MATH 4824C: Causal Inference

♠ Solution of Midterm Sample

Q1. In randomized trials; i.e. $(Y_i(1), Y_i(0), X_i) \perp Z_i$, when $E[Y(1)] = 0$, show that $Var(ZY) = P(Z = 1)Var(Y(1))$.

Answer

Proof. Observe that

$$\begin{aligned}
&\mathrm{Var}(ZY)\\
=&E(ZY^2) - (E[ZY])^2\\
=&E[Z((1-Z)Y(0) + ZY(1))^2] - (E[ZY(1)])^2\\
=&E[ZY^2(1)] - E[ZY(1)]^2\\
=&E(Z)\left(E[Y^2(1)]\right) - (EZ)^2 \cdot (EY(1))^2\\
=&E(Z)\left(E[Y^2(1)]\right)\\
=&P(Z=1)\,\mathrm{Var}(Y(1)).
\end{aligned}$$

□

Q2. Show that in observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, $E(Y(1) - X) = E[E(Y - X|Z = 1, X)]$.

Answer

Proof. Observe that

$$\begin{aligned}
&E[E(Y - X|Z = 1, X)]\\
=&E[E(ZY(1) + (1-Z)Y(0)|Z = 1, X) - X]\\
=&E[E(Y(1)|X) - X]\\
=&E(Y(1)) - E(X).
\end{aligned}$$

□

Q3. Show that in observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, $E(Y(1) - X) = E[Z(Y - X)/e(X)]$ where $e(X) = P(Z = 1|X)$.

Answer

Proof. Observe that

$$
\begin{aligned}
& E[Z(Y-X)/e(X)] \\
=& E[Z((1-Z)Y(0)+ZY(1)-X)/e(X)] \\
=& E[(ZY(1)-ZX)/e(X)] \\
=& E[E((ZY(1)-ZX)/e(X)|X)] \\
=& E[(1/e(X))(e(X)E[Y(1)|X]-e(X)X)] \\
=& E(Y(1))-E(X).
\end{aligned}
$$

□

Q4. Show that in observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, let $e(\mathrm{x}, \beta)$, be the logistic model of Z_i on X_i, and $\mu_1(\mathrm{x}, \alpha_1)$ be linear or logistic model of $Y_i(1)$ on X_i. Try to show that

$$
\tilde{\mu}_{1,\mathrm{DR}} = E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right\}}{e\left(\mathrm{X}_i, \beta\right)}+\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right]
$$

is doubly robust, that is, if either $\mu_1(\mathrm{x}, \alpha_1) = \mu_1(\mathrm{x}) := E[Y_i|Z_i = 1, \mathrm{X}_i = \mathrm{x}]$ or $e(\mathrm{x}, \beta) = e(\mathrm{x}) := P(Z_i|\mathrm{X}_i = \mathrm{x})$, then $\tilde{\mu}_{1,DR} = E\{Y_i(1)\}$.

Answer

Proof. It is not hard to verify that

$$
E[Y_i(1)] = E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i\right)\right\}}{e\left(\mathrm{X}_i\right)}+\mu_1\left(\mathrm{X}_i\right)\right].
$$

Then, we observe that

$$
\begin{aligned}
& \tilde{\mu}_{1,DR} - E\left\{Y_i(1)\right\} \\
=& E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right\}}{e\left(\mathrm{X}_i, \beta\right)}+\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right]-E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i\right)\right\}}{e\left(\mathrm{X}_i\right)}+\mu_1\left(\mathrm{X}_i\right)\right] \\
=& E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right\}}{e\left(\mathrm{X}_i, \beta\right)}+\mu_1\left(\mathrm{X}_i, \alpha_1\right)\right]-E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i\right)\right\}}{e\left(\mathrm{X}_i, \beta\right)}+\mu_1\left(\mathrm{X}_i\right)\right] \\
+& E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i\right)\right\}}{e\left(\mathrm{X}_i, \beta\right)}+\mu_1\left(\mathrm{X}_i\right)\right]-E\left[\frac{Z_i\left\{Y_i-\mu_1\left(\mathrm{X}_i\right)\right\}}{e\left(\mathrm{X}_i\right)}+\mu_1\left(\mathrm{X}_i\right)\right] \\
=& E\left[\left(1-\frac{Z_i}{e(X_i,\beta)}\right)(\mu_1(X_i,\alpha_1)-\mu_1(X_i))\right]+E\left[Z_i\{Y_i-\mu_1(X_i)\}\left(\frac{1}{e(X_i,\beta)}-\frac{1}{e(X_i)}\right)\right] \\
=& E\left[E\left[\left(1-\frac{e(X_i)}{e(X_i,\beta)}\right)(\mu_1(X_i,\alpha_1)-\mu_1(X_i))\middle| X_i\right]\right] \\
+& E\left[E\left[e(X_i)\{\mu_1(X_i)-\mu_1(X_i)\}\left(\frac{1}{e(X_i,\beta)}-\frac{1}{e(X_i)}\right)\middle| X_i\right]\right].
\end{aligned}
$$

The conclusion can be directly induced by the resulting form. □

Q5. In observational studies with binary response and binary unmeasured confounders U_i; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid (X_i, U_i)$, if $RR_{ZY}^{obs} = 1.2$, calculate the E-value and interpret it.

Answer

Proof. E-value is defined as

$$RR_{ZY}^{\text{obs}} + \sqrt{RR_{ZY}^{\text{obs}}\left(RR_{ZY}^{\text{obs}} - 1\right)}.$$

In this case, it equals

$$1.2 + \sqrt{1.2 \times 0.2}.$$

Interpretation: see page 20 of Chapter 5 of the lecture notes. □