# MATH4321 Game Theory

# Lecture Note 3 (Part 1)

# Games under incomplete information (Static Games)

## Introduction

One crucial step in making a decision in the games is to predict the action taken by the rivals. This requires the knowledge about the payoff functions of the rivals since all players should maximize their payoff in the games.

In the games that we discussed in the previous games, we assume that all players have *complete information* about other players such as strategic set, preference (quantified by payoff function) etc. So that all players can access the payoff functions of the rivals.

However, this assumption may not be true in reality. In fact, some of the player's information may not be known to the public.

- In the bidding games in which a group of bidders bids for an object, *a bidder does not know how other bidders value the object*.
- In the Cournot duopoly games in which two firms produce a common goods for a market, *a firm may not know the production cost of*

*another firm* since it does not have complete information on firm's background (e.g. manpower, technology etc.)

- In the battle of sexes games, a person (boy or girl) may not know precisely the preference of another person.

The above examples reveal that the importance of studying the incomplete information games. In fact, this class of games has a wide application in economics (R&D competition, patent race), finance (derivative pricing, capital financing problem) etc..

In this Chapter, we will study the games under incomplete information.

- A general setup
- Equilibrium concept: Bayesian Nash equilibrium (for static games) and Perfect Bayesian Nash equilibrium (for dynamic games, with possibility of information updating).
- Difference in equilibrium between complete information and incomplete information.

**General Setup (Static games under incomplete information)**

To start with, we consider the static games which all players make their decision simultaneously. The analysis of this games is easier since there is no information updating for any player: All players make their decisions based on the information acquired at the very beginning.

Roughly speaking, an $N$-person static games under incomplete information (also called Bayesian games) consists of the following elements:

1. Set of players $\{1,2, \dots, N\}$
2. Strategic set of players (denoted by $S_1, S_2, \dots, S_N$).
3. (New) *Type space* of player (denoted by $\Theta_1, \Theta_2, \dots, \Theta_N$)

   Each $\Theta_i = \left\{ \underbrace{\theta_{i1}}_{type\ 1}, \underbrace{\theta_{i2}}_{type\ 2}, \dots, \underbrace{\theta_{in_i}}_{type\ n_i} \right\}$ describes all possible types of player $i$. In general, "type" has different meaning depending on the games considered:

- In the bidding games, type space of player $i$ can be a collection of all possible player $i$'s valuation on the object. For example: $\Theta_i = \{60, 65, 70\}$ (discrete case) or $\Theta_i = [60,70]$ (continuous case).

  Although $\Theta_i$ is not important to player $i$ himself, it will be important to other players since they need to "predict" player $i$'s strategy in order to determine their optimal strategies.

4. (New) Player $i$'s belief on other players' types. Mathematically, it is quantified by the probability distribution which

$$p_i(\theta) = P(\theta_{-i} = \theta).$$

5. Player $i$'s payoff function under various outcomes, which is type-dependent. We denote it by $v_i(s_i; s_{-i}, \theta_i)$.

To incorporate incomplete information, we assume that player $i$'s type $\theta_i$ is *private information* of player $i$. Other players only know that player $i$'s type is drawn from the set $\Omega_i$ (associated with $\phi_j$).

In our static games model, we further assume that the true type of player $i$ ($\theta_i$) must be an element in $\Theta_i$.

**Example 1 (Bidding games)**

Two bidders (Player 1 and Player 2) compete each other to bid an object. They submit their bid simultaneously and independently and the bid must be an integer.

Bidder who places the highest bid can get an object and he has to pay for the object immediately. If both players submit the same bid, each of them wins with probability 50%.

It is given that

- Player 1 values the object at $30$ and player 2 values the object at $32$.
- Player 1 knows the player 2's valuation but player 2 does not know the player 1's valuation. Instead, player 2 only knows that player 1 values the object at $25$, $30$ and $35$ respectively.

Express the above games into normal form.

☺Solution

The normal form representation of the games can be expressed as follows:

1. Set of players $\{1,2\}$.
2. Strategic sets: $S_i = \{0,1,2,3,\dots\}$ for $i = 1,2$.
3. Type space of players: $\Theta_1 = \{25,\ 30,\ 35\}$, $\Theta_2 = \{32\}$.
4. Players' belief on another player's type:
$$p_1(32) = 1, \qquad p_2(25) = p_2(30) = p_2(35) = \frac{1}{3}.$$
5. Players' payoff function

   We let $b_1$ and $b_2$ be the bids submitted by player 1 and 2 respectively
$$v_1(b_1; b_2, \theta_1) = \begin{cases} \theta_1 - b_1 & if\ \ b_1 > b_2 \\ 0.5(\theta_1 - b_1) & if\ \ b_1 = b_2\,, \\ 0 & if\ b_1 < b_2 \end{cases} \qquad \theta_1 = 25,30,35.$$
$$v_2(b_2; b_1, \theta_2) = \begin{cases} \theta_2 - b_2 & if\ \ b_2 > b_1 \\ 0.5(\theta_2 - b_2) & if\ \ b_2 = b_1\,, \\ 0 & if\ b_2 < b_1 \end{cases} \qquad \theta_2 = 32.$$

**Example 2 (Cournot Duopoly)**

Two competing firms (firm 1 and firm 2) producing a common good for a market and each firm can choose the number of goods $(q_i)$ produced. It is given that

- The market price of the good is $p = 100 - q_1 - q_2$.
- The production costs of two firms are $c_1 q_1$ and $c_2 q_2$ respectively. The true values of $c_1$ and $c_2$ are $5$ and $9$ respectively.
- We assume that both firms do not know the true value of $c_j$ of another firm. Firm 1 only knows that $c_2$ can be either $6$, $9$, or $11$ with probabilities $0.3$, $0.2$ and $0.5$ respectively and Firm 2 know that $c_1$ can be any number from $4$ and $7$ with uniform distribution.

Express the above games into the normal form.

☺Solution

The normal form representation of the games can be expressed as follows:

1. Set of players $\{1,2\}$.
2. Strategic sets: $S_i = [0, \infty)$ for $i = 1,2$.
3. Type space of players: $\Theta_1 = \underbrace{[4,7]}_{interval}$ , $\Theta_2 = \{6,\ 9,\ 11\}$.
4. Players' belief on another player's type:

$$p_1(6) = 0.3,\ \ p_1(9) = 0.2,\ \ p_1(11) = 0.5.$$

   The player 2's belief $\phi_2$ will be expressed by the following *probability density function*:

$$p_2(x) = \begin{cases} \frac{1}{3} & if\ \ 4 \le x \le 7 \\ 0 & if\ \ otherwise \end{cases}.$$

5. Players' payoff functions

$$v_1(q_1; q_2, c_1) = (100 - q_1 - q_2)q_1 - c_1 q_1,\ \ c_1 \in [4,7];$$
$$v_2(q_2; q_1, c_2) = (100 - q_1 - q_2)q_2 - c_2 q_2,\ \ c_2 \in \{6, 9, 11\}.$$

**Equilibrium concept in static games under incomplete information: Bayesian Nash equilibrium (BNE)**

Recall that each player will choose a strategy which is a best response to opponents' optimal strategies. Under complete information games, the opponent's strategies can be determined sharply.

The story becomes complicated for incomplete information games. Since a player may not know the true type of another player and different types of player can adopt different strategies, therefore the player *must* predict the strategies chosen by *different* types of players before making his decision.

*Payoff functions of player under incomplete information: Expected payoff*

Given a strategy $s_i(\theta_i)$ chosen by player $i$ of type $\theta_i$ and the set of opponents' strategies $s_{-i} = \{s_{-i}(\theta_{-i}): \theta_{-i} \in \Theta_{-i}\}$, the player $i$'s expected payoff (his objective function) can be expressed as follows:

$$V_i(s_i; s_{-i}, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \underbrace{p_i(\theta_{-i})}_{P\left(\substack{opponents \\ are\ of\ type\ \theta_{-i}}\right)} \times v_i(s_i; s_{-i}(\theta_{-i}), \theta_i)$$

Each player should choose a strategy that maximizes the expected payoff. We say a strategy $s_i^*(\theta_i)$ is the *best response* of player $i$ of type $\theta_i$ to opponents' strategies $s_{-i}$ if and only if

$$V_i(s_i^*(\theta_i); s_{-i}, \theta_i) \geq V_i(s; s_{-i}, \theta_i), \qquad for\ any\ s \in S_i.$$

Similar to static games under complete information, we expect that each player's strategy $s_i^*$ must be the best response to rival strategies $s_{-i}^*$.

Then, we can state the following equilibrium concept for static games under incomplete information.

**Definition (Bayesian Nash equilibrium)**

We say a strategic profile $s^* = (s_1^*, \dots, s_N^*)$ is a Bayesian Nash equilibrium if and only if for any player $i$ and any type $\theta_i$, we have

$$V_i(s_i^*(\theta_i); s_{-i}^*, \theta_i) \geq V_i(s; s_{-i}^*, \theta_i), \qquad for\ any\ s \in S_i,$$

Here, $s_i^* = \left(s_i^*(\theta_{i1}), s_i^*(\theta_{i2}), \dots, s_i^*\left(\theta_{in_i}\right)\right)$ denotes the player $i$'s strategies of different types.

**Example 3 (Cournot Duopoly)**

Two competing firms (firm 1 and firm 2) producing a common good for a market and each firm can choose the number of goods $(q_i)$ produced. It is given that

- The market price of the good is $p = 100 - q_1 - q_2$.
- The production costs of two firms are $c_1 q_1$ and $c_2 q_2$ respectively. The true values of $c_1$ and $c_2$ are $10$ and $10$ respectively.
- Firm 2 knows the production cost of firm 1 and Firm 1 does not know the exact production cost of firm 2. Instead, Firm 1 knows that $c_2$ is either $c_{2L} = 5$, $c_{2M} = 10$ or $c_{2H} = 20$ with probabilities $\frac{1}{3}, \frac{1}{3}$ and $\frac{1}{3}$ respectively.

Based on the above information, determine all possible Bayesian Nash equilibrium for this duopoly games.

☺Solution

We let $q_1$ be quantity chosen by Firm 1. We let $q_{2L} = q_2(c_{2L})$, $q_{2M} = q_2(c_{2M})$ and $q_{2H} = q_2(c_{2H})$ be quantities chosen by different types of Firm 2.

Step 1: Find the best response of Firm 1

We first consider Firm 1's side. Given the strategies $(q_{2L}, q_{2M}, q_{2H})$ chosen by different types of Firm 2, the expected payoff of Firm 1 is expressed as

$$
\begin{aligned}
&V_1(q_1; q_2)\\
&= \frac{1}{3} \times \underbrace{[(100 - q_1 - q_{2L})q_1 - 10q_1]}_{v_1(q_1;q_{2L},c_1)} + \frac{1}{3} \times \underbrace{[(100 - q_1 - q_{2M})q_1 - 10q_1]}_{v_1(q_1;q_{2M},c_1)}\\
&\qquad + \frac{1}{3} \times \underbrace{[(100 - q_1 - q_{2H})q_1 - 10q_1]}_{v_1(q_1;q_{2H},c_1)}\\
&= \cdots = \left(90 - \frac{1}{3}q_{2L} - \frac{1}{3}q_{2M} - \frac{1}{3}q_{2H}\right)q_1 - q_1^2.
\end{aligned}
$$

By considering the first-order condition, we have

$$\frac{\partial V_1}{\partial q_1}\Big|_{q_1=q_1^*} = 0 \Rightarrow \left(90 - \frac{1}{3}q_{2L} - \frac{1}{3}q_{2M} - \frac{1}{3}q_{2H}\right) - 2q_1^* = 0$$

$$\Rightarrow q_1^* = \frac{90 - \frac{1}{3}q_{2L} - \frac{1}{3}q_{2M} - \frac{1}{3}q_{2H}}{2} \ldots\ldots(1).$$

<u>Step 2: Find the best response of Firm 2</u>

We proceed to determine the best response of Firm 2. Given Firm 1's strategy $q_1$ and the firm's type $c_{2i}$, the Firm 2's payoff is seen to be

$$V_2\left(q_{2i}; q_1, c_2^i\right) = (100 - q_1 - q_{2i})q_{2i} - c_{2i}q_{2i}, \qquad i = L, M, H.$$

The best response can be found by considering the corresponding first order condition:

$$\frac{\partial V_2}{\partial q_{2i}}\Big|_{q_{2i}^*} = 0 \Rightarrow (100 - c_{2i} - q_1) - 2q_{2i}^* = 0 \Rightarrow q_{2i}^* = \frac{100 - c_{2i} - q_1}{2}.$$

Substituting $c_{2i} = 5{,}10$ and 20, we get

$$q_2^{L^*} = \frac{95 - q_1}{2}, \qquad q_2^{M^*} = \frac{90 - q_1}{2}, \qquad q_2^{H^*} = \frac{80 - q_1}{2} \dots\dots (2).$$

Hence, the Bayesian Nash equilibrium can be identified by solving (1) and (2).

Substitute (2) into (1), we get

$$q_1^* = \frac{90 - \frac{1}{3}\left(\frac{95 - q_1^*}{2}\right) - \frac{1}{3}\left(\frac{90 - q_1^*}{2}\right) - \frac{1}{3}\left(\frac{80 - q_1^*}{2}\right)}{2} \Rightarrow q_1^* = 30.55556.$$

From (2), we get

$$q_{2L}^* = \frac{95 - q_1^*}{2} = 32.2222, \qquad q_{2M}^* = 29.7222, \qquad q_{2H}^* = 24.7222.$$

The corresponding *actual* profits made by two firms are given by

$$\begin{aligned} V_1 &= (100 - 30.55556 - 29.7222)(30.55556) - 10(30.55556) \\ &\approx 908.179. \end{aligned}$$

$$\begin{aligned} V_2(c_{2M}) &= (100 - 30.55556 - 29.7222)(29.7222) - 10(29.7222) \\ &\approx 883.4105. \end{aligned}$$

*Remark of Example 3* (Impact of incomplete information)

One can examine the impact of incomplete information by comparing the result with that under complete information.

Suppose that Firm 1 also knows the production cost of Firm 2 ($c_2^M = 10$), then the corresponding Nash equilibrium is found to be $q_1^{(C)} = q_2^{(C)} = 30$.

Since $q_1^* > q_i^{(C)}$ the Firm 1 acts more aggressively under incomplete information. It is probably because he predicts that the firm 2 is less cost efficient (i.e. higher production cost in average).

The corresponding profits are found to be

$$V_1 = (100 - 30 - 30)(30) - 10(30) = 900, \qquad V_2 = 900.$$

We summarize the result in the following table:

| | **Firm 1** | **Firm 2** |
|---|---|---|
| **Incomplete information** | 908.179 | 883.4105 |
| **Complete information** | 900 | 900 |

**Example 4 (Joint project, Impact of incomplete information)**

A firm (Firm 1) is operating an investment project. Recently, another firm (Firm 2) is interested to join the Firm 1 and work on the project. Each firm decides to put either high effort (H) or low efforts and they make their decisions simultaneously. It is given that

- Firm 2 does not the type of the investment project. It only knows that the project is of either high-potential type (with probability $p$) or low-potential type (with probability $1-p$).
- Firm 1 knows the project type.

Depending on the effort put by two firms and the type of the investment project, the payoffs are summarized by the following matrices:

| | | **Firm 2** | |
|---|---|---|---|
| | | **H** | **L** |
| **Firm 1** | **H** | $(10,10)$ | $(2,8)$ |
| | **L** | $(8,2)$ | $(1,1)$ |

(High-potential project, G)

| | | **Firm 2** | |
|---|---|---|---|
| | | **H** | **L** |
| **Firm 1** | **H** | $(-1,-1)$ | $(-3,3)$ |
| | **L** | $(3,-3)$ | $(1,1)$ |

(Low-potential project, B)

Take $p = 0.3$. Find the Bayesian Nash equilibrium for this games.

☺Solution

**Step 1: Find the best response of Firm 1**

We let $s_{1G}(s_2)$ and $s_{1B}(s_2)$ be the Firm 1's best response to Firm 2's strategy $s_2$. Here, $s_{1G}(s_2)$ denotes the best response when the project is of high-potential type.

Since Firm 1 knows the project's type, one can determine the best response using the payoff matrices directly.

$$s_{1G}(H) = H, \qquad s_{1G}(L) = H.$$

| High potential project | | **Firm 2** | |
|---|---|---|---|
| | | **H** | **L** |
| **Firm 1** | **H** | $(\bar{\mathbf{10}}, 10)$ | $(\bar{\mathbf{2}}, 8)$ |
| | **L** | $(8,2)$ | $(1,1)$ |

$$s_{1B}(H) = L, \qquad s_{1B}(L) = L.$$

| Low potential project | | **Firm 2** | |
|---|---|---|---|
| | | **H** | **L** |
| **Firm 1** | **H** | $(-1,-1)$ | $(-3,3)$ |
| | **L** | $(\bar{\mathbf{3}}, -3)$ | $(\bar{\mathbf{1}}, 1)$ |

**Step 2: Find the best response of Firm 2**

Given the strategies $(s_{1G}, s_{1B})$ chosen by different types of Firm 1, the expected payoff (with $p = 0.3$) of Firm 2 are given by the following matrix:

| | | Firm 2 | |
|---|---|---|---|
| | | $H$ | $L$ |
| Firm 1 | $(s_{1G}, s_{1B}) = (H, H)$ | $10(0.3) + (-1)(0.7) = 2.3$ | $8(0.3) + 3(0.7) = \overline{\mathbf{4.5}}$ |
| | $(H, L)$ | $10(0.3) + (-3)(0.7) = 0.9$ | $8(0.3) + 1(0.7) = \overline{\mathbf{3.1}}$ |
| | $(L, H)$ | $2(0.3) + (-1)(0.7) = -0.1$ | $1(0.3) + 3(0.7) = \overline{\mathbf{2.4}}$ |
| | $(L, L)$ | $2(0.3) + (-3)(0.7) = -1.5$ | $1(0.3) + 1(0.7) = \overline{\mathbf{1}}$ |

One can observe from the above table that Firm 2's best response is always $L$.

Combining the result in Step 1 and 2, we conclude that the Bayesian Nash equilibrium is given by $s_1^* = (s_{1G}, s_{1B}) = (H, L)$ and $s_2^* = L$.

*Remark of Example 4*

The equilibrium pattern changes when the probability $p$ changes. To see, we consider the case when $p = 0.7$ (Firm 2 believes that the project is likely to be of high-potential type). The corresponding payoff and best response (upper bar) of firm 2 is given by

| | | Firm 2 | |
|---|---|---|---|
| | | $H$ | $L$ |
| Firm 1 | $(H, H)$ | $\overline{\mathbf{6.7}}$ | $6.5$ |
| | $(H, L)$ | $\overline{\mathbf{6.1}}$ | $5.9$ |
| | $(L, H)$ | $1.1$ | $\overline{\mathbf{1.6}}$ |
| | $(L, L)$ | $0.5$ | $\overline{\mathbf{1}}$ |

Then the equilibrium becomes $s_1^* = (s_{1G}, s_{1B}) = (H, L)$ and $s_2^* = H$.

**Example 5 (Coordination games under incomplete information)**

Two people (Player 1 and 2) are going to meet each other at the airport. Each player can choose to go either by train (T) or by bus (B). Player 1 prefers to go by train and player 2 prefers to go by bus.

However, neither player knows whether another player wants to go together with him. It is given that each player conjectures that another player wishes to go with him with probability $2/3$.

The players' payoffs, which depends on the players' type $(\theta_1, \theta_2)$, are summarized by the following four matrices: (Here, $Y$ denoted the player's type which the player wishes to go together and $N$ denotes the player's type which the player wants to go alone.)

| $(\theta_1, \theta_2) = (Y, Y)$ | | **Player 2** | |
|---|---|---|---|
| | | **T** | **B** |
| **Player 1** | **T** | $(4,1)$ | $(0,0)$ |
| | **B** | $(0,0)$ | $(1,4)$ |

| $(\theta_1, \theta_2) = (Y, N)$ | | **Player 2** | |
|---|---|---|---|
| | | **T** | **B** |
| **Player 1** | **T** | $(4,0)$ | $(0,4)$ |
| | **B** | $(0,1)$ | $(1,0)$ |

| $(\theta_1,\theta_2) = (N,Y)$ | | Player 2 | |
|---|---|---|---|
| | | **T** | **B** |
| **Player 1** | **T** | $(0,1)$ | $(4,0)$ |
| | **B** | $(1,0)$ | $(0,4)$ |

| $(\theta_1,\theta_2) = (N,N)$ | | Player 2 | |
|---|---|---|---|
| | | **T** | **B** |
| **Player 1** | **T** | $(0,0)$ | $(4,4)$ |
| | **B** | $(1,1)$ | $(0,0)$ |

Based on the above information, find the Bayesian Nash equilibrium for this games.

☺Solution

We let $s_{1Y}^*, s_{1N}^*, s_{2Y}^*$ and $s_{2N}^*$ be the strategies chosen by different types of players. We first determine the best response of player 1 (type Y and type N).

Given the strategies $s_2 = (s_{2Y}, s_{2N})$ chosen by player 2, the expected payoff of player 1 under various outcomes can be summarized by the following matrix:

| | | $(T,T)$ | $(T,B)$ | $(B,T)$ | $(B,B)$ |
|---|---|---|---|---|---|
| Player 1 (Type Y) | $T$ | $\frac{2}{3}(4)+\frac{1}{3}(4)$ $=4$ | $\frac{2}{3}(4)+\frac{1}{3}(0)$ $=8/3$ | $\frac{2}{3}(0)+\frac{1}{3}(4)$ $=4/3$ | $\frac{2}{3}(0)+\frac{1}{3}(0)$ $=0$ |
| | $B$ | $\frac{2}{3}(0)+\frac{1}{3}(0)$ $=0$ | $\frac{2}{3}(0)+\frac{1}{3}(1)$ $=1/3$ | $\frac{2}{3}(1)+\frac{1}{3}(0)$ $=2/3$ | $\frac{2}{3}(1)+\frac{1}{3}(1)$ $=1$ |
| Best response | | $T$ | $T$ | $T$ | $B$ |

| | | $(T,T)$ | $(T,B)$ | $(B,T)$ | $(B,B)$ |
|---|---|---|---|---|---|
| Player 1 (Type N) | $T$ | $\frac{2}{3}(0)+\frac{1}{3}(0)$ $=0$ | $\frac{2}{3}(0)+\frac{1}{3}(4)$ $=4/3$ | $\frac{2}{3}(4)+\frac{1}{3}(0)$ $=8/3$ | $\frac{2}{3}(4)+\frac{1}{3}(4)$ $=4$ |
| | $B$ | $\frac{2}{3}(1)+\frac{1}{3}(1)$ $=1$ | $\frac{2}{3}(1)+\frac{1}{3}(0)$ $=2/3$ | $\frac{2}{3}(0)+\frac{1}{3}(1)$ $=1/3$ | $\frac{2}{3}(0)+\frac{1}{3}(0)$ $=0$ |
| Best response | | $B$ | $T$ | $T$ | $T$ |

Similarly, one can compute the expected payoff and best response of player 2:

| | | $(T,T)$ | $(T,B)$ | $(B,T)$ | $(B,B)$ |
|---|---|---|---|---|---|
| Player 2 (Type Y) | $T$ | 1 | 2/3 | 1/3 | 0 |
| | $B$ | 0 | 4/3 | 8/3 | 4 |
| Best response | | $T$ | $B$ | $B$ | $B$ |

| | | $(T,T)$ | $(T,B)$ | $(B,T)$ | $(B,B)$ |
|---|---|---|---|---|---|
| Player 2 (Type N) | $T$ | 0 | 1/3 | 2/3 | 1 |
| | $B$ | 4 | 8/3 | 4/3 | 0 |
| Best response | | $B$ | $B$ | $B$ | $T$ |

Given the best responses of the players, one can identify the Bayesian Nash equilibrium through the following steps:

Step 1: For each of the player 1's strategy $s_1$, we search for the best response of player 2 $s_2^* = (s_{2Y}^*, s_{2N}^*)$.

Step 2: We proceed to check whether $s_1$ is the best response to $s_2^*$. If yes, then $(s_1, s_2^*)$ is the desired equilibrium.

| $s_1 = (s_1(Y), s_1(N))$ | Best response $s_2^*$ | Is $s_1(Y)$ best response? | Is $s_1(N)$ best response? |
|---|---|---|---|
| $(T, T)$ | $(T, B)$ | Yes | Yes |
| $(T, B)$ | $(B, B)$ | No | No |
| $(B, T)$ | $(B, B)$ | Yes | Yes |
| $(B, B)$ | $(B, T)$ | No | No |

Hence, we conclude that there are two Bayesian Nash equilibria:

$$s_1^* = (s_{1Y}^*, s_{1N}^*) = (T, T), \qquad s_2^* = (s_{2Y}^*, s_{2N}^*) = (T, B);$$

and

$$s_1^* = (B, T), \qquad s_2^* = (B, B);$$

**Example 6 (The market for lemons – Adverse Selection Problem)**

Peter (Player 1) wishes to sell his car to another buyer (Player 2). Peter knows the condition of his car and the buyer does not know the car's quality. The buyer only knows that among all used cars traded in the market, 60% of them is in good condition and 40% of them is in bad condition.

The players' valuations on used cars in various conditions are summarized in the following table:

| | Car in good | Car in bad |
|---|---|---|
| Peter | $6$ | $1$ |
| Buyer | $7$ | $3$ |

Suppose that the market price of the car is $P$, Peter can choose whether to sell his car at this price $P$ and the buyer can choose whether to buy his car. We assume that they make their decision simultaneously.

- If the transaction is successful, Peter will get $P - V_{seller}$ and the buyer will get $V_{Buyer} - P$, where $V_{Buyer}, V_{seller}$ are the buyer's valuation and seller's valuation on the car respectively.
- If the transaction is not successful, both players will get nothing.

The payoffs to Peter and buyer are summarized by the following matrix:

| | | **Buyer (Player 2)** | |
|---|---|---|---|
| | | **Buy** | **Not buy** |
| **Peter (Player 1)** | **Sell** | $(P-6, 7-P)$ | $(0,0)$ |
| | **Not sell** | $(0,0)$ | $(0,0)$ |

(Car in good condition, G)

| | | **Buyer (Player 2)** | |
|---|---|---|---|
| | | **Buy** | **Not buy** |
| **Peter (Player 1)** | **Sell** | $(P-1, 3-P)$ | $(0,0)$ |
| | **Not sell** | $(0,0)$ | $(0,0)$ |

(Car in bad condition, B)

Suppose that Peter's car is in *good condition*. It is clear that Peter never sell his car if the market price $P$ is lower than his valuation (i.e. $6$) and the buyer will never buy the car at a price higher than 7 (i.e. largest valuation)

Is it possible to have a Bayesian Nash equilibrium in which Peter can sell his car at a price higher than 6 (i.e. $6 < P \leq 7$)?

☺Solution

To determine the Bayesian Nash equilibrium for $6 < P \leq 7$, we proceed to determine the best responses of the two players.

**Step 1: Finding Peter's best response under $6 < P \leq 7$**

We let $s_{1G}$ and $s_{1B}$ be the Peter's strategies when the car is in good condition and bad condition, respectively.

Under $P > 6$ and if the buyer buys the car (denoted by B), the best response of Peter is found to be

$$s_{1G}\underbrace{(B)}_{Buy} = S \quad and \quad s_{1B}(B) = \underbrace{S}_{Sell}.$$

If the buyer do not buy the car (denoted by NB), then the Peter's payoff is always 0 regardless of his strategy, Thus, the best response of Peter is simply $s_{1G}(NB) = s_{1B}(NB) = \{S, NS\}$.

**Step 2: Finding the buyer's best response under $6 < P \leq 7$**

Given the Peter's strategy $s_1 = (s_{1G}, s_{1B})$, the expected payoff of the buyer is

| | | Buyer (player 2) | |
|---|---|---|---|
| | | $B$ | $NB$ |
| Peter (Player 1) | $(S,S)$ | $(7-P)(0.6)+(3-P)(0.4)$ $= 5.4 - P$ | $\overline{\mathbf{0}}$ |
| | $(S,NS)$ | $(7-P)(0.6) = \overline{\mathbf{4.2-0.6}\boldsymbol{P}}$ | $\overline{\mathbf{0}}$ **(*)** |
| | $(NS,S)$ | $(3-P)(0.4) = 1.2 - 0.4P$ | $\overline{\mathbf{0}}$ |
| | $(NS,NS)$ | $\overline{\mathbf{0}}$ | $\overline{\mathbf{0}}$ |

(*Note: Provided that $P = 7$. When $P < 7$, the best response will be $\{B\}$ only.)

We observe from the above table that the buyer will only buy the car only when $s_1^* = (S, NS)$ or $s_1^* = (NS, NS)$. The second case is not interesting since nobody will sell the car. On the other hand, one can see from Step 1 that $(S, NS)$ is not the best response if the buyer wants to buy the car.

Therefore, we conclude that there is no Bayesian Nash which Peter can sell the car successfully at a price $P > 6$.

*Remark of Example 6 (Adverse selection problem)*

Under incomplete information, the buyer cannot access the quality of the car. Given that two types of buyers are willing to sell the car when $P > 6$, the buyer can only value the car by averaging the values of good condition car and bad condition car together (i.e. $0.6(7) + 0.4(3) = 5.4$).

Even though the car owned by Peter is in good condition, the buyer is willing to buy the car at any price lower than $5.4$ only. This explains why the buyer chooses not to buy the car when $P > 6$.

If the price $P$ is below $5.4$, we observe that only the person who own bad quality car is willing to sell his car. So the buyer can only get the bad car.

Due to the underestimation of car's value, we observe that the seller never sells the high-quality car (since it cannot be sold at a good price) and only low-quality car is traded in the market. This inefficient outcome is commonly known as *adverse selection problem*.

## Incomplete information games with continuum of types

### Example 7 (Contribution games)

3 students are working on a group project. Each student can choose whether to put effort in the project. The project can be completed only when *at least two students* contribute. It is given that the value of completing the project is $1$ (per student). The cost of contributing is known to be $c_i \in (0,1)$ $(i = 1,2,3)$.

- A student knows his own contribution cost $c_i$. He knows that the contribution cost $c_j$ of other students is randomly drawn from $(0,1)$ with uniform distribution.

Find a Bayesian Nash equilibrium for this game.

☺Solution

We let $s_i^*(c_i)$ be the student $i$'s strategy when his contribution cost is $c_i \in (0,1)$. Here, $s_i^*(c_i)$ can be either $C$ (contribute) or $N$ (no effort).

Given the strategies chosen by other two students (depending on his own type), student $i$ chooses to put effort in the project (with cost $c_i$) under the Nash equilibrium if and only if

$$s_i^*(c_i) = C \Leftrightarrow \underbrace{(1)\,P\binom{\geq 1\ student}{contribute} - c_i}_{Payoff\ (contribute)} \geq \underbrace{P\binom{2\ students}{contribute}}_{Payoff\ does\ not\ contribute}$$

$$\Rightarrow c_i \leq \underbrace{P\binom{\geq 1\ student}{contribute} - P\binom{2\ students}{contribute}}_{denoted\ by\ \widehat{c_i}} \ldots \ldots (1).$$

The above inequality reveals that student $i$ is willing to contribute only when his cost of contributing is sufficiently small. Thus we have the following claim:

For any player $i$, there is a critical threshold $\widehat{c_i} \in (0,1)$ such that player $i$ prefers to contribute if and only if $c_i \leq \widehat{c_i}$. That is, $s_i(c_i) = \begin{cases} C & if\ c_i \leq \hat{c}_i \\ N & if\ c_i > \hat{c}_i \end{cases}$

It remains to obtain the critical thresholds $\widehat{c_1}, \widehat{c_2}, \widehat{c_3}$.

When $c_1 = \widehat{c_1}$, the player 1 is indifferent between contributing and not contributing. So we have

$$P\binom{\geq 1\ student}{contribute} - \widehat{c_1} = P\binom{2\ students}{contribute}.$$

Using the fact that $c_j$ is uniformly distributed (so that $P\left(c_j \leq \widehat{c_j}\right) = \int_0^{\widehat{c_j}} 1\, dx = \widehat{c_j}$) and each $c_j$ is drawn independently, we get

$$[\widehat{c_2}(1 - \widehat{c_3}) + (1 - \widehat{c_2})\widehat{c_3} + \widehat{c_2}\widehat{c_3}] - \widehat{c_1} = \widehat{c_2}\widehat{c_3}$$

$$\Rightarrow \widehat{c_1} - \widehat{c_2} - \widehat{c_3} + 2\widehat{c_2}\widehat{c_3} = 0 \ldots\ldots (2)$$

Similarly, we consider player 2 and player 3 and deduce that

$$\widehat{c_2} - \widehat{c_1} - \widehat{c_3} + 2\widehat{c_1}\widehat{c_3} = 0 \ldots\ldots (3)$$

$$\widehat{c_3} - \widehat{c_1} - \widehat{c_2} + 2\widehat{c_1}\widehat{c_2} = 0 \ldots\ldots (4).$$

Suppose that we are seeking for symmetric equilibrium, we take $\hat{c} = \widehat{c_1} = \widehat{c_2} = \widehat{c_3}$, the above 3 equations are reduced into

$$-\hat{c} + 2\hat{c}^2 = 0 \Rightarrow \hat{c} = 0, \ \ \hat{c} = \frac{1}{2}.$$

The first equilibrium $\hat{c} = 0$ corresponds to the case that no person want to contribute. One can easily verify that it is the Nash equilibrium easily.

Next, we shall verify that $\hat{c} = \frac{1}{2}$ constitutes the PBE of the games. We consider two cases:

- If $c_i \le \hat{c} = \frac{1}{2}$ and $s_i^*(c) = C$, one can deduce that

$$V_i(C; s_{-i}(\cdot), c_i) = 1 \times (2\hat{c}(1-\hat{c}) + \hat{c}^2) - c_i = \frac{3}{4} - c_i$$

$$V_i(N; s_{-i}(\cdot), c_i) = \hat{c}^2 = \frac{1}{4} \overbrace{\le}^{as\ c_i \le \frac{1}{2}} \frac{3}{4} - c_i = V_i(C; s_{-i}(\cdot), c_i)$$

- If $c_i > \hat{c} = \frac{1}{2}$ and $s_i^*(c) = N$, one can easily verify that

$$V_i(N; s_{-i}(\cdot), c_i) = \hat{c}^2 = \frac{1}{4} > \frac{3}{4} - c_i = V_i(C; s_{-i}(\cdot), c_i).$$

This proves that player $i$ has no incentive to deviate and the proposed strategy is the desired PBE.

**Example 8 (Market Entry games under incomplete information)**

Two firms (Firm 1 and firm 2) consider to enter into a new market. Each firm can choose whether to enter into this market and they make the decisions simultaneously. It is given that

- The firm's payoff upon entering is $\frac{60}{n}$, where $n$ is the number of firms that enter;
- The cost of entering is $c_i$, $i = 1{,}2$.
- Each firm knows its own cost $c_i$ and does not know the cost of another firm $c_j$. The firm only knows that $c_j$ is randomly drawn from $[20{,}60]$ with uniform distribution.

Find the Bayesian Nash equilibrium for this games.

☺Solution

We let $s_i^*(c_i)$ be the strategy chosen by Firm $i$ with cost $c_i$. We proceed to compute the best response of Firm $i$.

Given the strategies chosen by its rival, Firm $i$ finds itself optimal to enter into the market under Nash equilibrium if and only if

$$\underbrace{\left[60(1-p_j)+\frac{60}{2}p_j\right]-c_i}_{Payoff\ (Entering)} > \underbrace{0}_{\substack{Payoff \\ (Do\ nothing)}} \quad \Rightarrow c_i < 60-30p_j.$$

Here, $p_j = P(Firm\ j\ enters\ the\ market)$. So we conclude that

| |
|---|
| For any player $i$, there is a critical threshold $\widehat{c_i} \in (0,1)$ such that player $i$ prefers to contribute if and only if $c_i \le \widehat{c_i}$. |

It remains to find the values of $\widehat{c_1}$ and $\widehat{c_2}$. We consider the player 1 with $c_1 = \widehat{c_1}$. Then player 1 must be indifferent between entering and not entering. That is,

$$\left[60(1-p_j)+\frac{60}{2}p_j\right]-\widehat{c_i} = 0 \dots\dots (1)$$

Given the critical threshold $\widehat{c_j}$, the probability $p_j$ is found to be

$$p_j = P(Firm\ j\ enters) = \Pr\left(c_j \leq \widehat{c_j}\right) = \int_{20}^{\widehat{c_j}} \frac{1}{40} dx = \frac{\widehat{c_j} - 20}{40}.$$

Then the equation $(1)$ becomes

$$\left(60 - 30\left(\frac{\widehat{c_j} - 20}{40}\right)\right) - \widehat{c_i} = 0 \Rightarrow 75 - \frac{3}{4}\widehat{c_j} - \widehat{c_i} = 0.$$

So we can deduce the following system of equations that governs $\widehat{c_1}, \widehat{c_2}$:

$$\begin{cases} 75 - \frac{3}{4}\widehat{c_1} - \widehat{c_2} = 0 \\ 75 - \frac{3}{4}\widehat{c_2} - \widehat{c_1} = 0 \end{cases} \Rightarrow \widehat{c_1} = \widehat{c_2} = \frac{300}{7}.$$

So under Bayesian Nash equilibrium, a firm chooses to enter the market only when its cost is less than $\widehat{c_i} = \frac{300}{7}$.

## Auctions and bidding games under incomplete information

In a bidding games (or auction), some bidders compete for acquiring a valuable object (such as property, painting, jewelry etc.) by submitting bids to the seller. Each player has his own valuation to the object.

There are two major types of bidding games.

1. *Sealed-bid auction (Static games setting)*

   Each bidder submit a single bid $b_i \in [0, \infty)$ confidentially and simultaneously so that the bidders do not know the bids submitted by their opponents. Once the seller collects all bids from the bidders, the bidder who submits the highest bid will win the auction and pays for the object:

   - First-price sealed-bid auction – The winner will pay his bid.
   - Second-price sealed-bid auction – The winner will buy the object by paying the *second highest bid* instead of his bid.

Example: There are 3 bidders and their bids are given in the following table

| | **Bidder 1** | **Bidder 2** | **Bidder 3** |
|---|---|---|---|
| **Bid** | 75 | 72 | 67 |

We see that bidder 1 is the winner of the auction.

- In the first-price auction, bidder 1 will pay $75 and buy the object.
- In the second-price auction, bidder will pay $72 and buy the object.

(*Tie-breaking rule: When there are more than 1 bidders submitting the highest bid, the object will be randomly assigned to one of these bidders with equal probability. Suppose that there are 3 bidders submitting the highest bid, then each of these bidders will get the object with probability $\frac{1}{3}$.)

2. *Open auction (Dynamic games setting)*

   The bidders take turn to submit their bids. Different from sealed-bid auction, bidders can observe the bid submitted by others previously. The games continue until the winner is determined. One classical example of such auction is *English auction*: The seller first set the *reserve price,* which is the minimum price of the object. Each bidder can choose to bid the object by raising the price. Usually the price increment is fixed, the games continues until there is no bidders raising the bid. The bidder who submits the last bid will pay the bid and acquire the object.

*Incomplete information in bidding games*

In reality, a bidder knows its own valuation but don't know the exact valuations of other bidders since a bidder's valuation on the object is a subjective assessment. At most, the bidder knows the *range* of the opponent's valuation.

**Second price sealed-bid auction under incomplete information**

We consider the following second price sealed-bid auction: There are $N$ bidders in the games and each bidder submits a single bid (denoted by $b_i$, where $b_i \in [0, \infty)$) to the seller. The bidder who submits the highest bid will be the winner and he will pay the second highest bid for the object. We assume in the model that

- the true object's valuation of player $i$ is $v_i \in (0, \infty)$.
- The value of $v_i$ is player $i$'s *private information* and other players only knows $v_i$ is randomly drawn from $[\underline{v}_i, \bar{v}_i]$ with some probability distribution $P(v_i = v) = f(v)$ (discrete or continuous distribution).
- If there are more than 1 bidders submitting the highest bid $b^*$, the object will be randomly assigned to one of these bidders with equal probability and the winner will pay $b^*$ for the object (since $b^*$ is also the second highest bid)

**Example 9 (A simple case)**

There aret wo bidders in a second price sealed-bid auction. It is given that

- The true valuations of bidder 1 and 2 are $v_1 = 60$ and $v_2 = 65$;
- Bidder 1 conjectures that bidder 2's valuation has the following probability distribution

$$P\left(v_2 = \underbrace{55}_{v_2^L}\right) = P\left(v_2 = \underbrace{65}_{v_2^M}\right) = P\left(v_2 = \underbrace{75}_{v_2^H}\right) = \frac{1}{3}.$$

- Bidder 2 knows the exact value of bidder 1's valuation $v_1 = 60$.

**(a)** Find the best responses of each bidder.

**(b)** Hence, find a (not all) Bayesian Nash equilibrium for this bidding games. (*Note: It will be very tedious to determine all Bayesian Nash equilibria)

☺Solution of (a)

We first obtain the best response of bidder 2 (denoted by $b_{2j}^*, j = L, M, H$) of different types. Given the bid $b_1$ submitted by bidder 1, the bidder 2's payoff can be expressed as

$$V_2(b_{2j}; b_1, v_{2j}) = \mathbf{1}_{\{b_{2j}<b_1\}}(0) + \mathbf{1}_{\{b_{2j}=b_1\}}\left[\frac{1}{2}(v_{2j} - b_1)\right] + \mathbf{1}_{\{b_{2j}>b_1\}}(v_{2j} - b_1).$$

We consider the following 3 cases:

Case 1a: If $v_{2j} > b_1$,

The bidder 2 can gain the winning status and enjoy a largest payoff $v_{2j} - b_1$ by submitting any bid strictly higher than $b_1$ (i.e. $b_{2j} > b_1$). So the best response is determined to be $b_{2j}^* \in (b_1, \infty)$.

Case 1b: If $v_{2j} = b_1$,

The bidder 2 gets zero payoff whenever he wins or loses the games. Thus the best response is $b_{2j}^* \in [0, \infty)$.

Case 1c: If $v_{2j} < b_1$,

The bidder 2 will get a negative payoff $v_{2j} - b_1 < 0$ if he submits a bid higher than $b_1$. The corresponding best response is $b_{2j}^* \in [0, b_1)$ (Here, bidder 2 will lose the games).

Next, we determine the best response of bidder 1.

Given the bids $b_2(\cdot) = (b_{2L}, b_{2M}, b_{2H})$ submitted by different types of bidder 2, the expected payoff of bidder 1 can be expressed as

$$V_1(b_1; b_2(\cdot), v_1)$$

$$= \sum_{j \in \{L,M,H\}} P(v_2 = v_{2j}) \left[ \mathbf{1}_{\{b_1 < b_{2j}\}}(0) + \mathbf{1}_{\{b_1 = b_{2j}\}} \left[ \frac{1}{2}(v_1 - b_{2j}) \right] \right.$$

$$\left. + \mathbf{1}_{\{b_1 > b_{2j}\}}(v_1 - b_{2j}) \right]$$

$$= \sum_{j \in \{L,M,H\}} \frac{1}{3} \left[ \mathbf{1}_{\{b_1 < b_{2j}\}}(0) + \mathbf{1}_{\{b_1 = b_{2j}\}} \left[ \frac{1}{2}(v_1 - b_{2j}) \right] + \mathbf{1}_{\{b_1 > b_{2j}\}}(v_1 - b_{2j}) \right].$$

To obtain the best response of bidder 1, we consider the following three cases:

Case 2a: If $\max(b_{2L}, b_{2M}, b_{2H}) < v_1$,

Bidder 1 can maximize his payoff by submitting any bid higher than $\max(b_{2L}, b_{2M}, b_{2H})$ since raising the bid can increase the chance of winning without harming the final payoff (why??). So the best response is $b_1^* \in (\max(b_{2L}, b_{2M}, b_{2H}), \infty)$.

Case 2b: If $\max(b_{2L}, b_{2M}, b_{2H}) > v_1 > \min(b_{2L}, b_{2M}, b_{2H})$

We consider the case when $b_{2L} < b_{2M} < v_1 < b_{2H}$ (the derivation of other cases is similar). Note that bidder 1 should get a negative payoff will be resulted if the bidder 2 is of type $H$ ($v_2 = v_{2H}$) as $v_1 - b_{2H} < 0$.

On the other hand, the bidder 1's payoff is maximized if it submits a bid higher than $b_{2M}$ (why). Thus the best response will be $b_1^* \in (b_{2M}, b_{2H})$.

(*Remark: In general, the bidder 1's best response is to submit a bid within $\left(\max_{j, b_{2j} < v_1} b_{2j}, \min_{j, b_{2j} > v_1} b_{2j}\right)$.)

Case 2c: If $\min(b_{2L}, b_{2M}, b_{2H}) > v_1$

Since $v_1 - b_{2i} < 0$ for all $i = L, M, H$, the negative payoff will be resulted if the bidder 1 chooses to offer a bid higher than $\min(b_{2L}, b_{2M}, b_{2H})$, so bidder 1 will lose the games for sure and the best responses will be $b_1^* \in [0, \min(b_{2L}, b_{2M}, b_{2H}))$.

☺Solution of **(b)**

A careful observation shows that submitting a bid that equal to the valuation (i.e. $b_i^* = v_i$) is always the best response of player $i$ with respect to the opponent's strategy. Hence, one possible Bayesian Nash equilibrium will be that all players submit the bid that equal to their own valuation. That is,

$$b_1^* = v_1, \qquad b_{2j}^* = v_2^j, \dots (*)$$

where $j = L, M, H$.

We first verify the optimality of bidder 2.

- If the bidder 2 is of type $L$ (with $v_2 = v_{2L} = 55$), he will submit a bid $b_{2L}^* = v_2 = 55$. Given that $b_1^* = v_1 = 60$, the result in case 1(c) reveals that the bidder 2's best response is any bid within $[0,60)$. This implies that ${b_2^L}^*$ is the best response to $b_1^*$.

- If the bidder 2 is of type $M$ (with $v_2 = v_{2M} = 65$), he will submit a bid $b_{2L}^* = v_2 = 65$. Given that $b_1^* = v_1 = 60$, the result in case 1(a) reveals that the bidder 2's best response is any bid within $[60, \infty)$. This implies that $b_{2L}^*$ is the best response to $b_1^*$.
- If the bidder 2 is of type $H$ (with $v_2 = v_2^H = 75$), one can apply the similar analysis in previous case and deduce that ${b_2^H}^* = v_2^H = 75$ is the best response to $b_1^* = 60$.

Next, we proceed to verify the optimality of bidder 1. Given that ${b_2^L}^* = 55$, ${b_2^M}^* = 65$ and ${b_2^H}^* = 75$, the result in case 2(b) reveals that the best response of bidder 1 is to submit a bid within $(55,65)$. So $b_1^* = 60$ is the best response to bidder 2's strategy.

Hence, we conclude that the strategic profile defined in $(*)$ is the Bayesian Nash equilibrium.

*How about the general case?*

We consider the general case which bidder $i$ conjectures that the valuation of bidder $j$ ($j \neq i$) is randomly drawn from $[\underline{v}_i, \bar{v}_i]$ with some probability density function $f_{v_i}(v)$. Since the full analysis of the equilibrium would be tedious, we focus on some special equilibrium.

As inspired by Example 9, we expect that all players should submit the bid that match their true valuation $v_i$. That is, $b_i^* = v_i$. The optimality of this strategy can be explained as follows:

- Increasing the bid $b_i$ can increase the chance of winning. In the context of second price auction, raising the bid does not harm the payoff upon winning the auction, i.e., $v_i - \max_{j \neq i} b_j$.
- The maximum bid that player $i$ is willing to pay is $v_i$, which is the player $i$'s valuation to the object. Player $i$ may get a negative payoff if others submits a bid higher than $v_i$ and player $i$ wins the games.

**Theorem 1**

In a second price auction games, there is a) Bayesian Nash equilibrium in which all players bid at their true valuations. That is $b_i^*(v_i) = v_i$ for any player $i$ and any type $v_i$.

Proof of the theorem 1

It suffices to check $b_i^*(v_i) = v_i$ is the best response to the opponent's strategies $b_{-i}^* = v_{-i}$. Given the opponent's strategy, the expected payoff of bidder $i$ is seen to be

$$V_i(b_i^*; b_{-i}^*, v_i) = V_i(v_i; v_{-i}, v_i)$$

$$= \sum_{v_{-i}} P\begin{pmatrix} Opponent's \\ valuation \\ is\ v_{-i} \end{pmatrix} \left[ \mathbf{1}_{\left\{v_i < \max\limits_{j \neq i} v_j\right\}}(0) + \mathbf{1}_{\left\{v_i = \max\limits_{j \neq i} v_j\right\}} \left[\frac{1}{n}\left(v_i - \max_{j \neq i} v_j\right)\right] \right.$$
$$\left. + \mathbf{1}_{\left\{v_i > \max\limits_{j \neq i} v_j\right\}} \left(v_i - \max_{j \neq i} v_j\right) \right].$$

where $n$ is number of players submitting the highest bid.

For any $b_i \neq v_i$, one can show that

$$\mathbf{1}_{\left\{v_i<\max\limits_{j\neq i} v_j\right\}}(0)+\mathbf{1}_{\left\{v_i=\max\limits_{j\neq i} v_j\right\}}\left[\frac{1}{n}\left(v_i-\max_{j\neq i} v_j\right)\right]+\mathbf{1}_{\left\{v_i>\max\limits_{j\neq i} v_j\right\}}\left(v_i-\max_{j\neq i} v_j\right)$$
$$\geq \mathbf{1}_{\left\{b_i<\max\limits_{j\neq i} v_j\right\}}(0)+\mathbf{1}_{\left\{b_i=\max\limits_{j\neq i} v_j\right\}}\left[\frac{1}{n'}\left(v_i-\max_{j\neq i} v_j\right)\right]$$
$$+\mathbf{1}_{\left\{b_i>\max\limits_{j\neq i} v_j\right\}}\left(v_i-\max_{j\neq i} v_j\right).$$

This implies that

$$V_i(v_i; v_{-i}, v_i)$$
$$\geq \sum_{v_{-i}} P\begin{pmatrix} Opponent's \\ valuation \\ is\ v_{-i} \end{pmatrix}\left[\mathbf{1}_{\left\{b_i<\max\limits_{j\neq i} v_j\right\}}(0)+\mathbf{1}_{\left\{b_i=\max\limits_{j\neq i} v_j\right\}}\left[\frac{1}{n'}\left(v_i-\max_{j\neq i} v_j\right)\right]\right.$$
$$\left.+\mathbf{1}_{\left\{b_i>\max\limits_{j\neq i} v_j\right\}}\left(v_i-\max_{j\neq i} v_j\right)\right]=V_i(b_i; v_{-i}, v_i).$$

The above inequality reveals that $b_i^* = v_i$ is the best response to $v_{-i}$.

**First price sealed-bid auction under incomplete information**

The games is similar to second price sealed-bid auction except that the winner will pay his own bid (highest bid) to buy the object. Recall that in second price sealed-bid auction, all players submit the bid that equals their own valuations, i.e. $b_i^* = v_i$. However, such strategy is no longer optimal under first price auction games since

- the bidder will get a payoff $v_i - b_i^* = v_i - v_i = 0$ even he can win the games. So the bidder is indifferent between winning and losing.
- By lowering the bid submitted $b_i^* < v_i$, the bidder may be able to receive some positive payoffs, provided $b_i^*$ is the highest bid in some situations.

The equilibrium analysis in first price auction is far more complicated then that in second price auction.

- When a bidder raises his bid, his payoff upon winning, given by $v_i - b_i$, will decrease although the chance of winning the auction increases.

Therefore, a bidder must strike a good balance between the probability of winning the games and the payoff upon winning.

In this section, we shall examine the equilibrium strategy of first-price auction games. Since the analysis for general case is very complicated, we will just focus on the following simplified model:

- There are $N$ bidders and each bidder can submit a bid $b_i$ for the object.
- Bidder $i$ ($i = 1,2, \dots, N$) values the object at $v_i$. The value of $v_i$ is the *private information* of the bidder $i$.
- Bidder $i$ does not know the true valuation of other bidders $v_j$. He only knows that each $v_j$ is a continuous random variable that is randomly drawn from the *common* interval $[\underline{v}, \bar{v}]$ with a *common* probability density function $f(v)$. We assume that all players have the common belief on the true valuations of other bidders.

Under the above assumptions, all bidders appear to be identical (in terms of valuations and belief). Thus, we can focus on the *symmetric Bayesian Nash equilibrium* in which the players having same valuations on the object will submit the same bid.

*Bidder's strategy and expected payoff of bidder $i$*

We let $b_i$ be the strategy of bidder $i$. Note that the bidder $i$'s valuation $v_i$ can be any real number within $[\underline{v}, \bar{v}]$ and different bids may be submitted for different values (types) of $v_i$. Therefore, $b_i = b_i(v_i)$ is a function that maps a valuation $v_i \in [0, \infty)$ to a single bidding strategy $b \in [0, \infty)$.

On the other hand, one may expect that the bidder should place a higher bid if his valuation on the object is higher. To simplify the forthcoming analysis, we impose the following assumption:

| |
|---|
| Assumption:<br>For any $v_i' < v_i''$, we have $b_i(v_i') < b_i(v_i'')$. |

Given the bids $b_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_N)$ submitted by the opponents, the expected payoff of bidder $i$ can be expressed as

$$V_i(b_i; b_{-i}, v_i)$$

$$= P\left(\max_{j\neq i} b_j(v_j) < b_i\right)(v_i - b_i) + \underbrace{P\left(\max_{j\neq i} b_j(v_j) = b_i\right)\left(\frac{1}{n}(v_i - b_i)\right)}_{\approx 0}$$

$$+ P\left(\max_{j\neq i} b_j(v_j) > b_i\right)(0)$$

$$= \left[\prod_{j\neq i} P\big(b_j(v_j) < b_i\big)\right](v_i - b_i) = \left[\prod_{j\neq i} P\left(v_j < b_j^{-1}(b_i)\right)\right](v_i - b_i)$$

where $b_j^{-i}$ is the inverse function of $b_j$,

$$= \left[\prod_{j\neq i}\left(\int_{\underline{v}}^{b_j^{-1}(b_i)} f(v)dv\right)\right](v_i - b_i)$$

$$= \left[\prod_{j\neq i} F\left(b_j^{-1}(b_i)\right)\right](v_i - b_i) \qquad (where\ F(x) = \int_{\underline{v}}^{x} f(v)dv)$$

$$\overset{\substack{b_1=\cdots=b_N \\ under\ symmetric \\ equilibrium}}{\triangleq} \left[\prod_{j\neq i} F\left(b^{-1}(b_i)\right)\right](v_i - b_i)$$

$$= \left[F\left(b^{-1}(b_i)\right)\right]^{N-1}(v_i - b_i).$$

To obtain the best response $b_i = b_i(v_i)$ of the bidder firm $i$, one can consider the first order condition as follows: (Assuming $b_j(v_i)$ is differentiable)

$$\frac{d}{db_i}V_i(b_i; b_{-i}, v_i) = 0 \Rightarrow \frac{d}{db_i}\left\{\left[F\left(b^{-1}(b_i)\right)\right]^{N-1}(v_i - b_i)\right\} = 0$$

$$\Rightarrow (N-1)\left[F\left(b^{-1}(b_i)\right)\right]^{N-2}\frac{d}{db_i}\left(F\left(b^{-1}(b_i)\right)\right)(v_i-b_i)$$
$$-\left[F\left(b^{-1}(b_i)\right)\right]^{N-1}=0$$

$$\Rightarrow (N-1)\left[F\left(b^{-1}(b_i)\right)\right]^{N-2}f\left(b^{-1}(b_i)\right)\frac{d}{db_i}\left(b^{-1}(b_i)\right)(v_i-b_i)$$
$$-\left[F\left(b^{-1}(b_i)\right)\right]^{N-1}=0$$

$$\Rightarrow (N-1)\left[F\left(b^{-1}(b_i)\right)\right]^{N-2}f\left(b^{-1}(b_i)\right)\left(\frac{1}{b'\left(b^{-1}(b_i)\right)}\right)(v_i-b_i)$$
$$-\left[F\left(b^{-1}(b_i)\right)\right]^{N-1}=0$$

Under symmetric equilibrium, we have $b_1(v_i)=b_2(v_i)=\cdots=b_N(v_i)=b(v_i)$. So the above equality can be simplified into

$$\Rightarrow (N-1)\left[F\left(b^{-1}(b)\right)\right]^{N-2}f\left(b^{-1}(b)\right)\left(\frac{1}{b'\left(b^{-1}(b)\right)}\right)(v_i-b(v_i))$$
$$-\left[F\left(b^{-1}(b)\right)\right]^{N-1}=0$$

$$\overset{b^{-1}(b)=b^{-1}(b(v_i))=v_i}{\Rightarrow} \quad (N-1)[F(v_i)]^{N-2} f(v_i) \left(\frac{1}{b'(v_i)}\right)(v_i - b(v_i)) - [F(v_i)]^{N-1} = 0.$$

To solve for $b(v_i)$ from the above differential equations, we rearrange the terms in the above equation as follows:

$$\Rightarrow (N-1)[F(v_i)]^{N-2} f(v_i)\big(v_i - b(v_i)\big) = b'(v_i)[F(v_i)]^{N-1}.$$

$$\Rightarrow b'(v_i)[F(v_i)]^{N-1} + (N-1)[F(v_i)]^{N-2} f(v_i) b(v_i) = (N-1)[F(v_i)]^{N-2} f(v_i) v_i.$$

$$\Rightarrow \frac{d}{dv_i}\left(b(v_i)[F(v_i)]^{N-1}\right) = (N-1)[F(v_i)]^{N-2} f(v_i) v_i$$

$$\Rightarrow \int_{\underline{v}}^{v_i} \frac{d}{dv_i}\left(b(v_i)[F(v_i)]^{N-1}\right) dv_i = \int_{\underline{v}}^{v_i} (N-1)[F(v_i)]^{N-2} f(v_i) v_i \, dv_i$$

$$\overset{F(\underline{v})=0}{\Rightarrow} \quad b(v_i) = \frac{1}{[F(v_i)]^{N-1}} \int_{\underline{v}}^{v_i} (N-1)[F(v_i)]^{N-2} f(v_i) v_i \, dv_i.$$

To simplify the integral on the R.H.S., one may apply integration by parts and obtain

$$b(v_i) = \frac{1}{[F(v_i)]^{N-1}} \int_{\underline{v}}^{v_i} v_i d([F(v_i)]^{N-1})$$

$$= \frac{1}{[F(v_i)]^{N-1}} \left\{ v_i [F(v_i)]^{N-1} \Big|_{\underline{v}}^{v_i} - \int_{\underline{v}}^{v_i} [F(v_i)]^{N-1} dv_i \right\}$$

$$\overset{F(\underline{v})=0}{\cong} \frac{v_i [F(v_i)]^{N-1} - \int_{\underline{v}}^{v_i} [F(v_i)]^{N-1} dv_i}{[F(v_i)]^{N-1}}.$$

So we deduce that the symmetric Bayesian Nash equilibrium is given by

$$b(v_i) = v_i - \frac{\int_{\underline{v}}^{v_i} [F(v_i)]^{N-1} dv_i}{[F(v_i)]^{N-1}}.$$