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Minimum Spanning Trees (Sometimes Greed Pays Off . . .)

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Once upon a time in a remote island kingdom there lived the Algo clan. The clansmen lived scattered all over the seven islands of the kingdom shown below.

The seven islands and the mainland were connected by several ferries allowing visits and excursions to the mainland. The ferry connections are plotted in dashed lines on the map (Fig. 33.1). The numbers indicate the length of the ferry connection in meters.

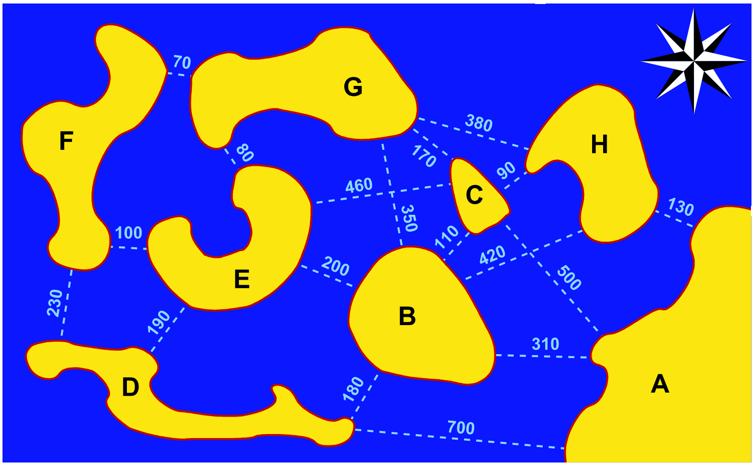

Fig. 33.1. The island kingdom of the Algos comprised the seven islands B, C, . . . , H. The dashed lines depict ferry connections. The numbers indicate the lengths of the ferry connections in meters. For example, one ferry cruised between the mainland A and the island D, covering a distance of 700 m one-way

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The Bridge Project of the Algos

Once in a while, during stormy weather, ferries capsized. Therefore, the Algos decided to replace certain ferry connections by bridges.

In the first year, one of the seven islands was going to be connected to the mainland by a new bridge. The Algos built a bridge of length 130 m between A and H, since all connections from A to other islands are longer.

During the second year, they wanted to connect another island to the mainland. Therefore, the construction of a bridge between A or H and another island was taken into consideration. The Algos built the shortest possible bridge of length 90 m between H and C.

In the third year, the construction of another bridge (starting from A, C, or H) should connect a third island to the mainland. This time, the shortest possible option was the bridge of length 110 m connecting B and C.

The status quo of the ongoing building project is shown in Fig. 33.2. As you can see, the Algos were not yet ready with their project.

In the following years, the bridge of length 170 m between C and G, then the bridge of length 70 m connecting F and G, next the bridge of length 80 m from G to E, and finally the bridge of length 180 m connecting B and D were built.

Finally, after seven years, all islands were connected to each other and to the mainland by bridges. The bridge project was therefore completed.

The final bridge system of the Algos is shown in Fig. 33.3.

The Algos were delighted. The time and effort for building the bridges had been enormous, but they were convinced that they had avoided the con-

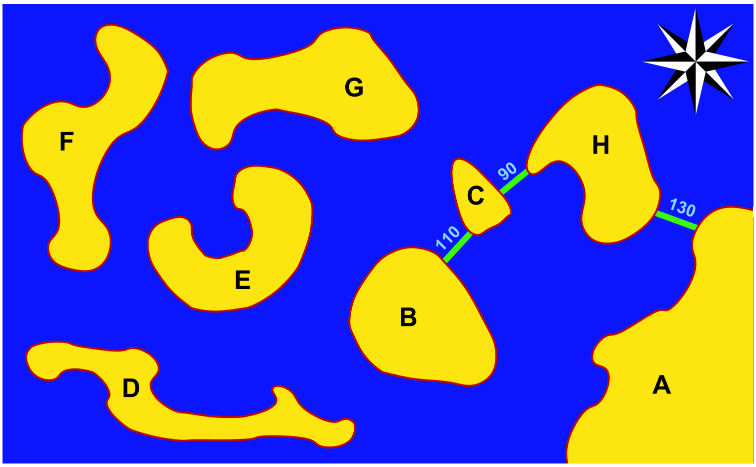

Fig. 33.2. The status quo of the building project after three years. The islands B, C, and H are already connected to the mainland

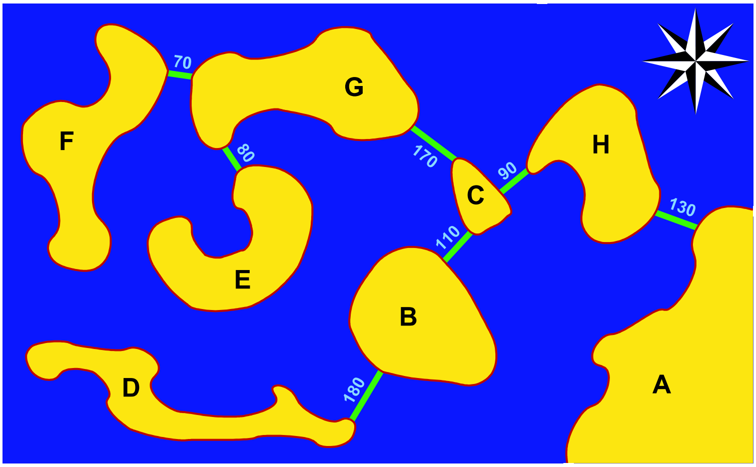

Fig. 33.3. The final bridge system of the Algos

struction of excessively long bridges as far as possible. The total length of all bridges added up, as you can easily check, to exactly 830 m.

Building Bridges After the Hurricane

Shortly after the completion of the last bridge, a terrible hurricane swept over the kingdom and completely destroyed the precious bridges. After recovering from the shock, the Algos decided to construct a new bridge system. Again, the bridges were supposed to connect all islands to each other and to the mainland.

Due to the hurricane, there was a lack of building material. It was agreed to first build a shortest possible bridge. Therefore, in the first year after the hurricane, the bridge of length 70 m connecting F and G was built.

Also in the second year building material was scarce, so that the next longer bridge of length 80 m from E to G was built.

According to this strategy, in the third year the bridge of length 90 m between C and H was built. After the construction of these three bridges the three islands E, F, and G and the two islands C and H were connected to each other (see Fig. 33.4).

In the fourth year, the shortest connection that had not been realized yet was the one of length 100 m between E and F. Because both islands were already connected to each other via G, they instead built the bridge of length 110 m connecting B and C.

During the fifth year, the bridge of length 130 m from A to H was added, then the bridge of length 170 m connecting C and G, and finally, in the seventh

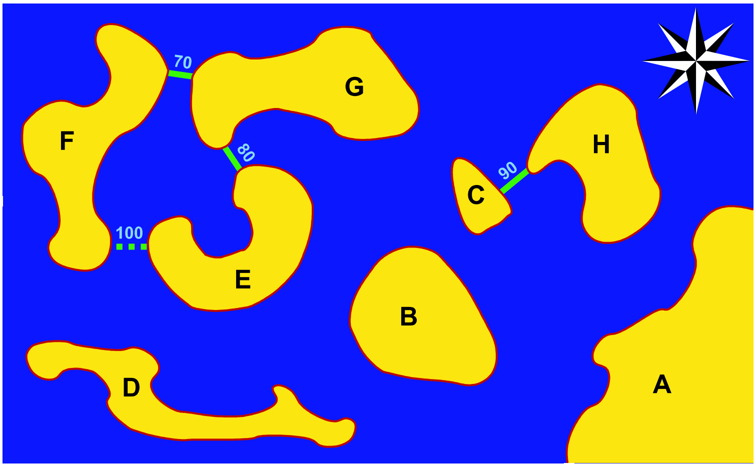

Fig. 33.4. The status quo of the second bridge project three years after the hurricane. The bridge system consists of three bridges of lengths 70 m, 80 m, and 90 m. The shortest connection that has not been realized yet is the one between E and F (marked by a dashed line on the map). In the fourth year, however, the Algos decided against building this bridge, because these two islands were already connected to each other via G

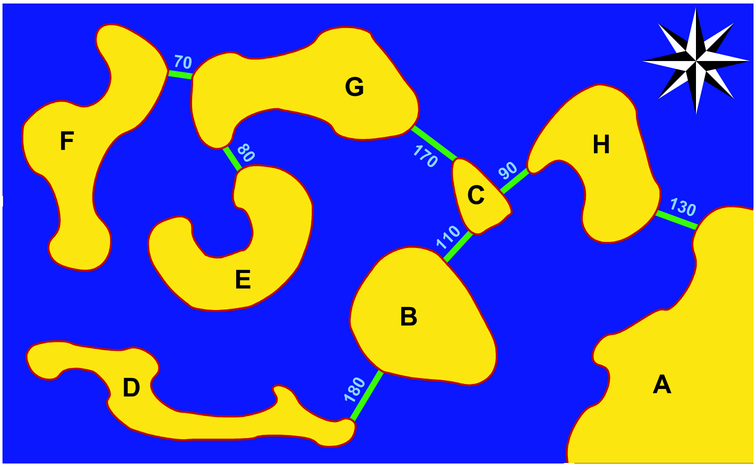

Fig. 33.5. The second bridge system of the Algos

year, the bridge of length 180 m from B to D. The new bridge system of the Algos is depicted in Fig. 33.5.

With great astonishment the Algos realized that despite their new strategy in building bridges, they had ended up with the same bridge system of total length 830 m (compare Fig. 33.3). This confirmed the Algos' belief that they had found the optimal bridge system for their islands. And unless no second hurricane has made trouble ever since, the Algos stroll happily and proudly over their bridges till today.

The Algorithms of Prim and Kruskal

Now you probably wonder, whether the Algos were rightly proud of their bridge system. Perhaps, there is still a better, thus shorter bridge system? Using trial and error you can find out that any other bridge system that connects the seven islands and the mainland to each other is longer than 830 m.

A bridge system of minimum total length that connects several places (here mainland A and islands B to H) to each other is called “minimum spanning tree.” The problem of finding a minimum spanning tree has many different practical applications aside from building bridges. For example, it arises when planning the sewage system of a new housing estate. The goal is to connect all estates to the sewage system at the lowest possible price. Other applications are the design of computer chips and the planning of traffic or communication networks (telephone, TV, internet, etc.).

Both strategies of the Algos exemplify well-known algorithms for solving the problem. The first strategy is known as “Prim’s Algorithm.” This algorithm connects places to the mainland one after another. In each step the shortest possible bridge is being built.

Prim’s Algorithm

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1   Choose a special place (mainland) and call it reachable.
2   Initially all other places are not reachable.
3   Repeat the following steps until all places are reachable:
    Build the shortest bridge between two places, of which one is reachable
    and one is not reachable, and call the one that had not been reachable
    yet reachable.
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The second strategy described above (after the hurricane) is known as “Kruskal’s Algorithm.” This algorithm builds in each step the shortest bridge connecting two places that have not been connected before. It differs from Prim’s Algorithm in the way that it not only takes bridges establishing a connection to the mainland into consideration, but allows for arbitrary bridges connecting two places that have not been connected yet.

Kruskal’s Algorithm

Repeat the following step until all places are connected to each other through bridges:
Build the shortest bridge that connects two places that are currently not yet reachable from each other.

The algorithms of Prim and Kruskal both compute a minimum spanning tree and they have something in common. Recall the strategy of the Algos. In both strategies, the Algos planned fairly nearsightedly from year to year. Every year they chose the best (i.e., shortest) bridge coming into question at that time. In doing so they did not take into consideration the effects of their decisions for the future of their construction project. They acted “greedily.” Algorithms with this property are also called “greedy,” because, at every step, they make the best possible choice under the current circumstances. As you can see, sometimes greediness pays off.

But such a greedy approach is not always successful. Imagine, for example, you were asked to connect only island D to the mainland A by a bridge system of minimum total length. The greedily designed bridge system of the Algos connects D to A via the islands H, C, and B. The total length of bridges along this path is 510 m. You can certainly find a shorter connection from mainland A to island D! If you want to learn more about how to find the shortest connection from mainland A to island D, go ahead and check Chap. 32.

In fact, the two algorithms described above have another interesting property. They always compute a solution, in which the length of the longest bridge is as small as possible. You can check this using the example of the island kingdom.

Further Reading

1. Chapter 32 (Shortest Paths)
   Not all Algos were happy with their bridges. For example, chief “Limping Leg” from island D, who regularly consulted the medicine man on the mainland, complained that the way from D to A via B, C, and H was obviously too long (510 bridge meters). One should have better built a bridge from A to B, which would have reduced the distance to 490 bridge meters. Find out in Chap. 32 how to find the shortest connection from the mainland to all bridges!
2. Chapter 40 (Travelling Salesman Problem)
   Also milkman “Whining Whey,” who had to deliver a bag of coconuts to each island day-to-day, kept complaining. In his opinion, one should have built the bridges in such a manner that they yield a shortest round trip starting from the mainland and passing by all islands. How to please the milkman is the topic of Chap. 40.
3. Chapter 3 (Fast Sorting Algorithms)
   To apply Kruskal’s Algorithm, it is advisable to first sort the possible connections according to their lengths and then process them in ascending order. You can find out how to sort fast in Chap. 3.
4. Chapter 9 (Cycles in Graphs)
   In the fourth year after the hurricane, the Algos did not build the shortest bridge connecting E to F, but instead the bridge from B to C. Because

the bridge from E to F would have connected two islands, which were already connected by several other bridges. In other words: building a bridge from E to F would have closed a cycle from E to G via F and back to E. More on cycles and how to detect them is discussed in Chap. 9.

5. T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein: *Introduction to Algorithms*. MIT Press, 2nd edition, 2001.
   How to implement Prim's and Kruskal's Algorithms on a computer, such that they find the solution as fast as possible, and many more things, can be found in this textbook, which is frequently used in courses for first-year computer science students.