Chapter 8 Seasonal Time Series Models

8.1 Introduction

Seasonal time series models are mainly used in some data sets which include seasonal components.

Section 8.2 Omitted

8.3 Seasonal ARIMA model

Seasonal models are very complicated, but only some simple cases are useful in practice.

Example 8.1. Let b_t be the quarterly series data of ice cream sales. It is possible that b_t follows the model:

$$(1 - \Phi B^4)b_t = a_t,$$

where a_t are white noises and $|\Phi| < 1$.

Maybe b_t follows the model:

$$b_t = (1 - \Psi B^4)a_t.$$

Maybe b_t follows the model:

$$(1 - \Phi B^4)b_t = (1 - \Psi B^4)a_t.$$

These are simple seasonal models and 4 is called seasonal period.

Example 8.2. Let b_t be the monthly series data of the U.S. employment figures for males. It is possible that b_t follows the model:

$$(1 - \Phi B^{12})b_t = a_t,$$

where a_t are white noises and $|\Phi| < 1$.

Maybe b_t follows the model:

$$b_t = (1 - \Psi B^{12})a_t.$$

Maybe b_t follows the model:

$$(1 - \Phi B^{12})b_t = (1 - \Psi B^{12})a_t.$$

These are simple seasonal models and 12 is called seasonal period.

Pure seasonal ARMA(P, Q) model:

$$\Phi_P(B^s)b_t = \Theta_Q(B^s)a_t,$$

where s is a positive integer and a_t are white noises with variance σ_a^2,

$$\Phi_P = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps},$$
$$\Theta_Q = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{Qs}.$$

b_t is called (pure) seasonal ARMA$(P,Q)_s$ model.

How to find s? Up to your problem !!

However, s usually is 4 (quarterly data) or 12(monthly data).

Properties:

If $\Phi_P(z)$ and $\Theta_Q(z)$ have not common root and all the roots lie outside of the unit circle, the ARMA(P,Q) model is stationary and invertible, with mean μ_b and variance σ_b^2, where

$$\mu_b = Eb_t, \qquad \sigma_b^2 = E(b_t - \mu_b)^2.$$

Definition:

$$\rho_{js} = \frac{E(b_{t+js} - \mu_b)(b_t - \mu_b)}{\sigma_b^2}$$

is called the between period (seasonal) correlations of b_t.

Example 8.3 Let

$$(1 - 0.9B^{12})b_t = a_t,$$

where a_t ~i.i.d. $\mathcal{N}(0, \sigma_a^2)$. Then $\mu_b = 0$ and $\rho_{12j} = 0.9^j$.

Pure seasonal ARIMA(P, D, Q) model:

$$\Phi_P(B^s)(1 - B^s)^D b_t = \Theta_Q(B^s)a_t,$$

b_t is called (pure) seasonal ARIMA$(P, D, Q)_s$ model.

b_t is not stationary.

Let $W_t = (1 - B^s)^D b_t$. Then W_t is stationary.

Box-Jenkins multiplicative seasonal ARIMA model

$$\Phi_P(B^s)\phi_p(B)(1 - B)^d(1 - B^s)^D \dot{Z}_t = \theta_q(B)\Theta_Q(B^s)a_t,$$

where

$$\dot{Z}_t = \begin{cases} Z_t - \mu, & \text{if } d = D = 0, \\ Z_t, & \text{otherwise.} \end{cases}$$

$\phi_p(B)$ and $\theta_q(B)$ are called the AR and MA factors and

$\Phi_P(B)$ and $\Theta_Q(B)$ are called the seasonal AR and MA factors, respectively.

$\dot{Z}_t$ is called (pure) seasonal ARIMA$(p, d, q) \times (P, D, Q)_s$ model.

Remark.

(1). Let $W_t = (1 - B)^d \dot{Z}_t$, W_t is not stationary if $D \neq 0$.

(2). Let $W_t = (1 - B^s)^D \dot{Z}_t$, W_t is not stationary if $d \neq 0$.

(3). Let $W_t = (1 - B)^d (1 - B^s)^D \dot{Z}_t$. W_t is stationary.

Example 8.4. Let us consider the ARIMA $(0,1,1) \times (0,1,1)_{12}$ model:

$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$,

where $a_t \sim$i.i.d. $\mathcal{N}(0, \sigma_a^2)$.

Let $W_t = (1-B)(1-B^{12})$. Then the autocovariance of W_t can be found to be

$$\begin{aligned}
\gamma_0 &= (1+\theta^2)(1+\Theta^2)\sigma_a^2, \\
\gamma_1 &= -\theta(1+\Theta^2)\sigma_a^2, \\
\gamma_{11} &= \theta\Theta\sigma_a^2, \\
\gamma_{12} &= -\Theta(1+\theta^2)\sigma_a^2, \\
\gamma_{13} &= \theta\Theta\sigma_a^2, \\
\gamma_j &= 0, \quad \text{otherwise.}
\end{aligned}$$

The ACF becomes:

$$\begin{aligned}
\rho_1 &= \frac{-\theta}{1+\theta^2}, \\
\rho_{11} &= \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} = \rho_{13}, \\
\rho_{12} &= \frac{-\Theta}{1+\Theta^2}, \\
\rho_j &= 0, \quad \text{otherwise.}
\end{aligned}$$

Section 8.4. Empirical Examples

Example 8.5. Simulating 150 values from an ARMA$(0,1,1) \times (0,1,1)_4$:

$(1-B)(1-B^4)Z_t = (1-\theta B)(1-\Theta B^4)a_t,$

with $\theta = 0.8$ and $\Theta = 0.6$ and $a_t \sim$i.i.d. $\mathcal{N}(0,1)$.

Given a real data set, how to build a model?

Example 8.6. International Airline Passengers data in Box and Jenkins (1976).

X_t= the number of Passengers in the $t-$th month.

Step 1. Make a transformation: $Z_t =$ log(X_t), and check whether or not the data are stationary by ACF.

Step 2. Remove nonstationary components:

$$W_t = (1-B)Z_t, \text{ or}$$
$$W_t = (1-B^{12})Z_t, \text{ or}$$
$$W_t = (1-B)(1-B^{12})Z_t.$$

Check whether or not W_t is stationary.

Step 3. Note that $W_t = (1 - B)(1 - B^{12})Z_t$ is stationary. So we use the seasonal ARIMA model to fit the data:

$$\Phi_P(B^{12})\phi_p(B)(1 - B)(1 - B^{12})Z_t$$
$$= \theta_q(B)\Theta_Q(B^{12})a_t, \text{ or}$$

$$\Phi_P(B^{12})\phi_p(B)W_t = \theta_q(B)\Theta_Q(B^{12})a_t.$$

Now, the problem is how to find p, q, P and Q !!!

Step 4. Look at the ACF and PACF of W_t or try some different p, q, P and Q.

For example, we try the model:

$$(1 - \phi B)(1 - \Phi B^{12})W_t = a_t.$$

Step 5. Estimate the parameters in $\Phi_P(B^{12})$, $\phi_p, \theta_q(B)$ and $\Theta_Q(B^{12})$.

How to estimate? CLSE, ULSE or MLE methods. The results are:

$$\phi = -0.38, \qquad \Theta = -0.5.$$

Step 6. Diagnostic checking.

Calculate the residuals:

$$e_t = (1 + 0.38B)(1 + 0.52B^{12})W_t.$$

As for ARIMA model, if $\{e_t\}$ are white noises, the model is correct.

Step 7. If the model is wrong, we should try other models. Even if it is correct, we still need to try some possible models.

For example, we try another model:

$$W_t = (1 - \theta B)(1 - \Theta B^{12})a_t.$$

Through **Step 5**, we obtain:

$$\theta = 0.4, \qquad \Theta = 0.61.$$

Through **Step 6**, we know this model is correct, too.

Step 8. Model selection: **AIC**, **BIC**, or **SBC**.

The final model is:

$$\begin{aligned} &W_t = (1 - 0.40B)(1 - 0.61B^{12})a_t, \ \ or \\ &(1 - B)(1 - B^{12}) \log(X_t) \\ &\qquad = (1 - 0.40B)(1 - 0.61B^{12})a_t. \end{aligned}$$

Step 9. Forecasting.