# Chapter 1: Overview

Observe two data sets:

| | | | | |
|---|---|---|---|---|
| Hang Seng Index | 12877 | 12850 | 13023 | $\cdots$ |
| Date | 30.8.04 | 31.8.04 | 01.9.04 | $\cdots$ |

| | | | | |
|---|---|---|---|---|
| Student's Weights | 130kg | 200kg | 45kg | $\cdots$ |
| Students | $A$ | $B$ | $C$ | $\cdots$ |

What is the difference between these two data sets?

**Definition**:

A time series (**TS**) is a sequence of random variables labeled by time $t$:

$$\{Z_1, Z_2, \cdots, Z_t, \cdots\}$$

or

$$\{\cdots, Z_{-1}, Z_0, Z_1, Z_2, \cdots, Z_t, \cdots\}.$$

Denote them by $\{Z_t\}$.

**Example**: Let $Z =$ weather.
Let $Z_t =$ weather on the $t$th day. Then $\{Z_1, Z_2, \cdots, Z_t, \cdots\}$ is a TS.

$Z_t =$exchange rate of USAD/HKD at the $t$th hour.
$Z_t =$daily Hang Seng Index on the $t$th day.
$Z_t =$average personal consumption in HongKong in the $t$th month.
$Z_t =$USA beer production at the $t$th quarter.
$Z_t =$USA tobacco production at the $t$th year.

**All these $\{Z_t\}$ are TS**.

**Time series data are observations of TS** $\{Z_t\}$.

**Example**: Let $Z_t$ =weather on the $t$th day.

| | | | | |
|---|---|---|---|---|
| Weather= | 29° | 30° | 9° | $\cdots$ |
| Date t= | 1 | 2 | 3 | $\cdots$ |
| Notation | $Z_1 = 29^\circ$ | $Z_2 = 30^\circ$ | $Z_3 = 9^\circ$ | $\cdots$ |

**The types of TS data**:

$$\text{Discrete time data} \begin{cases} \text{annual data} \\ \text{monthly data} \\ \text{quarterly data} \\ \text{daily data} \\ \text{hourly data} \end{cases}$$

Continuous time data

**Main objective of TS analysis**:

Past data $\Longrightarrow$TS r.v. $Z_t \Longrightarrow$ future of TS.

$$(a) \qquad E\left(Z_{n+l}|y_1,\cdots,y_n\right),$$

$$(b) \qquad P(a \le Z_{n+l} \le b|y_1,\cdots,y_n) \text{ for some } a < b.$$

## 2.1. **Strict stationarity and weak stationarity**

**Definition**: Let $\{Z_t\}$ be a TS.

When $t = t_1$, we have:

$$Z_{t_1} \to P(Z_{t_1} \le z).$$

When $t = t_{t_1+k}$, we have:

$$Z_{t_1+k} \to P(Z_{t_1+k} \le z).$$

If $P(Z_{t_1} \le z) = P(Z_{t_1+k} \le z)$ for $\forall t_1, k, z$, then we say: $Z_t$ is the first order stationary in distribution.

When $t = t_1, t_2$, we have:

$$(Z_{t_1}, Z_{t_2}) \to P(Z_{t_1} \le z_1, Z_{t_2} \le z_2)$$

When $t = t_1 + k, t_2 + k$, we have:

$$(Z_{t_1+k}, Z_{t_2+k}) \to P(Z_{t_1+k} \le z_1, Z_{t_2+k} \le z_2)$$

If

$$P(Z_{t_1} \le z_1, Z_{t_2} \le z_2) = P(Z_{t_1+k} \le z_1, Z_{t_2+k} \le z_2),$$

for $\forall t_1, t_2, k$ and $(z_1, z_2)$, then we say: $Z_t$ is the second order stationary in distribution.

When $t = t_1, \cdots, t_n$, we have:

$(Z_{t_1}, \cdots, Z_{t_n}) \rightarrow P(Z_{t_1} \leq z_1, \cdots, Z_{t_n} \leq z_n)$

When $t = t_1 + k, \cdots, t_n + k$, we have:

$(Z_{t_1+k}, \cdots, Z_{t_n+k}) \rightarrow P(Z_{t_1+k} \leq z_1, \cdots, Z_{t_n+k} \leq z_n)$

If

$$P(Z_{t_1} \leq z_1, \cdots, Z_{t_n} \leq z_n)$$

$$= P(Z_{t_1+k} \leq z_1, \cdots, Z_{t_n+k} \leq z_n),$$

for $\forall t_1, \cdots, t_n, k$ and $(z_1, \cdots, z_n)$ and $n$, we say: $\{Z_t\}$ is a **strictly stationary TS**.

**Definition**: Let $\{Z_t\}$ be a TS.

Mean function of $Z_t$: $\quad \mu_t = EZ_t$.

Variance function of $Z_t$: $\quad \sigma_t^2 = E(Z_t - \mu_t)^2$.

Covariance function between $Z_{t_1}$ and $Z_{t_2}$:

$$\gamma(t_1, t_2) = E[(Z_{t_1} - \mu_{t_1})(Z_{t_2} - \mu_{t_2})],$$

and their correlation function

$$\rho(t_1, t_2) = \frac{\gamma(t_1, t_2)}{\sqrt{\sigma_{t_1}^2}\sqrt{\sigma_{t_2}^2}}$$

**Definition**: Let $Z_t$ be a TS. If

$$\begin{aligned} \mu_t &= \mu < \infty, \\ \sigma_t^2 &= \sigma^2 < \infty, \\ \gamma(t, t+k) &= \gamma_k, \end{aligned}$$

for any $t$, then $\{Z_t\}$ is said **(second order) weakly stationary**.

**Property**: Assume $\{Z_t\}$ is strictly stationary.

If $E|Z_t| < \infty$, then $\mu_t = \mu < \infty$.

If $E|Z_t|^2 < \infty$, then $\sigma_t^2 = \sigma^2 < \infty$.

Furthermore

$$\gamma(t, t+k) = \gamma_k, \quad \rho(t, t+k) = \rho_k.$$

Thus, if $EZ_t^2 < \infty$, then strictly stationary $\Longrightarrow$ second-order weakly stationary.

**Example 2.2**: Consider the following time sequence

$$Z_t = A\sin(\omega t + \theta)\,,$$

where $A$ is a random variable with a zero mean and a unit variance and $\theta$ is a r.v. with a uniform distribution on the interval $[-\pi, \pi]$ independent of $A$. Then

$$E(Z_t) = 0, \quad E(Z_t Z_{t+k}) = \frac{1}{2}\cos(\omega k).$$

**Example 2.3**: Let $X_t \sim N(0, 1)$ be i.i.d. and $Y_t = \{1, -1\}$ be i.i.d so that $P(Y_t = 1) = P(Y_t = -1) = 1/2$ Let

$$Z_t = \begin{cases} X_t & \text{if } t \text{ is odd} \\ Y_t & \text{if } t \text{ is even} \end{cases},$$

where $\{X_t\}$ and $\{Y_t\}$ are independent. Then

$$EZ_t = 0\,, EZ_t^2 = 1,$$

$$E(Z_t Z_s) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases},$$

$$\rho(t, s) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases}.$$

**From now on**, the term "stationary" means " second-order weakly stationary".

## 2.2 Autocovariance and autocorrelation functions

Let $Z_t$ be a sequence of stationary TS r.v.s. Then $EZ_t = \mu$, a constant.

$\gamma_k = \textbf{cov}(Z_t, Z_{t+k}) = E[(Z_t - \mu)(Z_{t+k} - \mu)]$ only depends on $k$, $\gamma_k$ is called **autocovariance** (ACV) of $Z_t$.

Let

$$\rho_k = \frac{\textbf{cov}(Z_t, Z_{t+k})}{\sqrt{\textbf{var}(Z_t)}\sqrt{\textbf{var}(Z_{t+k})}} = \frac{\gamma_k}{\gamma_0}.$$

Then $\rho_k$ only depends on $k$. $\rho_k$ is called **autocorrelation function** (ACF) of $Z_t$.

Properties of $\gamma_k$ and $\rho_k$:

(1). $\gamma_0 = \sigma^2, \quad \rho_0 = 1.$

(2). $\gamma_k = \gamma_{-k}, \quad \rho_k = \rho_{-k}.$

(3). $\gamma_k \le \gamma_0, \quad \rho_k \le \rho_0.$

**Important point**: The smaller $\rho_k$, the less dependency between $Z_t$ and $Z_{t+k}$.

Intuitively, as $k \to \infty, \rho_k \to 0$, generally.

In general, $\rho_k \neq 0$, this is an important feature of TS r.v.s..

## 2.3 **Partial Autocorrelation function** (PACF).

**Definition**:
Let $Z_t$ be a stationary TS process. The conditional correlation

$$
\begin{aligned}
&Corr(Z_t, Z_{t+k}|Z_{t+1}, \cdots, Z_{t+k-1}) \\
&\qquad = \frac{Cov[(Z_t - \widehat{Z}_t)(Z_{t+k} - \widehat{Z}_{t+k})]}{\sqrt{Var(Z_t - \widehat{Z}_t)Var(Z_{t+k} - \widehat{Z}_{t+k})}}
\end{aligned}
$$

is called the PACF of $Z_t$ and $Z_{t+k}$, denoted by $\phi_{kk}$, where $\hat{Z}_t = E(Z_t|Z_{t+1}, \cdots, Z_{t+k-1})$.

**Formula**: $\phi_{11} = \rho_1$,

$$
\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}
$$

## 2.4 White noise processes

A process $\{a_t\}$ is called a white noise process if

$$
\begin{aligned}
&Ea_t = 0,\\
&\mathbf{var}(a_t) = \sigma_a^2,\\
&\gamma_k = \mathbf{cov}(a_t, a_{t+k}) = 0, \quad \text{if } k \neq 0.
\end{aligned}
$$

**Properties** of the white noise: if $a_t$ is a white noise, then

$$\text{(1).(ACV)} \quad \gamma_k = \begin{cases} \sigma_a^2 & k = 0 \\ 0 & k \neq 0 \end{cases},$$

$$\text{(2).(ACF)} \quad \rho_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases},$$

$$\text{(3).(PACF)} \quad \phi_{kk} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

## 2.5 Estimation of ACV and ACF

Given $Z_1, Z_2, \cdots, Z_n$, how to estimate $\mu, \sigma^2, \gamma_k$ and $\rho_k$?

### 2.5.1 Sample mean

$$\overline{Z} = \frac{1}{n} \sum_{t=1}^{n} Z_t$$

is called the sample mean of $Z_t$. $\overline{Z}$ is the estimator of the mean $\mu$. Is this estimator valid?

(1). $\overline{Z}$ is an unbiased estimator of $\mu$, i.e.

$$E\overline{Z} = \mu.$$

(2). $\overline{Z}$ is a consistent estimator of $\mu$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} Z_t = \mu\,,$$

almost surely, if $\rho_k \to 0$ as $k \to \infty$. (ergodic property)

### 2.5.2 Sample ACV

$$\widehat{\gamma}_k = \frac{1}{n-k} \sum_{t=1}^{n-k} (Z_t - \overline{Z})(Z_{t+k} - \overline{Z})$$

is called the sample ACV of $Z_t$.

$\widehat{\gamma}_k$ is the estimators of $\gamma_k$.

Are these estimators valid?

(1). $\widehat{\gamma}_k$ is biased estimator of $\gamma_k$, i.e.

$$E\widehat{\gamma}_k \neq \gamma_k,$$

(2). $\widehat{\gamma}_k$ is consistent estimator of $\gamma_k$, i.e.

$$\lim_{n\to\infty} \widehat{\gamma}_k = \gamma_k,$$

if $\rho_k \to 0$ as $k \to \infty$.

In particular,

$$\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n}(Z_t - \overline{Z})^2 = \frac{1}{n}\sum_{t=1}^{n} Z_t^2 - \overline{Z}^2$$

is called the sample variance of $Z_t$.

$\widehat{\sigma}_n^2$ is an estimator of $\sigma^2$, and

$$\lim_{n\to\infty} \widehat{\sigma}_n^2 = \sigma^2 \qquad \text{if} \quad \rho_k \to 0 \quad \text{as} \quad k \to \infty.$$

### 2.5.3 **Sample ACF**.

$$\widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0}$$

is called the sample ACF of $Z_t$.

$\widehat{\rho}_k$ is the estimator of $\rho_k$.

$\widehat{\rho}_k$ is the consistent estimator of $\rho_k$, i.e.

$$\lim_{n\to\infty} \widehat{\rho}_k = \rho_k \qquad \text{if} \quad \rho_k \to 0 \quad \text{as} \quad k \to \infty.$$

Bartlett(1946) showed that

$$\mathbf{Var}(\widehat{\rho}_k) \approx \frac{1}{n} \sum_{i=-\infty}^{\infty} (\rho_i^2 + \rho_{i+k}\rho_{i-k} - 4\rho_k\rho_i\rho_{i-k} + 2\rho_k^2\rho_i^2).$$

In particular, when $Z_t = a_t$ is a white noise, we have

$$\mathbf{Var}(\widehat{\rho}_k) \approx \frac{1}{n}.$$

**How to check whether $Z_t$ is a white noise or not?**

Let

$$S_{\widehat{\rho}_k} = \sqrt{\frac{1}{n}(1 + 2\widehat{\rho}_1^2 + \cdots + 2\widehat{\rho}_m^2)}\ ,$$

where $m$ is a fixed integer.

If $Z_t$ is a white noise, $S_{\widehat{\rho}_k} \approx \sqrt{\frac{1}{n}}$.

### 2.5.4 **Sample PACF**.

$\widehat{\phi}_{11} = \widehat{\rho}_1$,

$$\widehat{\phi}_{kk} = \frac{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & \widehat{\rho}_k \end{vmatrix}}{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-1} \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & 1 \end{vmatrix}}$$

is called the sample PACF of $Z_t$.

$\widehat{\phi}_{kk}$ is the estimator of $\phi_{kk}$ and is consistent.

## 2.6 Moving average and autoregressive representations of time series processes

**Definition**: Moving average representation of $Z_t$:

$$
\begin{aligned}
Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \\
&= \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}\,,
\end{aligned}
$$

where $\psi_0 = 1$, $a_t$ is a white noise, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. ( called Wold's representation or linear process )

**Notation** Backshift operator : $B^j x_t = x_{t-j}$.

Thus, $Z_t$ can be written as

$$
\begin{aligned}
Z_t &= \mu + B^0 a_t + \psi_1 B^1 a_t + \psi_2 B^2 a_t + \cdots \\
&= \mu + \sum_{j=0}^{\infty} \psi_j B^j a_t \\
&= \mu + \left( \sum_{j=0}^{\infty} \psi_j B^j \right) a_t.
\end{aligned}
$$

Denote $\dot{Z}_t = Z_t - \mu$ and $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$.

Then $\dot{Z}_t = \psi(B) a_t$.

**Some properties**:

$$EZ_t = \mu \,,$$

$$\textbf{Var}(Z_t) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2 \,,$$

$$E(a_t Z_{t-j}) = \begin{cases} \sigma_a^2 & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j > 0 \end{cases} \,,$$

$$\gamma_k = E(\dot{Z}_t \dot{Z}_{t-k}) = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \,,$$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{j=0}^{\infty} \psi_j^2}.$$

**Definition** Autoregressive representations of $Z_t$:

$$\begin{aligned}\dot{Z}_t &= \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \cdots + a_t \\ &= \sum_{j=0}^{\infty} \pi_j \dot{Z}_{t-j} + a_t\,,\end{aligned}$$

where $\dot{Z}_t = Z_t - \mu$, $1 + \sum_{j=0}^{\infty} |\pi_j| < \infty$.

Let $\pi(B) = 1 - \sum_{j=0}^{\infty} \pi_j B^j$. Then $\pi(B)\dot{Z}_t = a_t$.

**Relationship** of MA and AR representations:

(1) if the root of $\pi(z) = 0$ all lie outside the unit circle, then

$$\pi(B)\dot{Z}_t = a_t \Longrightarrow \dot{Z}_t = \frac{1}{\pi(B)} a_t = \psi(B) a_t.$$

(2) if the root of $\psi(z) = 0$ all lie outside the unit circle, then

$$\dot{Z}_t = \psi(B) a_t \Longrightarrow a_t = \frac{1}{\psi(B)} \dot{Z}_t = \pi(B)\dot{Z}_t.$$

## **2.7** Time Series Models

Let $\{\cdots, Z_{-t}, \cdots, Z_1, Z_0, Z_1, \cdots, Z_t, \cdots\}$ be a sequence of TS r.v.

How to describe the relationship between $Z_t$ and the past data $Z_{t-1}, Z_{t-2}, \cdots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \cdots) + a_t$$

$- - --$ is called time series models.

1. Autoregressive (AR(1)) model:

$$Z_t = \phi Z_{t-1} + a_t,$$

where $\phi$ is a constant and called the parameter.

2. AR(p) model:

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$

where $\phi_i$ is a constant and called the parameter and $p$ is called the order of the AR(p) model.

3. AR($\infty$) model:

$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t.$$

4. Moving-average (MA) model:

$$Z_t = \mu + a_t + \psi a_{t-1} + \psi_2 a_{t2} + \cdots .$$

5. ARMA model.

6. Threshold AR model (Tong 1977).

7. Long memory model (Granger (1980) and Hosking (1981)).

8. GARCH model (Engle, 1982/ Bolleslev 1986).

9 ARMA-GARCH model.

10. Vector ARMA-GARCH model.

...

...

many and many models.

Given a sequence of data, none knows its true model.

However, we can find a better model for the given data.

Chapter 3 Stationary Time Series Models

## 3.1 Autoregressive Processes

### 3.1.1 The first Order Autoregressive AR(1) process

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance $\sigma_a^2$. $\dot{Z}_t$ satisfies the following equation:

$$\dot{Z}_t = \phi \dot{Z}_{t-1} + a_t.$$

$\dot{Z}_t$ is called the AR(1) model.

$$\begin{aligned} \dot{Z}_{t+1} &= \phi \dot{Z}_t + a_{t+1}\,, \\ \dot{Z}_t &= \phi \dot{Z}_{t-1} + a_t\,, \\ \dot{Z}_{t-1} &= \phi \dot{Z}_{t-2} + a_{t-1}. \end{aligned}$$

**A.** Expansion of AR(1) model.

$\dot{Z}_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-1} a_1 + \phi^t \dot{Z}_0.$

When $\phi = 1$,

$$\dot{Z}_t = a_t + a_{t-1} + \cdots + a_1 + \dot{Z}_0.$$

$\dot{Z}_t$ is called the random walk or unstable process.

When $|\phi| > 1$, e.g. $\phi = 3$,

$\dot{Z}_t = a_t + 3a_{t-1} + 3^2 a_{t-2} + \cdots + 3^{t-1} a_1 + 3^t \dot{Z}_0.$

$\dot{Z}_t$ is called the explosive process.

When $|\phi| < 1$, e.g. $\phi = 0.5$,

$\dot{Z}_t = a_t + 0.5a_{t-1} + \cdots + 0.5^{t-1}a_1 + 0.5^t\dot{Z}_0$.

$\dot{Z}_t$ is called stable (?).

When the time $t$ goes very far away today, the impact of the past noises and the initial value on the current value $\dot{Z}_t$ almost disappear !!!

$$\begin{aligned}
\dot{Z}_0 &= \phi\dot{Z}_{-1} + a_0\,, \\
\dot{Z}_{-1} &= \phi\dot{Z}_{-2} + a_{-1}\,, \\
\dot{Z}_{-2} &= \phi\dot{Z}_{-3} + a_{-2}.
\end{aligned}$$

In general, we have the following expansion:

$$\dot{Z}_t = \sum_{i=0}^{m} \phi^i a_{t-i} + \phi^{m+1}\dot{Z}_{t-m-1}.$$

How far can the $m$ go?

Let $S_m = \sum\limits_{i=0}^{m} \phi^i a_{t-i}$.

Whether $\lim\limits_{n\to\infty} S_m$ exists or not?

**Definition**: If

$$E(\xi_m - \xi)^2 \to 0 \quad \text{as} \quad m \to \infty.$$

we say that the sequence $\xi_m$ of random variables converges to the random variable $\xi$ in mean square.

We can prove that

$$S_m \to \sum_{i=0}^{\infty} \phi^i a_{t-i} \quad \text{in mean square.}$$

if and only if $|\phi| < 1$.

The second term $\phi^{m+1}\dot{Z}_{t-m} \to 0$ (??).

Thus, we have the following result:

If and only if $|\phi| < 1$, $\dot{Z}_t$ in the AR(1) model has the following expansion:

$$\dot{Z}_t = \sum_{i=0}^{\infty} \phi^i a_{t-i}\,,$$

where the infinite sum converges in mean square.

**B.** ACF of the AR(1) Process.

When $|\phi| < 1$,

$$\mu = E\dot{Z}_t = E\left(\sum_{i=0}^{\infty} \phi^i a_{t-i}\right) = 0,$$

$$\sigma^2 = \mathbf{Var}(\dot{Z}_t) = \frac{\sigma_a^2}{1-\phi^2},$$

$$\gamma_k = E[(\dot{Z}_t - \mu)(\dot{Z}_{t+k} - \mu)] = \frac{\sigma_a^2 \phi^k}{1-\phi^2},$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k.$$

Thus, in this case, $\dot{Z}_t$ is stationary.

When $|\phi| \geq 1$, $\dot{Z}_t$ is not stationary.

**C.** Partial Autocorrelation function (PACF) of AR(1) Process

$$\phi_{kk} = \begin{cases} \rho_1 = \phi, & k = 1, \\ 0, & k \geq 2. \end{cases}$$

The AR(1) model can be written as:

$$(1 - \phi B)\dot{Z}_t = a_t\,,$$

**Example 3.1** Simulated 250 values from the model:

$$(1 - \phi B)(Z_t - 10) = a_t,$$

where $\phi = 0.9$ and $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

**Example 3.2** Simulated 250 values from the model:

$$(1 - \phi B)(Z_t - 10) = a_t,$$

where $\phi = -0.65$ and $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

### 3.1.2 **The Second Order Autoregressive AR(2) Model**

**A.** Model

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + a_t$$
$$\text{or} \quad \phi(B)\dot{Z}_t = a_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$.

**B.** Condition for stationarity:

**the roots of** $\phi(z) = 0$ **lie outside the unit circle**, or equivalently,

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$

all the roots of $(1 - \phi_1 z - \phi_2 z^2) = 0$ lie outside the unit circle.

Decompose $1 - \phi_1 z - \phi_2 z^2 = (1 - \alpha_1 z)(1 - \alpha_2 z)$.

Then $|\alpha_1| < 1$ and $|\alpha_2| < 1$.

$$(1 - \alpha_1 B)(1 - \alpha_2 B)\dot{Z}_t = a_t.$$

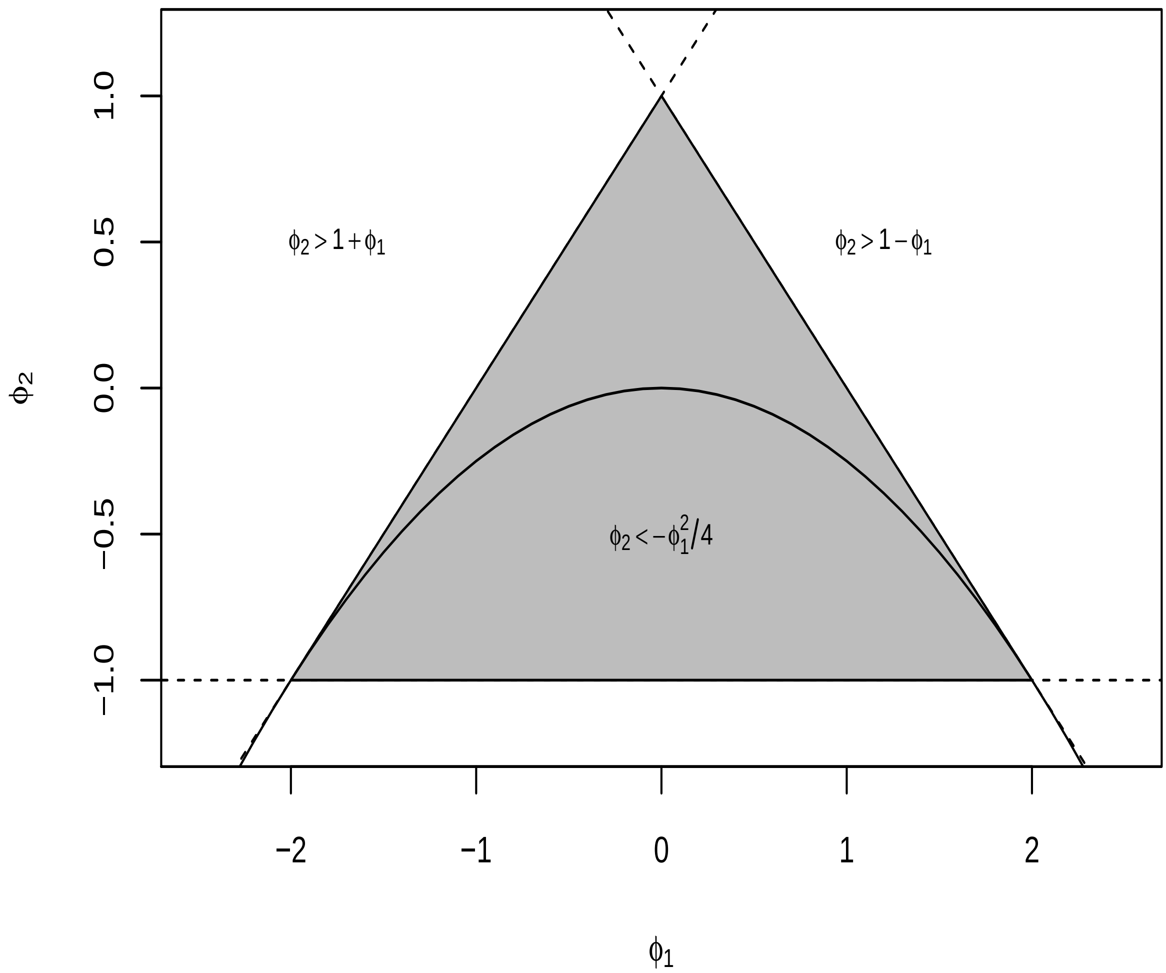
Let $u_t = (1 - \alpha_2 B)\dot{Z}_t$. Then

$$u_t = \alpha_1 u_{t-1} + a_t = a_t + \sum_{i=1}^{\infty} \alpha_1^i a_{t-i}.$$

$$\dot{Z}_t = \alpha_2 \dot{Z}_{t-1} + u_t = u_t + \sum_{j=1}^{\infty} \alpha_2^j u_{t-j}.$$

Stationarity condition is equivalent to

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$




Stationary region for AR(2) model.

**C.** ACF of the AR(2) model:

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}, \quad k \geq 1.$$
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}, \quad k \geq 1.$$

When $k = 1, 2$,

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} \quad \text{and} \quad \rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2}.$$

**D.** PACF of the AR(2) model:

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1 - \phi_2},$$
$$\phi_{22} = \phi_2,$$
$$\phi_{kk} = 0, \quad \text{as} \quad k \geq 3.$$

**Example 3.3** Simulated 250 values from the AR(2) model:

$$(1 - B + 0.5B^2) Z_t = a_t,$$

where $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

## 3.1.3. The General $p$th Order Autoregressive AR($p$) Model

**A.** Model:

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance $\sigma_a^2$.

$\dot{Z}_t$ is said to be an AR($p$) model, if

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t$$
$$\text{or} \quad \phi_p(B)\dot{Z}_t = a_t,$$

where $p$ is an positive integer, and $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.

**B.** Condition for Stationarity:

**the roots of $\phi_p(z) = 0$ lie outside the unit circle**, or equivalently,

**all the eigenvalues of the following matrix lie outside the unit circle**,

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

**C.** ACF of AR($p$) model:

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p}, \quad k > 0.$$

$$\rho_k = \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p}, \quad k > 0.$$

$---$ the difference equation of $\rho_k$.

Solve the following sets of equations:

$$\begin{cases} \rho_1 - \phi_1\rho_0 - \cdots + \phi_p\rho_{p-1} = 0, \\ \cdots\cdots \\ \rho_p - \phi_1\rho_{p-1} - \cdots + \phi_p\rho_0 = 0. \end{cases}$$

$---$ find $\rho_1, \cdots, \rho_p$.

When $k \geq p+1$, calculate $\rho_{p+1}, \rho_{p+2}, \cdots$ by:

$$\begin{aligned} &\rho_{p+1} - \phi_1\rho_p - \cdots - \phi_p\rho_1 = 0, \\ &\cdots\cdots \\ &\rho_k - \phi_1\rho_{k-1} - \cdots - \phi_p\rho_{k-p} = 0. \end{aligned}$$

**D.** PACF of AR($p$) Model:

$\psi_{kk}$ can be obtained from $\rho_1, \cdots, \rho_k$.

In particular, $\psi_{kk} = 0$ when $k > p$.