MATH4321 Game Theory
Problem Set 1

Chapter 1: Static games

Problem 1 (Location problems)

There are 2 department stores: A and B. Each of the two stores can choose a location for its new store in Kowloon. There are 4 available choices: Sham Shui Po (SSP), Mong Kok (MK), Kowloon Tong (KT) and Jordan (J). The profits made by the new stores are summarized in the following matrix:

		Store B (Player 2)			
		SSP	MK	KT	J
Store A (Player 1)	SSP	$(30,40)$	$(50,95)$	$(55,95)$	$(55,120)$
	MK	$(115,40)$	$(100,100)$	$(130,85)$	$(120,95)$
	KT	$(125,45)$	$(95,65)$	$(60,40)$	$(115,120)$
	J	$(105,50)$	$(75,75)$	$(95,95)$	$(45,50)$

Determine the final outcome using IESDS. Verify that this final outcome is the pure strategy Nash equilibrium.

Problem 2 (Discrete first-price auction)

An item is up for auction. Player 1 values the item at \$5 while player 2 values the item at \$8. Each player can bid either 0, 1, 2 or 3. If player i bids more than player j then i wins the games If both players place the same bid, then each player will win with probability 0.5. The winner of the games will win the good and pay his bid and the loser does not pay anything.

- **(a)** Express the games in normal form.
- **(b)** **(i)** Determine the final outcome using IESDS.
  (ii) Hence, determine all possible pure strategy Nash equilibrium.
- **(c)** Suppose that the player 1 values the item at \$1.5 (instead of \$5),
  (i) determine the final outcome using IESDS.
  (ii) Hence, determine all possible pure strategy Nash equilibrium.

Problem 3 (Election games)

Two candidates are engaging in an election. Each of them has three possible strategies:

1. focus on promoting the positive aspects of himself (denoted by "P");
2. promoting the positive aspects of himself and attacking another candidates at the same time (denoted by "B");
3. focus on attacking another candidates (denoted by "A").

It is given that

- If both candidates chooses the same strategies, each of them wins with equal probability.

- If one of the candidates chooses "P" and another candidate chooses "B", then the candidates who chooses "P" wins with probability 0.1.
- If one of the candidates chooses "P" and another candidate chooses "N", then the candidates who chooses "P" wins with probability 0.35.
- If one of the candidates chooses "B" and another candidate chooses "N", then the candidates who chooses "P" wins with probability 0.4.

The objective of the candidate is to choose a strategy in order to maximize the chance of winning the election. So we assume in this games that the payoff of a candidate is the probability of winning the election.

(a) Express the games in the normal form.
(b) Determine the final outcome using IESDS.

Problem 4 (Location games)

The following figure shows the route map of a subway station

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x------x------x-------x------x-------x-
1      2      3       4      5       6
(20)  (12)   (8)    (10)   (14)    (6)
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The number in the bracket are the number of tourists travelling in a particular station. Assume that the distances between any two consecutive stations are equal.

Two competing companies plan to build a hotel in one of these 6 stations. Their objectives are to attract as many tourists as possible. It is known that each tourist will choose to stay at the hotel that *is closest to* the place that he is travelling. For example, if company 1 builds the hotel (hotel 1) at station 2 and company 2 builds the hotel (hotel 2) at station 3. Then the tourists who will travel to station 1 and 2 will choose to stay at hotel 1 and the remaining tourists will stay at hotel 2. If tourists wish to travel to a particular station and the two hotels have the same distance from this station, these tourists will choose to stay at one of the two hotels with equal probability.

(a) Express the games in normal form.
(b) Simplify the games as much as possible using IESDS.
(c) Find all possible pure strategy Nash equilibrium.

Problem 5 (Roommates)

Two roommates each need to choose to clean their apartment (they share the room). Each of them can choose an amount of time $t_i \geq 0$ to clean the apartment. Suppose that they choose spend t_i and t_j units of time to clean the apartment respectively, the payoff to player i is given by

$$O_i(t_i;t_j) = \underbrace{t_j + (5 - t_j)t_i}_{benefit} - \underbrace{t_i^2}_{cost}.$$

Here, the first term $t_j + (5 - t_j)t_i$ represents the benefits of cleaning the apartment (The second function $(5 - t_j)t_i$ indicates that the more one roommate cleans (higher t_j), the less valuable is cleaning for the other roommates.). The second term t_i^2 represents the cost of cleaning the apartment.

Determine the final outcomes using IESDS using similar technique used in Example 8.

Problem 6 (Advertising games)

Company A and company B are rival pet food companies and are choosing among advertising media. They can choose to promote their products via one of the following three medias: Facebook, poster or television. Their profits under various scenarios are summarized in the following table:

		Company B		
		Facebook	Poster	Television
Company A	Facebook	$(7,7)$	$(2,4)$	$(3,3)$
	Poster	$(3,3)$	$(3,6)$	$(7,6)$
	Television	$(4,7)$	$(9,2)$	$(2,7)$

Determine all possible pure strategy Nash equilibrium in this games.

Problem 7 (Synergies)

Two division managers can invest time and effort in working on a joint investment project. Each can invest an effort e_i (any real value ≥ 0). We assume that the cost of investing effort e_i is $3e_i^2$. Given the efforts e_i, e_j chosen by the managers, the benefits to manager i is given by $(c_i + e_j)e_i + 2e_j$, where $i = 1,2, j \neq i$ and c_i is some positive constants. Thus the payoff functions of manager i can be express as

$$O_i(e_i; e_j) = (c_i + e_j)e_i + 2e_j - 3e_i^2.$$

We take $c_1 = 5$ and $c_2 = 10$. Determine the pure strategy Nash equilibrium of the games.

Problem 8 (Stag Hunt)

There are two hunters (A and B). Each of them can choose to hurt either rabbit or tiger. Each of them can hurt the rabbit for sure by himself. However, they need to cooperate in order to hunt the tiger successfully (he will fail if he hunts the tiger alone). It is given that

- The reward of hunting a rabbit is 7;
- The reward of hunting a tiger is 30 (the reward will be shared equally among two hunters)
- The reward will be -5 if the hunter fails to hunt his prey.

(a) Express the games in normal form.
(b) Identify all possible pure strategy Nash equilibria in this games.
(c) From (b), determine the Pareto-optimal equilibrium and the risk-dominant equilibrium.

Problem 9 (Driving games)

We consider the following games so called drive-on games. Two cars meet at the intersection of two roads, each of the drivers can choose to either wait (W) or go (G). If both choose to wait, both of them will receive zero payoff. If both choose to go, they will crash and each will suffer from a loss of 100. If one goes and the other waits, the one who goes will receive a payoff of 5 since he can move first and the one who waits will receive a payoff of 1.

- **(a)** Express the games in normal form.
- **(b)** **(i)** Identify all possible pure strategy Nash equilibria in this games.
  (ii) From **(b)(i)**, determine the Pareto-optimal equilibrium and the risk-dominant equilibrium.
- **(c)** Determine the mixed strategy Nash equilibrium of this games.

Problem 10 (Wars)

Country A and country B are rival nations, often at war, and both can produce and deploy nuclear weapon and poison gas on the battlefield. In a war in 2046, each country chooses to either deploy nuclear weapon or deploy poison gas or do nothing. The payoff matrix is shown below:

		Country B (Player 2)		
		Nuclear	Gas	Nothing
Country A (Player 1)	Nuclear	$(-18,-18)$	$(-10,-20)$	$(6,-30)$
	Gas	$(-20,-10)$	$(-9,-9)$	$(3,-15)$
	Nothing	$(-30,6)$	$(-15,3)$	$(0,0)$

- **(a)** **(i)** Determine if there is any Pure strategy Nash equilibrium.
  (ii) Hence, determine the Pareto-optimal equilibrium and the risk-dominate equilibrium.
- **(b)** **(Harder)** Determine if there is any mixed strategy Nash equilibrium (☺Hint: Simplify the games by identifying dominated pure strategy).

Problem 11

The following shows the payoff matrix of a two-person games

		Player 2		
		A	B	C
Player 1	A	$(0,0)$	$(7,2)$	$(1,-1)$
	B	$(2,7)$	$(6,6)$	$(0,5)$
	C	$(1,3)$	$(1,3)$	$(2,2)$

- **(a)** Find all possible pure strategy Nash equilibrium.
- **(b)** Simplify the games using IESDS. Hence, determine all possible mixed strategy Nash equilibria.

Problem 12 (Happy Hour)

Restaurant ABC and Restaurant XYZ compete for the same crowd in a district. Each can offer free drink during lunch hour, or not.

- If none of them offers free drink s, the profit of each bar will be \$35.
- If both of them offers free drinks, the profit of each bar will be reduced to \$25.
- If one offers free drinks and the other does not, the one who offers drinks will get most of the customers and gain a profit of \$70. Another restaurant will lose \$15.

(a) Express the games in normal form.
(b) Determine if there is any dominated strategies in this games.
(c) Find all possible pure strategy Nash equilibrium in this games.
(d) Find all possible mixed strategy Nash equilibrium in this games. How to interpret the mixed strategy equilibrium in reality?

Problem 13 (Coordination games)

A group of people are involved in some task that depends on efforts of each of them. Each of them can choose either “work” or “begin lazy”. If one person does not work, others need to increases effort. The payoffs are summarized in the following payoff matrix:

		Player 2	
		Work	Being lazy
Player 1	Work	$(10,10)$	$(2,16)$
	Being lazy	$(14,4)$	$(6,6)$

Determine all equilibrium (pure strategy equilibrium and mixed strategy equilibrium), if any, using indifference principle.

Problem 14 (Penalty Kick)

A kicker and a goalie confront each other in a penalty kick that will determine the outcome of a soccer games. The kicker can kick the ball left or middle or right, while the goalie can choose to jump left, stay at the middle or jump right. They need to make their decisions simultaneously.

- If the goalie jumps in the same direction as the kick, then the goalie wins and the kicker loses.
- Otherwise, the goalie loses and the kicker wins.
- The payoffs of winning and losing are 1 and -1 respectively

(a) Express the games in normal form.
(b) Show that there is no pure strategy Nash equilibrium in this games.
(c) Find all possible mixed strategy Nash equilibrium. (☺Hint: The games is quite similar to paper-scissor-rock games that is discussed in the lecture.)

Problem 15 (Contribution games)

Three players live in a building, and each can choose to contribute to fund a new mailbox. The value of having the mailbox is 8 for each player, and the value of not having it is 0. The mayor asks each player to contribute or not contribute. The cost of contributing is known to be $\frac{6}{N}$, where N is number of players who will contribute. If at least two players contribute then the mailbox will be installed. If one players or no players contribute, then the mailbox will not be installed, in which case any person who contributed will not get his money back.

(a) Express the games in the normal games.

(b) Find all possible pure strategy Nash equilibrium.

(c) Find all *symmetric* mixed strategy Nash equilibrium which each player chooses to contribute with the same probability p.
(From (c), indifference principle may be useful.)

Problem 16 (Market entry)

Three competing firms are considering entering a new market. The payoff for each firm that enters is $\frac{240}{n}$, where n is the number of firms that enter (The revenue to each firm will be lower when more firms are entering into the market). The cost of entering is 90.

(a) Find all possible pure-strategy Nash equilibria.

(b) Find the symmetric mixed strategy Nash equilibrium in which all three firms enter with the same probability.

Problem 17 (Dominated strategy and mixed strategy)

The payoff matrix of a two-person games is presented below:

		Player 2		
		L	C	R
Player 1	T	$(6,2)$	$(0,6)$	$(4,4)$
	M	$(2,12)$	$(4,3)$	$(2,5)$
	B	$(0,6)$	$(10,0)$	$(2,2)$

(a) Show that NO pure strategy dominated by any other pure strategy for any player.

(b) Show that "M" is dominated strategy for player 1 by considering the mixed strategy $\sigma_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$.

(c) Hence, show that "R" is also dominated strategy for player 2 by considering the mixed strategy $\sigma_2 = \left(\frac{5}{12}, \frac{7}{12}, 0\right)$.

(d) Determine all possible Nash equilibria (pure strategy and mixed strategy).

Harder Problems

Problem 18

We consider a two-person games. It is given that for any player i and any combination of opponent's pure strategies s_{-i} the player $i's$ best response s_i^* to s_{-i} is unique. Show that there is <u>no</u> Nash equilibrium in which one player (say player 1) uses pure strategy s_1^* and another player (say player 2) uses mixed strategy σ_2^*.

☺Hint: The uniqueness of best response implies that for any other strategy $s_i \in S_i$, we must have $O_i(s_i; s_{-i}) < O_i(s_i^*; s_{-i})$.)

Problem 19

Suppose in a n-person games, every player has a strategy $s_i^* \in S_i$ that strictly dominates all of his other strategies $s_i \in S_i$, Show that the strategic profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is the *unique* Nash equilibrium.

(☺Hint: Firstly, you need to verify that $s^*$is Nash equilibrium. To show the uniqueness, you need to show other strategic profile $s = (s_1, s_2, \ldots, s_n)$ cannot be Nash equilibrium. Recall the technique that we introduced in the lecture note.)

Problem 20

(a) We consider a two-person game with 2 pure strategies for each player. Suppose that the game has a *unique* pure-strategy Nash equilibrium $s^* = (s_1^*, s_2^*)$, show that s^* is the only survivor under IESDS.

(b) The statement in (a) becomes false if there are at least 3 pure strategies for each player. Verify this statement by constructing a suitable counter-example.

Problem 21

(a) Suppose that a strategic profile $s^* = (s_1, s_2, \ldots, s_n)$ Pareto-dominates all other strategic profile $s = (s_1, s_2, \ldots, s_n)$, show that $\vec{s}^*$ is the Nash equilibrium.

(b) Is s^* the unique Nash equilibrium? Prove it or disprove it by providing counter-example.

Problem 22

(a) We let $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ be the mixed strategy Nash equilibrium. Show that for any strategy $s_i \in S_i$ that $\sigma_i^*(s_i) > 0$, we have

$$V_i(s_i; \sigma_{-i}^*) = V_i(\sigma_i; \sigma_{-i}^*).$$

(☺Hint: Indifference principle will be useful)

(b) If there is a pure strategy $s_i^0 \in S_i$ such that $O_i(s_i^0; \sigma_{-i}^*) < O_i(\sigma_i; \sigma_{-i}^*)$ (That is, s_i^0 is not the best response to σ_{-i}^*), show that player i never choose s_i^0 under the equilibrium (i.e. $\sigma_i^*(s_i^0) = 0$).