1. Let $B_k \in R^{n\times n}$ symmetric positive definite. p_k solves $\min q_k(p) = \frac{1}{2}p^T B_k p + \nabla f(x_k)^T p + f(x_k)$. Try to prove $p_k$is a decent direction of f at x_k.

2. Let $q_k(p) = \frac{1}{2}p^T B_k p + \nabla f(x_k)^T p + f(x_k)$. Try to prove the Cauchy point is the minimizer of $q_k(p)$ along $\nabla f(x)$

$$p_k^C = -\tau_k \frac{\Delta_k}{\|\nabla f(x_k)\|}\nabla f(x_k),$$

where

$$\tau_k = \begin{cases} 1, & \text{if } \nabla f(x_k)^T B_k \nabla f(x_k) \le 0 \\ \min\left\{\|\nabla f(x_k)\|^3/(\nabla f(x_k)^T B_k \nabla f(x_k)), 1\right\} & \text{otherwise.} \end{cases}$$

3. If symmetric $B \in R^{n\times n}$ has factorization $B = Q\Lambda Q^T$ where $Q = (q_1, q_2, \cdots, q_n)$ is orthogonal, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Try to prove

$$\begin{cases} (B + \lambda I)p = -g \\ \|p\| = \Delta \end{cases}$$

solution can be expressed by

$$p(\lambda) = -\sum_{i=1}^{n} \frac{q_i^T g}{\lambda_i + \lambda} q_i.$$

Furthermore, try to prove

$$\frac{d}{d\lambda}\left(\|p(\lambda)\|^2\right) = -2\sum_{i=1}^{n} \frac{(q_i^T g)^2}{(\lambda_i + \lambda)^3}$$

4. Let $p_k = \text{argmin}\{m_k(p) : \|p\| \le \Delta_k, s \in \text{span}[g_k, B_k^{-1}g_k]\}$, where $m_k(p) = f(x_k) + g_k^T p + \frac{1}{2}p^T B_k p$, B_k is symmetric positive. Try to find the explicit solution p_k.

5. Let $r_1(x) = x_2 - x_1^2$, $r_2(x) = 1 - x_2$, $r(x) = (r_1(x), r_2(x))^T$, $f(x) = \frac{1}{2}[r_1(x)^2 + r_2(x)^2] = \frac{1}{2}r(x)^T r(x)$. It is known that the solution to $\min f(x)$ is $x^* = (1, 1)^T$.

   (a) For any initial guess x_0, give the Newton algorithm of

   $$\begin{cases} x_{k+1} = x_k + \alpha_k p_k, \\ p_k = -[\nabla^2 f(x)]\nabla f(x), \\ \alpha_k : f(x_k + \alpha_k p_k) = \min f(x_k + \alpha p_k). \end{cases}$$

   (b) Try to prove when $x \to x^*$,

   $$\nabla^2 f(x) \to \nabla r(x) \nabla r(x)^T.$$

6. Let x_1, x_2 are solutions to $(A^T A + \mu_i I)x = -A^T r$, $i = 1, 2$ with respect to μ_1 and μ_2, where $\mu_1 > \mu_2 > 0$, $A \in R^{m\times n}$, $r \in R^m$. Try to prove $\|Ax_2 + r\|_2^2 < \|Ax_1 + r\|_2^2$.