Exercises 3.3

1. Assuming that *sqrt* takes about ten times longer than each of the other operations in the innermost loop of *BruteForceClosestPoints*, which are assumed to take the same amount of time, estimate how much faster will the algorithm run after the improvement discussed in Section 3.3.

2. Can you design a more efficient algorithm than the one based on the brute-force strategy to solve the closest-pair problem for n points $x_1, ..., x_n$ on the real line?

3. Let $x_1 < x_2 < ... < x_n$ be real numbers representing coordinates of n villages located along a straight road. A post office needs to be built in one of these villages.

   a. Design an efficient algorithm to find the post-office location minimizing the average distance between the villages and the post office.

   b. Design an efficient algorithm to find the post-office location minimizing the maximum distance from a village to the post office.

4. a.$\triangleright$ There are several alternative ways to define a distance between two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. In particular, the ***Manhattan distance*** is defined as

$$d_M(p_1,\ p_2) = |x_1 - x_2| + |y_1 - y_2|.$$

   Prove that d_M satisfies the following axioms that every distance function must satisfy:

   (i) $d_M(p_1,\ p_2) \geq 0$ for any two points p_1 and p_2 and $d_M(p_1,\ p_2) = 0$ if and only if $p_1 = p_2$;

   (ii) $d_M(p_1, p_2) = d_M(p_2, p_1)$;

   (iii) $d_M(p_1, p_2) \leq d_M(p_1, p_3) + d_M(p_3, p_2)$ for any p_1, p_2, and p_3.

   b. Sketch all the points in the x, y coordinate plane whose Manhattan distance to the origin (0,0) is equal to 1. Do the same for the Euclidean distance.

   c.$\triangleright$ True or false: A solution to the closest-pair problem does not depend on which of the two metrics—d_E (Euclidean) or d_M (Manhattan)—is used.

5. The ***Hamming distance*** between two strings of equal length is defined as the number of positions at which the corresponding symbols are different. It is named after Richard Hamming (1915–1998), a prominent American scientist and engineer, who introduced it in his seminal paper on error-detecting and error-correcting codes.

(a) Does the Hamming distance satisfy the three axioms of a distance metric listed in Problem 4?

(b) What is the time efficiency class of the brute-force algorithm for the closest-pair problem if the points in question are strings of m symbols long and the distance between two of them is measured by the Hamming distance?

6. $\triangleright$ *Odd pie fight* There are $n \geq 3$ people positioned in a field (Euclidean plane) so that each has a unique nearest neighbor. Each person has a cream pie. At a signal, everybody hurls his or her pie at the nearest neighbor. Assuming that n is odd and that nobody can miss his or her target, true or false: There always remains at least one person not hit by a pie? [Car79].

7. The closest-pair problem can be posed in k-dimensional space in which the Euclidean distance between two points $p'(x'_1, ..., x'_k)$ and $p''(x''_1, ..., x''_k)$ is defined as

$$d(p', p'') = \sqrt{\sum_{s=1}^{k} (x'_s - x''_s)^2}.$$

What is the time-efficiency class of the brute-force algorithm for the k-dimensional closest-pair problem?

8. Find the convex hulls of the following sets and identify their extreme points (if they have any).
a. a line segment
b. a square
c. the boundary of a square
d. a straight line

9. Design a linear-time algorithm to determine two extreme points of the convex hull of a given set of $n > 1$ points in the plane.

10. What modification needs to be made in the brute-force algorithm for the convex-hull problem to handle more than two points on the same straight line?

11. Write a program implementing the brute-force algorithm for the convex-hull problem.

12. Consider the following small instance of the linear programming problem:

$$\begin{array}{rl} \text{maximize} & 3x + 5y \\ \text{subject to} & x + y \leq 4 \\ & x + 3y \leq 6 \\ & x \geq 0, \;\; y \geq 0 \end{array}$$

a. Sketch, in the Cartesian plane, the problem's ***feasible region*** defined as the set of points satisfying all the problem's constraints.

b. Identify the region's extreme points.

c. Solve this optimization problem by using the following theorem: A linear programming problem with a nonempty bounded feasible region always has a solution, which can be found at one of the extreme points of its feasible region.

Hints to Exercises 3.3

1. You may want to consider two versions of the answer: without taking into account the comparison and assignments in the algorithm's innermost loop and with them.

2. Sorting n real numbers can be done in $O(n \log n)$ time.

3. a. Solving the problem for $n = 2$ and $n = 3$ should lead you to the critical insight.

   b. Where would you put the post office if it would not have to be at one of the village locations?

4. a. Check requirements (i)–(iii) by using basic properties of absolute values.

   b. For the Manhattan distance, the points in question are defined by the equation $|x - 0| + |y - 0| = 1$. You can start by sketching the points in the positive quadrant of the coordinate system (i.e., the points for which $x, y \geq 0$) and then sketch the rest by using the symmetries.

   c. The assertion is false. You can choose, say, $p_1(0, 0)$ and $p_2(1, 0)$ and find p_3 to complete a counterexample.

5. a. Prove that the Hamming distance does satisfy the three axioms of a distance metric.

   b. Your answer should include two parameters.

6. True; prove it by mathematical induction.

7. Your answer should be a function of two parameters: n and k. A special case of this problem (for $k = 2$) was solved in the text.

8. Review the examples given in the section.

9. Some of the extreme points of a convex hull are easier to find than others.

10. If there are other points of a given set on the straight line through p_i and p_j, which of all these points need to be preserved for further processing?

11. Your program should work for any set of n distinct points, including sets with many colinear points.

12. a. The set of points satisfying inequality $ax + by \leq c$ is the half-plane of the points on one side of the straight line $ax + by = c$, including all the points on the line itself. Sketch such a half-plane for each of the inequalities and find their intersection.

    b. The extreme points are the vertices of the polygon obtained in part (a).

c. Compute and compare the values of the objective function at the extreme points.