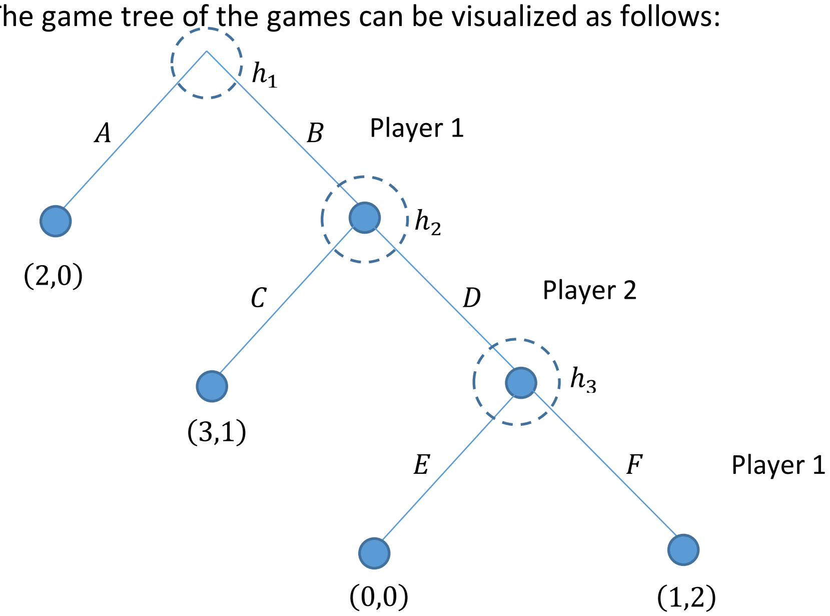
**MATH4321 Game Theory (2023 Spring)**
**Suggested Solution of Problem Set 2**

**(*Note: I skipped the solution of Problem 5. Also there is no Problem 13 in the problem set.)**

**Problem 1**

**(a)** The game tree of the games can be visualized as follows:




**(b)** We let $s_1 = (s_1(h_1), s_1(h_3))$ and $s_2 = s_2(h_2)$ be the strategies of two players. Here $s_i(h)$ denotes player $i$'s strategy when he/she is at information set $h$. The payoff matrix of the players is given as follows:

| | | Player 2's strategy | |
|---|---|---|---|
| | | $C$ | $D$ |
| Player 1's strategy | $(A, E)$ | (2,0) | (2,0) |
| | $(A, F)$ | (2,0) | (2,0) |
| | $(B, E)$ | (3,1) | (0,0) |
| | $(B, F)$ | (3,1) | (1,2) |

To find the Nash equilibrium, we find the best response of each player. We summarize the result below (the best response is highlighted by upperbar)

| | | Player 2's strategy | |
|---|---|---|---|
| | | $C$ | $D$ |
| Player 1's strategy | $(A, E)$ | $(2, \bar{0})$ | $(\bar{2}, \bar{0})$ |
| | $(A, F)$ | $(2, \bar{0})$ | $(\bar{2}, \bar{0})$ |
| | $(B, E)$ | $(\bar{3}, \bar{1})$ | $(0,0)$ |
| | $(B, F)$ | $(\bar{3}, 1)$ | $(1, \bar{2})$ |

We can conclude from the above analysis that there are 3 Nash equilibria in this case. That is, $(s_1, s_2) = \big((A, E), D\big)$, $(s_1, s_2) = \big((A, F), D\big)$ and $(s_1, s_2) = \big((B, E), C\big)$.

**(c)** We consider the 3rd stage (information node $h_3$). One can see that the player 1 is optimal to choose $F$ (yielding a payoff of 1) instead of $E$ (yielding a payoff of 0). Thus we deduce that $(s_1, s_2) = \big((A, E), D\big)$ and $(s_1, s_2) = \big((B, E), C\big)$ are not sequentially rational Nash equilibrium since $s_1(h_3) = E \neq F$ in these two equilibria.

It remains to consider the equilibrium $(s_1, s_2) = \big((A, F), D\big)$ which $s_1(h_3) = F$. We consider the 2nd stage (information set $h_2$):

- If player 2 chooses C, his payoff will be 1.
- If player 2 chooses D, then player 1 will choose F and the corresponding payoff will be 2.

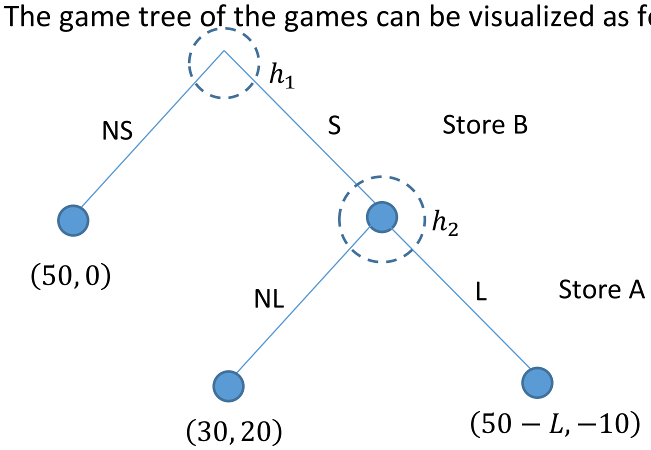
Therefore, player 2 is optimal to choose D so that the optimality at $h_2$ is verified. Finally, we consider the 1st stage (information set $h_1$)

- If player 1 chooses A, his payoff will be 2.
- If player 1 chooses B, then player 1 will choose F and player 2 will choose $D$ in future so that the corresponding payoff will be 1.

So it is optimal for player 1 to choose A and the optimality at $h_1$ is verified. Therefore $(s_1, s_2) = \big((A, F), D\big)$ is the desired sequentially rational Nash equilibrium.

**Problem 2**

**(a)** The game tree of the games can be visualized as follows:




(*Here, "S" and "NS" denotes "Sell the smartphone" and "Not sell the smartphone" respectively. "L" and "NL" denotes "Lower the selling price" and "Not lower the selling price" respectively.)

**(b)** We shall determine the Nash equilibrium through backward induction.

- We first consider the 2nd stage (information set $h_2$). Note that
  - The Store A will get 30 if it chooses NL and
  - The Store A will get $50 - L$ if it chooses $L$.

- As $30 > 50 - L \Leftrightarrow L > 20$, so we conclude that store A will choose NL when $L > 20$ and choose L if $L < 20$ (When $L = 20$, the store A is indifference between two options)
- Next, we consider the 1st stage (information set $h_1$). We shall consider the following three cases:
  - Case 1: $L < 20$.
    - If store B chooses NS, it will get a payoff of $0$.
    - If store B chooses S and store A will choose L afterwards, then the payoff of store B will be $-10$.

    Thus, it is optimal for store B to choose NS. So the sequentially rational Nash equilibrium will be $(s_A, s_B) = (L, NS)$.
  - Case 2: $L > 20$.
    - If store B chooses NS, it will get a payoff of $0$.
    - If store B chooses S and store A will choose NL afterwards, then the payoff of store B will be $20$.

    Thus, it is optimal for store B to choose S. So the sequentially rational Nash equilibrium will be $(s_A, s_B) = (NL, S)$.
  - Case 3: $L = 20$ (Speical Case)

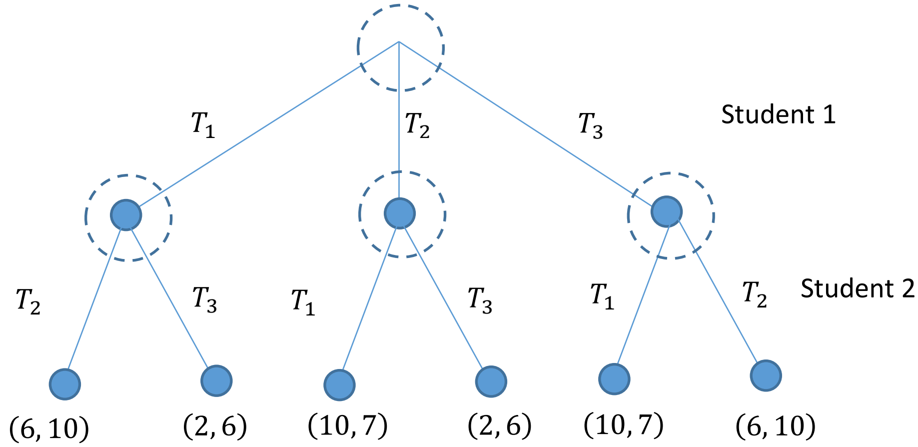
    As the store A will be indifference between choosing NL and L, so there will be two possibilities.
    - If store A chooses NL, then store B will choose $S$ according to the result of Case 2 so that the sequentially rational Nash equilibrium will be $(s_A, s_B) = (NL, S)$.
    - If store A chooses L, then store B will choose $NS$ according to the result of Case 1 so that the sequentially rational Nash equilibrium will be $(s_A, s_B) = (L, NS)$.

**Problem 3(a)**

Firstly, the whole games can be divided into two major stage:
Stage 1: Two students choose the topics simultaneously.
Stage 2: Two students takes turn to decide the final topic (i.e. Student 1 eliminate one topic and the student 2 chooses one topic from the remaining two topics) when there is no agreement in Stage 1.

To determine the subgame perfect equilibrium of the games, we shall determine the optimal strategy using backward induction.
We first consider the Stage 2 (i.e. two students do not pick the same choice in Stage 1).

Then the game tree of second stage can be described as follows (note that the players know the outcome of the first stage so the information set at initial node of stage 2 should be singteon):




We first determine student 2's optimal strategy:

- If student 1 chooses to delete $T_1$, then student 2 should choose $T_2$ (with payoff 10) instead of $T_3$ (with payoff 6).
- If student 1 chooses to delete $T_2$, then student 2 should choose $T_1$ (with payoff 7) instead of $T_3$ (with payoff 6).
- If student 1 chooses to delete $T_3$, then student 2 should choose $T_2$ (with payoff 10) instead of $T_1$ (with payoff 7).

Next, we proceed to determine student 1's optimal strategy

- If student 1 chooses to delete $T_1$, then student 2 should choose $T_2$ and student 1's payoff will be 6.
- If student 1 chooses to delete $T_2$, then student 2 should choose $T_1$ and the student 1's payoff will be 10.
- If student 1 chooses to delete $T_3$, then student 2 should choose $T_2$ and the student 1's payoff will be 6.

Therefore it is optimal for student 1 to delete $T_2$ (so the student 2 chooses $T_1$) and achieve a maximum payoff of 10. So the payoffs of the players is $(10, 7)$.

Next, we proceed to determine the optimal strategy of two students in the first stage. Note that the games will proceed to second stage if two students do not pick the same choice and players will receive $(10,7)$. Then the payoff matrix of 1st stage is given as follows:

| | | Player 2's strategy | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| Player 1's strategy | $T_1$ | $(10, 7)$ | $(10, 7)$ | $(10, 7)$ |
| | $T_2$ | $(10, 7)$ | $(6, 10)$ | $(10, 7)$ |
| | $T_3$ | $(10, 7)$ | $(10, 7)$ | $(2,6)$ |

To determine the equilibrium strategy, we determine the best response for each student. We summarize the result as follows (the best response is highlighted by upper bar)

| | | Player 2's strategy | | |
|---|---|---|---|---|
| | | $T_1$ | $T_2$ | $T_3$ |
| Player 1's strategy | $T_1$ | $(\overline{10},\overline{7})$ | $(\overline{10},\overline{7})$ | $(\overline{10},\overline{7})$ |
| | $T_2$ | $(\overline{10},7)$ | $(6,\overline{10})$ | $(\overline{10},7)$ |
| | $T_3$ | $(\overline{10},\overline{7})$ | $(\overline{10},\overline{7})$ | $(2,6)$ |

Therefore, the equilibria are $(T_1,T_1),(T_1,T_2),(T_1,T_3),(T_3,T_1),(T_3,T_2)$. In fact, the final outcome will be that two students will work on topic 1 in these 5 equilibria (Recall that two players will pick $T_1$ in the second stage if they do not pick the same choice in stage 1).

**Problem 3(b)**

Suppose that the roles of student 1 and student 2 are reversed, then one can compute the equilibrium using similar methods used in (a) (I omitted the details here). Here is the summary of the results:

- In 2nd stage, student 1 will pick $T_2$ if student 2 deletes $T_1$, pick $T_1$ if student 2 deletes $T_2$ and pick $T_1$ if student 2 deletes $T_3$. Given student 1's strategy, student 2 will choose to delete $T_1$ in 2nd stage (and student 1 chooses $T_2$). The payoffs of two players will be $(6,10)$
- In 1st stage, the equilibrium will be $(T_1,T_2)$, $(T_1,T_3)$, $(T_2,T_2)$, $(T_2,T_3)$ and $(T_3,T_2)$.

**Problem 4**

(a) We shall find the best response of each player using first order condition. That is,

$$\frac{\partial V_1}{\partial e_1}=0\Rightarrow\frac{\partial}{\partial e_1}(4e_1+e_1e_2-2e_1^2)=4+e_2-4e_1=0\Rightarrow e_1^*=\frac{4+e_2}{4}.$$

$$\frac{\partial V_2}{\partial e_2}=0\Rightarrow\frac{\partial}{\partial e_2}(4e_2+e_1e_2-2e_2^2)=4+e_1-4e_2=0\Rightarrow e_2^*=\frac{4+e_1}{4}.$$

Hence, the Nash equilibrium $(e_1^*,e_2^*)$ should satisfy

$$\begin{cases} e_1^*=\dfrac{4+e_2^*}{4} \\ e_2^*=\dfrac{4+e_1^*}{4} \end{cases}\Rightarrow\cdots\Rightarrow e_1^*=e_2^*=\frac{4}{3}.$$

(b) We shall obtain the equilibrium through the backward induction.

- Given player 1's choice $e_1$, the optimal player 2's strategy is governed by

$$\frac{\partial V_2}{\partial e_2}=0\Rightarrow\frac{\partial}{\partial e_2}(4e_2+e_1e_2-2e_2^2)=4+e_1-4e_2=0\Rightarrow e_2^*=\frac{4+e_1}{4}.$$

- Given player 2's optimal strategy, the player 1's payoff function can be expressed as

$$V_1 = 4e_1 + e_1\left(\frac{4+e_1}{4}\right) - 2e_1^2 = 5e_1 - \frac{7}{4}e_1^2.$$

Then the player 1's optimal strategy is governed by

$$\frac{dV_1}{de_1} = 0 \Rightarrow 5 - \frac{7}{2}e_1 = 0 \Rightarrow e_1 = \frac{10}{7}.$$

And the corresponding player 2's strategy is $e_2 = \frac{4+\frac{10}{7}}{4} = \frac{19}{14}$.

The following table summarizes the players' payoff under different equilibria obtained in (a) and (b)

| | Player 1's payoff | Player 2's payoff |
|---|---|---|
| (a) | 3.555556 | 3.555556 |
| (b) | 3.571429 | 3.683673 |

In this case, we observe that the payoffs of two players are improved in (b). In addition, we observe that player 2's payoff is higher than that of player 1, this reveals that there is no first-mover advantage in this case.

**Problem 5**
**(Skipped)**

**Problem 6**

(a) We first consider the second stage which players are going to decide how to split the pot 2. The payoff matrix of the players is given as follows:

| | | Player 2 | |
|---|---|---|---|
| | | T | S |
| Player 1 | T | $(6,6)$ | $(12,0)$ |
| | S | $(0,12)$ | $\left(\frac{12+B}{2},\frac{12+B}{2}\right)$ |

If $B < 12$, then the best response of two players are summarized as follows (highlighted by upper bar)

| | | Player 2 | |
|---|---|---|---|
| | | T | S |
| Player 1 | T | $(\overline{6},\overline{6})$ | $(\overline{12},0)$ |
| | S | $(0,\overline{12})$ | $\left(\frac{12+B}{2},\frac{12+B}{2}\right)$ |

So the only Nash equilibrium in second round is $(T,T)$.

Next, we consider the first stage which the players are going to split the pot 1 (and they will split pot 2 later). Given that two players must play $(T,T)$ in second round (regardless of players' strategy in the first round), then the payoff matrix of two players in the first round (including the payoff in second round) is summarized as follows:

| | | Player 2 | |
|---|---|---|---|
| | | T | S |
| Player 1 | T | $(6+6D, 6+6D)$ | $(12+6D, 0+6D)$ |
| | S | $(0+6D, 12+6D)$ | $\left(\frac{12+B}{2}+6D, \frac{12+B}{2}+6D\right)$ |

When $B < 12$, the best response of two players are given by (highlighted with upper bar)

| | | Player 2 | |
|---|---|---|---|
| | | T | S |
| Player 1 | T | $(\overline{6+6D}, \overline{6+6D})$ | $(\overline{12+6D}, 0+6D)$ |
| | S | $\left(0+6D, \overline{12+6D}\right)$ | $\left(\frac{12+B}{2}+6D, \frac{12+B}{2}+6D\right)$ |

Thus, the Nash equilibrium in the first round is also $(T, T)$.

Hence, we conclude that two players will always play $(T, T)$ in the games under subgame perfect equilibrium

(b) When $B \geq 12$, one can use the payoff matrix in (a) and check that $(S, S)$ and $(T, T)$ are the Nash equilibrium in the 2nd games (splitting Pot 2). It is left as exercise.

We let $s^{(k)} = (s_1^{(k)}, s_2^{(k)})$ denotes the players' strategy in $k^{th}$ stage of the games and consider the following strategic profiles

$$s^{(1)} = (S, S), \qquad s^{(2)} = \begin{cases} (S, S) & if \;\; s^{(1)} = (S, S) \\ (T, T) & if \;\; s^{(1)} \neq (S, S) \end{cases}.$$

We shall check the strategy is subgame perfect equilibrium.

- Since the players always play Nash equilibrium in second games, so no players have incentive to deviate in second games.
- Next, we consider the 1st round.
  - The payoff of choosing $s^{(1)} = (S, S)$ (yielding a payoff of $\frac{10+6}{2} = 8$) and $s^{(2)} = (S, S)$ is given by

$$V_i(S) = 8 + D\left(\frac{12+B}{2}\right)$$

  - If player $i$ deviate and chooses $T$ (he will get \$10 since another player chooses $S$) and two players will play $(T, T)$ in second round, then the corresponding payoff is

$$V_i(T) = 10 + 6D.$$

  So player $i$ has no incentive to deviate if and only if

$$V_i(S) \geq V_i(T) \Leftrightarrow 8 + D\left(6 + \frac{B}{2}\right) \geq 10 + 6D \Rightarrow D \geq \frac{4}{B}.$$

As $\frac{4}{B} \leq 1$ for $B \geq 12$, we conclude that the strategic profile is subgame perfecrt equilibrium when $D \in \left[\frac{4}{B}, 1\right]$.

**Problem 7**

(a) We consider a scenario that two firms cooperate and choose strategies to maximize their total payoff. That is,

$$V_1 + V_2 = 4e_1 + e_1e_2 - 2e_1^2 + 4e_2 + e_1e_2 - 2e_2^2.$$

Assuming that $e_1 = e_2 = e$, the optimal strategy can be determined by

$$\frac{\partial}{\partial e}(V_1 + V_2) = 0 \Rightarrow \frac{\partial}{\partial e}(8e - 2e^2) = 0 \Rightarrow 8 - 4e = 0 \Rightarrow e = 2.$$

In this case, the payoffs of two players are

$$V_1 = V_2 = 4(2) + 2(2) - 2(2)^2 = 4$$

which is greater than that under equilibrium strategy in (a).

(b) We let $s^{(k)} = \left(e_1^{(k)}, e_2^{(k)}\right)$ be the player's strategy adopted in $k^{th}$ round of the games. We consider the following strategy:

$$s^{(1)} = (2,2), \qquad s^{(k+1)} = \begin{cases} (2,2) & if\ s^{(k)} = (2,2) \\ \left(\frac{4}{3}, \frac{4}{3}\right) & if\ s^{(k)} \neq (2,2) \end{cases}.$$

We proceed to check if it is the subgame perfect equilibrium using one-stage deviation principle.

Suppose that the players are playing $k^{th}$ round of the games. We consider the following two cases:

Case 1: If two players do not play $(2,2)$ in previous round

Then two players will play $\left(\frac{4}{3}, \frac{4}{3}\right)$ in $k^{th}$ round and play $\left(\frac{4}{3}, \frac{4}{3}\right)$afterwards, the total discounted payoff is seen to be

$$V_i\left(\frac{4}{3}\right) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(e_i^{(n)}; e_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}\left(\frac{4}{3}; \frac{4}{3}\right)}_{=3.5556} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}\left(\frac{4}{3}; \frac{4}{3}\right)}_{=3.5556}$$

$$= C_i + \frac{3.5556 D^{k-1}}{1 - D}.$$

If player $i$ chooses to deviate and choose $e \neq \frac{4}{3}$ in $k^{th}$ round, then the total discounted payoff will be

$$V_i(e) = C_i + D^{k-1} V_i^{(k)}\left(e; \frac{4}{3}\right) + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}\left(\frac{4}{3}; \frac{4}{3}\right)}_{=3.5556}$$

$$\leq C_i + D^{k-1} V_i^{(k)}\left(\frac{4}{3}; \frac{4}{3}\right) + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}\left(\frac{4}{3}; \frac{4}{3}\right)}_{=3.5556}. = V_i\left(\frac{4}{3}\right)$$

since $e_i = \frac{4}{3}$ is best response to $e_j = \frac{4}{3}$ according to the result in (a).
It is clear that $V_i(e) < V_i\left(\frac{4}{3}\right)$ so that player $i$ has no incentive to deviate.

Case 2: If two players plays $(2,2)$ in previous round
Then two players will play $(2,2)$ in $k^{th}$ round and play $(2,2)$ afterwards, the total discounted payoff is seen to be

$$V_i(2) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(e_i^{(n)}; e_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}(2;2)}_{=4} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(2;2)}_{=4}$$

$$= C_i + \frac{4D^{k-1}}{1-D}.$$

If player $i$ chooses to deviate and choose $e \neq 2$ in $k^{th}$ round, then two players will play $\left(\frac{4}{3}, \frac{4}{3}\right)$ afterwards, the total discounted payoff will be

$$V_i(e) = C_i + D^{k-1} V_i^{(k)}(e; 2) + \sum_{n=k+1}^{\infty} D^{n-1}(3.5556)$$

$$= C_i + D^{k-1} V_i^{(k)}(e; 2) + \frac{3.5556 D^k}{1-D}.$$

Note that player $i$ will not deviate if and only if $V_i(2) \geq V_i(e)$.

- It is clear that the inequality hold if $4 \geq V_i^{(k)}(e; 2)$.
- If $V_i^{(k)}(e; 2) > 4$, one can show that

$$V_i(2) \geq V_i(e) \Leftrightarrow \frac{4D^{k-1}}{1-D} \geq D^{k-1} V_i^{(k)}(e; 2) + \frac{3.5556 D^k}{1-D}$$

$$\Rightarrow D \geq \frac{V_i^{(k)}(e; 2) - 4}{V_i^{(k)}(e; 2) - 3.5556} = 1 - \frac{0.4444}{V_i^{(k)}(e; 2) - 3.5556}.$$

Hence, the player 2 has no incentive to deviate if

$$D \geq \max_e \left(1 - \frac{0.4444}{V_i^{(k)}(e; 2) - 3.5556}\right) = \underbrace{1 - \frac{0.4444}{\max_e V_i^{(k)}(e; 2) - 3.5556}}_{denoted\ by\ D^*}$$

Summing up all cases, we conclude from one-stage deviation principe that the strategic profile is subgame perfect equilibrium if $D \geq D^*$.

**Problem 8**

Given the selling prices ($s_i$ and $s_j$) chosen by two stores, then the payoff of player $i$ can be expressed as

$$V_i(s_i; s_j) = \begin{cases} 100(0.9)s_i & if\ \ s_i < s_j \\ 100(0.5)s_i & if\ \ s_i = s_j \\ 100(0.1)s_i & if\ \ s_i > s_j \end{cases}$$

where $s_i, s_j \in \{20,25,30\}$.

Then the payoff matrix of the games can be expressed as

| | | Player 2's strategy | | |
|---|---|---|---|---|
| | | $s_2 = 20$ | $s_2 = 25$ | $s_2 = 30$ |
| Player 1's strategy | $s_1 = 20$ | $(1000, 1000)$ | $(1800, 250)$ | $(1800, 300)$ |
| | $s_1 = 25$ | $(250, 1800)$ | $(1250, 1250)$ | $(2250, 300)$ |
| | $s_1 = 30$ | $(300, 1800)$ | $(300, 2250)$ | $(1500, 1500)$ |

**(a)** (i) We first find the best response for each player. The result is summarized below (best response is highlighted by upper bar)

| | | Player 2's strategy | | |
|---|---|---|---|---|
| | | $s_2 = 20$ | $s_2 = 25$ | $s_2 = 30$ |
| Player 1's strategy | $s_1 = 20$ | $(\overline{1000}, \overline{1000})$ | $(\overline{1800}, 250)$ | $(1800, 300)$ |
| | $s_1 = 25$ | $(250, \overline{1800})$ | $(1250, 1250)$ | $\left(\overline{2250}, 300\right)$ |
| | $s_1 = 30$ | $(300, 1800)$ | $\left(300, \overline{2250}\right)$ | $(1500, 1500)$ |

We conclude that $(s_1, s_2) = (20, 20)$ is the Nash equilibrium.

(ii) For generality, we let $D \in (0,1]$ be the discounting factor over a period. We consider the following strategic profiles:

We let $s^{(k)} = \left(s_1^{(k)}, s_2^{(k)}\right)$ denotes the players' strategy in $k^{th}$ round of the contribution games. We consider the following strategic profiles:

$$s^{(1)} = (30,30), \qquad s^{(k+1)} = \begin{cases} (30,30) & if\ s^{(k)} = (30,30) \\ (20,20) & if\ s^{(k)} \neq (30,30) \end{cases}.$$

We shall argue that this strategic profile constitutes the subgame perfect equilibrium using one-stage deviation principle.

Case 1: If two players do not play $(30,30)$ in previous round

Then two players will play $(20,20)$ in $k^{th}$ round and play $(20,20)$ afterwards, the total discounted payoff is seen to be

$$V_i(20) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}(20; 20)}_{=1000}$$
$$+ \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(20; 20)}_{=1000}.$$

If player $i$ chooses to deviate and choose $s_i = 25$ or 30 in $k^{th}$ round, then the total discounted payoff will be

$$V_i(s_i) = C_i + D^{k-1} \underbrace{V_i^{(k)}(s_i; 20)}_{=250\ or\ 300} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(20; 20)}_{=1000} < V_i(20)$$

It is clear that $V_i(s_i) < V_i(20)$ so that player $i$ has no incentive to deviate.

Case 2: If two players plays (30,30) in previous round
Then two players will play (30,30) in $k^{th}$ round and play (30,30) afterwards, the total discounted payoff is seen to be

$$V_i(30) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}(C;C)}_{=1500} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(C;C)}_{=1500}$$
$$= C_i + \frac{1500D^{k-1}}{1-D}.$$

If player $i$ chooses to deviate and choose $s_i = 20$ or $s_i = 25$ in $k^{th}$ round, then two players will play (20,20) afterwards, the total discounted payoff will be

$$V_i(s_i) = C_i + D^{k-1} \underbrace{V_i^{(k)}(s_i;30)}_{=1800\ or\ 2250} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(20;20)}_{=1000}$$
$$= C_i + D^{k-1} \underbrace{V_i^{(k)}(s_i;30)}_{=1800\ or\ 2250} + \frac{1000D^k}{1-D}.$$

So player $i$ will not deviate if and only if

$$V_i(30) \geq \max_{s_i} V_i(s_i) \Leftrightarrow \frac{1500D^{k-1}}{1-D} \geq D^{k-1}(2250) + \frac{1000D^k}{1-D} \Rightarrow D \geq \frac{3}{5}.$$

Using one-stage deviation principle, we conclude that the strategic profile is subgame perfect equilibrium if $D \geq \frac{3}{5}$.

**(b)** We shall find the sequentially rational Nash equilibrium using backward induction.

- We first consider the second turn when store 2 makes the move after knowing the selling price chosen by store 1 ($s_1$)
  - If store 1 chooses $s_1 = 20$, one can see from the above payoff matrix that store 2 should choose $s_2 = 20$ (since 1000 v.s. 250, 300)
  - If store 1 chooses $s_1 = 25$, then store 2 should choose $s_2 = 20$ (since 1800 v.s. 1250 v.s. 300)
  - If store 1 chooses $s_1 = 30$, then store 2 should choose $s_2 = 25$ (since 1800 v.s. 2250 v.s. 1500)
- Next, we consider the first turn when store 1 makes the move
  - If store 1 chooses $s_1 = 20$, then store 2 should choose $s_2 = 20$ and the store 1's payoff is 1000.
  - If store 1 chooses $s_1 = 25$, then store 2 should choose $s_2 = 20$ and the store 1's payoff is 250.
  - If store 1 chooses $s_1 = 30$, then store 2 should choose $s_2 = 25$ and the store 1's payoff is 300.

Thus, the store 1 should choose $s_1 = 20$ in stage 1 to maximize the payoff and this is the sequentially rational Nash equilibrium.

**Problem 9**

**(a)** We shall find the sequentially rational Nash equilibrium by backward induction. We first consider player 2.

- If player 1 chooses to contribute $(C)$, then player 2 will choose $C$ (yielding a payoff of $4-c$) instead of $N$ (yielding a payoff of 2) if and only if
$$4 - c \geq 2 \Leftrightarrow c \leq 2$$
- If player 1 chooses not to contribute $(N)$, then player 2 will choose $C$ (yielding a payoff of $2-c$) instead of $N$ (yielding a payoff of 0) if and only if
$$2 - c \geq 0 \Leftrightarrow c \leq 2$$

Next, we consider player 1 and we consider two cases:

Case 1: If $c < 2$

- If player 1 chooses $C$ so that player 2 will choose $C$, then player 1's playoff is $4 - c$
- If player 1 chooses $N$ so that player 2 will choose $C$, then player 1's playoff is 2

Since $c < 2$ so that $4 - c > 2$, so player 1 will choose $C$.

Case 2: If $c > 2$

- If player 1 chooses $C$ so that player 2 will choose $N$, then player 1's playoff is $2 - c$
- If player 1 chooses $N$ so that player 2 will choose $N$, then player 1's playoff is $0$.

Since $c > 2$ so that $2 - c < 0$, so player 1 will choose $N$ instead.

Therefore, the sequentially rational Nash equilibrium can be described as follows:

- If $c < 2$, $s_1^* = C$ and $s_2^* = \left(s_2^*(C), s_2^*(N)\right) = (C, C)$. (*Here, $s_2^*(s_1)$ denotes player 2's strategy when player 1 chooses $s_1$ earlier)
- If $c > 2$, $s_1^* = N$ and $s_2^* = \left(s_2^*(C), s_2^*(N)\right) = (N, N)$.

(*Note: In the special case when $c = 2$, both players will be indifferent between contribute and not contribute)

**(b)** When $c = 3$, two players will choose not to contribute ($N$) in a single contribution games.

We let $s^{(k)} = \left(s_1^{(k)}, s_2^{(k)}\right)$ denotes the players' strategy in $k^{th}$ round of the contribution games. We consider the following strategic profiles:
$$s^{(1)} = (C, C), \qquad s^{(k+1)} = \begin{cases} (C, C) & if\ s^{(k)} = (C, C) \\ (N, N) & if\ s^{(k)} \neq (C, C) \end{cases}.$$

(*Here, we specify that player 2's strategy is independent of player 1's strategy in the same round)
We shall verify that the strategy is subgame perfect equilibrium using one-stage deviation principle.
Suppose that two players are playing $k^{th}$ round of the games. We consider the following two cases:

Case 1: If two players do not play $(C, C)$ in previous round
Then two players will play $(N, N)$ in $k^{th}$ round and play $(N, N)$ afterwards, the total discounted payoff is seen to be

$$V_i(N) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}(N; N)}_{=0} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(N; N)}_{=0}$$
$$= C_i.$$

If player $i$ chooses to deviate and choose $C$ (he/she will receive a payoff of $2 - c$ as another player chooses not to contribute) in $k^{th}$ round, then the total discounted payoff will be

$$V_i(C) = C_i + D^{k-1}(2 - c) + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(N; N)}_{=0} = C_i + D^{k-1} \underbrace{(2 - c)}_{=-1}.$$

It is clear that $V_i(N) < V_i(C)$ so that player $i$ has no incentive to deviate.

Case 2: If two players plays $(C, C)$ in previous round
Then two players will play $(C, C)$ in $k^{th}$ round (yield a payoff of $4 - 3 = 1$ each) and play $(C, C)$ afterwards, the total discounted payoff is seen to be

$$V_i(C) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C_i} + D^{k-1} \underbrace{V_i^{(k)}(C; C)}_{=1} + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(C; C)}_{=1}$$
$$= C_i + \frac{D^{k-1}}{1 - D}.$$

If player $i$ chooses to deviate and choose $N$ (he/she will receive a payoff of 2 as another player will contribute) in $k^{th}$ round, then two players will play $(N, N)$ afterwards, the total discounted payoff will be

$$V_i(N) = C_i + D^{k-1}(2) + \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(N; N)}_{=0} = C_i + 2D^{k-1}.$$

So player $i$ will not deviate if and only if

$$V_i(C) \geq V_i(N) \Leftrightarrow \frac{D^{k-1}}{1 - D} \geq 2D^{k-1} \Rightarrow D \geq \frac{1}{2}.$$

Using one-stage deviation principle, we conclude that the strategic profile is subgame perfect equilibrium if $D \geq \frac{1}{2}$.

**Problem 10**

(a) To find the Nash equilibrium, we determine the best response of each player. The best responses are highlighted by the upperbar:

| | | Player 2 | |
|---|---|---|---|
| | | A | B |
| Player 1 | A | $(3,3)$ | $(0,\overline{20})$ |
| | B | $(\overline{20},0)$ | $(\overline{1},\overline{1})$ |

Hence, we conclude that the Nash equilibrium of the games is $(B,B)$ and the payoffs of player is $(1,1)$

(b) (i) Note that the players' payoff if they play $(A,A)$ is $(3,3)$ which is greater than that of playing $(B,B)$. According to the Folk theorem, there exists $D^* \in (0,1]$ such that for $D > D^*$, there exists a subgame perfect equilibrium such that two players can achieve an average payoff of 3 in the infinite repeated games. According to the proof of Folk theorem, one possible subgame perfect equilibrium is given by

$$s^{(1)*} = (A,A), \qquad s^{(k+1)*} = \begin{cases} (A,A) & if\ s^{(k)*} = (A,A) \\ (B,B) & if\ s^{(k)*} \neq (A,A) \end{cases}.$$

Under this equilibrium, two players always play $(A,A)$ in all games.

(ii) We consider the following strategic profiles

- Two players play $(A,B)$ in odd rounds and play $(B,A)$ in even rounds.
- Two players will always play $(B,B)$ if two players did not play according to the first rule in some of previous rounds.

Next, we would like to verify that the strategic profile constitutes subgame perfect equilibrium using one-stage deviation principle. We consider the case when two players are playing $k^{th}$ game and we consider the following three cases:

Case 1: If two players do not play $(A,B)$ (in odd rounds) or $(B,A)$ (in even rounds) in previous games.

Then the players will play $(B,B)$ in this round and play $(B,B)$ afterwards. Then the payoff of player $i$ is given by

$$V_i(B) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C} + D^{k-1} \underbrace{V_i^{(k)}(B;B)}_{=1}$$

$$+ \sum_{n=k+1}^{\infty} D^{n-1} \underbrace{V_i^{(n)}(B;B)}_{=1} = C + \frac{D^{k-1}}{1-D}.$$

Suppose that player $i$ chooses $A$ instead of $B$ and two players will play $(B, B)$ afterwards. Then the corresponding payoff is

$$V_i(A) = C + D^{k-1}(0) + \sum_{n=k+1}^{\infty} D^{n-1}(1) = C + \frac{D^{k-2}}{1-D}.$$

As $V_i(A) < V_i(B)$, then player $i$ has no incentive to deviate.

Case 2: If two players play according to the first rule in previous games and suppose that two players play $(A, B)$ in $k^{th}$ round (i.e. $k$ is odd).

- We first consider player 1 (who plays $A$), the payoff can be computed as

$$V_1(A) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_1^{(n)}\left(s_1^{(n)}; s_2^{(n)}\right)}_{denoted\ by\ C} + D^{k-1}(0) + D^{k}(20) + D^{k+1}(0)$$

$$+ D^{k+2}(20) + \cdots = C + \frac{20D^{k}}{1-D^2}.$$

If the player 1 deviates and play $B$ in $k^{th}$ round, then two players will play $(B, B)$ afterwards, the corresponding payoffs is seen to be

$$V_1(B) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_1^{(n)}\left(s_1^{(n)}; s_2^{(n)}\right)}_{denoted\ by\ C} + D^{k-1}(1) + D^{k}(1) + D^{k+1}(1)$$

$$+ D^{k+2}(1) + \cdots = C + \frac{D^{k-1}}{1-D}.$$

The player 1 has no incentive to deviate if and only if

$$V_i(A) \geq V_i(B) \Leftrightarrow C + \frac{20D^{k}}{1-D^2} \geq C + \frac{D^{k-1}}{1-D} \Rightarrow D \geq \frac{1}{19}.$$

- Next, we consider player 2 (who plays $B$), the payoff can be computed as

$$V_2(B) = \underbrace{\sum_{n=0}^{k-1} D^{n-1} V_1^{(n)}\left(s_1^{(n)}; s_2^{(n)}\right)}_{denoted\ by\ C} + D^{k-1}(20) + D^{k}(0) + D^{k+1}(20)$$

$$+ D^{k+2}(0) + \cdots = C + \frac{20D^{k-1}}{1-D^2}.$$

If the player 2 deviates and play $A$ in $k^{th}$ round, then two players will play $(B, B)$ afterwards again, thus the corresponding payoff is

$$V_2(A) = C + D^{k-1}(3) + D^k(1) + D^{k+1}(1) + \cdots C = +3D^{k-1} + \frac{D^k}{1-D}.$$

Note that

$$V_2(B) \geq V_2(A) \Leftrightarrow \underbrace{2D^2 - D + 17}_{=2\left(D-\frac{1}{4}\right)^2+\frac{135}{8}} \geq 0.$$

As the inequality on the right holds trivial so that $V_2(B) \geq V_2(A)$ and player 2 has no incentive to deviate.

Case 3: If two players play according to the first rule in previous games and suppose that two players play $(B, A)$ in $k^{th}$ round (i.e. $k$ is even).
Since the players are symmetric in terms of strategy, thus the analysis is similar to that in Case 2 (interchange the role of player 1 and player 2) and we can conclude that no players have incentive to deviate if $D \geq \frac{1}{19}$.

It follows from one-stage deviation principle that the strategic profile is subgame perfect equilibrium when $D \geq \frac{1}{19}$.

**Problem 11**
We let $D \in (0,1]$ be the discounting factor over 1 period.
We shall find the subgame perfect equilibrium by backward induction.

We first consider period 3 when player 3 makes a proposal $x^{(3)} = \left(x_1^{(3)}, x_2^{(3)}, x_3^{(3)}\right)$.

- Since all players will receive zero payoff if the proposal is rejected, so player 1 (resp. player 2) will accept the proposal if and only if $x_1^{(3)} > 0$ (resp. $x_2^{(3)} > 0$).
- As the proposal will be accepted if at least 2 players (including player 3) support the proposal, thus the proposal will be accepted if and only if at least one of the player 1 and player 2 accept the proposal.

Given $x^{(3)}$, the player 3's payoff is $V_3 = x_3^{(3)} = 1 - x_1^{(3)} - x_2^{(3)}$ if the proposal is accepted and $V_3 = 0$ if otherwise. Thus player 3 can maximize his payoff by proposing $x^{(3)} = (\varepsilon, 0, 1-\varepsilon)$ (or $x^{(3)} = (0, \varepsilon, 1-\varepsilon)$) and player 1 (or player 2) accepts the proposal also. By taking $\varepsilon \to 0^+$, the optimal strategy of player 3 is $x^{(3)} = (0,0,1)$ and the players' payoff is $(0, 0, 1)$.

We then consider period 2 when player 2 makes a proposal $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right)$.

- If the proposal is rejected, then the players will receive the discounted payoff $D(0,0,1) = (0,0,D)$. Thus, player 1 (resp. player 3) will accept the proposal if and only if $x_1^{(2)} > 0$ (resp. $x_3^{(2)} > D$, be careful).

Given $x^{(2)}$, the player 2's payoff is $V_2 = x_2^{(2)} = 1 - x_1^{(2)} - x_3^{(2)}$ if the proposal is accepted and $V_2 = 0$ if otherwise. Thus player 2 can maximize his payoff by proposing $x^{(2)} = (\varepsilon, 1-\varepsilon, 0)$ and player 1 accepts the proposal also. By taking $\varepsilon \to 0^+$, the optimal strategy of player 2 is $x^{(2)} = (0,1,0)$ and the players' payoff at period 2 is $(0, 1, 0)$.

Finally, we consider period 1 when player 1 makes a proposal $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right)$.

- If the proposal is rejected, then the players will receive the discounted payoff $D(0,1,0) = (0, D, 0)$ as the deal will be settled at period 2. Thus, player 2 (resp. player 3) will accept the proposal if and only if $x_2^{(1)} > D$ (resp. $x_3^{(1)} > 0$), be careful).

Given $x^{(1)}$, the player 1's payoff is $V_1 = x_1^{(1)} = 1 - x_2^{(1)} - x_3^{(1)}$ if the proposal is accepted and $V_1 = 0$ if otherwise. Thus player 1 can maximize his payoff by proposing $x^{(1)} = (1-\varepsilon, 0, \varepsilon)$ and player 3 accepts the proposal also. By taking $\varepsilon \to 0^+$, the optimal strategy of player 1 is $x^{(1)} = (1,0,0)$ and the players' payoff at period 1 is $(1, 0, 0)$.

**Problem 12**

As suggested by the hint, the costs for two firms will become 80 (for company 1) and 160 (for company 2) at period 4 which the total cost is 240 which is greater than the revenue of the project \$150. It implies that at least one player will get a negative payoff when the deal is made at or after period 4. The bargaining should be settled at or before period 3 and the players should not bargain any further and both get 0 payoffs if otherwise.

We shall determine the subgame perfect equilibrium by backward induction again.

We first consider period 3 which $c_1 = 40$ and $c_2 = 80$. Suppose that player 1 makes a proposal $x^{(3)} = \big(150x_3, 150(1-x_3)\big)$.

Depending on the decision of player 2, the payoff matrix can be summarized as follows:

| | Player 2 Accept | Player 2 Reject |
|---|---|---|
| Payoff | $(150x_3 - 40, 150(1-x_3) - 80)$ $= (150x_3 - 40, 70 - 150x_3)$ | $(0,0)$ |

So player 2 accepts the proposal if and only if

$$70 - 150x_3 \geq 0 \Leftrightarrow x_3 \leq \frac{7}{15}.$$

As $150x_3 - 40$ is increasing with respect to $x_3$ and $150\left(\frac{7}{15}\right) - 40 = 30$ (when $x_3 = 70$), so player 1 can maximize its payoff by offering $x_3 = \frac{7}{15} - \varepsilon$ (where $\varepsilon \to 0^+$). So the final proposal is $x^{(3)} = \left(150\left(\frac{7}{15}\right), 150\left(1-\frac{7}{15}\right)\right) = (70{,}80)$ and the players' payoff is $V^{(3)} = (70-40, 80-80) = (30, 0)$.

Next, we consider period 2 which $c_1 = 20$ and $c_2 = 40$. Suppose that player 2 makes a proposal $x^{(2)} = \left(150x_2, 150(1-x_2)\right)$.
Depending on the decision of player 1, the payoff matrix can be summarized as follows:

| | Player 1 Accept | Player 1 Reject |
|---|---|---|
| Payoff | $(150x_2 - 20, 150(1-x_2) - 40)$ $= (150x_2 - 20, 110 - 150x_2)$ | $(30{,}0)$ |

So player 1 accepts the proposal if and only if

$$150x_2 - 20 \geq 30 \Leftrightarrow x_2 \geq \frac{1}{3}.$$

As $110 - 150x_2$ is decreasing with respect to $x_2$ and $110 - 150x_2 = 60 > 0$ when $x_2 = \frac{1}{3}$. So it is optimal for player 2 to submit a proposal $x_2 = \frac{1}{3} + \varepsilon$ where $\varepsilon \to 0^+$ and player 1 accepts the deal. Thus, the final proposal is $x^{(2)} = \left(150\left(\frac{1}{3}\right), 150\left(1-\frac{1}{3}\right)\right) = (50{,}100)$ and the players' payoff is $V^{(2)} = (50-20, 100-40) = (30, 60)$.

Finally, we consider period 1 with $c_1 = 10$ and $c_2 = 20$. Suppose that player 1 makes a proposal $x^{(1)} = \left(150x_1, 150(1-x_1)\right)$.
Depending on the decision of player 1, the payoff matrix can be summarized as follows:

| | Player 2 Accept | Player 2 Reject |
|---|---|---|
| Payoff | $(150x_1 - 10, 150(1-x_1) - 20)$ $= (150x_1 - 10, 130 - 150x_1)$ | $(30{,}60)$ |

So player 1 accepts the proposal if and only if

$$130 - 150x_1 \geq 60 \Leftrightarrow x_1 \leq \frac{7}{15}.$$

As $150x_1 - 10$ is increasing with respect to $x_1$ and $150x_1 - 10 = 60 > 0$ when $x_1 = \frac{7}{15}$. So it is optimal for player 1 to submit a proposal $x_1 = \frac{7}{15} - \varepsilon$ where $\varepsilon \to 0^+$ and player 2 accepts the deal. Thus, the final proposal is $x^{(1)} = \left(150\left(\frac{7}{15}\right), 150\left(1-\frac{7}{15}\right)\right) = (70{,}80)$ and the players' payoff is $V^{(1)} = (70-10, 80-20) = (60, 60)$.

**(No Problem 13 in the Problem Set)**

**Problem 14**

(a) Suppose that the games $G$ is played for $N$ times.
We let $s^{*(k)}$ denotes the players' strategy adopted in $k^{th}$ stage of the game (where We consider the following strategic profiles $s^* = \left(s^{*(1)}, s^{*(2)}, \dots, s^{*(N)}\right)$ defined as

$$s^{*(1)} = (s_1^0, s_2^0), \qquad s^{*(k+1)} = \begin{cases} (s_1^0, s_2^0) & if\ \ s^{*(k)} = (s_1^0, s_2^0) \\ (s_1'', s_2'') & if\ \ s^{*(k)} \neq (s_1^0, s_2^0) \end{cases} \quad for\ k+1 < N$$

$$s^{*(N)} = \begin{cases} (s_1', s_2') & if\ \ s^{*(k)} = (s_1^0, s_2^0) \\ (s_1'', s_2'') & if\ \ s^{*(k)} \neq (s_1^0, s_2^0) \end{cases}$$

Next, we would like to verify the strategic profile $s^*$ constitutes the subgame perfect equilibrium using one-stage deviation principle.

Since all players play Nash equilibrium strategy at $N^{th}$ stage regardless of previous strategies chosen by the players, no player has incentive to deviate in $N^{th}$ game.

Suppose that the games have been played for $k-1$ times (where $1 \leq k < N$), we consider the following two scenarios:

- *Case 1: If the players did not play $(s_1^0, s_2^0)$ in $(k-1)^{th}$ round;*
  Then the players will play $(s_1'', s_2'')$ in the $k^{th}$ stage and also the remaining stages of the game.
  Suppose that player $i$ tries to adopt another strategy $s \neq s_i''$ in $k^{th}$ stage and play $s_i''$, then two players will play the discounted payoff for player $i$ (denoted by $V_i(s)$) is seen to be

$$V_i(s) = \sum_{n=0}^{k-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right) + \underbrace{V_i^{(k)}\left(s; s_j''\right)}_{\substack{Payoff\ in \\ k^{th}\ stage}} + \sum_{n=k+1}^{N} V_i^{(n)}\left(s_i''; s_j''\right)$$

$$\overset{\substack{s_i''\ is\ best\ response \\ to\ s_j''\ in\ games\ G}}{\leq} \sum_{n=0}^{k-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right) + \underbrace{V_i^{(k)}\left(s_i''; s_j''\right)}_{\substack{Payoff\ in \\ k^{th}\ stage}}$$

$$+ \sum_{n=k+1}^{N} V_i^{(n)}\left(s_i''; s_j''\right) = V_i(s_i'').$$

  So player $i$ has no incentive to deviate.

- *Case 2: If the players plays $(s_1^0, s_2^0)$ in $(k-1)^{th}$ round*
  Then the players will play $(s_1^0, s_2^0)$ in the remaining rounds. Suppose that player $i$ tries to deviate and adopt other strategy $s \neq s_i^0$ in $k^{th}$ stage, then two players will play $(s_1'', s_2'')$ afterwards, then the corresponding payoffs can be expressed as

$$V_i(s) = \underbrace{\sum_{n=0}^{k-1} V_i^{(n)}\left(s_i^{(n)}; s_j^{(n)}\right)}_{denoted\ by\ C} + \underbrace{V_i^{(k)}\left(s; s_j^0\right)}_{\substack{Payoff\ in \\ k^{th}\ stage}} + \sum_{n=k+1}^{N} V_i^{(n)}\left(s_i''; s_j''\right)$$

$$\leq C + \max_s V_i^{(k)}(s; s_j^0) + \sum_{n=k+1}^{N} V_i^{(n)}(s_i''; s_j'')$$

$$< C + \left(V_i(s_i'; s_j') - V_i(s_i''; s_j'') + V_i(s_i^0, s_j^0)\right) + \sum_{n=k+1}^{N} V_i^{(n)}(s_i''; s_j'')$$

$$< C + \left(\underbrace{V_i(s_i'; s_j') - V_i(s_i''; s_j'')}_{>0} + V_i(s_i^0, s_j^0)\right) + \sum_{n=k+1}^{N} V_i^{(n)}(s_i^0; s_j^0)$$

$$= V_i(s_1^0).$$

So player $i$ has no incentive to deviate in this case too.

Therefore, it follows from one stage deviation principle that $s^*$ constitutes the subgame perfect equilibrium.

(b) We can consider a games $G$ with the following payoff matrix:

| | | Player 2 | | |
|---|---|---|---|---|
| | | A | B | C |
| Player 1 | A | $(5,5)$ | $(1,7)$ | $(-2,10)$ |
| | B | $(7,1)$ | $(4,4)$ | $(-1,2)$ |
| | C | $(10,-2)$ | $(2,-1)$ | $(0,0)$ |

One can check that $(s_1', s_2') = (B, B)$ and $(s_1'', s_2'') = (C, C)$ are Nash equilibrium (left as exercise) with

$$V_i(s_i'; s_j') = 4 > 0 = V_i(s_i''; s_j'')$$

We consider $(s_1^0, s_2^0) = (A, A)$. We have

$$V_i(s_i^0; s_j^0) = 5 > 4 = V_i(s_i'; s_j')$$

and

$$\max_s V_i(s; s_j^0) - V_i(s_i^0; s_j^0) = 10 - 5 = 5 > 4 = 4 - 0 = V_i(s_i'; s_j') = 4 > 0$$
$$= V_i(s_i'; s_j') - V_i(s_i''; s_j'').$$

The condition $(*)$ is violated.

Suppose that the games is played twice (i.e. $N = 2$). We proceed to argue that there is no subgame perfect equilibrium which the players will play $(A, A)$ in some stage of the game.

We suppose that such equilibrium $s^* = \left(s^{*(1)}, s^{*(2)}\right)$ exists. Since the players must play equilibrium strategies $(B, B)$ or $(C, C)$ in second games, so two players must play $s^{*(1)} = (A, A)$ in the first stage.

Next, we shall argue that player 1 have incentive to deviate in 1st stage. Suppose that player 1 plays $C$ in the 1st game instead (and player 2 plays $A$) , we let $s^{(2)}$

($(B,B)$ or $(C,C)$) be the corresponding strategy if players play $(C,A)$ in first stage. Then the corresponding payoff (denoted by $C$) is

$$V_1(C) = \underbrace{V_1^{(1)}(C;A)}_{=10} + \underbrace{V_1^{(2)}\left(s_1^{(2)};s_2^{(2)}\right)}_{=4\ or\ 0} \geq 10 + 0 > 5 + 4$$

$$\geq \underbrace{V_1^{(1)}(A;A)}_{=5} + \underbrace{V_1^{(2)}\left(s_1^{(2)*};s_2^{(2)*}\right)}_{=4\ or\ 0} = V_1(A).$$

So player 1 has strict incentive to deviate and it contradicts to the fact that $s^*$ is subgame perfect equilibrium.

**Problem 15**

Using similar idea as in Problem 14(a), we can construct the strategic profiles as follows:

- Players play $\left(s_1^{(1)}, s_2^{(1)}\right)$ in game $G_1$;
- For $j \neq 1\ or\ k$, players play $\left(s_1^{(j)*}, s_2^{(j)*}\right)$ in game $G_j$ (regardless of strategies adopted by players in previous games).
- For games $G_k$, we assume that players will play $\left(s_1^{(k)*}, s_2^{(k)*}\right)$ (yield bigger payoff) if the players play $\left(s_1^{(1)}, s_2^{(1)}\right)$ in game $G_1$ and play $\left(s_1^{(k)**}, s_2^{(k)**}\right)$ if otherwise.

We shall argue that this is the subgame perfect equilibrium using one-stage deviation principle.

- Note that the players always pay equilibrium strategies in each of $G_2, G_3, \dots, G_N$, it follows that the players have no incentive to adopt one-stage deviation at these games.
- It remains to check the optimality at games $G_1$. We let $V_i^{(m)}(\cdot)$ be the player $i$'s payoff in game $G_m$. Then the total payoff under this strategic profile (denoted by $V_i^*$) is seen to be

$$V_i^* = \sum_{m=1}^{n} V_i^{(m)}$$

$$= V_i^{(1)}\left(s_i^{(1)};s_j^{(1)}\right) + V_i^{(k)}\left(s_1^{(k)*}, s_2^{(k)*}\right) + \sum_{m\neq 1,k} V_i^{(m)}\left(s_i^{(m)*};s_j^{(m)*}\right)$$

Suppose that the player $i$ chooses to deviate and adopt another strategy $s \neq s_i^*$ in game $G_1$. Then players will play $\left(s_1^{(k)**}, s_2^{(k)**}\right)$ in Game $G_k$. The corresponding payoff is seen to be

$$V_i(s) = V_i^{(1)}\left(s;s_j^{(1)}\right) + V_i^{(k)}\left(s_1^{(k)**}, s_2^{(k)**}\right) + \sum_{m\neq 1,k} V_i^{(m)}\left(s_i^{(m)*};s_j^{(m)*}\right)$$

$$< \max_s V_i^{(1)}\left(s;s_j^{(1)}\right) + V_i^{(k)}\left(s_1^{(k)**}, s_2^{(k)**}\right) + \sum_{m\neq 1,k} V_i^{(m)}\left(s_i^{(m)*};s_j^{(m)*}\right)$$

$$< V_i^{(1)}\left(s_i^{(1)}; s_j^{(1)}\right) + \left[V_i^{(k)}\left(s_1^{(k)*}, s_2^{(k)*}\right) - V_i^{(k)}\left(s_1^{(k)**}, s_2^{(k)**}\right)\right] + V_i^{(k)}\left(s_1^{(k)**}, s_2^{(k)**}\right) + \sum_{m\neq 1,k} V_i^{(m)}\left(s_i^{(m)*}; s_j^{(m)*}\right) = V_i^*.$$

**Problem 16**

The answer is No.

To see this, we consider the following two-stage games (Game $G_1$ plays first and Game $G_2$ plays next)

| Game $G_1$ | | Player 2 | | | Game $G_2$ | | Player 2 | |
|---|---|---|---|---|---|---|---|---|
| | | A | B | | | | C | D |
| Player 1 | A | $(6,6)$ | $(8,2)$ | | Player 1 | C | $(10,7)$ | $(0,0)$ |
| | B | $(3,15)$ | $(4,10)$ | | | D | $(0,0)$ | $(1,8)$ |

One can observe that

- "B" is a dominated strategy for player 1 in Game $G_1$ since it is dominated by "A":
$$V_1(B;A) = 3 < 6 = V_1(A;A);$$
$$V_1(B;B) = 4 < 8 = V_1(A;B).$$
- $(C,C)$ and $(D,D)$ are Nash equilibria of Game $G_2$ since
$$V_1(C;C) = 10 \geq 0 = V_1(D;C) \quad and \quad V_2(C;C) = 7 \geq 0 = V_2(D;C);$$
(This implies $(C,C)$ is the Nash equilibrium)
$$V_1(D;D) = 1 \geq 0 = V_1(C;D) \quad and \quad V_2(D;D) = 8 \geq 0 = V_2(C;D);$$
(This implies $(D,D)$ is the Nash equilibrium)

We let $s_i^{(j)}$ be the player $i$'s strategy used in Game $G_j$. We consider the following strategic profiles:

$$\left(s_1^{(1)}, s_2^{(1)}\right) = (B,A), \qquad \left(s_1^{(2)}, s_2^{(2)}\right) = \begin{cases} (C,C) & if \ \left(s_1^{(1)}, s_2^{(1)}\right) = (B,A) \\ (D,D) & if \ \left(s_1^{(1)}, s_2^{(1)}\right) \neq (B,A) \end{cases}$$

We shall argue that this strategic profile constitutes subgames perfect equilibrium using one-stage deviation principle.

- Since two players always play equilibrium strategy in game $G_2$, so no players have incentive to deviate in 2nd stage.
- Next, we proceed to check if any player has incentive to deviate in stage 1.
  - If player 1 chooses to deviate and plays $A$, then two players will play $(D,D)$ in second round, then the total payoff of player 1 will be
$$V_1(A) = V_1^{(1)}(A;A) + V_1^{(2)}(D,D) = 6 + 1 = 7$$
$$< \underbrace{13 = 3 + 10 = V_1^{(1)}(B;A) + V_1^{(2)}(C,C) = V_1(B)}_{Payoff\ (no\ deviation)}$$

    So player 1 will not deviate

- If player 2 chooses to deviate and plays $B$, then two players will play $(D, D)$ in second round, then the total payoff of player 1 will be

$$V_2(B) = V_2^{(1)}(B; B) + V_2^{(2)}(D, D) = 10 + 8 = 18$$
$$< \underbrace{22 = 15 + 7 = V_2^{(1)}(A; B) + V_2^{(2)}(C, C) = V_2(A)}_{Payoff\ (no\ deviation)}$$

  So player 2 will not deviate too.

Thus, it follows from one stage deviation principle that the strategic profile constitutes the perfect Bayesian equilibrium. Under this equilibrium, player 1 plays dominated strategy $B$ at game $G_1$.