Exam Solution

Course: AE4874 I Fundamentals of Astrodynamics
Exam Source:ZZ Goldmine
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0 Introduction

This contains an exam solution. If you wish to contribute to this exam solution:

1. Create a github account, (you can create an ”anonymous” one).
2. git clone ...
3. edit your changes in the document.
4. open cmd, and browse to inside the folder you downloaded and edited
5. git pull (updates your local repository=copy of folder, to the latest version in github cloud)
6. git status shows which files you changed.
7. git add ”/some folder with a space/someFileYouChanged.tex”
8. git commit -m ”Included solution to question 1c.”
9. git push

It can be a bit initimidating at first, so feel free to click on ”issue” in the github browser of this repository and ask :) (You can also use that to say ”Hi, I'm having a bit of help with this particular equation, can someone help me out?”)

If you don't know how to edit a latex file on your own pc iso on overleaf, look at the ”How to use” section of `https://github.com/a-t-0/AE4872-Satellite-Orbit-Determination`.

0.1 Consistency

To make everything nice and structured, please use very clear citations:

1. If you copy/use an equation of some slide or document, please add the following data:
   (a) Url (e.g. if simple wiki or some site)
   (b) Name of document
   (c) (Author)
   (d) PAGE/SLIDE number so people can easily find it again
   (e) equation number (so people can easily find it again)
2. If you use an equation from the slides/a book that already has an equation number, then hardcode that equation number in this solution manual so people directly see which equation in the lecture material it is, this facilitates remembering the equations.
3. Here is an example is given in eq. (10.32[1]) (See file references.bib [1]).

$$R_n^E = \sum_{t=1}^{n} l_m(p_t, z_t) - \underset{i}{min} \sum_{t=1}^{n} {z_t}^i \tag{10.32[1]}$$

1 Exercise 1

Hill's Surfaces:

$$x^2 + Y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \tag{1}$$

a.

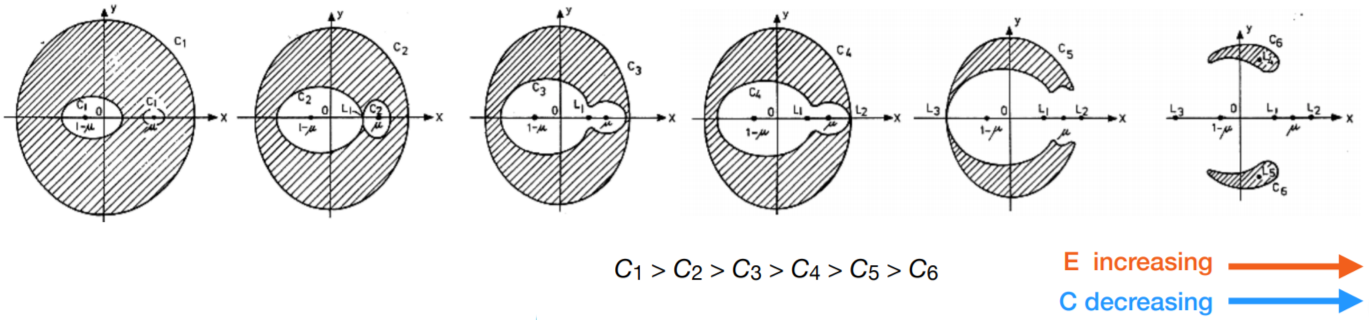
The parameters in the above equations mean the following:

- x,y: coordinates in the space-plane going through both large celestial bodies in the three-body problem. The origin is the barycenter of masses one and two.
- μ: parameter of the restricted circular three-body problem, equal to m_2, the mass of body two.
- r_1: distance from body one to body three, which is equal to $\sqrt{(\mu+x)^2+y^2+z^2}$
- r_2: distance from body two to body three, which is equal to $\sqrt{(1-\mu-x)^2+y^2+z^2}$
- C: Jacobi's constant, integration constant from the equations of motion

The physical interpretation of Hill's surfaces is that they represent surfaces in space which can not be crossed by the third body.

b.

Source picture: Wakker Figure 3.6

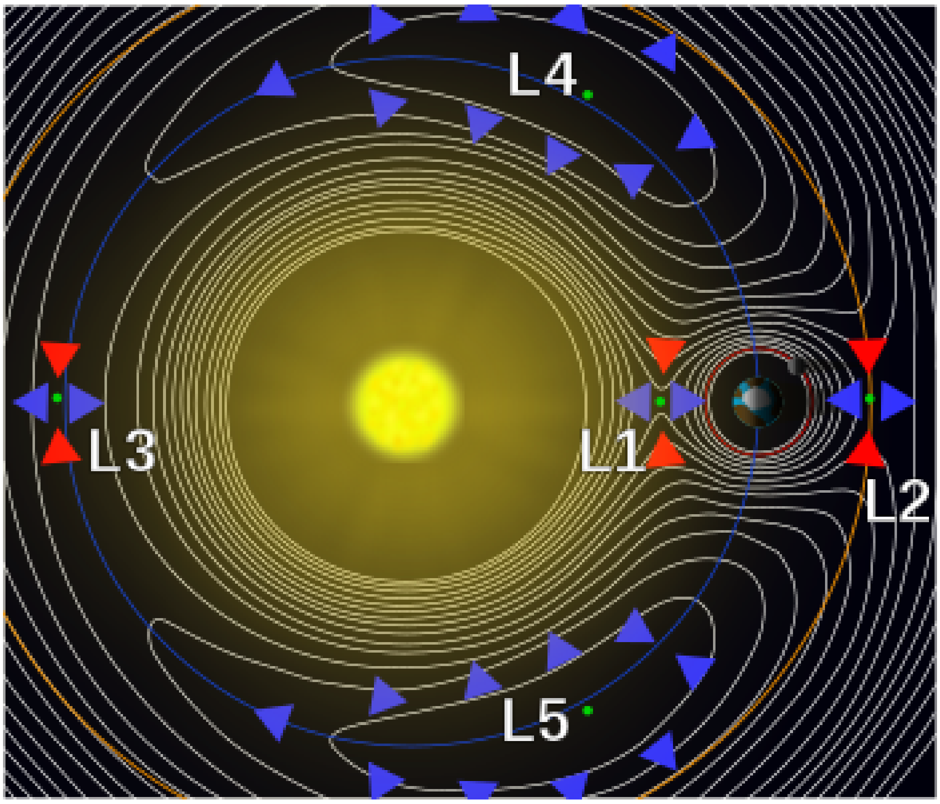


c.

The physical meaning of the Lagrange libration points is that a third body on this point will maintain its relative position to the other two bodies; they are equilibrium points. The gravitational force, the centripetal force and the Coriolis force are all in balance in these points. While they are equilibrium points, there are alas not stable equilibrium points.

d.

A Halo orbit is an orbit around Lagrangian libration points. Since the orbit is not naturally stable, it is called a slowly-changing elliptical orbit. A figure with the five Lagrangian points and their stability is shown below (credit: Wikipedia, Halo Orbit)



Regarding the derivation of the stability: for L1, L2 and L3 there is a restoring force towards the central axis through masses one and two due to axis-symmetry, but for all other perturbation directions the Lagrangian points are unstable, since gravitational mass is only attracting and never repulsive.

e.

Halo orbits are perfect for stable observations of space telescopes of a certain point in the celestial sky, or for deep-space observations without solar influence, or for solar observations while still being in permanent contact with the Earth. Similarly, Halo orbits are perfect for communication satellites for objects that are 'behind' the Sun or Moon, depending on which Lagrangian point is chosen.

2 Exercise 2

We have a body in an elliptical orbit with low eccentricity around the Sun. It is assumed only the Sun's pull and solar radiation pressure are acting on this body. The equations of motion then become:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= \frac{-\mu}{r^2} + \frac{F}{m}\sin(\delta) \\ \frac{d}{dt}(r^2\dot{\theta}) &= \frac{F}{m} r \cos(\delta) \end{aligned} \tag{2}$$

With

$$F = C_R \frac{WS}{c} \tag{3}$$

If it is assumed that the Sun emits energy radial-symmetric and that the body circling the Sun is spherical, we can write the equation for the acceleration of the body due to the radiation pressure using the above formula in the following way:

$$\frac{F}{m} = \frac{3}{4} \frac{C_R W_S R_S^2}{c\rho R} \frac{1}{r^2} \tag{4}$$

For a given body we get the following:

$$\frac{F}{m} = \frac{\alpha}{r} \tag{5}$$

With α a constant

a.

Asked is what the parameters in the equations mean.

- r: Radial distance between the Sun and the body
- θ: Angle between the periapsis and the body in orbit
- μ: Approximately Gm_{Sun} since the mass of the body is negligible compared to the mass of the Sun
- m: Mass of the body
- δ: Angle between the force and the normal to the radius vector
- C_R: Reflectivity of the satellite
- W: Radiation flux
- S: Effective cross-sectional area
- c: Speed of light
- R_S: Radius of the Sun
- ρ: Density of the body
- R: Radius of the body

b.

For the derivation of these formulae, we used the frequency of the solar light. However, if the body has a radial velocity with respect to the Sun, the frequency of the solar light is shifted through the Doppler effect to another frequency. Additionally, there is the light-time effect, which means that the speed of light is finite and that the solar light intercepted at a certain time is emitted by the Sun at an earlier time. hits leads to aberration of the incoming Solar light. These phenomena can be described by the following formulae:

$$\begin{aligned} W' &= W(1 - \frac{\dot{r}}{c}) \\ \gamma &= \frac{r\dot{\theta}}{c} \end{aligned} \tag{6}$$

c.

Substitution of this adapted radiation flux into equation (4) gives then

$$\frac{F}{m} = \frac{\alpha}{r^2}(1 - \frac{\dot{r}}{c}) \tag{7}$$

This since the radiation flux has to be multiplied with $1 - \frac{\dot{r}}{c}$

Now, δ is the angle between the direction of $\frac{F}{m}$ and the tangential, and γ is the aberration angle. Thus, we can say $\sin(\delta) = \cos(\gamma)$ and $\cos(\delta) = -\sin(\gamma)$. Since γ is small, we can approximately say (using first order approximation):

$$\begin{aligned} \sin(\delta) &\approx 1 \\ \cos(\delta) &\approx -\gamma = \frac{-r\dot{\theta}}{c} \end{aligned} \tag{8}$$

Substitution into equations (2) then automatically leads to:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu - \alpha}{r^2} - \frac{\alpha\dot{r}}{cr^2} \\ \frac{d}{dt}(r^2\dot{\theta}) &= -\frac{\alpha\dot{\theta}}{c} \end{aligned} \tag{9}$$

These equations show that the solar radiation pressure counteracts the gravitational acceleration by the Sun, and includes a term proportional to the circumferential velocity of the body. So on one end, the solar radiation pressure prevents the body from gradually falling into the Sun, but on the other end the other term has a minus sign and introduces drag, which shortens the length of the year for the body. This is called the Poynting-Robertson effect.

d.

With increasing values of α, the effect of radiation pressure on the orbit of the body increases. α is equal to $\frac{3C_R W_S R_S^2}{4c\rho R}$. Thus, this effect is the largest for a highly reflective, low density and small body. Ice particles fall in this category, and so do other small particles.

e.

For a values of $\alpha = \mu$, the first equation of section c is in good approximation equal to 0, since $\dot{r} << c$, meaning that only the drag term is still present. If a lot of such particles would exist in the inner system, a lot of solar light would be reflected or absorbed in various ways and we would see the Sun as blurry. This is not the case; we see it has a bright, sharp light source and thus little of these particles are in between the Earth and the Sun (possibly since the solar radiation pressure drag has lead to absorption of such particles in the Sun)

3 Exercise 3

We look at the movement of a satellite around the Earth. From observations of the movements of the satellite, we have derived that the orbit of the satellite has a period of $T = 1.85$ hours, or $T = 6660s$ en hat on a given time T_0 the radial and normal components of the velocity of the satellite are given by $\dot{r} = -2481m/s$ and $r\dot{Theta} = 6572.6m/s$. Furthermore, it is known that the satellite has a low eccentricity; $e < 0.3$.

a.

It is asked to derive expressions for the elliptical orbit from the general equations. The general equations are

$$H = r^2\dot{\theta}$$
$$r = \frac{H^2/\mu}{1 + e\cos(\theta)} \tag{10}$$

From this we can see:

$$r\dot{\theta} = \frac{H}{r} = \frac{H}{\frac{H^2/\mu}{1+e\cos(\theta)}} \tag{11}$$

Simplification gives:

$$r\dot{\theta} = \frac{\mu}{H}(1 + e\cos\theta) \tag{12}$$

Taking the derivative of r gives:

$$\dot{r} = \frac{H^2}{\mu}(e\sin(\theta))\frac{\dot{\theta}}{(1 + e\cos(\theta))} \tag{13}$$

Now since $\dot{\theta} = \frac{H}{r^2}$, we have:

$$\dot{r} = \frac{H^2}{\mu}(e\sin(\theta))\frac{1}{(1 + e\cos(\theta))}\frac{1}{\frac{(H^4/\mu^2)}{(1+e\cos(\theta))^2}} \tag{14}$$

This simplifies to:

$$\dot{r} = \frac{\mu}{H}e\sin(\theta) \tag{15}$$

b.

From Kepler's Third Law, we have

$$a = (\frac{T^2\mu}{4\pi^2})^{1/3} \tag{16}$$

Filling in the values gives $a = 7.651 \cdot 10^6$ m. We also know that $H = \sqrt{p\mu} = \sqrt{a(1-e^2)\mu}$. Together, these equations can be solved iteratively (or with a MatLab equation solver), but I did not manage to obtain values.

c.

With the eccentricity obtained, we can not that the perihelion is given by $r_p = a - ae = a(1-e)$. To compute the height above surface level, the formula $h_p = a(1-e) - R_e$ is used. Similarly, for the aphelion we have $r_a = a + ae = a(1+e)$ and $h_a = a(1+e) + R_p$.

d.

To get the parameter $t_0 - \tau$, we first introduce the eccentric anomaly E in the following formula:

$$r\cos(\theta) = a\cos(E) - ae \tag{17}$$

Alternatively, we have the expression

$$r\sin(\theta) = a\sqrt{1-e^2}\sin(E) \tag{18}$$

Summing these formulae squared then gives:

$$r = a(1 - e\cos(E)) \tag{19}$$

This is positive since the negative value has no meaning. Differentiation to time gives:

$$\dot{r} = ae\dot{E}\sin(E) \tag{20}$$

We also have still the relation

$$r = \frac{a(1-e^2)}{1+e\cos(\theta)} \tag{21}$$

From this, we can see that:

$$\dot{r} = \frac{\mu e\sin(\theta)}{\sqrt{\mu a(1-e^2)}} \tag{22}$$

And thus:

$$\frac{\mu e\sin(\theta)}{\sqrt{\mu a(1-e^2)}} = ae\dot{E}\sin(E) \tag{23}$$

Substituting $r\sin(\theta) = a\sqrt{1-e^2}\sin(E)$ gives:

$$\frac{\mu ea\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}r} = ae\dot{E} \tag{24}$$

Since $r = a(e - e\cos(E))$, this leads to:

$$\frac{\mu\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}(1-e\cos(E))} = a\dot{E} \tag{25}$$

Which gives:

$$\frac{\mu}{\sqrt{\mu a}} = a\dot{E}(1 - e\cos(E)) \tag{26}$$

Integration gives:

$$t - \tau = \sqrt{a^3}\mu(E - e\sin E) \tag{27}$$

Also, we have the transformation relation:

$$\tan(\theta/2) = \sqrt{\frac{1+e}{1-e}}\tan(\frac{E}{2}) \tag{28}$$

From this, the value of E can be obtained (if indeed you get values for e and θ).

4 Exercise 4

a.

The following terms need to be explained:

- Equatorial plane: Plane through the equator (of the Earth), perpendicular to the rotation axis and equidistant to both poles, extended into infinity.
- Meridian: Half of an imaginary great circle parallel to the Greenwich Meridian, with start and end points on the North and South Pole.

- Geographical Distance: distance on a sphere, so an arc distance. It is a distance measured on the surface of the Earth. Geographical length: length of an arc parallel to the equator on a sphere's surface.
- Geocentric Width: Length of an arc along a meridian on the celestial sphere, with the Earth as a center point.
- Standard Ellipsoid: Mathematical figure representing the approximation for the shape of a celestial body.
- Geodetic width: shortest distance between two points on a sphere with the same distance to the equator.

b.

The following terms need to be explained:

- Celestial sphere: abstract sphere with an arbitrary large radius and its origin at the Earth's center (or another celestial body).
- Hour circle: The great circle through both celestial poles and the body of interest.
- Declination: Angular distance between a body and its celestial equator.
- Right Ascension: Angular distance between a body and the vernal equinox.
- Ecliptic: Plane on which a celestial body moves on its path around another celestial body.
- Obliquity of the ecliptic: angle between the ecliptic and the celestial equator.
- Vernal Equinox: equinox of the Earth when the subsolar point crosses the equator northward.

c.

Due to the continuous precession of the Earth's rotation axis, the Earth's vernal equinox moves on the celestial plane. It is caused by the Sun and Moon pulling on the equatorial bulge of the Earth. The amplitude of the Sun-Moon precession is about 23 degrees, the amplitude of the Sun-Moon nutation is about 9". The Sun-Moon precession and the planetary precession do not influence the obliquity, it is the nutation which does.

d.

The following terms need to be explained:

- Sidereal Time: time system based on the Earth's rotation with respect to the stars
- Solar Time: time system based on the Sun's position in the sky
- Mean Solar Time: hour angle of the sun plus 12 hours
- Atomic Time: time system based on hyperfine transition frequency of atoms such as Caesium-123
- Universal Time: time system based on the Earth's rotation with respect to distant stars, but with additional corrections to make it closer to solar time
- Universal Time Coordinated: time system based on atomic time with leap seconds added to make it within 1 second of UT1.

5 Conclusion

Some general questions, but also a few outdated things. Some derivations were long and required much knowledge about equations and expressions. Overeal nt really representative, in my opinion.

References

[1] Some author. *Advanced tree dynamics*, volume lecture 5 of *AE2344 Some course*, page 15. Accessed: 2019-04-27.