1. Implement CG algorithm to solve linear systems in which A is the Hilbert matrix, whose elements are $A(i,j) = \frac{1}{i+j-1}$. Set the right-hand-side to $b = (1, 1, ..., 1)^T$ and the initial point to $x_0 = 0$. Try dimensions $n = 5, 8, 12, 20$ and show the performance of residual with respect to iteration numbers to reduce the residual below 10^{-6}.

2. Derive Preconditioned CG Algorithm by applying the standard CG method in the variables $\hat{x}$ and transforming back into the original variables x to see the expression of precondtioner M.

3. Try to prove that when $\phi = \phi_k^c = \frac{1}{1-\mu_k}$ where $\mu_k = \frac{(s_k^T B_k s_k)(y_k^T H_k y_k)}{(s_k^T y_k)^2}$, the Broyden class

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} + \phi_k (s_k^T B_k s_k) v_k v_k^T$$

where

$$v_k = \Big(\frac{y_k}{y_k^T s_k} - \frac{B_k s_k}{s_k^T B_k s_k}\Big)$$

becomes sigular.

4. Using BFGS method to minimize the extended Rosenbrock function

$$f(x) = \sum_{i=1}^{n-1} [100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2],$$

with $x_0 = [-1.2, 1, \cdots, -1.2, 1]^T$, $x^* = [1, 1, \cdots, 1, 1]^T$ and $f(x^*) = 0$. Try different $n = 6, 8, 10$ and $\epsilon = 10^{-5}$. Moreover, using BFGS method to minimize the Powell singular function

$$f(x) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4,$$

with $\epsilon = 10^{-5}$, $x_0 = [3, -1, 0, 1]^T$, $x^* = [0, 0, 0, 0]$ and $f(x^*) = 0$.