MATH4321 Game Theory
Suggested Solution to Problem Set 1

Chapter 1: Static games

Problem 1 (Location problems)

There are 2 department stores: A and B. Each of the two stores can choose a location for its new store in Kowloon. There are 4 available choices: Sham Shui Po (SSP), Mong Kok (MK), Kowloon Tong (KT) and Jordan (J). The profits made by the new stores are summarized in the following matrix:

		Store B (Player 2)			
		SSP	MK	KT	J
Store A (Player 1)	SSP	$(30,40)$	$(50,95)$	$(55,95)$	$(55,120)$
	MK	$(115,40)$	$(100,100)$	$(130,85)$	$(120,95)$
	KT	$(125,45)$	$(95,65)$	$(60,40)$	$(115,120)$
	J	$(105,50)$	$(75,75)$	$(95,95)$	$(45,50)$

Determine the final outcome using IESDS. Verify that this final outcome is the pure strategy Nash equilibrium.

☺Solution

Firstly, we note that both J and SSP are the dominated strategies for player 1 because both are dominated by another strategy MK, i.e.

$$\underbrace{V_1(SSP;SSP)}_{=30} < \underbrace{V_1(MK;SSP)}_{=115}, \qquad \underbrace{V_1(SSP;MK)}_{=50} < \underbrace{V_1(MK;MK)}_{=100},$$
$$\underbrace{V_1(SSP;KT)}_{=55} < \underbrace{V_1(MK;KT)}_{=130}, \qquad \underbrace{V_1(SSP;J)}_{=55} < \underbrace{V_1(MK;J)}_{=120},$$
$$\underbrace{V_1(J;SSP)}_{=105} < \underbrace{V_1(MK;SSP)}_{=115}, \qquad \underbrace{V_1(J;MK)}_{=75} < \underbrace{V_1(MK;MK)}_{=100},$$
$$\underbrace{V_1(J;KT)}_{=95} < \underbrace{V_1(MK;KT)}_{=130}, \qquad \underbrace{V_1(J;J)}_{=45} < \underbrace{V_1(MK;J)}_{=120}.$$

So we can rule out J and SSP from player 1's strategic set. On the other hand, we observe that SSP and KT are dominated strategies for player 2 since they are dominated by MK, that is

$$\underbrace{V_2(SSP;MK)}_{=40} < \underbrace{V_2(MK;MK)}_{=100}, \qquad \underbrace{V_2(SSP;KT)}_{=45} < \underbrace{V_2(MK;KT)}_{=65},$$
$$\underbrace{V_2(KT;MK)}_{=85} < \underbrace{V_2(MK;MK)}_{=100}, \qquad \underbrace{V_2(KT;KT)}_{=40} < \underbrace{V_2(MK;KT)}_{=65}.$$

So we can rule out SSP and KT from player 2's strategic set.

Next, we consider player 1 again. We note that KT is now the dominated strategy for player 1 since it is dominated by MK, i.e.

$$\underbrace{V_1(KT;MK)}_{=95} < \underbrace{V_1(MK;MK)}_{=100}, \qquad \underbrace{V_1(KT;J)}_{=115} < \underbrace{V_1(MK;J)}_{=120}.$$

So we rule out KT from player 1's strategic set. Finally, we note that $95 = V_2(J; MK) < V_2(MK; MK) = 100$. Hence, J is dominated strategy for player 2.

Therefore, we conclude that (MK, MK) is the only survivor under IESDS and is the final outcome of this games. To verify (MK, MK) is the Nash equilibrium , we note that

$$\underbrace{V_1(MK; MK)}_{=100} > \underbrace{V_1(KT; MK)}_{=95} > \underbrace{V_1(J; MK)}_{=75} > \underbrace{V_1(SSP; MK)}_{=55},$$

$$\underbrace{V_2(MK; MK)}_{=100} > \underbrace{V_2(J; MK)}_{=95} > \underbrace{V_2(KT; MK)}_{=85} > \underbrace{V_2(SSP; MK)}_{=40}$$

The above sets of inequalities reveal that MK is the best response of each player. So (MK, MK) is the Nash equilibrium.

Problem 2 (Discrete first-price auction)

An item is up for auction. Player 1 values the item at $5 while player 2 values the item at $8. Each player can bid either 0, 1, 2 or 3. If player i bids more than player j then i wins the games If both players place the same bid, then each player will win with probability 0.5. The winner of the games will win the good and pay his bid and the loser does not pay anything.

(a) Express the games in normal form.

(b) **(i)** Determine the final outcome using IESDS.

(ii) Hence, determine all possible pure strategy Nash equilibrium.

(c) Suppose that the player 1 values the item at $1.5 (instead of $5),

(i) determine the final outcome using IESDS.

(ii) Hence, determine all possible pure strategy Nash equilibrium.

☺Solution

(a) One can express the games in normal form as follows:

Set of players: $\{1,2\}$;

Strategic set $S_i = \{0,1,2,3\}$;

Payoff function:

For $i = 1,2$, we let b_i ($\in \{0,1,2,3\}$) be the bid submitted by player i. Then the player i's payoff function can be expressed as

$$V_i(b_i; b_j) = \begin{cases} v_i - b_i & if \;\; b_i > b_j \\ \frac{1}{2}(v_i - b_i) & if \;\; b_i = b_j, \\ 0 & if \;\; b_i < b_j \end{cases}$$

where v_i is player i's valuation on the object. Here, $v_1 = 5$ and $v_2 = 8$.

Alternatively, one can represent the payoffs by the following matrix:

		Player 2			
		0	**1**	**2**	**3**
Player 1	**0**	$(2.5, 4)$	$(0, 7)$	$(0, 6)$	$(0, 5)$
	1	$(4, 0)$	$(2, 3.5)$	$(0, 6)$	$(0, 5)$
	2	$(3, 0)$	$(3, 0)$	$(1.5, 3)$	$(0, 5)$
	3	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(1, 2.5)$

(b)(i) We apply IESDS and identify all dominated strategy. To save space, we will provide the outline of the procedure. The detailed derivation is similar to that in Problem 1.

(1) "0" is the dominated strategy for player 2 since it is strictly dominated by "3".
(2) "0" is the dominated strategy for player 1 since it is strictly dominated by "3".
(3) "1" is the domainted strategy for player 2 since it is strictly dominated by "3".
(4) "1" is the dominated strategy for player 1 since it is strictly dominated by "3".
(5) "2" is the dominated strategy for both players since it is strictly dominated by "3".

Thus, we can conclude that the only possible outcome is $(3,3)$.

(b)(ii) Recall that any dominated strategy must not be part of the Nash equilibrium, thus the only possible Nash equilibrium is $(3,3)$. In fact, one can show that

$$\underbrace{V_1(3;3)}_{=1} > \underbrace{V_1(x;3)}_{=0}, \quad for\ any\ x = 0,1,2;$$
$$\underbrace{V_2(3;3)}_{=2.5} > \underbrace{V_2(y;3)}_{=0}, \quad for\ any\ x = 0,1,2.$$

So "3" is the best response of each player and $(3,3)$ is the Nash equilibrium.

(c)(i) When $v_1 = 1.5$, the corresponding payoff matrix is given by

		Player 2			
		0	**1**	**2**	**3**
Player 1	**0**	$(0.75, 4)$	$(0, 7)$	$(0, 6)$	$(0, 5)$
	1	$(0.5, 0)$	$(0.25, 3.5)$	$(0, 6)$	$(0, 5)$
	2	$(-0.5, 0)$	$(-0.5, 0)$	$(-0.5, 3)$	$(0, 5)$
	3	$(-1.5, 0)$	$(-1.5, 0)$	$(-1.5, 0)$	$(-0.75, 2.5)$

We apply IESDS and identify all dominated strategy. To save space, we will provide the outline of the procedure. The detailed derivation is similar to that in Problem 1.

(1) "3" is the dominated strategy for player 1 since it is strictly dominated by "0".
(2) "0" is the dominated strategy for player 2 since it is strictly dominated by "3".

Thus, we can conclude that the possible outcomes is given by the set $\{(b_1, b_2): b_1 \in \{0,1,2\}, b_2 \in \{1,2,3\}\}$.

(c)(ii) We identify all possible Nash equilibria by finding the best responses of each player. The best responses are summarized in the following matrix:

		Player 2			
		0	**1**	**2**	**3**
Player 1	**0**	X	$(0, \overline{\mathbf{7}})$	$(\overline{\mathbf{0}}, 6)$	$(\overline{\mathbf{0}}, 5)$
	1	X	$(\overline{\mathbf{0.25}}, 3.5)$	$(\overline{\mathbf{0}}, \overline{\mathbf{6}})$	$(\overline{\mathbf{0}}, 5)$
	2	X	$(-0.5, 0)$	$(-0.5, 3)$	$(\overline{\mathbf{0}}, \overline{\mathbf{5}})$
	3	X	X	X	X

We conclude that there are two Nash equilibria: $(1,2)$ and $(2,3)$.

Problem 3 (Election games)

Two candidates are engaging in an election. Each of them has three possible strategies:

1. focus on promoting the positive aspects of himself (denoted by "P");
2. promoting the positive aspects of himself and attacking another candidates at the same time (denoted by "B");
3. focus on attacking another candidates (denoted by "A").

It is given that

- If both candidates chooses the same strategies, each of them wins with equal probability.
- If one of the candidates chooses "P" and another candidate chooses "B", then the candidates who chooses "P" wins with probability 0.1.
- If one of the candidates chooses "P" and another candidate chooses "A", then the candidates who chooses "P" wins with probability 0.35.
- If one of the candidates chooses "B" and another candidate chooses "A", then the candidates who chooses "B" wins with probability 0.4.

The objective of the candidate is to choose a strategy in order to maximize the chance of winning the election. So we assume in this games that the payoff of a candidate is the probability of winning the election.

(a) Express the games in the normal form.

(b) Determine the final outcome using IESDS.

☺Solution

(a) Set of players: $\{1,2\}$;

Strategic set $S_i = \{0,1,2,3\}$;

Payoff function: The payoff function O_i is the probability that player i wins the election. The payoff function under various scenarios is summarized by the following matrix:

		Player 2		
		P	**B**	**A**
Player 1	**P**	$(0.5, 0.5)$	$(0.1, 0.9)$	$(0.35, 0.65)$
	B	$(0.9, 0.1)$	$(0.5, 0.5)$	$(0.4, 0.6)$
	A	$(0.65, 0.35)$	$(0.6, 0.4)$	$(0.5, 0.5)$

(b) We apply IESDS and identify all dominated strategies of each player. We outline the procedure and the detail verification is similar to that in Problem 1.

(1) "P" is the dominated strategy for player 1 since it is strictly dominated by "B".

(2) "P" is the dominated strategy for player 2 since it is strictly dominated by "B".

(3) "B" is the dominated strategy for player 1 since it is strictly dominated by "A".

(4) "B" is the dominated strategy for player 2 since it is strictly dominated by "A".

Thus the only final outcome will be (A, A).

Problem 4 (Location games)

The following figure shows the route map of a subway station

x------x------x-------x------x-------x-

1 2 3 4 5 6

(20) (12) (8) (10) (14) (6)

The number in the bracket are the number of tourists travelling in a particular station. Assume that the distances between any two consecutive stations are equal.

Two competing companies plan to build a hotel in one of these 6 stations. Their objectives are to attract as many tourists as possible. It is known that each tourist will choose to stay at the hotel that *is closest to* the place that he is travelling. For example, if company 1 builds the hotel (hotel 1) at station 2 and company 2 builds the hotel (hotel 2) at station 3. Then the tourists who will travel to station 1 and 2 will choose to stay at hotel 1 and the remaining tourists will stay at hotel 2. If tourists wish to travel to a particular station and the two hotels have the same distance from this station, these tourists will choose to stay at one of the two hotels with equal probability.

- **(a)** Express the games in normal form.
- **(b)** Simplify the games as much as possible using IESDS.
- **(c)** Find all possible pure strategy Nash equilibrium.

☺Solution

(a) The games can be expressed in normal form as follows:

Set of players: $\{1,2\}$;

Strategic set $S_i = \{0,1,2,3\}$;

Payoff function: The payoff function under various scenarios is summarized by the following matrix:

		Player 2					
		1	**2**	**3**	**4**	**5**	**6**
Player 1	**1**	$(35,35)$	$(20,50)$	$(26,44)$	$(32,38)$	$(36,34)$	$(40,30)$
	2	$(50,20)$	$(35,35)$	$(32,38)$	$(36,34)$	$(40,30)$	$(45,25)$
	3	$(44,26)$	$(38,32)$	$(35,35)$	$(40,30)$	$(45,25)$	$(50,20)$
	4	$(38,32)$	$(34,36)$	$(30,40)$	$(35,35)$	$(50,20)$	$(57,13)$
	5	$(34,36)$	$(30,40)$	$(25,45)$	$(20,50)$	$(35,35)$	$(64,6)$
	6	$(30,40)$	$(25,45)$	$(20,50)$	$(13,57)$	$(6,64)$	$(35,35)$

(b) We apply IESDS and identify all dominated strategies of each player. We outline the procedure and the detail verification is similar to that in Problem 1.

(1) "1" is the dominated strategy for both players since it is strictly dominated by "2".

(2) "2" is the dominated strategy for both players since it is strictly dominated by "3".

(3) "6" is the dominated strategy for both players since it is strictly dominated by "3".

(4) "5" is the dominated strategy for both players since it is strictly dominated by "3".

(5) "4" is the dominated strategy for both players since it is strictly dominated by "3".

So we conclude that (3,3) is the final outcome of the games under IESDS.

(c) Since (3,3) is the only survivor under IESDS, it follows that (3,3) is the only Nash equilibrium of this games (since the players will not play dominated strategies which are ruled out from IESDS).

Problem 5 (Roommates)

Two roommates each need to choose to clean their apartment (they share the room). Each of them can choose an amount of time $t_i \geq 0$ to clean the apartment. Suppose that they choose spend t_i and t_j units of time to clean the apartment respectively, the payoff to player i is given by

$$O_i(t_i; t_j) = \underbrace{t_j + (5 - t_j)t_i}_{benefit} - \underbrace{t_i^2}_{cost}.$$

Here, the first term $t_j + (5 - t_j)t_i$ represents the benefits of cleaning the apartment (The second function $(5 - t_j)t_i$ indicates that the more one roommate cleans (higher t_j), the less valuable is cleaning for the other roommates.). The second term t_i^2 represents the cost of cleaning the apartment.

Determine the final outcomes using IESDS using similar technique used in Example 8.

☺Solution

We note that

$$\frac{\partial V_i}{\partial t_i} = (5 - t_j) - 2t_i \overbrace{\leqq}^{t_j \geq 0} 5 - 2t_i.$$

We note that $\frac{\partial V_i}{\partial t_i} < 0$ for all $t_j \geq 0$ when $t_i > \frac{5}{2}$. This implies that

$$V_i(t_i; t_j) < V_i\left(\frac{5}{2}; t_j\right), \quad for\ all\ t_j \geq 0, \qquad all\ t_i > \frac{5}{2}.$$

So any $t_i > \frac{5}{2}$ is dominated strategy for player i. Thus the strategic set becomes $\left[0, \frac{5}{2}\right]$. On the other hand, we note that

$$\frac{\partial V_i}{\partial t_i} = (5 - t_j) - 2t_i \overbrace{\geqq}^{t_j \leq \frac{5}{2}} \left(5 - \frac{5}{2}\right) - 2t_i = \frac{5}{2} - 2t_i.$$

We note that $\frac{\partial V_i}{\partial t_i} > 0$ for all $t_j \leq \frac{5}{2}$ when $t_i < \frac{5}{4}$. Then for any $t_i < \frac{5}{4}$, we have

$$O_i(t_i; t_j) < O_i\left(\frac{5}{4}; t_j\right), \quad for\ all\ t_j \in \left[0, \frac{5}{2}\right].$$

This implies any $t_i < \frac{5}{4}$ is also dominated strategy for player i. The strategic set becomes $\left[\frac{5}{4}, \frac{5}{2}\right]$. Therefore, the new strategic set after first round of elimination is seen to be

$$[a_1, b_1] = \left[\frac{5}{4}, \frac{5}{2}\right].$$

We let $S_i^{(k)} = [a_k, b_k]$ be the strategic set of player i after k^{th} round of elimination under IESDS, where $k \geq 1$. We consider

$$\frac{\partial V_i}{\partial t_i} = (5 - t_j) - 2t_i \overset{t_j \geq a_k}{\leq} (5 - a_k) - 2t_i.$$

We note that $\frac{\partial O_i}{\partial t_i} < 0$ for all $t_j \geq a_k$ when $t_i > \frac{5-a_k}{2}$. This implies that

$$V_i(t_i; t_j) < V_i\left(\frac{5 - a_k}{2}; t_j\right), \quad for\ all\ t_j \in [a_k, b_k], all\ t_i > \frac{5 - a_k}{2}.$$

So any $t_i > \frac{5-a_k}{2}$ is dominated strategy for player i. Thus the new strategic set becomes $\left[a_k, \frac{5-a_k}{2}\right]$. Next, we consider

$$\frac{\partial V_i}{\partial t_i} = (5 - t_j) - 2t_i \overset{t_j \leq \frac{5-a_k}{2}}{\geq} \left(5 - \frac{5 - a_k}{2}\right) - 2t_i = \frac{5 + a_k}{2} - 2t_i.$$

We note that $\frac{\partial V_i}{\partial t_i} > 0$ for all $t_j \leq \frac{5-a_k}{2}$ when $t_i > \frac{5+a_k}{4}$. Hence, the new strategic set becomes $[a_{k+1}, b_{k+1}] = \left[\frac{5+a_k}{4}, \frac{5-a_k}{2}\right]$.

By comparing coefficients, we get

$$\begin{cases} a_{k+1} = \dfrac{5 + a_k}{4} \\ b_{k+1} = \dfrac{5 - a_k}{2} \end{cases}, \quad a_1 = \frac{5}{4}, \quad b_1 = \frac{5}{2}.$$

One can show by induction (I omit the details here) that the sequence $\{a_k\}$ is increasing and $a_k \leq \frac{5}{3}$. This proves the convergence of $\{a_k\}$. On the other hand, one can show that b_k is increasing (since $b_{k+1} - b_k = -\frac{a_k - a_{k-1}}{2} < 0$) and $b_k = \frac{5-a_{k-1}}{2} \geq \frac{5-\frac{5}{3}}{2} = \frac{5}{3}$. So the sequence $\{b_k\}$ coverges to $\frac{5}{3}$.

To find the limits of a_k , b_k, we take $k \to \infty$ on both sides of the equations and let $a = \lim_{k\to\infty} a_k$ and $b = \lim_{k\to\infty} b_k$. We get

$$\begin{cases} a = \dfrac{5 + a}{4} \\ b = \dfrac{5 - a}{2} \end{cases} \Rightarrow a = b = \frac{5}{3}.$$

Therefore the player's strategic set will converge to $[a, b] = \left[\frac{5}{3}, \frac{5}{3}\right] = \left\{\frac{5}{3}\right\}$. Thus, we conclude that the final outcome is $(t_1, t_2) = \left(\frac{5}{3}, \frac{5}{3}\right)$ under IESDS.

Problem 6 (Advertising games)

Company A and company B are rival pet food companies and are choosing among advertising media. They can choose to promote their products via one of the following three medias: Facebook, poster or television. Their profits under various scenarios are summarized in the following table:

		Company B		
		Facebook	Poster	Television
Company A	Facebook	$(7{,}7)$	$(2{,}4)$	$(3{,}3)$
	Poster	$(3{,}3)$	$(3{,}6)$	$(7{,}6)$
	Television	$(4{,}7)$	$(9{,}2)$	$(2{,}7)$

Determine all possible pure strategy Nash equilibrium in this games.

☺Solution

We first find the best response s_i^* of each player. That is, we find s_i^* such that

$$V_i(s_i^*; s_{-i}^*) = \max_{s \in S_i} V_i(s; s_{-i}^*).$$

The best responses are marked in the following payoff matrix (with upper bar):

		Company B (Player 2)		
		Facebook	**Poster**	**Television**
Company A (Player 1)	**Facebook**	$(\overline{\mathbf{7}}, \overline{\mathbf{7}})$	$(2, 4)$	$(3, 3)$
	Poster	$(3, 3)$	$(3, \overline{\mathbf{6}})$	$(\overline{\mathbf{7}}, \overline{\mathbf{6}})$
	Television	$(4, \overline{\mathbf{7}})$	$(\overline{\mathbf{9}}, 2)$	$(2, \overline{\mathbf{7}})$

We conclude that there are two pure strategy Nash equilibria: $(s_1^*, s_2^*) = (Facebook, Facebook)$ and $(Poster, Television)$.

Problem 7 (Synergies)

Two division managers can invest time and effort in working on a joint investment project. Each can invest an effort e_i (any real value ≥ 0). We assume that the cost of investing effort e_i is $3e_i^2$. Given the efforts e_i, e_j chosen by the managers, the benefits to manager i is given by $(c_i + e_j)e_i + 2e_j$, where $i = 1{,}2, j \neq i$ and c_i is some positive constants. Thus the payoff functions of manager i can be express as

$$V_i(e_i; e_j) = (c_i + e_j)e_i + 2e_j - 3e_i^2.$$

We take $c_1 = 5$ and $c_2 = 10$. Determine the pure strategy Nash equilibrium of the games.

☺Solution

Given the strategy e_j chosen by the opponent, we find the best response of player i. This can be done by considering the first-order condition:

$$\frac{\partial V_i}{\partial e_i}\Big|_{e_i = e_i^*} = 0 \Rightarrow c_i + e_j - 6e_i^* = 0 \Rightarrow e_i^* = \frac{c_i + e_j}{6}.$$

Since $\frac{\partial^2 V_i}{\partial e_i^2}\Big|_{e_i = e_i^*} = -6 < 0$, so e_i^* is the required best response. We let (e_1^*, e_2^*) be the Nash equilibrium, then (e_1^*, e_2^*) must satisfy

$$\begin{cases} e_1^* = \dfrac{c_1 + e_2^*}{6} \\ e_2^* = \dfrac{c_2 + e_1^*}{6} \end{cases} \overset{\substack{c_1 = 5 \\ c_2 = 10}}{\Rightarrow} \begin{cases} e_1^* = \dfrac{5 + e_2^*}{6} \\ e_2^* = \dfrac{10 + e_1^*}{6} \end{cases} \Rightarrow (e_1^*, e_2^*) = \left(\frac{8}{7}, \frac{13}{7}\right).$$

Problem 8 (Stag Hunt)

There are two hunters (A and B). Each of them can choose to hurt either rabbit or tiger. Each of them can hurt the rabbit for sure by himself. However, they need to cooperate in order to hunt the tiger successfully (he will fail if he hunts the tiger alone). It is given that

- The reward of hunting a rabbit is 7;
- The reward of hunting a tiger is 30 (the reward will be shared equally among two hunters)
- The reward will be -5 if the hunter fails to hunt his prey.

(a) Express the games in normal form.
(b) Identify all possible pure strategy Nash equilibria in this games.
(c) From (b), determine the Pareto-optimal equilibrium and the risk-dominant equilibrium.

☺Solution

(a) The games can be expressed in normal form as follows:
Set of players: $\{A, B\}$
Strategic set: $S_i = \{R, T\}$. Here, "R" and "T" denote the strategies "hunting a rabbit" and "hunting a tiger" respectively.
Payoff function: The payoffs are summarized in the following matrix

		Hunter B	
		R	**T**
Hunter A	**R**	$(7,7)$	$(7,-5)$
	T	$(-5,7)$	$(15,15)$

(b) We first find the best response s_i^* of each player. That is, we find s_i^* such that

$$V_i(s_i^*; s_{-i}^*) = \max_{s \in S_i} V_i(s; s_{-i}^*).$$

The best responses are marked in the following payoff matrix (with upper bar):

		Hunter B	
		R	**T**
Hunter A	**R**	$(\overline{\mathbf{7}},\overline{\mathbf{7}})$	$(7,-5)$
	T	$(-5,7)$	$(\overline{\mathbf{15}},\overline{\mathbf{15}})$

We conclude that there are two pure strategy Nash equilibria: (R,R) and (T,T).

(c) It is obvious that

$$\underbrace{V_i(T;T)}_{=15} > \underbrace{V_i(R;R)}_{=7}, \quad i = A, B.$$

So (R,R) is Pareto-dominated by (T,T). Also, there is no other equilibrium that dominates (T,T). So (T,T) is the Pareto-optimal equilibrium.
On the other hand, we note that

$$\phi(R,R) = \big(7-(-5)\big)\big(7-(-5)\big) = 144.$$
$$\phi(T,T) = (15-7)(15-7) = 64.$$

Since (R,R) has larger deviation cost, thus it is risk-dominant equilibrium.

Problem 9 (Driving games)

We consider the following games so called drive-on games. Two cars meet at the intersection of two roads, each of the drivers can choose to either wait (W) or go (G). If both choose to wait, both of them will receive zero payoff. If both choose to go, they will crash and each will suffer from a loss of 100. If one goes and the other waits, the one who goes will receive a payoff of 5 since he can move first and the one who waits will receive a payoff of 1.

(a) Express the games in normal form.

(b) **(i)** Identify all possible pure strategy Nash equilibria in this games.

(ii) From **(b)(i)**, determine the Pareto-optimal equilibrium and the risk-dominant equilibrium.

(c) Determine the mixed strategy Nash equilibrium of this games.

☺Solution

(a) The games can be expressed in normal form as follows:

Set of players: $\{1,2\}$

Strategic set: $S_i = \{W, G\}$. Here, "W" and "G" denote the strategies "wait" and "go" respectively.

Payoff function: The payoffs are summarized in the following matrix

		Driver 2	
		W	**G**
Driver 1	**W**	$(0,0)$	$(1,5)$
	G	$(5,1)$	$(-100,-100)$

(b)(i) We first find the best response s_i^* of each player. That is, we find s_i^* such that

$$V_i(s_i^*; s_{-i}^*) = \max_{s \in S_i} V_i(s; s_{-i}^*).$$

The best responses are marked in the following payoff matrix (with upper bar):

		Driver 2	
		W	**G**
Driver 1	**W**	$(0,0)$	$(\bar{\mathbf{1}},\bar{\mathbf{5}})$
	G	$(\bar{\mathbf{5}},\bar{\mathbf{1}})$	$(-100,-100)$

There are two Nash equilibria: (G, W) and (W, G).

(b)(ii) Note that $\underbrace{V_1(G;W)}_{=5} > \underbrace{V_1(W;G)}_{=1}$ and $\underbrace{V_2(G;W)}_{=5} > \underbrace{V_2(W;G)}_{=1}$, (G, W) and (W, G) do not Pareto dominate each other. So both (G, W) and (W, G) are Pareto-optimal.

On the other hand,

$$\phi(G, W) = (5-0)\big(1-(-100)\big) = 505;$$
$$\phi(W, G) = \big(1-(-100)\big)(5-0) = 505.$$

Since $\phi(G, W) = \phi(W, G) = 505$, they have same deviation costs. So both equilibria are risk-dominant.

(c) Firstly, we claim that there is no mixed strategy Nash equilibrium in which one player adopt pure strategy and another player adopts mixed strategy. Suppose that player 1 adopts pure strategy (W or G), then player 2 should maximize its payoff by choosing G (if player 1 chooses W) or W (if player 1 chooses G) without mixing the two strategies. So the only mixed strategy Nash equilibrium is that both players mix their strategies.

We let $\sigma_i = (p_i, 1-p_i)$, $p_i \in (0,1)$, be the mixed strategy chosen by player i. Here, $p_i = P(player\ i\ chooses\ W)$. By indifference principle, one has

$$V_1(W;\sigma_2) = V_1(G;\sigma_2) \Rightarrow 0p_2 + 1(1-p_2) = 5p_2 - 100(1-p_2) \Rightarrow p_2 = \frac{101}{106}.$$

Similarly, we have

$$V_2(W;\sigma_2) = V_2(G;\sigma_2) \Rightarrow 0p_1 + 1(1-p_1) = 5p_1 - 100(1-p_1) \Rightarrow p_1 = \frac{101}{106}.$$

Problem 10 (Wars)

Country A and country B are rival nations, often at war, and both can produce and deploy nuclear weapon and poison gas on the battlefield. In a war in 2046, each country chooses to either deploy nuclear weapon or deploy poison gas or do nothing. The payoff matrix is shown below:

		Country B (Player 2)		
		Nuclear	Gas	Nothing
Country A (Player 1)	Nuclear	$(-18,-18)$	$(-10,-20)$	$(6,-30)$
	Gas	$(-20,-10)$	$(-9,-9)$	$(3,-15)$
	Nothing	$(-30,6)$	$(-15,3)$	$(0,0)$

(a) **(i)** Determine if there is any Pure strategy Nash equilibrium.

(ii) Hence, determine the Pareto-optimal equilibrium and the risk-dominate equilibrium.

(b) **(Harder)** Determine if there is any mixed strategy Nash equilibrium (☺Hint: Simplify the games by identifying dominated pure strategy).

☺Solution

(a)(i) Firstly, we find the player's best response to each opponent's strategy. We summarize the result in the following payoff matrix (the best responses are marked with upper bar)

		Player 2		
		Nuclear (Nu)	Gas (G)	Nothing (N)
Player 1	Nuclear (Nu)	$(\overline{\mathbf{-18}},\overline{\mathbf{-18}})$	$(-10,-20)$	$(\overline{\mathbf{6}},-30)$
	Gas (G)	$(-20,-10)$	$(\overline{\mathbf{-9}},\overline{\mathbf{-9}})$	$(3,-15)$
	Nothing (N)	$(-30,\overline{\mathbf{6}})$	$(-15,3)$	$(0,0)$

So we conclude that there are two pure strategy Nash equilibria: $(s_1^*, s_2^*) = (Nu, Nu)$ and (G, G)

(a)(ii) It is seen that (Nu, Nu) is Pareto-dominated by (G, G) since

$$-18 = V_i(Nu; Nu) < V_i(G, G) = -9$$

for all players. Thus, (G, G) is the Pareto-optimal equilibrium.

On the other hand, we note that

$$\phi(Nu, Nu) = \big(-18 - (-20)\big)\big(-18 - (-20)\big) = 4,$$
$$\phi(G, G) = \big(-9 - (-10)\big)\big(-9 - (-10)\big) = 1.$$

Since (Nu, Nu) has higher deviation cost, thus (Nu, Nu) is risk dominant equilibrium.

(b) Firstly, we observe that "N" is the dominated strategy for both players since it is strictly dominated by "G". That is,

$$\underbrace{V_i(N; Nu)}_{=-30} < \underbrace{V_i(G; Nu)}_{=-20}, \quad \underbrace{V_i(N; G)}_{=-15} < \underbrace{V_i(G; G)}_{=-9}, \quad \underbrace{V_i(N; N)}_{=0} < \underbrace{V_i(G; N)}_{=3}$$

So both players will not choose "N" in their mixed strategy. On the other hand, one can argue that there is NO Nash equilibrium which one player plays pure strategy and another player plays mixed strategy. To see this, we assume that player 1 plays pure strategy and chooses "G". It is clear that player 2 must choose "G" for sure without mixing its strategies.
Hence, the only mixed strategy Nash equilibrium is that each player mixes between "G" and "Nu". We let $\sigma_i = (p_i, 1-p_i, 0)$ be the mixed strategy of player , where $p_i = \Pr(player\ i\ chooses\ Nu)$ and $1 - p_i = \Pr(player\ i\ chooses\ G)$. Since $p_i \in (0,1)$, one can apply indifference and obtain

$$V_i(Nu; \sigma_j) = V_i(G; \sigma_j) \Rightarrow -18p_j - 10(1-p_j) = -20p_j - 9(1-p_j) \Rightarrow p_j = \frac{1}{3},$$

for $i = 1,2$ and $j \neq i$.
Thus, we conclude that $\sigma_i^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $i = 1,2$, is the unique mixed strategy Nash equilibrium.

Problem 11

The following shows the payoff matrix of a two-person games

		Player 2		
		A	B	C
Player 1	A	$(0,0)$	$(7,2)$	$(1,-1)$
	B	$(2,7)$	$(6,6)$	$(0,5)$
	C	$(1,3)$	$(1,3)$	$(2,2)$

(a) Find all possible pure strategy Nash equilibrium.
(b) Simplify the games using IESDS. Hence, determine all possible mixed strategy Nash equilibria.

☺Solution

(a) Firstly, we find the player's best response to each opponent's strategy. We summarize the result in the following payoff matrix (the best responses are marked with upper bar)

		Player 2		
		A	B	C
Player 1	A	$(0,0)$	$(\bar{\mathbf{7}}, \bar{\mathbf{2}})$	$(1,-1)$
	B	$(\bar{\mathbf{2}}, \bar{\mathbf{7}})$	$(6,6)$	$(0,5)$
	C	$(1, \bar{\mathbf{3}})$	$(1, \bar{\mathbf{3}})$	$(\bar{\mathbf{2}}, 2)$

So we conclude that there are two pure strategy Nash equilibria: $(s_1^*, s_2^*) = (B, A)$ and (A, B).

(b) We first simplify the games using IESDS. Firstly, we observe that "C" is dominated strategy for player 2 since it is strictly dominated by "A". Once "C" is removed from player 2's strategic set, we note that "C" is also dominated strategy for player 1 since it is dominated by "B". Therefore the games can be simplified into

		Player 2	
		A	**B**
Player 1	**A**	$(0,0)$	$(7,2)$
	B	$(2,7)$	$(6,6)$

On the other hand, there is no Nash equilibrium in which one player adopt pure strategy and another player adopts mixed strategy. To see, we assume that player 1 adopt pure strategy and chooses A, it is obvious that player 2 must choose B for sure (pure strategy).

So both players must mix between A and B. We let $\sigma_1^* = (p_i^*, 1-p_i^*, 0)$ be the player i's mixed strategy, where $p_i^* = \Pr(player\ i\ chooses\ A)$ and $1 - p_i^* = \Pr(player\ i\ chooses\ B)$. Using indifference principle, we get

$$V_1(A;\sigma_2^*) = V_1\,(B;\sigma_2^*) \Rightarrow 0p_2^* + 7(1-p_2^*) = 2p_2^* + 6(1-p_2^*) \Rightarrow p_2^* = \frac{1}{3}.$$

$$V_2(A;\sigma_1^*) = V_2\,(B;\sigma_1^*) \Rightarrow 0p_1^* + 7(1-p_2^*) = 2p_2^* + 6(1-p_2^*) \Rightarrow p_1^* = \frac{1}{3}.$$

So we conclude that the unique mixed strategy Nash equilibrium is $\sigma_1^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ for $i = 1,2$.

Problem 12 (Happy Hour)

Restaurant ABC and Restaurant XYZ compete for the same crowd in a district. Each can offer free drink during lunch hour, or not.

- If none of them offers free drink s, the profit of each bar will be \$35.
- If both of them offers free drinks, the profit of each bar will be reduced to \$25.
- If one offers free drinks and the other does not, the one who offers drinks will get most of the customers and gain a profit of \$70. Another restaurant will lose \$15.

(a) Express the games in normal form.

(b) Determine if there is any dominated strategies in this games.

(c) Find all possible pure strategy Nash equilibrium in this games.

(d) Find all possible mixed strategy Nash equilibrium in this games. How to interpret the mixed strategy equilibrium in reality?

☺Solution

(a) The games can be expressed in normal form as follows:

Set of players: $\{ABC, XYZ\}$

Strategic set: $S_i = \{F, N\}$. Here, "F" and "N" denote the strategies “offer free drink” and “not offer free drink” respectively.

Payoff function: The payoffs are summarized in the following matrix

		XYZ (Player 2)	
		F	**N**
ABC (Player 1)	**F**	$(25, 25)$	$(70, -15)$
	N	$(-15, 70)$	$(35, 35)$

(b) One can observe that N is the dominated strategy for both players since it is strictly dominated by "F". That is,

$$35 = V_i(N;N) < V_i(F;N) = 70,$$
$$-15 = V_i(N;F) < V_i(F;F) = 25.$$

(c) Since N is dominated strategy, so it is not chosen by any player under equilibrium. Thus, the only possible Nash equilibrium will be (F, F). This can be verified by the inequalities in (b).

(d) Since N will not be chosen (with positive probability) under equilibrium (from (c)), thus it is impossible for both players to mix between F and N. Since F and N are the only strategies that can be chosen by the players, thus there is no mixed strategy Nash equilibrium.

Problem 13 (Coordination games)

A group of people are involved in some task that depends on efforts of each of them. Each of them can choose either "work" or "begin lazy". If one person does not work, others need to increases effort. The payoffs are summarized in the following payoff matrix:

		Player 2	
		Work (W)	Being lazy (L)
Player 1	Work (W)	$(10,10)$	$(2,16)$
	Being lazy (L)	$(14,4)$	$(6,6)$

Determine all equilibrium (pure strategy equilibrium and mixed strategy equilibrium), if any, using indifference principle.

☺Solution

Firstly, we determine the best response of each player using indifference principle:

Best response of player 1

We let $\sigma_2 = (q, 1-q)$ be the strategy of player 2. We consider the equation

$$\underbrace{10q + 2(1-q)}_{V_1(W;\sigma_2)} = \underbrace{14q + 6(1-q)}_{V_1(L;\sigma_2)} \Rightarrow 8q + 2 = 8q + 6 \Rightarrow 0 = 4.$$

So the equation has no solution in q. In fact, for any $q \in [0,1]$, we have

$$V_1(L;\sigma_2) = 8q + 6 > 8q + 2 = V_1(W;\sigma_2).$$

Then it follows from the proof of indifference principle that the player 1 should adopt the strategy "L" with 100% probability. Thus, the best response of player 1 is

$$\sigma_1^* = (p^*, 1-p^*) = (0,1) \quad for \ \ all \ \ q \in [0,1].$$

Best response of player 2

We let $\sigma_1 = (p, 1-p)$ be the strategy of player 2. We consider the equation

$$\underbrace{10p + 4(1-p)}_{V_2(W;\sigma_1)} = \underbrace{16p + 6(1-p)}_{V_2(L;\sigma_1)} \Rightarrow 6p + 4 = 10p + 6 \Rightarrow p = -\frac{1}{2}.$$

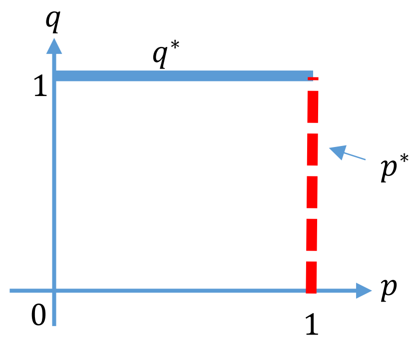
In fact, for any $p \in [0,1]$, we have

$$V_2(W;\sigma_1) = 6p + 4 < 10p + 6 = V_2(L;\sigma_1).$$

Then it follows from the proof of indifference principle that the player 2 should adopt the strategy "L" with 100% probability. Thus, the best response of player 1 is

$$\sigma_2^* = (q^*, 1-q^*) = (0,1) \quad for \ \ all \ \ p \in [0,1].$$

Then the best responses of two players can be visualized by the following graph:



We observe that two lines intersects at $(p, q) = (1,1)$. Thus, the only Nash equilibrium of the games is $\sigma_1^* = \sigma_2^* = (0,1)$ and two players chooses to play L with 100% probability (pure strategy).

Remark:
In fact, one can show that W is the dominated strategy for both players.

Problem 14 (Penalty Kick)
A kicker and a goalie confront each other in a penalty kick that will determine the outcome of a soccer games. The kicker can kick the ball left or middle or right, while the goalie can choose to jump left, stay at the middle or jump right. They need to make their decisions simultaneously.

- If the goalie jumps in the same direction as the kick, then the goalie wins and the kicker loses.
- Otherwise, the goalie loses and the kicker wins.
- The payoffs of winning and losing are 1 and -1 respectively

(a) Express the games in normal form.
(b) Show that there is no pure strategy Nash equilibrium in this games.
(c) Find all possible mixed strategy Nash equilibrium. (☺Hint: The games is quite similar to paper-scissor-rock games that is discussed in the lecture.)

☺Solution
(a) Set of players: $\{Kicker, Goalie\}$
Strategic set: $S_i = \{Left, Middle, Right\}$. Payoff function: The payoffs are summarized in the following matrix

		Goalie (Player 2)		
		L	M	R
Kicker (Player 1)	Left (L)	$(-1,1)$	$(1,-1)$	$(1,-1)$
	Middle (M)	$(1,-1)$	$(-1,1)$	$(1,-1)$
	Right (R)	$(1,-1)$	$(1,-1)$	$(-1,1)$

(b) We argue that (L, L) is not Nash equilibrium since the player 1 can regain his winning status by choosing M or R. Similarly, (M, M) and (R, R) cannot be Nash equilibrium.

On the other hand, (L, M) is not Nash equilibrium, since the player 2 can regain his winning status by choosing L. Similarly, we can deduce that (L, R), (M, L), (M, R), (R, L) and (R, M) are not Nash equilibria. Therefore, there is no pure strategy Nash equilibrium.

(c) The derivation is very similar to that of Example 29 in Lecture Note 1. We omit the details here. The mixed strategy Nash equilibrium is given by $\sigma_1 = \sigma_2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Problem 15 (Contribution games)

Three players live in a building, and each can choose to contribute to fund a new mailbox. The value of having the mailbox is 8 for each player, and the value of not having it is 0. The mayor asks each player to contribute or not contribute. The cost of contributing is known to be $\frac{6}{N}$, where N is number of players who will contribute. If at least two players contribute then the mailbox will be installed. If one players or no players contribute, then the mailbox will not be installed, in which case any person who contributed will not get his money back.

(a) Express the games in the normal games.

(b) Find all possible pure strategy Nash equilibrium.

(c) Find all *symmetric* mixed strategy Nash equilibrium which each player chooses to contribute with the same probability p.
(From (c), indifference principle may be useful.)

☺Solution

(a) Set of players: $\{1, 2, 3\}$
Strategic set: $S_i = \{C, N\}$. Here, "C" and "N" denote the strategies “contribute” and “not contribute” respectively.
Payoff function:

$$V_i(C; C, C) = 8 - \frac{6}{3} = 6, \;\; V_i(C; N, C) = V_i(C; C, N) = 8 - \frac{6}{2} = 5, \;\; V_i(N; C, C) = 8,$$
$$V_i(C; N, N) = -6, \;\; V_i(N; C, N) = V_i(N; N, C) = V_i(N, N, N) = 0.$$

(b) We consider the following four cases.
Case 1: If none of the players contribute,
In this case, one can show that

$$\underbrace{V_i(N; N, N)}_{=0} > \underbrace{V_i(C; N, N)}_{=-6}.$$

So $(s_1^*, s_2^*, s_3^*) = (N, N, N)$ is the Nash equilibrium.
Case 2: If there is only one player (say player k) contributes, one can show that player k has incentive to deviate, i.e.

$$\underbrace{V_k(C; N, N)}_{=-6} < \underbrace{V_k(N; N, N)}_{=0}.$$

Hence, (C, N, N), (N, C, N) and (N, N, C) are not Nash equilibria.
Case 3: If there are two players (say player 1 and 2) contribute, one can show that

$$\underbrace{V_i(C; N, C)}_{=5} > \underbrace{V_i(N; N, C)}_{=0} \quad for \;\; i = 1,2 \; and$$
$$\underbrace{V_3(N; C, C)}_{=8} > \underbrace{V_3(C; C, C)}_{=6}.$$

This proves that (C, C, N), (C, N, C) and (N, C, C) are all Nash equilibria.

Case 4: If all players contribute, one can show that any player has incentive to deviate and choose not to contribute, i.e.

$$\underbrace{V_i(C;C,C)}_{=6} < \underbrace{V_i(N;C,C)}_{=8}.$$

Hence, (C,C,C) is not the Nash equilibrium.

(c) We let $\sigma_i^* = (p^*, 1-p^*)$ be the mixed strategy of player I, where $p^* = P(player\ i\ chooses\ to\ contribute)$. (Note: Since we are seeking for symmetric equilibrium, so all players share the common p^*). Since $p^* \in (0,1)$, one can apply indifference principle and obtain

$$V_i(C;\sigma_{-i}^*) = V_i(N;\sigma_{-i}^*) \Rightarrow p^2(6) + 2p(1-p)(5) + (1-p)^2(-6) = 8p^2$$

$$\Rightarrow 18p^2 - 22p + 6 = 0 \Rightarrow p^* = \frac{22 \pm \sqrt{22^2 - 4(18)(6)}}{2(18)} \approx 0.81142 \ \ or \ \ 0.410803.$$

Hence, there are two mixed strategy Nash equilibrium.

Problem 16 (Market entry)

Three competing firms are considering entering a new market. The payoff for each firm that enters is $\frac{240}{n}$, where n is the number of firms that enter (The revenue to each firm will be lower when more firms are entering into the market). The cost of entering is 90.

(a) Find all possible pure-strategy Nash equilibria.

(b) Find the symmetric mixed strategy Nash equilibrium in which all three firms enter with the same probability.

☺Solution

(a) We consider the following four scenarios:

Case 1: If none of the players enter the market,
In this case, one can show that

$$\underbrace{V_i(N;N,N)}_{=0} > \underbrace{V_i(E;N,N)}_{=240-90=150}.$$

So $(s_1^*, s_2^*, s_3^*) = (N,N,N)$ is not the Nash equilibrium.

Case 2: If there is only one player (say player k) enters, one can show that other players has incentive to deviate, i.e.

$$\underbrace{V_i(N;E,N)}_{=0} < \underbrace{V_i(E;E,N)}_{=\frac{240}{2}-90=30}, \qquad i \neq k.$$

Hence, (E,N,N), (N,E,N) and (N,N,E) are not Nash equilibria.

Case 3: If there are two players (say player 1 and 2) enter, one can show that

$$\underbrace{V_i(E;N,E)}_{=30} > \underbrace{V_i(N;N,E)}_{=0} \quad and$$

$$\underbrace{V_3(N;E,E)}_{=0} > \underbrace{V_3(E;E,E)}_{=\frac{240}{3}-90=-10}.$$

This proves that (E,E,N), (E,N,E) and (N,E,E) are all Nash equilibria.

Case 4: If all players enter one can show that any player has incentive to deviate, i.e.

$$\underbrace{V_i(E;E,E)}_{=-10} < \underbrace{V_i(N;E,E)}_{=0}.$$

Hence, (E,E,E) is not the Nash equilibrium.

(b) We let $\sigma_1^* = (p_i^*, 1-p_i^*)$ be player i's mixed strategy, where $p_i^* = P(player\ i\ chooses\ to\ enter)$. Assume that $p_i^* \in (0,1)$, we can deduce from indifference principle that

$V_i(C;\sigma_{-i}^*) = V_i(N;\sigma_{-i}^*) \Rightarrow p^2(-10) + 2p(1-p)(30) + (1-p)^2(150) = 0$

$\Rightarrow 80p^2 - 240p + 150 = 0$

$\Rightarrow p^* = \dfrac{240 \pm \sqrt{240^2 - 4(80)(150)}}{2(80)} = 2.112372\ (rej.)\ \ and\ \ 0.887628.$

So there is an unique mixed strategy Nash equilibrium.

Problem 17 (Dominated strategy and mixed strategy)

The payoff matrix of a two-person games is presented below:

		Player 2		
		L	C	R
Player 1	T	$(6,2)$	$(0,6)$	$(4,4)$
	M	$(2,12)$	$(4,3)$	$(2,5)$
	B	$(0,6)$	$(10,0)$	$(2,2)$

(a) Show that NO pure strategy dominated by any other pure strategy for any player.

(b) Show that "M" is dominated strategy for player 1 by considering the mixed strategy $\sigma_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$.

(c) Hence, show that "R" is also dominated strategy for player 2 by considering the mixed strategy $\sigma_2 = \left(\frac{5}{12}, \frac{7}{12}, 0\right)$.

(d) Determine all possible Nash equilibria (pure strategy and mixed strategy).

☺Solution

(a) One can show directly from payoff matrix that that

$$\underbrace{V_1(T;L)}_{=6} > \underbrace{V_1(M;L)}_{=2},\quad \underbrace{V_1(T;L)}_{=6} > \underbrace{V_1(B;L)}_{=0},$$
$$\underbrace{V_1(M;C)}_{=4} > \underbrace{V_1(T;C)}_{=0},\quad \underbrace{V_1(M;L)}_{=2} > \underbrace{V_1(B;L)}_{=0},$$
$$\underbrace{V_1(B;C)}_{=10} > \underbrace{V_1(M;C)}_{=4},\quad \underbrace{V_1(B;C)}_{=10} > \underbrace{V_1(T;C)}_{=0}.$$

The first set of inequalities show that T is not dominated strategy. The second set of inequalities show that M is not dominated strategy. The third set of inequalities show that B is not dominated strategy. Hence, there is no dominated strategy for player 1. The case for player 2 can be derived in a similar fashion.

(b) One can establish that

$$2 = V_1(M;L) < V_1(\sigma_1;L) = \frac{1}{2}(6) + \frac{1}{2}(0) = 3,$$
$$4 = V_1(M;C) < V_1(\sigma_1;C) = \frac{1}{2}(0) + \frac{1}{2}(10) = 5,$$
$$2 = V_1(M;R) < V_1(\sigma_1;R) = \frac{1}{2}(4) + \frac{1}{2}(2) = 3.$$

Hence M is the dominated strategy and should be ruled out from player 1's strategic set.

(c) Provided that player 1 only chooses between T and B, one can show that

$$4 = V_2(R;T) < V_2(\sigma_2;T) = \frac{5}{12}(2) + \frac{7}{12}(6) = 4.5,$$
$$2 = V_2(R;B) < V_2(\sigma_2;B) = \frac{5}{12}(6) + \frac{7}{12}(0) = 2.5.$$

So R is dominated strategy and should be ruled out from player 2's strategic set.

(d) Using the result from (b) and (c), the games can be reduced into

		Player 2	
		L	**C**
Player 1	**T**	(6,2)	(0,6)
	B	(0,6)	(10,0)

We let $\sigma_1 = (p, 1-p)$ and $\sigma_2 = (q, 1-q)$ be the players' mixed strategy. Here, $p = P(player\ 1\ chooses\ T)$ and $q = P(player\ 2\ chooses\ L)$. The payoff functions can be expressed as

$$\begin{aligned} V_1(p;q) &= pq(6) + p(1-q)(0) + (1-p)q(0) + (1-p)(1-q)(10) \\ &= 10 - 10p - 10q + 16pq. \\ V_2(q;p) &= pq(2) + p(1-q)(6) + (1-p)q(6) + (1-p)(1-q)(0) \\ &= 6p + 6q - 10pq. \end{aligned}$$

We first obtain the best response (denoted by p^* and q^* respectively) of each player. We consider

$$\frac{\partial V_1}{\partial p} = -10 + 16q = \begin{cases} < 0 & if\ \ q < \frac{5}{8} \\ = 0 & if\ \ q = \frac{5}{8} \\ > 0 & if\ \ q > \frac{5}{8} \end{cases} \Rightarrow p^* = \begin{cases} 0 & if\ \ q < \frac{5}{8} \\ x & if\ \ q = \frac{5}{8} \\ 1 & if\ \ q > \frac{5}{8} \end{cases},$$

$$\frac{\partial V_2}{\partial q} = 6 - 10p = \begin{cases} > 0 & if\ \ p < \frac{3}{5} \\ = 0 & if\ \ p = \frac{3}{5} \\ < 0 & if\ \ p > \frac{3}{5} \end{cases} \Rightarrow q^* = \begin{cases} 1 & if\ \ p < \frac{3}{5} \\ y & if\ \ p = \frac{3}{5} \\ 0 & if\ \ p > \frac{3}{5} \end{cases}.$$

where x and y are real constant between 0 and 1. By plotting the best responses on p-q plane, one can find that p^* and q^* intersect at $(p,q) = \left(\frac{3}{5}, \frac{5}{8}\right)$. Thus we conclude that the unique Nash equilibrium is

$$\sigma_1^* = \left(\frac{3}{5}, \frac{2}{5}\right), \qquad \sigma_2^* = \left(\frac{5}{8}, \frac{3}{8}\right).$$

Problem 18

We consider a two-person games. It is given that for any player i and any combination of opponent's pure strategies s_{-i} the player $i's$ best response s_i^* to s_{-i} is unique. Show that there is no Nash equilibrium in which one player (say player 1) uses pure strategy s_1^* and another player (say player 2) uses mixed strategy σ_2^*.

☺Hint: The uniqueness of best response implies that for any other strategy $s_i \in S_i$, we must have $V_i(s_i; s_{-i}) < V_i(s_i^*; s_{-i})$.)

☺Solution

Suppose that such equilibrium exists, we assume that player 1 adopts the pure strategy s_1^*. Since player 2 will choose mixed strategy, we let s_2^1, s_2^2, ..., s_2^m be strategies that are mixed by player 2 in his mixed strategy (i.e. $\sigma_2\left(s_2^j\right) > 0$). Using the indifference principle, one has

$$V_2(s_2^1; s_1^*) = V_2(s_2^2; s_1^*) = \cdots = V_2(s_2^m; s_1^*).$$

Since the best response to s_1^* is unique (call it s_2^*), this implies that any strategy s_2^j cannot be the best response to s_1^*. That is,

$$V_2\left(s_2^j; s_1^*\right) < V_2(s_2^*; s_1^*).$$

This implies that

$$V_2(\sigma\_2\,; s_1^*) = \sum_j \sigma_2\left(s_2^j\right) V_2\left(s_2^j; s_1^*\right) < \sum_j \sigma_2\left(s_2^j\right) V_2(s_2^*; s_1^*) \overset{\sum_j \sigma_2\left(s_2^j\right)=1}{\widehat{=}} V_2(s_2^*; s_1^*).$$

This implies that σ_2 cannot be the best response to s_1^* and this leads to contradiction.

Problem 19

Suppose in a n-person games, every player has a strategy $s_i^* \in S_i$ that strictly dominates all of his other strategies $s_i \in S_i$, Show that the strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is the *unique* Nash equilibrium.

(☺Hint: Firstly, you need to verify that $s^*$is Nash equilibrium. To show the uniqueness, you need to show other strategic profile $s = (s_1, s_2, \dots, s_n)$ cannot be Nash equilibrium. Recall the technique that we introduced in the lecture note.)

☺Solution

Firstly, s_i^* should satisfy

$$V_i(s_i^*; s_{-i}) > V_i(s_i; s_{-i})$$

for any $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

By taking $s_{-i} = s_{-i}^*$, one can deduce that

$$V_i(s_i^*; s_{-i}^*) > V_i(s_i; s_{-i}^*) \quad for\ any\ s_i.$$

This shows that s_i^* is the best response to s_{-i}^* and s^* is the Nash equilibrium.

Suppose that there is another Nash equilibrium s^0 ($\neq s^*$), we have $s_j^0 \neq s_j^*$ for some j. Since s_j^* dominates s_j^0, one can deduce that (take $s_{-j} = s_{-j}^0$ in the first inequality)

$$V_j\left(s_j^0; s_{-j}^0\right) < V_j\left(s_j^*; s_{-j}^0\right).$$

This show that s_j^0 is not the best response to s_{-j}^0 and s^0 cannot be Nash equilibrium.

Problem 20

(a) We consider a two-person game with 2 pure strategies for each player. Suppose that the game has a *unique* pure-strategy Nash equilibrium $s^* = (s_1^*, s_2^*)$, show that s^* is the only survivor under IESDS.

(b) The statement in (a) becomes false if there are at least 3 pure strategies for each player. Verify this statement by constructing a suitable counter-example.

☺Solution

(a) We let $S_1 = \{A, B\}$ and $S_2 = \{a, b\}$ be the strategic sets of two players. Without loss of generality, we assume that (A, a) is the unique Nash equilibrium of the games.

We argue that either B is dominated strategy for player 1 or b is dominated strategy for player 2. Suppose not, we must have

$$V_1(B; s_2) \geq V_1(A; s_2), \qquad for\ some\ s_2 \in \{a, b\} \ldots (*)$$

Since (A, a) is Nash equilibrium and A is the unique best response to a, we must have $V_1(A; a) > V_1(B; a)$. So it follows that $s_2 = b$.

Using similar argument, we can deduce that

$$V_2(b; B) \geq V_2(a; B) \ldots \ldots (**)$$

Combining $(*)$ and $(**)$, we conclude that (B, b) is also Nash equilibrium. There is a contradiction. So either B or b is dominated strategy.

Assume that B is dominated strategy for player 1, then B is ruled out from player 1's strategic set. Since a is the unique best response to A, we have

$$V_2(b; A) < V_2(a; A).$$

So b is also dominated strategy for player 2. Thus (A, a) is the unique survivor under IESDS.

(b) The answer is negative. We consider the following counter-example

		Player 2		
		a	b	c
Player 1	A	(6,5)	(5,4)	(6,4)
	B	(5,3)	(6,6)	(4,5)
	C	(4,6)	(8,0)	(7,5)

One can verify that

- The best response to any opponent's strategy is unique;
- There is a unique Nash equilibrium (A, a).
- There is no dominated strategy for each player so that the survivor of IESDS is not unique.

Problem 21

(a) Suppose that a strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ Pareto-dominates all other strategic profile $s = (s_1, s_2, \dots, s_n)$, show that s^* is the Nash equilibrium.

(b) Is s^* the unique Nash equilibrium? Prove it or disprove it by providing counter-example.

☺Solution

(a) For any player i, note that $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ Pareto-dominates the strategic profile $s = (s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$ (where s_i is another pure strategy for player i), we have

$$V_i(s_i^*; s_{-i}^*) \geq V_i(s_i; s_{-i}^*).$$

As the above inequality holds for any s_i, so s_i^* is the best response to s_{-i}^*. Therefore, $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is the Nash equilibrium.

(b) No, one can consider the following 2-person finite games with the following payoff matrix:

		Player 2	
		X	**Y**
Player 1	**A**	(2,2)	(0,0)
	B	(0,0)	(1,1)

- One can verify that (X, A) Pareto dominates all other strategies since
  $V_1(A; X) = 2 > 0 = V_1(A; Y), \qquad V_2(X; A) = 2 > 0 = V_2(Y; A)$
  $V_1(A; X) = 2 > 0 = V_1(B; Y), \qquad V_2(X; A) = 2 > 0 = V_2(Y; B)$
  $V_1(A; X) = 2 > 1 = V_1(B; Y), \qquad V_2(X; A) = 2 > 1 = V_2(Y; B)$
- Note that (A, X) is Nash equilibrium as shown in (a). On the other hand, one can show that (B, Y) is also Nash equilibrium since
  $$V_1(B; Y) = 1 > 0 = V_1(A; Y),$$
  $$V_2(Y; B) = 1 > 0 = V_2(X; B).$$

Therefore, the Nash equilibrium is not unique for this games.

Problem 22

(a) We let $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ be the mixed strategy Nash equilibrium. Show that for any strategy $s_i \in S_i$ that $\sigma_i^*(s_i) > 0$, we have

$$V_i(s_i; \sigma_{-i}^*) = V_i(\sigma_i^*; \sigma_{-i}^*).$$

(☺Hint: Indifference principle will be useful)

(b) If there is a pure strategy $s_i^0 \in S_i$ such that $V_i(s_i^0; \sigma_{-i}^*) < V_i(\sigma_i^*; \sigma_{-i}^*)$ (That is, s_i^0 is not the best response to σ_{-i}^*), show that player i never choose s_i^0 under the equilibrium (i.e. $\sigma_i^*(s_i^0) = 0$).

☺Solution

(a) According to indifference principle, one must have

$$V_i(s_i^1; \sigma_{-i}^*) = V_i(s_i^2; \sigma_{-i}^*) = \cdots = V_i(s_i^m; \sigma_{-i}^*) \overset{def}{\triangleq} M$$

for $s_i^1, s_i^2, \dots, s_i^m$ which $\sigma_i^*(s_i^k) > 0$ for $k = 1,2, \dots, M$.

It follows that

$$V_i(\sigma_i^*;\sigma_{-i}^*) =^{(*)} \sum_{k=1}^{m} \sigma_i(s_i^k) V_i(s_i;\sigma_{-i}^*) = M \underbrace{\sum_{k=1}^{m} \sigma_i(s_i^k)}_{=1} = M.$$

(*Technically, we may omit the terms with $\sigma_i(s_i^k) = 0$)

(b) Suppose that $\sigma_i^*(s_i^0) > 0$, it follows from (a) that

$$V_i(s_i^0;\sigma_{-i}^*) = V_i(\sigma_i^*;\sigma_{-i}^*).$$

However, it contradicts to the given condition. Thus, we conclude that $\sigma_i^*(s_i^0) = 0$.