1. Find the Fourier series expansion of the functions on the specified intervals.

   i. $f(x) = e^x$, on $[-1, 1]$

   ii. $f(x) = x^4$, on $[-1, 1]$. Sketch the periodic function.

   iii. The rectified sine wave $f(t) = |\sin t|$ on $0 < t < \pi$. Sketch the sine wave function.

2. i. Give reasons why the functions: (a) $\frac{1}{3-t}$ (b) $\sin\left(\frac{1}{t-2}\right)$ do not satisfy Dirichlet's conditions in the interval $0 < t < 2\pi$.

   ii. How should $f(x)$ be defined at discontinuities for convergence?

$$f(x) = \begin{cases} 5\sin x & -2\pi < x < -\frac{\pi}{2} \\ 4 & x = -\frac{\pi}{2} \\ x^2 & -\frac{\pi}{2} < x < 2 \\ 8\cos x & 2 < x < \pi \\ 4x & \pi < x < 2\pi \end{cases}$$

3. i. Find the Fourier Series expansion of

$$f(x) = \begin{cases} -x & -5 < x < 0 \\ 1 + x^2 & 0 < x < 5 \end{cases}$$

   ii. A periodic function $f(x)$ of period 2π is defined within the period $0 < t < 2\pi$ by

$$f(x) = \begin{cases} 2 - \frac{t}{\pi} & 0 < t < \pi \\ \frac{t}{\pi} & \pi < t < 2\pi \end{cases}$$

   Sketch a graph for $-4\pi < t < 4\pi$ and obtain its Fourier series expansion.

   iii. A periodic function $f(x)$ of period 2π is defined within the period $0 < t < 2\pi$ by

$$f(x) = \begin{cases} t & 0 < t < \frac{\pi}{2} \\ \frac{\pi}{2} & \frac{\pi}{2} < t < \pi \\ \pi - \frac{t}{2} & \pi < t < 2\pi \end{cases}$$

   Sketch a graph for $-2\pi < t < 3\pi$. Find the Fourier series expansion of the function.

   iv. Determine the Fourier series representation of the periodic voltage $e(t)$ shown in figure 6.7:

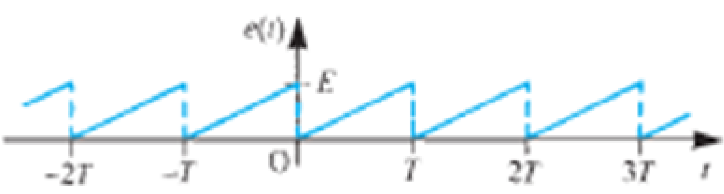


Fig. 6.7:

4. i. Write the Fourier cosine and Fourier sine expansion of $f(x) = e^{2x}$ for $x0 < x < 1$.

   ii. A tightly stretched flexible uniform string has its ends fixed at the points $x = 0$ and $x = l$. The midpoint of the string is displaced a distance a, shown in figure 6.8. If $f(x)$ denotes the displaced profile of the string, express $f(x)$ as a Fourier series expansion consisting only of sine terms.

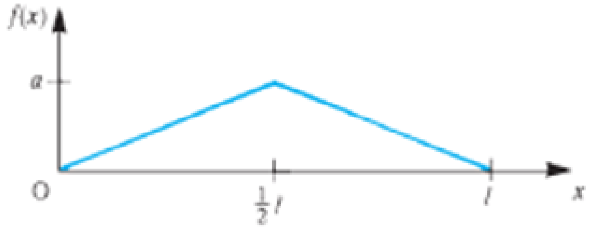


Fig. 6.8:

5. Find the complex form of the Fourier series expansion of:

   i. $f(x) = x$ for $-1 < x < 1$, $f(x+2) = f(x)$.

   ii. $f(t) = \cos(t/2)$ for $-\pi < t < \pi$, $f(t+2\pi) = f(t)$

Exercise Four

1. Specify the domain of the given functions.

   a. $f(x, y) = \sqrt{xy}$ b. $f(x, y) = \dfrac{xy}{x^2 - y^2}$

   c. $f(x, y) = \dfrac{1}{\sqrt{x^2 - y^2}}$ d. $f(x, y) = \sin^{-1}(x + y)$

   e. $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$

2. Show that the given functions satisfy the corresponding partial differential equations.

   a. $z = \dfrac{x + y}{x - y}, x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$

   b. $w = x^2 + yz, x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{partialy} + z\dfrac{\partial w}{\partial z} = 2w$

   c. $z = f(x^2 + y^2),$ where f is any differentiable function of one variable
   $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$

3. Find and classify the critical points of the given functions.
   a. $f(x, y) = x^4 + y^4 - 4xy$
   b. $f(x, y) = \cos x + \cos y$

4. Find the minimum value of $f(x, y) = x + 8y + \frac{1}{xy}$ in the first quadrant $x > 0, y > 0$. How do you know that a minimum exists?

5. The material used to make the bottom of a rectangular box is twice as expensive per unit area as the material used to make the top or side walls. Find the dimension of the box of given volume V for which the cost of the materials is minimum.

6. Find the three positive numbers a, b and c, whose sum is 30 and for which the expressionab^2c^3 is maximum.

References

The list of references is not exhaustive. Any other book on the topics above may be suitable as well.

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