Math 4824C - HW2

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1、 $\widehat{ACE}_{block} = \sum_{k=1}^{k=K} \frac{n_k}{n} \widehat{ACE}_K$

$= \sum_{k=1}^{k=K} \frac{n_k}{n} (\bar{Y}_{k,1} - \bar{Y}_{k,0})$

$= \sum_{k=1}^{k=K} \frac{n_k}{n} \left(\frac{\sum_{X_i=k \wedge Z_i=1} Y_i}{n_{k,1}} - \frac{\sum_{X_i=k \wedge Z_i=0} Y_i}{n_{k,0}} \right)$

$\widehat{ACE} = \bar{Y}_1 - \bar{Y}_0 = \frac{1}{n}\left(\sum_{Z_i=1} Y_i - \sum_{Z_i=0} Y_i\right)$

So we try to prove

$\sum_{k=1}^{K} \frac{n_k \sum_{X_i=k \wedge Z_i=1} Y_i}{n_{k,1}} = \sum_{Z_i=1} Y_i$

sorry, I failed...

(2) $\widehat{ACE}(\gamma, \gamma) = (\bar{Y}_1 - \gamma^T \bar{X}_1) - (\bar{Y}_0 - \gamma^T \bar{X}_0)$

$Var(\widehat{ACE}(\gamma,\gamma) | Z) = E(ACE(\gamma,\gamma)|Z)^2 - [E(ACE(\gamma,\gamma)|Z)]^2$

the latter part $= (E\bar{Y}_1 - E\bar{Y}_2)^2$ has nothing to do with γ,
so we focus on the first part:

$E(\widehat{ACE}(\gamma,\gamma)|Z)^2 = E(\bar{Y}_1 - \gamma^T \bar{X}_1)^2 + E(\bar{Y}_0 - \gamma^T \bar{X}_0)^2$

$\frac{\partial Var(\widehat{ACE}(\gamma,\gamma))}{\partial \gamma} = 2E(\bar{X}_1^2 + \bar{X}_2^2)\gamma^T - 2E(\bar{X}_1\bar{Y}_1 + \bar{X}_2\bar{Y}_2) = 0$

$\Rightarrow \gamma = \frac{E(\bar{X}_1\bar{Y}_1 + \bar{X}_2\bar{Y}_2)}{E(\bar{X}_1^2 + \bar{X}_2^2)}$

$\alpha = Y - \gamma X$

$E(Y - \alpha - \gamma X)^2$

$= E(Y^2 + (\alpha + \gamma X)^2 - 2Y(\alpha + \gamma X)) = f(\alpha, \gamma)$

$\begin{cases} \frac{\partial f(\alpha,\gamma)}{\partial \gamma} = 2X^2\gamma + (2\alpha\gamma - 2YX) = 0 \\ \frac{\partial f(\alpha,\gamma)}{\partial \alpha} = 2\alpha + 2\gamma X - 2Y = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{-Y^2 + 2YX + Y^2 X}{X} \\ \gamma = \frac{Y^2 - YX - Y^2 X}{X^2} \end{cases}$

Thus $\underset{\alpha,\gamma}{\operatorname{argmin}} E(Y - \alpha - \gamma X)^2 = E\left(Y - \frac{-Y^2 + 2YX + Y^2 X}{X} - \frac{Y^2 - YX - Y^2 X}{X}\right) = 0$

3. In observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, if $Y = \alpha + \beta Z + \gamma X + \epsilon$, write down $E(Y(1)|X)$ and ACE;

$$ACE = E(Y(1) - Y(0)|X) \xlongequal{\text{ignorability}} E(Y|X, Z=1) - E(Y|X, Z=0)$$

$$= E(\alpha + \beta + \gamma X + \varepsilon | X, Z=1) - E(\alpha + \gamma X + \varepsilon | X, Z=0)$$

$$= \beta$$

$$E(Y(1)|X) \xlongequal{\text{ignorbility}} E(Y|X, Z=1)$$

$$= E(\alpha + \beta + \gamma X + \varepsilon | X, Z=1) = \alpha + \beta + \gamma E(X)$$

4. In observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, if $0 < e(X) = P(Z = 1|X) < 1$, show that $EY_i(0) = E\dfrac{(1 - Z_i)Y_i}{1 - e(X_i)}$ and $E\dfrac{1 - Z_i}{1 - e(X_i)} = 1$.

First Equation:

$$E\frac{(1-Z_i)Y_i}{1-e(X_i)} = E\left[E\left(\frac{(1-Z_i)Y_i}{1-e(X_i)} \Big| X\right)\right] = E\left[\frac{Y_i(0)}{1-e(X_i)} \cdot P(Z=0|X) + 0 \cdot P(Z=0|X)\right]$$

$$= E\left[\frac{Y_i(0) \cdot (1 - P(Z=1|X))}{1 - P(Z=1|X)}\right] = EY_i(0)$$

Second Equation:

$$E\frac{1-Z_i}{1-e(X_i)} = E\left[E\left(\frac{1-Z_i}{1-e(X_i)} \Big| X\right)\right]$$

$$= E\left[\frac{1}{1-e(X_i)} P(Z=0|X) + 0 \cdot P(Z=1|X)\right]$$

$$= E\left[\frac{1-P(Z=1|X)}{1-P(Z=1|X)}\right] = 1$$