MATH4425 (T1A) – Tutorial 4

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Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

1 Time series models

Let $\dots, Z_{-t}, \dots, Z_{-1}, Z_0, Z_1, \dots, Z_t, \dots$ be a sequence of TS r.v.

How to describe the relationship between Z_t and the past data $Z_{t-1}, Z_{t-2}, \dots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \dots) + a_t$$

It is called the **time series model**.

1. AR(p) model:
$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$
where ϕ_p is a constant and called the **parameter**, and p is called the **order** of the AR(p) model
2. $AR(\infty)$ model:
$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t$$
3. Moving-average (MA) model:
$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots.$$
4. ARMA model
5. ARIMA model

2 Stationary time series models. Autoregressive Moving-average ARMA(p, q) Model

2.1 The general ARMA(p, q) process

2.1.1 Model

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be **an ARMA(p, q) process**, if

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} + a_t,$$

where p and q are positive integers.

(p, q) is called **the order** or **lag** of the process.

The ARMA(p, q) process can be written as

$$\phi_p(B)\dot{Z}_t = \theta_q(B)a_t,$$

where

$$\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$
$$\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q.$$

2.1.2 Condition for identifiability

If $\phi_p(z)$ and $\theta_q(z)$ have **no common roots**, then the process can be identified **uniquely**. Otherwise, the order of ARMA process can be reduced.

2.1.3 Condition for stationarity

All roots of $\phi_p(z) = 0$ lie outside the unit circle. Then

$$\phi_p(B)^{-1}\theta_q(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$$

$$\dot{Z}_t = \left(1 + \sum_{i=1}^{\infty} \psi_i B^i\right) a_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots,$$

where $\psi_i = O(h^i)$ with $|h| < 1$.

2.1.4 Condition for invertibility

All roots of $\theta_q(z) = 0$ lie outside the unit circle. Then

$$\theta_q(B)^{-1}\phi_p(B) = 1 - \sum_{i=1}^{\infty} \pi_i B^i$$

$$a_t = \left(1 - \sum_{i=1}^{\infty} \pi_i B^i\right) \dot{Z}_t = \dot{Z}_t - \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \ldots,$$

where $\pi_i = O(h^i)$ with $|h| < 1$.

2.1.5 ACF of ARMA(p, q) process

Here, we assume that stationarity condition is fulfilled. Then

$$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}, \qquad k \geq \max(p, q+1)$$
$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}, \qquad k \geq \max(p, q+1)$$

with initial conditions $k < \max(p, q+1)$

$$\gamma_k - \phi_1 \gamma_{k-1} - \cdots - \phi_p \gamma_{k-p} = \sigma_a^2(\theta_k \psi_0 + \theta_{k+1}\psi_1 + \cdots + \theta_q \psi_{q-k})$$

Note:

1. ρ_{q+1}, ρ_{q+2}, ... **do not directly depend** on the coefficients in the MA part
2. ρ_1, ρ_2, ..., ρ_q **depend** on the coefficients in both AR and MA parts
3. $\rho_k = O(h^k)$, where $|h| < 1$, i.e. ρ_k goes down to zero, **exponentially**

2.1.6 PACF of ARMA(p, q) process

ϕ_{kk} can be obtained from $\rho_1, \rho_2, \ldots, \rho_k$.

Note: $\phi_{kk} = O(h^k)$, where $|h| < 1$, i.e., ϕ_{kk} goes down to zero, **exponentially**.

2.2 The ARMA(1, 1) process

2.2.1 Model

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t - \theta_1 a_{t-1},$$

or, equivalently,

$$(1 - \phi_1 B)\dot{Z}_t = (1 - \theta_1 B)a_t$$

2.2.2 Condition for stationarity

The root of $\phi(z)$ should lie outside the unit circle $\Longrightarrow |\phi_1| < 1$. Then

$$(1 - \phi_1 B)^{-1}(1 - \theta_1 B) = 1 + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} B^i$$

$$\dot{Z}_t = \left[1 + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} B^i\right] a_t = a_t + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} a_{t-i}$$

$\Longrightarrow$ MA(∞) process.

Some properties of general MA process $\boldsymbol{Z_t = \mu + \sum_{j=0}^{\infty} \theta_j a_{t-j}}$:

$$\mathbb{E}Z_t = \mu < \infty$$

$$\text{var}(Z_t) = \sigma_a^2 \sum_{j=0}^{\infty} \theta_j^2 < \infty$$

$$\mathbb{E}(a_t Z_{t-j}) = \begin{cases} \sigma_a^2, & \text{if } j = 0 \\ 0, & \text{if } j > 0 \end{cases}$$

$$\gamma_k = \mathbb{E}(\dot{Z}_t \dot{Z}_{t-k}) = \sigma_a^2 \sum_{i=0}^{\infty} \theta_i \theta_{i+k}$$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \theta_k \theta_{i+k}}{\sum_{j=0}^{\infty} \theta_j^2}.$$

2.2.3 Condition for invertibility

The root of $\theta(z)$ should lie outside the unit circle $\Longrightarrow |\theta_1| < 1$. Then

$$(1 - \theta_1 B)^{-1}(1 - \phi_1 B) = 1 + (\theta_1 - \phi_1) \sum_{i=1}^{\infty} \theta_1^{i-1} B^i$$

$$a_t = \left[1 + (\theta_1 - \phi_1) \sum_{i=1}^{\infty} \theta_1^{i-1} B^i\right] \dot{Z}_t$$

2.2.4 Properties of ARMA(1, 1) process

Again, we assume that stationarity condition is fulfilled. Then

$$\mathbb{E}\dot{Z}_t = 0$$

$$\gamma_0 = \sigma_a^2 \left[1 + (\phi_1 - \theta_1)^2 \sum_{i=1}^{\infty} \phi_1^{2(i-1)}\right] = \sigma_a^2 \left[1 + \frac{(\phi_1 - \theta)^2}{1 - \phi_1^2}\right]$$

$$\gamma_1 = \mathbb{E}\dot{Z}_t\dot{Z}_{t+1} = \mathbb{E}\left[\dot{Z}_t(\phi_1\dot{Z}_t + a_{t+1} - \theta_1 a_t)\right] = \phi_1\gamma_0 - \theta_1\sigma_a^2$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \phi_1 - \frac{\theta_1\sigma_a^2}{\gamma_0}$$

$$\gamma_2 = \mathbb{E}\dot{Z}_t\dot{Z}_{t+2} = \mathbb{E}\left[\dot{Z}_t(\phi_1\dot{Z}_{t+1} + a_{t+2} - \theta_1 a_{t+1})\right] = \phi_1\gamma_1$$

$$\gamma_k = \phi_1\gamma_{k-1}, \qquad k \geq 2$$

$$\rho_k = \phi_1\rho_{k-1} = \cdots = \phi^{k-1}\rho_1$$

Note:

1. ρ_1 **depends** on ϕ_1, θ_1
2. ρ_k $(k \geq 2)$ **do not directly depend** on θ_1
3. ρ_k goes down to zero, **exponentially**

2.3 AR, MA and ARMA process with drift

2.3.1 The AR(p) process with drift

Let Z_t be an AR process:

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$

where all roots of $1 - \sum_{i=1}^{p} \phi_i z^i = 0$ lie outside the unit circle.

When $\theta_0 = 0$,

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t$$

When $\theta_0 \neq 0$, let $\mu = \frac{\theta_0}{1-\phi_1-\cdots-\phi_p}$. Then we have

$$Z_t - \mu = \phi_1(Z_{t-1} - \mu) + \cdots + \phi_p(Z_{t-p} - \mu) + a_t$$

If $\dot{Z}_t = Z_t - \mu$, then

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \cdots + \phi_p\dot{Z}_{t-p} + a_t$$

Properties of AR(p) process with drift

1. $\mathbb{E}Z_t = \mu$
2. Condition for stationarity, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$

2.3.2 MA(q) process with drift

Let Z_t be an MA process:

$$Z_t = \theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t$$

When $\theta_0 = 0$,

$$Z_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t$$

When $\theta_0 \neq 0$, let $\mu = \theta_0$. Then we have

$$Z_t - \mu = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

If $\dot{Z}_t = Z_t - \mu$, then

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

Properties of MA(q) process with drift

1. $\mathbb{E}Z_t = \mu$
2. Condition for invertibility, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$

2.3.3 The ARMA(p, q) process with drift

Let Z_t be an ARMA(p, q) process:

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

where all roots of $1 - \sum_{i=1}^{p} \phi_1 z^i = 0$ lie outside the unit circle.

When $\theta_0 = 0$, then

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

When $\theta_0 \neq 0$, let $\mu = \frac{\theta_0}{1-\phi_1-\cdots-\phi_p}$. Then we have

$$Z_t - \mu = \phi_1(Z_{t-1} - \mu) + \cdots + \phi_p(Z_{t-p} - \mu) + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

If $\dot{Z}_t = Z_t - \mu$, then

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

Properties of ARMA(p, q) process with drift

1. $\mathbb{E}Z_t = \mu$
2. Conditions for stationarity and invertibility, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$

3 Non-stationary time series models

3.1 Non-stationarity in mean

3.1.1 Deterministic trend models

Let $\{x_t\}$ be a sequence of *stationary* time series.

Z_t is called **a deterministic trend model**, if

$$Z_t = f(t) + x_t, \quad \alpha_1 \neq 0.$$

Note: Z_t is **not** stationary. However, after the following transformation

$$\dot{Z}_t := Z_t - f(t) = x_t,$$

$\dot{Z}_t$ becomes stationary.

Deterministic trend models:

- $Z_t = \alpha_0 + \alpha_1 t + x_t$
- $Z_t = \alpha_0 + \alpha_1 t + \alpha_1 t^2 + x_t$
- $Z_t = \gamma_0 + \gamma_1 \cos(t\omega + \theta) + x_t$
- $Z_t = \gamma_0 + \sum_{j=1}^{m} [\alpha_j \cos(t\omega_j) + \beta \sin(t\omega_j)] + x_t$

3.1.2 Stochastic trend models

Let $\{x_t\}$ be a sequence of *stationary* time series.

Z_t is called **a stochastic trend model**, if

$$Z_t = Z_{t-1} + x_t, \quad \text{or} \quad (1-B)Z_t = x_t$$

Note: Z_t is **not** stationary. In particular, when $x_t = a_t$,

$$Z_t = Z_{t-1} + a_t$$

is the random walk.

However, after differenced

$$\dot{Z}_t := Z_t - Z_{t-1} = x_t,$$

$\dot{Z}_t$ becomes stationary.

General stochastic trend models:

$$(1-B)^d Z_t = x_t, \qquad d \geq 1$$

After d times differenced

$$\dot{Z}_t := (1-B)^d Z_t = x_t,$$

$\dot{Z}_t$ becomes stationary.

3.2 The autoregressive integrated moving-average ARIMA process

3.2.1 The general ARIMA process

Let Z_t is *a general stochastic trend model*:

$$(1-B)^d Z_t = x_t, \qquad d \geq 1$$

If x_t is a *stationary* and *invertible* ARMA process

$$\phi_p(B) x_t = \theta_q(B) a_t,$$

where $\phi_p(B)$ and $\theta_q(B)$ have *no common* roots. Then:

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t,$$

Z_t is called **ARIMA(p, d, q) process without constant**.

If x_t is the following ARMA process,

$$\phi_p(B)x_t = \theta_0 + \theta_q(B)$$

Then

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t,$$

Z_t is called **ARIMA(p, d, q) process with constant**.

3.2.2 The random walk model

ARIMA(0, 1, 0) process without constant

$$(1-B)Z_t = a_t,$$

or, equivalently,

$$Z_t = Z_{t-1} + a_t$$

is the random walk.

ARIMA(0, 1, 0) process with constant

$$(1-B)Z_t = \theta_0 + a_t,$$

or, equivalently,

$$Z_t = \theta_0 + Z_{t-1} + a_t$$

is the random walk with drift.

Properties of the random walk (with drift)

Z_0 is usually a deterministic, fixed number. Let's assume this holds. Then

1. $Z_t = \theta_0 + a_t + Z_{t-1} = \theta_0 + a_t + \theta_0 + a_{t-1} + Z_{t-2} = \theta_0 + a_t + \theta_0 + a_{t-1} + \cdots + \theta_0 + a_1 + Z_0 = t\theta_0 + \sum_{i=1}^t a_i + Z_0$
2. $\mathbb{E}Z_t = t\theta_0 + Z_0$
3. $\mathrm{var}(Z_t) = t\sigma_a^2$
4. $\mathrm{cov}(Z_t, Z_{t-k}) = (t-k)\sigma_a^2$
5. $\mathrm{corr}(Z_t, Z_{t-k}) = \dfrac{\mathrm{cov}(Z_t, Z_{t-k})}{\sqrt{\mathrm{var}(Z_t)}\sqrt{\mathrm{var}(Z_{t-k})}} \approx 1, \quad$ if t is large enough

3.2.3 Other examples of the simple ARIMA models

- **ARIMA(1, 0, 0)** (without/with constant) = AR(1) (without/with drift)

$$(1-\phi_1 B)Z_t = \theta_0 + a_t,$$

  where $|\phi_1| < 1$

- **ARIMA(1, 1, 0)** (without/with constant) = differenced AR(1) (without/with drift)

$$(1-\phi_1 B)(1-B)Z_t = \theta_0 + a_t \Leftrightarrow$$

$$(1-\phi_1 B)(Z_t - Z_{t-1}) = \theta_0 + a_t,$$

where $|\phi_1| < 1$

- **ARIMA(0, 1, 1) without constant** $\sim$ simple exponential smoothing

$$(1-B)Z_t = (1-\theta_1 B)a_t \Leftrightarrow$$

$$Z_t = Z_{t-1} + a_t - \theta_1 a_{t-1},$$

where $|\theta_1| < 1$.

If we define $a_t := Z_t - \hat{Z}_t$ and $\hat{Z}_{t+1} := Z_t - \theta_1 a_t$, then

$$\hat{Z}_t = Z_{t-1} - \theta_1(Z_{t-1} - \hat{Z}_{t-1}) \Leftrightarrow$$

$$\hat{Z}_t = Z_{t-1} - \hat{Z}_{t-1} - \theta_1 Z_{t-1} + \theta_1 \hat{Z}_{t-1} + \hat{Z}_{t-1} \Leftrightarrow$$

$$\hat{Z}_t = (1-\theta_1)(Z_{t-1} - \hat{Z}_{t-1}) + \hat{Z}_{t-1} \Leftrightarrow$$

$$\hat{Z}_t = (1-\theta_1)Z_{t-1} + \theta_1 \hat{Z}_{t-1}$$

- **ARIMA(0, 1, 1) with constant** $\sim$ simple exponential smoothing with growth

$$(1-B)Z_t = \theta_0 + (1-\theta_1 B)a_t \Leftrightarrow$$

$$Z_t = Z_{t-1} + \theta_0 + a_t - \theta_1 a_{t-1},$$

where $|\theta_1| < 1$.

Now we define $a_t := Z_t - \hat{Z}_t$ and $\hat{Z}_{t+1} := \theta_0 + Z_t - \theta_1 a_t$. Then we get similar result.

Expansion of ARIMA(0, 1, 1):

$$Z_t = (1-\theta_1)\sum_{j=1}^{\infty} \theta_1^{j-1} Z_{t-j} + a_t$$

$\Longrightarrow$ ARIMA(0, 1, 1) is not stationary but invertible.