MATH4321 -- Game Theory

Lecture Note 1

Static games of complete information and Nash equilibrium

Introduction: Static games and Games in normal forms

Static games (of complete information) is the simplest but the most fundamental games in game theory. The games can be described as follows:

- Each player makes a single (once-and-for-all) decision. They make their decisions *simultaneously* so that the players cannot observe the actions chosen by other players. Also, the player can choose his/her action *independently* without any influence from other players.
- Once all players have chosen their actions, an outcome will be revealed based on the players' actions. Payoffs (depending on the outcomes) will be distributed to the players.
- (Complete information) Each player knows everything in the games including (1) available strategies/actions of all players, (2) all possible outcomes generated by various combinations of actions and (3) payoff functions of all players.

In this Chapter, we will examine the static games in details and study some basic concepts that helps us to identify the players' optimal strategies and expected outcomes of the games.

- Static games in normal form: Mathematical formulation
- Solving the games – Identify the optimal strategy chosen by players
  - Dominated strategy: Strategies that players do not choose
  - Nash equilibrium
  - Equilibrium refinement: Pareto-dominance and risk-dominance
- Pure strategy and Mixed strategy
- Existence of Nash equilibrium
- Application of static games

Mathematical formulation of static games in normal (strategic) form

Definition (Games in normal form)

A game in normal form is a games consisting of the following 3 elements:

(i) A finite set of players, denoted by $P = \{1,2, \dots, n\}$;

(ii) A set of strategies (can be finite or infinite) that can be chosen by player i, denoted by S_i. S_i is called *strategic set* of player i. Each player chooses its strategy $s_i \in S_i$ simultaneously and independently (without the influence from other players);

(iii) *Payoff function* of each player, denoted by $V_i(s_1, s_2, \dots, s_n)$ or $V_i(s_i; s_{-i})$, describing the payoff received by player i given the strategies chosen by the players. Here, $s_{-i} = (s_1, s_2, .., s_{i-1}, s_{i+1}, \dots, s_n)$ denotes the strategies chosen by other players.

Mathematically, the games in normal form can be described as an order triple $G = (P, \{S_i\}_{i \in P}, \{V_i\}_{i \in P})$.

Some examples of static games

Example 1 (Paper-Scissor-rock games)

Two players are playing paper-scissor-rock games. Players chooses their actions (paper, scissor or rock) simultaneously. The player will win \$1 if he wins and lose \$1 if he loses. Both players will receive nothing if it is a draw. Express the above games into normal form.

☺Solution

The normal form of this games can be expressed as follows:

- Set of players: $N = \{1,2\}$;
- Strategic set of player i: $S_i = \left\{ \underbrace{Paper}_{\substack{denoted \\ by\ "P"}}, \underbrace{Scissor}_{\substack{denoted \\ by\ "S"}}, \underbrace{Rock}_{\substack{denoted \\ by\ "R"}} \right\}$
- Payoff function:

$$V_i(R;R) = V_i(S;S) = V_i(P;P) = 0,$$
$$V_i(R;S) = V_i(P;R) = V_i(S;P) = 1,$$
$$V_i(R;P) = V_i(P;S) = V_i(S;R) = -1.$$

Remark of Example 1 (Matrix representation of two-person games)

It may be a bit tedious to present the payoff function in this way. For the case of two-person games with finite strategic set, one can represent the player's payoff functions in a matrix form. As an example, the payoff functions in the above example can be expressed as

The column denotes the strategies of player 1

The row denotes the strategies of player 2

		Player 2		
		Paper	Scissor	Rock
Player 1	Paper	$(0,0)$	$(-1,1)$	$(1,-1)$
	Scissor	$(1,-1)$	$(0,0)$	$(-1,1)$
	Rock	$(-1,1)$	$(1,-1)$	$(0,0)$

Each entry in the matrix denotes the payoffs to the players. (i^{th} entry = payoff of player i)

Such games is sometimes called *matrix games*.

Example 2 (Prisoner's dilemma)

Two suspects in a robbery and a murder are put into separate cells. They are known to be guilty in the robbery but the police have no evidence for the murder. Now two suspects are given chances to confess.

- If both confess the crime, each of them will spend 10 years in jails.
- If only one of them confesses, he will act as a witness against the other, who will spend 20 years in jail. In this case, he will receive no punishment.
- If none of them confess, they will be judged for the robbery and each of them will spend 1 year in jail.

We define the payoff functions of each suspect as

$$Payoff = -(term\ of\ imprisonment)$$

Express the above games into normal games form.

☺Solution

The normal form of this games can be expressed as follows:

- Set of players: $N = \{1,2\}$;
- Strategic set of player i: $S_i = \left\{ \underbrace{Confess}_{\substack{denoted \\ by\ "C"}},\ \underbrace{Keep\ slient}_{\substack{denoted \\ by\ "S"}} \right\}$
- Payoff function:

$$V_i(S;S) = -1, \quad V_i(S;C) = -20,$$
$$V_i(C;S) = 0, \quad V_i(C;C) = -10.$$

We can express the payoff functions by the following matrix:

		Player 2	
		Keep silent	Confess
Player 1	Keep silent (S)	$(-1,-1)$	$(-20,0)$
	Confess (C)	$(0,-20)$	$(-10,-10)$

Remark of Example 2

Apparently, one will guess that two suspects chooses to "cooperate" and not to confess the murder (keeping silent) since they will just need to spend 1 year in the jail which is almost the best outcome of the games. However, one can show that if both players act rationally, both will receive 10-year imprisonment.

The reason behind this discouraging fact is that if a player (say player 1) knows that another player keeps silent, he has strict incentive to confess to avoid 1-year imprisonment.

		Player 2	
		Keep silent	Confess
Player 1	Keep silent (S)	$(-1,-1)$	$(-20,0)$
	Confess (C)	$(0,-20)$	$(-10,-10)$

This is a classic example of games which players do not cooperate.

Example 3

An object is to be auctioned. There are n potential bidders (players) bidding for the object. The rules of the auction are as follows:

- Each player submit his/her own bid (denoted by a_i, where $a_i > 0$) and all players submit bids simultaneously.
- The player who offers the highest bid will receive the object. He/she must pay his bid.
- If two or more players offer the highest bid, the object will be randomly assigned to these players with equal probability.

Suppose that player i's valuation of the object is v_i, where $v_1 > v_2 > \cdots > v_n > 0$, express the games into normal form.

☺Solution

The normal form of this games can be expressed as follows:

- Set of players: $P = \{1,2, \dots, n\}$;
- Strategic set of player i: $S_i = [0, \infty)$. (Here, we assume that there is no budget constraint for each player)
- Payoff function: We let $a_i \in S_i$ be the bid chosen by player i.

$$V_i(a_i; \overrightarrow{a_{-i}}) = \begin{cases} v_i - a_i & if \;\; a_i > \max\limits_{j \neq i} a_j \\ \dfrac{1}{M}(v_i - a_i) & if \;\; a_i = \max\limits_{j \neq i} a_j \\ \quad 0 & if \; otherwise \end{cases}.$$

In the second case, M denotes the number of players who offer the highest bid.

Finding "optimal" strategy in a game

Given a static games, one would like to ask what would be the expected outcome of the games. One should evaluate the "optimal strategies" adopted by each player, which is commonly known as *equilibrium* of the games.

To facilitate the equilibrium analysis, we need to make the following assumptions that describe the characteristics of the players:

- **(i)** All players are *rational*: Each player should choose his strategy/action in order to *maximize his payoff* as much as possible.
- **(ii)** All players are *intelligent*: All players should know all information available in the games including available strategies of each player, payoff functions of each player, preferences of each player and possible outcomes of the games.
- **(iii)** All players should engage in *non-cooperative ("selfish") behavior:* Each player is in control of his own actions, and he should choose an action that can achieve his best interest.

If there is only one person in a games (single-person problem), it is easy to identify the player's optimal strategy that maximizes his payoff functions. The story becomes more complicated when there are more than one player involved in the games because *the payoff function of a player will also be affected by the strategies /actions adopted by other players*.

Therefore, it is essential for a player to predict the strategy adopted by other players in order to determine his/her own best strategy.

Mathematically, there are two different approaches:

- Eliminating dominated strategies – Rule out all strategies which the rivals will *not* choose.
- Best response and Nash equilibrium – Predict what will rivals do based on their payoff functions.

To facilitate the analysis, we shall first introduce some additional notations.

Pure strategy & Mixed strategy

In a n-person games, each player usually chooses a *single strategy* s_i from his strategic set S_i. This strategy is known as *pure strategy*.

There is a possibility that a player may *randomize* his strategy. That is, a player chooses a strategy *randomly* from his strategic set with preset probability distribution. This strategy is known as *mixed strategy*.

To simplify our forthcoming discussion, we shall first assume that *each player adopts pure strategy*. The discussion on mixed strategy is postponed to later section.

Strategic profile for pure strategy

Given the strategies chosen by the player. we denote the vector $\vec{s} = (s_1, s_2, \dots, s_n)$, $s_i \in S_i$, be *strategic profile* which collects a set of pure strategies chosen by the players.

Approach #1 – Identifying dominated strategies

To motivate this concept, we consider the following two-person games with the following matrix representation:

		Player 2: A	Player 2: B	Player 2: C
Player 1	A	$(4,2)$	$(1,5)$	$(5,4)$
	B	$(3,3)$	$(1,4)$	$(3,2)$
	C	$(2,4)$	$(6,5)$	$(6,7)$

- We can observe that player 2 is always worse-off if he chooses strategy A instead of strategy B (regardless of player 1's strategy).
- Given that player 2 is rational, he must not choose strategy A. In this case, we say the strategy A is *strictly dominated by* strategy B for player 2 and the strategy A is called *dominated strategy*.

Definition (Dominated strategy)

We let $s_i \in S_i$ and $s_i' \in S_i$ be two possible strategies for player i. We say that s_i is <u>strictly dominated</u> by the strategy s_i' if for <u>any</u> strategies adopted by other players (denoted by $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$), one has

$$V_i(s_i; s_{-i}) < V_i(s_i'; s_{-i})$$

That is, player i is always worse-off by adopting the strategy s_i than s_i', regardless of the strategies adopted by other players.

We say $s_i \in S_i$ is a *dominated strategy* for player i if there is another strategy $s_i' \in S_i$ such that s_i is strictly dominated by s_i'.

Remark (Less important)

We say s_i is *weakly* dominated by the strategy s_i' if and only if $V_i(s_i; s_{-i}) \leq V_i(s_i'; s_{-i}),\ for\ any\ s_{-i}$ and $V_i(s_i; s_{-i}) < V_i(s_i'; s_{-i})$ for some s_{-i}.

Importance of identifying dominated strategy

Since player's payoff will always be lowered if he chooses to adopt dominated strategy, thus a rational player *should not* choose such strategy. Since all players know this, the dominated strategy can be ruled out from the equilibrium analysis and this can simplify the solution procedure.

We consider the above example, we have identified that strategy A is a dominated strategy for player 2. Since player 2 never choose strategy A and this strategy can be ruled out from player 2's strategic set. Thus the games can be reduced into

Player 2

		~~A~~	B	C
	A	~~(4,2)~~	(1,5)	(5,4)
Player 1	B	~~(3,3)~~	(1,4)	(3,2)
	C	~~(2,4)~~	(6,5)	(6,7)

Example 4 (Prisoner Dilemma)

We consider the prisoner dilemma games discussed in Example 2. Show that "keep silent" is strictly dominated strategy for any suspect (player). Hence, determine the possible outcome of the games.

☺Solution

		Player 2	
		Keep silent	Confess
Player 1	Keep silent (S)	$(-1,-1)$	$(-20,0)$
	Confess (C)	$(0,-20)$	$(-10,-10)$

Without loss of generality, we consider the case for player 1 and the derivation for player 2 is similar. One can establish that

$$-1 = \underbrace{V_1(S;S)}_{payoff\ (slient)} < \underbrace{V_1(C;S)}_{payoff\ (confess)} = 0,$$

$$-20 = \underbrace{V_1(S;C)}_{payoff\ (slient)} < \underbrace{V_1(C;C)}_{payoff\ (confess)} = -10.$$

So player 1 finds himself better off to confess instead of keeping silent, regardless of the action taken by another player. Thus, the strategy "keeping silent" is dominated strategy for the player 1 (so does player 2).

		Player 2	
		Keep silent	Confess
Player 1	Keep silent (S)	~~$(-1,-1)$~~	~~$(-20,0)$~~
	Confess (C)	~~$(0,-20)$~~	$(-10,-10)$

Since none of the players will keep silent, the only possible outcome of the games is that both players confess and spend 10 years in jail. In later section, we can show that this is the equilibrium of the games.

Example 5 (Contribution games)

Three members are working on a group project.

- Each member can choose to contribute (denoted by "C") or not contribute (denoted by "N"). Members who contribute needs to pay a cost 3 units.
- The group project can be completed if at least 2 members contribute. If exactly 2 members contribute, each member (whenever he contribute or not) will receive a reward of 2 from the completed group project. If 3 members contribute, the corresponding reward is 4.
- If the group project is incomplete, the reward of each player will be 0.

One can observe that each member can receive a positive payoff $4 - 3 = 1$ if all of them chooses to contribute and this should be a nice outcome. However, we are going to show that it is not the case.

Show that "contribute" is a dominated strategy for all players.

☺Solution

Without loss of generality, we just need to consider the case for player 1. To show "contribute (C)" is a dominated strategy, we need to show

$$\underbrace{V_1(C; s_2, s_3)}_{payoff\ (contribue)} \quad < \quad \underbrace{V_i(N; s_2, s_3)}_{payoff\ (do\ nothing)}$$

for any $(s_2, s_3) \in \{N, C\} \times \{N, C\}$

By some simple calculation, one can verify that

$$-3 = 0 - 3 = V_1(C; N, N) < V_i(N; N, N) = 0;$$
$$-1 = 2 - 3 = V_1(C; C, N) < V_i(N; C, N) = 0;$$
$$-1 = 2 - 3 = V_1(C; N, C) < V_i(N; N, C) = 0;$$
$$1 = 4 - 3 = V_1(C; C, C) < V_i(N; C, C) = 2.$$

One can see that player 1 is always worse-off if he/she chooses to contribute. Therefore, "contribute" is a dominated strategy for any player. Hence, we conclude that none of the players is willing to contribute to the group project.

Example 6 (Dominated strategy does not exist)

We consider the paper-scissor-rock games considered in Example 1. Show that there is no dominated strategy for any player.

☺Solution

The corresponding matrix form of the games is given by

		Player 2		
		Paper	**Scissor**	**Rock**
Player 1	**Paper**	$(0,0)$	$(-1,1)$	$(1,-1)$
	Scissor	$(1,-1)$	$(0,0)$	$(-1,1)$
	Rock	$(-1,1)$	$(1,-1)$	$(0,0)$

To verify that there is no dominated strategy, one needs to show that any given strategy will not be strictly dominated by other strategies.

Without loss of generality, we consider the case for player 1 and the case for player 2 can be derived in a similar fashion. We note that

$$\underbrace{V_i(Paper; Rock)}_{=1} > \underbrace{V_i(Rock; Rock)}_{=0} > \underbrace{V_i(Scissor; Rock)}_{=-1}.$$

The above inequality shows that the strategy "paper" cannot be dominated by other two strategies (scissor and rock). Similarly, one can show that

$$\underbrace{V_i(Rock; Scissor)}_{=1} > \underbrace{V_i(Scissor; Scissor)}_{=0} > \underbrace{V_i(Paper; Scissor)}_{=-1},$$

$$\underbrace{V_i(Scissor; Paper)}_{=1} > \underbrace{V_i(Paper; Paper)}_{=0} > \underbrace{V_i(Rock; Paper)}_{=-1}.$$

These imply that the strategies "rock" and "scissor" cannot be strictly dominated by any other strategy respectively. Hence, we can conclude that there is no dominated strategy in this case.

Iterative Elimination of Strictly Dominated Strategies (IESDS)

The purpose of identifying dominated strategies is to rule out all strategies which are never chosen by players and narrow down the strategic sets. In some cases, one may able to obtain an unique solution [see Example 4].

Even if this is not successful, one can still simplify the games and the equilibrium analysis. In this section, we would like to develop a systematic scheme for eliminating dominated strategy, inspired by previous simple examples.

Roughly speaking, the scheme is as follows:

Step 1: We first consider player 1 and his strategic set S_1. We remove all dominated strategies from his strategic set. We then repeat this process for player 2, 3, .. , n sequentially. We let $S_i^{(1)}$ denote the new strategic set of player i after the first round of elimination.

Step 2: Given strategic sets $S_i^{(k)}$ after k rounds ($k = 1,2, \ldots$) of eliminations, we define a new games with these strategic sets $S_i^{(k)}$, we repeat Step 1 and rule out dominated strategies for each player sequentially.

Step 3: Repeat the elimination process until $S_i^{(k+1)} = S_i^{(k)}$ for all i. That is, there is no more dominated strategy identified. The remaining strategies in the strategic set $S_i^{(k)}$ are reasonable forecast of players' behavior in a games. Any combination of these strategies is known as *iterated-elimination equilibrium*.

Definition (Iterated-elimination equilibrium)
A strategic profile $\vec{s} = (s_1, s_2, \ldots, s_n)$ is said to be iterated-eliminated equilibrium if it *survives* the process of IESDS.

Example 7

We consider the following two-person games with the following matrix representation:

		Player 2		
		C	D	E
Player 1	A	$(1,0)$	$(1,3)$	$(0,2)$
	B	$(-1,5)$	$(0,1)$	$(4,0)$

Simplify the games as much as possible using above iterative scheme and determine the final outcomes if possible.

☺Solution

<u>First stage</u>

To start with, we first consider player 1. One cannot identify any dominated strategy since

$$\underbrace{V_1(A;C)}_{=1} > \underbrace{V_1(B;C)}_{=-1} \quad and \quad \underbrace{V_1(B;E)}_{=4} > \underbrace{V_1(A;E)}_{=0}$$

Here, first inequality states that A cannot be strictly dominated by B and second inequality states that B cannot be strictly dominated by A.

We move on to player 2, we note that strategy E is a dominated strategy since it is strictly dominated by the strategy D. That is,

$$\underbrace{V_2(E;A)}_{=2} < \underbrace{V_2(D;A)}_{=3} \quad and \quad \underbrace{V_2(E;B)}_{=0} < \underbrace{V_2(D;B)}_{=1}.$$

Thus, we eliminate E from player 2's strategic set. After first round of elimination, the games is now reduced into

		Player 2		
		C	D	E
Player 1	A	$(1,0)$	$(1,3)$	X
	B	$(-1,5)$	$(0,1)$	X

<u>Second stage</u>

Now we come back to player 1 again, we observe that strategy B is strictly dominated by strategy A since

$$\underbrace{V_1(B;C)}_{=-1} < \underbrace{V_1(A;C)}_{=1} \quad and \quad \underbrace{V_1(B;D)}_{=0} < \underbrace{V_1(A;D)}_{1}.$$

So we rule out strategy B from player 1's strategy set and player 1 must play strategy A. The games is further reduced into

		Player 2		
		C	D	E
Player 1	A	$(1,0)$	$(1,3)$	X
	B	X	X	X

We consider player 2 again. Knowing that the player 1 chooses A, strategy C is now strictly dominated by strategy D since $\underbrace{V_2(C;A)}_{=0} < \underbrace{V_2(D;A)}_{=3}$. So we eliminate strategy C from the player 2's strategic set.

Since there is only one strategy remaining for each player and there are no more eliminations, thus the strategic profile (A, D) should be the final outcome of the games.

Example 8 (Harder, Cournot Duopoly)

There are two identical firms (Firm 1 and Firm 2) producing some products for a same market. Two firms need to decide the number of products (denoted by $q_i \in [0,\infty)$, $i = 1,2$) produced. It is given that

- The total cost of production of firm i is $c(q_i) = 10q_i$.
- The market price (selling price) of the product, which is driven by supply and demand, is assumed to be $p(q_1, q_2) = 100 - q_1 - q_2$.

Suppose that two firms choose to produce q_1 and q_2 units of goods respectively, the profits of the two firms. denoted by $V_1(q_1, q_2)$ and $V_2(q_1, q_2)$ respectively, can be expressed as

$$V_1(q_1, q_2) = \underbrace{q_1(100 - q_1 - q_2)}_{\substack{revenue \\ =price\times quantity}} - \underbrace{10q_1}_{cost} = 90q_1 - q_1^2 - q_1q_2,$$

$$V_2(q_1, q_2) = q_2(100 - q_1 - q_2) - 10q_2 = 90q_2 - q_2^2 - q_1q_2.$$

By applying IESDS, find the final strategies adopted by each firm.

☺Solution

<u>Step 1</u>

We first consider firm 1 and rule out dominated strategy from his strategic set. Intuitively, one should expect that it is always sub-optimal for the firms to produce too many products (q_1 is too large) or too less products (q_1 is too small). To see this, we note that

$$\frac{\partial}{\partial q_1}V_1(q_1,q_2) = \underbrace{90-2q_1}_{<0\ for\ q_i>45} - q_2 \ldots .(1)$$

It is clear that $\frac{\partial V_1(q_1,q_2)}{\partial q_1} < 0$ for $q_1 > 45$ and $q_2 \geq 0$. This implies the firm 1's payoff will decrease if it chooses to increases q_i beyond 45. Using this fact, one can deduce that for any $q_1 > 45$,

$$\underbrace{V_1(q_1,q_2)}_{\substack{payoff \\ (choose\ q_1>45)}} < \underbrace{V_1(45,q_2)}_{\substack{payoff \\ (choose\ q_1=45)}}, \quad for\ any\ q_2 \geq 0.$$

Since any $q_1 > 45$ is strictly dominated by the strategy $q_1 = 45$, so $q_1 > 45$ is dominated strategy for player 1. Similarly, $q_2 > 45$ is dominated strategy for firm 2. Both strategic sets are narrowed down to $S_1^{(1)} = S_2^{(1)} = [0,45]$.

Step 2

Given the new strategic set, we proceed to identify more dominated strategies. We consider player 1 again. Using the fact that $q_2 \leq 45$, one can observe that

$$\frac{\partial}{\partial q_1} V_1(q_1, q_2) = 90 - 2q_1 - q_2 \geq 90 - 2q_1 - 45 = \underbrace{45 - 2q_1}_{>0\ for\ q_1 < 22.5}.$$

We observe that $\frac{\partial}{\partial q_1} V_1(q_1, q_2) > 0$ for $q_1 < 22.5$. This implies that

$$\underbrace{V_1(q_1, q_2)}_{\substack{payoff \\ (choose\ q_1 < 22.5)}} < \underbrace{V_1(22.5, q_2)}_{\substack{payoff \\ (choose\ q_1 = 22.5)}} \quad for\ any\ \ q_1 < 22.5.$$

Therefore, $q_1 < 22.5$ is also dominated strategy for firm 1. Similarly, one can deduce that $q_2 < 22.5$ is dominated strategy for firm 2. After the second stage of elimination, the strategy sets of both firms are narrowed down to

$$S_1^{(2)} = S_2^{(2)} = [22.5, 45].$$

*Remark: One may further reduce the strategic set using $q_i \geq 0$ (since $S_i^{(1)} = [0, 45]$). However, this only implies that $q_i > 45$ is dominated strategy (same as Step 1).

Step 3 (Recursive relation for elimination process)

We let $S_i^{(k)} = [a_k, b_k]$ be the strategic set of player 1 after k^{th} round of elimination. We consider the $(k+1)^{th}$ round of the elimination process:

- Since $q_j \geq a_k$, we can deduce that
$$\frac{\partial}{\partial q_i} V_i(q_i, q_j) = 90 - 2q_i - q_j \leq 90 - 2q_i - a_k.$$
This implies that $\frac{\partial V_i}{\partial q_i} < 0$ when $q_i > \frac{90-a_k}{2}$. We get
$$V_i(q_i; q_j) < V_i\left(\frac{90 - a_k}{2}; q_j\right) \;\; for\; q_i > \frac{90 - a_k}{2} \;\; and \;\; q_j \geq a_k.$$
So any $q_i > \frac{90-a_k}{2}$ is dominated strategy for player i.
- Similarly, using the fact that $q_j \leq b_k$, we obtain
$$\frac{\partial}{\partial q_i} V_i(q_i, q_j) = 90 - 2q_i - q_j \geq 90 - 2q_i - b_k.$$
- This implies that $\frac{\partial V_i}{\partial q_i} > 0$ when $q_i < \frac{90-b_k}{2}$. We get
$$V_i(q_i; q_j) < V_i\left(\frac{90 - b_k}{2}; q_j\right) \;\; for\; q_i < \frac{90 - b_k}{2} \;\; and \;\; q_j \leq b_k.$$
So any $q_i < \frac{90-b_k}{2}$ is dominated strategy for player i.

Combining the results, we deduce that $S_i^{(k+1)} = [a_{k+1}, b_{k+1}]$ where
$$a_{k+1} = \frac{90 - b_k}{2}, \qquad b_{k+1} = \frac{90 - a_k}{2}$$

By continuing the elimination process, one can narrow down the strategy sets continuously and we summarize the result in the following table:

k	$S_1^{(k)} = S_2^{(k)}$
1	$[0,\ \ 45]$
2	$[22.5,\ \ 45]$
3	$[22.5,\ \ 33.75]$
4	$[28.125,\ \ 33.75]$
5	$[28.125,\ \ 30.9375]$
6	$[29.53125,\ \ 30.9375]$
7	$[29.53125,\ \ 30.23438]$
$\vdots$	$\vdots$
∞	$[30,\ \ 30]$

We observe from the above table that the upper bound and lower bound of the strategic set are getting close to 30. In fact, one can show that the strategic sets will become a singleton. Thus, we conclude that both firms will produce 30 units of products respectively in equilibrium.

Mathematically, one can prove this using the recursive relation derived in Step 3. We summarize the key steps below and omit the detailed verification.

*Recall that $S_i^{(1)} = [0,45]$ and $S_i^{(2)} = [22.5, 45]$ from step 1 and 2, we take $a_1 = 0$, $a_2 = 22.5$, $b_1 = b_2 = 45$.

- ✓ Step 1: Argue that $22.5 \leq a_k \leq b_k \leq 45$ for $k \geq 2$. This can be done using mathematical induction
- ✓ Step 2: Argue that the sequence $\{a_i\}$ is increasing, i.e. $a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$, using mathematical induction.
- ✓ Step 3: Combining the above results, one can deduce that
  - $S_i^{(k+1)} \subseteq S_i^{(k)} \subseteq [0, 45]$
  - Both sequences $\{a_1, a_2, \dots\}$ and $\{b_1, b_2, \dots\}$ converges since the sequence $\{a_k\}$ is increasing and bounded from above and the sequence $\{b_k\}$ is decreasing and bounded from below.
- ✓ Step 4: We let $\lim\limits_{k\to\infty} a_k = a$ and $\lim\limits_{k\to\infty} b_k = b$. By taking limit on the above recursive relation, one can show that $a = b = 30$. So that $S_i^{(\infty)} = \bigcap_{k=1}^{\infty} S_i^{(k)} = [a, b] = [30,30]$ which is a singleton.

Example 9

We consider a two-person games represented by the following payoff matrix:

		Player 2		
		X	Y	Z
Player 1	A	$(6,-2)$	$(5,0)$	$(3,3)$
	B	$(4,3)$	$(2,4)$	$(2,2)$
	C	$(3,4)$	$(3,5)$	$(7,1)$

Using iterative scheme of eliminating dominated strategy, determine all possible outcomes of the games.

☺Solution

<u>First stage</u>

We first consider player 1. One can see that "B" is dominated strategy for player 1 since it is strictly dominated by another strategy "A". That is,

$$\underbrace{V_1(B;X)}_{=4} < \underbrace{V_1(A;X)}_{=6}, \qquad \underbrace{V_1(B;Y)}_{=2} < \underbrace{V_1(A;Y)}_{=5}, \qquad \underbrace{V_1(B;Z)}_{=2} < \underbrace{V_1(A;Z)}_{=3}.$$

On the other hand, we observe that "X" is dominated strategy for player 2 since it is dominated by strategy "Y". That is,

$$\underbrace{V_2(X;A)}_{=-2} < \underbrace{V_2(Y;A)}_{=0}, \qquad \underbrace{V_2(X;B)}_{=3} < \underbrace{V_2(Y;B)}_{=4}, \qquad \underbrace{V_2(X;C)}_{=4} < \underbrace{V_2(Y;C)}_{=5}.$$

So we eliminate strategies "B" and "X" from players' strategic sets and the games is reduced to

Player 2

		Y	Z
Player 1	A	$(5,0)$	$(3,3)$
	C	$(3,5)$	$(7,1)$

Second stage

One can observe that none of the strategies is dominated strategy since

$$\underbrace{V_1(A;Y)}_{=5} > \underbrace{V_1(C;Y)}_{=3}, \qquad \underbrace{V_1(C;Z)}_{=7} > \underbrace{V_1(A;Z)}_{=3} \qquad (player\ 1)$$

$$\underbrace{V_2(Y;C)}_{=5} > \underbrace{V_2(Z;C)}_{=1}, \qquad \underbrace{V_2(Z;A)}_{=3} > \underbrace{V_2(Y;A)}_{=0} \qquad (player\ 2)$$

Since there is no more strategies that can be dominated, so all possible outcomes (players' strategies) of the games is given by

$$S^* = \{(s_1, s_2): s_1 \in \{A, C\},\ \ s_2 \in \{Y, Z\}\}.$$

Remark of Example 9

- In this case, there will be 4 iterated-elimination equilibrium. That is, $(A, Y), (A, Z), (C, Y)$ and (C, Z)
- The above example reveals that the iterative scheme does not necessarily generate an unique solution. It is because the scheme only rule out all strategies that are <u>obviously (or always)</u> sub-optimal for the players. In order to predict a more precise outcomes, one needs to develop a new notation. This notation is called "Nash equilibrium".

Approach #2 – Best response and Nash equilibrium

Different from the first approach, this approach aims to identify the outcome by investigating the strategy which will be likely adopted by the players.

Belief and Best response

In a n-person games, a player needs to figure out all possible strategies that will be adopted by other players in order to determine his own optimal strategy. We call such prediction as *belief* (denoted by B_i) of the player.

Definition (Belief)

In a n-person games, a player's belief B_i is a subset of $S_{-i} = S_1 \times S_2 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ that collects a set of possible strategies adopted by other players.

Initially, a player's belief will be the entire strategic sets of other players. Since all players are rational and intelligent, the belief can be sharpen by considering the *optimality* of his opponents. One possible way is to eliminate all strategies that won't be chosen by the opponents such as dominated strategy.

It is clear that each player will pick a strategy that can maximize his/her payoff function. This strategy is commonly known as *best response*.

Definition (Best response)
Given a set of opponents' strategy $s_{-i} \in S_{-i}$, a player i's strategy $s_i^* \in S_i$ is said to be *best response* to the strategy s_{-i} if and only if the corresponding payoffs of player i is maximized over his strategic set:

$$V_i(s_i^*; s_{-i}) = \max_{s_i \in S_i} V_i(s_i; s_{-i}).$$

Here, the collection of best response (w.r.t. s_{-i}) is denoted by $BR_i(s_{-i})$.

Remark

1. The best response to the given opponents' strategies s_{-i} may not be unique.
2. The best response depends on the rival's strategy.

Example 10

We consider the following 2-person games with the following payoff matrix

		Player 2		
		X	Y	Z
Player 1	A	$(6,-2)$	$(5,3)$	$(3,3)$
	B	$(4,4)$	$(3,2)$	$(2,1)$

(a) For each of player 1's strategy, determine the best response of player 2.

(b) For each of player 2's strategy, determine the best response of player 1.

☺Solution

(a) We consider two scenarios:

- If player 1 chooses "A", the player 2's best response will be $\{Y, Z\}$ since both strategies can yield the highest payoff 3.

- If player 1 chooses "B", the player 2's best response will be $\{X\}$ as it can generate the highest payoff 4.

(b) Similar to (a), we shall consider three cases:

- Suppose player 2 chooses "X", the player 1's best response will be $\{A\}$ (yielding a payoff of 6)
- Suppose player 2 chooses "Y", the player 1's best response will be $\{A\}$ (yielding a payoff of 5).
- Suppose player 2 chooses "Z", the player 1's best response will be $\{A\}$ (yielding a payoff of 3).

Remark of Example 10(b) (Dominated strategy v.s. best response)

One can verify that strategy B is a dominated strategy (i.e. dominated by A). We observed the strategy B is never a best response of player 1 in any scenario. This result is very natural since the player 1 will always be worse-off if he/she chooses strategy A.

Nash Equilibrium

A strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is the desired outcome of the games only when

- Player i has made correct prediction on opponents' strategies. That is,
$$B_i = \{s_{-i}^*\}, \dots \dots (**)$$
where $s_{-i}^* = (s_1^*, s_2^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$.
- The chosen strategy s_i^* should be player i's *best response* to opponents' strategies in his belief. That is,
$$V_i(s_i^*; s_{-i}^*) = \max_{s \in S_i} V_i(s; s_{-i}^*) \, . \dots \dots (*)$$

Combining $(*)$ and $(**)$, we get the following definition of *Nash equilibrium*.

Definition (Nash equilibrium for pure strategy)
A strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is called pure strategy Nash equilibrium if and only if for any player i
$$V_i(s_i^*; s_{-i}^*) = \max_{s_i \in S_i} V_i(s_i; s_{-i}^*) \geq V_i(s_i; s_{-i}^*), \;\; for\ any\ s_i \in S_i \dots (***)$$
In other words, no players has strict incentive to deviate from the strategy s_i^* and adopt other strategies.

Example 11 (Prisoner Dilemma)

We consider the prisoner dilemma games considered in Example 4

		Player 2: Keep silent	Player 2: Confess
Player 1	Keep silent (S)	$(-1,-1)$	$(-20,0)$
	Confess (C)	$(0,-20)$	$(-10,-10)$

In Example 4, we have found that strategic profile $s^* = (C, C)$ (i.e. both prisoners confess) is the only possible outcome of the games. Show that this strategic profile is the Nash equilibrium of the games.

☺Solution

Based on the definition of Nash equilibrium, we need to verify the ineq. (***) [see p.42] for both players.

Since two players are symmetric (in terms of available strategies and payoff functions), we first consider player 1's side (the derivation for the case of player 2 will be similar). We note that

$$\underbrace{V_1(C;C)}_{=-10} > \underbrace{V_1(S;C)}_{-20}.$$

So player 1 has no incentive to change his strategy to "keep silent" (S). Similarly, one can deduce that player 2 has no incentive to "keep silent" instead of "confess". That is,

$$\underbrace{V_2(C;C)}_{=-10} > \underbrace{V_2(S;C)}_{-20}$$

Thus the optimality condition (***) is satisfied and the strategic profile (C, C) is the desired Nash equilibrium.

How to identify Nash equilibria of the games?

In the following examples, we shall present some methodologies to identify the Nash equilibrium of a given games.

Example 12 (Location games)

There are two competing restaurants in a country. Each of them wishes to open a new branch. There are four possible locations (city A, city B, city C and city D) that they can choose. The profits can be made by the restaurants depends on location of the new branches and whether two branches are located at the same place. The expected profits are summarized in the following table:

		Player 2			
		City A	City B	City C	City D
Player 1	City A	$(3,3)$	$(10,9)$	$(11,6)$	$(8,8)$
	City B	$(8,11)$	$(5,5)$	$(12,5)$	$(6,8)$
	City C	$(6,9)$	$(7,10)$	$(4,3)$	$(6,12)$
	City D	$(5,10)$	$(6,10)$	$(8,11)$	$(4,7)$

Find all possible Nash equilibrium of the games.

☺Solution

Step 1: Find the best response of player 1

We first determine the player 1's best response to player 2's strategies. For each possible strategy of player 2 (s_2), we determine the strategy s_1^* that maximizes the player 1's payoff. That is,

$$V_1(s_1^*; s_2) = \max_{s \in S_1} V_1(s; s_2).$$

The player 1's best responses are marked in the payoff matrix as shown below:

		Player 2			
		City A	City B	City C	City D
	City A	$(3,3)$	$(\overline{\mathbf{10}}, 9)$	$(11,6)$	$(\overline{\mathbf{8}}, 8)$
Player 1	City B	$(\overline{\mathbf{8}}, 11)$	$(5,5)$	$(\overline{\mathbf{12}}, 5)$	$(6,8)$
	City C	$(6,9)$	$(7,10)$	$(4,3)$	$(6,12)$
	City D	$(5,10)$	$(6,10)$	$(8,11)$	$(4,7)$

Step 2: Find the best response of player 2

Next, we proceed to determine the player 2's best response to player 1's strategies. For every possible strategy of player 1 (s_1), we determine the strategy s_2^* that maximizes the player 2's payoff, i.e.

$$V_2(s_2^*; s_1) = \max_{s \in S_2} V_2(s; s_1).$$

We mark the player 2's best responses in the payoff matrix as follows:

		Player 2			
		City A	City B	City C	City D
	City A	$(3,3)$	$(\overline{\mathbf{10}}, \overline{\mathbf{9}})$	$(11,6)$	$(\overline{\mathbf{8}}, 8)$
Player 1	City B	$(\overline{\mathbf{8}}, \overline{\mathbf{11}})$	$(5,5)$	$(\overline{\mathbf{12}}, 5)$	$(6,8)$
	City C	$(6,9)$	$(7,10)$	$(4,3)$	$(6, \overline{\mathbf{12}})$
	City D	$(5,10)$	$(6,10)$	$(8, \overline{\mathbf{11}})$	$(4,7)$

Thus the Nash equilibria are $(s_1^*, s_2^*) = (8,11)$ and $(s_1^*, s_2^*) = (10,9)$.

Example 13 (Cournot Duopoly)

We revisit Cournot Duopoly games considered in Example 8 which two competing firms decide the number of products produced in order to maximize their own profits. Given the quantities q_1, q_2 chosen by the firms, the payoffs of the two firms are given by

$$V_1(q_1, q_2) = 90q_1 - q_1^2 - q_1 q_2,$$
$$V_2(q_1, q_2) = 90q_2 - q_2^2 - q_1 q_2.$$

Find all possible Nash equilibrium of the games.

☺Solution:

We first determine the player 1's best response (denoted by $q_1^*(q_2)$, given a strategy q_2 chosen by its rival. This can be done by considering the first-order condition. That is,

$$\frac{\partial V_1(q_1, q_2)}{\partial q_1}|_{q_1=q_1^*(q_2)} = 0 \Rightarrow (90 - 2q_1 - q_2)|_{q_1=q_1^*(q_2)} = 0$$
$$\Rightarrow q_1^*(q_2) = \frac{90 - q_2}{2} \dots (1)$$

Since $\frac{\partial^2 V_1}{\partial q_1^2}|_{q_1=q_1^*(q_2)} = -2 < 0$, thus the player 1's payoff is maximized when $q_1 = q_1^*(q_2)$ and $q_1^* = q_1^*(q_2)$ is the best response of player 1.

We proceed to determine the player 2's best response (denoted by $q_2^*(q_1)$), given a player 1's strategy. Using similar method, we get

$$\frac{\partial V_2(q_1,q_2)}{\partial q_2}\Big|_{q_2=q_2^*(q_1)} = 0 \Leftrightarrow (90-2q_2-q_1)|_{q_2=q_2^*(q_1)} = 0$$

$$\Rightarrow q_2^*(q_1) = \frac{90-q_1}{2} \ldots (2)$$

Note that $\frac{\partial^2 V_1}{\partial q_2^2}|_{q_2=q_2^*(q_1)} = -2 < 0$, the player 2's best response is $q_2^* = q_2^*(q_1) = \frac{90-q_1}{2}$.

Under Nash equilibrium, the player's strategy must be the best response to its rival's strategy. So (q_1^*, q_2^*) must satisfy the eqs. (1) and (2).

$$\begin{cases} q_1^* = \dfrac{90-q_2^*}{2} \\ q_2^* = \dfrac{90-q_1^*}{2} \end{cases}$$

$$\Rightarrow (q_1^*, q_2^*) = (30{,}30).$$

Therefore, we conclude that $(q_1^*, q_2^*) = (30{,}30)$ is the (unique) Nash equilibrium of the games.

Example 14 (Paper-scissor-rock games)

We consider paper-scissor-rock games with the following matrix representation [see Example 1]:

		Player 2		
		Paper	Scissor	Rock
Player 1	Paper (P)	$(0,0)$	$(-1,1)$	$(1,-1)$
	Scissor (S)	$(1,-1)$	$(0,0)$	$(-1,1)$
	Rock (R)	$(-1,1)$	$(1,-1)$	$(0,0)$

Suppose that each player is allowed to use pure strategy, show that there is *no* pure strategy Nash equilibrium in this games.

(*Remark: This example reveals the pure-strategy Nash equilibrium does not always exist)

☺Solution

To do so, one can first try to identify the best response of each player against each of the rival possible strategy. We summarize the result in the following matrix and the best response is highlighted by upper-bar.

		Player 2		
		Paper	Scissor	Rock
Player 1	Paper (P)	$(0,0)$	$(-1,\overline{\mathbf{1}})$	$(\overline{\mathbf{1}},-1)$
	Scissor (S)	$(\overline{\mathbf{1}},-1)$	$(0,0)$	$(-1,\overline{\mathbf{1}})$
	Rock (R)	$(-1,\overline{\mathbf{1}})$	$(\overline{\mathbf{1}},-1)$	$(0,0)$

Recall that the strategy of every player must be the best response of rival's strategy. However, we observe from the above matrix that there is no entry with two upper bounds (i.e. the strategy of at least one player is *not* the best response to rival's strategy).

Therefore, we deduce that there is no pure strategy Nash equilibrium in this games.

Example 15 (Dominated strategy v.s. Nash equilibrium)

We let $s_i^0 \in S_i$ be a *dominated strategy* of player i. Show that, for any opponents' strategy $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$, the strategic profile $s = \left(s_1, s_2, \dots, s_{i-1}, s_i^0, s_{i+1}, \dots, s_n\right)$ cannot be Nash equilibrium.

☺Solution

It is sufficient to show that s_i^0 is not best response to the strategy s_{-i}. Since s_i^0 is a dominated strategy, there is another strategy $s_i' \in S_i$ such that

$$V_i\left(s_i^0; s_{-i}\right) < V_i(s_i'; s_{-i}),$$

for any s_{-i}. The inequality implies that player i will be better off if he chooses strategy s_i'. So s_i^0 is not a best response to the strategy s_{-i}. Thus $\vec{s}$ cannot be Nash equilibrium.

Remark of Example 15

This example also reveals any dominated strategy is never a best response.

Example 16 (IESDS v.s. Nash equilibrium)

Suppose that a strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ uniquely survives IESDS in a finite normal-form game, then s^* is Nash equilibrium.

(*Note: A finite games is a game with a finite number of players and the number of strategies in S_i is finite for all players i.)

☺Solution

We shall prove by contradiction. Suppose that there is a player i which s_i^* is not the best response to s_{-i}^*, then there exists a pure strategy $s_i^{(1)} \neq s_i^*$ such that $V_i\left(s_i^{(1)}; s_{-i}^*\right) > V_i(s_i^*, s_{-i}^*)$.

Since $s_i^{(1)}$ was eliminated from IESDS (as s_i^* is the *unique* survivor in IESDS), so there exists another strategy $s_i'' \neq s_i'$ such that $V_i\left(s_i^{(2)}; s_{-i}^*\right) > V_i\left(s_i^{(1)}, s_{-i}^*\right)$.

By repeating this process, one can deduce that there exists a set of strategy $s_i^{(1)}, s_i^{(2)}, \dots, s_i^{(n)} \in S_i$ such that

$$V_i\left(s_i^{(n)}; s_{-i}^*\right) > \cdots > V_i\left(s_i^{(2)}, s_{-i}^*\right) > V_i\left(s_i^{(1)}, s_{-i}^*\right) > V_i(s_i^*, s_{-i}^*)$$

As n is arbitrary, it implies that S_i has infinitely many elements and it leads to contradiction.

Equilibrium Refinement

We observe that there may exist multiple equilibria in some games. One may ask which equilibrium will be the desired outcome of the games. In order to determine the final outcome of the games from the set of equilibria, one needs to impose some *addition* conditions to rule out the equilibria that are unlikely to occur. This process is called *equilibrium refinement*.

In this section, we shall examine two of the criterions that are commonly used in selecting the final equilibrium.

- Pareto-dominance: Equilibrium selection with respect to the players' payoff.
- Risk-dominance: Equilibrium selection with respect to the "stability" of the equilibrium.

Pareto Dominance

Example 17 (Coordination games)

To motivate this concept, we consider the following coordination games: Two drivers are driving along a narrow road. Unfortunately, the road is blocked by a tree. Two drivers can choose to either lifting the tree or doing nothing. The tree can only be removed when both drivers work together.

- If both drivers choose to lift the tree, the tree is then removed and two drivers can continue their journey, The payoff (in form of utility) to each driver is assumed to be 3.
- If there is only one driver lifting the tree, this driver will get injured and another driver needs to take this injured driver to the hospital. The payoffs to the injured driver and the uninjured driver are assumed to be -15 and -1 respectively.
- If both drivers do nothing, both drivers take a U-turn and return home. The payoffs to each driver is assumed to be 0.

The payoff functions can be expressed by the following matrix:

		Player 2	
		Lift (L)	Nothing (N)
Player 1	Lift (L)	$(3,3)$	$(-15,-1)$
	Nothing (N)	$(-1,-15)$	$(0,0)$

One can deduce that there are two Nash equilibria in this games. That is, $(s_1^*, s_2^*) = (L, L)$ and $(s_1^*, s_2^*) = (N, N)$.

One can observe that the payoffs of two drivers are strictly higher under the equilibrium (L, L), we expect that these two drivers should choose to lift the tree (instead of doing nothing) since it can yield a better payoff and create a "win-win" outcomes.

We say the equilibrium (L, L) *Pareto dominates* the equilibrium (N, N) and the equilibrium (L, L) is said to be *Pareto optimal*.

Definition (Pareto-dominance and Pareto-optimality)

In a n-person games, we say a strategic profile (or equilibrium) $s = (s_1, s_2, \ldots, s_n)$ <u>Pareto dominates</u> another strategic profile $s' = (s_1', s_2', \ldots, s_n')$ if and only if

$$V_i(s_i; s_{-i}) \geq V_i(s_i'; s_{-i}'), \quad for\ all\ i = 1,2, \ldots, n,$$

and

$$V_i(s_i; s_{-i}) > V_i(s_i'; s_{-i}'), \quad for\ some\ i = 1,2, \ldots, n.$$

On the other hand, we say a strategic profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is <u>Pareto optimal</u> if there is no strategy s' ($\neq s^*$) that Pareto dominates it.

Remark:

1. The first inequality states that none of the players will be worse off if all players chooses the strategy s instead of s'.
2. The second inequality states that at least one of the players will be better off if all players chooses the strategy s instead of s'

Example 18 (Battle of the sexes, coordination games)

A couple needs to choose a place for their dinner. There are three locations that they can choose: McDonalds, sushi and buffet. Each of them can make their own choice independently. If both of them choose the same place, they will go to that place for dinner happily. Otherwise, they cancel the dating and return home immediately. The payoff matrix (in terms of utility) is given below:

		Girl (Player 2)		
		McDonald (M)	Sushi (S)	Buffet (B)
Boy (Player 1)	McDonald (M)	$(5,2)$	$(-1,-1)$	$(-1,-1)$
	Sushi (S)	$(-1,-1)$	$(6,6)$	$(-1,-1)$
	Buffet (B)	$(-1,-1)$	$(-1,-1)$	$(2,5)$

It can be shown that (M,M), (S,S) and (B,B) are Nash equilibria of the games (left as exercise). Find the equilibrium that is Pareto-optimal.

☺Solution

We observe from the above payoff matrix that

$$\underbrace{V_1(S;S)}_{=6} > \underbrace{V_1(M;M)}_{=5}, \qquad \underbrace{V_2(S,S)}_{=6} > \underbrace{V_2(M,M)}_{=2};$$

$$\underbrace{V_1(S,S)}_{=6} > \underbrace{V_1(B,B)}_{=2}, \qquad \underbrace{V_2(S,S)}_{=6} > \underbrace{V_2(B,B)}_{=5};$$

- The first set of inequalities reveals that the strategy (S,S) Pareto dominates (M,M). So (M,M) is *not* Pareto optimal.
- The second set of inequalities reveals that the strategy (S,S) Pareto dominates (B,B). So (B,B) is *not* Pareto optimal.

Therefore, (S,S) is the desired Pareto-optimal equilibrium since there is no equilibrium that can dominate it.

Example 19 (Game of chicken)

In this games, two players (A and B) drive their cars directly to each other in a straight road. There will be a head-on collision and both players will be killed if none of the players choose to turn away. Each player has 2 choices to choose: drive towards (D) or turn away (T).

- If both players chooses to drive towards, there will be a collusion and both players will be killed. The payoff to each player is assumed to be -200.
- If one of the player chooses to turn away, this player (chicken or coward) will be the loser. Another player, who chooses to drive forward, will be the winner. The payoffs to the winner and the loser are assumed to be 100 and -20, respectively
- If both players chooses to turn away, both players will receive nothing in this case.

The payoff matrix of this games can be expressed as follows:

		Player 2	
		Drive (D)	Turn (T)
Player 1	Drive (D)	$(-200,-200)$	$(100,-20)$
	Turn (T)	$(-20,100)$	$(0,0)$

Some analysis reveals that there are two Nash equilibria (left as exercise) in this games. That is, (D, T) and (T, D). Show that both equilibria are Pareto-optimal.

☺Solution

It is suffice to show that each of the equilibria is not dominated by another equilibrium. From the payoff matrix, we observe that

- (D, T) is not Pareto-dominated by (T, D) since
$$\underbrace{V_1(D;T)}_{=100} > \underbrace{V_1(T;D)}_{=-20}.$$
In other words, player 1 will be worse-off under the equilibrium (T, D).
- (T, D) is not Pareto-dominated by (D, T) since
$$\underbrace{V_2(D;T)}_{=100} > \underbrace{V_2(T;D)}_{=-20}$$
In other words, player 2 will be worse off under the equilibrium (D, T).

Therefore, both equilibria are Pareto optimal since any equilibrium is not Pareto-dominated by any other equilibrium.

Remark of Example 19

This examples reveals that the Pareto optimal equilibria is not necessarily to be unique.

Risk Dominance

To motivate this concept, we revisit the coordination games in Example 17.

		Player 2	
		Lift (L)	Nothing (N)
Player 1	Lift (L)	$(3,3)$	$(-15,-1)$
	Nothing (N)	$(-1,-15)$	$(0,0)$

Recall that there are two Nash equilibria [(L, L) and (N, N)] in this games. Apparently, one will expect that both players choose to lift the tree (instead of doing nothing) since it yields a strictly higher payoff.

However, this equilibrium depends heavily on the rationality of the player. That is, all players wish to maximize their payoff and each player fully believe that his opponents should adopt the strategy that maximizes his payoff (i.e. "L").

In reality, it may be risky for the player to choose "L" since he may suffer from a big loss (getting the payoff -15) if another player prefers to follow another equilibrium (N, N) and chooses to play N.

		Player 2	
		Lift (L)	Nothing (N)
Player 1	Lift (L)	$(3,3)$	$(-15, -1)$
	Nothing (N)	$(-1, -15)$	$(0,0)$

Player 1's payoff drops from 3 to -15 if player 2 chooses "Do nothing" (N) (prefers the second Nash equilibrium).

Being alerted by the existence of potential risk, two players may prefer to "do nothing" and the Nash equilibrium (N, N) is resulted.

Given a pair of Nash equilibria $[(s_1^*, s_2^*)$and $(s_1^{**}, s_2^{**})]$ in a two-person games, we would like to compare the stabilities between two equilibria and see which equilibrium is likely to occur.

For the Nash equilibrium (s_1^*, s_2^*), we define the *deviation cost* as the product of players' losses if a player chooses to play another strategy s_i^{**} instead of s_i^* (moving from (s_1^*, s_2^*) to (s_1^{**}, s_2^{**}). That is,

$$\Phi(s_1^*, s_2^*) = \underbrace{\left(V_1(s_1^*; s_2^*) - V_1(s_1^{**}; s_2^*)\right)}_{\substack{player\ 1's\ loss \\ (choose\ s_1^{**}\ instead\ of\ s_1^*)}} \underbrace{\left(V_2(s_2^*; s_1^*) - V_2(s_2^{**}; s_1^*)\right)}_{\substack{player\ 2's\ loss \\ (choose\ s_2^{**}\ instead\ of\ s_2^*)}}.$$

Similarly, one can define the corresponding deviation cost for (s_1^{**}, s_2^{**}). That is,

$$\Phi(s_1^{**}, s_2^{**}) = \underbrace{\left(V_1(s_1^{**}; s_2^{**}) - V_1(s_1^*; s_2^{**})\right)}_{\substack{player\ 1's\ loss \\ (choose\ s_1^*\ instead\ of\ s_1^{**})}} \underbrace{\left(V_2(s_2^{**}; s_1^{**}) - V_2(s_2^*; s_1^{**})\right)}_{\substack{player\ 2's\ loss \\ (choose\ s_2^*\ instead\ of\ s_2^{**})}}.$$

Definition (Risk-dominance, Harsanyi and Selten (1988))
We let (s_1^*, s_2^*) and (s_1^{**}, s_2^{**}) be two pure strategy Nash equilibria in a 2-person games. (s_1^*, s_2^*) is said to *risk-dominant* against another equilibrium (s_1^{**}, s_2^{**}) if and only if

$$\Phi(s_1^*, s_2^*) > \Phi(s_1^{**}, s_2^{**}).$$

Remark

- When $\Phi(s_1^*, s_2^*) > \Phi(s_1^{**}, s_2^{**})$, the deviation cost of (s_1^*, s_2^*) is higher. The players will suffer from a greater loss if the player "moves" to another equilibrium (s_1^{**}, s_2^{**}). So the players have less incentive to deviate and the equilibrium is (s_1^*, s_2^*) is more stable (safer) than another equilibrium $\Phi(s_1^{**}, s_2^{**})$.
- In the above example, we observe that (N, N) has larger deviation cost ($\Phi(N, N) = \big(0 - (-15)\big)\big(0 - (-15)\big) = 225$) than that of (L, L) ($\Phi(L, L) = \big(3 - (-1)\big)\big(3 - (-1)\big) = 16$). So (N, N) is risk-dominant against (L, L).
- In Harsanyi and Selten (1988), they justified the concept of Harsanyi and Selten using probabilistic approach and the above criterion is the mathematical consequence from the approach.

Example 20 (Battle of sexes)

A couple decide their activity in the coming date. There are two activities that they can choose: playing video games or going to concert. Although they wish to go together, they cannot contact each other since their phones are out-of-battery. So each of them will join one of these activities. The following matrix summarizes their payoffs under different scenarios.

		Girl (Player 2)	
		Games	Concert
Boy	Games (G)	$(6,3)$	$(-20,2)$
(Player 1)	Concert (C)	$(0,0)$	$(3,6)$

Apparently, boy prefers video games and girl prefers concert. If girl goes to concert, the girl will be "very angry" if the boy goes to play video games and there is a great loss (?) to boy's payoff. Find the risk-dominant pure Nash equilibrium of this games.

☺Solution

We first determine all possible pure Nash equilibrium of the games. For each possible pure strategy s_j adopted by the opponent, we determine the best response s_i^* of the player to s_j. That is, $V_i\left(s_i^*; s_j\right) = \max_{s_i \in S_i} V_i\left(s_i; s_j\right)$.

The best responses of the players are summarized in the following matrix:

		Girl (Player 2)	
		Games	Concert
Boy (Player 1)	Games (G)	$(\bar{\mathbf{6}}, \bar{\mathbf{3}})$	$(-20,1)$
	Concert (C)	$(0,0)$	$(\bar{\mathbf{3}}, \bar{\mathbf{6}})$

So (G, G) and (C, C) are pure strategy Nash equilibrium.

The deviation costs of these two equilibria are found to be

$$\Phi(G, G) = (6-0)(3-1) = 12,$$

$$\Phi(C, C) = \left(3-(-20)\right)(6-0) = 138.$$

Since (C, C) has higher deviation cost, so (C, C) is risk dominant Nash equilibrium.

Example 21 (Pareto dominance v.s. Risk dominance)

We consider the following 2-person matrix games with the following payoff matrix.

		Player 2	
		C	D
Player 1	A	$(\overline{\mathbf{6}},\overline{\mathbf{5}})$	$(-20,1)$
	B	$(0,-20)$	$(\overline{\mathbf{3}},\overline{\mathbf{4}})$

One can verify that both (A,C) and (B,D) are Nash equilibria. Also note that

- (B,D) is Pareto dominated by (A,C) and (A,C) is Pareto optimal equilibrium.
- In terms of risk dominance, the deviation costs of these two equilibria are

$$\Phi(A,C)=(6-0)(5-1)=24,$$
$$\Phi(B,D)=\big(3-(-20)\big)\big(4-(-20)\big)=552.$$

  Hence, (B,D) is the risk-dominant equilibrium since it has higher deviation cost.

Therefore, it is not necessary that these two criterions yield the same conclusion. It is because they perform the equilibrium selection based on different consideration.

Mixed strategy: Randomizing the strategies

In previous examples, we assumed that each player chooses a single strategy (pure strategy). In this section, we relax this assumption and consider the possibility that a player chooses a strategy *randomly* from a set of available strategies. This action is called *mixed strategy*.

Definition (Mixed strategy for finite strategic set)
We let $S_i = \{s_{i1}, s_{i2}, \dots, s_{iN_i}\}$ be strategic set of player i. A *mixed strategy* of player i is a vector $\sigma_i = \left(\sigma_i(s_{11}), \sigma_i(s_{i2}), \cdots, \sigma_i(s_{iN_i})\right)$ that describes the probability distribution over the strategic set S_i, where $\sigma_i(s_{ij}) = P(s_i = s_{ij})$ denotes the probability that player i chooses s_{ij} (*Note: N_i denotes number of pure strategies in S_i)

Here, the probability $\sigma_i(s_{ij})$ should possess the following properties:

- $\sigma_i(s_{ij}) \geq 0$ and
- $\sum_{j=1}^{N_i} \sigma_i(s_{ij}) = 1.$

Why does players choose mixed strategy?

Technically, the mixed strategy can *extend* the available actions of the players. It can help us to identify the equilibrium of the games when there is no pure strategy Nash equilibrium (such as paper-scissor-rock games). Some rigorous mathematical arguments show that there always exists a mixed strategy Nash equilibrium in any n-player normal-form game with finite strategic sets S_i.

Conceptually, the main reason of using mixed strategy is that the players are *uncertain* about the strategies of his opponents. As an example, we consider paper-scissor-rock games. Suppose that a player is very sure that his rival chooses "scissor", the player should choose "rock" for sure without mixing his strategies.

However if the player does not know the move of his rival, the player cannot fix his strategy since the best responses are different for different rival's strategy.

Player's payoff: Expected payoff

Given the mixed strategies $\sigma_1, \sigma_2, \dots, \sigma_n$ chosen by players, the player i's expected payoff, denoted by $V_i(\sigma_i; \sigma_{-i})$, can be expressed as

$$V_i(\sigma_i; \sigma_{-i}) = \sum_{\vec{s} \in S_1 \times \ldots \times S_n} P\big(players\ chooses\ s = (s_1, s_2, \dots, s_n)\big)\, V_i(s_i; s_{-i})$$

$$= \sum_{\vec{s} \in S_1 \times \ldots \times S_n} \left(\prod_{k=1}^{n} P\left(\begin{matrix} player\ k \\ chooses\ s_k \end{matrix} \right) \right) V_i(s_i; s_{-i}) = \sum_{\vec{s} \in S_1 \times \ldots \times S_n} \left(\prod_{k=1}^{n} \sigma_k(s_k) \right) V_i(s_i; s_{-i})$$

$$= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} (\sigma_i(s_i)) \underbrace{\left(\prod_{\substack{k=1,\dots,n \\ k \neq i}} \sigma_k(s_k) \right)}_{denoted\ by\ \sigma_{-i}(s_{-i})} V_i(s_i; s_{-i})$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \underbrace{\left(\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) V_i(s_i; s_{-i}) \right)}_{V_i(s_i; \sigma_{-i})}$$

The second equality follows from the assumption that each player chooses his strategy independently.

Example 21

The payoff matrix of a two-person games is given as follows:

		Player 2		
		A	B	C
Player 1	A	(2,5)	(4,4)	(6,3)
	B	(3,0)	(1,4)	(2,2)
	C	(4,2)	(3,3)	(2,4)

Suppose that the mixed strategies chosen by player 1 and player 2 are $\sigma_1 = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ and $\sigma_2 = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)$ respectively.

Calculate the expected payoff of player 1.

☺Solution

The player 1's expected payoffs can be computed as

$$V_1(\sigma_1;\sigma_2)=\sum_{s_1\in\{A,B,C\}}\sum_{s_2\in\{A,B,C\}}\left(\underbrace{\sigma_1(s_1)\sigma_2(s_2)}_{\Pr\left(\substack{players\ plays\\ s=(s_1,s_2)}\right)}\right)\times V_1(s_1;s_2)$$

$$\begin{aligned}
&=\sigma_1(A)\sigma_2(A)V_1(A;A)+\sigma_1(A)\sigma_2(B)V_1(A;B)+\sigma_1(A)\sigma_2(C)V_1(A;C)\\
&\quad+\sigma_1(B)\sigma_2(A)V_1(B;A)+\sigma_1(B)\sigma_2(B)V_1(B;B)+\sigma_1(B)\sigma_2(C)V_1(B;C)\\
&\quad+\sigma_1(C)\sigma_2(A)V_1(C;A)+\sigma_1(C)\sigma_2(B)V_1(C;B)+\sigma_1(C)\sigma_2(C)V_1(C;C)
\end{aligned}$$

$$\begin{aligned}
&=\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)(2)+\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)(4)+\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)(6)+\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)(3)+\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)(1)\\
&\qquad+\left(\frac{1}{2}\right)\left(\frac{1}{5}\right)(2)+\left(\frac{1}{6}\right)\left(\frac{3}{5}\right)(4)+\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)(3)+\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)(2)
\end{aligned}$$

$$=\frac{17}{6}.$$

Mixed strategy Nash equilibrium

We are ready to state the definition of Nash equilibrium for mixed strategy. The definition is very similar to that for pure strategy. A strategic profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ is said to be Nash equilibrium if and only if

- Player i makes a correct guess on the opponents' strategy. That is, $B_i = \{\sigma_{-i}^*\}$.
- The player i's strategy σ_i^* is the best response to the strategies $\sigma_{-i}^* \in B_i$ of the opponents under his belief.

Definition (Nash equilibrium for mixed strategy)

The strategic profile $\sigma = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ is a Nash equilibrium if and only if for any player i,

$$V_i(\sigma_i^*; \sigma_{-i}^*) \geq V_i(\sigma_i; \sigma_{-i}^*),$$

for any mixed strategy σ_i of player i.

Example 22

We consider the Battle of sexes games with the following payoff matrix

		Player 2	
		Sushi (S)	Buffet (B)
Player 1	Sushi (S)	(3,2)	(0,1)
	Buffet (B)	(0,0)	(2,3)

Find all possible mixed strategy Nash equilibrium of this games.

☺Solution

We let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$ be the mixed strategies of two players. The expected payoffs of two players are given by

$$\begin{aligned} V_1(p;q) &= pq(3) + p(1-q)(0) + (1-p)(q)(0) + (1-p)(1-q)(2) \\ &= 5pq - 2p - 2q + 2; \end{aligned}$$

$$\begin{aligned} V_2(q;p) &= pq(2) + p(1-q)(1) + (1-p)(q)(0) + (1-p)(1-q)(3) \\ &= 4pq - 2p - 3q + 3. \end{aligned}$$

Given the mixed strategy adopted by player 2 ($\sigma_2 = (q, 1-q)$), we determine the player 1's best response ($\sigma_1^* = (p^*, 1-p^*)$). We note that

$$\frac{\partial V_1(p;q)}{\partial p} = 5q - 2 \Rightarrow \frac{\partial V_1(p;q)}{\partial p} = \begin{cases} > 0 & if \;\; q > \frac{2}{5} \\ = 0 & if \; q = \frac{2}{5} \\ < 0 & if \; q < \frac{2}{5} \end{cases}.$$

So the player 1 best response $\sigma_1^* = (p^*, 1-p^*)$ is seen to be

$$p^* = \begin{cases} 1 & if \;\; q > \frac{2}{5} \\ x & if \; q = \frac{2}{5} \\ 0 & if \; q < \frac{2}{5} \end{cases}, \qquad where\ x\ is\ any\ number\ between\ 0\ and\ 1.$$

(Note: When $q = \frac{2}{5}$, the payoff function $V_1(p;q) = V_1\left(p; \frac{2}{5}\right) = \frac{6}{5}$ is independent of p.)

Given the mixed strategy adopted by player 1 ($\sigma_1 = (p, 1-p)$), we determine the player 2's best response ($\sigma_2^* = (q^*, 1-q^*)$). We note that

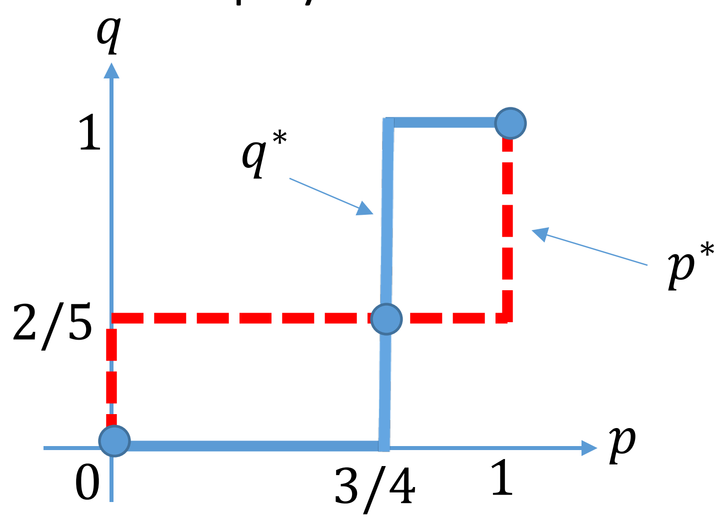
$$\frac{\partial V_2(q;p)}{\partial q} = 4p - 3 \Rightarrow \frac{\partial V_2(q;p)}{\partial q} = \begin{cases} > 0 & if \;\; p > \frac{3}{4} \\ = 0 & if \; p = \frac{3}{4} \\ < 0 & if \; p < \frac{3}{4} \end{cases}.$$

So the player 2 best response $\sigma_2^* = (q^*, 1-q^*)$ is seen to be

$$q^* = \begin{cases} 1 & if \;\; p > \frac{3}{4} \\ y & if \; p = \frac{3}{4} \\ 0 & if \; p < \frac{3}{4} \end{cases}, \qquad where\; y\; is\; any\; number\; between\; 0\; and\; 1.$$

(Note: When $p = \frac{3}{4}$, the payoff function $V_2(q;p) = V_1\left(q;\frac{3}{4}\right) = \frac{3}{2}$ is independent of q.)

To identify the Nash equilibrium, one can consider the following figures which shows the best responses of two players.



We observe that the two curves p^* and q^* intersect at $(p^*, q^*) = (0,0)$, $\left(\frac{3}{4}, \frac{2}{5}\right)$ and $(1,1)$ respectively. Note that the first and third intersection points corresponds to the pure strategy Nash equilibrium (B, B) and (S, S) respectively. The only mixed strategy Nash equilibrium is found to be

$$\sigma_1^* = (p^*, 1 - p^*) = \left(\frac{3}{4}, \frac{1}{4}\right), \qquad \sigma_2^* = (q^*, 1 - q^*) = \left(\frac{2}{5}, \frac{3}{5}\right).$$

Example 23 (Welfare games)

In a country, the unemployment rate is very high since many people in the country do not want to work. To encourage them to work, government may choose to aid them by providing some subsidies. However, the people may just rely on government's subsidy and do not find jobs. They search for work only when he cannot depend on the help from government. This leaves us a question: Should the government aid those people?

The welfare games is constructed as follows:

- There are two players: Government and unemployed person.
- Government can choose to either aid the person (denoted by A) or does not aid the person (denoted by NA).
- On the other hand, the person can choose to either find a job or work (W) or do nothing (denoted by NW).

The payoff functions of the games are summarized in the following matrix

		Unemployed person (Player 2)	
		Work (W)	Do nothing (NW)
Government (Player 1)	Aid (A)	$(3,2)$	$(-1,3)$
	Does not aid (NA)	$(1,1)$	$(0,0)$

One can show that there is no pure strategy Nash equilibrium in this games. Determine if there is any mixed strategy Nash equilibrium in this games.

☺Solution

We let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$ be the mixed strategies of two players. The expected payoffs of two players are given by

$$V_1(p;q) = pq(3) + p(1-q)(-1) + (1-p)(q)(1) + (1-p)(1-q)(0)$$
$$= 3pq - p + q;$$

$$V_2(q;p) = pq(2) + p(1-q)(3) + (1-p)(q)(1) + (1-p)(1-q)(0)$$
$$= 3p + q - 2pq.$$

Given the mixed strategy adopted by player 2 ($\sigma_2 = (q, 1-q)$), we determine the player 1's best response ($\sigma_1^* = (p, 1-p)$). We note that

$$\frac{\partial V_1(p;q)}{\partial p} = 3q - 1 \Rightarrow \frac{\partial V_1(p;q)}{\partial p} = \begin{cases} > 0 & if\ \ q > \frac{1}{3} \\ = 0 & if\ q = \frac{1}{3} \\ < 0 & if\ q < \frac{1}{3} \end{cases}.$$

So the player 1 best response $\sigma_1^* = (p^*, 1-p^*)$ is seen to be

$$p^* = \begin{cases} 1 & if\ \ q > \frac{1}{3} \\ x & if\ q = \frac{1}{3} \\ 0 & if\ q < \frac{1}{3} \end{cases}, \qquad where\ x\ is\ any\ number\ between\ 0\ and\ 1.$$

(Note: When $q = \frac{1}{3}$, the payoff function $V_1(p;q) = V_1\left(p;\frac{1}{3}\right) = \frac{1}{3}$ is independent of p.)

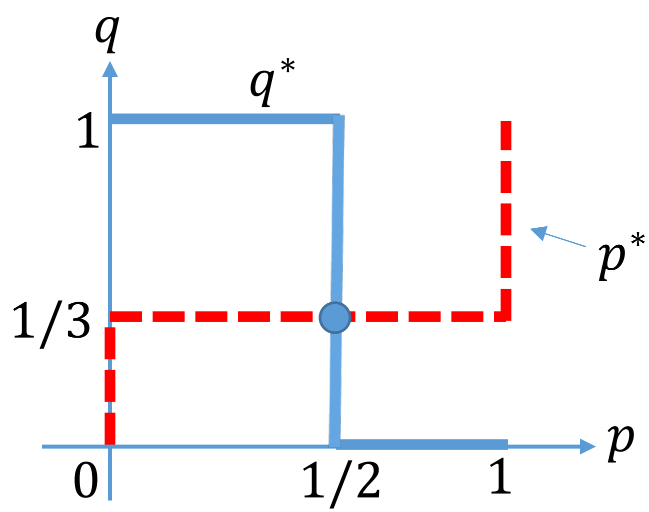
Given the mixed strategy adopted by player 1 ($\sigma_1 = (p, 1-p)$), we determine the player 2's best response ($\sigma_2^* = (q^*, 1-q^*)$). We note that

$$\frac{\partial V_2(q;p)}{\partial q} = 1 - 2p \Rightarrow \frac{\partial V_1(q;p)}{\partial p} = \begin{cases} < 0 & if\ \ p > \dfrac{1}{2} \\ = 0 & if\ p = \dfrac{1}{2} \\ > 0 & if\ p < \dfrac{1}{2} \end{cases}.$$

So the player 2 best response $\sigma_2^* = (q^*, 1-q^*)$ is seen to be

$$q^* = \begin{cases} 0 & if\ \ p > \dfrac{1}{2} \\ y & if\ p = \dfrac{1}{2} \\ 1 & if\ p < \dfrac{1}{2} \end{cases}, \qquad where\ y\ is\ any\ number\ between\ 0\ and\ 1.$$

To identify the Nash equilibrium, one can consider the following figures which shows the best responses of two players.



We observe that the curves intersect at $(p^*, q^*) = \left(\frac{1}{2}, \frac{1}{3}\right)$. We conclude that the unique mixed strategy Nash equilibrium is

$$\sigma_1^* = (p^*, 1-p^*) = \left(\frac{1}{2}, \frac{1}{2}\right), \qquad \sigma_2^* = (q^*, 1-q^*) = \left(\frac{1}{3}, \frac{2}{3}\right).$$

Computing mixed strategy equilibrium – Indifference principle

In practice, it is tedious to compute mixed strategy equilibrium by finding the player's best response to each of the possible strategies chosen by opponents especially when the strategic set has more than 2 elements. We should seek for alternate approach that allows us to identify mixed strategy equilibrium efficiently. This principle is known as *indifference principle*.

We let $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ be a mixed strategy Nash equilibrium. The indifference principle states that given a mixed (or pure) strategy adopted by other players, the player i's payoff remains constant over *all* pure strategy s_i chosen by the player in his mixed strategy.

Theorem 1 (Indifference principle)

We let $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ be the mixed strategy Nash equilibrium. For any pair of pure strategies $s_i^1 \in S_i$ and $s_i^2 \in S_i$ that $\sigma_i^*(s_i^1) > 0,\ \sigma_i^*(s_i^2) > 0$, we have

$$V_i(s_i^1; \sigma_{-i}^*) = V_i(s_i^2; \sigma_{-i}^*).$$

Proof of the indifference principle

Suppose that $V_i(s_i^1; \sigma_{-i}^*) \neq V_i(s_i^2, \sigma_{-i}^*)$ for some s_i^1 and s_i^2, we assume, without loss of generality, that $V_i(s_i^1; \sigma_{-i}^*) > V_i(s_i^2, \sigma_{-i}^*)$.

Our goal is to show that σ_i^* is *not* a *best response* to the opponents' mixed strategy σ_{-i}^*, which will contradict to the fact that $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ is Nash equilibrium.

To see this, we *construct* another player i's strategy (σ_i') as follows:

$$\sigma_i'(s_i) = \begin{cases} \sigma_i^*(s_i) & if \;\; s_i \neq s_i^1, s_i^2 \\ 0 & if \;\; s_i = s_i^2 \\ \sigma_i^*(s_i^1) + \sigma_i^*(s_i^2) & if \;\; s_i = s_i^1 \end{cases} \quad \dots.(*)$$

Note: Knowing that the payoff of choosing s_i^1 is smaller than that of choosing s_i^2, player i is able to enhance his payoff by "giving up" s_i^2 and transferring the weight to the strategy s_i^1.

The payoffs of choosing the mixed strategy σ_i can be computed as

$$V_i(\sigma_i; \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i(s_i) V_i(s_i; \sigma_{-i})$$

$$= \sum_{\substack{s_i \in S_i \\ s_i \neq s_i^1, s_i^2}} \sigma_i(s_i) V_i(s_i; \sigma_{-i}) + \left(\sigma_i^*(s_i^1) + \sigma_i^*(s_i^2)\right) V_i(s_i^1; \sigma_{-i}) + 0 \times V_i(s_i^2; \sigma_{-i})$$

$$\underset{V_i(s_i^1;\sigma_{-i}^*) > V_i(s_i^2;\sigma_{-i}^*)}{\overset{by\ assumption}{\gtrless}} \sum_{\substack{s_i \in S_i \\ s_i \neq s_i^1, s_i^2}} \sigma_i^*(s_i) V_i(s_i; \sigma_{-i}) + \sigma_i^*(s_i^1) V_i(s_i^1; \sigma_{-i}) + \sigma_i^*(s_i^2) V_i(s_i^2; \sigma_{-i})$$

$$= V_i(\sigma_i^*; \sigma_{-i}^*).$$

The above inequality reveals that player i can get a better payoff by adopting the strategy σ_i' (instead of σ_i^*). This contradicts to the fact that σ_i^* is the best response to σ_{-i}^*. So we conclude that

$$V_i(s_i^1; \sigma_{-i}^*) = V_i(s_i^2, \sigma_{-i}^*).$$

Example 24 (Example 23 revisited)

We revisit the welfare games considered in Example 23.

		Unemployed person (Player 2)	
		Work (W)	Do nothing (NW)
Government	Aid (A)	$(3,2)$	$(-1,3)$
(Player 1)	Does not aid (NA)	$(1,1)$	$(0,0)$

Identify all Nash equilibrium using indifference principle.

☺Solution

Similar to Example 23, we shall determine the best response of each player. This can be done using indifference principle.

For government (Player 1), we let $\sigma_2 = (q, 1-q)$ be the mixed strategy adopted by player 2 and we let $\sigma_1 = (p^*, 1-p^*)$ be the player 1's best response to σ_2.

We consider the equation.

$$V_1(A;\sigma_2) = V_1(NA;\sigma_2) \Leftrightarrow 3q + (-1)(1-q) = q + (0)(1-q) \Leftrightarrow q = \frac{1}{3}.$$

Next, we consider the following 3 scenarios:

- If $q = \frac{1}{3}$, one can deduce from indifference principle that the best response of player 1 is that the player 1 will mix the strategies A and NA (i.e. $p \in (0,1)$). Note that the expected payoff of player 1 is

$$V_1(\sigma_1;\sigma_2) = pV_1(A;\sigma_2) + (1-p)V_1(NA;\sigma_2) \overset{V_1(A;\sigma_2)=V_1(NA;\sigma_2)}{\cong} V_1(A;\sigma_2)$$

  and is independent of p. Thus, the best response of player 1 is $\sigma_1^* = (p, 1-p)$, where $p \in [0,1]$.
- If $q < \frac{1}{3}$, we have $V_1(A;\sigma_2) < V_1(NA;\sigma_2)$. According to the proof of the indifference principle (why?), the player 1 should adopt "NA" surely. Thus, the corresponding best response is $\sigma_1^* = (0,1)$ (i.e. $p = 0$).

- If $q > \frac{1}{3}$, we have $V_1(A;\sigma_2) < V_1(NA;\sigma_2)$. Similar argument implies that the player 1 should adopt "A" surely. Thus, the corresponding best response is $\sigma_1^* = (1,0)$ (i.e. $p = 1$).

Next, we proceed to obtain the player 2's best response σ_2^* with respect to a player 1's strategy $\sigma_1 = (p, 1-p)$. We first consider

$$V_2(W;\sigma_1) = V_2(NW;\sigma_1) \Leftrightarrow 2p + 1(1-p) = 3p + (0)(1-p) \Leftrightarrow p = \frac{1}{2}.$$

Similarly, we consider the following 3 scenarios:

- If $p = \frac{1}{2}$, one can deduce from indifference principle that the player 2 will mix the strategies W and NW (i.e. $q \in (0,1)$). Note that the expected payoff of player 1 is

$$V_2(\sigma_2;\sigma_1)$$
$$= qV_2(W;\sigma_1) + (1-q)V_2(NW;\sigma_1) \overset{V_2(W;\sigma_1)=V_2(NW;\sigma_1)}{\cong} V_2(W;\sigma_1)$$

and is independent of q. Thus, the best response of player 2 is $\sigma_2^* = (q, 1-q)$, where $q \in [0,1]$.

- If $p < \frac{1}{2}$, we have $V_2(W;\sigma_1) > V_2(NW;\sigma_1)$. According to the proof of the indifference principle, the player 2 should adopt "W" surely. Thus, the corresponding best response is $\sigma_2^* = (1,0)$ (i.e. $q = 1$).
- If $p > \frac{1}{2}$, we have $V_2(W;\sigma_1) < V_2(NW;\sigma_1)$. Similar argument implies that the player 1 should adopt "A" surely. Thus, the corresponding best response is $\sigma_2^* = (0,1)$ (i.e. $q = 0$).

Hence, it follows from similar argument in Example 23 that the desired Nash equilibrium is

$$\sigma_1^* = (p^*, 1-p^*) = \left(\frac{1}{2}, \frac{1}{2}\right), \qquad \sigma_2^* = (q^*, 1-q^*) = \left(\frac{1}{3}, \frac{2}{3}\right).$$

Example 25 (Volunteer's Dilemma)

Two people are arrested after committing a crime. The police do not have enough manpower to carry out the in-depth investigation. So the chief investigator presents the suspects with the following proposal:

- If at least one of them confesses, every suspect who confessed will receive a one-year imprisonment and other suspects will be released.
- If none of them confesses, the police will continue the investigation and each suspect will finally receive five-year imprisonment.

		Player 2	
		Confess (C)	Say nothing (N)
Player 1	Confess (C)	$(-1,-1)$	$(-1,0)$
	Say nothing (N)	$(0,-1)$	$(-5,-5)$

Using the indifference principle, find all possible mixed strategy Nash equilibrium which both players randomize their strategies.

☺Solution

We let $\sigma_1^* = (p^*, 1-p^*)$ and $\sigma_2^* = (q^*, 1-q^*)$ be the mixed strategies where $p^* = P(player\ 1\ confesses)$ and $q^* = P(player\ 2\ confesses)$.

Since we assume that $p^* \in (0,1)$ and $q^* \in (0,1)$, we can applying indifference principle on the payoff functions of the two players and get

$$V_1(C;\sigma_2^*) = V_1(S;\sigma_2^*) \Rightarrow -q^* + (-1)(1-q^*) = 0(q^*) + (-5)(1-q^*)$$
$$\Rightarrow q^* = 4/5;$$

and

$$V_2(C;\sigma_1^*) = V_1(S;\sigma_1^*) \Rightarrow -p^* + (-1)(1-p^*) = 0(p^*) + (-5)(1-p^*)$$
$$\Rightarrow p^* = 4/5.$$

Thus we deduce that the unique mixed strategy Nash equilibrium is found to be

$$\sigma_1^* = \sigma_2^* = \left(\frac{4}{5}, \frac{1}{5}\right).$$

Example 26 (Market Entry)

Three firms are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$, where n denotes the number of firms that enter. The cost of entering is 62.

Find the symmetric mixed strategy equilibrium in which all three players enter the market with same probability.

😊Solution

We let $p > 0$ be the probability that a firm will enter into the market and let $\sigma_1^* = \sigma_2^* = \sigma_3^* = (p, 1-p)$ be the proposed mixed strategy equilibrium.

If each firm mixes its strategy, it follows from indifference principle that

$$V_i(Enter: \sigma_{-i}^*) = V_i(No\ enter; \sigma_{-i}^*)$$

$$\Rightarrow p^2\left(\frac{150}{3} - 62\right) + 2p(1-p)\left(\frac{150}{2} - 62\right) + (1-p)^2(150-62) = 0$$

$$\Rightarrow 50p^2 - 150p + 88 = 0 \Rightarrow (5p-4)(5p-11) = 0$$

$$\Rightarrow p = 0.8 \quad \text{or} \quad p = 2.2\ (rejected).$$

Example 27 (Prisoner dilemma revisited)

We revisit the Prisoner dilemma games considered in Example 4.

		Player 2	
		Keep silent	Confess
Player 1	Keep silent (S)	$(-1,-1)$	$(-20,0)$
	Confess (C)	$(0,-20)$	$(-10,-10)$

Recall that there is an unique pure equilibrium Nash equilibrium which both players confess and none of the players will keep silent, we would like to investigate whether there is any mixed strategy Nash equilibrium that each player randomize the two strategies.

Show that such equilibrium does not exist using indifference principle.

☺Solution

We suppose that there is a mixed strategy Nash equilibrium $\sigma_i^* = (p_i^*, 1-p_i^*), i = 1,2$, where $p_i^* = P(player\ i\ keeps\ silent)$.

Applying indifference principle, we get

$$V_1(S;\sigma_2^*) = V_1(C;\sigma_2^*)$$

$$\Rightarrow (-1)q^* + (-20)(1-q^*) = 0(q^*) + (-10)(1-q^*) \Rightarrow q^* = \frac{10}{9}.$$

Since q^* should lie between 0 and 1, the above result reveals that $O_1(S;\sigma_2^*) \neq O_1(C;\sigma_2^*)$ for any $q^* \in (0,1)$. By considering the contrapositive of the indifference principle, we conclude that σ_i^* should not be Nash equilibrium and contradiction occurs.

Therefore, we conclude that there is no mixed strategy Nash equilibrium.

Remark of Example 27

Recall that "keeping silent" is a dominated strategy and the player's payoff is always lowered if he chooses to play this strategy, so a rational player should not consider such strategy in the mixed strategy. This explains why there is no mixed strategy Nash equilibrium in this games (see also Example 28).

Example 28 (Dominated strategy v.s. Mixed strategy equilibrium)

In a n-person games, we let $s_i^0 \in S_i$ be the dominated strategy of player i. Show that player i *never* choose to play s_i^0 for any possible mixed strategy Nash equilibrium $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$.

☺Solution

We need to prove that $\sigma_i^*(s_i^0) = 0$. Suppose that $\sigma_i^*(s_i^0) > 0$, we proceed to show that σ_i^* cannot be the best response.

Since s_i^0 is the dominated strategy, there is another strategy s_i' that strictly dominates s_i^0. That is,

$$V_i(s_i^0; s_{-i}) < V_i(s_i'; s_{-i}) \quad for\ \ any\ s_{-i} \in S_{-i}$$

We construct the following mixed strategy for player i:

$$\sigma_i'(s_i) = \begin{cases} \sigma_i^*(s_i) & if\ \ s_i \neq s_i^0, s_i' \\ 0 & if\ \ s_i = s_i^0 \\ \sigma_i^*(s_i') + \sigma_i^*(s_i^0) & if\ \ s_i = s_i' \end{cases} \quad \dots.(*).$$

Then the expected payoff to player i is seen to be

$$V_i(\sigma_i'; \sigma_{-i}^*) = \sum_{s_{-i}\in S_{-i}} \sum_{s_i\in S_i} \sigma_i'(s_i)\sigma_{-i}^*(s_{-i})V_i(s_i; s_{-i})$$

$$= \sum_{s_{-i}\in S_{-i}} \left[\sum_{\substack{s_i\in S_i \\ s_i\neq s_i^0, s_i'}} \sigma_i'(s_i)\sigma_{-i}^*(s_{-i})V_i(s_i; s_{-i}) + \sum_{s_i=s_i^0, s_i^i} \sigma_i'(s_i)\sigma_{-i}^*(s_{-i})V_i(s_i; s_{-i}) \right]$$

$$\overset{from\ (*)}{\cong} \sum_{s_{-i}\in S_{-i}} \left[\sum_{\substack{s_i\in S_i \\ s_i\neq s_i^0, s_i'}} \sigma_i^*(s_i)\sigma_{-i}^*(s_{-i})O_i(s_i; \overrightarrow{s_{-i}}) \right.$$

$$\left. + \left(\sigma_i^*(s_i') + \sigma_i^*(s_i^0)\right)\sigma_{-i}^*(s_{-i})V_i(s_i'; s_{-i}) \right]$$

$$\overset{\substack{by\ assumption\\ V_i(s_i';s_{-i})>V_i(s_i^0;s_{-i})}}{\overbrace{>}} \sum_{\overrightarrow{s_{-i}}\in S_{-i}} \left[\sum_{\substack{s_i\in S_i\\ s_i\neq s_i^1,s_i^2}} \sigma_i^*(s_i)\sigma_{-i}^*(s_{-i})V_i(s_i;s_{-i}) \right.$$

$$\left. + \sigma_i^*(s_i')\sigma_{-i}^*(s_{-i})V_i(s_i';s_{-i}) + \sigma_i^*\left(s_i^0\right)\sigma_{-i}^*(s_{-i})V_i\left(s_i^0;s_{-i}\right) \right]$$

$$= \sum_{\overrightarrow{s_{-i}}\in S_{-i}} \sum_{s_i\in S_i} \sigma_i^*(s_i)\sigma_{-i}^*(s_{-i})V_i(s_i;s_{-i}) = V_i(\sigma_i;\sigma_{-i}^*).$$

The above inequality reveals that player i can get a better payoff by adopting the strategy σ_i' (instead of σ_i^*). This contradicts to the fact that σ_i^* is the best response to σ_{-i}^*. So we conclude that $\sigma_i^*\left(s_i^0\right) = 0$.

Example 29 (Paper-scissor-rock games)

We consider the paper-scissor-rock games with following payoff matrix:

Player 2

		Paper	Scissor	Rock
Player 1	Paper (P)	$(0,0)$	$(-1,1)$	$(1,-1)$
	Scissor (S)	$(1,-1)$	$(0,0)$	$(-1,1)$
	Rock (R)	$(-1,1)$	$(1,-1)$	$(0,0)$

We have seen from previous example that there is no pure strategy Nash equilibrium. We would like to see whether there is mixed strategy Nash equilibrium in this games. Find all possible mixed strategy equilibria.

☺Solution

We let $\sigma_1^* = (p_1^*, p_2^*, 1 - p_1^* - p_2^*)$ and $\sigma_2^* = (q_1^*, q_2^*, 1 - q_1^* - q_2^*)$ be the mixed strategies chosen by each player, where

$p_1^* = P(player\ 1\ chooses\ paper),\ p_2^* = P(player\ 1\ chooses\ scissor)$

$q_1^* = P(player\ 2\ chooses\ paper),\ q_2^* = P(player\ 2\ chooses\ scissor)$

We first argue that each player should "mix" all three strategies under mixed strategy Nash equilibrium. This can be done in two steps:

- **Step 1: Show that there is no equilibrium in which player i uses pure strategy and player j ($j \neq i$) uses mixed strategy (i.e. mixing 2 or more pure strategies).**
  Proof: Suppose that player i uses pure strategy (say, for example, paper) under equilibrium, then player j can maximum his payoff by choosing "scissor" instead of using the mixed strategy. In fact, one can show that
  $$\underbrace{V_j(S;P)}_{=1} > \underbrace{V_j(P;P)}_{=0} > \underbrace{V_j(R,P)}_{-1}$$
  It follows from indifference principle that the player j should not adopt mixed strategy by mixing 2 or more pure strategies (why?). This leads to contradiction.
- **Step 2: Show that there is no Nash equilibrium in which at least one player mixes only two pure strategies.**
  Proof: Suppose such equilibrium exists and player i mixes two pure strategies. Without loss of generality, we assume that player i mixes between "paper" and "scissor" only ($p_1^* > 0, p_2^* > 0$ but $1 - p_1^* - p_2^* = 0$).

Next, we consider player j's side (rival). One can show that

$$V_j(P;\sigma_i^*) = 0p_1^* + (-1)p_2^* < 1p_1^* + 0p_2^* = V_j(S;\sigma_i^*)$$

It follows from indifference principle that the player j must not mix "P" and "S".

Furthermore, $V_j(S;\sigma_i^*) > V_j(P;\sigma_i^*)$. It follows that player j should not adopt "P" in the mixed strategy (i.e. $q_1^* = 0$).

Going back to player i's side, we can deduce that

$$V_i(S;\sigma_i^*) = q_2^*(0) + (1-q_2^*)(-1) < q_2^*(1) + (1-q_2^*)(0) = V_i(R;\sigma_i^*).$$

It follows that

$$\underbrace{V_i(P;\sigma_i^*) = V_i(S;\sigma_i^*)}_{\substack{indifference\ principle \\ (mixing\ P.and\ S)}} < V_i(R;\sigma_i^*).$$

It implies that player i will be better off to adopt R instead of mixing "P" and "S".

So that the mixing strategy is not the best response and it leads to contraidcition.

Combining the result, we can conclude that each player must mix between all three strategies under Nash equilibrium.

Using indifference principle, we obtain

$$\begin{cases} V_1(P;\sigma_2^*) = V_1(S;\sigma_2^*) \\ V_1(P;\sigma_2^*) = V_1(R;\sigma_2^*) \end{cases} \Rightarrow \begin{cases} -q_2^* + (1-q_1^*-q_2^*) = q_1^* - (1-q_1^*-q_2^*) \\ -q_2^* + (1-q_1^*-q_2^*) = -q_1^* + q_2^* \end{cases}$$

$$\Rightarrow \begin{cases} 3q_1^* + 3q_2^* = 2 \\ 3q_2^* = 1 \end{cases} \Rightarrow (q_1^*, q_2^*) = \left(\frac{1}{3}, \frac{1}{3}\right).$$

Similarly, one can obtain

$$\begin{cases} V_2(P;\sigma_1^*) = V_2(S;\sigma_1^*) \\ V_2(P;\sigma_1^*) = V_2(R;\sigma_1^*) \end{cases} \Rightarrow \begin{cases} -p_2^* + (1-p_1^*-p_2^*) = p_1^* - (1-p_1^*-p_2^*) \\ -p_2^* + (1-p_1^*-p_2^*) = -p_1^* + p_2^* \end{cases}$$

$$\Rightarrow \begin{cases} 3p_1^* + 3p_2^* = 2 \\ 3p_2^* = 1 \end{cases} \Rightarrow (p_1^*, p_2^*) = \left(\frac{1}{3}, \frac{1}{3}\right).$$

Hence, we conclude that the unique mixed strategy Nash equilibrium is given by

$$\sigma_1^* = \sigma_2^* = \left(\frac{1}{3}, \frac{1}{3}, 1 - \frac{1}{3} - \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

That is, each player will choose one of the three strategies with equal probability.

Dominated strategy: Extended definition

Recall that identifying dominated strategy allows us to simplify the games and perform equilibrium analysis in an easier way. With the possibility of adopting mixed strategy, one can strengthen the definition of dominated strategy. This allows us to identify more dominated strategies and simplify the games effectively.

Definition (Dominated strategies: Extended version)

A strategy $s_i \in S_i$ is said to be *strictly dominated* by mixed strategy σ_i if and only if

$$V_i(s_i; s_{-i}) < V_i(\sigma_i; s_{-i})$$

for any combination of opponents' pure strategies $s_{-i} \in S_{-i}$.

A strategy $s_i \in S_i$ is said to be a dominated strategy if it is strictly dominated by some mixed strategy σ_i.

If s_i is dominated strategy, one can show that the player i never choose s_i with positive probability under any mixed strategy Nash equilibrium.

Example 30

We consider the following two person games with following payoff matrix:

		Player 2 U	Player 2 M	Player 2 D
Player 1	U	$(5,1)$	$(1,4)$	$(1,0)$
	M	$(3,2)$	$(0,0)$	$(3,5)$
	D	$(4,3)$	$(4,4)$	$(0,3)$

One can show that no pure strategy is strictly dominated by other pure strategy so that no strategies can be ruled out by using the old definition of dominated strategy.

(a) Show that for player 2, the strategy U is strictly dominated by mixed strategy $\sigma_2 = \left(\sigma_2(U), \sigma_2(M), \sigma_2(D)\right) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

(b) Hence, show that U is a dominated strategy for player 1 by considering mixed strategy $\sigma_1 = \left(\sigma_1(U), \sigma_1(M), \sigma_1(D)\right) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

(c) Using the result of **(a)** and **(b)**, identify all possible Nash equilibria for these games.

☺Solution of **(a)**

By some calculation (taking $s_2 = U, M, D$ respectively), we obtain

$$1 = V_2(U;U) < V_2(\sigma_2;U) = \frac{1}{2}(4) + \frac{1}{2}(0) = 2,$$

$$2 = V_2(U;M) < V_2(\sigma_2;M) = \frac{1}{2}(0) + \frac{1}{2}(5) = 2.5,$$

$$3 = V_2(U;D) < V_2(\sigma_2;D) = \frac{1}{2}(4) + \frac{1}{2}(3) = 3.5.$$

The above inequalities reveal that the player 2's payoff of choosing U is strictly lower than that of choosing σ_2. So we conclude that U is strictly dominated by σ_2.

☺Solution of **(b)**

We observe from the result of **(a)** that U is dominated strategy and can be ruled out from player 2's strategic set (since player 2 will not choose it).

So the games can be reduced into

		Player 2		
			M	D
	U	X	$(1,4)$	$(1,0)$
Player 1	M	X	$(0,0)$	$(3,5)$
	D	X	$(4,4)$	$(0,3)$

By some calculation (taking $s_2 = U, M, D$ respectively), we obtain

$$1 = V_1(U; M) < V_1(\sigma_1; M) = \frac{1}{2}(0) + \frac{1}{2}(4) = 2,$$

$$1 = V_1(U; D) < V_1(\sigma_1; D) = \frac{1}{2}(3) + \frac{1}{2}(0) = 1.5.$$

We observe that player 1's payoff of choosing U is strictly lower than that of choosing σ_1. Thus U is strictly dominated by σ_1 and U is dominated strategy for player 1.

☺Solution of **(c)**

Since U is dominated strategy for player 1, so we can rule out it from player 1's strategic set and the games can be further reduced into

			Player 2	
			M	D
		X	X	X
Player 1	M	X	$(0,0)$	$(3,5)$
	D	X	$(4,4)$	$(0,3)$

We let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$ be mixed strategies chosen by two players, where $p = P(player\ 1\ plays\ M)$ and $q = P(player\ 2\ plays\ M)$. The expected payoffs of two players are given by

$$V_1(p;q) = pq(0) + p(1-q)(3) + (1-p)(q)(4) + (1-p)(1-q)(0) \\ = 3p + 4q - 7pq;$$

$$V_2(q;p) = pq(0) + p(1-q)(5) + (1-p)(q)(4) + (1-p)(1-q)(3) \\ = 3 + 2p + q - 6pq.$$

Given the mixed strategy adopted by player 2 ($\sigma_2 = (q, 1-q)$), we determine the player 1's best response ($\sigma_1^* = (p^*, 1-p^*)$). We note that

$$\frac{\partial V_1(p;q)}{\partial p} = 3 - 7q \Rightarrow \frac{\partial V_1(p;q)}{\partial p} = \begin{cases} < 0 \quad if \;\; q > \dfrac{3}{7} \\ = 0 \quad if \; q = \dfrac{3}{7} \\ > 0 \quad if \; q < \dfrac{3}{7} \end{cases}.$$

So the player 1 best response $\overrightarrow{\sigma_1}^* = (p^*, 1-p^*)$ is seen to be

$$p^* = \begin{cases} 0 \quad if \;\; q > \dfrac{3}{7} \\ x \quad if \; q = \dfrac{3}{7} \\ 1 \quad if \; q < \dfrac{3}{7} \end{cases}, \qquad where\ x\ is\ any\ number\ between\ 0\ and\ 1.$$

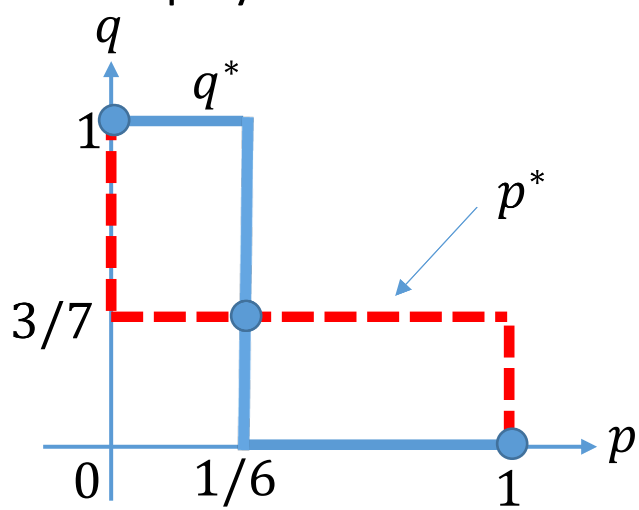
Given the mixed strategy adopted by player 1 ($\sigma_1 = (p, 1-p)$), we determine the player 2's best response ($\sigma_2^* = (q^*, 1-q^*)$). We note that

$$\frac{\partial V_2(q;p)}{\partial q} = 1 - 6p \Rightarrow \frac{\partial V_1(q;p)}{\partial p} = \begin{cases} < 0 & if \;\; p > \dfrac{1}{6} \\ = 0 & if \; p = \dfrac{1}{6} \\ > 0 & if \; p < \dfrac{1}{6} \end{cases}.$$

So the player 2 best response $\sigma_2^* = (q^*, 1-q^*)$ is seen to be

$$q^* = \begin{cases} 0 & if \;\; p > \dfrac{1}{6} \\ y & if \; p = \dfrac{1}{6}, \\ 1 & if \; p < \dfrac{1}{6} \end{cases} \qquad where\; y\; is\; any\; number\; between\; 0\; and\; 1.$$

To identify the Nash equilibrium, one can consider the following figures which shows the best responses of two players.



We observe that the two curves p^* and q^* intersect at $(p^*, q^*) = (0,1), \left(\frac{1}{6}, \frac{3}{7}\right)$ and $(1,0)$ respectively. Note that the first and third intersection points correspond to the pure strategy Nash equilibria (D, M) and (M, D) respectively. The second one corresponds to mixed strategy Nash equilibrium $\sigma_1^* = (p^*, 1 - p^*) = \left(\frac{1}{6}, \frac{5}{6}\right)$, $\sigma_2^* = (q^*, 1 - q^*) = \left(\frac{3}{7}, \frac{4}{7}\right)$.

Existence of Nash equilibrium in n-person finite games

We have seen from the previous examples that the pure strategy Nash equilibrium may not exist in some games such as paper-scissor-rock games. With the inclusion of mixed strategy, we are able to identify the equilibrium for these games.

One would like to ask if the mixed strategy Nash equilibrium always exists in any n-person static games. The following theorem confirms that such Nash equilibrium always exist provided that the strategic sets S_i is finite.

Theorem (Nash's existence theorem)
For any n-person normal form games with finite strategic sets S_i for every player, there exists at least one mixed strategy Nash equilibrium.

(*Note: Here, pure strategy Nash equilibrium is treated as a special case of mixed strategy Nash equilibrium)

Proof of Nash's existence theorem

Given strategic set of each player S_i, we define ΔS_i be a collection of all mixed strategies that can be adopted by player i. That is,

$$\Delta S_i = \left\{ \underbrace{\big(\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{iN_i})\big)}_{\sigma_i} \middle| \sigma_i(s_{i1}) + \cdots + \sigma_i(s_{iN_i}) = 1 \right\}$$

We let $\Delta S = \Delta S_1 \times \Delta S_2 \times \cdots \times \Delta S_n$ be the set of mixed strategies of n players.

Given a strategic profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, recall that $BR_i(\sigma)$ denotes that player i's best response (mixed strategy) to the rival's strategy σ_{-i} and $BR(\sigma) = \big(BR_1(\sigma), BR_2(\sigma), \ldots, BR_n(\sigma)\big)$ denotes the best response of n players to the strategic profile σ.

Recall that a strategic profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is Nash equilibrium if and only if σ_i^* is the best response to the rival's strategy σ_{-i}^* for any player i (i.e. $BR_i(\sigma^*) = \sigma_i^*$). We get

$$BR(\sigma^*) = \sigma^*.$$

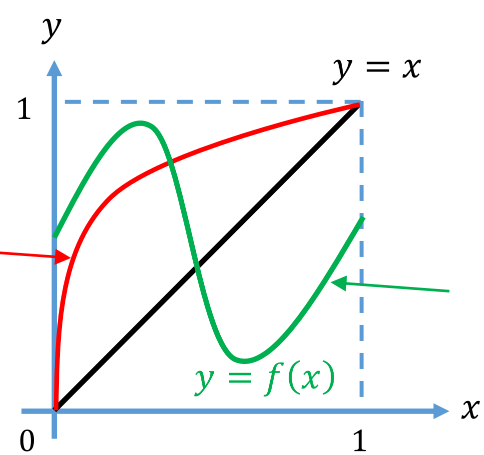
So σ^* is seen to be the *fixed point* of the correspondence $BR(\sigma)$. Therefore, it suffices to prove that the correspondence $BR\colon \Delta S \rightrightarrows \Delta S$ has a fixed point.

Fixed point theorem

There are a wide selection of fixed point theorems which provides different sufficient conditions which a given function has a fixed point. The most elementary version of fixed point theorem is known as *Brouwer's fixed point theorem*.

Theorem (Brouwer's fixed point theorem, simple version)
We let $f: [0,1] \to [0,1]$ be a continuous function, then there exists $x^* \in [0,1]$ such that $f(x^*) = x^*$, i.e. x^* is a fixed point of f.

If $\boldsymbol{f(0) = 0}$ or $\boldsymbol{f(1) = 1}$, then $\boldsymbol{x = 0}$ or $\boldsymbol{x = 1}$ is the fixed point of $\boldsymbol{x}$ and we are done.



Even if $\boldsymbol{f(0) \neq 0}$ and $\boldsymbol{f(1) \neq 1}$, we observe that $\boldsymbol{f(x)}$ will eventually intersect the line $\boldsymbol{y = x}$ at some point $\boldsymbol{x^*}$ as $\boldsymbol{f(x)}$ is continuous.

Unfortunately, this theorem cannot be applied directly to our case. It is because $BR(\sigma)$ may have *multiple outputs* given σ (i.e. $BR(\sigma)$ is a set-valued function).Therefore, one would need a more powerful fixed point theorem for set-valued function as follows:

Theorem (Kakutani's fixed point theorem)

We let $C: X \rightrightarrows X$ be a correspondence where $X \subseteq \mathbb{R}^p$, where $C(x) \subseteq X$ for all $x \in X$. Suppose that

1. X is non-empty, closed, bounded and convex;
   (*Note: A set X is said to be closed if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ (where $x_n \in X$ which $\lim_{n\to\infty} x_n = x_0$, we have $x_0 \in X$)
2. $C(x)$ is non-empty for all $x \in X$;
   (*Note: A set B is said to be convex if and only if for any $x, y \in B$, we have $\alpha x + (1-\alpha)y \in B$ for any $\alpha \in [0,1]$.)
3. $C(x)$ is convex for all x;
4. $C(x)$ has closed graph. That is, the set $\{(x, C(x)) : x \in X\}$ is closed.

Then there exists $x^* \in X$ such that

$$C(x^*) = x^*.$$

Proof of Nash's existence theorem

Based on Kakutani's fixed point theorem, it suffices to verify that the conditions (1)-(4) hold for the best response correspondence $BR: \Delta S \rightrightarrows \Delta S$.

(*Note: We let $p_{ij} = \sigma_i(s_{ij})$, where $s_{ij} \in S_i$)

- For condition 1,
  - ✓ It is clear that ΔS_i is non-empty for all i so that ΔS is non-empty.
  - ✓ Since $0 \leq p_{ij} \leq 1$ for all i, j. It follows that $\Delta S_i \subseteq [0,1]^{N_i}$ and ΔS_i is bounded.
  - ✓ We let $\sigma_i^{(n)} \in \Delta S_i$ (where $\sigma_i^{(n)} = \left(p_{i1}^{(n)}, p_{i2}^{(n)}, \dots, p_{iN_i}^{(n)}\right)$) be a sequence of vectors which $\lim_{n\to\infty} \sigma_i^{(n)} = \sigma_i^* = (p_{i1}^*, p_{i2}^*, \dots, p_{iN_i}^*)$)
    - As $p_{ij}^{(n)} \geq 0$ for all n, it follows that $p_{ij}^* = \lim_{n\to\infty} p_{ij}^{(n)} \geq 0$.
    - On the other hand, $\sigma_i^{(n)} \in \Delta S_i$ so that
$$p_{i1}^{(n)} + p_{i2}^{(n)} + \cdots + p_{iN_i}^{(n)} = 1.$$
      Taking limit on both sides, we deduce that
$$p_{i1}^* + p_{i2}^* + \cdots + p_{iN_i}^* = 1.$$

    It follows that $\sigma_i^* \in \Delta S_i$. So that ΔS_i is closed.

✓ For any $\sigma_i^{(1)}, \sigma_i^{(2)} \in \Delta S_i$ and $\alpha \in [0,1]$, we let $\sigma = \alpha\sigma_i^{(1)} + (1-\alpha)\sigma_i^{(2)}$. One can verify that

- $\alpha p_{ij}^{(1)} + (1-\alpha)p_{ij}^{(2)} \geq 0$ as $p_{ij}^{(1)}, p_{ij}^{(2)} \geq 0$
- $\sum_{j=1}^{N_i}\left(\alpha p_{ij}^{(1)} + (1-\alpha)p_{ij}^{(2)}\right) = \alpha \underbrace{\sum_{j=1}^{N_i} p_{ij}^{(1)}}_{=1} + (1-\alpha)\underbrace{\sum_{j=1}^{N_i} p_{ij}^{(2)}}_{=1} = 1.$

So $\sigma \in \Delta S_i$ and ΔS_i is convex.

- For condition 2,
  Note that the (expected) payoff function

$$V_i(\sigma_i;\sigma_{-i}) = \sum_{j=1}^{N_i} p_{ij} V_i(s_{ij};\sigma_{-i})$$

is continuous with respect to p_{ij} (as it is linear in p_{ij}) and the domain ΔS_i is closed and bounded. It follows from extreme value theorem that the function $V_i(\sigma_i;\sigma_{-i})$ has a maximum in ΔS_i. So the best response exists and $BR_i(\sigma)$ (and hence $BR(\sigma)$) is non-empty.

- For condition 3,
  Given a strategic profile σ, we let σ_i^* and σ_i^{**} be two best responses to the rival's strategy σ_{-i}. We let
  $$M^* = V_i(\sigma_i^*;\sigma_{-i}) = V_i(\sigma_i^{**};\sigma_{-i}).$$
  As σ_i^* is best response, we must have
  $$M^* = V_i(\sigma_i^*;\sigma_{-i}) \geq V_i(\sigma_i;\sigma_{-i}) \quad for\ all\ \sigma_i \in \Delta S_i,$$

  Next, we consider the strategy $\sigma_0 = \beta\sigma_i^* + (1-\beta)\sigma_i^{**}$, then the corresponding payoff function is found to be
  $$\begin{aligned} V_i(\sigma_0;\sigma_{-i}) &= \sum_{j=1}^{N_i}\left(\beta p_{ij}^* + (1-\beta)p_{ij}^{**}\right)V_i\left(s_{ij};\sigma_{-i}\right) \\ &= \beta\underbrace{\sum_{j=1}^{N_i} p_{ij}^* V_i\left(s_{ij};\sigma_{-i}\right)}_{=M^*} + (1-\beta)\underbrace{\sum_{j=1}^{N_i} p_{ij}^{**} V_i\left(s_{ij};\sigma_{-i}\right)}_{=M^*} = M^* \geq V_i(\sigma_i;\sigma_{-i}) \end{aligned}$$
  for all $\sigma_i \in \Delta S_i$. Hence, σ_0 is also the best response and $\sigma_0 \in BR(\sigma)$. Thus, $BR(\sigma)$ is convex.

- For condition 4,

  We let $\lim_{n\to\infty}\left(\sigma^{(n)}, BR\left(\sigma^{(n)}\right)\right) = (\sigma^*, \sigma^{**})$. To prove the graph is closed, it remains to show $\sigma^{**} = BR(\sigma^*)$, i.e. $\sigma^{**} = \lim_{n\to\infty} BR\left(\sigma^{(n)}\right)$ is the best response to σ^*.

  ✓ Recall that $BR\left(\sigma^{(n)}\right)$ is best response to $\sigma^{(n)}$, one must have

  $$V_i\left(BR_i\left(\sigma^{(n)}\right); \sigma_{-i}^{(n)}\right) \geq V_i\left(\sigma_i; \sigma_{-i}^{(n)}\right) \dots\dots (*)$$

  for any σ_i.

  ✓ Note that $V_i(\sigma_i; \sigma_{-i}) = \sum_{s_i\in S_i}\sum_{s_{-i}\in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})V_i(s_i; s_{-i})$ is a continuous function of $\sigma_i(s_i)$ and $\sigma_{-i}(s_{-i})$ (and hence σ_i and σ_{-i}). By taking $n \to \infty$ in $(*)$, we get

  $$\lim_{n\to\infty} V_i\left(BR_i\left(\sigma^{(n)}\right); \sigma_{-i}^{(n)}\right) \geq \lim_{n\to\infty} V_i\left(\sigma_i; \sigma_{-i}^{(n)}\right)$$

  $$\Rightarrow V_i(\sigma_i^{**}; \sigma_{-i}^*) \geq V_i(\sigma_i; \sigma_{-i}^*).$$

  Hence, we conclude that σ_i^{**} is the best response to σ_{-i}^* and $\sigma^{**} = BR(\sigma^*)$.

Hence, it follows from Kakutani's fixed point theorem that there exists σ^* such that $BR(\sigma^*) = \sigma^*$ and the mixed strategy Nash equilibrium exists.

Additional topic: Existence of pure strategy Nash equilibrium in 2-person finite games

We have seen earlier that the pure strategy Nash equilibrium may not exist in some static games. Although we manage to show that there is always a mixed strategy Nash equilibrium in any finite games. It is still worthwhile to examine the conditions which the pure strategy equilibrium exists.

To facilitate the analysis, we shall focus on 2-person zero-sum games. Roughly speaking, the zero-sum games is the games which the sum of players' payoffs is always 0 for any strategic profile. That is,

$$V_1(s_1; s_2) + V_2(s_2; s_1) = 0.$$

for any $(s_1, s_2) \in S_1 \times S_2$.

In other words, what one player gains equals what another player loses. One example of zero-sum games is Paper-Scissor-Rock Games.

In this section, we shall examine the conditions which a 2-person zero sum games has a pure strategy Nash equilibrium. To do so, we shall review how do the players make decisions in the games.

For player 1,

- Since all players are assumed to be rational, the player 1 would expect that the rival (player 2) should choose a strategy which maximizes player 2's payoff $V_2(s_2; s_1)$ given player 1's strategy.
- As $V_2(s_2; s_1) = -V_1(s_1; s_2)$, this is equivalent to minimizing player 1's payoff. Therefore, the player 1 expects that the payoff of choosing strategy s_1 is
$$Payoff = \min_{s_2} V_1(s_1; s_2).$$
- On the other hand, the player 1 should choose a strategy that maximizes his/her payoff, therefore the payoff of player 1 is expected to be
$$v^- = \max_{s_1}\left(\min_{s_2} V_1(s_1; s_2)\right).$$
Hence, v^- is called lower value of the games which indicates the minimum guarantee cutoff that player 1 can receive from the games.

For player 2,

- Using similar logic, the player 2 would expect that the rival (player 1) should choose a strategy which maximizes player 1's payoff $V_1(s_1; s_2)$ given player 2's strategy.
- Therefore, the player 2 expects that the payoff of choosing strategy s_1 is
$$Payoff = -\max_{s_1} V_1(s_1; s_2).$$
- On the other hand, the player 2 should choose a strategy that maximizes his/her payoff. That is, maximizing $-\max_{s_1} V_1(s_1; s_2)$. Mathematically, it is equivalent to minimizing $\max_{s_1} V_1(s_1; s_2)$. Hence, the player 2's payoff is expected to be
$$Payoff = \max_{s_2}\left(-\max_{s_1} V_1(s_1; s_2)\right) = -\min_{s_2}\left(\max_{s_1} V_1(s_1; s_2)\right).$$
Here, $v^+ = \min_{s_2}\left(\max_{s_1} V_1(s_1; s_2)\right)$ is called upper value of the games which indicates the maximum loss that player 2 can suffer from the games.

Recall that under Nash equilibrium (s_1^*, s_2^*), s_1^* is the best response to s_2^* and s_2^* is the best response to s_1^*.

- We consider player 1's side.
  - ✓ Note that player 2 choose s_2^* to *minimize player 1's payoff* (recall that $V_2(s_2; s_1) = -V_1(s_1; s_2)$), so we have
$$V_1(s_1^*; s_2^*) = \min_{s_2} V_1(s_1^*; s_2) \leq \max_{s_1}\left(\min_{s_2} V_1(s_1; s_2)\right) = v^-.$$
  - ✓ On the other hand, the player 1 has no incentive to deviate from adopting s_1^*, then it follows that
$$V_1(s_1^*; s_2^*) = \max_{s_1} V_1(s_1^*; s_2^*) \geq \max_{s_1}\left(\min_{s_2} V_1(s_1; s_2)\right) = v^-$$

  Combining two inequalities, we get $V_1(s_1^*; s_2^*) = v^-$.
- We consider player 2's side.
  - ✓ Using similar argument (by replacing V_1 by $V_2 = -V_1$), we can deduce that
$$V_2(s_2^*; s_1^*) = \max_{s_2}\left(\min_{s_1} V_2(s_2; s_1)\right)$$
$$\Rightarrow -V_1(s_1^*; s_2^*) = \max_{s_2}\left(-\max_{s_1} V_1(s_1; s_2)\right) = -\min_{s_2}\left(\max_{s_1} V_1(s_1; s_2)\right)$$
$$\Rightarrow V_1(s_1^*; s_2^*) = \min_{s_2}\left(\max_{s_1} V_1(s_1; s_2)\right) = v^+$$

Combining the result, we deduce that $v^+ = V_1(s_1^*; s_2^*) = v^-$ if (s_1^*, s_2^*) is Nash equilibrium.

Next, we would like to show that the pure strategy Nash equilibrium exists if $v^+ = v^-$.

- We let s_1^* be a strategy such that $\min\limits_{s_2} V_1(s_1^*; s_2) = v^-$. On the other hand, we let s_2^* be a strategy such that $\max\limits_{s_1} V_1(s_1; s_2^*) = v^+$. Our goal is to argue that (s_1^*, s_2^*) is the Nash equilibrium.
- We consider player 1. Note that
$$V_1(s_1^*, s_2^*) \geq \min_{s_2} V_1(s_1^*; s_2) = v^- = v^+ = \max_{s_1} V_1(s_1; s_2^*) \geq V_1(s_1; s_2^*)$$
for any $s_1 \in S_1$. This implies that s_1^* is the player 1's best response to s_2^*.
- Next, we consider player 2. Using similar argument, we can deduce that
$$\begin{aligned} V_2(s_2^*; s_1^*) = -V_1(s_1^*; s_2^*) &= -\max_{s_1} V_1(s_1; s_2^*) = -v^+ = -v^- = -\min_{s_2} V_1(s_1^*; s_2) \\ &\geq -V_1(s_1^*; s_2) = V_2(s_2; s_1^*). \end{aligned}$$
for any $s_2 \in S_2$. This implies that s_2^* is the player 2's best response to s_1^*.

Hence, we conclude that (s_1^*, s_2^*) is the desired pure strategy Nash equilibrium.

Hence, we deduce the following theorem:

Theorem (Existence of pure strategy Nash equilibrium in 2-person finite games)

A two-person finite games has the pure strategy Nash equilibrium if and only if

$$v^- = \max_{s_1}\left(\min_{s_2} V_1(s_1; s_2)\right) = \min_{s_2}\left(\max_{s_1} V_1(s_1; s_2)\right) = v^+.$$

Example 31 (Paper-Scissor-Rock Games)

We consider the paper-scissor-rock games with the following payoff matrix:

		Player 2		
		Paper	**Scissor**	**Rock**
Player 1	**Paper**	$(0,0)$	$(-1,1)$	$(1,-1)$
	Scissor	$(1,-1)$	$(0,0)$	$(-1,1)$
	Rock	$(-1,1)$	$(1,-1)$	$(0,0)$

One can see that the games is zero-sum games, so one can check the existence of Nash equilibrium by computing v^- and v^+:

- Note that $\min_{s_2} V_1(s_1; s_2) = -1$ for every $s_1 \in S_1$, we have $v^- = \max_{s_1} \underbrace{\left(\min_{s_2} V_1(s_1; s_2)\right)}_{=-1} = -1.$
- On the other hand, we note that $\max_{s_1} V_1(s_1, s_2) = 1$ for every $s_2 \in S_2$. So $v^+ =$ $\min_{s_2} \underbrace{\left(\max_{s_1} V_1(s_1; s_2)\right)}_{=1} = 1.$

As $v^+ \neq v^-$, we can deduce from the theorem that the pure strategy Nash equilibrium does not exist in this game.

	Paper	**Scissor**	**Rock**	
Paper	$(0,0)$	$(-1,1)$	$(1,-1)$	$\Rightarrow \min_{s_2} V_1(s_1; s_2) = -1$
Scissor	$(1,-1)$	$(0,0)$	$(-1,1)$	$\Rightarrow \min_{s_2} V_1(s_1; s_2) = -1$
Rock	$(-1,1)$	$(1,-1)$	$(0,0)$	$\Rightarrow \min_{s_2} V_1(s_1; s_2) = -1$
	$\max_{s_1} V_1(s_1; s_2) = 1$	$\max_{s_1} V_1(s_1; s_2) = 1$	$\max_{s_1} V_1(s_1; s_2) = 1$	