MATH4321 Game Theory
Problem Set 3

Chapter 3: Games of incomplete information

Problem 1

Two universities offer two MSc. programs (Program A and B) and each MSc. program has only one vacancy. Peter (Player 1) and Ben (Player 2) are interested in applying MSc. program. Each of them can choose to apply one of these two programs. If they apply different program, they will be enrolled in their chosen program. If they apply the same program, they will compete and only one of them can get the offer and another will get nothing. We assume that Peter (player 1) can get the offer with probability p ($p \in (0,1)$). The value of p depends on Ben's (player 2) ability:

- If Ben is better than Peter, then $p = 0.2$ (Peter has smaller chance to win).
- If Ben is worse than Peter, then $p = 0.7$ (Peter has larger chance to win).

It is given that the payoffs of enrolling into program A and program B are 5 and 2 respectively. Thus the player's (expected) payoff can then be described by the following matrix:

		Player 2	
		A	**B**
Player 1	**A**	$\left(5p, 5(1-p)\right)$	$(5,2)$
	B	$(2,5)$	$\left(2p, 2(1-p)\right)$

(a) Suppose that each player knows the ability of another player, find the Nash equilibrium for each value of p.

(b) We assume that Ben (player 2) knows Peter's (player 1) ability and Peter does not. Peter only knows that there is a probability q that Ben is better than him.
  (i) Take $q = 0.5$, find the Bayesian Nash equilibrium.
  (ii) Repeat the calculation for $q = 0.1$ and $q = 0.8$ and examine whether there is any difference in the final outcome of the games.

Problem 2 (Contribution games)

We consider the following contribution games: Two companies are working on an investment project. Each company can decide the amount of effort (denoted by e_i, $i = 1,2$) to be put in the project. The contribution cost of company i is known to be $c_i e_i^2$, where c_i is some positive constant. It is given that the project's revenue (common for two companies) is known to be

$$Revenue = (R + e_1 + e_2)^2,$$

where $R > 0$ is a constant.

It is given that the true values of R, c_1 and c_2 are 2, 3 and 4 respectively.

(a) (Complete information) Suppose that two companies have *complete information* about the games, find the pure strategy Nash equilibrium.

(b) **(Information asymmetry)** We now assumed that company 1 does not know the true value of c_2 and it conjectures that c_2 is either $c_2^L = 2$ (with probability 0.3) or $c_2^H = 4$ (with probability 0.7). Find the Bayesian Nash equilibria by completing the following:

(i) For each of company 2's type (type L and type H), find the company 2's best response $e_2^{j^*}$ $(j = L, H)$ to player 1's strategy e_1.

(ii) Given the strategies (e_2^L and e_2^H) chosen by different types of company 2, find the best responses of player 1 by considering the player 1's expected payoff.

(iii) Hence, find the Bayesian Nash equilibrium.

(iv) Compare the companies' payoffs between (a) and (b).

(c) **(Harder) (Incomplete information, double sides)** We extend the model in (b) and further assumed that company 2 does not know the true value of R. It conjectures that R is either 1 (with probability 0.2) or 2 (with probability 0.8). On the other hand, company 1 knows the true value of R. Find the corresponding Bayesian Nash equilibrium using similar method as in (b). (☺Hint: Be careful that player 1 also has two types: Type L (with $R = 1$) and Type H (with $R = 2$)).

Problem 3

Two restaurants compete each other and serve a group of 100 customers in an area. Each restaurant can decide whether to offer set dinner to the customers. It is given that

- If two restaurants do not offer any set dinner, each restaurant will get 50% of the customers.
- If only one restaurant offers set dinner, this restaurant will get 85% of the customers and another restaurant will get 15%.
- If both restaurants offers set dinner, each restaurant will get 50% of the customers.

It is given that

- A customer will spend \$100 in average;
- The costs of offering set dinner are $c_1 = 3000$ (for restaurant 1) and $c_2 = 3200$ (for restaurant 2) respectively.

(a) We suppose that restaurant 1 knows the true value of c_2 and the restaurant 2 does not know the true value of c_1. The restaurant 2 knows that c_1 is either 2000 (with probability 0.3) or 3000 (with probability 0.3) or 4000 (with probability 0.4). Find the corresponding Bayesian Nash equilibrium.

Problem 4

We consider the following two-person dynamic games: There are two players in the game (Player 1 and Player 2). You are given that

- Player 1 has two possible types (type θ_1 and type θ_2). Player 1 knows his type but Player 2 only knows that the probability that player 1 is of type θ_1 is p.

The games consist of two stages:

- In the first stage, player 1 (of any type) can choose either a_{11} or a_{12}
    - If player 1 chooses a_{11}, both player 1 and player 2 will get a payoff of 0 and the games end.

- If player 1 chooses a_{12}, player 2 knows player 1's decision and the games will move to second stage.

- In the second stage, player 2 chooses either a_{21}, a_{22} or a_{23} and the games end afterwards. The final payoffs of two players, which depend on player 1's type and player 2's decision, is summarized in the following table:

Player 1's type	**Player 2's strategy**	**Payoff to player 1**	**Payoff to player 2**
θ_1	a_{21}	-1	3
	a_{22}	-1	2
	a_{23}	2	0
θ_2	a_{21}	-3	0
	a_{22}	-1	2
	a_{23}	3	3

(a) Draw the game tree for this game.

(b) Take $p = \frac{1}{3}$, find the Perfect Bayesian equilibrium for the game.

(c) (Harder) Find the range of p (if exists) such that there exists an equilibrium which player 2 is able to figure out the player 1's true type by observing the decision made by player 1.

Problem 5 (Harder)

We consider the following market entry games. There are two firms in the game: Incumbent firm (Player 1) and Entrant firm (Player 2). You are given that

- The incumbent firm monopolizes a market initially. The total value of having 100% of the market is 10, which the incumbent will receive if no one enters.
- The entrant firm has two possible type: Tough (T) or Weak (W)
  - A weak entrant can choose one of the three options: Small entry (S), Big entry (B) or no entry (N);
  - A tough entrant can choose only between Small entry (S) and Big Entry (B).

  The entrant firm knows its own type but the incumbent firm only know that there is 50% chance that the entrant is tough and 60% chance that the entrant is weak.

- The games consists of two stages:
  - Stage 1: The entrant chooses a strategy from its strategic set.
    - ✓ There is no cost for a tough entrant to enter at any level.
    - ✓ It costs a weak entrant 4 to enter at small level (S) and 6 to enter at big level (B).

    We assume that the incumbent firm can observe the decision made by the entrant.
  - Stage 2: If the entrant firm enters, the incumbent can choose to accommodate (A) or fight (F).
    - ✓ Accommodating an entrant imposes no cost. Independent of the entrant's type, accommodating small entry (S) gives the incumbent 60% of the market

and the entrant 40%. On the other hand, accommodating big entry (B) gives the incumbent 40% of the market and the entrant 60%.

- ✓ Fighting a tough entrant will cost incumbent 6. However, the incumbent can get 80% of the market and the entrant gets 20%.
- ✓ Fighting a weak entrant that chose small entry (S) will cost incumbent 2. And the incumbent will get 100% of the market.
- ✓ Fighting a weak entrant that chose small entry (B) will cost incumbent 8. And the incumbent will get 100% of the market.

- The payoff of each firm (incumbent firm & entrant firm) is defined as total revenue received minus the cost.

(a) Draw the game tree for this dynamic game.

(b) Find the Bayesian Nash equilibrium.

(c) Hence, determine all perfect Bayesian Nash equilibrium of the games.