3.2. **Moving-Average Models**

$$\dot{Z}_t = \sum_{i=1}^{\infty} \phi_i \dot{Z}_{t-i} + a_t.$$

$- - -$ called the AR(∞) model.

A special case:

When $\phi_k = -\theta^k$ and $|\theta| < 1$,

$$\dot{Z}_t = \sum_{i=1}^{\infty} (-\theta^i) \dot{Z}_{t-i} + a_t.$$

$$(1 + \sum_{i=1}^{\infty} \theta^i B^i) \dot{Z}_t = a_t.$$

$$(1 - \theta B)^{-1} \dot{Z}_t = a_t$$

$\Longrightarrow$

$$\dot{Z}_t = (1 - \theta B) a_t = a_t - \theta a_{t-1}.$$

3.2.1. **MA(1) Model**

A. Model:

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a MA(1) model, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1}.$$

Notation: $\dot{Z}_t = (1 - \theta_1 B)a_t$.

B. Properties:

If $|\theta_1| < 1$, then a_t can be written as

$$a_t = \dot{Z}_t + \sum_{i=1}^{\infty} \theta_1^i \dot{Z}_{t-i}.$$

Definition:

Given $\dot{Z}_t$, $\dot{Z}_{t-1}$, $\cdots$, if we can calculate a_t, then the model is said to be invertible.

By the definition,

$$\left.\begin{array}{c}\text{AR}(1)\\[2em]\text{AR}(2)\\[2em]\text{AR}(p)\end{array}\right\}\quad \text{are invertible.}$$

MA(1) model is invertible if $|\theta_1| < 1$.

C. ACF of MA(1) model:

$$\begin{aligned}
&\mu = 0,\\
&\text{Var}(\dot{Z}_t) = \sigma_a^2(1+\theta_1^2),\\
&\gamma_1 = -\theta_1\sigma_a^2,\\
&\gamma_k = 0, \quad \text{if } k > 1,\\
&\rho_1 = -\frac{\theta_1}{1+\theta_1^2},\\
&\rho_k = 0, \quad \text{if } k > 1.
\end{aligned}$$

Feature: MA(1) is always stationary.

D. PACF of MA(1) Model:

$$\begin{aligned}
\phi_{11} = \rho_1 &= \frac{-\theta_1(1-\theta_1^2)}{1-\theta_1^4}, \\
\phi_{22} &= \frac{-\theta_1^2(1-\theta_1^2)}{1-\theta_1^6}, \\
\phi_{kk} &= \frac{-\theta_1^k(1-\theta_1^2)}{1-\theta_1^{2(k+1)}}, \quad k > 1.
\end{aligned}$$

ϕ_{kk} goes down to 0, exponentially.

Example 3.5. Simulated 250 values from:

$$Z_t = (1 - 0.5B)a_t,$$

where $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

3.2.2. **The Second Order Moving-average MA(2) Model**

A. Model:

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a MA(2) model, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}.$$

Notation: $\dot{Z}_t = \theta(B) a_t$, where $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$.

B. Condition for invertibility:

all the root of $\theta(z) = 0$ lie outside the unit circle, or equivalently,

$$\begin{cases} \theta_2 + \theta_1 < 1, \\ \theta_2 - \theta_1 < 1, \\ -1 < \theta_2 < 1. \end{cases}$$

C. ACF of MA(2) model:

$$
\begin{aligned}
&\mu = 0, \\
&\gamma_0 = E\dot{Z}_t^2 = \sigma_a^2(1 + \theta_1^2 + \theta_2^2), \\
&\gamma_1 = -\theta_1(1 - \theta_2)\sigma_a^2, \\
&\gamma_2 = -\theta_2\sigma_a^2, \\
&\gamma_k = 0, \quad k > 2. \\
&\rho_1 = \frac{-\theta_1(1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2}, \\
&\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}, \\
&\rho_k = 0, \quad k > 2.
\end{aligned}
$$

Important feature:

MA(2) model is always stationary.

D. PACF of MA(2) model:

$$\begin{aligned}
\phi_{11} &= \rho_1, \\
\phi_{22} &= \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}, \\
&\cdots\cdots .
\end{aligned}$$

Important feature:

$$\phi_{kk} = O(\rho^k), \quad \text{where} \quad |\rho| < 1.$$

i.e. ϕ_{kk} goes down to zero, exponentially.

Example 3.6. Simulated 250 values from MA(2) model:

$$Z_t = (1 - 0.65B - 0.24B^2)a_t.$$

Show that the sample ACF and PACF.

3.2.3. **The General q^{th} Order Moving-average MA(q) Model**

A. Model

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a MA(q) model, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},$$

where q is an positive integer.

q is called the order or lag of the model.

Notation: $\dot{Z}_t = \theta_q(B) a_t$,

where $\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$.

B. Condition for invertibility:

all the roots of $\theta_q(z) = 0$ lie outside the unit circle, or equivalently,

all the eigenvalues of the following matrix lie outside the unit circle,

$$\begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_q \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

C. ACF of MA(q) model:

$$\mu = 0,$$
$$\gamma_0 = \sigma_a^2(1 + \theta_1^2 + \cdots + \theta_q^2),$$
$$\gamma_k = \begin{cases} \sigma_a^2(-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q), & k = 1, \cdots, \\ 0, & k > q. \end{cases}$$
$$\rho_k = \begin{cases} \frac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2}, & k = 1, \cdots, q; \\ 0, & k > q. \end{cases}$$

Important feature:

MA(q) model is always stationary.

D. PACF of MA(q) model:

ϕ_{kk} can be obtained from $\rho_1, \rho_2, \cdots, \rho_k$.

Important feature:

$$\phi_{kk} = O(\rho^k), \quad \text{where} \quad |\rho| < 1.$$

i.e. ϕ_{kk} goes down to zero, exponentially.

3.3. The Dual Relationship Between AR(p) and MA(q) Models

Basic Lemma:

Let $f(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_s z^s$. If all the roots of $f(z) = 0$ lie outside the unit circle, the $f^{-1}(B)$ has the following expansion:

$$f^{-1}(B) = \frac{1}{1 + b_1 B + \cdots + b_s B^s} = \sum_{i=0}^{\infty} c_i B^i,$$

where $c_0 = 1$ and $c_i = O(h^i)$ with $|h| < 1$.

A. AR(p) Model:

Let $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.

$$\phi_p(B)\dot{Z}_t = a_t,$$
$$\dot{Z}_t = \phi_p^{-1}(B)a_t.$$

If all the roots of $\phi_p(z) = 0$ lie outside the unit circle, then

$$\phi_p^{-1}(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$$

$\Longrightarrow$

$$\begin{aligned} \dot{Z}_t &= \left(1 + \sum_{i=1}^{\infty} \psi_i B^i\right) a_t \\ &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots . \end{aligned}$$

where $\psi_i = O(h^i)$ with $|h| < 1$.

Remark:

A stationary AR(p) model has an MA(∞) expansion. This expansion tells us that originally, $\dot{Z}_t$ comes from a sequence of white noises.

B. MA(q) Model:

Let $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$.

$$\dot{Z}_t = \theta(B) a_t,$$
$$a_t = \theta^{-1}(B)\dot{Z}_t.$$

If all the roots of $\theta_q(z) = 0$ lie outside the unit circle, then

$$\theta^{-1}(B) = 1 - \sum_{i=1}^{\infty} \pi_i B^i$$

$\Longrightarrow$

$$\begin{aligned} a_t &= \left(1 - \sum_{i=1}^{\infty} \pi_i B^i\right) \dot{Z}_t \\ &= \dot{Z}_t - \pi_1 \dot{Z}_{t-1} - \pi \dot{Z}_{t-2} - \cdots . \end{aligned}$$

where $\pi_i = O(h^i)$ with $|h| < 1$, i.e.,

$$\dot{Z}_t = \sum_{i=1}^{\infty} \pi_i \dot{Z}_{t-i} + a_t.$$

Remark:

An invertible MA(q) model is a special AR(∞) model.