MATH4321 Game Theory (2023 Spring)
Suggested Solution of Problem Set 3

Problem 1

(a) If each player knows the ability of another player, then the true value of p is known to each problem

When $p = 0.2$, the payoff matrix and the corresponding best responses (highlighted in upperbar) are given by

		Player 2	
		A	B
Player 1	A	$(1, \bar{4})$	$(\bar{5}, 2)$
	B	$(\bar{2}, \bar{5})$	$(0.4, 1.6)$

So the pure strategy Nash equilibrium is (B, A)

When $p = 0.7$, the payoff matrix and the corresponding best responses (highlighted in upperbar) are given by

		Player 2	
		A	B
Player 1	A	$(\overline{3.5}, 1.5)$	$(\bar{5}, \bar{2})$
	B	$(2, \bar{5})$	$(1.4, 0.6)$

So the pure strategy Nash equilibrium becomes (A, B)

(b) Firstly, the Peter's belief on the value of p is

$$P(p = 0.2) = q, \qquad P(p = 0.7) = 1 - q.$$

To facilitate the later analysis, I will present the general procedure for finding BNE in this case.

We let s_1 and $s_2 = (s_{2L}, s_{2H})$ be the strategies of two players. Here, s_{2L} denotes player 2's strategy when he knows $p = 0.2$ and s_{2H} denotes player 2's when he knows $p = 0.7$

Step 1: Find the best response of player 2 (Ben)

- If Ben is of type L (with $p = 0.2$), then the player 2's payoff matrix becomes

		Player 1	
		A	B
Player 2 $p = 0.2$	A	4	5
	B	2	1.6
	Best response	A	A

- If Ben is of type L (with $p = 0.2$), then the player 2's payoff matrix becomes

		Player 1	
		A	B
Player 2 $p = 0.7$	A	1.5	5
	B	2	0.6
	Best response	B	A

Step 2: Find the best response of player 1 (Peter)

Given the value of q (belief), the expected payoff of player 1 under different strategic profile is given by

	Player 2's strategy			
	(A, A)	(A, B)	(B, A)	(B, B)
A	$q + 3.5(1-q)$ $= 3.5 - 2.5q$	$q + 5(1-q)$ $= 5 - 4q$	$5q + 3.5(1-q)$ $= 3.5 + 1.5q$	$5q + 5(1-q)$ $= 5$
B	$2q + 2(1-q)$ $= 2$	$2q + 1.4(1-q)$ $= 0.6q + 1.4$	$0.4q + 2(1-q)$ $= 2 - 1.6q$	$0.4q$ $+ 1.4(1-q)$ $= 1.4 - q$
Best response	$\begin{cases} A & if\ q = 0.1 \\ A & if\ q = 0.5 \\ B & if\ q = 0.8 \end{cases}$	$\begin{cases} A & if\ q = 0.1 \\ A & if\ q = 0.5 \\ B & if\ q = 0.8 \end{cases}$	A	A

Finally, we determine BNE by doing the following equilibrium analysis:

Player 1's strategy	Player 2's strategy	Best response for player 1	Best response for player 2
A	(A, A)	Yes if $q = 0.1$ Yes if $q = 0.5$ No if $q = 0.8$	No
	(A, B)	Yes if $q = 0.1$ Yes if $q = 0.5$ No if $q = 0.8$	Yes
	(B, A)	Yes	No
	(B, B)	Yes	No
B	(A, A)	No if $q = 0.1$ No if $q = 0.5$ Yes if $q = 0.8$	Yes
	(A, B)	No if $q = 0.1$ No if $q = 0.5$ Yes if $q = 0.8$	No
	(B, A)	No	No
	(B, B)	No	No

Therefore, we conclude that

- When $q = 0.1$, the BNEs will be $(s_1, s_2) = \big(A, (A, B)\big)$
- When $q = 0.5$, the BNEs will be $(s_1, s_2) = \big(A, (A, B)\big)$
- When $q = 0.8$, the BNEs will be $(s_1, s_2) = \big(B, (A, A)\big)$

Generally speaking, Peter consider to apply program B (yielding lower payoff) if he has strong belief that Ben is stronger than him (i.e. q is very high).

Problem 2

Firstly, the payoff function of player i is given by the profit earned by the company. That is,

$$V_i(e_i; e_j) = \left(R + e_i + e_j\right)^2 - c_i e_i^2$$

(a) Given $R = 2, c_1 = 3$ and $c_2 = 4$, the payoff functions of two firms are

$$V_1(e_1; e_2) = (R + e_1 + e_2)^2 - c_1 e_1^2 = (2 + e_1 + e_2)^2 - 3e_1^2$$
$$V_2(e_2; e_1) = (R + e_1 + e_2)^2 - c_2 e_2^2 = (2 + e_1 + e_2)^2 - 4e_2^2$$

The best responses of two players can be found by considering the first order conditions:

$$\frac{\partial V_1}{\partial e_1} = 2(2 + e_1 + e_2) - 6e_1 = 0 \Rightarrow e_1^* = \frac{2 + 2e_2}{2}$$

$$\frac{\partial V_1}{\partial e_2} = 2(2+e_1+e_2) - 8e_1 = 0 \Rightarrow e_2^* = \frac{2+e_1}{3}$$

Then the Nash equilibrium (w_1^*, w_2^*) can be found by solving

$$\begin{cases} e_1^* = \dfrac{2+2e_2^*}{2} \\ e_2^* = \dfrac{2+e_1^*}{3} \end{cases} \Rightarrow e_1^* = \frac{8}{5}, \qquad e_2^* = \frac{6}{5}.$$

(b) Suppose that firm 1 only knows $c_2 = c_{2L} = 2$ with probability 0.3 and $c_2 = c_{2H} = 4$ with probability 0.7.

- Firm 2's best response (denoted by e_{2L} and e_{2H} respectively, where e_{2k} denotes firm 2's strategy when its cost is $c_2 = c_{2k}$) can be found using similar method as in (a).
  - If the firm's cost is $c_{2L} = 2$, the payoff function becomes
  $$V_2(e_{2L}; e_1, c_{2L}) = (2+e_1+e_{2L})^2 - 2e_{2L}^2.$$
  The best response e_{2L}^* can be found by solving
  $$\frac{\partial V_2}{\partial e_{2L}} = 0 \Rightarrow 2(2+e_1+e_{2L}) - 4e_{2L} = 0 \Rightarrow e_{2L}^* = 2+e_1.$$
  - If the firm's cost is $c_{2H} = 4$, the payoff function becomes
  $$V_2(e_{2H}; e_1, c_{2H}) = (2+e_1+e_{2H})^2 - 4e_{2H}^2.$$
  The best response e_{2L}^* can be found by solving
  $$\frac{\partial V_2}{\partial e_{2H}} = 0 \Rightarrow 2(2+e_1+e_{2H}) - 4e_{2H} = 0 \Rightarrow e_{2H}^* = \frac{2+e_1}{3}.$$
- Given firm 1's belief and firm 2's strategy (e_{2L} and e_{2H}), the expected payoff of firm 1 can be expressed as
$$\begin{aligned} V_1(\cdot) &= 0.3V_1(e_1; e_{2L}) + 0.7V_1(e_1; e_{2H}) \\ &= 0.3(2+e_1+e_{2L})^2 + 0.7(2+e_1+e_{2H})^2 - 3e_1^2 \end{aligned}$$
Then the best response can be found using first order condition:
$$\frac{\partial V_1}{\partial e_1} = 0 \Rightarrow 0.6(2+e_1+e_{2L}) + 1.4(2+e_1+e_{2H}) - 6e_1 = 0$$
$$\Rightarrow e_1^* = \frac{2+0.3e_{2L}+0.7e_{2H}}{2}$$

Therefore, the Bayesian Nash equilibrium can be found by solving

$$\begin{cases} e_1^* = \dfrac{2+0.3e_{2L}^*+0.7e_{2H}^*}{2} \\ e_{2L}^* = 2+e_1^* \\ e_{2H}^* = \dfrac{2+e_1^*}{3} \end{cases} \Rightarrow e_1 = \frac{85}{44}, \qquad e_{2L} = \frac{173}{44}, \qquad e_{2H} = \frac{173}{132}.$$

(c) The calculation is very similar to those in (b) except that firm 1 also has two type: Type L (when $R = R_L = 1$) and Type H (when $R = R_H = 2$)

- Given that firm 2 knows $R = 1$ with probability 0.2 and $R = 2$ with probability 0.8, the expected payoff of firm 2 (with $c_2 = c_{2k}$) is
$$V_2(e_{2k}; e_1, c_{2k}) = 0.2(1+e_{1L}+e_{2k})^2 + 0.8(2+e_{1H}+e_{2k})^2 - c_{2k}e_{2k}^2$$
Then the best response of firm 2 can be found by solving
$$\frac{\partial V_2}{\partial e_{2k}} = 0 \Rightarrow 0.4(1+e_{1L}+e_{2k}) + 1.6(2+e_{1H}+e_{2k}) - 2c_{2k}e_{2k} = 0$$
$$\Rightarrow e_{2k}^* = \frac{1.8+0.2e_{1L}+0.8e_{1H}}{c_{2k}-1}.$$

So we have

$$e_{2L}^* = 1.8 + 0.2e_{1L} + 0.8e_{1H}$$
$$e_{2H}^* = \frac{1.8 + 0.2e_{1L} + 0.8e_{1H}}{3}$$

- On the other hand, the expected payoff of firm 1 (of type k, $R = R_k$) can be expressed as (see (b))

$$V_1(\cdot) = 0.3(R_k + e_{1k} + e_{2L})^2 + 0.7(R_k + e_{1k} + e_{2H})^2 - 3e_{1k}^2$$

So the best response is found to be

$$\frac{\partial V_1}{\partial e_1} = 0 \Rightarrow 0.6(R_k + e_{1k} + e_{2L}) + 1.4(R_k + e_{1k} + e_{2H}) - 6e_{1k} = 0$$
$$\Rightarrow e_{1k}^* = \frac{R_k + 0.3e_{2L} + 0.7e_{2H}}{2}.$$

Therefore, we have

$$e_{1L}^* = \frac{1 + 0.3e_{2L} + 0.7e_{2H}}{2}, \qquad e_{1H}^* = \frac{2 + 0.3e_{2L} + 0.7e_{2H}}{2}.$$

Therefore, the Nash equilibrium can be obtained by solving

$$\begin{cases} e_{1L}^* = \dfrac{1 + 0.3e_{2L}^* + 0.7e_{2H}^*}{2} \\ e_{1H}^* = \dfrac{2 + 0.3e_{2L}^* + 0.7e_{2H}^*}{2} \\ e_{2L}^* = 1.8 + 0.2e_{1L}^* + 0.8e_{1H}^* \\ e_{2H}^* = \dfrac{1.8 + 0.2e_{1L}^* + 0.8e_{1H}^*}{3} \end{cases} \Rightarrow \cdots.$$

Problem 3

We let $s_1 = (s_{1L}, s_{1M}, s_{1H})$ and s_2 be strategies of player 1 and plater 2 respectively. Here, s_{1k} denotes the player 1's operation cost is c_{1k}, where $c_{1L} = 2000$, $c_{1M} = 3000$ and $c_{1H} = 4000$. We also let "D" denotes the strategy which the restaurant offers set dinner and "N" denotes the strategy which the restaurant does not offer set dinner.

Given the players' strategies, the payoff function of player i can be expressed as

$$v_i(s_i; s_j) = \begin{cases} 100(100)(0.5) = 5000 & if\ (s_i, s_j) = (N, N) \\ 100(100)(0.15) = 1500 & if\ (s_i, s_j) = (N, D) \\ 100(100)(0.85) - c_i = 8500 - c_i & if\ (s_i, s_j) = (D, N) \\ 100(100)(0.5) - c_i = 5000 - c_i & if\ (s_i, s_j) = (D, D) \end{cases}.$$

<u>Step 1: Find the best response of player 1</u>

If player 1 is of type L (with $c_L = 2000$), then the player 1's payoff matrix becomes

		Player 2	
		N	D
Player 1	N	5000	1500
	D	6500	3000
	Best response	D	D

If player 1 is of type M (with $c_M = 3000$, then the player 1's payoff matrix becomes

		Player 2	
		N	D
Player 1	N	5000	1500
	D	5500	2000
	Best response	D	D

If player 1 is of type H (with $c_H = 4000$, then the player 1's payoff matrix becomes

		Player 2	
		N	D
Player 1	N	5000	1500
	D	4500	1000
	Best response	N	N

Step 2: Find the best response of player 2

Recall that player 2's belief on player 1's type is

$$P(c_1 = c_L) = 0.3, \qquad P(c_1 = c_M) = 0.3, \qquad P(c_1 = c_H) = 0.4.$$

Then the player 2's expected payoff can be expressed as

$$V_2(s_2; s_1) = 0.3v_2(s_2; s_{1L}) + 0.3v_2(s_2; s_{1M}) + 0.4v_2(s_2; s_{1H})$$

For example, if $s_2 = N$ and $s_1 = (N, N, N)$, then the expected payoff of player 2 is

$$V_2(s_2; s_1) = 0.3\underbrace{v_2(N; N)}_{=5000} + 0.3v_2(N; N) + 0.4v_2(N; N) = 5000$$

The payoff matrix of player 2 is given as follows:

	Player 1's strategy							
	(N, N, N)	(N, N, D)	(N, D, N)	(N, D, D)	(D, N, N)	(D, N, D)	(D, D, N)	(D, D, D)
N	5000	3600	3950	2550	3950	2550	2900	1500
D	5300	3900	4250	2850	4250	2850	3200	1800
Best Response	D	D	D	D	D	D	D	D

Step 3: Determine the BNEs

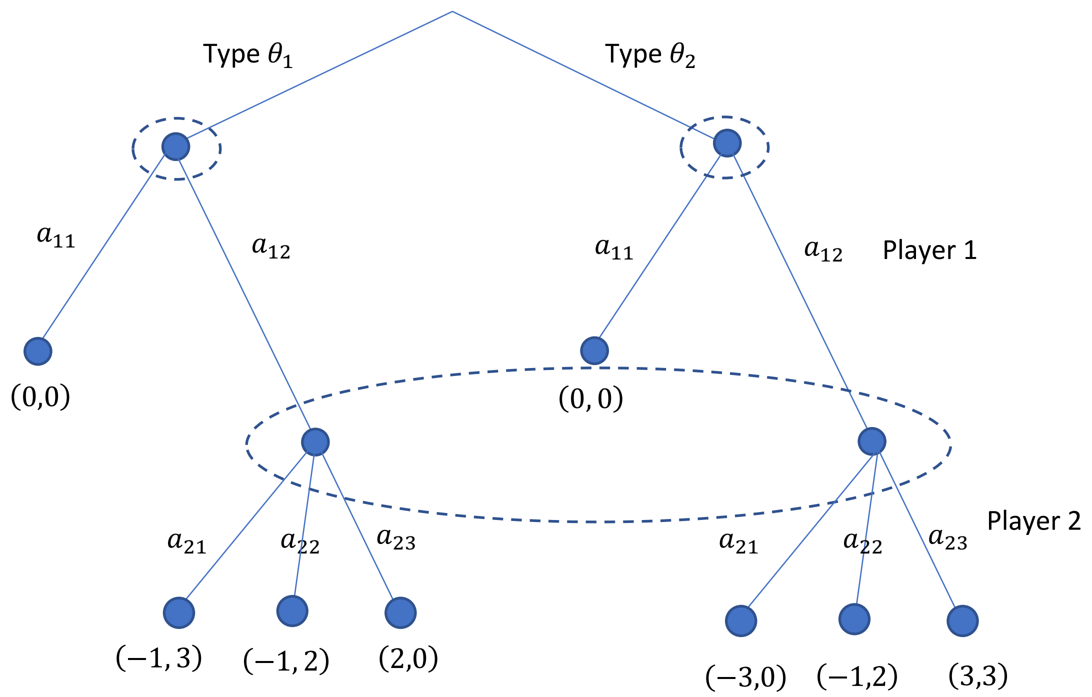
One can do the equilibrium analysis as follows

Player 1's strategy	Player 2's strategy	Best response for player 1	Best response for player 2
(N, N, N)	N	No	No
	D	No	Yes
(N, N, D)	N	No	No
	D	No	Yes
(N, D, N)	N	No	No
	D	No	Yes
(N, D, D)	N	No	No
	D	No	Yes
(D, N, N)	N	No	No
	D	No	Yes
(D, N, D)	N	No	No
	D	No	Yes
(D, D, N)	N	Yes	No
	D	Yes	Yes
(D, D, D)	N	No	No
	D	No	Yes

So there is only one BNE in this games. That is, $(s_{1L}, s_{1M}, s_{1H}) = (D, D, N)$ and $s_2 = D$.

Problem 4

(a) The game tree for this games can be visualized as follows:



(b) We let $s_1 = (s_1(\theta_1), s_1(\theta_2))$ and s_2 be the strategies of player 1 and player 2. Hence $s_1(\theta_j)$ denotes the player 1's strategy when its true type is θ_j.

Step 1: Determine the best response of player 1

If player 1 is of type θ_1 then the player 1's payoff matrix becomes

		Player 2		
		a_{21}	a_{22}	a_{23}
Player 1	a_{11}	0	0	0
	a_{12}	-1	-1	2
	Best response	a_{11}	a_{11}	a_{12}

If player 1 is of type θ_2 then the player 1's payoff matrix becomes

		Player 2		
		a_{21}	a_{22}	a_{23}
Player 1	a_{11}	0	0	0
	a_{12}	-3	-1	3
	Best response	a_{11}	a_{11}	a_{12}

Next, we determine the best response for player 2. Given the player 2's belief on player 1's type (i.e. $p_2(\theta_1) = p$ and $p_2(\theta_2) = 1 - p$), the payoff matrix of player 2 and the best response are given as follows:

	(a_{11}, a_{11})	(a_{11}, a_{12})	(a_{12}, a_{11})	(a_{12}, a_{12})
a_{21}	$0p + 0(1 - p)$ $= 0$	$0p + 0(1 - p)$ $= 0$	$3p + 0(1 - p)$ $= 3p$	$3p + 0(1 - p)$ $= 3p$

a_{22}	$0p + 0(1-p)$ $= 0$	$0p + 2(1-p)$ $= 2(1-p)$	$2p + 0(1-p)$ $= 2p$	$2p + 2(1-p) = 2$
a_{23}	$0p + 0(1-p)$ $= 0$	$0p + 3(1-p)$ $= 3(1-p)$	$0p + 0(1-p)$ $= 0$	$0p + 3(1-p)$ $= 3(1-p)$
Best response	Any	a_{23}	a_{21}	$\begin{cases} a_{23} \ \ if\ p \leq \frac{1}{3} \\ a_{22} \ \ if \frac{1}{3} \leq p \leq \frac{2}{3} \\ a_{21} \ \ if\ p \geq \frac{2}{3} \end{cases}$

(*Note: (1) $3p \geq 2 \Leftrightarrow p \geq \frac{2}{3}$, (2) $2 \geq 3(1-p) \Leftrightarrow p \geq \frac{1}{3}$ and (3) $3p \geq 3(1-p) \Leftrightarrow p \geq \frac{1}{2}$)

Step 3: Find BNEs

One can do so by executing the following equilibrium analysis:

Player 1's strategy	Player 2's strategy	Best response for player 1	Best response for player 2
(a_{11}, a_{11})	a_{21}	Yes	Yes
	a_{22}	Yes	Yes
	a_{23}	No	Yes
(a_{11}, a_{12})	a_{21}	No	No
	a_{22}	No	No
	a_{23}	No	Yes
(a_{12}, a_{11})	a_{21}	No	Yes
	a_{22}	No	No
	a_{23}	No	No
(a_{12}, a_{12})	a_{21}	No	Yes if $p \geq \frac{2}{3}$ No if $p < \frac{2}{3}$
	a_{22}	No	Yes if $\frac{1}{3} \leq p \leq \frac{2}{3}$ No if $p > \frac{1}{2}\ or\ \ p < \frac{2}{3}$
	a_{23}	Yes	Yes if $p \leq \frac{1}{3}$ No if $p > \frac{1}{3}$

Take $p = \frac{1}{3}$, the BNEs are $(s_1, s_2) = \big((a_{11}, a_{11}), a_{21}\big)$, $(s_1, s_2) = \big((a_{11}, a_{11}), a_{22}\big)$, and $(s_1, s_2) = \big((a_{12}, a_{12}), a_{23}\big)$

Step 4: For each BNE obtained in Step 3, check if it is PBE

1. For equilibrium $(s_1, s_2) = \big((a_{12}, a_{12}), a_{23}\big)$, one can observe from the game tree that all information set (player 1 and player 2) can be reached with a positive probability. Thus it follows from Theorem 1 that this equilibrium is also PBE.

2. For equilibrium $(s_1, s_2) = \big((a_{11}, a_{11}), a_{21}\big)$, then the player 2's belief when player 1 chooses a_{12} will be

$$p_2(\theta_1|a_{12}) = q, \qquad p_2(\theta_2|a_{12}) = 1 - q$$

- If player 1 chooses a_{12} and player 2 chooses a_{21}, then the player 2's payoff will be

$$\mathbb{E}[v_2(a_{21}; a_{12})] = q(3) + (1 - q)(0) = 3q$$

Suppose that player 2 deviate and chooses a_{22} or a_{23}, then the player 2's payoff becomes

$$\mathbb{E}[v_2(a_{22}; a_{12})] = q(2) + (1 - q)(2) = 2.$$

$$\mathbb{E}[v_2(a_{23}; a_{12})] = q(0) + (1 - q)(3) = 3(1 - q).$$

Then player 2 has no incentive to deviate if and only if

$$\begin{cases} 3q \geq 2 \\ 3q \geq 3(1-q) \end{cases} \Leftrightarrow \begin{cases} q \geq 2/3 \\ q \geq 1/2 \end{cases} \Leftrightarrow q \geq \frac{2}{3}.$$

By setting $q \geq \frac{2}{3}$, then player 2's decision will be optimal.

Finally, we check the optimality of player 1.

- If player 1 (of type θ_k) chooses a_{11}, the payoff will be 0
- If player 1 chooses a_{12} (and player 2 chooses a_{21}), the payoff will be

$$\begin{cases} -1 & if\ he\ is\ type\ \theta_1 \\ -3 & if\ he\ is\ type\ \theta_2 \end{cases}.$$

So player 1 has no incentive to deviate too.

Therefore, $(s_1, s_2) = \big((a_{11}, a_{11}), a_{21}\big)$ is the PBE.

3. For equilibrium $(s_1, s_2) = \big((a_{11}, a_{11}), a_{22}\big)$, then the player 2's belief when player 1 chooses a_{12} will be

$$p_2(\theta_1|a_{12}) = q, \qquad p_2(\theta_2|a_{12}) = 1 - q$$

- If player 1 chooses a_{12} and player 2 chooses a_{22}, then the player 2's payoff will be

$$\mathbb{E}[v_2(a_{22}; a_{12})] = q(2) + (1 - q)(2) = 2.$$

Suppose that player 2 deviate and chooses a_{22} or a_{23}, then the player 2's payoff becomes

$$\mathbb{E}[v_2(a_{21}; a_{12})] = q(3) + (1 - q)(0) = 3q$$

$$\mathbb{E}[v_2(a_{23}; a_{12})] = q(0) + (1 - q)(3) = 3(1 - q).$$

Then player 2 has no incentive to deviate if and only if

$$\begin{cases} 2 \geq 3q \\ 2 \geq 3(1-q) \end{cases} \Leftrightarrow \begin{cases} q \leq 2/3 \\ q \geq 1/3 \end{cases} \Leftrightarrow \frac{1}{3} \leq q \leq \frac{2}{3}.$$

By setting $q \in \left[\frac{1}{3}, \frac{2}{3}\right]$ in the off-the-equilibrium belief, then player 2's decision will be optimal.

Finally, we check the optimality of player 1.

- If player 1 (of type θ_k) chooses a_{11}, the payoff will be 0
- If player 1 chooses a_{12} (and player 2 chooses a_{22}), the payoff will be

$$\begin{cases} -1 & if\ he\ is\ type\ \theta_1 \\ -1 & if\ he\ is\ type\ \theta_2 \end{cases}.$$

So player 1 has no incentive to deviate too.

Therefore, $(s_1, s_2) = \big((a_{11}, a_{11}), a_{22}\big)$ is also the PBE.

(c) It suffices to check if there is any separating equilibrium which player 1 of different types adopt different strategies. As shown in (b), it appears player 1 of different types adopts the same strategy (either a_{11} or a_{12}) under any equilibrium for any p. Thus, such separating equilibrium does not exist.

Problem 5
One can obtain the Bayesian Nash equilibrium by finding the best response of two players. We let $s_1 = (s_{1S}, s_{1B})$ and $s_2 = (s_{2T}, s_{2W})$ be the strategies of two players. Here

- s_{1k} denotes player 1's strategy when player 2 chooses action k.
- s_{2k} denotes player 2 's strategy if his own type is k.

We consider player 1 first. Given his belief on player 2's type, the expected payoff of player 1 can be computed as

$$V_1(s_1; s_2) = 0.5v_1(s_1; s_{2T}) + 0.5v_1(s_1; s_{2W})$$

under different actions are given as follows:

	Player 2's strategy								
	(N,N)	(N,S)	(N,B)	(S,N)	(S,S)	(S,B)	(B,N)	(B,S)	(B,B)
(F,F)	10	9	6	6	5	2	6	5	2
(F,A)	10	9	7	6	5	3	7	6	4
(A,F)	10	8	6	8	6	4	6	4	2
(A,A)	10	8	7	8	6	4	7	5	4
Best Response	Any	(F,F) (F,A)	(F,A) (A,A)	(A,F) (A,A)	(A,F) (A,A)	(A,F) (A,A)	(F,A) (A,A)	(F,A)	(F,A) (F,A)

*Note: Take second column as example:

- If $s_1^* = (F,F)$ and $s_2^* = (N,S)$, the expected payoff is
$$V_1(s_1; s_2) = 0.5\underbrace{v_1(s_1; s_{2T} = N)}_{=10} + 0.5\underbrace{v_1(s_1; s_{2W} = S)}_{=v_1(F;S)=8} = 9$$
- If $s_1^* = (F,A)$ and $s_2^* = (N,S)$, the expected payoff is
$$V_1(s_1; s_2) = 0.5\underbrace{v_1(s_1; s_{2T} = N)}_{=10} + 0.5\underbrace{v_1(s_1; s_{2W} = S)}_{=v_1(F;S)=8} = 9$$
- If $s_1^* = (A,F)$ and $s_2^* = (N,S)$, the expected payoff is
$$V_1(s_1; s_2) = 0.5\underbrace{v_1(s_1; s_{2T} = N)}_{=10} + 0.5\underbrace{v_1(s_1; s_{2W} = S)}_{=v_1(A;S)=8} = 8$$
- If $s_1^* = (A,A)$ and $s_2^* = (N,S)$, the expected payoff is
$$V_1(s_1; s_2) = 0.5\underbrace{v_1(s_1; s_{2T} = N)}_{=10} + 0.5\underbrace{v_1(s_1; s_{2W} = S)}_{=v_1(A;S)=6} = 8$$

Next, we proceed to determine Firm 2's best response

If player 2 is of type T, the payoff matrix is given by

	Player 1's strategy			
Type T	(F,F)	(F,A)	(A,F)	(A,A)
N	0	0	0	0
S	2	2	4	4
B	2	6	2	6
Best response	**S, B**	**B**	**S**	**B**

If player 2 is of type W, the payoff matrix is given by

	Player 1's strategy			
Type W	(F,F)	(F,A)	(A,F)	(A,A)
N	0	0	0	0
S	-4	-4	0	0

B	-6	0	-6	0
Best response	**N**	**N, B**	**N, S**	**Any**

Hence, one can determine the Nash equilibrium through the following ad-hoc checking

Player 1's strategy s_1^*	Player 2's best response with respect to s_1^*	Player 1 playing best response?	Equilibrium?
(F,F)	(S,N)	No	
	(B,N)	No	
(F,A)	(B,N)	Yes	BNE
	(B,B)	Yes	BNE
(A,F)	(S,N)	Yes	BNE
	(S,S)	Yes	BNE
(A,A)	(B,N)	Yes	BNE
	(B,S)	No	
	(B,B)	Yes	BNE

There are 6 BNEs identified.

To determine the PBE, we let μ_1 be the player 1's updated belief on player 2's type. We consider 6 cases:

<u>Case 1: $s_1^* = (F,A)$ and $s_2^* = (B,N)$</u>
Then the updated belief under various action is given by

$$\mu_1(N) = (0,1), \qquad \mu_1(S) = (q, 1-q), \qquad \mu_1(B) = (1,0)$$

Next, we first check the sequential rationality of player 1 (after observing player 2's move):

- If player 2 plays B (so $\mu_1(B) = (1,0)$), then we have
$$\underbrace{\mathbb{E}[v_1(A;B)|\mu_1]}_{Play\ s_{1B}^*=A} = 4 > 2 = \mathbb{E}[v_1(F;B)|\mu_1]$$
- If player 2 plays S (so $\mu_1(S) = (q,1-q)$), then we have
$$\underbrace{\mathbb{E}[v_1(F;S)|\mu_1]}_{Play\ s_{1S}^*=F} = 2q + 8(1-q) = 8 - 6q \ \ and.\ \mathbb{E}[v_1(A;S)|\mu_1] = 6q + 6(1-q) = 6.$$
s_{1S}^* is optimal if and only if $8 - 6q \geq 6 \Rightarrow q \leq \frac{1}{3}$.

Finally, we check the optimality of player 2. Given that $s_1^* = (F,A)$, one can deduce from the above tables that the best responses of player 2 are $s_{2T}^* = B$ and $s_{2W}^* = B\ \ or\ N$. So $s_2^* = (B,N)$ is optimal with respect to $s_1^* = (F,A)$.
Therefore, $s_1^* = (F,A)$ and $s_2^* = (B,N)$ constitutes the PBE.

<u>Case 2: $s_1^* = (F,A)$ and $s_2^* = (B,B)$</u>
Then the updated belief under various action is given by

$$\mu_1(N) = (p, 1-p), \qquad \mu_1(S) = (q, 1-q), \qquad \mu_1(B) = (1,0)$$

The analysis of sequential rationality of player 1 is same as that of Case 1. I omit the details here
Next, we check the optimality of player 2. Given that $s_1^* = (F,A)$, one can deduce from the above tables that the best responses of player 2 are $s_{2T}^* = B$ and $s_{2W}^* = B\ \ or\ N$. So $s_2^* = (B,B)$ is optimal with respect to $s_1^* = (F,A)$.
Therefore, $s_1^* = (F,A)$ and $s_2^* = (B,B)$ constitutes the PBE.

Case 3: $s_1^* = (A, F)$ and $s_2^* = (S, N)$

Then the updated belief under various action is given by

$$\mu_1(N) = (0,1), \qquad \mu_1(S) = (1,0), \qquad \mu_1(B) = (q, 1-q)$$

Next, we first check the sequential rationality of player 1 (after observing player 2's move):

- If player 2 plays B (so $\mu_1(B) = (q, 1-q)$), then we have

$$\underbrace{\mathbb{E}[v_1(F; B)|\mu_1]}_{Play\ s_{1B}^*=F} = 2q + 2(1-q) = 2 \ \ and\ \mathbb{E}[v_1(A; B)|\mu_1] = 4q + 4(1-q) = 4$$

  It is clear that $\mathbb{E}[v_1(A; B)|\mu_1] > \mathbb{E}[v_1(F; B)|\mu_1]$ and player 1 must deviate.

So the sequential rationality of player 1 failed and the strategic profile is not PBE.

Case 4: $s_1^* = (A, F)$ and $s_2^* = (S, S)$

Then the updated belief under various action is given by

$$\mu_1(N) = (0,1), \qquad \mu_1(S) = (0.5, 0.5), \qquad \mu_1(B) = (q, 1-q)$$

Similar to Case 3, one can check the sequential rationality of player 1 failed when player 2 plays B. Thus the strategic profile is not PBE.

Case 5: $s_1^* = (A, A)$ and $s_2^* = (B, N)$

Then the updated belief under various action is given by

$$\mu_1(N) = (0,1), \qquad \mu_1(S) = (q, 1-q), \qquad \mu_1(B) = (1,0)$$

Next, we first check the sequential rationality of player 1 (after observing player 2's move):

- If player 2 plays B (so $\mu_1(B) = (1,0)$), then we have

$$\underbrace{\mathbb{E}[v_1(A; B)|\mu_1]}_{Play\ s_{1B}^*=A} = 4 > 2 = \mathbb{E}[v_1(F; B)|\mu_1]$$

- If player 2 plays S (so $\mu_1(S) = (q, 1-q)$), then we have

$$\mathbb{E}[v_1(F; S)|\mu_1] = 2q + 8(1-q) = 8 - 6q \ \ and.\ \mathbb{E}[v_1(A; S)|\mu_1] = 6q + 6(1-q) = 6.$$

  $s_{1S}^* = A$ is optimal if and only if $6 \geq 8 - 6q \Rightarrow q \geq \frac{1}{3}$.

Finally, we check the optimality of player 2. Given that $s_1^* = (A, A)$, one can deduce from the above tables that the best responses of player 2 are $s_{2T}^* = B$ and $s_{2W}^* = B\ or\ S\ or\ N$. So $s_2^* = (B, N)$ is optimal with respect to $s_1^* = (A, A)$.

Therefore, $s_1^* = (A, A)$ and $s_2^* = (B, N)$ constitutes the PBE.

Case 6: $s_1^* = (A, A)$ and $s_2^* = (B, B)$

Then the updated belief under various action is given by

$$\mu_1(N) = (p, 1-p), \qquad \mu_1(S) = (q, 1-q), \qquad \mu_1(B) = (0.5, 0.5)$$

Next, we first check the sequential rationality of player 1 (after observing player 2's move):

- If player 2 plays B (so $\mu_1(B) = (0.5\ ,0.5)$), then we have

$$\underbrace{\mathbb{E}[v_1(A; B)|\mu_1]}_{Play\ s_{1B}^*=A} = 4(0.5) + 4(0.5) = 4 > 2 = 2(0.5) + 2(0.5) = \mathbb{E}[v_1(F; B)|\mu_1]$$

- If player 2 plays S (so $\mu_1(S) = (q, 1-q)$), then we have

$$\mathbb{E}[v_1(F; S)|\mu_1] = 2q + 8(1-q) = 8 - 6q \ \ and.\ \mathbb{E}[v_1(A; S)|\mu_1] = 6q + 6(1-q) = 6.$$

  $s_{1S}^* = A$ is optimal if and only if $6 \geq 8 - 6q \Rightarrow q \geq \frac{1}{3}$.

Finally, we check the optimality of player 2. Given that $s_1^* = (A, A)$, one can deduce from the above tables that the best responses of player 2 are $s_{2T}^* = B$ and $s_{2W}^* = B\ or\ S\ or\ N$. So $s_2^* = (B, N)$ is optimal with respect to $s_1^* = (A, A)$.

Therefore, $s_1^* = (A, A)$ and $s_2^* = (B, B)$ constitutes the PBE.