Problem1

(a)

$$||x-\tilde{x}||_\infty = 0.5$$
$$A\tilde{x} = (1,-1.3,1.8)^t$$
$$||A\tilde{x}-b||_\infty = 0.3$$

(b)

$$||x-\tilde{x}||_\infty = 0.9$$
$$A\tilde{x}-b = (1.27,-1.16,2.21)^t$$
$$||A\tilde{x}-b||_\infty = 0.27$$

Problem2

先有引理一：(来自课本的定理)

$$A是n\times n矩阵，则||A||_2=[\rho(A^TA)]^{\frac{1}{2}}$$

Let $\mu = [\rho(\mathbf{A}^T\mathbf{A})]^{1/2}$,

$$||A\mathbf{x}||_2^2 = \mathbf{x}^TA^TA\mathbf{x} \le \mu^2\mathbf{x}^T\mathbf{x}$$

Thus,

$$||A||_2 = \max_{||\mathbf{x}||_2=1} ||A\mathbf{x}||_2 \le \mu$$

If $\mathbf{u}$ is an eigenvector of A^TA corresponding to μ^2, then

$$\mathbf{u}^TA^TA\mathbf{u} = \mu^2\mathbf{u}^T\mathbf{u},$$

which shows that equality holds.

先证明引理二：

$$A是n\times n矩阵，则\rho(A^2)=[\rho(A)]^2$$

证明：

$$设\lambda是A的特征值，则Ax=\lambda x \Rightarrow A^2x=\lambda Ax=\lambda^2x$$
$$则\lambda^2是A^2的特征值。$$
$$那么\rho(A^2)=max(\lambda^2)=[max(\lambda)]^2=[\rho(A)]^2$$
$$Q.E.D.$$

因为A是对称矩阵，故$A=A^T$, 由引理:

$$||A||_2=[\rho(A^TA)]^{\frac{1}{2}}=[\rho(A^2)]^{\frac{1}{2}}=\rho(A)$$
$$Q.E.D.$$

综上，结论证毕

Problem3

> 代码见`Pro1.cpp`

(a)

```
输入:
2
0.03 58.9 59.2
5.31 -6.10 47.0
输出:
10 1
```

(b)

```
输入:
3
3.03 -12.1 14 -119
-3.03 12.1 -7 120
6.11 -14.2 21 -139
输出:
0 10 0.142857
```

Problem4

> 代码见文件夹里的两个Python文件

思路：通过$x^k = D^{-1}(L+U)x^{(k-1)} + D^{-1}b$迭代。

(a)

```
[ 1.25       -1.33333333  0.2       ]
[ 1.63333333 -0.98333333  0.23333333]
[ 1.55416667 -0.86666667 -0.06      ]
final:  [ 1.44776124 -0.83582088 -0.04477614]
```

(b)

```
[-2.  2.  0.]
[-1.  1. -1.]
[-1.75  1.75 -0.5 ]
final:  [-1.45408793  1.45408793 -0.72760766]
```

Problem5

> 代码见文件夹里的Python文件.

本题分别用两种迭代方法进行寻根，并采用$|x_k - x_{k+1}|_\infty < TOL$作为循环的终止的条件。

从结果来看，果然Guass-Seidel比Jacobi快一点诶。

(a)

Jacobi Method :

```
After 7 iterations,we get the final root:
[ 0.03502399 -0.23732106  0.65737656]
```

Guass-Seidel Method :

```
After 5 iterations,we get the final root:
[ 0.03535107 -0.23678863  0.65775895]
```

(b)

Jacobi Method :

```
After 5 iterations,we get the final root:
[0.995725 0.957775 0.79145 ]
```

Guass-Seidel Method :

```
After 3 iterations,we get the final root:
[0.9957475  0.95787375 0.79157475]
```

Problem6

$$\text{假设有} c_1x_1 + c_2x_2 + ... + c_kx_k = 0$$
$$\text{左乘} A, \text{由} Ax_i = \rho_i x_i, \text{有:}$$
$$c_1\rho_1x_1 + c_2\rho_2x_2 + ... + c_k\rho_kx_k = 0$$
$$\text{将}(1) * \rho_k - (2):$$
$$c_1(\rho_k - \rho_1)x_1 + c_2(\rho_k - \rho_2)x_2 + ... + c_k(\rho_k - \rho_{k-1})x_{k-1} = 0$$
$$\text{将} c_i(\rho_k - \rho_i) \text{记为} d_k, \text{则:}$$
$$d_1x_1 + d_2x_2 + ... + d_{k-1}x_{k-1} = 0$$
$$\text{如此操作下去，有:}$$
$$m_1x_1 = 0 \quad m_1 = 0$$
$$\text{在代回，可以得到} m_1 = m_2 = ... = m_k = 0$$
$$\text{进一步，有} c_1 = c_2 = ... = c_k = 0\text{，即} x_1, x_2, ..., x_k \text{线性无关。}$$
$$Q.E.D.$$

Problem7

> 证明**strictly diagonally dominant matrix**（简记为SDD）阵可逆

回顾SDD阵的定义：$|a_{kk}| > |\sum_{j\neq k} a_{kj}|$，$k = 1，2，...，n$

设A为SDD阵，假设A不可逆，即$|A| = 0$。那么$AX = 0$有非零解，记为$x_1, x_2, ..., x_n$。令$|x_k| = max\{|x_i|\}$。

因为$AX = 0$，则有

$$\sum_{j=1}^{n} a_{kj}x_j = 0 \Rightarrow \sum_{j\neq k} a_{kj}x_j = -a_{kk}x_k$$

$$\text{即} |\sum_{j\neq k} a_{kj}x_j| = |a_{kk}||x_k|$$

而由SDD阵的定义:

$$|a_{kk}||x_k| > |\sum_{j\neq k} a_{kj}||x_k| > |\sum_{j\neq k} a_{kj}||x_j| \geq |\sum_{j\neq k} a_{kj}x_j|$$

$$\text{即} |a_{kk}||x_k| > |a_{kk}||x_k|, \text{矛盾。}$$

故strictly diagonally dominant matrix阵可逆。