Chapter 1

Question 1 Newtons law of gravitation august 27 '03

a) Mutual gravitational attraction

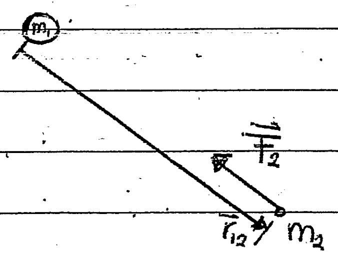
$$\vec{F}_2 = -G\frac{m_1 m_2}{r_{12}^2}\left(\frac{\vec{r}_{12}}{r_{12}}\right)$$

Field strenght (force per unit mass)

$$\vec{g}_2 = -G\frac{m_1}{r_{12}^3}\vec{r}_{12}$$

defining potential U

$$\vec{g}_2 = -\vec{\nabla}U_2$$



gives for U

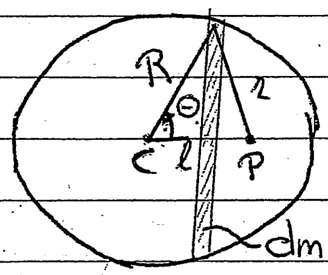
$$U_2 = -G\frac{m_1}{r_{12}} + U_{2,0}$$

Setting $U_{2,0} = 0$, the general case is

$$U = -G\frac{m_1}{r}$$

b) Mass in ring

$$dm = (2\pi R\sin\Theta)(Rd\Theta)t\rho$$



Potential

$$dU_p = -\frac{1}{r}G2\pi R^2 t\rho\sin\Theta d\Theta$$

with

$$r^2 = R^2 + l^2 - 2Rl\cos\Theta$$

Total potential

$$U_p = -\frac{1}{2}GM\int_{\Theta=0}^{\pi}\frac{\sin\Theta d\Theta}{\sqrt{R^2+l^2-2Rl\cos\Theta}}$$

Since

$$r = (R^2 + \ell^2 - 2R\ell\cos\Theta)^{1/2}$$

$$dr/d\Theta = \frac{1}{2}(R^2+\ell^2-2R\ell\cos\Theta)^{-1/2} \cdot 2R\ell\sin\Theta$$

$$= \frac{R\ell\sin\Theta}{\sqrt{R^2+\ell^2-2R\ell\cos\Theta}}$$

Substitution gives

$$U_p = -\frac{1}{2}\frac{GM}{R\ell}\int_{R-\ell}^{R+\ell} dr$$

Evaluating the integral

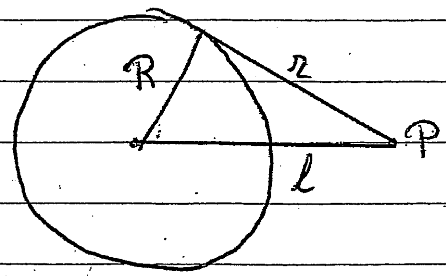
$$U_p = -\frac{1}{2}\frac{GM}{R\ell}\left[(R+\ell)-(R-\ell)\right] = -\frac{GM}{R} = \text{const} ; \vec{F} = \vec{0}$$

c) Outside we have the integral

$$U_p = -\frac{1}{2}\frac{GM}{R\ell}\int_{r=\ell-R}^{\ell+R} dr$$

$$= -\frac{1}{2}\frac{GM}{R\ell}\left[(\ell+R)-(\ell-R)\right]$$

$$= -\frac{GM}{\ell}$$



$$\vec{F}_\ell = -\frac{\partial U}{\partial \ell} m_p = -G\frac{M m_p}{\ell^2}\left(\frac{\vec{\ell}}{\ell}\right)$$

d) $U = -\frac{G}{\ell}\sum M_\ell = -\frac{GM_T}{\ell}$

e) 1^{st} orde approximation

$$U = -\frac{GM}{l} - \frac{1}{2}\frac{G}{l^3}(A+B+C-3D)$$

U : gravity potential
M : Total mass body
G : Universal gravity constant
l : Distance from center of gravity
A : moment of inertia around x-axis
B : " " " " y-axis
C : " " " " z-axis
D : " " " " line OP

if radial symmetric: $A = B = C = D$

$$U = -\frac{GM}{l}$$

f) rotational symmetric

$$U = -\frac{GM}{l}\left[1 + \frac{1}{2}\frac{C-A}{MR^2}\left(\frac{R}{l}\right)^2(1-3\sin^2\varphi)\right]$$

φ : latitude

$$\frac{C-A}{MR^2} \simeq 10^{-3}$$

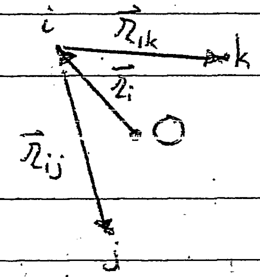
$$F = \frac{\partial U}{\partial l} = -\frac{GM}{l^2}\left[1 + \frac{3}{2}\frac{C-A}{MR^2}\left(\frac{R}{l}\right)^2(1-3\sin^2\varphi)\right]$$

gen 3 body

Question 2: General three body problem   august 27 '03

a) Euler formulering

$$\frac{d^2\vec{r}_i}{dt^2} = G\frac{m_j}{r_{ij}^3}\vec{r}_{ij} + G\frac{m_k}{r_{ik}^3}\vec{r}_{ik}$$

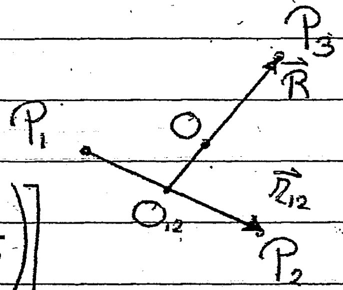


$\vec{r}_i$: Vector from origin of inertial reference frame to i
G: Universal gravitational constant
$\vec{r}_{ij}$: Vector from body i to body j
$\vec{r}_{ik}$: Vector from body i to body k

Jacobian formulation

$$\frac{d^2\vec{R}}{dt^2} = -GM\left[\alpha\frac{\vec{r}_{13}}{r_{13}^3} + (1-\alpha)\frac{\vec{r}_{23}}{r_{23}^3}\right]$$

$$\frac{d^2\vec{r}_{12}}{dt^2} = -G\left[(m_1+m_2)\frac{\vec{r}_{12}}{r_{12}^3} + m_3\left(\frac{\vec{r}_{13}}{r_{13}^3} - \frac{\vec{r}_{23}}{r_{23}^3}\right)\right]$$



$\vec{R}$: Vector from O_{12} to P_3 through O
M: total mass of all bodies
$\alpha = m_1/m_1+m_2$

b) Eulerian: order 18 $(3\times3\times2)$

three bodies, three components, 2^{nd} order diff. eq

c) Jacobian: order 12 $(2\times3\times2)$

two equations, three components, 2^{nd} order diff eq

Reduction due to implicit use of the six center of mass integrals

For both reduction to order 6 possible
- 6: center of mass integrals
- 1: invariable plane Laplace
- replace time by angular coord.
- 3: angular momentum
- 1: energy

d) Earth (P_1) moon (P_2), sun (P_3) system

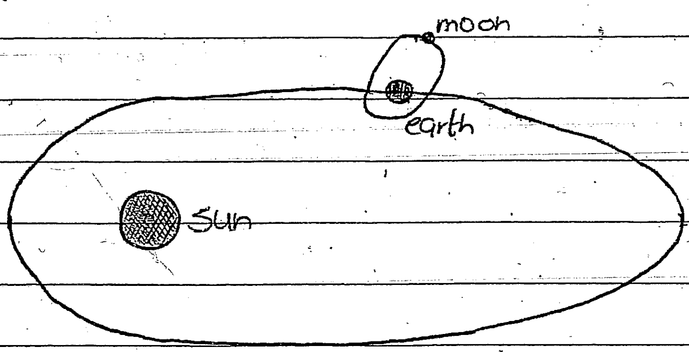
~~$\alpha \simeq 1$~~

$$\vec{r}_{13} \simeq \vec{r}_{23} = \vec{R}$$

$$\frac{d^2\vec{R}}{dt^2} = -GM\left[1\,\frac{\vec{R}}{R^3} + 0\,\frac{\vec{R}}{R^3}\right] = -GM\frac{\vec{R}}{R^3}$$

$$\frac{d^2\vec{r}_{12}}{dt^2} = -G\left[(m_1+m_2)\frac{\vec{r}_{12}}{r_{12}^3} + m_3\left(\frac{\vec{R}}{R^3} - \frac{\vec{R}}{R^3}\right)\right] = -G(m_1+m_2)\frac{\vec{r}_{12}}{r_{12}^3}$$

e) motions may be aproximated with two two body problems. Two slowly pertubated Keplerian orbits



elorbimp

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Question 2: elliptical orbit & impulsive shot

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$h_p = 1500$ [km]
$h_a = 5000$ [km]

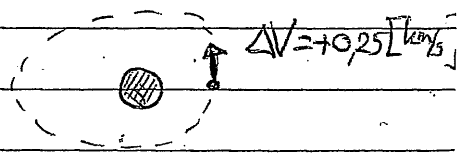
a) $r_p = h_p + R_\oplus = 1500 + 6378{,}14 = 7878{,}14$ [km]

$r_a = h_a + R_\oplus = 5000 + 6378{,}14 = 11.378{,}14$ [km]

$2a = r_p + r_a$

$$\begin{cases} a = \frac{1}{2}(r_p + r_a) = \frac{1}{2}(7878{,}14 + 11378{,}14) = 9628{,}14 \text{ [km]} \\ e = \dfrac{r_a - r_p}{r_a + r_p} = \dfrac{3500}{2 \cdot 9628{,}14} = 0{,}18{,}18 \text{ [-]} \\ T = 2\pi \sqrt{\dfrac{a^3}{\mu}} = 2\cdot\pi\cdot\sqrt{\dfrac{9628{,}14^3}{398600.4}} = 9402 \text{ [s]} \\ \quad = 2{,}61 \text{ [hr]} \end{cases}$$

b) $\mathcal{E}_{tot} = \mathcal{E}_{pot} + \mathcal{E}_{kin} = -\frac{\mu}{r} + \frac{V^2}{2} = -\frac{\mu}{2a}$



in perigee

$$V_{p,original}^2 = V_c^2(1+e) = \frac{\mu}{r_p}(1+e) = \frac{398600.4}{7878{,}14}(1+0{,}18{,}18) = 59{,}79$$

$V_{p,original} = 7{,}73$ [km/s]
$V_p = 7{,}98$ [km/s]

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow a = \frac{1}{2/r - V^2/\mu}$$

$$= \frac{1}{2/7878{,}14 - 7{,}98^2/\mu} = 10626 \text{ [km]}$$

$r_p = a(1-e) \Rightarrow e = \frac{a - r_p}{a} = 0{,}259 \; [-]$

$r_a = a(1+e) = 13378 \; [km]$

$h_p = r_p - R_\oplus = 1500 \; [km]$

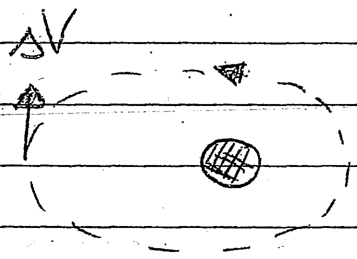
$h_a = r_a - R_\oplus = 13378 - 6378 = 7000 \; [km]$

$V_p = 7{,}98 \; [km/s]$

$V_a = \frac{1-e}{1+e} V_p = 4{,}70 \; [km/s]$

$T = 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi \cdot \sqrt{\frac{10626}{398600{,}4}} = 10900 \, [s] = 3{,}0 \, [hr]$

) $r_a = 11378{,}14$



$h_a = 5000 \; [km]$

$V_{a\,orig.} = V_c \sqrt{1-e} = \sqrt{\frac{\mu}{r_a}(1-e)} = 5{,}35 \; [km/s]$

$V_a = 5{,}35 - 0{,}25 = 5{,}10 \, [km/s]$

$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow a = \frac{1}{2/r - V^2/\mu} = 9048 \, [km]$

$T = 2\pi \sqrt{\frac{a^3}{\mu}} = 8565 \, [s] = 2{,}38 \, [hr]$

$$r_a=(1+e)a \Rightarrow e=\frac{r_a-a}{a}=0{,}258\ [-]$$

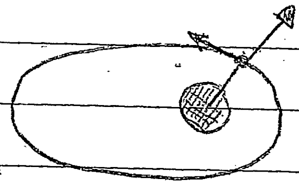
$$h_p = r_p - R_\oplus = a(1-e)-R_\oplus = 335{,}47\ [km]$$

$$h_a = r_a - R_\oplus = 11378{,}14 - R_\oplus = 5000\ [km]$$

$$V_p = V_a\left(\frac{1+e}{1-e}\right) = 8{,}65\ [km/s]$$

d.) $\mathcal{E} = \mathcal{E}_{kin} + \mathcal{E}_{pot}$

$= V^2/2 - \mu/r = -\mu/2a$



Since v and r are still the same so is a.
The orbital period remains 2,61 [hr]

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

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Question 4 Asteroid

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hyperbolic orbit $e>1$

t_0: $r = 900000$ [km]

$\dot{r} = -5{,}4638$ [km/s]

$r\dot{\theta} = -0{,}0920$ [km/s]

a) $$r = \frac{H^2/\mu}{1+e\cos\theta} = \frac{H^2}{\mu}\left[1+e\cos\theta\right]^{-1}$$

$$\dot{r} = \frac{H^2/\mu}{(1+e\cos\theta)^2}\sin\theta\,\dot{\theta}\,e\cdot\frac{r^2}{r^2}$$

$$= \frac{H^2}{\mu}\frac{1}{(1+e\cos\theta)}\sin\theta\,\dot{\theta}\,e\,r^2\,\frac{\mu}{H^2}\frac{\mu}{H^2}(1+e\cos\theta)^2$$

$$= \frac{\mu}{H^2}\sin\theta\,\dot{\theta}\,e\,r^2$$

$$= \frac{\mu e}{H}\sin\theta$$

$$\left[\begin{array}{l} V_r = V\sin\gamma = \dot{r} \\ V_\theta = V\cos\gamma = r\dot{\theta} \\ H = rV\cos\gamma = r^2\dot{\theta} \end{array}\right.$$

$$r = \frac{H/\mu}{1+e\cos\theta}\,r^2\dot{\theta}$$

$$r\dot{\theta} = \frac{\mu}{H}(1+e\cos\theta)$$

b) $$\mathcal{E}_{tot} = \mathcal{E}_{pot} + \mathcal{E}_{kin}$$

$$-\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{V^2}{2} \Longrightarrow a = \frac{1}{2/r - V^2/\mu}$$

$$V = \sqrt{V_r^2 + V_\theta^2} = 5{,}46\ [\text{km/s}]$$

c) $r = \dfrac{H^2/\mu}{1 + e\cos\Theta}$

closest if $\Theta = 0 \Rightarrow \cos\Theta = 1$

$r_{min} = \dfrac{H^2/\mu}{1+e} = 6880$ [km]

$h_{min} = 501$ [km]

$\dfrac{V^2}{2} - \dfrac{\mu}{r} = -\dfrac{\mu}{2A} \Rightarrow r = \dfrac{2}{V^2/\mu + 1/A}$

$V = \sqrt{-\dfrac{\mu}{A} + 2\dfrac{\mu}{r}} = 12{,}03$ [km/s]

e) $\dfrac{V^2}{2} - \dfrac{\mu}{r} = -\dfrac{\mu}{2A}$

op $r \Rightarrow \infty$ $\quad \dfrac{V^2}{2} = -\dfrac{\mu}{2A}$

$V_\infty = \sqrt{-\dfrac{\mu}{a}} =$

$$a = \frac{1}{2/900.000 - 5{,}46^2/398600.4} = -13780 \text{ [km]}$$

$$H = r V_\Theta = r \cdot r\dot{\Theta} = 900000 \cdot -0{,}0920 = -82800 \left[{}^{m^2}/_{s^2}\right]$$

$$\dot{r} = \frac{\mu}{H} e \sin\Theta$$

$$-5{,}4638 = -4{,}8140\, e \sin\Theta$$

$$\boxed{1{,}1350 = e \sin\Theta}$$

$$r\dot{\Theta} = \frac{\mu}{H}(1 + e\cos\Theta)$$

$$-0{,}0920 = -4{,}8140\,(1 + e\cos\Theta)$$

$$\boxed{-0{,}9809 = e\cos\Theta}$$

$$\tan\Theta = \frac{1{,}1350}{0{,}9809} \Rightarrow \Theta = 130^\circ$$

$$e = \sqrt{1{,}1350^2 + 0{,}9809^2} = 1{,}50 \text{ [-]}$$