Chapter 4 Non-stationary TS Models

4.1. **Nonstationarity in Mean**

4.1.1. Deterministic Trend Models

Let $\{x_t\}$ be a sequence of stationary time series.

Z_t is called a deterministic trend model, if

$$Z_t = \alpha_0 + \alpha_1 t + x_t, \quad \alpha_1 \neq 0.$$

Z_t is not stationary.

Feature: If Z_t is a deterministic trend model, then after transformed:

$$\dot{Z}_t = Z_t - \alpha_0 - \alpha_1 t.$$

Then $\dot{Z}_t = x_t$ and hence $\dot{Z}_t$ is stationary.

Other Deterministic Trend Models:

$$Z_t = \alpha_0 + \alpha_1 t + \alpha_1 t^2 + x_t,$$
$$Z_t = \gamma_0 + \gamma_1 \cos(t\omega + \theta) + x_t,$$
$$Z_t = \gamma_0 + \sum_{j=1}^{m} (\alpha_j \cos t\omega_j + \beta \sin(t\omega_j) + x_t.$$

4.1.2. Stochastic Trend Models

Let $\{x_t\}$ ba a sequence of stationary time series.

Z_t is called a Stochastic Trend Model, if

$Z_t = Z_{t-1} + x_t, \qquad$ or $\qquad (1 - B)Z_t = x_t$

Z_t is not stationary.

In particular, when $x_t = a_t$,

$Z_t = Z_{t-1} + a_t$ is a random walk.

Feature: If Z_t is a Stochastic Trend Model, then after differenced:

$$\dot{Z}_t = Z_t - Z_{t-1}$$

$\dot{Z}_t$ is stationary.

General Stochastic Trend Models:

$$(1 - B)^d Z_t = x_t, \qquad d \geq 1.$$

Let $\dot{Z}_t = (1 - B)^d Z_t$. Then $\dot{Z}_t$ is stationary.

4.2. **Autoregressive Integrated Moving-average Model**

4.2.1. The General ARIMA Model

Let Z_t is a General Stochastic Trend Model:

$$(1 - B)^d Z_t = x_t, \qquad d \geq 1.$$

If x_t is a weakly stationary and invertible ARMA model,

$$\phi_p(B) x_t = \theta_q(B) a_t\,,$$

where $\phi_p(B)$ and $\theta_q(B)$ have no common roots. Then:

$$\phi_p(B)(1 - B)^d Z_t = \theta_q(B) a_t\,,$$

Z_t is called ARIMA(p, d, q) model.(?)

If x_t is the following ARMA model,

$$\phi_p(B) x_t = \theta_0 + \theta_q(B) a_t\,,$$

then,

$$\phi_p(B)(1 - B)^d Z_t = \theta_0 + \theta_q(B) a_t\,,$$

Z_t is called ARIMA(p, d, q) model.

4.2.2. The Random Walk model

ARIMA(0, 1, 0) model:

$(1-B)Z_t = a_t$ or

$Z_t = Z_{t-1} + a_t$ (random walk)

$Z_t = \theta_0 + Z_{t-1} + a_t$

—- called the random walk with drift.

Example 4.1: Simulated 100 values from

$$(1-B)Z_t = a_t\,,$$

and

$$(1-B)Z_t = 4 + a_t\,,$$

Show the sample ACF and PACF.

4.2.3. The ARIMA(0, 1, 1) or IMA(1, 1) Model

ARIMA(0, 1, 1) model:

$$(1-B)Z_t = (1-\theta B)a_t\,.$$

or

$$Z_t = Z_{t-1} - \theta a_{t-1} + a_t\,,$$

where $|\theta| < 1$.

Expansion:

$$a_t = Z_t - \alpha \sum_{j=1}^{\infty} (1-\alpha)^{j-1} Z_{t-j}.$$

or

$$Z_t = \alpha \sum_{j=1}^{\infty} (1-\alpha)^{j-1} Z_{t-j} + a_t\,,$$

where $\alpha = 1 - \theta$.

Example 4.2: Simulated 100 values from three models:

ARIMA(1, 1, 0) model:

$(1 - 0.8B)(1 - B)Z_t = a_t$,

ARIMA(0, 1, 1) model:

$(1 - B)Z_t = (1 - 0.75B)a_t$,

ARIMA(1, 1, 1) model:

$(1 - 0.9B)(1 - B)Z_t = (1 - 0.5B)a_t$,

a. Show the sample ACF and PACF.

b. Let $W_t = (1 - B)Z_t$. Show the sample ACF and PACF of W_t.