MATH4425 (T1A) – Tutorial 10

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Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

Seasonal ARIMA model

Seasonal time series models are mainly used for datasets which include seasonal components, or **seasonality**. By seasonality, we mean periodic fluctuations.

Example: Retail sales tend to peak for the Christmas season and then decline after the holidays. So time series of retail sales will typically show increasing sales from September through December and declining sales in January and February.

Pure seasonal ARMA(P, Q) model

Definition

The model **(pure) seasonal ARMA$(P, Q)_s$** is defined as

$$\Phi_P(B^s)Z_t = \Theta_Q(B^s)a_t,$$

where s is a positive integer (**seasonal period**), a_t — white noise with variance σ_a^2,

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps},$$

$$\Theta_Q(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{Qs}$$

Usually we also require $\Phi_P(z)$ and $\Theta_Q(z)$ to have no common roots for **model identifiability**.

Note: Choice of s depends on the problem setting. Usually it might be 4 or 12.

Properties

If all roots of $\Phi_P(z)$ and $\Theta_Q(z)$ lie outside the unit circle, the ARMA$(P, Q)_s$ is **stationary** and **invertible** with $\mathbb{E}Z_t =: \mu_Z$, $\mathbb{E}(Z_t - \mu_Z)^2 =: \sigma_Z^2$.

The **between period (seasonal) correlations** of Z_t are given as

$$\rho_{js} = \frac{\mathbb{E}(Z_{t+js} - \mu_Z)(Z_t - \mu_Z)}{\sigma_Z^2}$$

Example

Let

$$(1 - 0.9B^{12})Z_t = a_t,$$

where $a_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_a^2)$.

Let's calculate expectation first:

$$Z_t = (1 - 0.9B^{12})^{-1} a_t = \sum_{i=0}^{\infty} (0.9B^{12})^i a_t = \sum_{i=0}^{\infty} 0.9^i a_{t-12i} \Rightarrow$$

$$\mathbb{E}Z_t = \sum_{i=0}^{\infty} 0.9^i \mathbb{E}a_{t-12i} = 0$$

Now let's find variance:

$$\sigma_Z^2 = \mathbb{E}Z_t^2 = \sum_{i=0}^{\infty} 0.9^{2i} \mathbb{E}a_{t-12i}^2 = \frac{1}{1 - 0.9^2}$$

Now let's find seasonal correlations:

$$\gamma_1 = \mathbb{E}Z_t Z_{t+1} = \mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^j a_{t+1-12j}\right) = 0$$

$$\gamma_2 = \mathbb{E}Z_t Z_{t+2} = \mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^j a_{t+2-12j}\right) = 0$$

$$\vdots$$

$$\gamma_{12} = \mathbb{E}Z_t Z_{t+12} = \mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^j a_{t+12-12j}\right) =$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(a_{t+12} + \sum_{j=1}^{\infty} 0.9^j a_{t+12-12j}\right) =$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^{j+1} a_{t+12-12(j+1)}\right) = \mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^{j+1} a_{t-12j}\right) =$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^{2i+1} a_{t-12i}^2\right) = \frac{0.9}{1 - 0.9^2}$$

$$\vdots$$

$$\gamma_{12k} = \mathbb{E}Z_t Z_{t+12k} = \mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty} 0.9^j a_{t+12k-12j}\right) =$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty} 0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{k-1} 0.9^j a_{t+12k-12j} + \sum_{j=k}^{\infty} 0.9^j a_{t+12k-12j}\right) =$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty}0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty}0.9^{j+k}a_{t+12k-12(j+k)}\right)=\mathbb{E}\left(\sum_{i=0}^{\infty}0.9^i a_{t-12i}\right)\left(\sum_{j=0}^{\infty}0.9^{j+k}a_{t-12j}\right)=$$

$$\mathbb{E}\left(\sum_{i=0}^{\infty}0.9^{2i+k}a_{t-12i}^2\right)=\frac{0.9^k}{1-0.9^2}$$

$$\rho_{12}=\frac{\gamma_{12}}{\sigma_Z^2}=0.9$$

$$\vdots$$

$$\rho_{12k}=\frac{\gamma_{12k}}{\sigma_Z^2}=0.9^k$$

Pure seasonal ARIMA(P, D, Q) model

Definition

The model **(pure) seasonal ARIMA$(\boldsymbol{P}, \boldsymbol{D}, \boldsymbol{Q})_{\boldsymbol{s}}$** is defined as

$$\Phi_P(B^s)(1-B^s)^D Z_t=\Theta_Q(B^s)a_t,$$

where s is a positive integer, a_t — white noise with variance σ_a^2.

Note: Z_t is **not** stationary. Although, $W_t=(1-B^s)^D Z_t$ will be stationary if $\Phi_P(z)$ has all roots outside the unit circle.

Properties

W_t shares the same properties as pure seasonal ARMA$(P, Q)_s$.

Box-Jenkins multiplicative seasonal ARIMA model

Definition

The model **(pure) seasonal ARIMA$(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{q}) \times (\boldsymbol{P}, \boldsymbol{D}, \boldsymbol{Q})_{\boldsymbol{s}}$** is defined as

$$\Phi_P(B^s)\phi_p(B)(1-B)^d(1-B^s)^D\dot{Z}_t=\theta_q(B)\Theta_Q(B^s)a_t,$$

where s is a positive integer, a_t — white noise with variance σ_a^2,

$$\dot{Z}_t:=\begin{cases}Z_t-\mu, & \text{if } d=D=0,\\ Z_t, & \text{otherwise}\end{cases}$$

Note: $\phi_p(B), \theta_q(B)$ are called **AR and MA factors**. $\Phi_P(B), \Theta_Q(B)$ are called **seasonal AR and MA factors**.

Remark

1) $(1-B)^d\dot{Z}_t$ is **not** stationary
2) $(1-B^s)^D\dot{Z}_t$ is **not** stationary
3) $(1-B)^d(1-B^s)^D\dot{Z}_t$ **is** stationary

Example

Let
$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t,$$
where $a_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,\sigma_a^2)$.

Let's define $W_t := (1-B)(1-B^{12})Z_t$.

Let's calculate expectation first:
$$W_t = (1-\theta B)(1-\Theta B^{12})a_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13} \Rightarrow$$
$$\mathbb{E}W_t = 0$$

Now let's find variance:
$$\mathbb{E}W_t^2 = \sigma_a^2(1+\theta^2+\Theta^2+\theta^2\Theta^2) = \sigma_a^2(1+\theta^2)(1+\Theta^2)$$

Now let's find ACF:
$$\gamma_1 = \mathbb{E}W_tW_{t+1} = \mathbb{E}\left(a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\right)\left(a_{t+1} - \theta a_t - \Theta a_{t-11} + \theta\Theta a_{t-12}\right) =$$
$$(-\theta - \theta\Theta)\sigma_a^2 = -\theta(1+\Theta^2)\sigma_a^2$$
$$\gamma_2 = \mathbb{E}W_tW_{t+2} = \mathbb{E}\left(a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\right)\left(a_{t+2} - \theta a_{t+1} - \Theta a_{t-10} + \theta\Theta a_{t-11}\right) = 0$$
$$\vdots$$
$$\gamma_{11} = \mathbb{E}W_tW_{t+11} = \mathbb{E}\left(a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\right)\left(a_{t+11} - \theta a_{t+10} - \Theta a_{t-1} + \theta\Theta a_{t-2}\right) =$$
$$\theta\Theta\sigma_a^2$$
$$\gamma_{12} = \mathbb{E}W_tW_{t+12} = \mathbb{E}\left(a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\right)\left(a_{t+12} - \theta a_{t+11} - \Theta a_t + \theta\Theta a_{t-1}\right) =$$
$$(-\Theta - \theta^2\Theta)\sigma_a^2 = -\Theta(1+\theta^2)\sigma_a^2$$
$$\gamma_{13} = \mathbb{E}W_tW_{t+13} = \mathbb{E}\left(a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}\right)\left(a_{t+13} - \theta a_{t+12} - \Theta a_{t+1} + \theta\Theta a_t\right) =$$
$$\theta\Theta\sigma_a^2$$
$$\rho_1 = \frac{-\theta}{1+\theta^2}$$
$$\rho_2 = 0$$
$$\vdots$$
$$\rho_{11} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} = \rho_{13}$$
$$\rho_{12} = \frac{-\Theta}{1+\Theta^2}$$

Note: ρ_1 is equal to ρ_1 of MA(1) model $Y_t = (1-\theta)a_t$. Similarly, ρ_{12} is equal to ρ_1 of MA(1) model $Y_t = (1-\Theta)a_t$. Moreover, $\rho_{11} = \rho_{13} = \rho_1\rho_{12}$.

GARCH model

In ARMA(p,q) model

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j}$$

white noise a_t has constant variance σ_a^2. Usually, the conditional variance (conditioned on *past information*) of a_t is also σ_a^2.

However, we can develop a new model which will let conditional variance **include some *past information***. Since this model is going to be more powerful, it might improve statistical inference and forecasting.

ARCH-based models

ARCH model

The model **ARCH(r) (Autoregressive Conditional Heteroscedasticity)** is defined as

$$a_t = \eta_t \sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2,$$

where $\eta_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

Note: In finance, h_t is called **volatility at time t**.

GARCH model

The model **GARCH(r,s) (Generalized Autoregressive Conditional Heteroscedasticity)** is defined as

$$a_t = \eta_t \sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j},$$

where $\eta_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

Note: There are many related models such as Exp-GARCH, Argument-GARCH, IGARCH, TGARCH, etc.

ARMA-GARCH model

The model **ARMA(p,q)-GARCH(r,s)** is defined as

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j},$$

$$a_t = \eta_t \sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j},$$

where $\eta_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

ARIMA-GARCH model

The model **ARIMA(p,d,q)-GARCH(r,s)** is defined as

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t,$$

$$a_t = \eta_t\sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r}\alpha_i a_{t-i}^2 + \sum_{j=1}^{s}\beta_j h_{t-j},$$

where $\eta_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

Basic Properties of GARCH model

Let's consider GARCH model

$$a_t = \eta_t\sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r}\alpha_i a_{t-i}^2 + \sum_{j=1}^{s}\beta_j h_{t-j},$$

where $\alpha_0 > 0$, $\alpha_i, \beta_j \geq 0$.

In case **GARCH(1, 1)** the definition reduces to

$$a_t = \eta_t\sqrt{h_t},$$

$$h_t = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 h_{t-1}$$

Then

$$\mathbb{E}a_t = \mathbb{E}\mathbb{E}(a_t \mid F_{t-1}) = \mathbb{E}\mathbb{E}(\eta_t\sqrt{h_t} \mid F_{t-1}) = \mathbb{E}[\sqrt{h_t}\mathbb{E}(\eta_t \mid F_{t-1})] = \mathbb{E}[\sqrt{h_t}\mathbb{E}\eta_t] = 0$$
$$\mathbb{E}(a_t^2 \mid F_{t-1}) = \mathbb{E}(\eta_t^2 h_t \mid F_{t-1}) = h_t\mathbb{E}(\eta_t^2 \mid F_{t-1}) = h_t\mathbb{E}\eta_t^2 = h_t$$
$$\mathbb{E}a_t^2 = \mathbb{E}\mathbb{E}(a_t^2 \mid F_{t-1}) = \mathbb{E}h_t$$

Moreover, the model has the solution

$$h_t = \alpha_0\left[1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j}(\alpha_1\eta_{t-i}^2 + \beta_1)\right]$$

if and only if

$$\mathbb{E}\ln(\alpha_1\eta_t^2 + \beta_1) < 0$$

In this case the solution is unique, **strictly stationary**, and ergodic.

The necessary and sufficient condition for **strict stationarity and finite variance** is

$$\sum_{i=1}^{r}\alpha_i + \sum_{j=1}^{s}\beta_j < 1,$$

which is a stronger condition than $\mathbb{E}\ln(\alpha_1\eta_t^2 + \beta_1) < 0$.

Here is some intuition behind these conditions:

$$h_t = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 h_{t-1} = \alpha_0 + \alpha_1\eta_{t-1}^2 h_{t-1} + \beta_1 h_{t-1} = \alpha_0 + (\alpha_1\eta_{t-1}^2 + \beta_1)h_{t-1}$$

$$h_{t-1} = \alpha_0 + (\alpha_1 \eta_{t-2}^2 + \beta_1) h_{t-2} \Rightarrow$$

$$h_t = \alpha_0 + (\alpha_1 \eta_{t-1}^2 + \beta_1)(\alpha_0 + (\alpha_1 \eta_{t-2}^2 + \beta_1) h_{t-2}) =$$

$$\alpha_0 + \alpha_0(\alpha_1 \eta_{t-1}^2 + \beta_1) + (\alpha_1 \eta_{t-1}^2 + \beta_1)(\alpha_1 \eta_{t-2}^2 + \beta_1) h_{t-2} =$$

$$\alpha_0 + \alpha_0(\alpha_1 \eta_{t-1}^2 + \beta_1) + \alpha_0(\alpha_1 \eta_{t-1}^2 + \beta_1)(\alpha_1 \eta_{t-2}^2 + \beta_1) + \cdots + \alpha_0 \prod_{i=1}^{m} (\alpha_1 \eta_{t-i}^2 + \beta_1) + \prod_{i=1}^{m+1} (\alpha_1 \eta_{t-i}^2 + \beta_1) h_{t-m-1} =$$

$$\alpha_0 \left[1 + \sum_{j=1}^{m} \prod_{i=1}^{j} (\alpha_1 \eta_{t-i}^2 + \beta_1) \right] + \prod_{i=1}^{m+1} (\alpha_1 \eta_{t-i}^2 + \beta_1) h_{t-m-1}$$

Now let $m \to \infty$ and explore the term $\prod_{i=1}^{j} (\alpha_1 \eta_{t-i}^2 + \beta_1)$:

$$\prod_{i=1}^{j} (\alpha_1 \eta_{t-i}^2 + \beta_1) = \exp\left(\ln \prod_{i=1}^{j} (\alpha_1 \eta_{t-i}^2 + \beta_1) \right) = \exp\left(\sum_{i=1}^{j} \ln(\alpha_1 \eta_{t-i}^2 + \beta_1) \right)$$

Notice that $X_i := \ln(\alpha_1 \eta_{t-i}^2 + \beta_1)$ are i.i.d. So, $\mathbb{E} X_i = \mathbb{E} \ln(\alpha_1 \eta_{t-i}^2 + \beta_1) = \mu$.

According to Law of Large Numbers, $\frac{1}{j} \sum_{i=1}^{j} X_i \longrightarrow \mu$ meaning that for large j we can assume $\frac{1}{j} \sum_{i=1}^{j} X_i \approx \mu$.

So,

$$\sum_{j=1}^{\infty} \exp\left(\sum_{i=1}^{j} \ln(\alpha_1 \eta_{t-i}^2 + \beta_1) \right) = \sum_{j=1}^{\infty} \exp\left(\sum_{i=1}^{j} X_i \right) \approx \sum_{j=1}^{\infty} \exp(j\mu) = \begin{cases} const, & \text{if } \mu < 0 \\ +\infty, & \text{otherwise} \end{cases}$$

Note: This explanation **cannot** be treated as a rigorous proof.

According to this expansion,

$$\mathbb{E} a_t^2 = \mathbb{E} h_t = \alpha_0 \left[1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} (\alpha_1 \mathbb{E} \eta_{t-i}^2 + \beta_1) \right] = \alpha_0 \left[1 + \sum_{j=1}^{\infty} (\alpha_1 + \beta_1)^j \right] = \alpha_0 \sum_{j=0}^{\infty} (\alpha_1 + \beta_1)^j =$$

$$\begin{cases} \frac{\alpha_0}{1 - \alpha_1 - \beta_1}, & \text{if } \alpha_1 + \beta_1 < 1 \\ +\infty, & \text{otherwise} \end{cases}$$