# MATH4425 (T1A) – Tutorial 2

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## Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

## 1 Time series models

Let $\ldots, Z_{-t}, \ldots, Z_{-1}, Z_0, Z_1, \ldots, Z_t, \ldots$ be a sequence of TS r.v.

How to describe the relationship between $Z_t$ and the past data $Z_{t-1}, Z_{t-2}, \ldots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \ldots) + a_t$$

It is called the **time series model**.

1. Autoregressive (AR(1)) model:
$$Z_t = \phi Z_{t-1} + a_t,$$
where $\phi$ is a constant and called the **parameter**

2. AR(p) model:
$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$
where $\phi_p$ is a constant and called the **parameter**, and $p$ is called the **order** of the AR(p) model

3. $AR(\infty)$ model:
$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t$$

4. Moving-average (MA) model:
$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots.$$

5. ARMA model

# 2 Stationary time series models. Autoregressive processes

## 2.1 The first order autoregressive AR(1) process

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance $\sigma_a^2$. If $\dot{Z}_t$ satisfies the following equation:

$$\dot{Z}_t = \phi \dot{Z}_{t-1} + a_t,$$

$\dot{Z}_t$ is called the **AR(1)** .

By the definition we also have

$$\dot{Z}_{t+1} = \phi \dot{Z}_t + a_{t+1}$$
$$\dot{Z}_t = \phi \dot{Z}_{t-1} + a_t$$
$$\dot{Z}_{t-1} = \phi \dot{Z}_{t-2} + a_{t-1}$$

### 2.1.1 Expansion of AR(1) process

$$\dot{Z}_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-1} a_1 + \phi^t \dot{Z}_0,$$

since

$$\begin{gathered}\dot{Z}_t = \phi \dot{Z}_{t-1} + a_t = \phi(\phi \dot{Z}_{t-2} + a_{t-1}) + a_t = \\ \phi^2(\phi \dot{Z}_{t-3} + a_{t-2}) + \phi a_{t-1} + a_t = \\ \phi^3(\phi \dot{Z}_{t-4} + a_{t-3}) + \phi^2 a_{t-2} + \phi a_{t-1} + a_t = \\ \phi^k \dot{Z}_{t-k} + \phi^{k-1} a_{t-k+1} + \cdots + \phi^2 a_{t-2} + \phi a_{t-1} + a_t = \dots\end{gathered}$$

When $\phi = 1$,

$$\dot{Z}_t = \dot{Z}_{t-1} + a_t = a_t + a_{t-1} + a_{t-2} + \cdots + a_1 + \dot{Z}_0$$

is called the **random walk** or **unstable process**.

When $|\phi| > 1$, e.g. $\phi = 3$,

$$\dot{Z}_t = a_t + 3a_{t-1} + 3^2 a_{t-2} + \cdots + 3^{t-1} a_1 + 3^t \dot{Z}_0$$

is called the **explosive process**.

When $|\phi| < 1$, e.g. $\phi = 0.5$,

$$\dot{Z}_t = a_t + 0.5a_{t-1} + 0.5^2 a_{t-2} + \cdots + 0.5^{t-1} a_1 + 0.5^t \dot{Z}_0$$

is called the **stable process**[1].
Intuitively, when the time $t$ goes very far away, the impact of the past noises and the initial value on the current value $\dot{Z}_t$ becomes negligible, almost disappears.

By the definition we also have

$$\dot{Z}_0 = \phi \dot{Z}_{-1} + a_0$$
$$\dot{Z}_{-1} = \phi \dot{Z}_{-2} + a_{-1}$$
$$\dot{Z}_{-2} = \phi \dot{Z}_{-3} + a_{-2}$$

In general, we have the following expansion:

$$\dot{Z}_t = \sum_{i=0}^{m} \phi^i a_{t-i} + \phi^{m+1} \dot{Z}_{t-m-1}$$

[1]Stable process is a process whose associated probability distributions are stable distributions.

Let's consider the first term $S_m = \sum_{i=0}^{m} \phi^i a_{t-i}$. What will happen if we let $m \to \infty$?

**Definition:** If

$$\mathbb{E}(\xi_m - \xi)^2 \to 0 \quad \text{as} \quad m \to \infty.$$

we say that the sequence $\xi_m$ of random variables **converges** to the random variable $\xi$ **in mean square** (or **in $L^2$-sense**).

Actually, we can prove that

$$S_m \to \sum_{i=0}^{\infty} \phi^i a_{t-i} \quad \text{in mean square,}$$

if and only if $|\phi| < 1$.

Moreover, the second term $\phi^{m+1} \dot{Z}_{t-m} \to 0$ in mean square.

Thus, we have the following result: *if and only if $|\phi| < 1$, $\dot{Z}_t$ in the AR(1) process has the following expansion:*

$$\dot{Z}_t = \sum_{i=0}^{\infty} \phi^i a_{t-i},$$

where the infinite sum converges in mean square.

### 2.1.2 ACF of the AR(1) process

When $|\phi| < 1$,

1)

$$\mu_t = \mathbb{E}\dot{Z}_t = \mathbb{E}\left(\sum_{i=0}^{\infty} \phi^i a_{t-i}\right) = 0 = \mu < \infty$$

2)

$$\sigma_t^2 = \text{var}(\dot{Z}_t) = \mathbb{E}\dot{Z}_t^2 = \mathbb{E}\dot{Z}_t(\phi\dot{Z}_{t-1} + a_t) =$$

$$\phi\mathbb{E}\dot{Z}_t\dot{Z}_{t-1} + \mathbb{E}Z_t a_t = \phi\mathbb{E}\left[\sum_{i=0}^{\infty} \phi^i a_{t-i} \times \sum_{i=0}^{\infty} \phi^i a_{t-1-i}\right] + \mathbb{E}(\sum_{i=0}^{\infty} \phi^i a_{t-i})a_t =$$

$$\phi \sum_{i=0}^{\infty} \phi^{i+1}\phi^i \sigma_a^2 + \sigma_a^2 = \phi\sigma_a^2 \sum_{i=0}^{\infty} \phi^{2i+1} + \sigma_a^2 = [|\phi| < 1] =$$

$$\phi\sigma_a^2 \frac{\phi}{1-\phi^2} + \sigma_a^2 = \sigma_a^2\left(\frac{\phi^2}{1-\phi^2} + 1\right) = \frac{\sigma_a^2}{1-\phi^2} = \sigma^2 < \infty$$

3)

$$\gamma(t, t+k) = \mathbb{E}[(\dot{Z}_t - \mu)(\dot{Z}_{t+k} - \mu)] =$$

$$\mathbb{E}\left[\dot{Z}_t\left(\sum_{i=0}^{k-1} \phi^i a_{t+k-i} + \phi^{k-1+1}\dot{Z}_{t+k-(k-1)-1}\right)\right] =$$

$$\mathbb{E}\left[\dot{Z}_t\left(\sum_{i=0}^{k-1} \phi^i a_{t+k-i} + \phi^k \dot{Z}_t\right)\right] = 0 + \phi^k\sigma^2 = \frac{\sigma_a^2\phi^k}{1-\phi^2} = \gamma_k$$

4)

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k$$

Thus, in this case, $\dot{Z}_t$ is *stationary.*

**Note:** when $|\phi| \geq 1$, $\dot{Z}_t$ is **not** stationary.

### 2.1.3 Partial Autocorrelation function (PACF) of AR(1) process

$$\phi_{kk} = \begin{cases} \rho_1 = \phi, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

Let's recall:

**Formula**: $\phi_{11} = \rho_1$,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

Notice that last 2 rows of the matrix in the nominator are linearly dependant. So, $\phi_{kk} = 0$ if $k \neq 1$.
The AR(1) process can be written as:

$$(1 - \phi B)\dot{Z}_t = a_t,$$

## 2.2 The second order autoregressive AR(2) process

### 2.2.1 Model

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + a_t$$

or

$$\phi(B)\dot{Z}_t = a_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$.

### 2.2.2 Condition for stationarity

All roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, or equivalently,

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1 \end{cases}$$

Note that

$$1 - \phi_1 z - \phi_2 z^2 = 0 \Leftrightarrow$$
$$z^{-2} - \phi_1 z^{-1} - \phi_2 = 0 \Leftrightarrow$$
$$x^2 - \phi_1 x - \phi_2 = 0 \Leftrightarrow$$
$$(x - \alpha_1)(x - \alpha_2) = 0 \Leftrightarrow$$
$$\left(\frac{1}{z} - \alpha_1\right)\left(\frac{1}{z} - \alpha_2\right) = 0 \Leftrightarrow$$
$$(1 - \alpha_1 z)(1 - \alpha_2 z) = 0$$

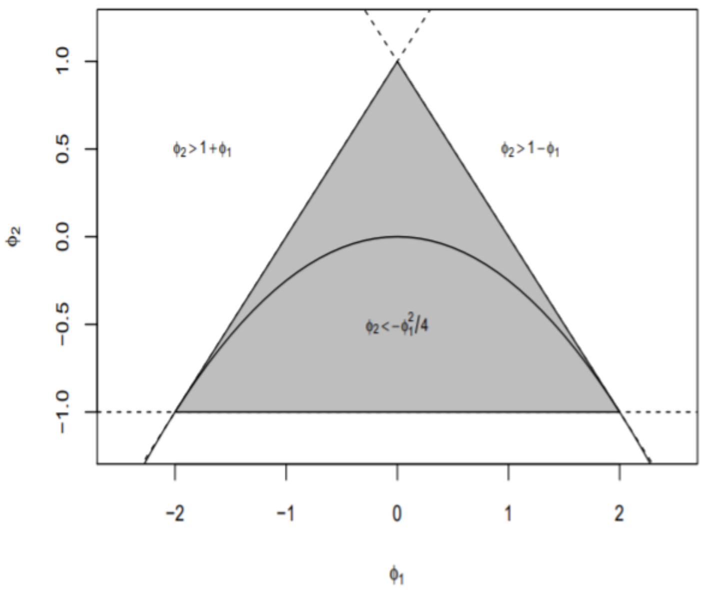


Figure 1: The stationarity triangle of an AR(2) process.

Then $|\alpha_1| < 1$ and $|\alpha_2| < 1$, since $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}$ are roots.

Thus, we have

$$(1-\alpha_1 B)(1-\alpha_2 B)\dot{Z}_t = a_t$$

Let $u_t = (1-\alpha_2 B)\dot{Z}_t$. Then

$$u_t = \dot{Z}_t - \alpha_2 \dot{Z}_{t-1}$$

$$(1-\alpha_1 B)u_t = u_t - \alpha_1 u_{t-1} = a_t$$

and

$$u_t = \alpha_1 u_{t-1} + a_t = \alpha_1^2 u_{t-2} + \alpha_1 a_{t-1} + a_t = \cdots = a_t + \sum_{i=1}^{\infty} \alpha_1^i a_{t-i}$$

$$\dot{Z}_t = \alpha_2 \dot{Z}_{t-1} + u_t = \alpha_2^2 \dot{Z}_{t-2} + \alpha_2 u_{t-1} + u_t = \cdots = u_t + \sum_{j=1}^{\infty} \alpha_2^j u_{t-j}$$

So, the final expansion is

$$\dot{Z}_t = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \alpha_1^i \alpha_2^j a_{t-i-j} = \sum_{k=0}^{\infty} \psi_k a_{t-k}$$

for some $\psi_k$. This is MA $\Rightarrow$ stationary.

If $\alpha_1, \alpha \in \mathbb{R}$ (i.e. $\phi_1^2 + 4\phi_2 \geq 0$), then

$$\alpha_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

Stationarity condition is

$$-1 < \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1 \Leftrightarrow$$

$$-2 < \phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2} < 2$$

Then

$$\max(\alpha_1, \alpha_2) = \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2 \Leftrightarrow$$

$$\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1 \Leftrightarrow$$

$$\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2 \Leftrightarrow$$

$$\phi_2 < 1 - \phi_1 \Leftrightarrow$$

$$\phi_2 + \phi_1 < 1$$

Similarly

$$\phi_2 < 1 + \phi_1 \Leftrightarrow$$

$$\phi_2 - \phi_1 < 1$$

If $\alpha_1, \alpha \in \mathbb{C}$ (i.e. $\phi_1^2 + 4\phi_2 < 0$), then

$$\alpha_{1,2} = \frac{\phi_1 \pm i\sqrt{-(\phi_1^2 + 4\phi_2)}}{2}$$

Then

$$|\alpha_{1,2}|^2 = \frac{\phi_1^2}{4} - \frac{\phi_1^2 + 4\phi_2}{4} = -\phi_2$$

Stationarity condition is

$$-\phi_2 = |\alpha_{1,2}|^2 < 1 \Leftrightarrow$$

$$\phi_2 > -1$$

### 2.2.3 ACF of the AR(2) process

Here, we assume that stationarity condition is fulfilled. Then

$$\gamma_k = \mathbb{E}\dot{Z}_t\dot{Z}_{t+k} = \mathbb{E}\dot{Z}_t(\phi_1\dot{Z}_{t+k-1} + \phi_2\dot{Z}_{t+k-2} + a_{t+k}) =$$

$$\phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} + 0, \quad k \geq 1$$

And consequently[2]

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \quad k \geq 1$$

Let's consider simple cases $k = 1, 2$:

$$\gamma_1 = \mathbb{E}\dot{Z}_t\dot{Z}_{t+1} = \mathbb{E}\dot{Z}_t(\phi_1\dot{Z}_t + \phi_2\dot{Z}_{t-1} + a_{t+1}) = \phi_1\gamma_0 + \phi_2\gamma_1 + 0 \Rightarrow$$

$$\gamma_1 = \frac{\phi_1\gamma_0}{1 - \phi_2}$$

Let's notice that

$$\mathbb{E}\dot{Z}_t^2 = \mathbb{E}(\phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} + a_t)^2 \Leftrightarrow$$

$$\gamma_0 = \mathbb{E}(\phi_1^2\dot{Z}_{t-1}^2) + \mathbb{E}(\phi_2^2\dot{Z}_{t-2}^2) + \mathbb{E}a_t^2 +$$

$$2\left(\mathbb{E}(\phi_1\phi_2\dot{Z}_{t-1}\dot{Z}_{t-2}) + \mathbb{E}(\phi_1\dot{Z}_{t-1}a_t) + \mathbb{E}(\phi_2\dot{Z}_{t-2}a_t)\right) =$$

$$\phi_1^2\gamma_0 + \phi_2^2\gamma_0 + \sigma_a^2 + 2(\phi_1\phi_2\gamma_1 + 0 + 0) =$$

$$(\phi_1^2 + \phi_2^2)\gamma_0 + \sigma_a^2 + 2\phi_1\phi_2\frac{\phi_1\gamma_0}{1 - \phi_2} \Leftrightarrow$$

[2] To find general solution, check *Linear recurrence with constant coefficients*

$$(1-\phi_1^2-\phi_2^2)(1-\phi_2)\gamma_0 = \sigma_a^2(1-\phi_2) + 2\phi_1^2\phi_2\gamma_0 \Leftrightarrow$$
$$(1-\phi_2-\phi_1^2+\phi_1^2\phi_2-\phi_2^2+\phi_2^3-2\phi_1^2\phi_2)\gamma_0 = \sigma_a^2(1-\phi_2) \Leftrightarrow$$
$$\gamma_0 = \frac{\sigma_a^2(1-\phi_2)}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$

Then

$$\gamma_1 = \frac{\phi_1\gamma_0}{1-\phi_2} = \frac{\sigma_a^2\phi_1}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$

and

$$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0 = \left(\frac{\phi_1^2}{1-\phi_2}+\phi_2\right)\gamma_0 = \frac{\phi_1^2+\phi_2-\phi_2^2}{1-\phi_2}\gamma_0 =$$
$$\frac{\sigma_a^2(\phi_1^2+\phi_2-\phi_2^2)}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$$

It is also easy to see that

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1-\phi_2}$$
$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi_1^2+\phi_2-\phi_2^2}{1-\phi_2}$$

#### 2.2.4 PACF of the AR(2) process

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1-\phi_2}$$
$$\phi_{22} = \phi_2$$
$$\phi_{kk} = 0 \quad \text{as} \quad k \geq 3$$

## 2.3 The general p-th order autoregressive AR(p) process

### 2.3.1 Model

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance $\sigma_a^2$.

$\dot{Z}_t$ is said to be an **AR(p)** , if

$$\dot{Z}_t = \phi_1 Z_{t-1} + \phi_2\dot{Z}_{t-2} + \cdots + \phi_p\dot{Z}_{t-p} + a_t,$$

or

$$\phi_p(B)\dot{Z}_t = a_t,$$

where $p$ is an positive integer, and $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.

### 2.3.2 Condition for stationarity

All roots of $\phi_p(Z) = 0$ lie outside the unit circle,

or equivalently,

all eigenvalues of the following matrix lie outside the unit circle,

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

### 2.3.3 ACF of AR(p) process

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p}, \quad k > 0$$
$$\rho_k = \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p}, \quad k > 0$$

— the difference equation of $\rho_k$.

Solve the following sets of equations:

$$\begin{cases} \rho_1 - \phi_1\rho_0 - \cdots - \phi_p\rho_{p-1} = 0, \\ \cdots \\ \rho_p - \phi_1\rho_{p-1} - \cdots - \phi_p\rho_0 = 0. \end{cases}$$

— find $\rho_1, \ldots, \rho_p$.

When $k \geq p+1$, calculate $\rho_{p+1}, \rho_{p+2}, \ldots$ by:

$$\begin{cases} \rho_{p+1} - \phi_1\rho_p - \cdots - \phi_p\rho_1 = 0, \\ \cdots \\ \rho_k - \phi_1\rho_{k-1} - \cdots - \phi_p\rho_{k-p} = 0. \end{cases}$$

### 2.3.4 PACF of AR(p) process

$\psi_{kk}$ can be obtained from $\rho_1, \ldots, \rho_k$.

In particular, $\psi_{kk} = 0$ when $k > p$.