MATH4321 Game Theory

Lecture Note 2

Dynamic games: Games in extensive form

Introduction

Another important class of games in game theory is called *dynamic games* (games in extensive form) which the players make their decision *sequentially* (instead of simultaneously). One unique feature in dynamic games is that some players can acquire more information about the games (e.g. strategies made by players previously) and these information can help those players to choose a better strategy. Therefore, this feature can drastically affect the equilibrium derivation in dynamic games.

In this chapter, we shall introduce some important concepts and solution procedures in solving equilibrium for dynamic games:

- Structure of dynamic games: games in extensive form
- Games of perfect information: Sequentially rational Nash equilibrium
- Game of imperfect information: Subgame perfect equilibrium
- Multi-stage games and repeated games

Dynamic games: Extensive games

Roughly speaking, a dynamic game consists of the following elements:

1. *A set of players*: $N = \{1,2,\dots,n\}$.
2. *Order of moves of the players*

   Different from static games which the players move their move simultaneously, one needs to specify the order of player's move in the games: Which player moves first? Which player moves second? and so on.

3. *Information set of the players*

   Some players (especially those who move later) can acquire more updated information about the games such as the strategies chosen by the players who have made their moves and the latest status of the games.

4. *Strategic set of the players*
   The set that describes all possible actions that can be chosen by a player when it is his/her turn to move. Different from the case under static games, a player's strategic set may be affected by the status of the games.

5. *Payoffs of the players under various outcomes* (depends on the strategies chosen by the player.)
   Given a strategic profile $s = (s_1, s_2, \ldots, s_n)$ chosen by the players, we denote $V_i(s_i; s_{-i})$ be the payoff function of player i.
   (*Note: Here, s_i represents a sequence of actions (instead of single action in a static games) taken by player i in different stages of the games)

Example 1

Peter has encountered some difficulties in doing his final year project. He can choose whether to seek help from his friend Johnny.

- If Peter chooses to work alone without seeking help (choosing N), both Peter and Johnny have zero payoff.
- If Peter decides to seek help from Johnny (choosing Y), Johnny can choose to put either large effort (H), medium effort (M) or small effort (S) to help Peter. It is clear that more effort can increase the quality of the project and increase the cost at the same time. The payoffs to Peter and Johnny under various cases are summarized in the following table:

Effort	Peter	John
High (H)	3	1
Medium (M)	1	2
Low (L)	-1	3

The above games is a kind of dynamic games which Peter makes his decision first and Johnny makes his decision after knowing the peter's preference. The above games can be described formally as follows:

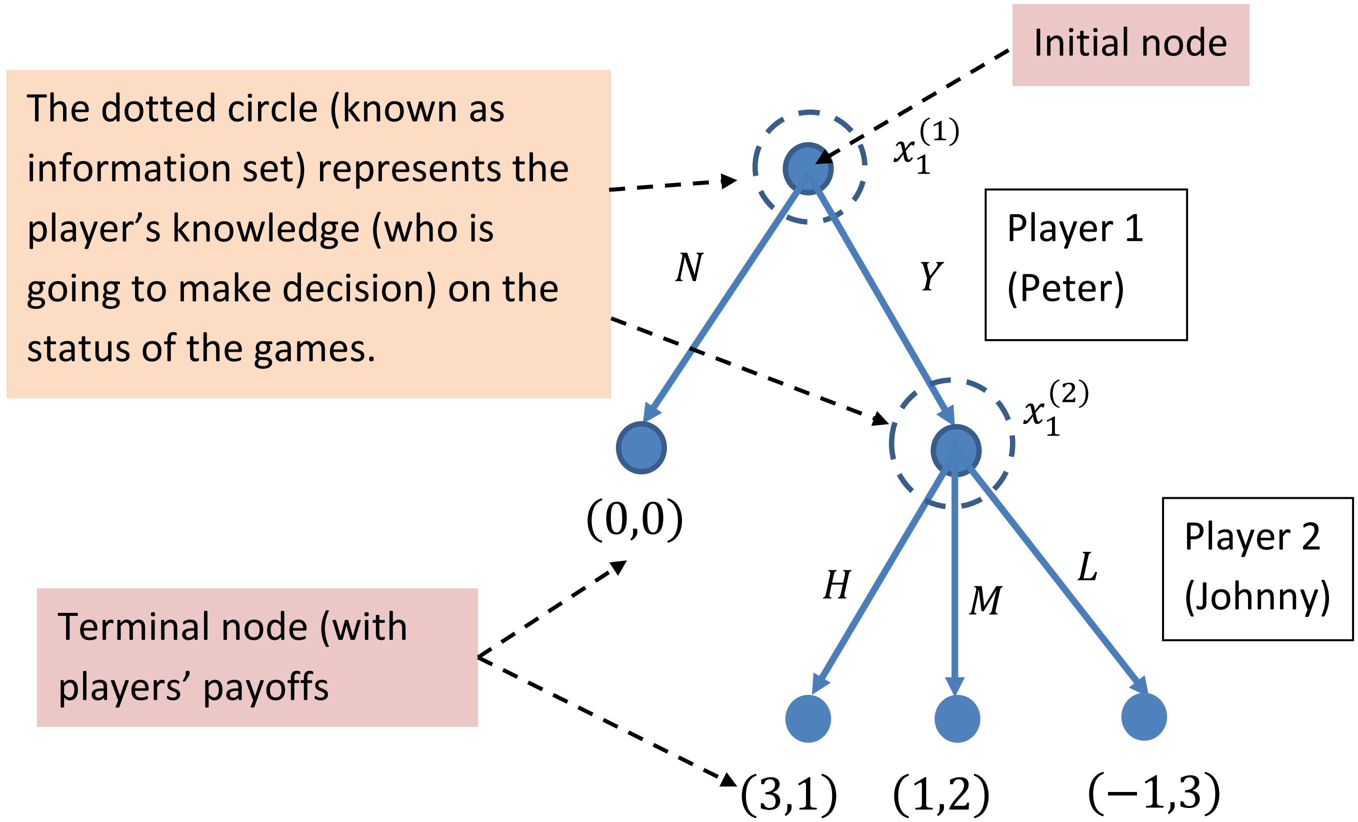
- *Set of players*: $\{Peter,\ John\}$;
- *Order of moves*: Peter moves first and Johnny moves next.
- *Information set at the time of making decision*: Peter (First-mover) cannot observe Johnny's choice. Johnny can observe Peter's strategy.
- Strategic set of the players:

$$S_{Peter} = \{Y, N\}, \qquad S_{Johnny} = \{H, M, L\}.$$

- Payoff functions (V_1, V_2)

Peter	**Johnny**	**Payoff**
N	---	$(0,0)$
Y	H	$(3,1)$
Y	M	$(1,2)$
Y	L	$(-1,3)$

Practically, one will present such games in the form of tree (games tree) as shown below.



The games starts from node at the top (initial node) to the node at the bottom (terminal node).

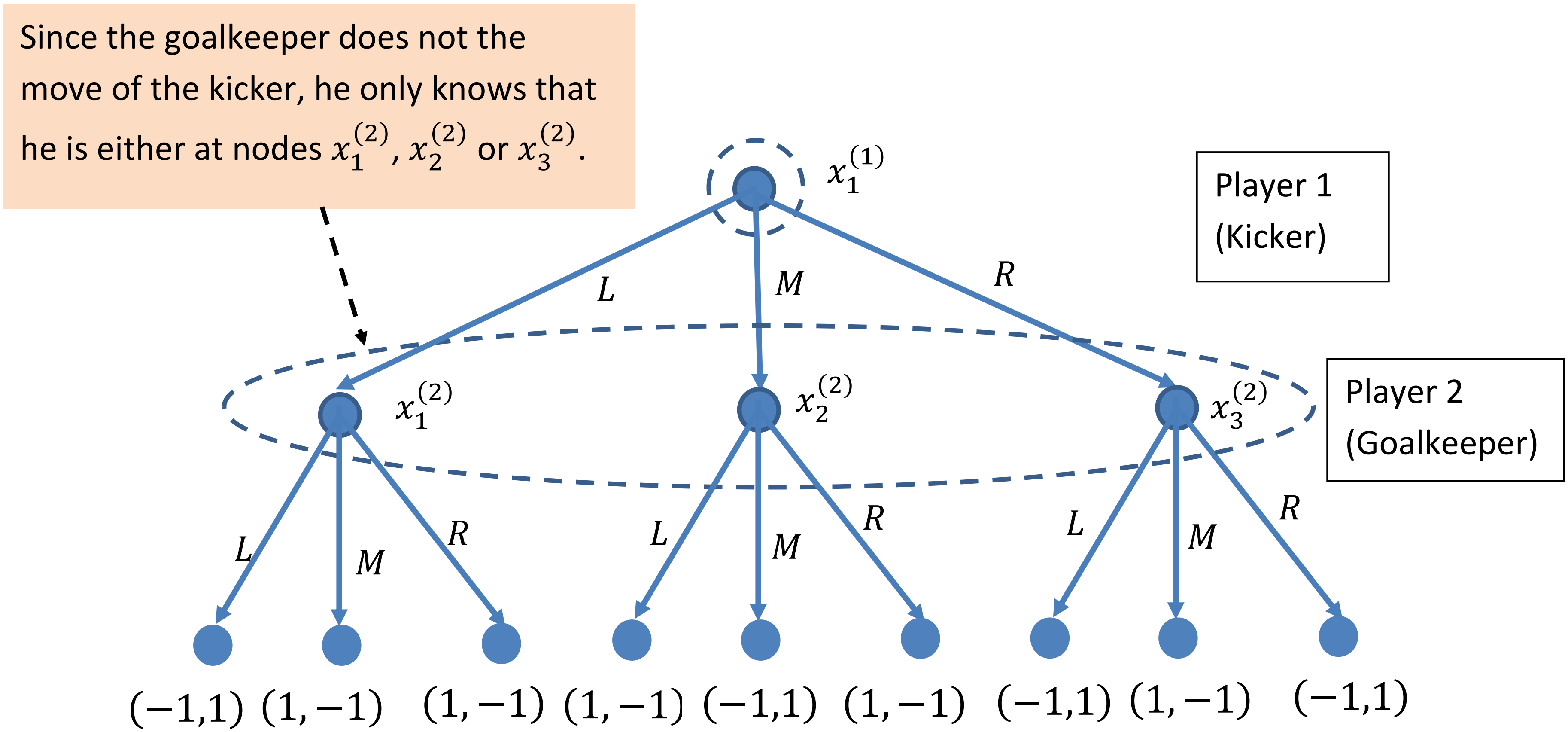
Example 2 (Penalty Kick)

We consider the Penalty kick games that a kicker (Player 1) and a goalkeeper (Player 2) will play a penalty kick. The kicker can kick the ball left or middle or right and the goalkeeper can choose to jump left or stay at the middle or jump right.

The kicker will be the winner if there is a goal. Otherwise, the goalkeeper will be the winner. The winner can get a payoff of 1 and the loser can get a payoff of -1.

The games belongs to a kind of dynamic games which the kicker moves first and the goalkeeper moves later. Different from the games from Example 1, the goalkeeper does not have enough time to figure out the move chosen by the kicker.

The corresponding games tree of this games is presented below:



Information updating and decision making in dynamic games

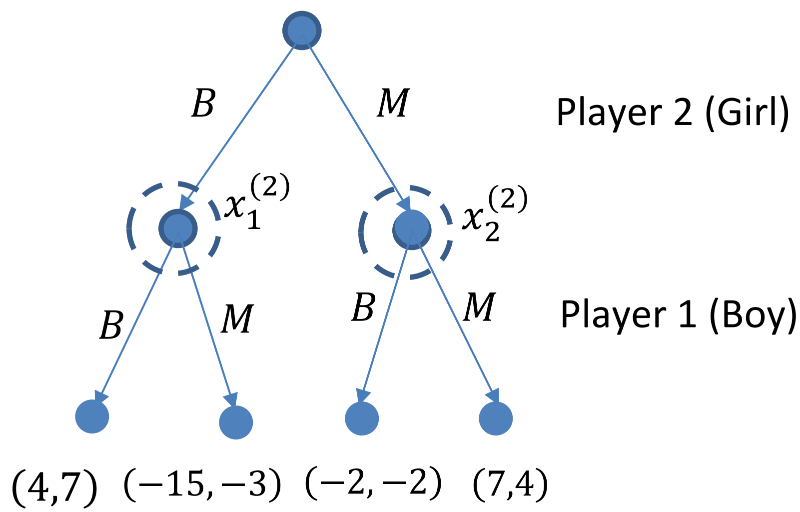
Different from the static games in which the players make decision based on the initial information, some players in the dynamic games may be able to acquire more information about the games such as the strategies made by the players who move earlier.

Example 3 (Battle of sexes)

We consider the following Battle of sexes games: A couple decides a place for their dinner. There are two possible places to choose: McDonald (M) and Buffet (B). Each of them can make his/her decision. If both of them makes the same decision, they will go to that place happily. Otherwise, the dinner will be cancelled. The payoffs are given by the following matrix:

		Girl (Player 2)	
		B	M
Boy (Player 1)	B	$(4,7)$	$(-2,-2)$
	M	$(-15,-3)$	$(7,4)$

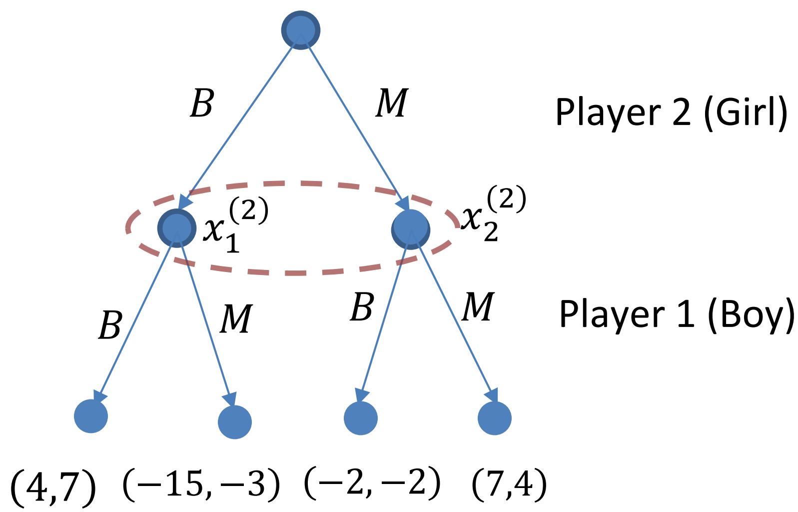
Suppose that the girl first tells her decisions (preference) to the boy, then the games becomes



Then the boy can make his decision optimally by observing the decision chosen by the girl.

- If the girl chooses B, then the boy should choose B;
- If the girl chooses M, then the boy should choose M.

Suppose that the girl has made her decision and choose not to tell to the boy (ask the boy to guess it, perhaps), the games will become



Without knowing precisely the strategy made by the girl, the boy can only make his decision "blindly" (either B or M). In fact, the games is similar to the cases which two players made their decision simultaneously.

Information set of a player i: A formal definition

Since all players are assumed to be intelligent and rational, a player should try to acquire the most updated information (e.g. decisions made by the opponents in the past) when it is his turn to make the decision. This is called *information set* of player i. When the games is expressed into game tree form, such information can be captured by the <u>player's knowledge</u> to his current position.

Player 1 knows that he is at node $x_1^{(2)}$ since he knows that player 2 has chosen strategy B.

Information set of player 1: $\{x_1^{(2)}\}$

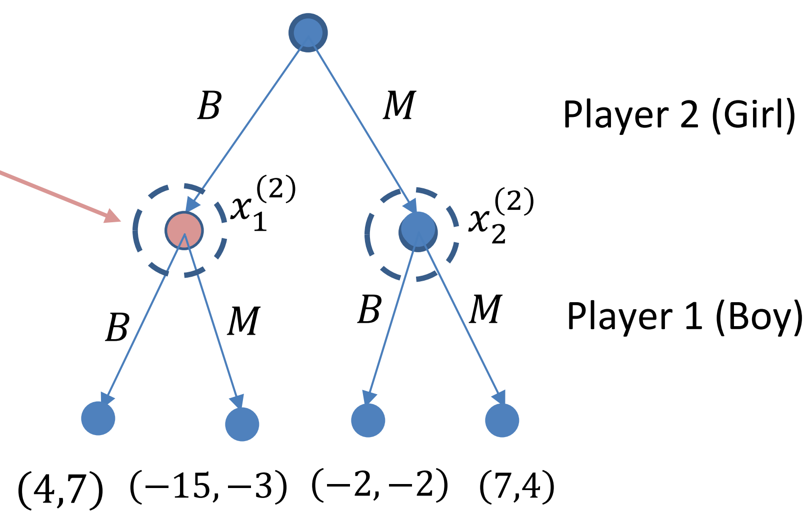


Figure 1: Game tree of Battle of sexes (Example 3)

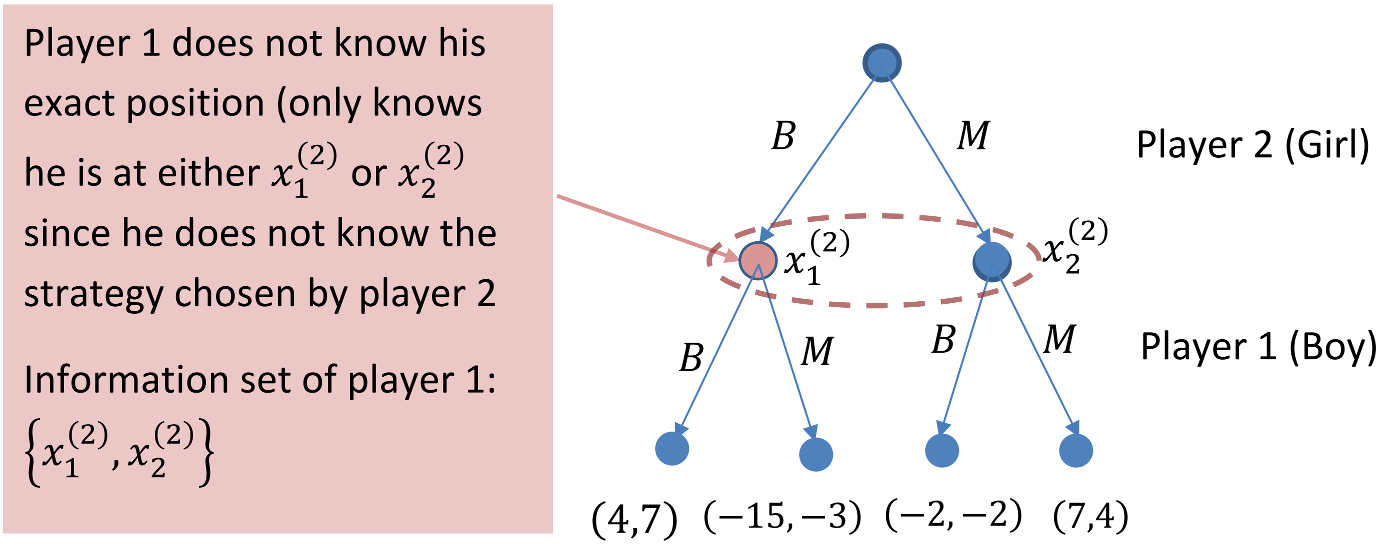


Figure 2: Game tree of Battle of sexes (Example 3)

Inspired by the above example, we define the information set as follows:

Definition (Information set)

We let N_i be the set of nodes that player i needs to make his next decision at a particular stage. The *information set* of a player i at node x, denoted by $h_i(x)$, is an element of the partition of N_i that describe the player i's knowledge about his position at that stage.

Example 4 (Penalty Kick)

We consider the Penalty kick games that a kicker (Player 1) and a goalkeeper (Player 2) will play penalty kick for two rounds. In each round, the kicker can kick the ball left or middle or right and the goalkeeper can choose to jump left or stay at the middle or jump right.

- Since the kick is speedy, so goalkeeper cannot figure not the move made by the kicker.
- The result of each round can be observed by both players.

The winner of each round will get a payoff 2 and the loser will get a payoff -1. The payoff of each player in the games will be the sum of total payoffs gained in these two rounds.

By considering the games tree of the model, determine the information sets for each player at each stage.

☺Solution

The corresponding game tree can be found in Example 2.

The information sets of the players in the *first round* are given by

$$h_1\left(x_1^{(1)}\right)=\left\{x_1^{(1)}\right\}\quad and$$

$$h_2\left(x_1^{(1)}\right)=h_2\left(x_2^{(2)}\right)=h_2\left(x_3^{(2)}\right)=\left\{x_1^{(2)},x_2^{(2)},x_3^{(2)}\right\}.$$

Since all players know the result and the players' move in first round, the information sets of the players in the *second round* are given by

$$h_1\left(x_i^{(3)}\right)=\left\{x_i^{(3)}\right\},$$

$$h_2\left(x_{3i-2}^{(4)}\right)=h_2\left(x_{3i-1}^{(4)}\right)=h_2\left(x_{3i}^{(4)}\right)=\left\{x_{3i-2}^{(4)},x_{3i-1}^{(4)},x_{3i}^{(4)}\right\},$$

where $i=1,2,\cdots,9$.

Example 5

There are two investment opportunities: producing new smartphone (S) and producing new computer (C). Two companies (one of them is leader) decide to cooperate and work on one of these projects. First, the leader chooses the investment project and decides the amount of resource (High (H) or low (L)) to be put in the project. Another company (follower) knows the project chosen by the leader but does not know the amount of resource contributed by the leader. It then chooses the amount of resource (High (H) or low (L)) to be put in the project. The payoffs to the two companies are summarized in the following two matrices:

		Follower (Player 2)	
		High	Low
Leader (Player 1)	High	$(8,8)$	$(6,10)$
	Low	$(12,5)$	$(4,4)$

(Producing smartphone)

		Follower (Player 2)	
		High	Low
Leader (Player 1)	High	$(7,7)$	$(6,11)$
	Low	$(10,6)$	$(5,5)$

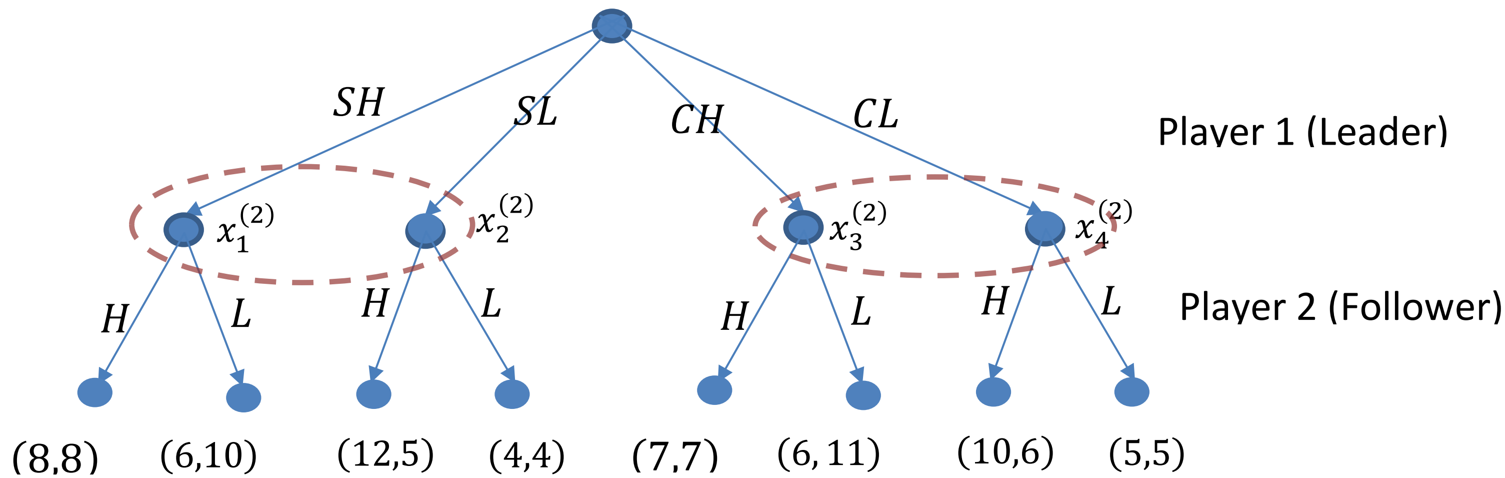
(Producing computer)

(a) Express the games in extensive form.

(b) Determine the information set of each player at different stages.

☺Solution of (a)

The games can be represented by the following games tree:



Note that player 2 can observe the project chosen (S or C) but does not know the effort contributed (H or L) by the leader.

☺Solution of (b)

In the first stage, the information set of player 1 is found to be

$$h_1\left(x_1^{(1)}\right) = \left\{x_1^{(1)}\right\}.$$

In the second stage, the information set of player 2 is found to be

$$h_2\left(x_1^{(2)}\right) = h_2\left(x_2^{(2)}\right) = \left\{x_1^{(2)}, x_2^{(2)}\right\},$$

$$h_2\left(x_3^{(2)}\right) = h_2\left(x_4^{(2)}\right) = \left\{x_3^{(2)}, x_4^{(2)}\right\}.$$

Perfect information and Imperfect information

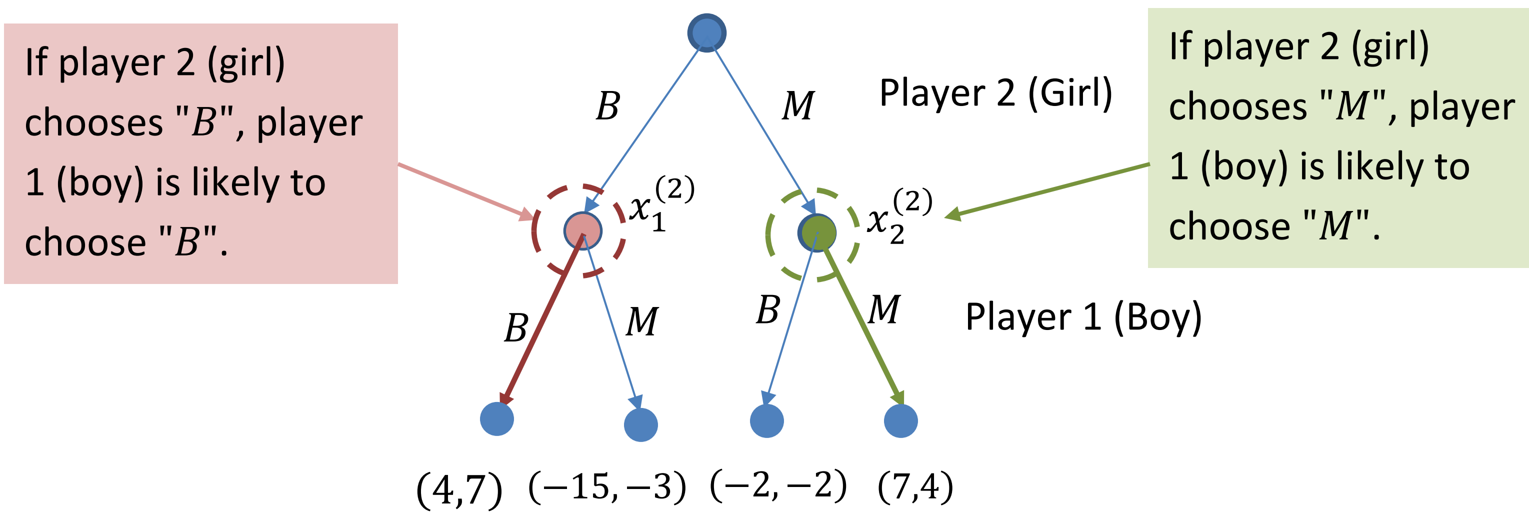
We say a games is of perfect information if all players can acquire the full information about strategies chosen by the players in earlier rounds and the events happened in the past. With the notion of information set, we can now present the formal definition of "perfect information" and "imperfect definition".

Definition (Perfect information and imperfect information)
A game of extensive form is said to be of *perfect information* if *every* information set consists of a single element.
On the other hand, a games is said to be of *imperfect information* if some information sets consists of two or more elements.

Strategies and Nash equilibrium in dynamic games

Different from static games in which each player chooses a single strategy a player chooses his decision based on his *updated information* (described by his information set) of the current status of the games in dynamic games.

As an example, we consider the Battle of sexes games.



We observe that player 2 may take different strategies in two cases, depending on his knowledge on the status of the games (information set).

Therefore, a player's strategy should be defined as a *plan* of actions chosen by the player under various scenarios. Mathematically, we define the player's strategy as follows:

Definition (Pure strategy in dynamic games)

We let $I = \{h_i(x): x \in N_i\}$ be a collection of possible information sets of player i and S_i be the strategic set of player i. A player i's pure strategy is defined as a function $s_i: I \to S_i$ which assigns a single strategy in S_i to every information set $h_i(\cdot) \in I$.

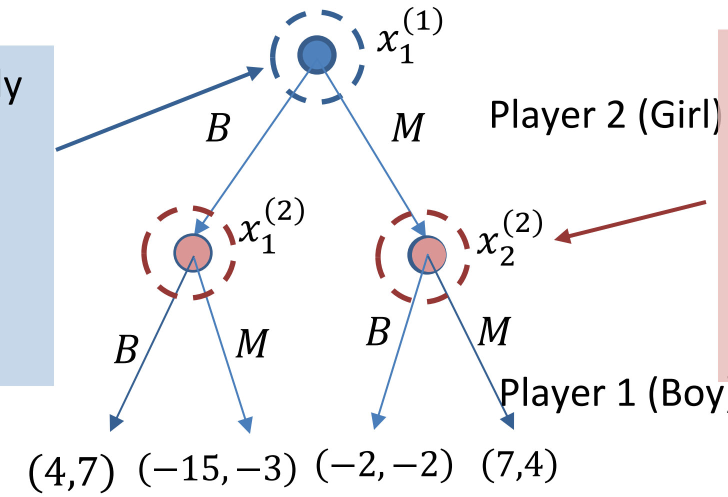
Remark

If $h_i(x) = h_i(x')$, we must have $s_i\big(h_i(x)\big) = s_i\big(h_i(x')\big)$. Recall that the player makes his decision based on his *information set.* In this case, player i cannot distinguish between the nodes x and x' so that two nodes share the common information set ($h_i(x) = h_i(x')$). Therefore, the player should choose the same strategy in these two nodes.

Example 6: We consider the two versions of "Battle of sexes" games again. (Top: Perfect information, Bottom: Imperfect information)

Player 2 (Girl) has only one information set. So her pure strategy can be described as

$$s_2 = s_2\left(\left\{x_1^{(1)}\right\}\right).$$

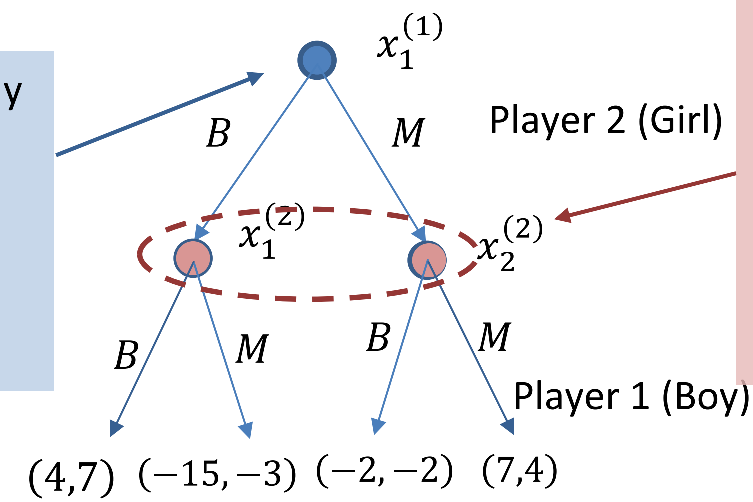


There are two information sets for player 1. So his pure strategy is represented by an ordered pair

$$s_1 = \left(s_1\left(\left\{x_1^{(2)}\right\}\right), s_1\left(\left\{x_2^{(2)}\right\}\right)\right).$$

Player 2 (Girl) has only one information set. So her pure strategy can be described as

$$s_2 = s_2\left(\left\{x_1^{(1)}\right\}\right).$$



There is only one information set for player 1. So his pure strategy becomes

$$s_1 = s_1\left(\underbrace{\left\{x_1^{(2)}, x_2^{(2)}\right\}}_{h_1\left(x_1^{(2)}\right) = h_1\left(x_2^{(2)}\right)}\right).$$

Given the new definition of pure strategy, one can extend the definition of Nash equilibrium to dynamic games as follows:

Definition (Nash equilibrium for pure strategy)

A strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ constitutes pure strategy Nash equilibrium if and only if for any player i

$$V_i(s_i^*; s_{-i}^*) = \max_{s_i \in S_i} V_i(s_i; s_{-i}^*)$$

or equivalently,

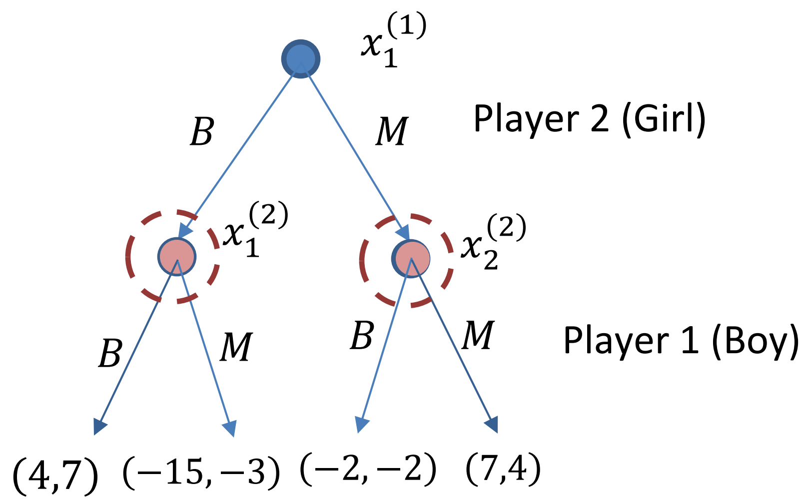
$$V_i(s_i^*; s_{-i}^*) \geq V_i(s_i; s_{-i}^*), \quad for\ any\ s_i \in S_i \dots (*)$$

The determination of pure strategy Nash equilibrium for dynamic games is quite similar to that for static games:

- For each player, we compute his best response to every possible combination of the strategies adopted by opponents.
- We deduce the Nash equilibrium by searching the "intersection" of the best responses of all players.

Example 7

We consider the battle of sexes games (perfect information case) again.



Determine all possible pure strategy Nash equilibrium for this games.

☺Solution

Note that the pure strategy of two players can be expressed as

$$s_1 = \left(s_1\left(\left\{x_1^{(2)}\right\}\right), s_1\left(\left\{x_2^{(2)}\right\}\right)\right), \qquad s_2 = s_2\left(\left\{x_1^{(1)}\right\}\right)$$

where $s_1\left(\left\{x_1^{(2)}\right\}\right), s_1\left(\left\{x_2^{(2)}\right\}\right), s_2\left(\left\{x_1^{(1)}\right\}\right) \in \{B, M\}$.

One can express the games into the following equivalent matrix form:

		Player 2 (Girl)	
		B	M
Player 1 (Boy)	(B, B)	$(4,7)$	$(-2,-2)$
	(B, M)	$(4,7)$	$(7,4)$
	(M, B)	$(-15,-3)$	$(-2,-2)$
	(M, M)	$(-15,-3)$	$(7,4)$

For each player i, we determine his best response s_i^* to every opponent's (player j, $j \neq i$) strategy by solving

$$V_i(s_i^*; s_j) = \max_{s_i} V_i(s_i; s_j).$$

The best responses of the players are summarized in the matrix below: (with upper bar)

		Player 2 (Girl)	
		B	M
Player 1 (Boy)	(B,B)	$(\bar{\mathbf{4}},\bar{\mathbf{7}})$	$(-2,-2)$
	(B,M)	$(\bar{\mathbf{4}},\bar{\mathbf{7}})$	$(\bar{\mathbf{7}},4)$
	(M,B)	$(-15,-3)$	$(-2,\overline{\mathbf{-2}})$
	(M,M)	$(-15,-3)$	$(\bar{\mathbf{7}},\bar{\mathbf{4}})$

We observe from the above matrix that there are 3 entries which all coordinates are marked with upper bar. Thus, we conclude that there are 3 pure strategy Nash equilibria of this games. Namely,

$$s^{*(1)} = \big((B,B),B\big), \qquad s^{*(2)} = \big((B,M),B\big) \quad and \;\; s^{*(3)} = \big((M,M),M\big).$$

- The first two equilibria induce the same outcome which player 2 first chooses B and the player 1 (who knows player 2's strategy) also chooses B.
- The last equilibrium corresponds to the outcome that player 2 first chooses M and the player 1 also chooses M.

Example 8 (Voting games)

Tony and Sunny will have a short-trip in the coming summer. They choose to visit one of the following countries: Singapore (S), Japan (J) or Taiwan (T). It is known that

- Tony's preference is given by $\underbrace{S}_{\substack{1st\\choice}} > \underbrace{J}_{\substack{2nd\\choice}} > \underbrace{T}_{\substack{3rd\\choice}}$.
- Sunny's preference is given by $\underbrace{J}_{\substack{1st\\choice}} > \underbrace{T}_{\substack{2nd\\choice}} > \underbrace{S}_{\substack{3rd\\choice}}$.

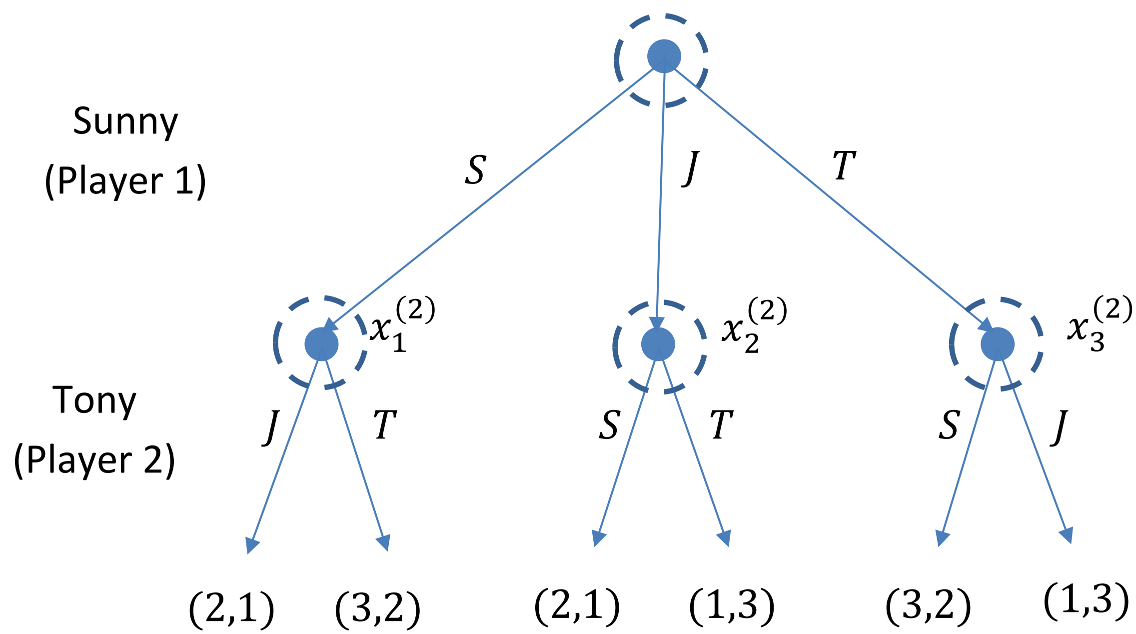
Two people must choose among these 3 choices. The games is as follows: Sunny first decides and rule out one of the three choices. Then Tony chooses to rule out one of the remaining two choices. Depending on the final outcome F. The player's payoff is assumed to be

$$V_i(1st\ choice) = 3, \ \ V_i(2nd\ choice) = 2, \ \ V_i(3rd\ choice) = 1.$$

Determine all possible pure strategy Nash equilibrium for this games.

☺Solution

We first express the games into games tree form. Since Tony will rule out one of the choices from the remaining two, he should know the choice that is eliminated by Sunny (Leader). Thus the games is of perfect information. Given the payoffs profile, the game tree can be expressed as



The pure strategies of these two players can be expressed as

$$s_1 = s_1\left(x_1^{(1)}\right), \qquad s_2 = \left(s_2\left(x_1^{(2)}\right), s_2\left(x_2^{(2)}\right), s_2\left(x_3^{(2)}\right)\right),$$

where $s_1\left(x_1^{(1)}\right) \in \{S, J, T\}$, $s_2\left(x_1^{(2)}\right) \in \{J, T\}$, $s_2\left(x_2^{(2)}\right) \in \{S, T\}$ and $s_2\left(x_3^{(2)}\right) \in \{S, J\}$. The corresponding matrix form of this games is given by

		Tony (Player 2)							
		(J,S,S)	(J,S,J)	(J,T,S)	(J,T,J)	(T,S,S)	(T,S,J)	(T,T,S)	(T,T,J)
Sunny (Player 1)	S	(2,1)	(2,1)	(2,1)	(2,1)	(3,2)	(3,2)	(3,2)	(3,2)
	J	(2,1)	(2,1)	(1,3)	(1,3)	(2,1)	(2,1)	(1,3)	(1,3)
	T	(3,2)	(1,3)	(3,2)	(1,3)	(3,2)	(1,3)	(3,2)	(1,3)

For each player i, we determine his best response s_i^* to every opponent's (player j, $j \neq i$) strategy. The best responses of the players are summarized in the matrix below (with upper bar).

		Tony (Player 2)							
		(J,S,S)	(J,S,J)	(J,T,S)	(J,T,J)	(T,S,S)	(T,S,J)	(T,T,S)	(T,T,J)
Sunny (Player 1)	S	$(2,1)$	$(\bar{\mathbf{2}},1)$	$(2,1)$	$(\bar{\mathbf{2}},1)$	$(\bar{\mathbf{3}},\bar{\mathbf{2}})$	$(\bar{\mathbf{3}},\bar{\mathbf{2}})$	$(\bar{\mathbf{3}},\bar{\mathbf{2}})$	$(\bar{\mathbf{3}},\bar{\mathbf{2}})$
	J	$(2,1)$	$(\bar{\mathbf{2}},1)$	$(1,\bar{\mathbf{3}})$	$(1,\bar{\mathbf{3}})$	$(2,1)$	$(2,1)$	$(1,\bar{\mathbf{3}})$	$(1,\bar{\mathbf{3}})$
	T	$(\bar{\mathbf{3}},2)$	$(1,\bar{\mathbf{3}})$	$(\bar{\mathbf{3}},2)$	$(1,\bar{\mathbf{3}})$	$(\bar{\mathbf{3}},2)$	$(1,\bar{\mathbf{3}})$	$(\bar{\mathbf{3}},2)$	$(1,\bar{\mathbf{3}})$

We observe that there are four Nash equilibria in this games: $s_1^* = \big(S,(T,S,S)\big)$, $s_2^* = \big(S,(T,S,J)\big)$, $s_3^* = \big(S,(T,T,S)\big)$ and $s_4^* = \big(S,(T,T,J)\big)$.

Furthermore, these equilibria induces a common outcome: Sunny first rule out the choice "S" (Singapore) and Tony rule out the choice "T" (Taiwan). So the remaining choice will be "J" (Japan).

Equilibrium refinement: Sequential Rationality

We observe from the above examples that there may be multiple equilibria in a games. Although each player is choosing a strategy that is best response to opponent's strategy in every Nash equilibrium, not all equilibria are "truly optimal" in general.

To see this, we consider the Battle of sexes games in Example 7 and concentrate on the third equilibrium $s_3^* = \big((M, M), M\big)$. In this equilibrium, the boy (player 1) chooses the strategy "M" regardless of the decision made by the girl (player 2).

However, it is not optimal for the boy to choose "M" if he knows that the girl chooses the strategy "B". It goes against our intuition that the players should make their decisions optimally based on their updated information.

Here, $\big((M, M), M\big)$ is still a Nash equilibrium because the girl chooses "M" so that the strategy $s_1\left(x_2^{(1)}\right) = M$ is irrelevant.

Suppose that the boy plays strategy B (instead of M) when the girl chooses the strategy B, it is no longer optimal for the girl to choose M since she can achieve a better payoff 7 by choosing "B". Under the Nash equilibrium $s_3^* = \big((M, M), M\big)$, choosing "$M$" is optimal for the girl because of the assumption that the boy plays "M" (act sub-optimally) for sure even when the girl plays B. Hence, the Nash equilibrium $s_3^* = \big((M, M), M\big)$ is not reasonable in this games.

When determining the Nash equilibrium in the above examples, we have implicitly *ignored* the possibility that players (Player 1 in this example) always choose its strategy optimal by observing the current status of the games (information set), which is crucial in dynamic games

Therefore, the traditional Nash equilibrium is not good enough to describe the "true" equilibrium in dynamic games. Thus, we need to modify the notion in order to incorporate this important feature. This concept is called *sequential rationality*.

Definition (Sequential rationality)

We let s_{-i} be the strategy chosen by the opponents, we say that a player i's strategy s_i is *sequentially rational* if and only if s_i is a best response to s_{-i} in any information set $h_i(x)$ of player i. That is,

$$V_i(s_i; s_{-i})|_{h_i(x)} = \max_s V_i(s; s_{-i})\,|_{h_i(x)}.$$

In other words, sequential rationality requires all players should make their decisions optimally at *every stage* of the games (not just the whole games).

Definition (Sequentially rational Nash equilibrium)

A pure strategy Nash equilibrium $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is said to be *sequentially rational Nash equilibrium* if and only if s_i^* is sequentially rational (with respect to s_{-i}^*) for any player i. That is,

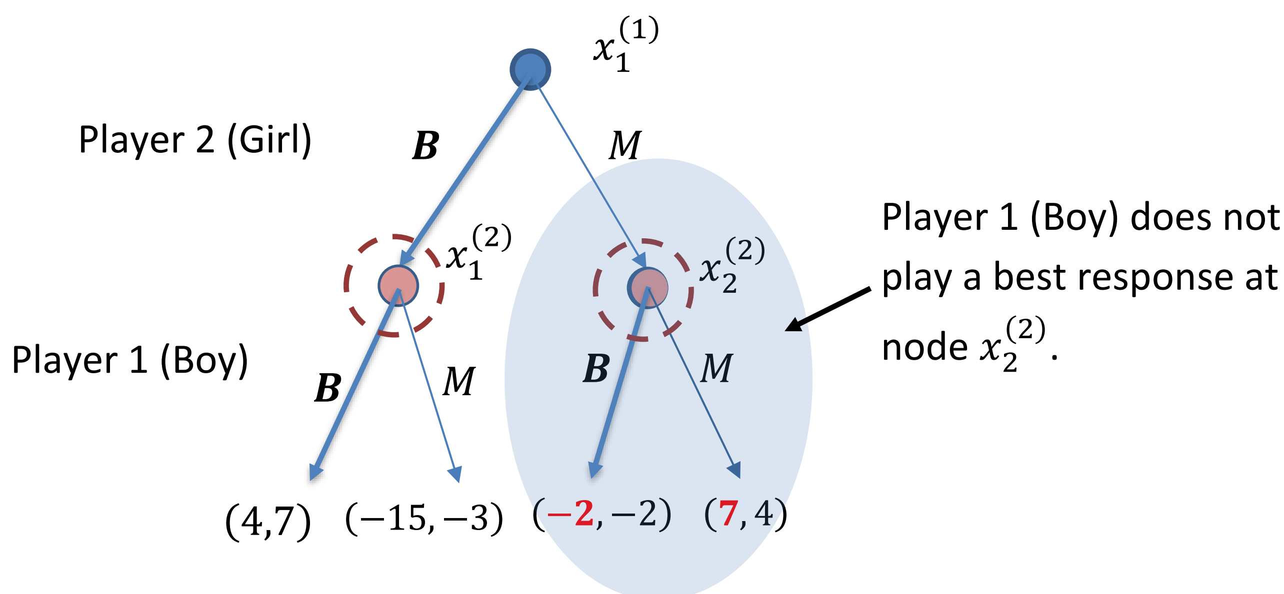
$$V_i(s_i^*; s_{-i}^*)|_{h_i(x)} = \max_s V_i(s; s_{-i}^*)\,|_{h_i(x)}$$

for any information set $h_i(x)$.

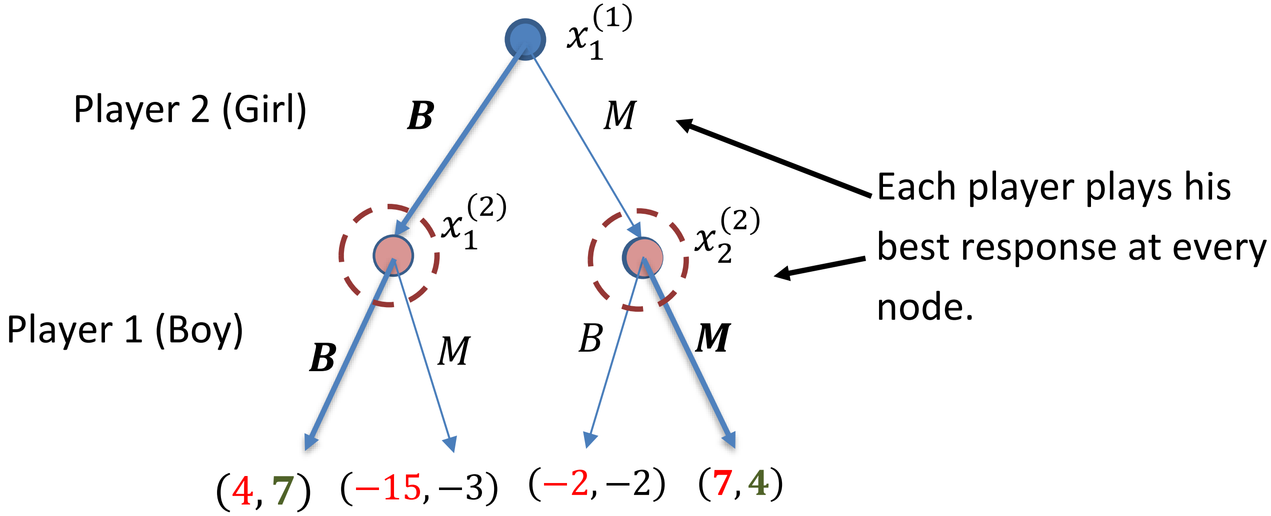
Example 9

We consider the battle of sexes games in Example 3. There are three Nash equilibria in this games: $s^{*(1)} = \big((B, B), B\big)$, $s^{*(2)} = \big((B, M), B\big)$ and $s^{*(3)} = \big((M, M), M\big)$. We proceed to check the sequential rationality of these equilibria.

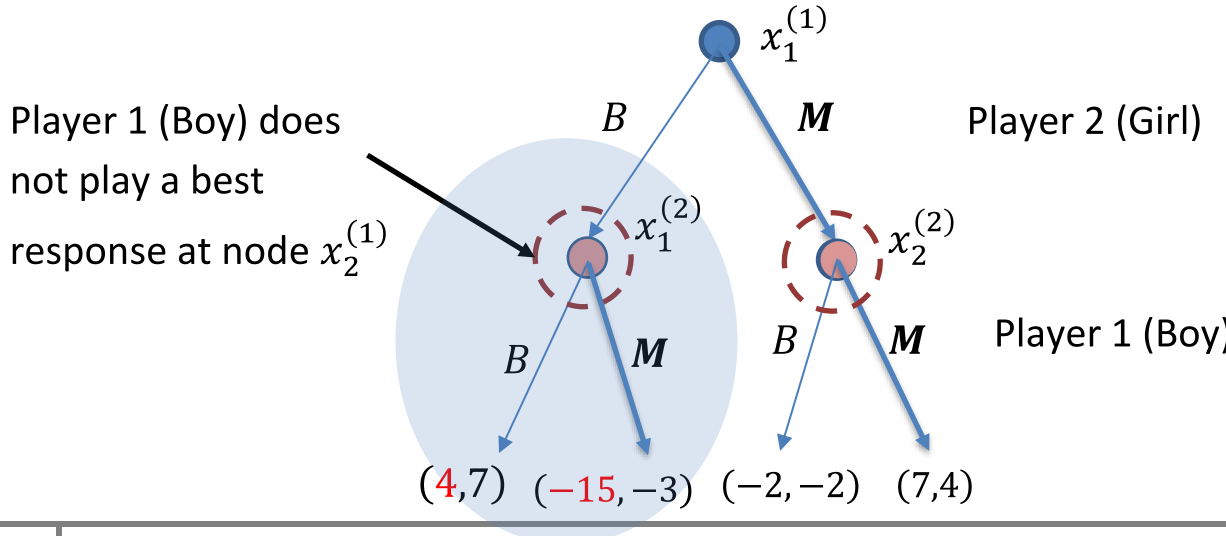
First equilibrium: $s^{*(1)} = \big((B, B), B\big)$ is NOT sequentially rational.



Second equilibrium: $s^{*(2)} = \big((B, M), B\big)$ is sequentially rational.



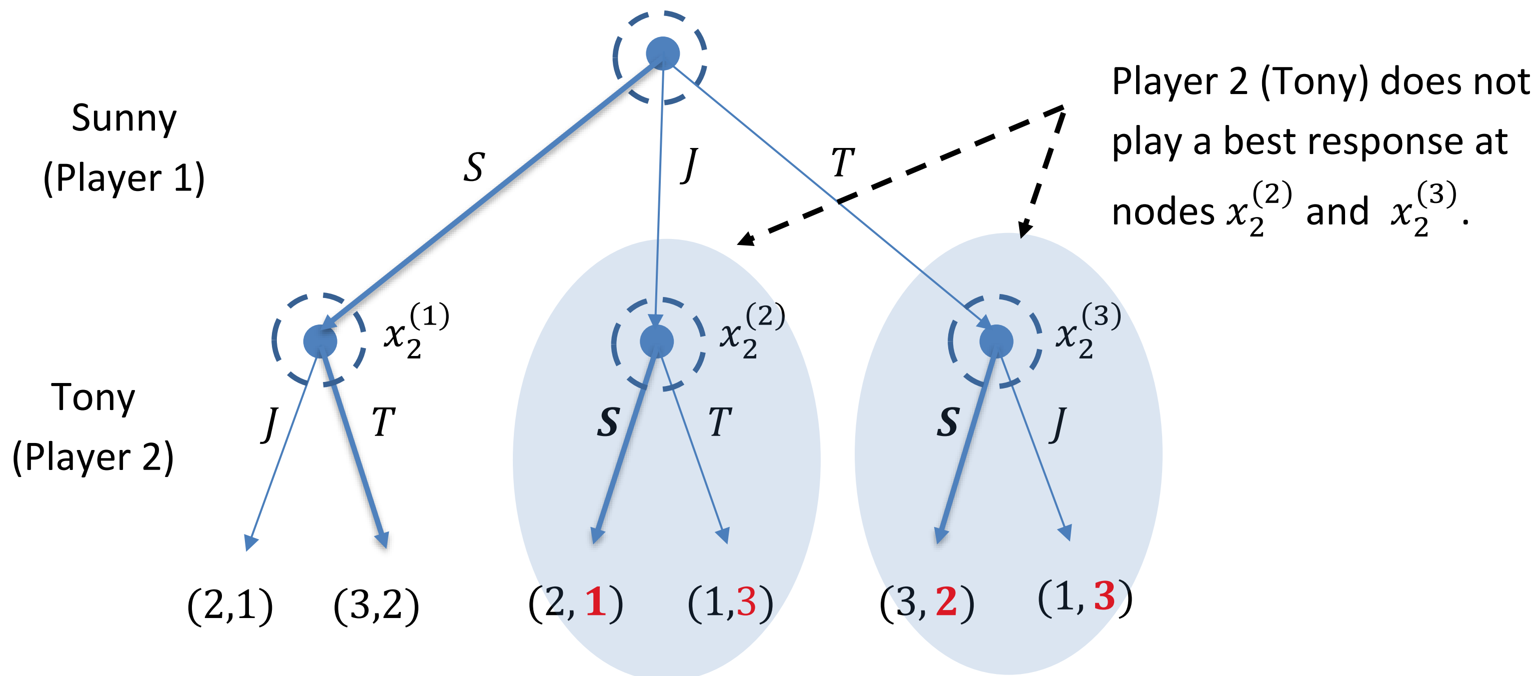
Third equilibrium: $s^{*(3)} = \big((M, M), M\big)$ is NOT sequentially rational.



Example 10

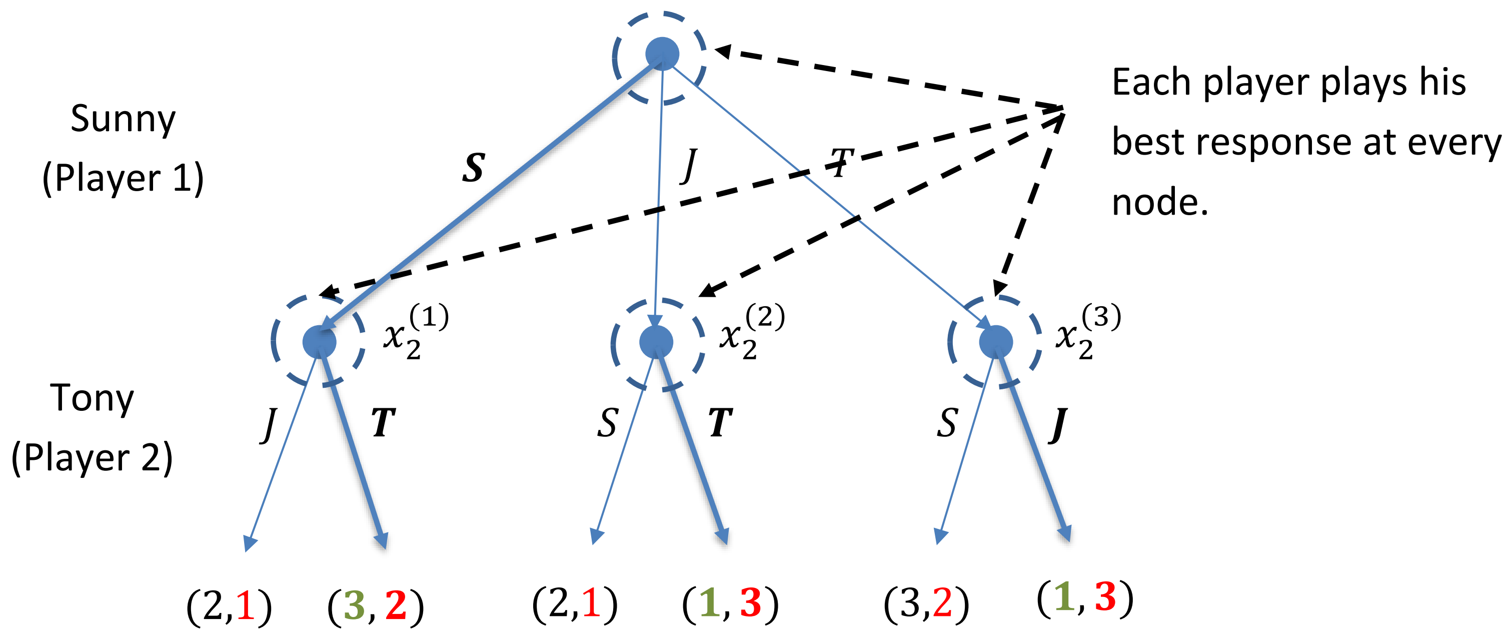
We consider the voting games in Example 8. There are four equilibria in this games: $s^{*(1)} = \big(S,(T,S,S)\big)$, $s^{*(2)} = \big(S,(T,S,J)\big)$, $s^{*(3)} = \big(S,(T,T,S)\big)$ and $s^{*(4)} = \big(S,(T,T,J)\big)$.

First equilibrium: $s^{*(1)} = \big(S,(T,S,S)\big)$ is NOT sequentially rational.



Similarly, one can show that the equilibria $s^{*(2)} = \big(S,(T,S,J)\big)$ and $s^{*(3)} = \big(S,(T,T,S)\big)$ are NOT sequentially rational (left as exercise).

Fourth equilibrium: $s^{*(4)} = \big(S,(T,T,J)\big)$ is sequentially rational.



Finding sequentially rational Nash equilibrium in dynamic games

In this section, we introduce some methods in finding sequentially rational Nash equilibrium. For simplicity, we first consider the games of perfect information since the derivation is relatively easier.

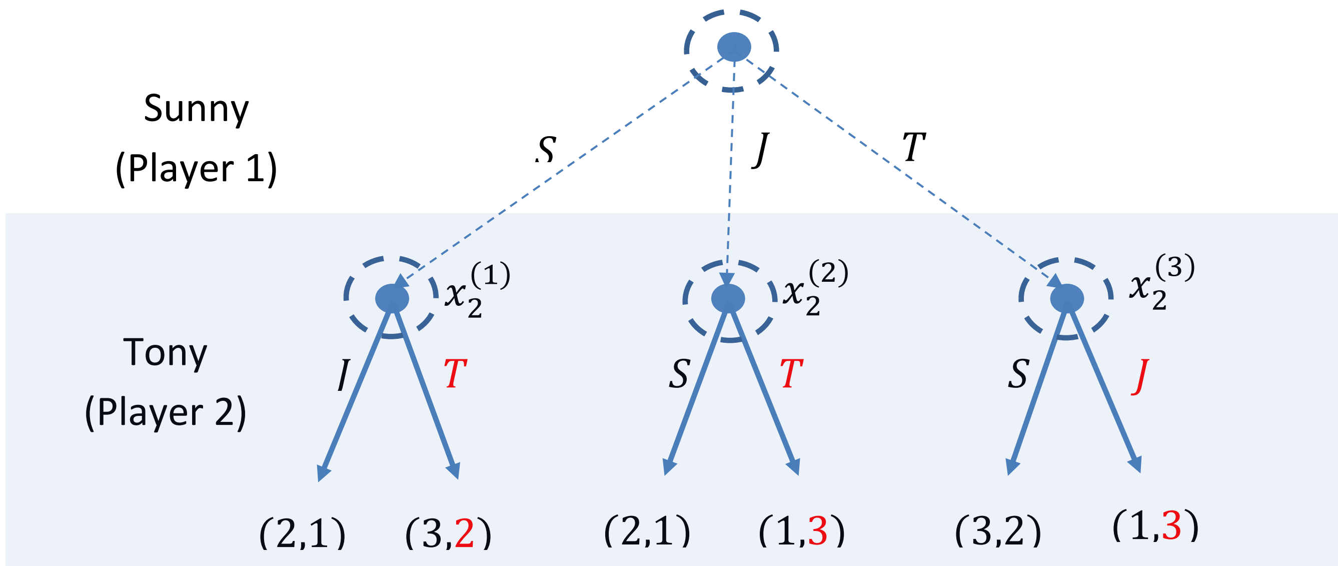
Games of perfect information: Method of dynamic programming

We first note that each player needs to play optimally at each stage of the games Furthermore, all players know the moves made by themselves and other opponents in the previous stage in the perfect information games. Hence, we can obtain the equilibrium through the dynamic programming.

The idea of this method is to break down the entire games into several stages and solve for the optimal strategy at each stage. In our case, this can be done through *backward induction*.

As an example, we revisit the voting games in Example ?8. One can use backward induction and find the Nash equilibrium as follows:

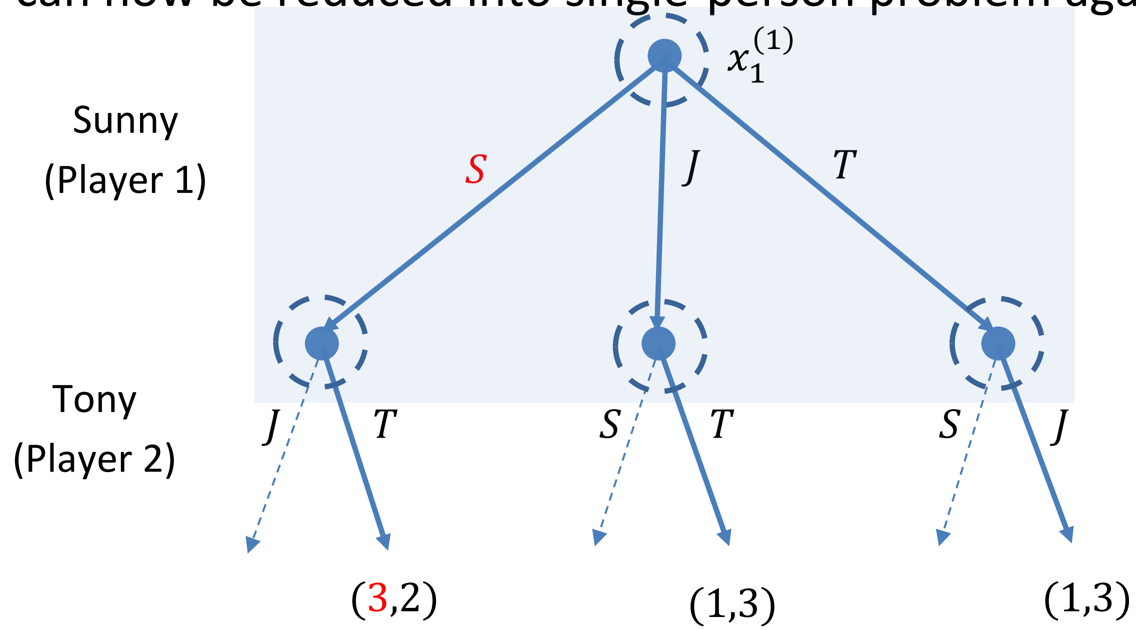
Step 1: We first consider the second stage where Tony makes the last move. Knowing the strategy made by Sunny previously, Tony's decision problem is equivalent to a single-person optimization problem.



From the game tree, we deduce that Tony's optimal strategy is given by

$$s_2^*\left(x_2^{(1)}\right) = T, \qquad s_2^*\left(x_2^{(2)}\right) = T, \qquad s_2^*\left(x_2^{(3)}\right) = J.$$

Step 2: We move to first stage and determine the Sunny's optimal strategy. Knowing that Tony should make his decision rationally, Sunny can "forecast" Tony's strategy under various scenarios. So his optimization problem can now be reduced into single-person problem again.



We deduce from the above tree that Sunny's optimal strategy is $s_1^*\left(x_1^{(1)}\right) = S$. Thus, $\vec{s}^* = \big(S, (T, T, J)\big)$ is the potential sequentially rational Nash equilibrium.

Backward induction versus Nash equilibrium

It is clear that the optimal strategic profile $\vec{s}^* = (s_1^*, s_2^*, \dots, s_n^*)$ obtained from backward induction is sequentially rational. Furthermore, we also observe that each player always choose a strategy that is best response to other players' strategy. Therefore, we should expect that $\vec{s}^*$ is the Nash equilibrium. This can be confirmed by the following theorem:

Theorem

For any games of perfect information, the strategic profile s^* (in pure strategy) obtained through backward induction is the sequentially rational Nash equilibrium. Furthermore, such equilibrium always exist for every games of perfect information.

Remark (Uniqueness of solution)

The equilibrium is uniquely determined if there do not exist two terminal nodes that yields the same payoffs to any player.

Example 11 (Duopoly games)

There are two competing firms (firm 1 and firm 2) in a same market. Each of them decides the number of goods q_i ($q_i \geq 0$) produced for the market. It is given that

- the cost of producing q_i units of goods is $c_i q_i$. Here, c_i is some positive constants. In this example, we take $c_1 = 9$ and $c_2 = 20$ so that Firm 1 can produce the goods at lower cost.
- The market price of the goods is given by $100 - q_1 - q_2$.

The profit made by firm i can be expressed as

$$V_i(q_i; q_j) = (100 - q_i - q_j)q_i - c_i q_i$$

(a) Suppose that the two firms make their decision simultaneously, find the pure strategy Nash equilibrium of the games.

(b) Suppose that Firm 1 (leader) first chooses the value of q_1 and Firm 2 (follower), after observing q_1, chooses the value of q_2, find the sequentially rational Nash equilibrium of this games.

☺Solution of (a)

Firstly, the payoff functions of two players are given by

$$V_1(q_1; q_2) = (100 - q_1 - q_2)q_1 - 9q_1 = 91q_1 - q_1^2 - q_1q_2;$$
$$V_2(q_2; q_1) = (100 - q_1 - q_2)q_2 - 20q_2 = 80q_2 - q_2^2 - q_1q_2.$$

To determine the Nash equilibrium, we first obtain the best response of each player, this can be done by considering the first-order condition:

$$\frac{\partial V_1}{\partial q_1}\Big|_{q_1=q_1^*} = 0 \Rightarrow 91 - 2q_1^* - q_2 = 0 \Rightarrow q_1^* = \frac{91 - q_2}{2}.$$
$$\frac{\partial V_2}{\partial q_2}\Big|_{q_2=q_2^*} = 0 \Rightarrow 80 - 2q_2^* - q_1 = 0 \Rightarrow q_2^* = \frac{80 - q_1}{2}.$$

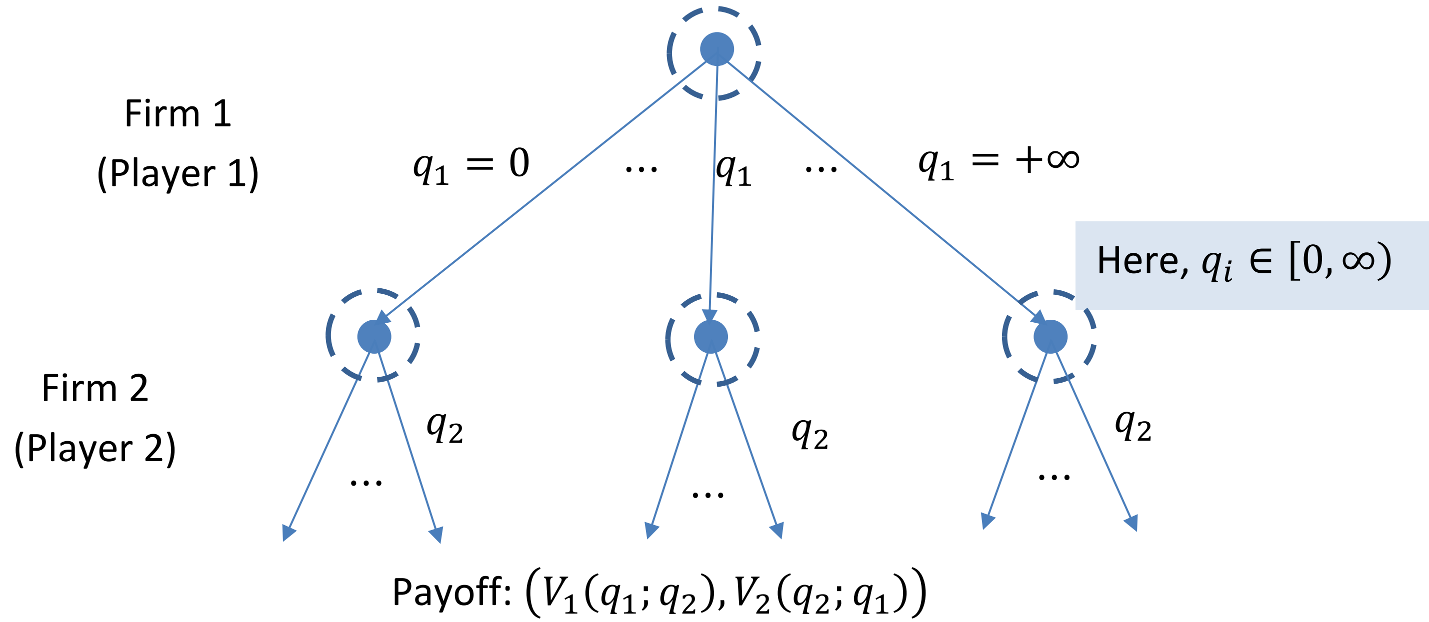
Then the Nash equilibrium (q_1^*, q_2^*) can be found by solving

$$\begin{cases} q_1^* = \dfrac{91 - q_2^*}{2} \\ q_2^* = \dfrac{80 - q_1^*}{2} \end{cases} \Rightarrow (q_1^*, q_2^*) = (34,23).$$

We conclude that $(q_1^*, q_2^*) = (34,23)$ is the unique Nash equilibrium.

☺Solution of (b)

The games can be represented by the following tree:



To determine the corresponding Nash equilibrium, we apply backward induction and consider the last stage where Firm 2 makes its decision. Given the value of q_1 chosen by firm 1, the optimal strategy of firm 2 can be found by considering

$$\frac{\partial V_2}{\partial q_2}|_{q_2=q_2^*(q_1)} = 0 \Rightarrow 80 - 2q_2^* - q_1 = 0 \Rightarrow q_2^*(q_1) = \frac{80 - q_1}{2}.$$

Next, we proceed to determine the optimal strategy of firm 1. Knowing the firm 2's optimal strategy $q_2^*(q_1)$ under various scenarios, the payoff function of Firm 1 can be expressed as

$$V_1\big(q_1; q_2^*(q_1)\big) = 91q_1 - q_1^2 - q_1\left(\frac{80 - q_1}{2}\right) = 51q_1 - \frac{q_1^2}{2}.$$

Its optimal strategy q_1^* can be obtained by solving

$$\frac{\partial V_1}{\partial q_1}|_{q_1=q_1^*} = 0 \Rightarrow 51 - q_1^* = 0 \Rightarrow q_1^* = 51.$$

The corresponding strategy of firm 2 will be

$$q_2^* = \frac{80 - q_1^*}{2} = \frac{80 - 51}{2} = \frac{29}{2}.$$

We conclude that the sequentially rational Nash equilibrium is given by

$$(q_1^*, q_2^*) = \left(51, \frac{29}{2}\right).$$

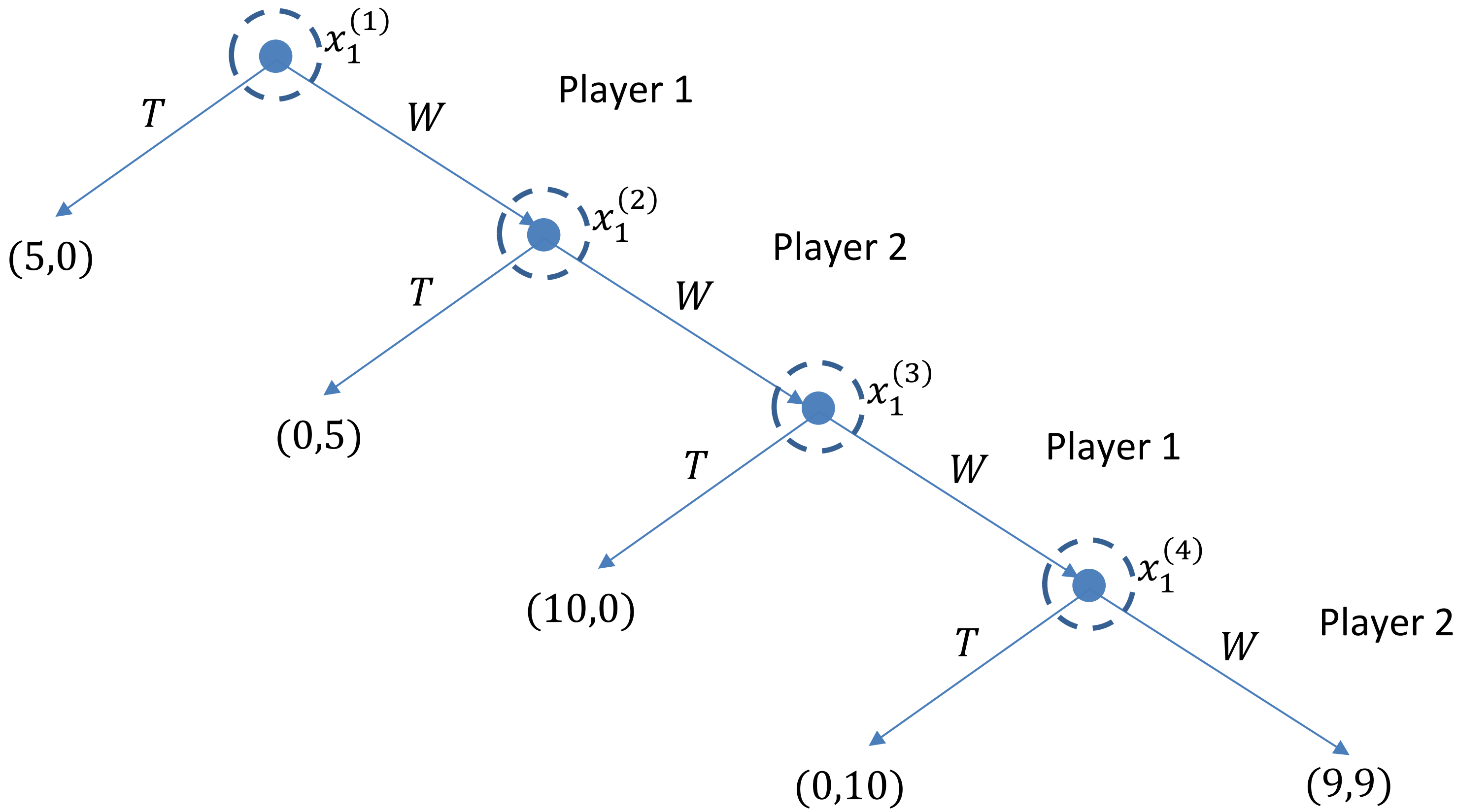
Example 12

We consider the following two-person games as follows: Two players (Player 1 and 2) are going to share the money in a pot. Initially, the pot has \$5. Player 1 moves first and can choose to either take all money (T) or wait (W). If he chooses to take all money, he will get \$5 and the game is over. If he chooses to wait, it will be player 2's turn and choose to either take all money or wait. If he chooses to take all money, he will get \$5. Otherwise, the money in the pot will be doubled and the games moves to second round.

The second round is similar to the first round which two players takes turn to choose to either take all money or wait. If none of the players choose to take the money in the second round, additional \$8 will be deposited into the pot and the players share the pot equally (i.e. each player will receive $\$\frac{10+8}{2} = \$9$). Find the sequentially rational Nash equilibrium for this games.

☺Solution

Firstly, one express the games by the following game tree:



We proceed to determine the equilibrium by backward induction. We first consider the node $x_1^{(4)}$ which the player 2 makes the last move. It is clear that player 2 should choose "T" and a payoff of 10.

We then consider the node $x_1^{(3)}$. If player 1 chooses to wait (W), his payoff will be 0 since player 2 will choose to take all in next turn. Thus player 1 will choose "T" and get a payoff of 10.

We consider the node $x_1^{(2)}$. If player 2 chooses to wait (W), he will get 0 since player 1 will choose to take all in the next round. Thus player 2 chooses "T" and get a payoff of 5.

Applying similar argument, one can argue that player 1 will choose T at node $x_1^{(1)}$ and get a payoff of 5. Therefore the desired Nash equilibrium will be $s_1^* = \{T, T\}$ and $s_2^* = \{T, T\}$.

Example 13 (Contribution games)

Peter, Mark and John would like to buy a new computer together. The cost of buying a new computer is \$5,000. The games goes as follows: Peter first decides the amount of money contributed (denoted by c_1). Knowing the value of c_1, Mark then decides the amount of money contributed (denoted by c_2). Lastly, John decides the amount of money contributed (denoted by c_3). It is also given that the player's valuation of the computer is \$3500.

Assuming the games is of perfect information, find the sequentially rational Nash equilibrium for this games.

☺Solution

Since the games is of perfect information, we can obtain the equilibrium using backward induction. Given the values of c_1 and c_2, we first determine the optimal strategies for player 3. The player 3's payoff can be expressed as

$$V_3(c_3; c_1, c_2) = \begin{cases} 3500 - c_3 & if\ c_1 + c_2 + c_3 \geq 5000 \\ 0 & if\ otherwise \end{cases}.$$

It is clear that player 3 should minimize c_3 as much as possible. We consider the following three cases:

Case 1: If $c_1 + c_2 \geq 5000$, player 3 does not need to contribute anything so $c_3^* = 0$.

Case 2: If $1500 \leq c_1 + c_2 < 5000$, player 3 should choose c_3^* such that $c_1 + c_2 + c_3^* = 5000 \Rightarrow c_3^* = 5000 - c_1 - c_2$.

Case 3: If $c_1 + c_2 < 1500$, player 3 needs to pay more than \$3500 ($c_3 = 5000 - c_1 - c_2 > 3500$) in order to buy a computer. Player 3 will get a negative payoff. Thus the player 3 will choose to give up and $c_3^* = 0$.

In summary, c_3^* can be written as

$$c_3^* = \begin{cases} 0 & if\ c_1 + c_2 \geq 5000 \\ 5000 - c_1 - c_2 & if\ 1500 \leq c_1 + c_2 < 5000 \ldots (*) \\ 0 & if\ c_1 + c_2 < 1500 \end{cases}$$

We consider the player 2's side. Knowing the strategy chosen by player 1 (c_1) and the optimal strategies chosen by player 3 (c_3^*). Similar to the case for player 3, player 2 should choose c_2 such that $c_1 + c_2 + c_3^* = 5000$.

From $(*)$, player 2 knows that player 3 will contribute the remaining amount provided that $c_1 + c_2 \geq 1500$. In order to minimize c_2, player 2 should choose c_2^* such that

$$c_1 + c_2^* = 1500 \Rightarrow c_2^* = 1500 - c_1 \ldots \ldots (**)$$

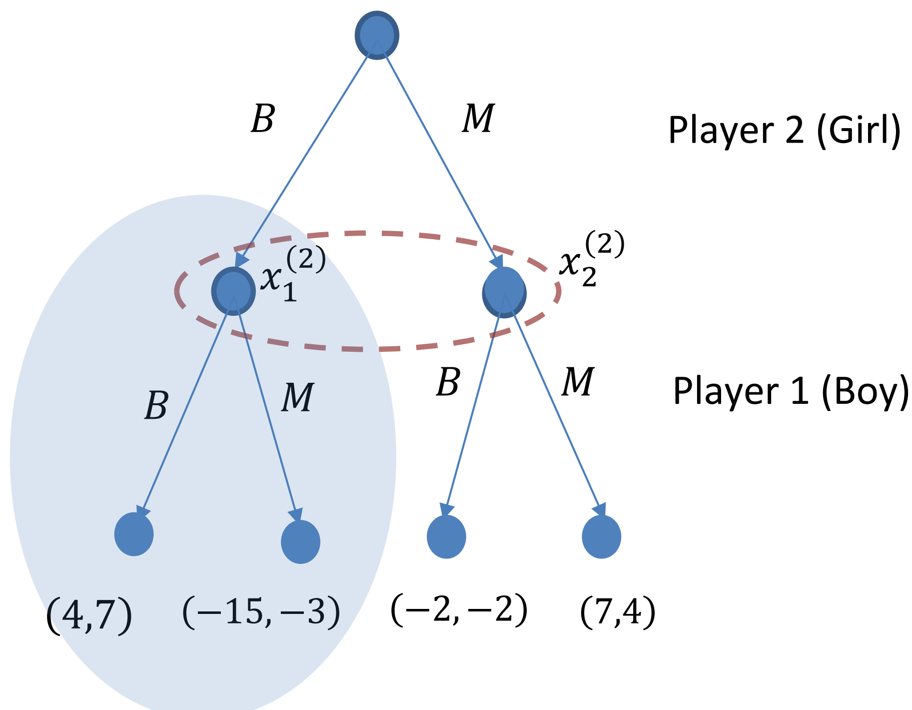
(Here, player 3 will have to contribute 3500.)

Finally, we consider player 1's side. From the above analysis, Player 1 knows that the Player 2 will contribute $1500 - c_1$ and player 3 will contribute 3500. Since player 2 will still get a positive payoff if he chooses to contribute 1500, player 1 find himself better off to contribute nothing, $c_1^* = 0$. So we conclude that the sequentially rational Nash equilibrium will be

$$(c_1^*, c_2^*, c_3^*) = (0,\ \ 1500,\ \ 3500).$$

Finding equilibrium in games of imperfect information – Subgame perfect Nash equilibrium

Although dynamic programming provides an efficient approach to find the equilibrium that is sequentially rational, one cannot apply this directly to the games of imperfect information. To see, we consider the battle of sexes in Example 3 again.



At node $x_1^{(2)}$, the boy cannot determine his strategy based on the information at the sub-stage (shaded portion) since he <u>does not know</u> that the girl has chosen "B".

In order to determine the boy's strategy, he needs to consider the possibility that the girl may choose the strategy and needs the information at $x_2^{(2)}$. Equivalently, the boy chooses his optimal strategy by considering the *bigger games* instead of just a sub-stage. This leads to the concept of *subgames.*

Definition (Subgames)

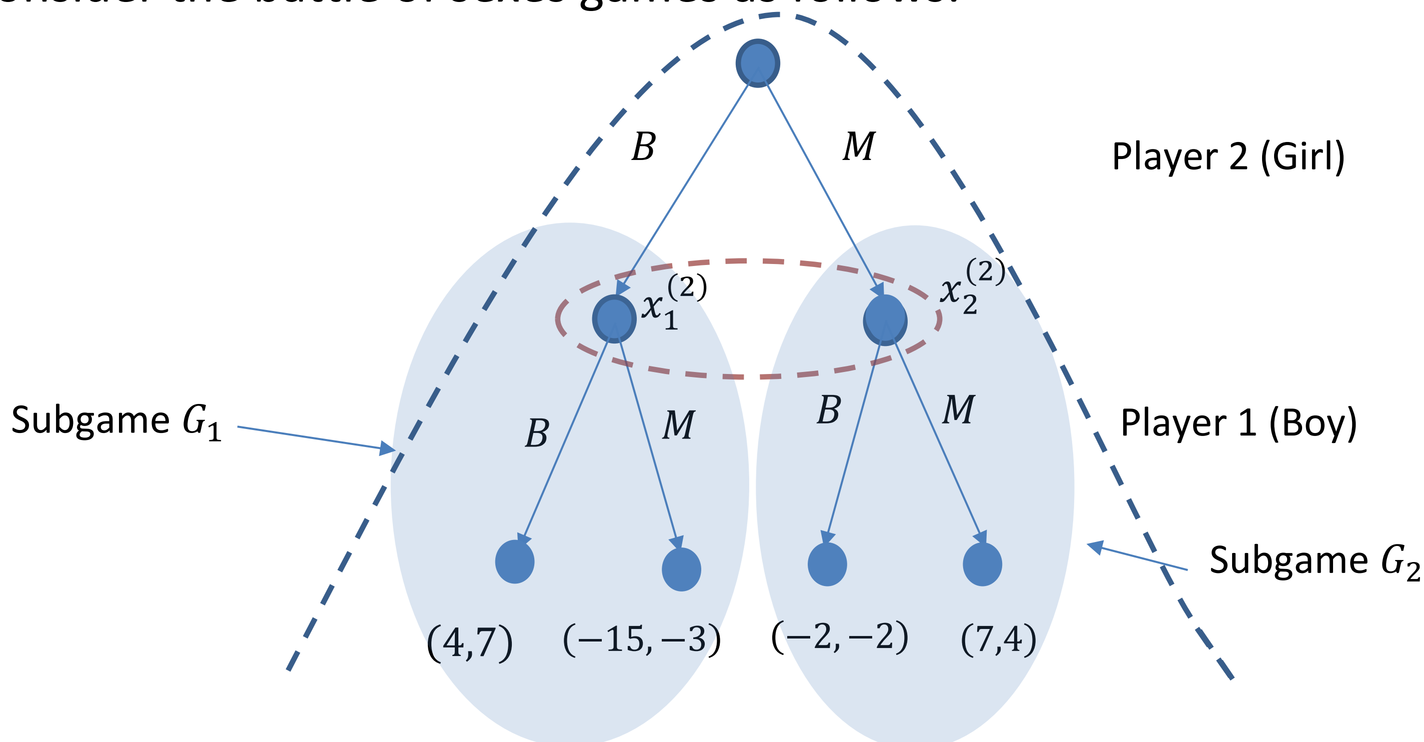
We let G be a games in extensive form. We say G' is a *proper subgame* of G if it has the following properties:

(1) It has a single initial node;

(2) For every node $x \in G'$ and any $y \in h(x)$, we must have $y \in G'$. (In other words, then player's optimal strategies in the subgames does not depend on any information outside the subgames.)

Example 14

We consider the battle of sexes games as follows:



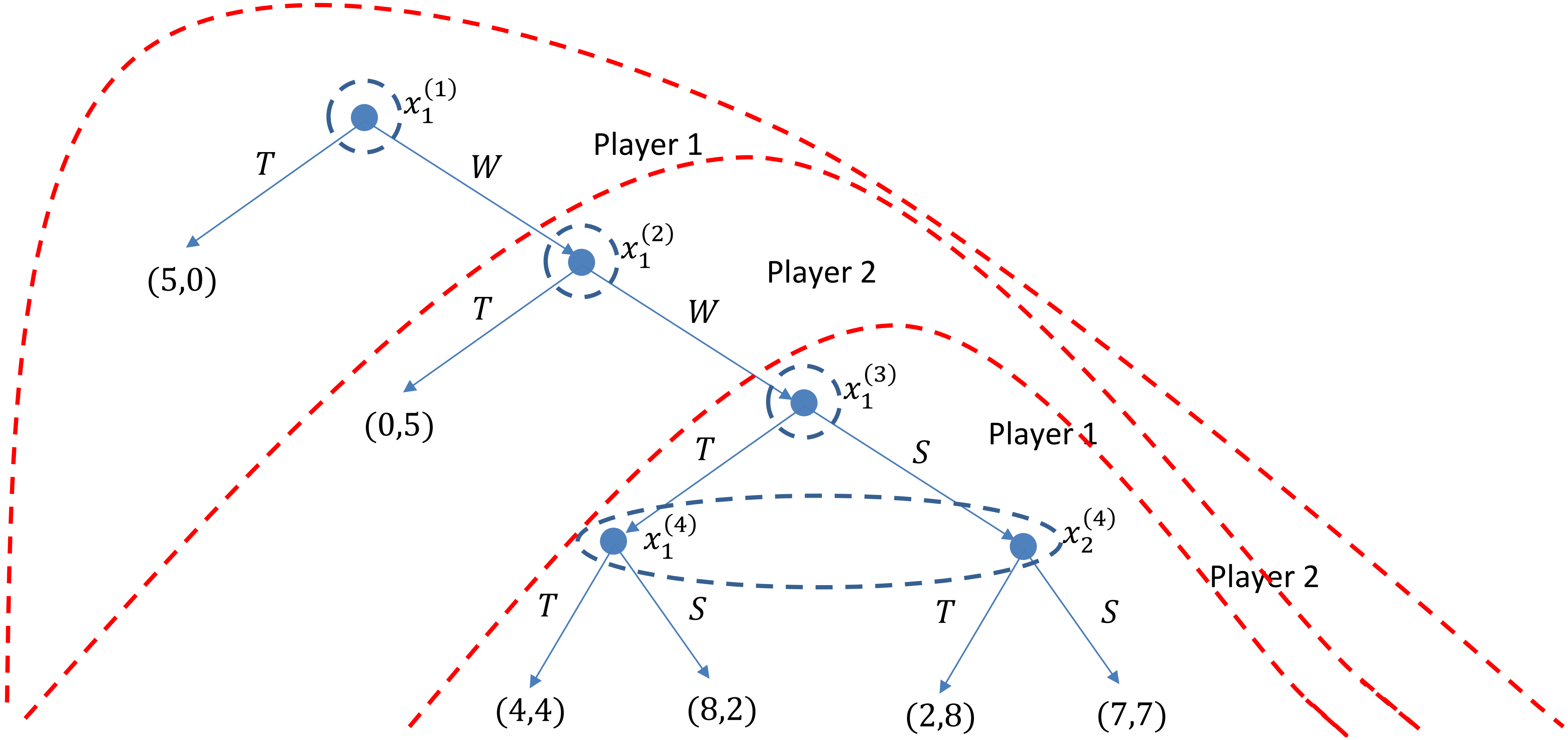
We note that neither G_1 nor G_2 are subgames since $x_2^{(2)} \notin h\left(x_1^{(2)}\right)$ (for subgame G_1) and $x_1^{(2)} \notin h\left(x_2^{(2)}\right)$ (for subgame G_2). The only subgame in this example is the entire games.

Example 15

We consider the following two-person games as follows: Two players (Player 1 and 2) are going to share the money in a pot. Initially, the pot has \$5. Player 1 moves first and can choose to either take all money (T) or wait (W). If he chooses to take all money, he will get \$5 and the game is over. If he chooses to wait, it will be player 2's turn and choose to either take all money or wait. If he chooses to take all money, he will get \$5. Otherwise, the money in the pot will be doubled and the games moves to second round. In the second round, two players simultaneously decide to either take all money (T) or share (S). The games is over once they make their decisions and the payoff is given by the following matrix:

		Player 2	
		T	S
Player 1	T	$(4,4)$	$(8,2)$
	S	$(2,8)$	$(7,7)$

The games can be expressed as follows:



We observe that there are 3 proper subgames (including the entire games, highlighted in red dash line) in this example.

In a dynamic games (or games in extensive form) of perfect information, we require that all players should choose their decisions optimally at *every stage of the games* (i.e. sequentially rational).

However, this solution concept cannot be applied directly to the games of imperfect information since some players may not able to choose their decisions optimally at some stages due to the lack of information on the past events.

Therefore, we relax the requirement by just requiring that all players choose their decisions optimally at *every subgame* (not every stage). This solution concept is known as subgame perfect equilibrium.

Definition (Subgame perfect equilibrium)

A strategic profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is said to be a subgame perfect equilibrium in a dynamic games G if and only if s^* is Nash equilibrium in *every proper subgames* G' in G.

Remark

- Since the entire games is one of the proper subgames, thus the subgame perfect equilibrium is a Nash equilibrium as defined earlier.
- To determine subgame perfect equilibrium in the games of imperfect information, one need to apply a modified version of dynamic programming algorithm. Instead of solving solutions backward in stage, we solve for equilibrium backward in "*proper subgames*": We start with the smallest proper subgames and solve for the Nash equilibrium. Using the Nash equilibrium obtained, we proceed to solve for the equilibrium for second smallest proper subgames and so on.
- For the games of perfect information, the set of subgame perfect equilibrium is same as the set of sequentially rational Nash equilibrium obtained by standard backward induction.

Example 16 (Solving the simplest type of imperfect information games)

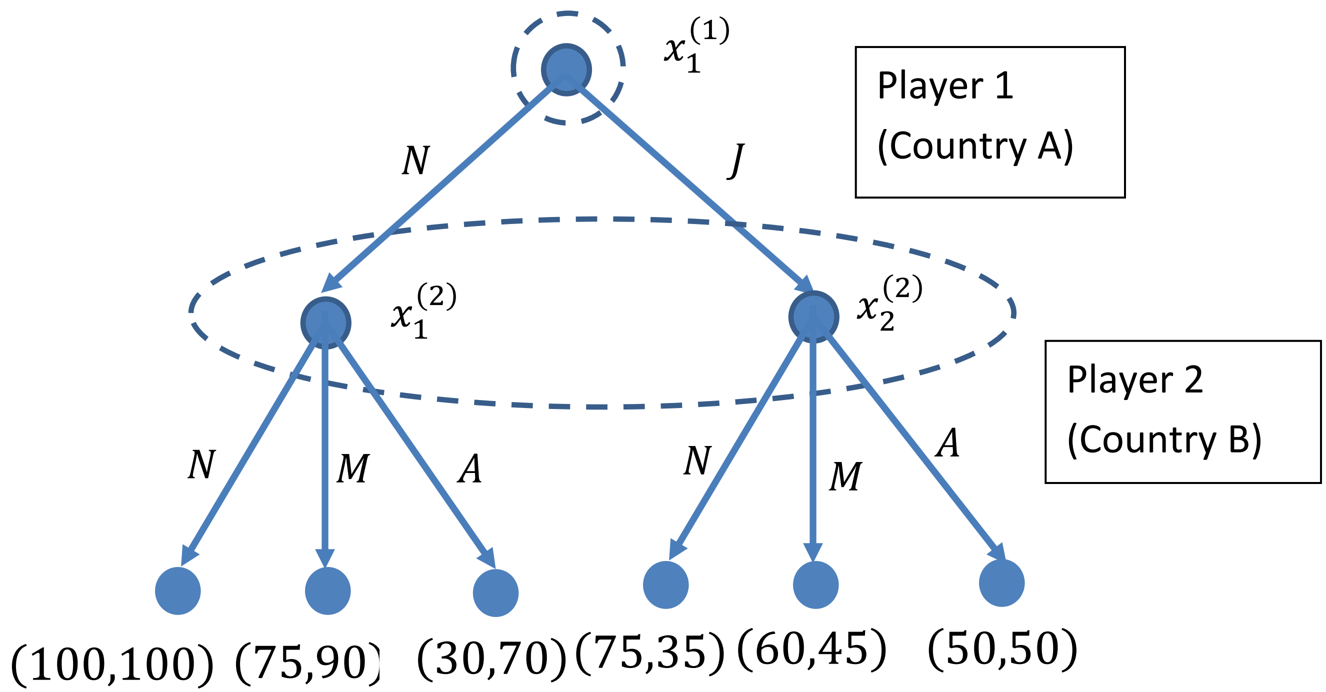
We consider the games called Security Dilemma: Country A and country B are rivals of each other. Each of them can choose to develop some weapons to protect itself against the potential attack from another country. It is known that

- Country A can develop fighter jets (J).
- Country B can develop either missile (M) or atomic bomb (A).

However, developing weapons is very costly and may convey negative message to other countries in the world. Thus the best outcome will be the one in which both countries do not produce any weapons (N). On the other hand, a country will be at disadvantage side if another country has developed some weapons. We assume that

- Each country can produce at most 1 weapon due to the limitation of its budget. Country A will make the decision first.
- Each country does not know the action taken by another country.

The payoffs and structure of the games are summarized by the following game tree:



It is obvious that the only subgame is the entire game. Find the subgame perfect Nash equilibrium in this games.

☺Solution

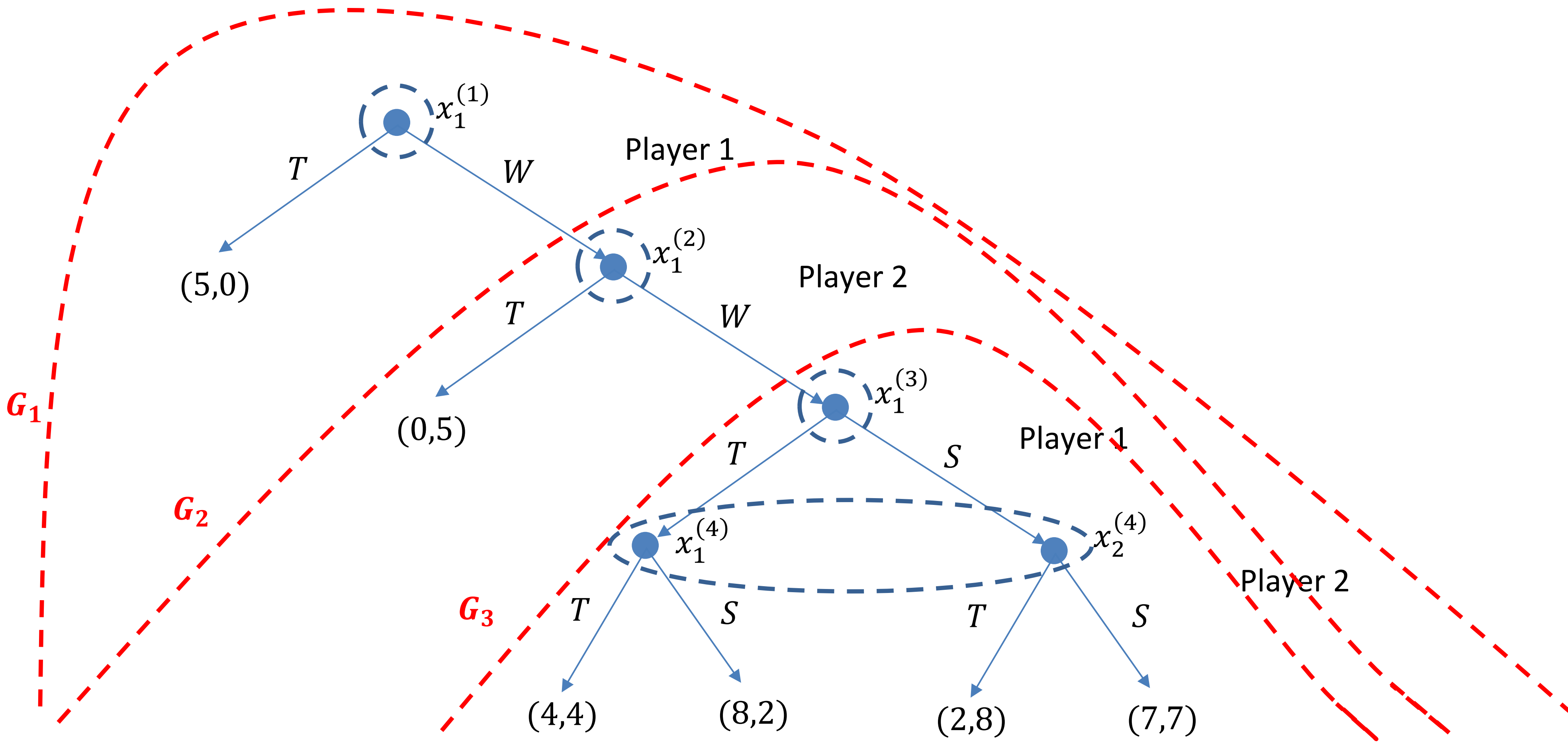
Since player 2 does not know the action taken by player 1, it chooses its action just based on the initial information. Thus, the games is equivalent to a static games which two players make their decision simultaneously and independently. So we transform the games into the following matrix:

		Player 2 (Country B)		
		N	M	A
Player 1 (Country A)	N	$(100,100)$	$(75,90)$	$(30,70)$
	J	$(75,35)$	$(60,45)$	$(50,50)$

Since the whole games is the only proper games, thus the subgame perfect equilibrium is simply the Nash equilibrium of this matrix games. The equilibrium can be determined easily by first finding the best responses of each player to opponent's strategy and seeking for the intersection of the best responses. After some calculations (left as exercise), there are two equilibria: (N, N) and (J, A).

Example 17 (Example 15 revisited)

We consider the games considered in Example 15.



Find the subgame perfect equilibrium for this games.

☺Solution

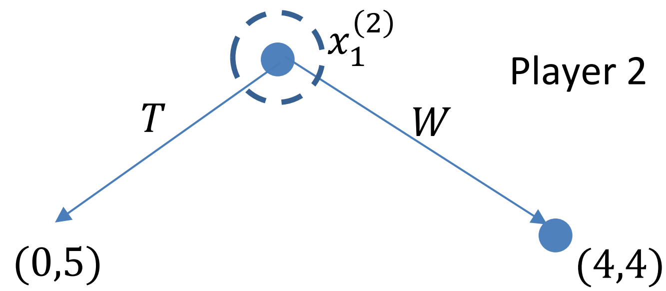
There are 3 proper subgames in this games. We shall find the required equilibrium using backward induction.

We start with the smallest subgames G_3. As seen in Example 16, such subgames can be treated as a static games with the following payoff matrix:

		Player 2	
		T	S
Player 1	T	$(4,4)$	$(8,2)$
	S	$(2,8)$	$(7,7)$

After some calculations (left as exercise), there is an unique Pure strategy Nash equilibrium (T, T) in this subgames.

Knowing the equilibrium in the games G_3, we proceed to consider the bigger subgames G_2. Given the players' optimal strategies, the subgames is equivalent to a single-person problem:



It is clear that player 2 should choose T and get a payoff of 5.

Finally, we consider the last subgame G_1. Using similar method, one can find that player 1 should choose T in G_1 and he will get zero payoff if he chooses W. Therefore, the unique subgame perfect equilibrium $s^* = (s_1^*, s_2^*)$ is given by

$$s_1^* = \left(\underbrace{T}_{s_1\left(\left\{x_1^{(1)}\right\}\right)}, \underbrace{T}_{s_1\left(\left\{x_1^{(3)}\right\}\right)} \right), \qquad s_2^* = \left(\underbrace{T}_{s_2\left(\left\{x_1^{(2)}\right\}\right)}, \underbrace{T}_{s_2\left(\left\{x_1^{(4)}, x_2^{(4)}\right\}\right)} \right)$$

Example 18

There are two investment opportunities: producing new smartphone (S) and producing new computer (C). Two companies (one of them is leader) decide to cooperate and work on one of these project. First, the leader chooses the investment project and decides the amount of resource (High (H) or low (L)) to be put in the project. Another company (follower) knows the project chosen by the leader but does not know the amount of resource contributed by the leader. It then chooses the amount of resource (High (H) or low (L)) to be put in the project. The payoffs to the two companies are summarized in the following two matrices:

		Follower (Player 2)	
		High	Low
Leader (Player 1)	High	$(8,8)$	$(5,10)$
	Low	$(12,5)$	$(4,4)$

(Producing smartphone)

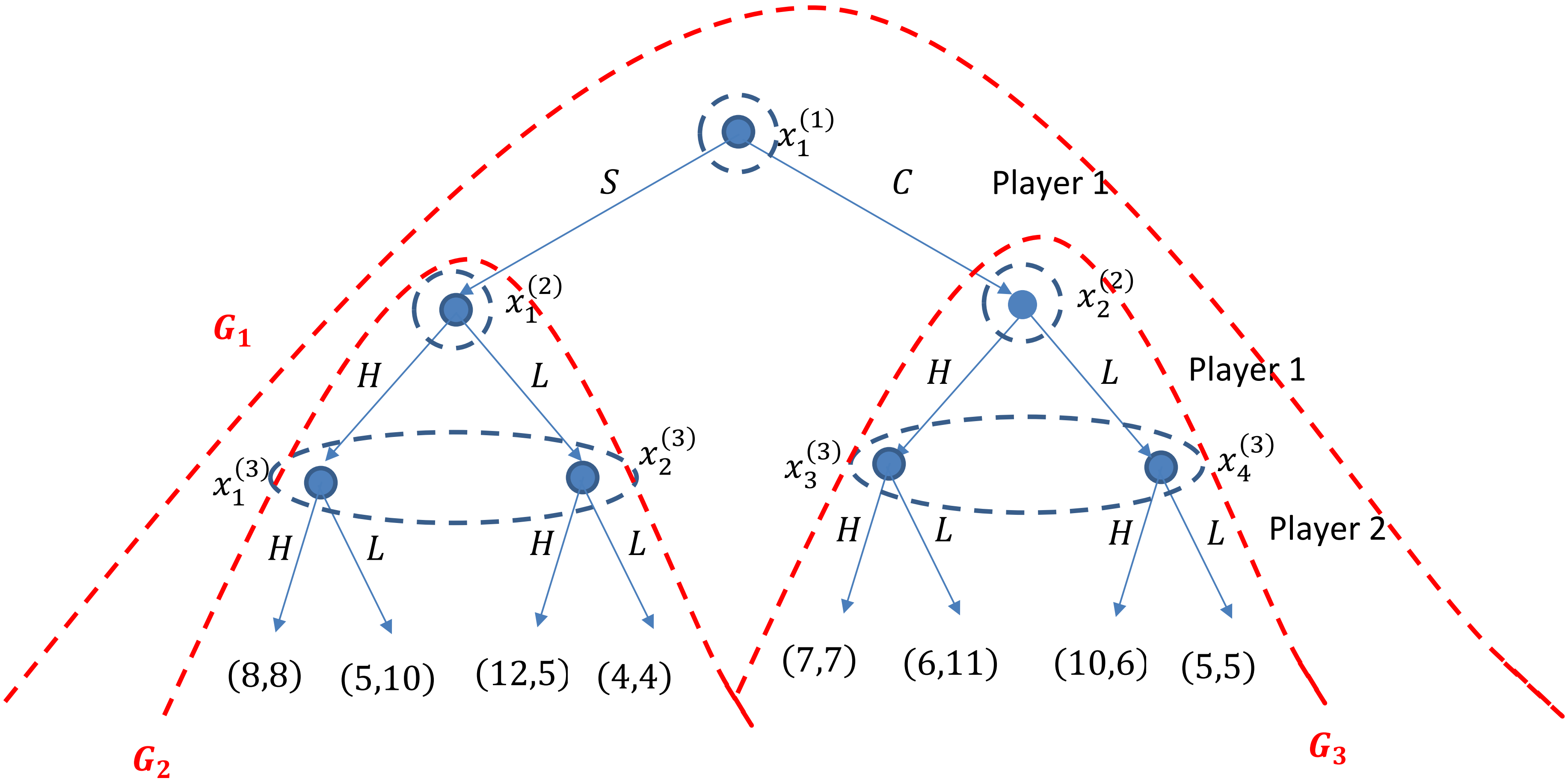
		Follower (Player 2)	
		High	Low
Leader (Player 1)	High	$(7,7)$	$(6,11)$
	Low	$(10,6)$	$(5,5)$

(Producing computer)

Find all possible subgame perfect equilibrium for this games.

☺Solution:

Firstly, we express the games into the following game trees:



We observe that there are 3 proper subgames.

Firstly, we solve for Nash equilibrium for the subgames G_2 and G_3.

For G_2, the games is equivalent to static games with following payoff matrix

		Player 2	
		H	L
Player 1	H	$(8,8)$	$(\overline{\mathbf{5}}, \overline{\mathbf{10}})$
	L	$(\overline{\mathbf{12}}, \overline{\mathbf{5}})$	$(4,4)$

There are two Nash equilibria in this matrix games: (L, H) and (H, L).

For G_3, the games is equivalent to static games with following payoff matrix

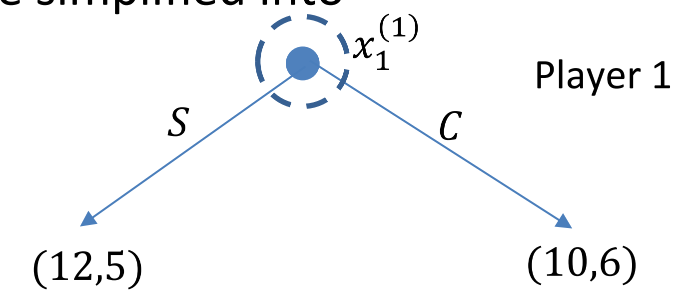
		Player 2	
		H	L
Player 1	H	$(7,7)$	$(\overline{\mathbf{6}}, \overline{\mathbf{11}})$
	L	$(\overline{\mathbf{10}}, \overline{\mathbf{6}})$	$(5,5)$

There are two Nash equilibria in this matrix games: (L, H) and (H, L).

To determine the optimal strategy in the subgame G_1, we consider the following four cases (It is because the players adopt any one of two equilibria in each of subgames G_2, G_3):

Case 1: If players adopt (L, H) in G_2 and players adopt (L, H) in G_3.

The subgames can be simplified into

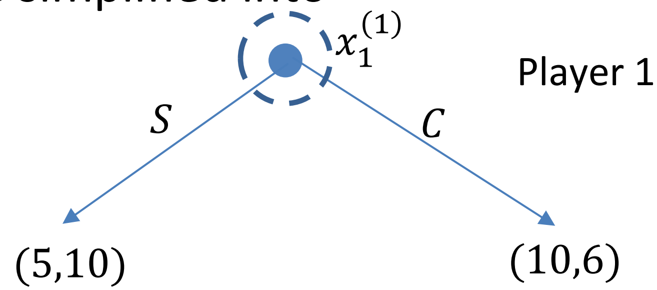


So player 1 will choose S to receive a payoff of 12. Thus the equilibrium $s = (s_1^*, s_2^*)$ will be

$$s_1^* = \left(\underbrace{S}_{s_1\{x_1^{(1)}\}}, \underbrace{L}_{s_1\{x_1^{(2)}\}}, \underbrace{L}_{s_1\{x_2^{(2)}\}} \right), \qquad s_2^* = \left(\underbrace{H}_{s_2\{x_1^{(3)}, x_2^{(3)}\}}, \underbrace{H}_{s_2\{x_3^{(3)}, x_4^{(3)}\}} \right).$$

Case 2: If players adopt $(\boldsymbol{H}, \boldsymbol{L})$ in $\boldsymbol{G_2}$ and $(\boldsymbol{L}, \boldsymbol{H})$ in $\boldsymbol{G_3}$

The subgames can be simplified into



So player 1 will choose C to receive a payoff of 10. The corresponding equilibrium will be $s_1^* = (C, H, L)$ and $s_2^* = (L, H)$.

The derivation of the last two cases is similar. We omit the detail and summarize the result below:

Case 3: If players adopt $(\boldsymbol{L}, \boldsymbol{H})$ in $\boldsymbol{G_2}$ and $(\boldsymbol{H}, \boldsymbol{L})$ in $\boldsymbol{G_3}$,

The equilibrium is $s_1^* = (S, L, H)$ and $s_2^* = (H, L)$.

Case 4: If players adopt $(\boldsymbol{H}, \boldsymbol{L})$ in $\boldsymbol{G_2}$ and $(\boldsymbol{H}, \boldsymbol{L})$ in $\boldsymbol{G_3}$,

There are two equilibria: (1) $s_1^* = (C, H, L)$ and $s_2^* = (H, L)$.

Application 1: Multi-stage games and repeated games

In this section, we shall study a special class of dynamic games called *multi-stage games*. Roughly speaking, the games consists of a finite number of normal-form games $G_1, G_2, G_3, \dots, G_N$ with the following properties:

- Players play the games $G_1, G_2, \dots, G_N$ (G_1 is played first and G_N is played last) sequentially.
- All subgames ($G_1, G_2, \dots, G_N$) are independent (with respect to payoffs, available strategy of each player etc.)
- Each subgame G_i is a static games in which all players choose their decisions simultaneously and independently.
- The result of the games G_i is known to all players immediately after G_i is played.

Payoff structure of multi-stage games

Given the players' strategy $s = (s_1, s_2, \dots, s_n)$ in a multi-stage games, we let $V_i^{(j)} = V_i^{(j)}(s_i; s_{-i})$ be the payoff received by player i in the games G_j.

Intuitively, one can define the player i's payoff in the entire games as the total payoffs received by the player in the games $G_1, G_2, \dots, G_N$. That is,

$$V_i(s_i; s_{-i}) = V_i^{(1)} + V_i^{(2)} + V_i^{(3)} + \cdots + V_i^{(N)}.$$

In most of the games, it would be more realistic to put less weight on the payoff at the games that played later. This adjustment is important when the games G_1 , $G_2, .., G_N$ are played at different time periods.

We assume that there are N periods and the games G_i is played at period i. We let $D \in (0,1]$ be the *discounted factor over 1 period*. Then the total discounted payoff of player i can be expressed as

$$V_i(s_i; s_{-i}) = V_i^{(1)} + DV_i^{(2)} + D^2V_i^{(3)} + \cdots + D^{N-1}V_i^{(N)} = \sum_{j=1}^{N} D^{j-1}V_i^{(j)}.$$

Some examples of multi-stage games

Example 19 (Investment race)

There are two competing firms in a market. Now the firms decide to promote their services through internet and television. To start with, they first privately decides the amount of resources (High (H) or Low (L)) to be put in promotion through internet. After knowing the responses from the customers, they proceed to decide privately the amount of resources (High (H) or Low (L)) to be put in promotion through television. The profits made from the two promotion channels are summarized in the following payoff matrix:

		Firm 2	
		H	**L**
Firm 1	**H**	$(3,3)$	$(7,2)$
	L	$(2,7)$	$(6,6)$

(Internet: First stage)

		Firm 2	
		H	**L**
Firm 1	**H**	$(4,3)$	$(9,0)$
	L	$(0,9)$	$(7,7)$

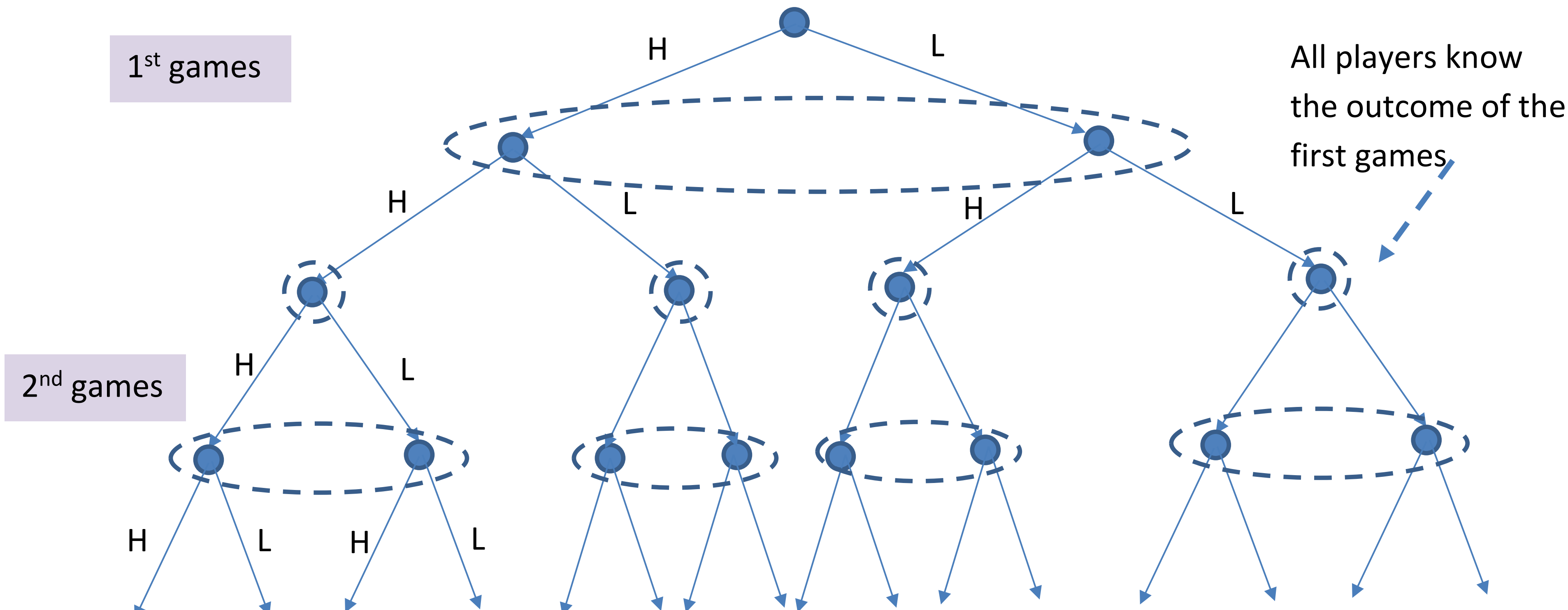
(Television: Second stage)

The total profits made by Firm i in these two stages can be defined as

$$V_i(s_i; s_{-i}) = \underbrace{V_i^{(1)}(s_i; s_{-i})}_{payoff\ at\ stage\ 1} + D\ \underbrace{V_i^{(2)}(s_i; s_{-i})}_{payoff\ at\ stage\ 2},$$

where $D \in [0,1]$ is the discount factor over 1 period.

Remark: Game tree representation



Example 20

Two firms cooperate and work on an investment project. The duration of the project is 10 years. Every year, each firm decides (simultaneously) the amount of efforts (high (H) or low (L)) to be put in the project. The annual profit made by the firms (denoted by $O_1^{(j)}, O_2^{(j)}$) from this project is summarized in the following payoff matrix:

		Firm 2	
		High (H)	**Low (L)**
Firm 1	**High (H)**	$(10,7)$	$(4,6)$
	Low (L)	$(12,4)$	$(6,5)$

The firms know the profit made at the end of each year. Assuming the riskfree interest rate is 5% per year so that $D = \frac{1}{1+0.05} = \frac{20}{21}$, then the total profit of Firm i is $V_i = DV_i^{(1)} + D^2V_i^{(2)} + \cdots + D^{10}V_i^{(10)}$.

Strategies and information set in multi-stage games

In a multi-stage games, all players know the result of the games G_j once the games G_j is played. This includes the payoffs received and the strategies made by the players in G_j.

For $k = 1,2, \dots, N$, we let $h_i^{(k)}$ be the information set of player i at k^{th} stage in which the games $G_1, G_2, \dots, G_{k-1}$ have been played. Then $h_i^{(k)}$ can be (informally) described as

$$h_i^{(k)} = h^{(k)} = \{outcomes\ of\ games\ G_1, G_2, \dots, G_{k-1}\} = \{s^{(1)}, s^{(2)}, \dots, s^{(k-1)}\},$$

where $s^{(j)} = \left(s_1^{(j)}, s_2^{(j)}, \dots, s_n^{(j)}\right)$ denotes the strategy chosen by the players in games G_j.

Since the player chooses his strategy based on the updated information, the player i's strategy in games G_k, denoted by $s_i^{(k)}$, is a function of his information set at that stage. That is $s_i^{(k)} = s_i^{(k)}\left(h_i^{(k)}\right)$.

Solution concept for multi-stage games

Since the multi-stage games is seen to be a dynamic games with *imperfect information*, we adopt the concept of *subgame perfect equilibrium* to obtain the players' optimal strategies in the games.

The equilibria can be found through the standard backward induction:

- We first consider the last game G_N. Since the remaining $N-1$ games have been played, all players just concentrate on G_N and choose a strategy that maximizes the payoffs $V_i^{(N)}$. The equilibrium strategy in games G_N is simply the Nash equilibrium in the games G_N.
- Suppose the equilibrium strategy in games $G_{k+1}, G_{k+2}, \dots, G_N$ has been obtained, we consider the games G_k. In this stage, each player chooses his strategy in games G_k to maximize the total payoffs $V_i^{(k)} + DV_i^{(k+1)} + \cdots + D^{N-k}V_i^{(N)}$, given the equilibrium strategy adopted in games $G_{k+1}, \dots, G_N$.

Example 21 (Investment games revisited)

We revisit the investment games in Example 19. Find all possible subgame perfect equilibrium for this games.

☺Solution

We first consider the second games (Promotion through television) and solve for the Nash equilibrium of the games. To do this, we find the players' best response to each of the opponent's strategy. The result is summarized in the following matrix:

		Firm 2	
		H	**L**
Firm 1	**H**	$(\overline{\mathbf{4}},\overline{\mathbf{3}})$	$(\overline{\mathbf{9}},0)$
	L	$(0,\overline{\mathbf{9}})$	$(7,7)$

We observe that there is an unique pure strategy Nash equilibrium (H,H). So each firm will choose H in the second game.

Next, we consider the first games (Promotion through internet). Each firm chooses its optimal strategy in first games in order to maximize the total discounted payoff $V_i^{(1)} + DV_i^{(2)}$.

Knowing that all firms must choose $\left(s_1^{(2)}, s_2^{(2)}\right) = (H, H)$ in the second games, we have $\left(V_1^{(2)}, V_2^{(2)}\right) = (4, 3)$. Then the equilibrium strategy in the first games can be found by solving the following matrix games:

		Firm 2	
		H	**L**
Firm 1	**H**	$(\overline{\mathbf{3+4D}}, \overline{\mathbf{3+3D}})$	$(\overline{\mathbf{7+4D}}, 2+3D)$
	L	$(2+4D, \overline{\mathbf{7+3D}})$	$(6+4D, 6+3D)$

(*Here, the best responses are marked with upper bar)

We conclude from the above matrix that the unique equilibrium strategy in the first games is (H, H). In summary, the subgame perfect equilibrium is that all firm chooses to put high effort in both advertising channels.

What is the equilibrium strategy in multi-stage games? An intuitive guess

Since all subgames $G_i s$ in the multi-stage games are independent, one would expect that each player should play optimally in *each games* G_i in order to maximize his total payoff in the entire games. We let $s^{(j)*}$be the Nash equilibrium in j^{th} games G_j, we expect that all players should adopt the equilibrium strategies $s^{(j)*}$ in every games G_j, *regardless of the outcomes of the games played previously*.

The following theorem confirms that such strategic profile $\vec{s}^* = \left(s^{(1)*}, s^{(2)*}, \dots, s^{(N)*}\right)$ constitutes the Nash equilibrium.

Theorem 1

We let $s^{(j)*} = \left(s_1^{(j)}, s_2^{(j)}, \dots, s_n^{(j)}\right)$ be a Nash equilibrium in the subgame G_j $(j = 1,2, \dots, n)$. The strategic profile $s^* = \left(s^{(1)*}, s^{(2)*}, \dots, s^{(N)*}\right)$ is the subgame perfect equilibrium of the multi-stage games.

Proof of the Theorem 1

We need to show that $s^* = \left(s^{(1)*}, s^{(2)*}, \dots, s^{(N)*}\right)$ is Nash equilibrium in all stages of the games. This can be done by backward induction.

We consider the last games in which only games G_N is not played, one can show easily that

$$V_i(s_i^*; s_{-i}^*)|_{Nth\ stage} = \underbrace{V_i^{(1)} + \ldots + D^{N-2} V_i^{(N-1)}}_{fixed} + D^{N-1} V_i^{(N)}\left(s_i^{(N)*}; s_{-i}^{(N)*}\right)$$

$$\geq V_i^{(1)} + \ldots + D^{N-2} V_i^{(N-1)} + D^{N-1} V_i^{(N)}\left(s_i^{(N)}; s_{-i}^{(N)*}\right),$$

$$for\ any\ s_i^{(N)}.$$

The above inequality shows that s^* is the Nash equilibrium at N^{th} stage.

Assuming s^* is the Nash equilibrium at $(k+1)^{th}$ stage, we proceed to show s^* is also the Nash equilibrium at k^{th} stage in which the games $G_1, \dots, G_{k-1}$ have been played.

Using the property of s^* and the assumption that all players always play $s^{(j)*}$ in games G_j under all scenarios, we deduce that

$$V_i(s_i^*; s_{-i}^*)|_{kth\ stage} = \underbrace{V_i^{(1)} + \cdots + D^{k-2}V_i^{(k-1)}}_{denoted\ by\ C_k} + \sum_{j=k}^{N} D^{j-1}V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right)$$

$$\geq C_k + D^{k-1}V_i^{(k)}\left(s_i^{(k)*}; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{N} D^{j-1}V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right)$$

$$\geq C_k + D^{k-1}V_i^{(k)}\left(s_i^{(k)*}; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{N} D^{j-1}V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right),$$

$$\geq C_k + D^{k-1}V_i^{(k)}\left(s_i^{(k)*}; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{N} D^{j-1}V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right).$$

for any $s_i^{(k)}, s_i^{(k+1)}, \dots, s_i^{(N)}$. So s^* is also Nash equilibrium at k^{th} stage and it completes the induction.

Is that all? Conditional strategy

Although such strategic profile $s^* = \left(s^{(1)*}, s^{(2)*}, \ldots, s^{(N)*}\right)$ constitutes the Nash equilibrium, this strategy has implicitly assumed that all players treat each stage independently without considering the outcomes of the games played previously.

In reality, the players may adopt different strategies for different outcomes of the previous games (captured by information set). That is, "If … happens in the games $G_1, G_2, \ldots, G_{k-1}$, the player will play … in games G_k". This strategy is called *conditional strategy*. Theoretically, it is feasible since player can assign different strategies to different information sets.

The conditional strategy creates a link between the strategies made in different games. In fact, such "strategic link" may allow the players to achieve a better payoff in the entire games.

Example 22 (Two stage battle-of-sex games)

We consider the following Battle-of-sex games. A couple have a date today.

- (1^{st} game) First, they choose to either go shopping (S) or play video games (V) in the afternoon. Boy loves video games and girl loves shopping. Although they do not need to join the same activities, the girl will be very happy if the boy goes shopping with her. The payoff matrix is given as follows:

		Girl (Player 2)	
		S	**V**
Boy (Player 1)	**S**	$(0,12)$	$(-2,-1)$
	V	$(2,3)$	$(3,-2)$

- (2^{nd} games) After that, they choose a place for their dinner. They can choose between Buffet (B) and McDonald (M). If they choose the same place, they have their dinner happily. Otherwise, the date is finished.

The payoff matrix is given as follows:

		Girl (Player 2)	
		B	**M**
Boy (Player 1)	**B**	$(4,7)$	$(-2,-2)$
	M	$(-15,-1)$	$(7,4)$

The payoff of each player in the entire games is simply the total payoffs in these two games (without discounting, $D = 1$). That is,

$$V_i(s_i; s_{-i}) = V_i^{(1)}(s_i; s_{-i}) + V_i^{(2)}(s_i; s_{-i}).$$

(a) Assuming all players will adopt the same strategies in 2nd games for all possible outcomes resulted in the first games. Find all possible subgame perfect Nash equilibrium in which all players follow equilibrium strategies in each of the two games. Find the players' payoffs.

(b) Find all possible subgame perfect Nash equilibrium in which the girl chooses M in the second games if both of them choose S in the first games and chooses B if otherwise. Find the corresponding payoffs.

☺Solution of **(a)**

One can show that

- there is an unique Pure strategy Nash equilibrium in game 1: (V,S).
- There are two Pure strategy Nash equilibrium in game 2: (B,B) and (M,M).

Using the theorem in P.81 and the assumption that the players adopt a common equilibrium strategy in second games for all information sets, we deduce that there are two subgame perfect Nash equilibrium.

1st equilibrium: Players play (V,S) in the first games and play (B,B) in second games. The corresponding payoff is $(V_1,V_2)=(2+4,3+7)=(6,10)$.

2nd equilibrium: Players play (V,S) in the first games and play (M,M) in second games. The corresponding payoff is $(V_1,V_2)=(2+7,\ 3+4)=(9,7)$.

☺Solution of **(b)**

We shall obtain the corresponding equilibrium using backward induction. We first consider the 2nd games (last games). Note that each player must play equilibrium strategy in 2nd games. Under the given restriction, the players' strategies in the 2nd games is seen to be

$$\left(s_1^{(2)*}\left(h^{(1)}\right), s_2^{(2)*}\left(h^{(1)}\right)\right) = \begin{cases} (M, M) & if \ \ (S, S) \ is \ played \ in \ 1^{st} \ game \\ (B, B) & if \ \ otherwise \end{cases}.$$

We proceed to consider the 1st games. Given the equilibrium strategies chosen by the players in the second games, the equilibrium strategies in the 1st games can be found by solving the following matrix games:

		Girl (Player 2)	
		S	**V**
Boy (Player 1)	**S**	$(0 + 7, 12 + 4) = (\overline{\mathbf{7}}, \overline{\mathbf{16}})$	$(-2 + 4, -1 + 7) = (2,6)$
	V	$(2 + 4, \ \ 3 + 7) = (6, \overline{\mathbf{10}})$	$(3 + 4, -2 + 7) = (\overline{\mathbf{7}}, 5)$

Here, the players' best responses are highlighted with upper bar. We deduce from the above matrix that (S,S) is the players' equilibrium strategies in the first games.

In summary, the required subgame perfect equilibrium is given by

$$\left(s_1^{(1)*}, s_2^{(1)*}\right) = (S,S)$$

$$\left(s_1^{(2)*}\left(h^{(1)}\right), s_2^{(2)*}\left(h^{(1)}\right)\right) = \begin{cases} (M,M) & if \quad \left(s_1^{(1)*}, s_2^{(1)*}\right) = (S,S) \\ (B,B) & if \quad otherwise \end{cases}.$$

The corresponding payoffs of the players are $(V_1, V_2) = (0+7, 12+4) = (7,16)$.

*Remark of Example 22**(b)***

- In **(b)**, we observe that the couples does not play the equilibrium strategy (play (S,S) instead of (V,S) in the first games.) This shows that the players do not play optimally in *every* subgames.

- At girl's side, she can enjoy a very high payoff (12) in the first games if both of them go shopping (S). If they just play the first games alone, such outcome never occur since the boy will be better off by choosing video games (V). With the inclusion of the second games, the girl can "convince" the boy to choose S in the first games by giving him some bonus in the second games. That is, the girl agrees to go McDonald together if the boy agrees to go shopping with the girl. It is optimal for the girl to do so since she can get an additional payoff 6 ($= 9 - 3$).
- At boy's side, he agrees the girl's proposal. Although he suffers from a loss of 2 (short term loss) if he chooses to go shopping with the girl in the first games, he can gain a payoff of 3 in the second games (long term gain). This gives him an addition payoff 1.
- By striking the balance between the short-term gain (loss) and the long-term loss (gain), the players may not always follow the equilibrium strategy in every subgames.

Example 23 (Investment games)

Two companies (Company 1 and 2) work on two investment projects (P_1, P_2) together. Two projects are conducted in different periods: P_1 is conducted at period 1 and P_2 is conducted at period 2.

- In the first project P_1, two companies decide simultaneously the amount of effort (high (H) or low (L)) to be invested in the project. The companies' payoffs in P_1 are summarized by the following matrix:

		Company 2	
		H	**L**
Company 1	**H**	$(5, 5)$	$(2, 6)$
	L	$(6, 2)$	$(3, 3)$

- In the second project P_2, two companies decide whether to put effort (E) in the project. The project is successfully only when both companies choose to put effort. The companies' payoffs in P_2 are summarized by the following matrix:

		Company 2	
		E	**N**
Company 1	**E**	$(5,3)$	$(-2,0)$
	N	$(0,-2)$	$(0,0)$

(*Note: "N" denotes "no effort")

The total payoffs of the company i is assumed to be

$$V_i = V_i^{(1)} + DV_i^{(2)},$$

where $V_i^{(1)}$ and $V_i^{(2)}$ are the payoffs received by company i in P_1 and P_2 respectively. Here, $D \in [0,1]$ denotes the discount factor over 1 period.

If the company conducts the first project P_1 only, one can show that $(s_1^*, s_2^*) = (L, L)$ and both companies choose to put small effort in P_1.

This example examine whether the inclusion of second project P_2 can give companies some incentives to increase their effort in the first project. Show that <u>when D is sufficiently large</u>, there is a subgame perfect equilibrium which two companies choose H in P_1.

☺Solution

To obtain the required subgame perfect equilibrium, we apply backward induction and first find all Nash equilibrium in the last games (project P_2).

The best responses of the companies in P_2 is marked (with upper bar) in the following matrix

		Company 2	
		E	**N**
Company 1	**E**	$(\bar{\mathbf{5}}, \bar{\mathbf{3}})$	$(-2,0)$
	N	$(0,-2)$	$(\bar{\mathbf{0}}, \bar{\mathbf{0}})$

We observe that there are two Nash equilibria [namely, (E, E) and (N, N)] in the last games.

Next, we proceed to determine the optimal strategies in the first project P_1. In order to “encourage” two companies to choose (H, H) in the first games, we take the companies’ strategies in P_2 to be

$$\left(s_1^{(2)*}(h^{(1)}), s_2^{(2)*}(h^{(1)})\right) = \begin{cases} (E, E) & if \ \left(s_1^{(1)}, s_2^{(1)}\right) = (H, H) \\ (N, N) & if \ otherwise \end{cases} \quad \dots (1)$$

In other word, each company tries to threaten another company by choosing N (not to put effort in the second project) if another company chooses to put low effort in the first project.

Given the strategic profile in P_2, the optimal strategies can be solved by considering the following matrix games:

		Company 2	
		H	**L**
Company 1	**H**	$(5 + 5D, 5 + 3D)$	$(2 + 0D, 6 + 0D)$
	L	$(6 + 0D, 2 + 0D)$	$(3 + 0D, 3 + 0D)$

In order that (H, H) is the Nash equilibrium in P_1. It must be that

- "H" is company 1's best response to company 2's strategy (H). That is,

$$5 + 5D \geq 6 \Rightarrow D \geq \frac{1}{5}.$$

- "H" is company 1's best response to company 2's strategy (H). That is,

$$5 + 3D \geq 6 \Rightarrow D \geq \frac{1}{3}.$$

So when $D \geq \max\left\{\frac{1}{3}, \frac{1}{5}\right\} = \frac{1}{3}$, $\left(s_1^{(1)*}, s_2^{(1)*}\right) = (H, H)$ is the Nash equilibrium in the first games. Hence, we conclude that $\left(s_1^{(1)*}, s_2^{(1)*}\right) = (H, H)$ and $\left(s_1^{(2)*}, s_2^{(2)*}\right)$ defined in (1) is the required subgame perfect equilibrium.

Remark of Example 23

When $D < \frac{1}{3}$, $\left(s_1^{(1)*}, s_2^{(1)*}\right) = (H, H)$ is no longer to be the Nash equilibrium in P_1. Since the long-term gain is smaller than short-term loss in P_1, some players have no incentive to deviate from the equilibrium strategy (L, L) and choose H. So the discount factor D greatly affect the equilibrium in multi-stage games.

From the above two examples, we observe that players' behavior in earlier subgames may not necessarily to be Nash equilibrium in that subgames.

One would like to ask for the conditions which such equilibrium exists. Inspired by the previous examples, some major conditions are

1. There exists two or more Nash equilibria in some subgames at later stage. This condition is necessary so that the players can "force" other players to adopt a particular strategy at earlier stage by offering some "rewards" or "penalty" in later games: Adopt different strategies for different outcomes in earlier games.
2. The value of discount factor D. Recall that the discount factor affects the players' payoff at later games, this would affect the players' incentives to deviate from the equilibrium strategies at the earlier subgames.

One-stage deviation principle

To check whether a given strategy is a subgame perfect equilibrium, one needs to check that all players act optimally in every stage (or subgames) of the games. In general, it is very tedious since the games may contain several (more than 2) subgames.

In this section, we introduce a useful shortcut to overcome this difficulty. This technique is called *one-stage deviation principle*. This principle states that if a strategic profile $\vec{s}^*$ is subgame perfect equilibrium, no players have strict incentive to deviate from the $\vec{s}^*$ and adopt other pure strategy s_i in a particular subgames (one-shot deviation).

Some notations

For any player i's strategy s_i and information set $h_i(x)$ at a particular node, we define $s_i^{(s,h_i)}$ be a strategy as

$$s_i^{(s,h_i)} = s_i^{(s,h_i)}(h) = \begin{cases} s & if \;\; h = h_i \\ s_i & if \;\; h \neq h_i \end{cases}.$$

In other words, $s_i^{(s,h_i)}$ is another player i's strategy which player i chooses to play s (instead of s_i) at a particular information set (node/stage) and strictly follows s_i in other nodes. In fact, $s_i^{(s,h_i)}$ represents a "one-stage deviation" strategy for player i.

Definition (One-stage unimprovable)

A player i's strategy s_i is said to be <u>one-stage unimprovable</u> with respect to the opponents' strategy s_{-i} if there is no information set h_i and strategy $s \in S_i$ such that $V_i\left(s_i^{(s,h_i)}; s_{-i}\right)|_{h_i} > V_i(s_i; s_{-i})|_{h_i}$. In other words, player i cannot get any additional benefit by adopting any one-stage deviation strategy.

For any subgame perfect equilibrium s^* and all players act optimally at every subgames, it is obvious that no players have incentive to deviate from $\vec{s}^*$ and adopt one-shot deviation strategy at any stage. Thus s_i^* is always one-stage unimprovable with respect to s_{-i}^*.

How about the converse?

The following theorem shows that the converse is also true.

Theorem 2 (One-stage deviation principle)
If a strategy s_i^* is one-stage unimprovable with respect to s_{-i}^* for any player i, then the strategic profile $s^* = (s_1^*, s_2^*, \dots, s_N^*)$ is the subgame perfect equilibrium in a dynamic games.

Therefore, the concept of "one-stage unimprovable" is useful in checking whether a given strategic profile is a subgame perfect equilibrium.

Proof of Theorem 2

We need to check the strategic profile s^* is a Nash equilibrium at any stage of the multi-stage games. That is,

$$V_i(s_i^*; s_{-i}^*)|_{h_i^{(k)}(x)} > V_i(s_i; s_{-i}^*)|_{h_i^{(k)}(x)}$$

for any information set $h_i^{(k)}(x)$. The above inequality is equivalent to

$$V_i\left(s_i^{(1)},\dots,s_i^{(k-1)},s_i^{(k)*},\dots,s_i^{(N)*};s_{-i}^*\right)$$
$$\geq V_i\left(s_i^{(1)},\dots,s_i^{(k-1)},s_i^{(k)},\dots,s_i^{(N)};s_{-i}^*\right),\quad k=1,2,\dots,N$$

for any past strategies $s_i^{(1)},\dots,s_i^{(k-1)}$ and any future strategies $s_i^{(k)},\dots,s_i^{(N)}$.

Since s_i^* is one-stage unimprovable, we must have

$$V_i\left(s_i^{(1)^*},\dots,s_i^{(j-1)^*},s_i^{(j)*},\dots,s_i^{(N)*};s_{-i}^*\right)|_{h_i^{(j)}(x)}$$
$$\geq V_i\left(s_i^{(1)^*},\dots,s_i^{(j-1)^*},s_i^{(j)},s_i^{(j+1)*},\dots,s_i^{(N)};s_{-i}^*\right)|_{h_i^{(j)}(x)}$$

for any $j=1,2,\dots,N$, $s_i^{(j)}\in S_i$ and any information set $h_i^{(j)}(x)$.

Suppose that x is the node at j^{th} stage that is reached by the strategies $s_i^{(1)},\dots,s_i^{(j-1)}$ (any player i's strategy) and s_{-i}^*, then the above inequaity can be rewritten as

$$V_i\left(s_i^{(1)}, \dots, s_i^{(j-1)}, s_i^{(j)*}, \dots, s_i^{(N)*}; s_{-i}^*\right)$$
$$\geq V_i\left(s_i^{(1)}, \dots, s_i^{(j-1)}, s_i^{(j)}, s_i^{(j+1)*}, \dots, s_i^{(N)*}; s_{-i}^*\right)$$

for any $s_i^{(1)}, \dots, s_i^{(j-1)}$.

Using the inequality, we establish that

$$V_i\left(s_i^{(1)}, \dots, s_i^{(k-1)}, s_i^{(k)*}, s_i^{(k+1)*}, \dots, s_i^{(N)*}; s_{-i}^*\right)$$
$$\overbrace{\geq}^{j=k} V_i\left(s_i^{(1)}, \dots, s_i^{(k-1)}, s_i^{(k)}, s_i^{(k+1)*}, \dots, s_i^{(N)*}; s_{-i}^*\right)$$
$$\overbrace{\geq}^{j=k+1} V_i\left(s_i^{(1)}, \dots, s_i^{(k-1)}, s_i^{(k)}, s_i^{(k+1)}, s_i^{(k+2)*} \dots, s_i^{(N)*}; s_{-i}^*\right)$$
$$\overbrace{\geq}^{j=k+2} \dots \overbrace{\geq}^{j=N} V_i\left(s_i^{(1)}, \dots, s_i^{(k-1)}, s_i^{(k)}, \dots, s_i^{(N)}; s_{-i}^*\right).$$

This implies that s_i^* is the best response to s_{-i}. Thus s^* is the Nash equilibrium at k^{th} stage.