Problem1

> 代码见`pro1.cpp`

```
1: -0.880333
2: -0.865684
3: -0.865474
4: -0.865474
final root: -0.865474
```

不能取$p_0 = 0$，因为$f'(x) = -3x^2 - sin(x), f'(0) = 0$，代入迭代公式中分母为0，不可取。

Problem2

1. 由**Newton-Raphson方法**，有：

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{b - \frac{1}{x_k}}{\frac{1}{x_k^2}} = 2x_k - bx_k^2$$

$$\text{于是}|\epsilon_{k+1}| = |\frac{\frac{1}{b} - x_{k+1}}{\frac{1}{b}}| = |\frac{(\frac{1}{b} - x_k)^2}{\frac{1}{b^2}}| = \epsilon_k^2$$

2. 当$0 < x_0 < \frac{2}{b}$时，有$|\epsilon_0| < 1$,而：

$$\epsilon_0 = (\epsilon_1)^{\frac{1}{2}} = (\epsilon_2)^{\frac{1}{2^2}} = \ldots = (\epsilon_k)^{\frac{1}{2^k}}$$

$$\lim_{k \to +\infty} \epsilon_k = \lim_{k \to +\infty} (\epsilon_0)^{\frac{1}{2^k}} = 0$$

$$\text{故最后会收敛到}\frac{1}{b}$$

Problem3

A.

$$J(x_1, x_2, x_3) = \begin{pmatrix} 3 & x_3 sin(x_2 x_3) & x_2 sin(x_2 x_3) \\ 8x_1 & -1250x_2 + 2 & 0 \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{pmatrix}$$

$$x^{(0)} = 0 \Rightarrow J(x^{(0)}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

$$\text{解} \quad J^{x(0)} y^{(0)} = -F(x^{(0)})$$

$$\text{得到} \quad y^{(0)} = (\frac{1}{2}, \frac{1}{2}, \frac{-1\pi}{6})^t$$

$$\text{则} \quad x^{(1)} = x^{(0)} + y^{(0)} = (\frac{1}{2}, \frac{1}{2}, \frac{-1\pi}{6})^t$$

$$\text{同样解} \quad J^{x(1)} y^{(1)} = -F(x^{(1)})$$

$$x^{(2)} = y^{(1)} + x^{(1)}$$

$$\text{最终得到} \quad x^{(2)} = (0.50016669, 0.25080364, -0.51738743)$$

B.

$$J(x_1,x_2,x_3)=\begin{pmatrix}2x_1 & 1 & 0\\ 1 & 2x_2 & 0\\ 1 & 1 & 1\end{pmatrix}$$

$$x^{(0)}=0\Rightarrow J(x^{(0)})=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 1 & 1 & 1\end{pmatrix}$$

$$J^{x(0)}y^{(0)}=-F(x^{(0)})\Rightarrow y^{(0)}=(5,37,-39)^t$$

$$x^{(1)}=x^{(0)}+y^{(0)}\Rightarrow x^{(1)}=(5,37,-39)^t$$

$$\text{同样解}\quad J^{x(1)}y^{(1)}=-F(x^{(1)})$$

$$x^{(2)}=y^{(1)}+x^{(1)}$$

$$\text{最终得到}\quad x^{(2)}=(4.35087719,18.49122807,-19.84210526)$$

Problem4

> 这俩题的程序见压缩包里的Python文件~

代码的思路主要参考课本P658,659的伪代码，其中矩阵的运算采用Python中常用的模块`numpy`操作。$\nabla g(x_1,x_2,x_3)$的计算较为繁琐，展开再合并后可以用$2J'(x)F(x)$来计算，能较为简便的得到结果。最后通过$x^{(k+1)}=x^k-\alpha\nabla g(x_1,x_2,x_3)$不断迭代得到结果。

(a)

$$f_1(x_1,x_2,x_3)=15x_1+x_2^2-4x_3-13$$

$$f_2(x_1,x_2,x_3)=x_1^2+10x_2-x_3-11$$

$$f_3(x_1,x_2,x_3)=x_2^3-25x_3+22$$

$$g(x_1,x_2,x_3)=\sum_{i=1}^{n}f_i^2(x_1,x_2,x_3)$$

$$J(x_1,x_2,x_3)=\begin{pmatrix}15 & 2x_2 & -4\\ 2x_1 & 10 & -1\\ 0 & 3x_2^2 & -25\end{pmatrix}$$

$$\nabla g(x_1,x_2,x_3)=2J'(x_1,x_2,x_3)\begin{pmatrix}f_1(x_1,x_2,x_3)\\ f_2(x_1,x_2,x_3)\\ f_3(x_1,x_2,x_3)\end{pmatrix}$$

$$x^{(k+1)}=x^k-\alpha\nabla g(x_1,x_2,x_3)$$

以上是数学部分

结果展示

```
x1 =  1.043465654410717
x2 =  1.063646526417548
x3 =  0.9256954364554192
```

(b)

$$f_1(x_1,x_2,x_3)=10x_1-2x_2^2+x_2-2x_3-5$$
$$f_2(x_1,x_2,x_3)=8x_2^2+x_3^2-9$$
$$f_3(x_1,x_2,x_3)=8x_2x_3+4$$

结果展示

```
x1 =  0.8996604615655656
x2 =  -0.9791330381049683
x3 =  0.5364333126942094
```

Problem5

(a)

Theorem (Fixed Point Theorem)

Let $\mathbf{D}=\{(x_1,x_2,\ldots,x_n)^t | a_i \le x_i \le b_i, \text{for each} i=1,2,\ldots,n\}$ *for some collection of constants* $a_1,a_2,\ldots,a_n$ *and* $b_1,b_2,\ldots,b_n$*. Suppose* $\mathbf{G}$ *is a continuous function from* $\mathbf{D}\in\mathbb{R}^n$ *into* $\mathbb{R}^n$ *with the property that* $\mathbf{G}(\mathbf{x})\in\mathbf{D}$ *whenever* $\mathbf{x}\in\mathbf{D}$*. Then* $\mathbf{G}$ *has a fixed point in* $\mathbf{D}$*.*
Moreover, suppose that all the component functions of $\mathbf{G}$ *have continuous partial derivatives and a constant* $K<1$ *exists with*

$$\left|\frac{\partial g_i(\mathbf{x})}{\partial x_j}\right| \le \frac{K}{n}, \quad whenever \ \ \mathbf{x}\in\mathbf{D},$$

for each $j=1,2,\ldots,n$ *and each component function* g_i*. Then the sequence* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *defined by an arbitrarily selected* $\mathbf{x}^{(0)}$ *in* $\mathbf{D}$ *and generated by*

$$\mathbf{x}^{(k)}=\mathbf{G}(\mathbf{x}^{(k-1)}), \quad for\ each \ \ k\ge 1$$

converges to the unique fixed point $\mathbf{p}\in\mathbf{D}$ *and*

$$||\mathbf{x}^{(k)}-\mathbf{p}||_\infty \le \frac{K^k}{1-K}||\mathbf{x}^{(1)}-\mathbf{x}^{(0)}||_\infty$$

$$J(x_1,x_2)=\begin{pmatrix}\frac{x_1}{5} & \frac{x_2}{5}\\ \frac{1+x_2^2}{10} & \frac{x_1x_2}{5}\end{pmatrix}$$

$$\text{取}K=0.95\text{，则：}$$

$$\text{当}x_i\in D\text{时，}|\frac{\partial g_i(x)}{\partial x_j}|\le|\frac{\partial g_2(x)}{\partial x_2}|=|\frac{x_1x_2}{5}|\le\frac{9}{20}<\frac{0.95}{2}$$

$$\text{由上述定理，在}D\text{上不动点唯一。}$$

(b)

直接代入计算:

$$x^{(0)} = [0, 1]^t$$
$$x^{(1)} = [\frac{9}{10}, \frac{8}{10}]^t$$
$$x^{(2)} = [1.045, 0.9046]^t$$