Mathematical Modeling of Mechanical and Electrical Systems

Control Systems,

Unit 1

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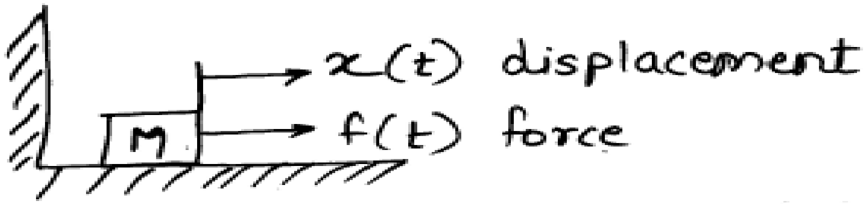
Mathematical Modeling of Mechanical & Electrical Systems

★ Mechanical Systems :-

Translational Motion :-

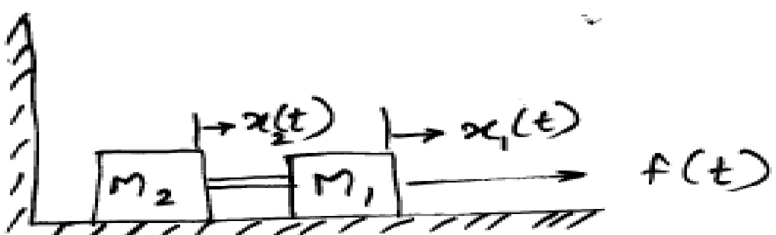
elements → mass, spring, friction.



[consider zero friction]

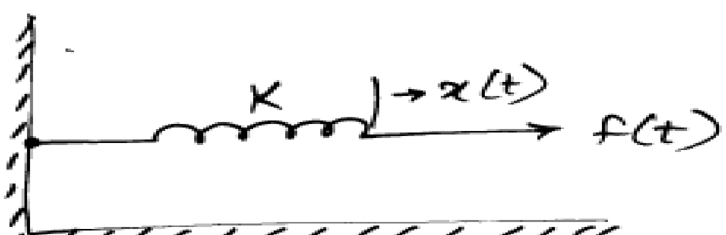
$$\boxed{f(t) = M \times accl^n = M\frac{d^2x(t)}{dt^2}}$$

Newton's Law
$F = m \cdot a$



mass can't store potential energy.
∴ consumption of force in mass is zero.

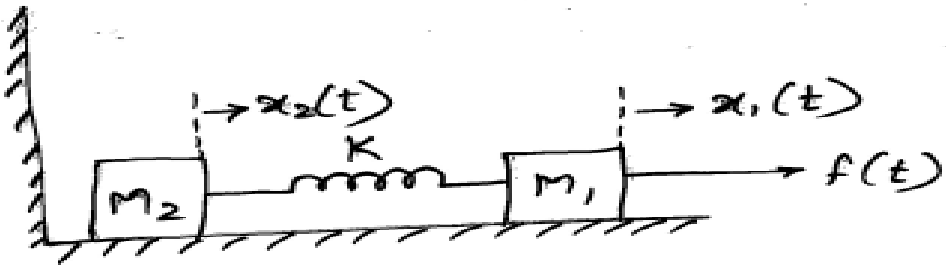
$$\therefore\ x_1(t) = x_2(t) = x(t)$$



$f(t) \propto x(t)$

$$\boxed{f(t) = K \cdot x(t)}$$

where K = spring constant



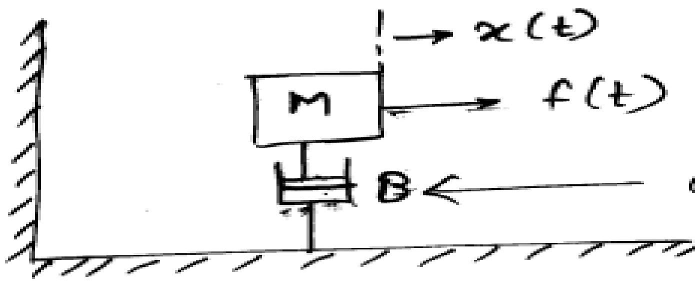
$x_1(t)$ & $x_2(t)$ are not same as spring stores some potential energy.

net displacement in spring $= x_1(t) - x_2(t)$ &

$$F_{spring} \propto [x_1(t) - x_2(t)]$$

$$\therefore \boxed{F_{spring} = K[x_1(t) - x_2(t)]}$$

Friction is represented by dash-pot or damper.



damper / dash pot

frictional force $\propto$ velocity of mass M.

$$F_{frictional} = B \cdot \frac{dx(t)}{dt}$$

B ← frictional constant.

$$\therefore \boxed{F_{frictional} = B \cdot \frac{dx(t)}{dt}}$$

if friction is betⁿ 2 Masses → $F = B\left[\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right]$

(3)

★ Electrical Systems :-

elements → R, L, C

i) resistor ⇒ $\boxed{V = I \cdot R}$ or $V = R \cdot \frac{dq}{dt}$

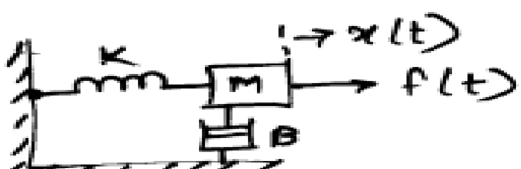
ii) Inductor ⇒ $\boxed{V = L \frac{dI}{dt}}$ or $V = L \cdot \frac{d^2q}{dt^2}$

iii) Capacitor ⇒ $\boxed{V = \frac{1}{C} \int I\, dt}$ $V = \frac{1}{C} \cdot q$

Analogy betn Mechanical & Electrical systems

Mechanical = $F = \underbrace{M \frac{d^2x(t)}{dt^2}}_{\text{mass}} + \underbrace{B \frac{dx(t)}{dt}}_{\text{friction}} + \underbrace{K \cdot x(t)}_{\text{spring}}$

Electrical = $V = \underbrace{L \frac{d^2q(t)}{dt^2}}_{\text{inductor}} + \underbrace{R \cdot \frac{dq(t)}{dt}}_{\text{resistor}} + \underbrace{\frac{1}{C} \cdot q(t)}_{\text{capacitor}}$



force voltage analogy

from above 2 eqns

$F \Leftrightarrow V$, $M \Leftrightarrow L$, $B \Leftrightarrow R$, $K \Leftrightarrow \frac{1}{C}$, $x \Leftrightarrow q$

(4)

For Electrical system,
we can write

$$I = \frac{V}{R}, \quad I = \frac{1}{L}\int V\,dt, \quad I = C\frac{dV}{dt}$$

we can write eqⁿs in terms of flux $\phi(t)$

$$V = \frac{d\phi}{dt}.$$

$$\therefore \quad I = \frac{1}{R}\cdot\frac{d\phi}{dt}$$

$$I = \frac{1}{L}\phi(t)$$

$$I = C\cdot\frac{d^2\phi(t)}{dt^2}$$

$$\therefore \quad I = C\cdot\frac{d^2\phi(t)}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\cdot\phi(t)$$

analogy with Mechanical system

$$\boxed{F \Leftrightarrow I, \quad M \Leftrightarrow C, \quad B \Leftrightarrow \frac{1}{R}, \quad K \Leftrightarrow \frac{1}{L}, \quad x \Leftrightarrow \phi}$$

force current analogy

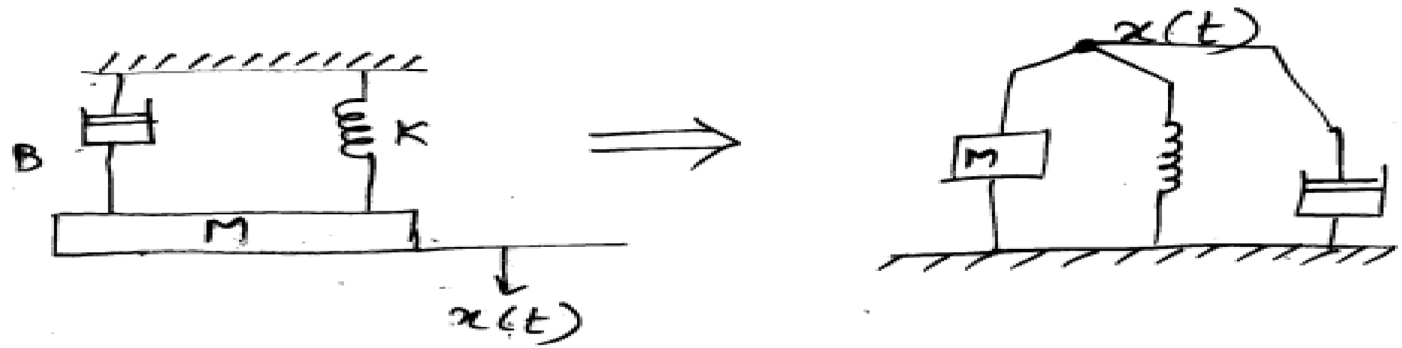
Force Voltage Analogy — Direct Analogy (loop analysis)
Force Current Analogy — Inverse Analogy (Node analysis)

Equivalent Mechanical system :-

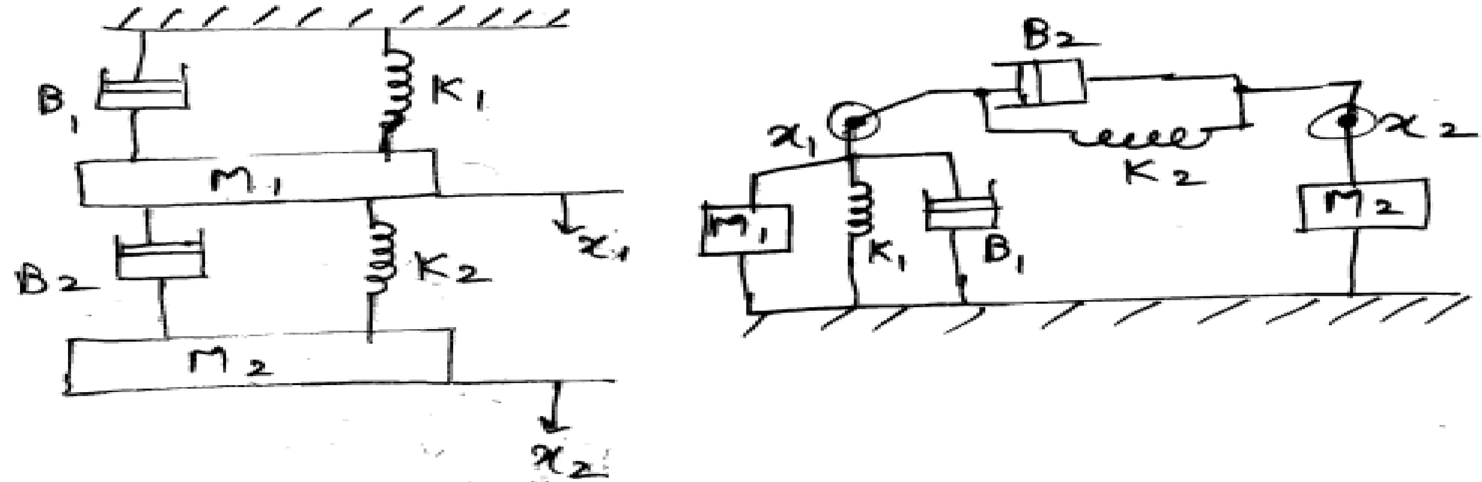
① represent each displacement by a separate node
② all elements under the influence of same displacement are in parallel.
③ elements causing <u>same change in displacement</u> are connected in parallel betⁿ respective nodes.

examples :-

①



②

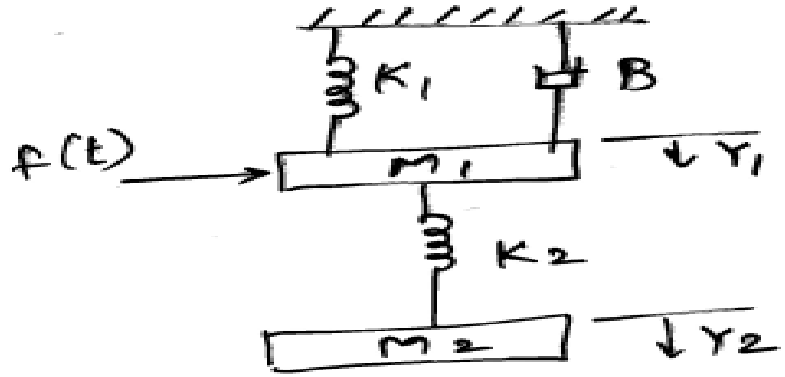


⑥ <u>Steps to derive an analogous system:-</u>

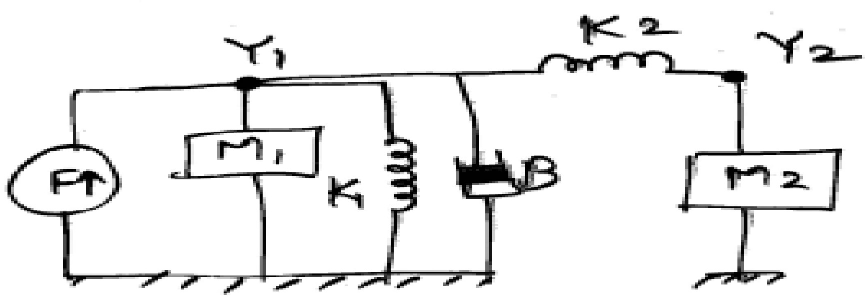
1. Draw equivalent mechanical system with no. of nodes = no. of displacements
2. Write equilibrium equations. At each node, algebraic sum of all the forces acting at the node is zero.
3. Use F-V or F-I analogy & rewrithe the eqns.
4. Simulate the eqns using loop (F-V) method or node (F-I) method.

Pb 7

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown.



<u>Soln</u> :- equivalent mechanical system:-



eqns :-

$$F = M_1 \frac{d^2 Y_1}{dt^2} + K_1 Y_1 + B \frac{dY_1}{dt} + K_2 (Y_1 - Y_2)$$

$$0 = K_2 (Y_2 - Y_1) + M_2 \frac{d^2 Y_2}{dt^2}$$

Taking Laplace Transform,

$$F(s) = M_1 s^2 Y_1(s) + K_1 Y_1(s) + B \cdot s \cdot Y_1(s) + K_2 (Y_1(s) - Y_2(s))$$

$$0 = K_2 (Y_2(s) - Y_1(s)) + M_2 \cdot s^2 \cdot Y_2(s)$$

$$F(s) = Y_1(s)\left[M_1 s^2 + K_1 + B\cdot s + K_2\right] - K_2 Y_2(s)$$

$$\therefore \quad F(s) + K_2 Y_2(s) = Y_1(s)\left[M_1 s^2 + K_1 + Bs + K_2\right]$$

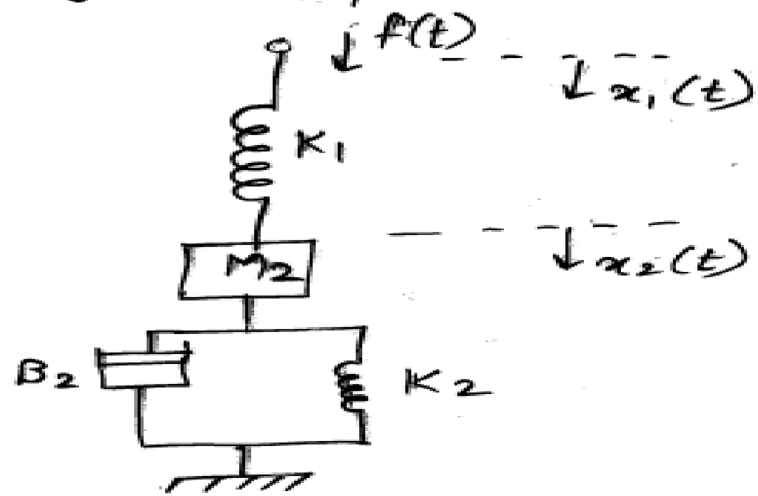
$$\therefore \quad Y_1(s) = \frac{F(s) + K_2 Y_2(s)}{M_1 s^2 + B\cdot s + K_1 + K_2}$$

substitute for $Y_1(s)$ in 2^{nd} eq^n.

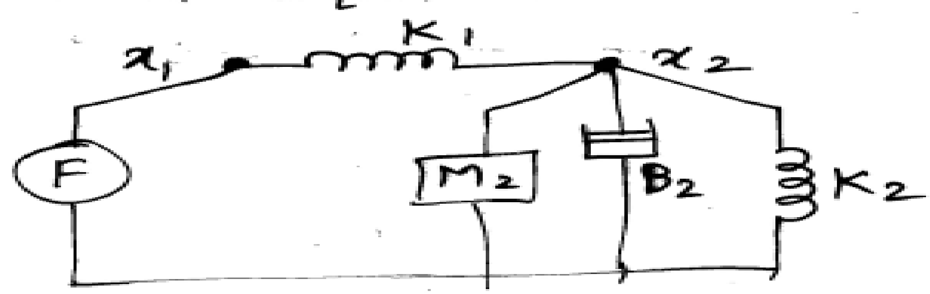
& fing

$$\boxed{\frac{Y_2(s)}{F(s)} = \frac{K_2}{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2}}$$

(9) Pb Draw equivalent mechanical system and analogous systems based on F-V and F-I methods for the given system.



Sol^n:- Mechanical Equivalent -



eq^{n}s:- at x_1 node,

$$F = K_1 (x_1 - x_2) \quad \text{--- (1)}$$

at node x_2,

$$0 = K_1 (x_2 - x_1) + M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{d x_2}{dt} + K_2 x_2 \quad \text{--- (2)}$$

3) Force- Voltage Analogy,

$F \rightarrow V,\ M \rightarrow L,\ B \rightarrow R,\ K \rightarrow \frac{1}{C},\ x \rightarrow q,\ \frac{dx}{dt} \rightarrow i$

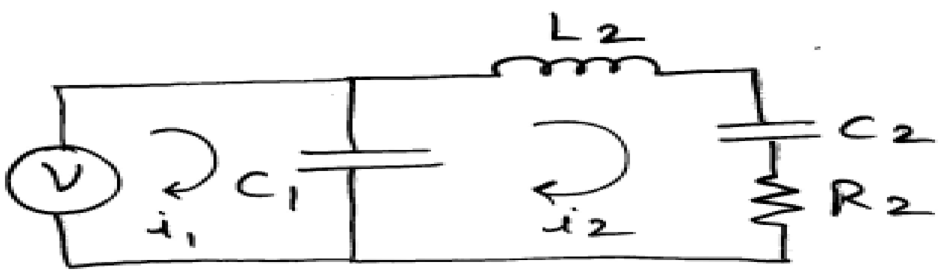
eq^{n}s ① & ② can be written as

$$V(t) = \frac{1}{C_1}(q_1 - q_2) = \frac{1}{C_1}\int (i_1 - i_2)\,dt \quad \text{③}$$

$$0 = \frac{1}{C_1}\int (i_2 - i_1)\,dt + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2\,dt + R_2 i_2 \quad \text{④}$$

from eq^n ③ & ④, n/w can be formed as

F-V
- loop
analysis



Force- current analogy,

$F \rightarrow I,\ M \rightarrow C,\ B \rightarrow \frac{1}{R},\ K \rightarrow \frac{1}{L},\ x \rightarrow \phi,\ \frac{dx}{dt} \rightarrow V$

$$I(t) = \frac{1}{L_1}\int (V_1 - V_2)\,dt$$

$$0 = \frac{1}{L_1}\int (V_2 - V_1)\,dt + C_2\frac{dV_2}{dt} + \frac{V_2}{R_2} + \frac{1}{L_2}\int V_2\,dt$$