MATH 4824C - Sample Final

May 15, 2024

Please (1) show the work to the questions and (2) return the answer sheet on time.

1. In randomized trials; i.e. $(Y_i(1), Y_i(0), X_i) \perp Z_i$, show that $Var(Y(1)) = Var(Y|Z = 1)$;

2. Show that in observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, show that $\alpha_0 = E(Y(1) - X\beta_0)$ where $(\alpha_0, \beta_0) = \arg\min_{\alpha,\beta} E[Z(Y - \alpha - X\beta)^2/e(X)]$ and $e(X) = P(Z = 1|X)$;

3. Show that in observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, let $e(\mathrm{x}, \beta)$, be the logistic model of Z_i on X_i, and $\mu_1(\mathrm{x}, \alpha_1)$ be linear or logistic model of $Y_i(1)$ on X_i. Try to show that

$$\tilde{\mu}_{1,\mathrm{DR}} = \frac{E(Z_i \{Y_i - \mu_1(\mathrm{X}_i, \alpha_1)\} / e(\mathrm{X}_i, \beta))}{E(Z/e(\mathrm{X}_i, \beta))} + E(\mu_1(\mathrm{X}_i, \alpha_1))$$

is doubly robust, that is, if either $\mu_1(\mathrm{x}, \alpha_1) = \mu_1(\mathrm{x}) := E[Y_i|Z_i = 1, \mathrm{X}_i = \mathrm{x}]$ or $e(\mathrm{x}, \beta) = e(\mathrm{x}) := P(Z_i|\mathrm{X}_i = \mathrm{x})$, then $\tilde{\mu}_{1,DR} = E\{Y_i(1)\}$.

4. Consider linear models $Y_i = \beta_0 + \beta_1 D_i + X_i^\top \beta_X + \epsilon_i$ and $D_i = \gamma_0 + \gamma_1 Z_i + X_i^\top \gamma_X + \eta_i$ where Z_i and X_i are endogeneous, ϵ_i is independent of Z_i and η_i is independent of X_i, write down the reduced form of $Y_i \sim Z_i + X_i$ and show that both Z and X are exogeneous in the reduced model.

5. Consider the true linear model $Y = \beta_0 + \beta_1 D + \beta_2 U + \eta$ and $D = \alpha_0 + \alpha_1 Z + \epsilon$, show that $\beta_1 = \dfrac{cov(Y, Z)}{cov(D, Z)}$ if $cov(D, Z) \neq 0$ and Z is independent of U and η.

6. In regression discontinuity design with forcing variable X and treatment indicator $Z = I_{X \geq c}$ where c is the cutoff, show that $E(Y_i(0) \mid X_i = c) = \lim_{x \uparrow c} E(Y_i \mid X_i = x)$ if $E(Y_i(0) \mid X_i = c)$ is left continuous at c.

7. In the non-compliance observational study where the exclusion and the monotonicity assumption are satisfied and $Z_i \perp \{Y_i(1), Y_i(0), D_i(1), D_i(0)\} \mid \mathbf{X}_i$, show that $E\{Y_i(1) - Y_i(0) \mid complier, X = x\} = (E(Y \mid Z = 1, X = x) - E(Y \mid Z = 0, X = x))/(E(D \mid Z = 1, X = x) - E(D \mid Z = 0, X = x))$

8. In mediation analysis where there exist no treatment-outcome confounding, no mediator-outcome confounding and no treatment-mediator confounding and the cross-world assumption is satisfied, write down the identifiable quantity of $E(Y(0, M(1)) - E(Y(0, M(0))$.