# MATH4321 Game Theory

## Lecture Note 3 (Part 2)

## Games under incomplete information (Dynamic Games)

## Introduction

In this Chapter, we would like to study Dynamic Games under incomplete information which the players take turn to choose their strategies.

Different from the static games, the players who act at later stages may be able to observe the strategies chosen by players previously. So they can know the current status of the games. In addition, they may be able to obtain more information about the type of other players and they can make a better decision.

- We first introduce the formulation of the dynamic games (in extensive form) and introduce the relevant solution concept (i.e. Perfect Bayesian equilibrium).
- We shall study some types of dynamic games and their applications: Signaling games and repeated games (reputation effect).

**General formulation of dynamic games in extensive form**

Similar to the dynamic games under complete information, the formulation specifies (1) Set of players, (2) Order of moves, (3) information set of the players (i.e. player's knowledge about the status of the games, (4) Strategic set of the players and (5) players' payoffs.

Besides, it will also specify the player's belief on other players' type as well as the belief at different stages of the game (information set).

To demonstrate the idea, we consider the following example:

**Example 1 (Market Entry Game)**

There are two players in the games: Player 1 is a potential entrant to an industry and Player 2 is an incumbent who is monopolistic in this industry.

- Player 1 first decides whether it will enter the industry (E) or will not enter the industry (NE). The game ends if player 1 chooses not enter the industry.

- If player 1 decides to enter the industry, the player 2 (incumbent) will decide either fight against the entry (F) or accommodate the entry (A). The game ends afterwards.

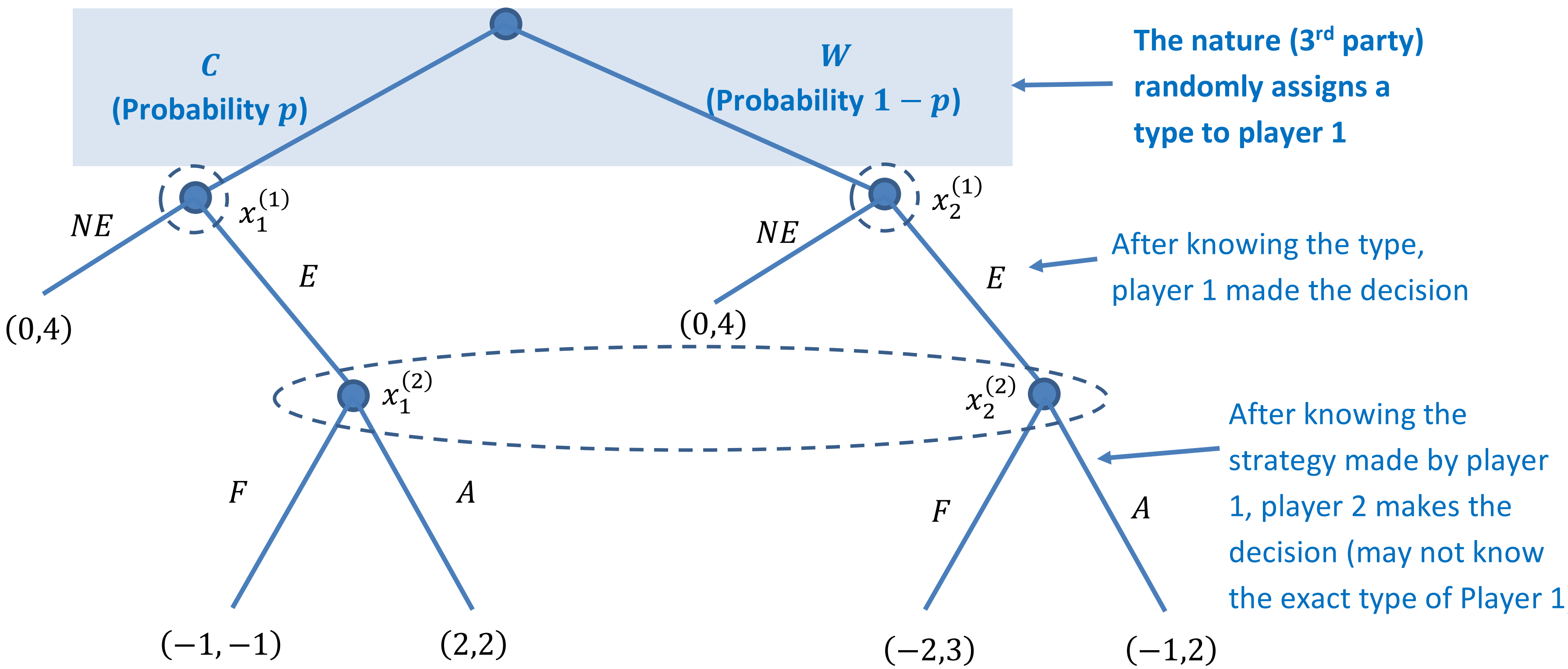
We assume that the incumbent does not know the competitiveness of the entrant. Instead, it conjectures that the entrant is competitive (type $C$) with probability $p$ and is weak (type $W$) with probability $1-p$. The entrant knows its type.

*Payoff structure*

- If the entrant does not enter into the industry, the incumbent will receive a payoff of $4$ and the entrant receives nothing.
- If the entrant enters into the industry, the players' payoff, depends on entrant's type and incumbent strategy, is summarized as follows:

| | Competitive (C) | Weak (W) |
|---|---|---|
| Fight (F) | $(-1,-1)$ | $(-2,3)$ |
| Accommodate (A) | $(2,2)$ | $(-1,2)$ |

One can express the games in extensive form using the game trees as follows:




*Comparing with the dynamic games under complete information, there is an extra step (1st step) which the players' type is assigned.

## Strategic profile and equilibrium concept

In dynamic games, a strategy for a player $i$ (denoted by $s_i(\cdot)$) is a *complete plan of play* that describe which strategy (pure or mixed) player $i$ will choose at each of his/her information sets $h_i$.

- In the above example, player 1's has two information sets (induced from his type). So player 1's strategy can be expressed as
$$s_1 = \left(s_1\left(\left\{x_1^{(1)}\right\}\right), s_1\left(\left\{x_1^{(2)}\right\}\right)\right) \quad or \quad \left(s_1(C), s_1(W)\right).$$
On the other hand, player 2 has only one information set (i.e. $h_2 = \left\{x_1^{(2)}, x_2^{(2)}\right\}$), so his strategy is seen to be $s_2 = \left(s_2(h_2)\right)$.

Under Bayesian Nash equilibrium, we expect that each player of type $\theta_i$ should choose a strategy $s_i^* = s_i^*(\theta_i)$ which is the best response to rival's strategy $s_{-i}^*$. That is, for any $\theta_i \in \Theta_i$, $s_i^*$ should satisfy

$$V_i(s_i^*; s_{-i}^*, \theta_i) = \sum_{\theta_{-i} \in \Theta_i} p_i(\theta_{-i}) v_i(s_i^*; s_{-i}^*, \theta_i) \geq \sum_{\theta_{-i} \in \Theta_i} p_i(\theta_{-i}) v_i(s_i; s_{-i}^*, \theta_i)$$
$$= V_i(s_i; s_{-i}^*, \theta_i) \;\; for\ any\ strategy\ s_i.$$

**Example 2**

Determine all Bayesian Nash equilibrium of the games in Example 1 by taking $p = \frac{1}{2}$

😊Solution

**Step 1: Find the best response of player 1**

- If player 1 is of type $C$, then the payoff matrix is given as follows:

| | | Player 2 | |
|---|---|---|---|
| | | $F$ | $A$ |
| Player 1 | $E$ | $(\mathbf{-1}, -1)$ | $(\mathbf{2}, 2)$ |
| | $NE$ | $(\mathbf{0}, 4)$ | $(\mathbf{0}, 4)$ |

Then the best response of player 1 is $s_1^*(C) = \begin{cases} NE & if\ s_2 = F \\ E & if\ s_2 = A \end{cases}$

- If player 1 is of type $W$, then the payoff matrix is given as follows:

| | | Player 2 | |
|---|---|---|---|
| | | $F$ | $A$ |
| Player 1 | $E$ | $(\mathbf{-2}, 3)$ | $(\mathbf{-1}, 2)$ |
| | $NE$ | $(\mathbf{0}, 4)$ | $(\mathbf{0}, 4)$ |

Then the best response of player 1 is $s_1^*(W) = \begin{cases} NE & if\ s_2 = F \\ NE & if\ s_2 = A \end{cases}$

**Step 2: Find the best response of player 2**

Given his belief, the player 2's expected payoff under different strategic profiles can be computed as:

| | Player 1's strategy $s_1 = \left(s_1(C), s_1(W)\right)$ | | | |
|---|---|---|---|---|
| | $(E,E)$ | $(E,NE)$ | $(NE,E)$ | $(NE,NE)$ |
| $A$ | $\frac{1}{2}(2)+\frac{1}{2}(2) = \overline{\mathbf{2}}$ | $\frac{1}{2}(2)+\frac{1}{2}(4) = \overline{\mathbf{3}}$ | $\frac{1}{2}(4)+\frac{1}{2}(2) = 3$ | $\frac{1}{2}(4)+\frac{1}{2}(4) = \overline{\mathbf{4}}$ |
| $F$ | $\frac{1}{2}(-1) + \frac{1}{2}(3) = \mathbf{1}$ | $\frac{1}{2}(-1) + \frac{1}{2}(4) = \mathbf{1.5}$ | $\frac{1}{2}(4)+\frac{1}{2}(3) = \overline{\mathbf{3.5}}$ | $\frac{1}{2}(4)+\frac{1}{2}(4) = \overline{\mathbf{4}}$ |

By comparing the payoffs, the player 2's best response is found to be

$$s_2^* = \begin{cases} A & if\ \ s_1 = (E,E) \\ A & if\ \ s_1 = (E,NE) \\ F & if\ \ s_1 = (NE,E) \\ A\ \ or\ F & if\ \ s_1 = (NE,NE) \end{cases}$$

Step 3: Determine the Bayesian Nash equilibrium

| $s_1$ | $s_2$ | Best response for player 1? | Best response for player 2? |
| --- | --- | --- | --- |
| $(E, E)$ | $F$ | No | No |
| | $A$ | No | Yes |
| $(E, NE)$ | $F$ | No | No |
| | $A$ | Yes | Yes |
| $(NE, E)$ | $F$ | No | Yes |
| | $A$ | No | No |
| $(NE, NE)$ | $F$ | Yes | Yes |
| | $A$ | No | Yes |

We conclude from the above table that the Bayesian Nash equilibria are $(s_1^*, s_2^*) = \big((E, NE), A\big)$ or $(s_1^*, s_2^*) = \big((NE, NE), F\big)$.

**Equilibrium refinement and Perfect Bayesian Equilibrium (PBE)**

In dynamic games, one expects that each player should choose his/her decision optimally based on the information that he/she has at that time.

As we have observed in dynamic games of complete information, classical Nash equilibrium only guarantees the optimality of players' decision along the *on-the-equilibrium path* and does *not* guarantee the optimality along the *off-the-equilibrium path*.

Hence, one would expect that similar problem may also occur in Bayesian Nash equilibrium. To see this, we revisit the Example 2.

- We consider the equilibrium $(s_1^*, s_2^*) = \big((NE, NE), F\big)$ and consider the off-the-equilibrium path which player 1 chooses $E$.
  - Assuming that there is no information updating on player's type, then the player 2's payoff of choosing $F$ will be $\frac{1}{2}(-1) + \frac{1}{2}(3) = 1$ and the player 2's payoff of choosing $A$ will be $\frac{1}{2}(2) + \frac{1}{2}(2) = 2$.
  - In this case, the player 2 should choose $A$ if he/she observe that player 1 has chosen $E$. It indicates the $s_2 = F$ is not optimal.

On the other hand, the players who act later can acquire additional information about the strategy chosen by players previously. Suppose that the player of different types adopt different strategies (i.e. $s_i(\theta_{ij}) \neq s_i(\theta_{ik})$), then it is possible for other players to learn about the player $i$'s type by observing the strategy chosen by player $i$.

- We consider the equilibrium $s_1 = (E, NE)$ and $s_2 = A$ in Example 2 again. Suppose that the player 1 (the entrant firm) chooses to enter into the industry, then the player 2 (the incumbent firm) will know immediately that player 1 must be competitive (type $C$) since *it is the only type who choose $E$ under equilibrium*.
- In the calculation of Bayesian Nash equilibrium, we apparently "ignored" the possibility that the player 2 can update the belief on player's type.

Based on the above discussions, it appears the Bayesian Nash equilibrium is not an appropriate equilibrium concept for such dynamic games.

*Perfect Bayesian Equilibrium (PBE)*

Recall that in the dynamic games under complete information, we develop the concept of *sequentially rational Nash equilibrium* and *subgame perfect equilibrium* which both require that all players should make the decision optimally at *every* step of the games given his knowledge on the status of the games.

One would have similar expectation on players' strategy in the dynamic games of incomplete information. In particular, we expect that

1. Based on the strategy made by the players' previously, each player should update his/her information on (1) the status of the games and (2) the types of other players as much as possible.
2. Based on the updated belief on the player type and the current status of the games (captured by information set), each player chooses the strategy that maximizes his/her play at every his/her turn.

*Updating player's belief over the game– Baye's Rule*

We consider a player $i$ which has $m$ possible types (denoted by $\theta_{i1}, \theta_{i2}, \dots, \theta_{im}$ respectively). We let

- $p_j(\theta_{ik}) = P(\theta_i = \theta_{ik})$ be the player $j$'s belief on player $i$'s type and
- $s_i(\theta_{ij})$ be the pure strategy (or mixed strategy in general) chosen by player $i$ of type $j$.

Suppose that player $j$ observes that player $i$ has chosen a strategy $A \in S_i$, then player $j's$ updated belief can be estimated as

$$p_j(\theta_{ik}|A) = P(\theta_i = \theta_{ik}|A) = \frac{P(\theta_i = \theta_{ik} \ \ and \ \ A)}{P(A)}$$
$$= \frac{P(A|\theta_i = \theta_{ik})p_j(\theta_{ik})}{\sum_{k=1}^{m} P(A|\theta_i = \theta_{ik})p_j(\theta_{ik})}.$$

The above formula is also known as *Baye's rule*.

(*Note: $P(A|\theta_i = \theta_{ik}) = 0$ or 1 if player $i$ adopts pure strategy.)

**Example 3**

We consider the games in Example 2.

- If the players adopt the equilibrium $s_1^* = (E, NE)$ and $s_2^* = A$
  - If the player 1 chooses $E$, then the player 2's belief is given by

$$p_2(C|E) = \frac{1(p)}{1(p) + 0(1-p)} \overset{p=0.5}{\cong} 1, \qquad p_2(W|E) = \frac{0(1-p)}{1(p) + 0(1-p)} \overset{p=0.5}{\cong} 0$$

  - If the player 1 chooses $NE$, then the corresponding belief is given by

$$p_2(C|NE) = \frac{0(p)}{0(p) + 1(1-p)} \overset{p=0.5}{\cong} 0, \qquad p_2(W|NE) = \frac{1(1-p)}{0(p) + 1(1-p)} \overset{p=0.5}{\cong} 1$$

- If the players adopt equilibrium $s_1^* = (NE, NE)$ and $s_2^* = F$.
  - If the player 1 chooses $NE$, then the corresponding belief is given by

$$p_2(C|NE) = \frac{1(p)}{1(p) + 1(1-p)} \overset{p=0.5}{\cong} \frac{1}{2}, \qquad p_2(W|NE) = \frac{1(1-p)}{1(p) + 1(1-p)} \overset{p=0.5}{\cong} \frac{1}{2}.$$

    (*Note: There is no information updating in this case.)
  - If the player 1 chooses $E$, the player 2's belief cannot be defined by Baye's rule as

$$p_2(C|E) = \frac{0(p)}{0(p) + 0(1-p)} = \frac{0}{0}.$$

So the belief cannot be determined using Baye's rule along *off-the-equilibrium* path.

**Belief system of the players**

In order to determine the equilibrium strategy for the players, each player should have well-defined belief about player's type at each of his/her information set.

**Definition (Belief System)**

A system belief $\mu$ of a dynamic games assigns a probability distribution over decision nodes to every information set. That is, for every information set $h$ and for every decision node $x$, $\mu(x|h)$ denotes the probability that the player is at node $x \in h$. In addition, $\sum_{x \in h} \mu(x|h) = 1$ for every information set $h$.

Given a strategic profile $(s_1, s_2, \dots, s_n)$, we require that

- The player's belief on every information set that are on the equilibrium path should be determined based on the Bayes' rule as shown above.
- For other information set that are off the equilibrium path, the Bayes' rule cannot be applied. In the case, any belief can be assigned. (*Note: In equilibrium analysis, we will set the off-the-equilibrium path in such a way that the players have no incentive to deviate from on-the-equilibrium path)

Given the belief system $\mu$, we expect that the players' strategies must be *sequentially rational* in the sense that in every information set $h$, the players will choose a strategy which is a best response to their beliefs and the rival's strategy. That is, given the rival's strategy $s_{-i}$, the player $i$ chooses the strategy $s_i^*$ such that

$$\underbrace{\mathbb{E}[v_i(s_i^*; s_{-i}, \theta_i)|\mu, h]}_{\substack{Expected\ payoff \\ (given\ belief\ and \\ information\ set)}} \geq \mathbb{E}[v_i(s_i; s_{-i}, \theta_i)|\mu, h] \quad for\ \ any\ s_i \in S_i \ldots \ldots (*)$$

for any information set $h$.

Now, we can state the formal definition of Perfect Bayesian Equilibrium (PBE)

**Definition (Perfect Bayesian Equilibrium)**

A strategic profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ and a belief system $\mu$ constitute Perfect Bayesian Equilibrium if and only if

- Players' strategies are sequentially rational which the inequality $(*)$ holds for all players.
- The belief system $\mu$ that are on-the equilibrium path should be determined according to strategic profile and Bayes' rule.

**Example 4 (Example 2 revisited)**

**(a)** Determine if $(s_1^*, s_2^*) = \big((E, NE), A\big)$ constitutes the Perfect Bayesian Equilibirum (PBE).

**(b)** Determine if $(s_1^*, s_2^*) = \big((NE, NE), F\big)$ constitutes the PBE.

☺Solution

**(a)** We first determine the belief system of the player 2 (the only player who do not know the type of another player).

If the players adopt the equilibrium $s_1^* = (E, NE)$ and $s_2^* = A$

- If the player 1 chooses $E$, then the player 2's belief is given by

$$\underbrace{p_2(C|E)}_{\mu\left(x_1^{(2)}\right)} = \frac{1(p)}{1(p) + 0(1-p)} \overset{p=0.5}{\cong} 1, \qquad \underbrace{p_2(W|E)}_{\mu\left(x_2^{(2)}\right)} = \frac{0(1-p)}{1(p) + 0(1-p)} \overset{p=0.5}{\cong} 0$$

- If the player 1 chooses $NE$, then the corresponding belief is given by

$$p_2(C|NE) = \frac{0(p)}{0(p) + 1(1-p)} \overset{p=0.5}{\cong} 0, \qquad p_2(W|NE) = \frac{1(1-p)}{0(p) + 1(1-p)} \overset{p=0.5}{\cong} 1$$

(*Note: Player 1 knows its own type and there is no uncertainty on player 2's belief, so it is not necessary to define the belief of player 1.)

Next, we proceed to check the sequential rationality of each player.

- We first consider player 2. Note that the player 2's belief is $p_2(C|E) = 1$ and $p_2(W|E) = 0$ if player 1 chooses $E$.
  It follows that
  $$\mathbb{E}[v_2(A; s_1^*)|\mu, h] = \underbrace{p_2(C|E)}_{=1}\underbrace{v_2(A;E)_C}_{=2} + \underbrace{p_2(W|E)}_{=0}\underbrace{v_2(A;E)_W}_{=2} = 2.$$
  $$\mathbb{E}[v_2(F; s_1^*)|\mu, h] = \underbrace{p_2(C|E)}_{=1}\underbrace{v_2(F;E)_C}_{=-1} + \underbrace{p_2(W|E)}_{=0}\underbrace{v_2(F;E)_W}_{=3} = -1.$$
  Since $\mathbb{E}[v_2(A; s_1^*)|\mu, h] > \mathbb{E}[v_2(F; s_1^*)|\mu, h]$, so $s_2^* = A$ is sequentially rational.
- Next, we consider player 1. Since the player 1 knows that player 2 will choose $A$ if it chooses to enter into the industry. It follows that
  - If player 1 is of type $C$ and $s_1^*(C) = E$, we have
    $$v_1(E; A, C) = 2 > 0 = v_1(NE; A, C).$$
  - If player 1 is of type $W$ and $s_1^*(W) = NE$, we have
    $$v_1(NE; A, W) = 0 > -1 = v_1(E; A, W).$$

So the player 1 (of any type) has no incentive to deviate and its strategy is also sequentially rational. Therefore $(s_1^*, s_2^*)$ constitutes the PBE.

**(b)** Next, we consider the strategic profile $(s_1^*, s_2^*) = \big((NE, NE), F\big)$ and determine the belief system for player 2:

- If the player 1 chooses $NE$, then the corresponding belief is given by

$$p_2(C|NE) = \frac{1(p)}{1(p)+1(1-p)} \overset{p=0.5}{=} \frac{1}{2}, \qquad p_2(W|NE) = \frac{1(1-p)}{1(p)+1(1-p)} \overset{p=0.5}{=} \frac{1}{2}.$$

- If the player 1 chooses $E$, the belief cannot be defined using Bayes' rule since it is on off-the-equilibrium path. So we write

$$p_2(C|E) = q, \qquad p_2(W|E) = 1-q,$$

  where $q \in [0,1]$ is some constant.

Suppose that player 1 has chosen $E$ and $s_2^* = F$, one can verify that

$$\mathbb{E}[v_2(F; s_1^*)|\mu, h] = \underbrace{p_2(C|E)}_{=q}\,\underbrace{v_2(F; E, C)}_{=-1} + \underbrace{p(W|E)}_{=1-q}\,\underbrace{v_2(F; E, W)}_{=3} = 3 - 4q$$

$$\mathbb{E}[v_2(A; s_1^*)|\mu, h] = \underbrace{p_2(C|E)}_{=q}\,\underbrace{v_2(A; E, C)}_{=2} + \underbrace{p(W|E)}_{=1-q}\,\underbrace{v_2(A; E, W)}_{=2} = 2$$

Note that $\mathbb{E}[v_2(F; s_1^*)|\mu, h] \geq \mathbb{E}[v_2(A; s_1^*)|\mu, h] \Leftrightarrow 3 - 4q \geq 2 \Leftrightarrow q \leq \frac{1}{4}$

By choosing the off-the-equilibrium belief with some $q \leq \frac{1}{4}$, then the player 2 has no incentive to deviate.

Finally, we consider player 1. Since the player 1 knows that player 2 will choose $F$ if it chooses to enter into the industry. It follows that

- If player 1 is of type $C$ and $s_1^*(C) = NE$, we have
$$v_1(NE; F, C) = 0 > -1 = v_1(E; F, C).$$
- If player 1 is of type $W$ and $s_1^*(W) = NE$, we have
$$v_1(NE; F, W) = 0 > -2 = v_1(E; F, W).$$

**Remarks of Example 4(b)**

- When checking the optimality along the off-the-equilibrium path, we note that the optimality of player's action may depend on the exact off-the-equilibrium path belief (i.e. the player 2's action is optimal only when $q \geq \frac{1}{4}$ in (b)).
- Since the off-the-equilibrium belief can be arbitrary, one just need to argue that the player has no incentive to deviate for *some* off-the-equilibrium path. This indicates that the player's action *may* be optimal in some case.
- However, if there is no off-the-equilibrium belief that supports the optimality of the action. This implies that the player will *always* deviate. Then the proposed strategic profile is no longer PBE.

**How to determine the Perfect Bayesian Equilibrium?**

There are two possible ways to obtain the PBE of a dynamic games:

Method 1: By Ad-hoc searching

- Step 1: Obtain all Bayesian Nash equilibrium (BNE) by definition.
- Step 2: For each BNE, we construct the corresponding belief system.
- Step 3: Check the sequential rationality of players.

(*This method may be tedious if the players' strategic set is large)

Method 2: By Backward induction

Similar to finding subgames perfect equilibrium, one may obtain the PBE through backward induction. At each step, we obtain the optimal strategies of the players for every possible updated belief. Finally, we determine the optimal strategies and the belief system that is consistent to proposed optimal strategies.

**Example 5**

Determine the PBE of Example 2 using backward induction (Method 2).

😊Solution

We will first determine the optimal strategy of player 2.

We let $\mu(E) = (q, 1-q)$ be player 2's updated belief on player 1's type if he knows player 1 chose $E$. Recall that

- The expected payoff of choosing $F$ is $(-1)q + 3(1-q) = 3 - 4q$;
- The expected payoff of choosing $A$ is $2q + 2(1-q) = 2$.

Hence, we conclude that the player 2 will play $F$ if $3 - 4q \geq 2 \Leftrightarrow q \leq \frac{1}{4}$ and play $A$ if $3 - 4q \leq 2 \Leftrightarrow q \geq \frac{1}{4}$.

Next, we determine the player 1's optimal strategy. We consider 2 cases:

*Case 1: $q \leq \frac{1}{4}$ and player 2 plays $F$ if player 1 chooses E*

Then the optimal strategy of player 1 will be $s_1(C) = NE$ (0 v.s. -1) if player 1 is of type $C$ and $s_1(W) = NE$ (0 v.s. -2) if player 1 is of type $W$ respectively.

Since player 1 never chooses $E$, the updated player 2's belief when player 1 chooses $E$ can be arbitrary ($q$ can be any number from 0 to 1) since it is belief on off-the-equilibrium path. By choosing $q \leq \frac{1}{4}$ in the off-the-equilibrium path, the strategic profile $\big((NE, NE), F\big)$ is the PBE.

*Case 2:* $q \geq \frac{1}{4}$ *and player 2 plays* $A$ *if player 1 chooses E*

Then the optimal strategy of player 1 will be $s_1(C) = E$ (0 v.s. 2) if player 1 is of type $C$ and $s_1(W) = NE$ (0 v.s. -1) if player 1 is of type $W$ respectively.

If player 1 chooses $E$, then the player 2 knows that player 1 must be of type $C$ so that the corresponding belief is $\mu(E) = \big(p_2(C), p_2(W)\big) = (1,0)$ which corresponding to $q = 1$.

Since the player 2's optimality is consistent to the belief system, so we conclude that the strategic profile $\big((E, NE), A\big)$ is the PBE.

*Note: Since player 2's updated belief cannot be determined until the player 1's strategy is known, so this is the reason why we need to determine player 2's optimal strategy for any possible belief when using backward induction.

**Example 6**

There are two candidates competing for president of a student society in coming year. You are given that

- Candidate 1 (Player 1) is former president of the society in last year. He has either broad base of support (B) or small base of support (S).
- Candidate 2 (Player 2) is a new candidate. He does not know the level of support that Candidate 1 has. He conjectures that there is probability $p = 0.6$ that the Candidate 1 has broad base of support and there is probability 0.4 that the Candidate 1 has small base of support.
- The Candidate 1 will first decide the amount of effort to spend on election: low amount (L) or high amount (H). The decision can be observed by the rival.
- The Candidate 2 will then decide whether to race (R) or not to race (N).

*Payoff structure*

Suppose that the Candidate 1 chooses to spend low amount (L) on election, then given the support base and the decision made by candidate 2, the payoffs are given by the following payoff matrix:

| | | Player 2's decision | |
|---|---|---|---|
| | | $R$ | $N$ |
| Support Base | $B$ | (6, 4) | (10, 0) |
| | $S$ | (4, 4) | (6, 0) |

If the Candidate 1 chooses to spend high effort (H) on election, the additional cost will be 2 if he have broad base of support and will be 4 if otherwise.

- If the candidate 2 chooses to race against the candidate 1, he will receive a payoff of $-10$ if candidate 1 has broad base of support and the payoff will be $-4$ if candidate has small base of support.

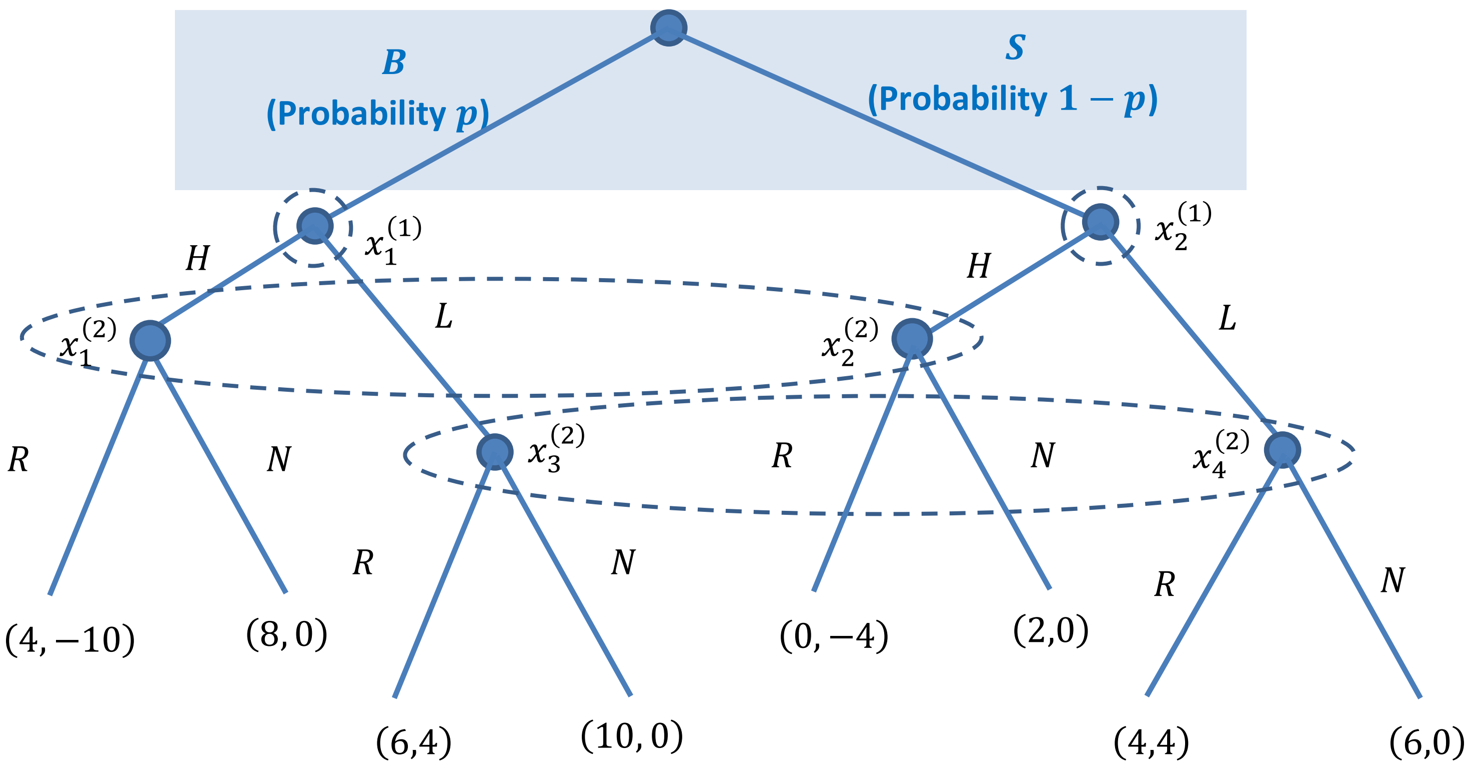
So the payoff matrix will be

| | | Player 2's decision | |
|---|---|---|---|
| | | $R$ | $N$ |
| Support Base | $B$ | (4, -10) | (8, 0) |
| | $S$ | (0, -4) | (2, 0) |

Determine the Perfect Bayesian Nash equilibrium of the games.

☺Solution

Firstly, we draw the game tree for this game:



We let $s_1 = (s_{1B}, s_{1S})$ and $s_2 = (s_{2H}, s_{2L})$ be the strategies of player 1 and player 2. Here, $s_{1j}$ denotes the player 1's strategy if he is of type $j$ and $s_{2k}$ denotes the player 2's strategy if player 1 choose the strategy $k$.

To compute the PBEs, we adopt the first method and obtain all Bayesian Nash equilibria of the games.

<u>Step 1: Find Player 1's best response</u>

If the player is of type B, then his payoff under different strategies is given by

| | $s_2^* = (R, R)$ | $s_2^* = (R, N)$ | $s_2^* = (N, R)$ | $s_2^* = (N, N)$ |
|---|---|---|---|---|
| H | 4 | 4 | 8 | 8 |
| L | **6** | **10** | 6 | 10 |
| **Best response** | **L** | **L** | **H** | **L** |

If the player is of type S, then his payoff under different strategies is given by

| | $s_2^* = (R, R)$ | $s_2^* = (R, N)$ | $s_2^* = (N, R)$ | $s_2^* = (N, N)$ |
|---|---|---|---|---|
| H | 0 | 0 | 2 | 2 |
| L | 4 | 6 | 4 | 6 |
| **Best response** | **L** | **L** | **L** | **L** |

Step 2: Find Player 2's best response

Given $p = 0.6$, the expected payoff of player 2 under different strategies are

| | $s_1^* = (H, H)$ | $s_1^* = (H, L)$ | $s_1^* = (L, H)$ | $s_1^* = (L, L)$ |
|---|---|---|---|---|
| $s_2^* = (R, R)$ | $-10p - 4(1-p) = -7.6$ | $-10p + 4(1-p) = -4.4$ | $4p - 4(1-p) = 0.8$ | $4p + 4(1-p) = 4$ |
| $(R, N)$ | $-10p - 4(1-p) = -7.6$ | $-10p + 0(1-p) = -6$ | $0p - 4(1-p) = -1.6$ | $0p + 0(1-p) = 0$ |
| $(N, R)$ | $0p + 0(1-p) = 0$ | $0p + 4(1-p) = 1.6$ | $4p + 0(1-p) = 2.4$ | $4p + 4(1-p) = 4$ |
| $(N, N)$ | $0p + 0(1-p) = 0$ | $0p + 0(1-p) = 0$ | $0p + 0(1-p) = 0$ | $0p + 0(1-p) = 0$ |
| **Best response** | $(\boldsymbol{N}, \boldsymbol{R}), (\boldsymbol{N}, \boldsymbol{N})$ | $(\boldsymbol{N}, \boldsymbol{R})$ | $(\boldsymbol{N}, \boldsymbol{R})$ | $(\boldsymbol{R}, \boldsymbol{R}), (\boldsymbol{N}, \boldsymbol{R})$ |

By checking every strategic profile (I skipped the detail here), there are two Bayesian Nash equilibrium in this game. That is,

$$(s_1^*, s_2^*) = \big((H, L), (N, R)\big), \qquad (s_1^*, s_2^*) = \big((L, L), (R, R)\big).$$

Next, we check if these two equilibria constitutes the PBE.

We first consider the equilibrium $(s_1^*, s_2^*) = \big((H, L), (N, R)\big)$.

- The belief system of player 2 can be determined using the Baye's rule:
  - If player 1 chooses $H$, then the updated belief becomes

$$p_2^*(B|H) = \frac{p(1)}{p(1) + (1-p)(0)} = 1, \qquad p_2^*(S|H) = \frac{(1-p)(0)}{p(1) + (1-p)(0)} = 0$$

  - If player 1 chooses $L$, then the updated belief becomes

$$p_2^*(B|L) = \frac{p(0)}{p(0) + (1-p)(1)} = 0, \qquad p_2^*(S|L) = \frac{(1-p)(1)}{p(0) + (1-p)(1)} = 1$$

- We first check the optimality of player 2,

  If player 1 chooses $H$ and $s_2^* = N$, one can check that

$$\mathbb{E}[v_2(N; s_1^*)|H] = 0 > -10 = \mathbb{E}[v_2(R; s_1^*)|H]$$

  If player 1 chooses $L$ and $s_2^* = R$, one can verify that

$$\mathbb{E}[v_2(R; s_1^*)|L] = 4 > 0 = \mathbb{E}[v_2(N; s_1^*)|L]$$

  So player 2 has no incentive to deviate from adopting $s_2^*$ under all scenarios and $s_2^*$ is optimal.

- Next, we proceed to check the optimality of player 1

  If player 1 is of type $B$ and chooses $s_{1B} = H$, then we can deduce that

$$v_1(H; s_2^*, B) \overset{s_2^*=N}{\cong} v_1(H; N, B) = 8 > 6 \overset{s_2^*=R}{\cong} v_1(L; R, B).$$

  If player 1 is of type $S$ and chooses $s_{1S} = L$, then we can verify that

$$v_1(L; s_2^*, S) \overset{s_2^*=R}{\cong} v_1(L; R, S) = 4 > 2 \overset{s_2^*=N}{\cong} v_1(H; N, S)$$

  So the player 1 of any type has no incentive to deviate too.

Therefore, we conclude that $(s_1^*, s_2^*) = \big((H, L), (N, R)\big)$ is sequentially rational and is the desired PBE.

Next, we consider the equilibrium $(s_1^*, s_2^*) = \big((L, L), (R, R)\big)$

- The belief system of player 2 can be determined using the Baye's rule:
  - If player 1 chooses $L$, then the updated belief becomes

$$p_2^*(B|L) = \frac{p(1)}{p(1) + (1-p)(1)} = p, \qquad p_2^*(S|L) = \frac{(1-p)(1)}{p(1) + (1-p)(1)} = 1 - p$$

  - If player 1 chooses $H$, then the belief system cannot be determined by Baye's rule since it is off-the-equilibrium path, so we write

$$p_2^*(B|H) = q, \qquad p_2^*(S|L) = 1 - q.$$

    for some $q \in [0,1]$.

- Next, we consider the optimality of player 2,
  If the player 1 chooses $H$ (off-the-equilibrium path) and $s_2^* = R$, one can verify that for *any* $q \in [0,1]$,
  $$\mathbb{E}[v_2(R; s_1^*)|H] = -10q - 4(1-q) < 0 = \mathbb{E}[v_2(N; s_1^*)|H]$$
  The above inequality reveals that the player 2 has incentive to deviate *regardless of the belief.*
  Therefore, the player 2's strategy is not sequentially rational. So $(s_1^*, s_2^*) = \big((L,L),(R,R)\big)$ is not the PBE.

Summarizing all analysis, we conclude the only PBE of the games is $(s_1^*, s_2^*) = \big((H,L),(N,R)\big)$.

*Remark of Example 6*

Alternatively, one can find the same set of PBEs using backward induction. (*I provided a brief calculation and may skip some details.)

We let $\mu(H) = (q_H, 1-q_H)$ and $\mu(L) = (q_L, 1-q_L)$ be the player 2's updated belief when player 1 plays $H$ and plays $L$ respectively.

We first determine the optimal strategy of player 2.

If player 1 plays $H$, then the expected payoff of playing $R$ and $N$ are given by

$$\mathbb{E}[v_2(R;H)] = (-10)q_H + (-4)(1-q_H) = -4 - 6q_H,$$
$$\mathbb{E}[v_2(N;H)] = (0)q_H + (0)(1-q_H) = 0.$$

Since $\mathbb{E}[v_2(R;H)] < \mathbb{E}[v_2(N;H)]$ for all $q_H \in [0,1]$, so player 2 must play $N$.

If player 1 plays $L$, then the expected payoff of playing $R$ and $N$ are given by

$\mathbb{E}[v_2(R;L)] = (4)q_L + (4)(1-q_L) = 4$, $\mathbb{E}[v_2(N;L)] = (0)q_L + (0)(1-q_L) = 0$.

Since $\mathbb{E}[v_2(R;L)] > \mathbb{E}[v_2(N;L)]$ for all $q_L \in [0,1]$, so player 2 must play $R$.

Finally, we proceed to determine the optimal strategy of player 1. Given that $s_{2H}^* = N$ and $s_{2L}^* = R$. The payoff matrix of player 1 can be summarized as follows:

| | **Play H** | **Play L** | **Optimal strategy $s_1^*$** |
|---|---|---|---|
| **Type B** | 8 | 6 | H |
| **Type S** | 2 | 4 | L |

If player 1 plays H, the updated belief will be $\mu(H) = (1,0)$ (i.e. $q_H = 1$) and if player 1 plays $L$, the corresponding updated belief will be $\mu_L = (0,1)$ (i.e. $q_L = 0$). So the optimality of player 2 is guaranteed. So the PBE is seen to be $s_1^* = (H, L)$ and $s_2^* = (N, R)$.

As discussed earlier, the Bayesian Nash equilibrium does not guarantee the optimality of off-the-equilibrium path. So one would conjecture that the equilibrium may also be PBE if there is no off-the-equilibrium path in the sense that every information set can be reached with a positive probability.

**Theorem 1**

We let $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ be a Bayesian Nash equilibrium of a dynamic games. Suppose that the strategic profile $s^*$ induces that every information node can be reached with a positive probability, then $s^*$ and the belief system $\mu^*$ derived uniquely from $s^*$ by Bayes' rule constitutes the Perfect Bayesian Equilibrium.

*Proof of the Theorem 1 (Rough proof)*

Recall that if $s^*$ constitutes the Bayesian Nash equilibrium, it follows that for any player $i$, we have

$$\underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} p_i(\theta_{-i}) v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i)}_{\mathbb{E}[v_i(s_i^*;s_{-i}^*,\theta_i)]} \geq \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} p_i(\theta_{-i}) v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i)}_{\mathbb{E}[v_i(s_i;s_{-i}^*,\theta_i)]} \ldots (*)$$

for any strategy $s_i$.

Suppose that $s^*$ is not PBE, then there exists a player and an information set $h_i$ such that he has incentive to deviate to another strategy $s_i'$ at $h_i$. That is,

$$\mathbb{E}[v_i(s_i'; s_{-i}^*, \theta_i)|\mu^*, h_i] > \mathbb{E}[v_i(s_i^*; s_{-i}^*, \theta_i)|\mu^*, h_i] \ldots \ldots (**)$$

We let $s_{-i}^{h_i}$ denoted the strategies chosen by the rival previously so that the information set $h_i$ is reached. Since $h_i$ is on-the-equilibrium path, so the belief $\mu^*$ can be computed by

$$p_i(\theta_{-i}|s_{-i}^{h_i}) = \frac{P(s_{-i}^{h_i}|\theta_{-i})p_i(\theta_{-i})}{P(s_{-i}^{h_i})} = \frac{p_i(\theta_{-i})}{P(s_{-i}^{h_i})}\mathbf{1}_{\{s_{-i}^{past*}(\theta_{-i})=s_{-i}^{h_i}\}}$$

(*Here, $P(s_{-i}^{h_i})$ denotes the probability that the rival chooses $s_{-i}^{h_i}$ and $s_{-i}^{past*}(\theta_{-i})$ denotes the strategies chosen by rival before $h_i$ is reached)

We let $A$ be the set of rival types $\theta_{-i}$ which $s_{-i}^{past*}(\theta_{-i}) = s_{-i}^{h_i}$. Then the inequality $(**)$ can be rewritten as

$$\underbrace{\sum_{\theta_{-i}\in A}\left(\frac{p_i(\theta_{-i})}{P(s_{-i}^{h_i})}\right)v_i(s_i'; s_{-i}^*(\theta_{-i}), \theta_i)}_{\mathbb{E}[v_i(s_i'; s_{-i}^*, \theta_i)|\mu^*, h_i]} > \underbrace{\sum_{\theta_{-i}\in A}\left(\frac{p_i(\theta_{-i})}{P(s_{-i}^{h_i})}\right)v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i)}_{\mathbb{E}[v_i(s_i^*; s_{-i}^*, \theta_i)|\mu^*, h_i]}$$

$$\Rightarrow \sum_{\theta_{-i}\in A} p_i(\theta_{-i})v_i(s_i'; s_{-i}^*(\theta_{-i}), \theta_i) > \sum_{\theta_{-i}\in A} p_i(\theta_{-i})v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i).$$

Then one can construct a strategy $s^{**}$ for player $i$ as follows:

- Play $s_i'$ at information node $h_i$ and play $s_i^*$ at other information node.

So we can deduce that the expected payoff under new strategy is

$$\begin{aligned}
&\underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} p_i(\theta_{-i})v_i(s_i^{**}; s_{-i}^*(\theta_{-i}), \theta_i)}_{\mathbb{E}[v_i(s_i^{**};s_{-i}^*,\theta_i)]} \\
&= \sum_{\theta_{-i}\in A} p_i(\theta_{-i})v_i(s_i'; s_{-i}^*(\theta_{-i}), \theta_i) + \sum_{\theta_{-i}\in\Theta_{-i}\setminus A} p_i(\theta_{-i})v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i) \\
&> \sum_{\theta_{-i}\in A} p_i(\theta_{-i})v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i) + \sum_{\theta_{-i}\in\Theta_{-i}\setminus A} p_i(\theta_{-i})v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i) \\
&= \sum_{\theta_{-i}\in\Theta_{-i}} p_i(\theta_{-i})v_i(s_i^*; s_{-i}^*(\theta_{-i}), \theta_i)
\end{aligned}$$

So this implies $s_i^*$ is no longer to be the best response with respect to $s_{-i}^*$ and $(s_1^*, \dots, s_n^*)$ is no longer to be Bayesian Nash equilibrium and there is a contradiction.

**Example 7 (Signaling games)**

In this example, we shall investigate the signaling value of the education. That is, whether acquiring a professional qualification can convey a "positive message" to the employers so that the candidate can receive a better job offer.

There are 2 people in this games: Graduated Student (Player 1) and Employer (Player 2).

- The student (Player 1) just graduated from his UG study and wishes to pursue career in certain industry. The student decides whether to pursue an additional professional qualification before getting the job (Q) or apply for the job immediately without pursuing the qualification (N).
    - Getting the qualification requires some effort that is type-dependent: The cost will be $c_H = 2$ if the student has high productivity (Type H) and will be $c_L = 5$ if the student has low productivity (Type L). We assume that the student knows his own type.
- The student applies the job afterwards. After knowing the student's qualification, the employer (Player 2) can assign the student to be either a

manager (M) or a trainee (T). It is given that the market wage for a manager is $w_M = 10$ and the market wage for a trainee is $w_T = 6$.

- However, the employer does not know the actual productivity of the student at the time when he gives the offer to the student. Instead, the employer conjectures that there is a probability $p = 0.25$ that the student has high productivity and the probability that the student has low productivity is $0.75$.
- Player 2's payoff (i.e. profit made by the employer) is determined by the combination of employer's productivity and job assignment. The payoffs are summarized in the following payoff matrix.

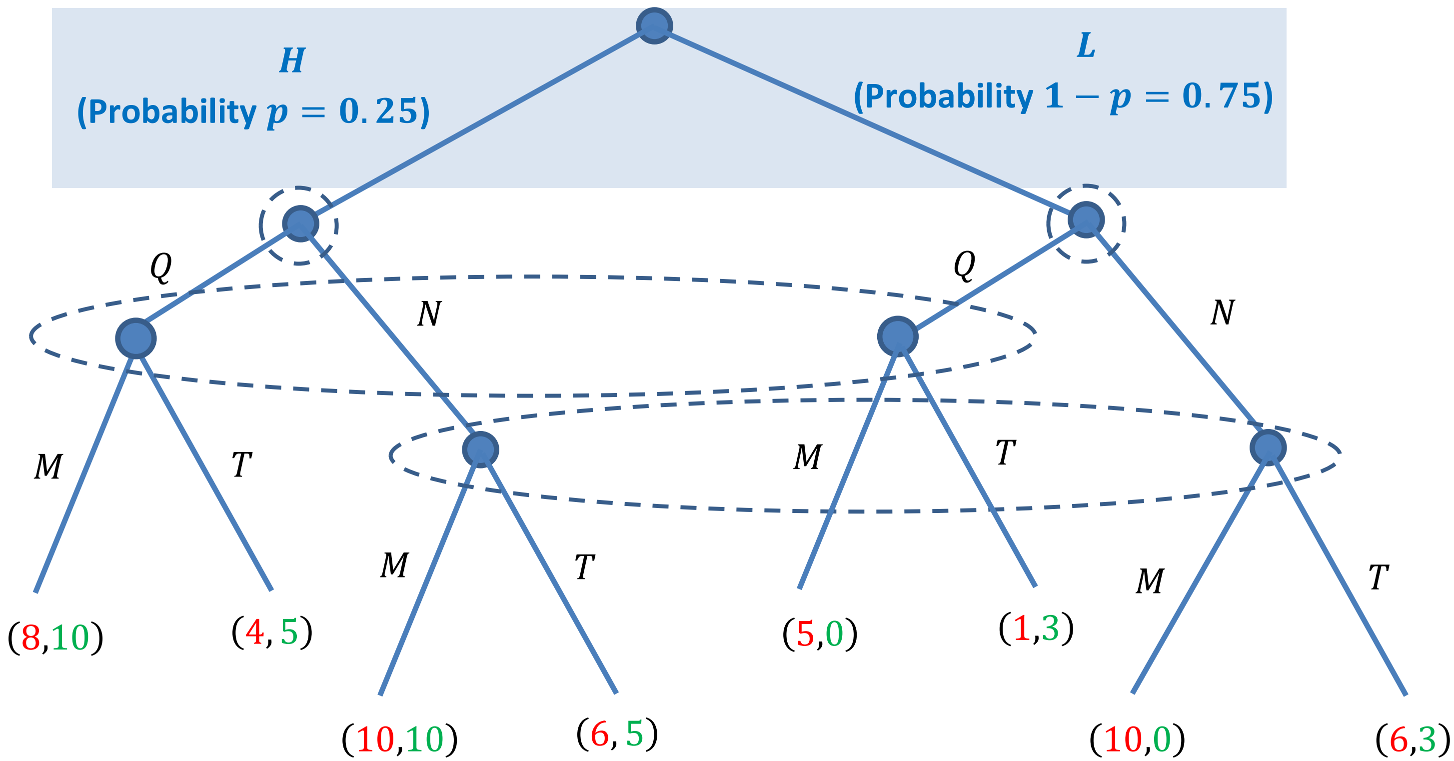
| | | Player 2's decision | |
|---|---|---|---|
| | | $M$ | $T$ |
| Productivity | $H$ | 10 | 5 |
| | $L$ | 0 | 3 |

On the other hand, *the payoff of student (player 1) is simply the wage received minus the cost required for acquiring professional qualification* (Note: The cost will be 0 if the student decides not to get the qualification.)

Determine all Perfect Bayesian Nash equilibrium of the games.

☺Solution

Firstly, one can draw the game tree as follows:



We let $s_1^* = (s_{1H}, s_{1L})$ and $s_2^* = \left(s_{2Q,} s_{2N}\right)$ be the strategic profile of player 1 and player 2 (at different information node). Here, $s_{1k}$ denotes the player 1's strategy given that he is of type $k$ and $s_{2j}$ denotes player 2's strategy if he observes that player 1 plays $j \in \{Q, N\}$.

We first start with finding all Bayesian Nash equilibrium of the games:

**Step 1: Find the best response of player 1**

If the player is of type H, then his payoff under different strategies is given by

| | $s_2^* = (M, M)$ | $s_2^* = (M, T)$ | $s_2^* = (T, M)$ | $s_2^* = (T, T)$ |
|---|---|---|---|---|
| Q | 8 | 8 | 4 | 4 |
| N | 10 | 6 | 10 | 6 |
| **Best response** | **N** | **Q** | **N** | **N** |

If the player is of type L, then his payoff under different strategies is given by

| | $s_2^* = (M, M)$ | $s_2^* = (M, T)$ | $s_2^* = (T, M)$ | $s_2^* = (T, T)$ |
|---|---|---|---|---|
| Q | 5 | 5 | 1 | 1 |
| N | 10 | 6 | 10 | 6 |
| **Best response** | **N** | **N** | **N** | **N** |

Step 2: Find Player 2's best response

Given $p = 0.25$, the expected payoff of player 2 under different strategies are

| | $s_1^* = (Q, Q)$ | $s_1^* = (Q, N)$ | $s_1^* = (N, Q)$ | $s_1^* = (N, N)$ |
|---|---|---|---|---|
| $s_2^* = (M, M)$ | $10p + 0(1-p) = 2.5$ | $10p + 0(1-p) = 2.5$ | $10p + 0(1-p) = 2.5$ | $10p + 0(1-p) = 2.5$ |
| $(M, T)$ | $10p + 0(1-p) = 2.5$ | $10p + 3(1-p) = 4.75$ | $5p + 0(1-p) = 1.25$ | $5p + 3(1-p) = 3.5$ |
| $(T, M)$ | $5p + 3(1-p) = 3.5$ | $5p + 0(1-p) = 1.25$ | $10p + 3(1-p) = 4.75$ | $10p + 0(1-p) = 2.5$ |
| $(T, T)$ | $5p + 3(1-p) = 3.5$ | $5p + 3(1-p) = 3.5$ | $5p + 3(1-p) = 3.5$ | $5p + 3(1-p) = 3.5$ |
| Best response | $(T, M), (T, T)$ | $(M, T)$ | $(T, M)$ | $(M, T), (T, T)$ |

Combining the result (I skipped the details here), there are two Bayesian Nash equilibrium. That is, $(s_1^*, s_2^*) = \big((Q, N), (M, T)\big)$, $(s_1^*, s_2^*) = \big((N, N), (T, T)\big)$.

For the first equilibrium $(s_1^*, s_2^*) = \big((Q,N),(M,T)\big)$,

Since all information set (player 1 and player 2) can be reached with positive probability, so the equilibrium is also Perfect Bayesian Equilibrium by theorem 1.

For the second equilibrium $(s_1^*, s_2^*) = \big((N,N),(T,T)\big)$

- The belief system of player 2 can be determined using the Baye's rule:
  - If player 1 chooses $N$, then the updated belief becomes
  $$p_2^*(H|N) = \frac{p(1)}{p(1) + (1-p)(1)} = 0.25, \qquad p_2^*(L|N) = \frac{(1-p)(1)}{p(1) + (1-p)(1)} = 0.75$$
  - If player 1 chooses $Q$, then the belief system cannot be determined by Baye's rule since it is off-the-equilibrium path, so we write
  $$p_2^*(H|Q) = q, \qquad p_2^*(L|Q) = 1 - q.$$
  for some $q \in [0,1]$.
- We first check the optimality of player 2.
  - If player 1 chooses $N$ and $s_2^* = T$, then one can verify that
  $$\begin{aligned}\mathbb{E}[v_2(T; s_1^*)|N] &= 5p + 3(1-p) = 3.5 > 2.5 = 10p + 0(1-p) \\ &= \mathbb{E}[v_2(M; s_1^*)|N]\end{aligned}$$

- If player 1 chooses $Q$ and $s_2^* = T$, the expected payoffs under different player 2's strategies ($T$ and $M$) are given by
  $\mathbb{E}[v_2(T; s_1^*)|Q] = 5q + 3(1-q) = 3 + 2q$ and
  $\mathbb{E}[v_2(M; s_1^*)|N] = 10q + 0(1-q) = 10q.$
  One can observe that the player 2 has no incentive to deviate from choosing $T$ if and only if
  $$\mathbb{E}[v_2(T; s_1^*)|Q] \geq \mathbb{E}[v_2(M; s_1^*)|N] \Leftrightarrow 3 + 2q \geq 10q \Leftrightarrow q \leq \frac{3}{8}.$$
  Hence, one can set $q \leq \frac{3}{8}$ and the player 2's strategy will be the best response to player 1's strategy under this belief.

▪ Finally, we check the optimality of player 1.
- If the player is of type $H$ and $s_{1H}^* = N$, one can show that
  $$v_1(N; s_2^*, H) \overset{s_{2N}^*=T}{\triangleq} 6 > 4 \overset{s_{2Q}^*=T}{\triangleq} v_1(Q; s_2^*, H)$$
- If the player is of type $L$ and $s_{1L}^* = N$, one can show that
  $$v_1(N; s_2^*, L) = 6 > 1 = v_1(Q; s_2^*, H)$$

So player 1 has no incentive to deviate too.

Therefore, $(s_1^*, s_2^*) = \big((N, N), (T, T)\big)$ is also the PBE.

*Remark of Example 7*

- We consider the equilibrium $(s_1^*, s_2^*) = \big((Q, N), (M, T)\big)$. Note that the player 1 of different types adopt different strategies. The student of high productivity (type H) chooses to pursue additional qualification (i.e. $Q$) and the student of low productivity (type L) chooses not to pursue the qualification (i.e. $N$).
  - In this case, the employer (player 2) can learn the player 1's type by observing the player 1's strategy. The result suggests that getting the qualification can "convince" the employer that the student has high productivity and get a good job offer.
  - However, such equilibrium is no longer to be credible if the cost of acquiring qualification for low productivity student is lower (say $c_L = 3$) To see this, suppose that $(s_1^*, s_2^*) = \big((Q, N), (M, T)\big)$ remains to be PBE when $c_L = 3$. One can show that for player 1 with type $L$
$$v_1(Q; s_2^*, L) \overset{s_2^*(Q)=M}{\triangleq} 7 > 6 \overset{s_2^*(N)=T}{\triangleq} v_1(N; s_2^*).$$
    Thus, the player 1 with type $L$ has incentive to deviate and $(s_1^*, s_2^*)$ is no longer to be PBE.

## Signaling Games

The games in Example 7 is commonly known as *signaling games* in which a player can choose a strategy and signal his/her type to another player who does not know the true type of this player.

The simplest type of the signaling games can be described as follows: There are two players (Player 1 and player 2).

- Player 1 knows his/her own type $\theta_1$. However, player 2 does not know the exact type of player 1. Instead, player 2 conjectures that player 1 can be one of the types in the type space $\Theta_1 = \{\theta_{11}, \theta_{12}, \dots, \theta_{1n_1}\}$ with certain belief function $p_2(\theta_{1k}) = P(\theta_1 = \theta_{1k})$.

- The games consists of two stages:
    - In Stage 1, player 1 chooses a strategy $s_1$ from his/her strategic set $S_1$ and we assume that player 2 knows player 1's strategy chosen.
    - In Stage 2, player 2 updates his/her belief on player 1's strategy and choose his/her strategy $s_2 \in S_2$.

**Classification of equilibrium**

There are three possible types of equilibrium (PBE) in a signaling games.

1. *Separating Equilibrium*

   Under this equilibrium, player 1 of different types adopt different strategies. That is, $s_1(\theta_{1j}) \neq s_1(\theta_{1k})$ for any $\theta_{1j} \neq \theta_{1k}$. Then the player 2 can completely figure out the true type of player 1 by observing the strategy chosen by player 1.

2. *Pooling Equilibrium*

   Under this equilibrium, player 1 of all types will adopt a common strategy. That is, $s_1(\theta_1) = s_1^*$ for all $\theta_1 \in \Theta_1$. In this case, the player 2 fails to update his/her belief on player 1's type by observing the strategy.

3. *Semi-pooling Equilibrium*

   Under this equilibrium, some types of player 1 will choose a common strategy and other types of player 1 will choose other different strategy. In this scenario, the player 2 may be able to obtain partial/complete information on player 1's type by observing player 1's action.

**Example 8 (Limit Pricing and Entry Deterrence)**

There are two firms in this games: The incumbent firm (player 1) who is monopolist in the market and the entrant firm (player 2) who is new to this field.

You are given that

- The incumbent firm has two types: Low-cost type (type $\theta_L$) and High-cost type (type $\theta_H$). The incumbent knows its own type but the entrant only knows that the incumbent firm can be one of these types with 50% probability.
- The games consists of two periods:
  - In the first period ($t = 1$), the incumbent is monopolist and it can set one of two prices $p_L$ and $p_H$. Its profit earned in this period, which depends on its type, is summarized in the following table:

| Type | Profit from $p_L$ | Profit from $p_H$ |
|---|---|---|
| $\theta_L$ | 6 | 8 |
| $\theta_H$ | 1 | 5 |

- In the second period $(t = 2)$, the entrant firm (player 2) will decide whether to enter the market $(E)$ and not to enter the market $(N)$. The payoffs of both players in second period is summarized in the following tables:

| Incumbent's type | Entrant Decision | Incumbent's payoff | Entrant's Payoff |
|---|---|---|---|
| $\theta_L$ | $E$ | 0 | -2 |
| | $N$ | 8 | 0 |
| $\theta_H$ | $E$ | 0 | 3 |
| | $N$ | 5 | 0 |

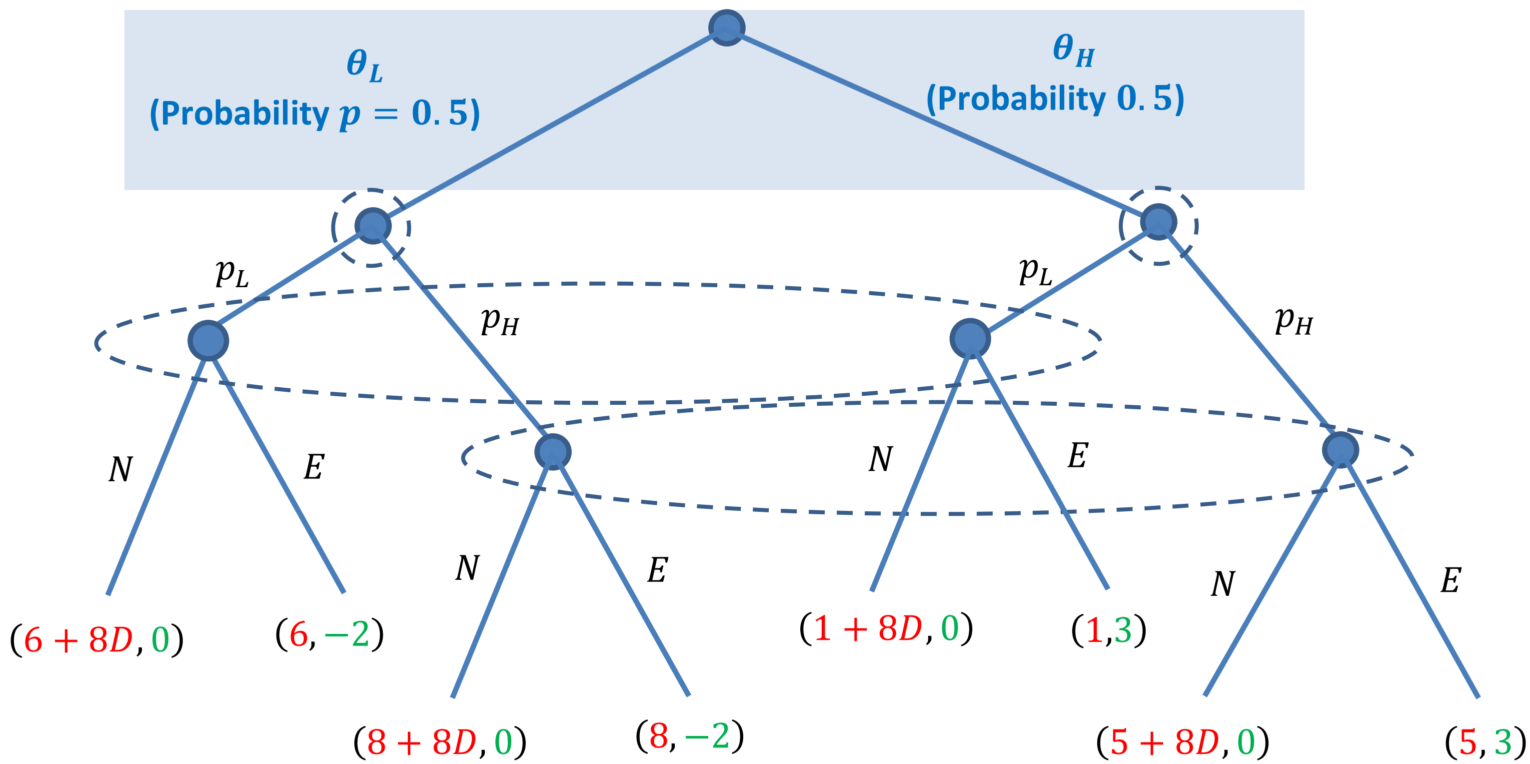
(*Note: We observe that the incumbent firm suffers from a big loss in period 2 if the entrant enters into the market.)

- The payoffs of two players is simply the sum of discounted payoffs received in these two periods. We let $D \in (0,1]$ be the discounting factor over 1 period.

Question: Determine all Perfect Bayesian Equilibrium. Under what values of $D$ that there will be a separating equilibrium which the incumbent's firm can be revealed to the entrant firm?

☺Solution

We can first draw the games tree as follows:



We first start with finding all Bayesian Nash equilibrium for the games. We let $s_{1k}^*$ be the player 1's strategy if it is of type $\theta_k$. We let $s_2^* = \left(s_2^*(p_L), s_2^*(p_H)\right)$ be the player 2's strategy where $s_2^*(p_i)$ denotes the strategy when player 2 knows that player 1 chooses $p_i$.

**Step 1: Find the best response of player 1**

If the player is of type $\theta_L$, then his payoff under different strategies is given by

| | $s_2^* = (E,E)$ | $s_2^* = (E,N)$ | $s_2^* = (N,E)$ | $s_2^* = (N,N)$ |
|---|---|---|---|---|
| $p_L$ | $6$ | $6$ | $6+8D$ | $6+8D$ |
| $p_H$ | $8$ | $8+8D$ | $8$ | $8+8D$ |
| Best response | $p_H$ | $p_H$ | $\begin{cases} p_L & if\ D \geq \frac{1}{4} \\ p_H & if\ D \leq \frac{1}{4} \end{cases}$ | $p_H$ |

If the player is of type $\theta_H$, then his payoff under different strategies is given by

| | $s_2^* = (E,E)$ | $s_2^* = (E,N)$ | $s_2^* = (N,E)$ | $s_2^* = (N,N)$ |
|---|---|---|---|---|
| $p_L$ | $1$ | $1$ | $1+8D$ | $1+8D$ |
| $p_H$ | $5$ | $5+8D$ | $5$ | $5+8D$ |
| Best response | $p_H$ | $p_H$ | $\begin{cases} p_L & if\ D \geq \frac{1}{2} \\ p_H & if\ D \leq \frac{1}{2} \end{cases}$ | $p_H$ |

## Step 2: Find Player 2's best response

The expected payoff of player 2 under different strategies are

| | $s_1^* = (p_L, p_L)$ | $s_1^* = (p_L, p_H)$ | $s_1^* = (p_H, p_L)$ | $s_1^* = (p_H, p_H)$ |
|---|---|---|---|---|
| $s_2^* = (E, E)$ | $0.5(-2) + 0.5(3) = 0.5$ | $0.5(-2) + 0.5(3) = 0.5$ | $0.5(-2) + 0.5(3) = 0.5$ | $0.5(-2) + 0.5(3) = 0.5$ |
| $(E, N)$ | $0.5(-2) + 0.5(3) = 0.5$ | $0.5(-2) + 0.5(0) = -1$ | $0.5(0) + 0.5(3) = 1.5$ | $0.5(0) + 0.5(0) = 0$ |
| $(N, E)$ | $0.5(0) + 0.5(0) = 0$ | $0.5(0) + 0.5(3) = 1.5$ | $0.5(-2) + 0.5(0) = -1$ | $0.5(-2) + 0.5(3) = 0.5$ |
| $(N, N)$ | $0.5(0) + 0.5(0) = 0$ | $0.5(0) + 0.5(0) = 0$ | $0.5(0) + 0.5(0) = 0$ | $0.5(0) + 0.5(0) = 0$ |
| Best response | $(E, E), (E, N)$ | $(N, E)$ | $(E, N)$ | $(E, E), (N, E)$ |

## Step 3: Determine the equilibrium

One has to be careful that the best response of player 1 varies under different level of discounting factors. One has to divide the analysis into 3 major cases. (*Note: I skipped the analysis details here and summarize the result below).

*Summary of Bayesian Nash equilibria:*

**Case 1: $D < \frac{1}{4}$**

Then the player 1's best response to $s_2^* = (N, E)$ is $s_1^* = \left(s_1^*(\theta_L), s_1^*(\theta_H)\right) = (p_H, p_H)$. After the checking, there are two Bayesian Nash equilibria. They are $(s_1^*, s_2^*) = \left((p_H, p_H), (N, E)\right)$ and $(s_1^*, s_2^*) = \left((p_H, p_H), (E. E)\right)$.

**Case 2: $\frac{1}{4} < D < \frac{1}{2}$**

Then the player 1's best response to $s_2^* = (N, E)$ is $s_1^* = \left(s_1^*(\theta_L), s_1^*(\theta_H)\right) = (p_L, p_H)$. After the checking, there are two Bayesian Nash equilibria. They are $(s_1^*, s_2^*) = \left((p_L, p_H), (N, E)\right)$ and $(s_1^*, s_2^*) = \left((p_H, p_H), (E. E)\right)$.

**Case 3: $D > \frac{1}{2}$**

Then the player 1's best response to $s_2^* = (N, E)$ is $s_1^* = \left(s_1^*(\theta_L), s_1^*(\theta_H)\right) = (p_L, p_L)$. After the checking, there is only one Bayesian Nash equilibria. They are $(s_1^*, s_2^*) = \left((p_H, p_H), (E. E)\right)$.

According to the above analysis, there is a potential separating equilibrium when $\frac{1}{4} < D < \frac{1}{2}$. That is, $s_1^* = (p_L, p_H)$ and $s_2^* = (N, E)$.

Since all information sets (player 1 and player 2) can be reached with a positive probability under this strategic profile, it follows that $(s_1^*, s_2^*)$ is also the PBE by theorem 1.

Under this equilibrium, the player 2's belief in different information sets can be computed as

- If player 1 chooses $p_L$, then player 2's belief is found to be

$$p_2(\theta_L|p_L) = \frac{0.5(1)}{0.5(1) + 0.5(0)} = 1, \qquad p_2(\theta_H|p_L) = \frac{0.5(0)}{0.5(1) + 0.5(0)} = 0.$$

- If player 1 chooses $p_H$, then player 2's belief is found to be

$$p_2(\theta_L|p_H) = \frac{0.5(0)}{0.5(0) + 0.5(1)} = 0, \qquad p_2(\theta_H|p_H) = \frac{0.5(1)}{0.5(0) + 0.5(1)} = 1.$$

So player 1's true type can be revealed completely to player 2 under this separating equilibrium.

*Remark of Example 7*

- (*Characterization of other PBEs)*
  For completeness, we shall investigate if there are other equilibria that constitute PBEs.
  - We first consider the equilibrium $(s_1^*, s_2^*) = \big((p_H, p_H), (E.E)\big)$ (which exists for all $D \in (0,1]$).
    If player 1 chooses $p_H$, the player 2's belief is seen to be
    $$p_2(\theta_L|p_H) = \frac{0.5(1)}{0.5(1) + 0.5(1)} = 0.5, \qquad p_2(\theta_H|p_H) = \frac{0.5(1)}{0.5(1) + 0.5(1)} = 0.5.$$
    If player 1 chooses $p_L$, as it is off-the-equilibrium path, so we set
    $$p_2(\theta_L|p_L) = q, \qquad p_2(\theta_H|p_H) = 1 - q.$$
    We first check the optimality of player 2:
    - If player 1 chooses $p_H$ and $s_2^* = E$, we consider
      $$\mathbb{E}[v_2(E; s_1^*)|p_H] = 0.5(-2) + 0.5(3) = 0.5 > 0 = 0.5(0) + 0.5(0) = \mathbb{E}[v_2(N; s_1^*)|p_H]$$
    - If player 1 chooses $p_L$ and $s_2^* = E$, we can deduce that the player 2 has no incentive to deviate if

$$\mathbb{E}[v_2(E; s_1^*)|p_L] = q(-2) + (1-q)(3) = 0 \geq 0 = 0.5(0) + 0.5(0)$$
$$= \mathbb{E}[v_2(N; s_1^*)|p_L] \Leftrightarrow q \leq \frac{3}{5}.$$

By setting $q < \frac{3}{5}$, we see that player 2 has no incentive to deviate given its belief.

- For player 1 (first-mover), one can see from the result in p.44 that $s_1^* = (p_H, p_H)$ is the best response to $s_2^* = (E, E)$.

  So we deduce that $(s_1^*, s_2^*) = \big((p_H, p_H), (E.E)\big)$ constitutes PBE if $q < \frac{3}{5}$.

- Next, we consider the equilibrium$(s_1^*, s_2^*) = \big((p_H, p_H), (N, E)\big)$ (valid when $D < \frac{1}{4}$)

  As player 1 always play $p_H$, so the player 2's belief is same as above.

  If player 1 chooses $p_H$,

$$p_2(\theta_L|p_H) = 0.5, \qquad p_2(\theta_H|p_H) = 0.5.$$

  If player 1 chooses $p_L$, we have

$$p_2(\theta_L|p_L) = q, \qquad p_2(\theta_H|p_H) = 1 - q,$$

  for some $q \in [0,1]$.

We proceed to check the optimality of player 2:

- If player 1 chooses $p_H$ and $s_2^* = E$, we consider

$$\mathbb{E}[v_2(E; s_1^*)|p_H] = 0.5(-2) + 0.5(3) = 0.5 > 0 = 0.5(0) + 0.5(0)$$
$$= \mathbb{E}[v_2(N; s_1^*)|p_H]$$

- If player 1 chooses $p_L$ and $s_2^* = N$ , we can deduce that the player 2 has no incentive to deviate if

$$\mathbb{E}[v_2(E; s_1^*)|p_L] = 0.5(0) + 0.5(0) = 0 \geq q(-2) + (1-q)(3)$$
$$= \mathbb{E}[v_2(E; s_1^*)|p_L] \Leftrightarrow q \geq \frac{3}{5}.$$

By setting $q \geq \frac{3}{5}$, we see that player 2 has no incentive to deviate given its belief.

- For player 1 (first-mover), one can see from the previous result that $s_1^* = (p_H, p_H)$ is the best response to $s_2^* = (N, E)$.

Thus, we conclude that $(s_1^*, s_2^*) = \big((p_H, p_H), (N, E)\big)$ is also PBE if $q \geq \frac{3}{5}$ and $D < \frac{1}{4}$.

- (Importance of Separating Equilibrium and Signaling)
  In this example, the incumbent firm suffers from potential loss at period 2 due to the potential entry of the entrant firm.
  - When the entrant is uncertain about the type of firm 1, the expected payoff of entering into the market is
  $$\mathbb{E}[v_2(E; s_1^*)] = \frac{1}{2}(-2) + \frac{1}{2}(3) = 0.5 > 0 = \mathbb{E}[v_2(N; s_1^*)].$$
    Therefore, the entrant will enter the market for sure at period 2.
- One would like to ask what can the incumbent firm do to block the entry of the entrant firm. Knowing that the entrant will not join the market (as it will receive a payoff of $-3$ if it knows the incumbent firm is of type $\theta_L$. The incumbent firm (of type $\theta_L$) can block the entry by signaling its type to the entrant firm.
  - The above analysis reveals that when $\frac{1}{4} < D < \frac{1}{2}$, there exists a separating equilibrium in which the incumbent firm of type $\theta_L$ may do signal its type by choosing $p_L$ in the first period. Such strategy is known *signaling strategy*.

- However, the incumbent firm's payoff at period 1 will suffer if it chooses $p_L$ to signal its type (yielding a payoff of $6$) instead of choosing $p_H$ (yielding a payoff of $8$) in period 1. Such deviation cost is known *signaling costs*.
- On the other hand, such signaling strategy is only effective if the following conditions hold:
  1. The incumbent firm of type $\theta_H$(cost inefficient firm) has no incentive to *mimic* the firm of type $\theta_L$ and play $p_L$ also.
     - ✓ In this example, we see the firm of type $\theta_H$ has no incentive to do so since when $D < \frac{1}{2}$,

$$\underbrace{v_1(p_L; s_2^* = (N,E), \theta_L)}_{\substack{Player\ 1' spayoff \\ (Mimicing)}} = 1 + 8D < 5 = \underbrace{v_1(p_H; s_2^* = (N,E), \theta_L)}_{\substack{Player\ 1's\ payoff \\ (equilibrium)}}$$

  2. The incumbent firm of type $\theta_L$ (cost efficient firm) is optimal to play $p_L$ and has no incentive to deviate. This happens when

$$v_1(p_L; s_2^*, \theta_L) = 6 + 8D > 8 = v_1(p_H; s_2^*, \theta_L) \Leftrightarrow D > \frac{1}{4}.$$

Then such signaling strategy is credible.

**Example 8 (Harder)**

A company (Player 1) introduced a new smartphone and would like to sell it to customers (Player 2).

- The smartphone may be either of high quality (type $H$) or of medium quality (type $M$) of low quality (type $L$). The company knows the quality of the smart phone but the customer does not. Instead, the customer conjuctures that probabilities that the smartphone is of high quality, medium quality and low quality are $p_H = \frac{1}{3}$, $p_M = \frac{1}{3}$, $p_L = \frac{1}{3}$ respectively.
- To promote the product, the company can choose either to advertise the smartphone ($A$) at a cost $c > 0$ or not to advertise ($NA$).
- (Customer's payoff) After that the consumers decides whether or not to buy the product. It is given that the net payoff to the consumers from buying the smartphone is $2b > 0$ if the smartphone is of high quality, $b$ if the smartphone is of medium quality and $-4b$ if it is of low quality.
- (Company's payoff) It is clear that the revenue of the company will be 0 if the consumer chooses not to buy the smartphone.

On the other hand, suppose that the consumers buy the smartphone,

- If the consumers find that the smartphone is of high quality $(H)$ or medium quality $(M)$, then they will buy more smartphones from this company in future, the company will earn a revenue of $R_H$ if the smartphone is of high quality and $R_M$ if the smartphone is of medium quality. Here, $R_H > R_M > c$,
- If the consumers find that the smartphone of low quality $(L)$, then they will not buy smartphones from this company anymore in future, the company will then earn a small revenue $R_L < c$.

Issue: Suppose that the company is selling the smartphones of high quality $(H)$ but the customers are uncertain about the quality of the smartphone, then the customer may not want to buy it since the expected payoff of buying is

$$Payoff = \frac{1}{3}(2b) + \frac{1}{3}(b) + \frac{1}{3}(-4b) = -\frac{b}{3} < 0.$$

This is certainly not desirable to the company since it will make a zero profit. One would like to ask if it is possible for the company to resolve this problem by advertising the product. Determine all possible Perfect Bayesian equilibrium.

☺Solution

We let $s_1 = (s_{1H}, s_{1M}, s_{1L})$ and $s_2 = \big(s_2(A), s_2(N)\big)$ be the strategies of two players. Here, $s_{1k} \in \{A, N\}$ denotes the strategy of player 1 of type $k$ and $s_2(k) \in \{B, NB\}$ denotes player 2's strategy if player 1 plays the strategy $k$.

We will determine the Perfect Bayesian equilibrium using backward induction by first characterizing the optimal strategies of player 2.

<u>Step 1: Optimal strategies of player 2</u>

We let $\mu(A) = (p_H, p_M, p_L)$ and $\mu(N) = (q_H, q_M, q_L)$ be the updated belief of player 2 when player 1 chooses to play A and N respectively.

- If player 1 plays $A$, then the expected payoff of player 2 under different actions are found to be
$$\mathbb{E}[v_2(B; s_1^*)|A] = 2bp_H + bp_M - 4bp_L \quad and \quad \mathbb{E}[v_2(NB; s_1^*)|A] = 0$$
Thus, player 2 will play $B$ if and only if
$$\mathbb{E}[v_2(B; s_1^*)|A] \geq \mathbb{E}[v_2(NB; s_1^*)|A] \Leftrightarrow 2p_H + p_M - 4p_L \geq 0 \ldots (*)$$
- Similarly if player 1 plays $N$, one can show that player 2 will play B when
$$2q_H + q_M - 4q_L \geq 0 \ldots \ldots (**)$$

Step 2: Optimal strategies of player 1 and consistency check

We consider the following 4 cases:

**Case 1: If $s_2^* = (B, B)$**

Then the player 1's payoff under different actions are given by the following table:

| Action | **Type H** | **Type M** | **Type L** |
|---|---|---|---|
| A | $R_H - c$ | $R_M - c$ | $R_L - c$ |
| N | $R_H$ | $R_M$ | $R_L$ |
| Best response | N | N | N |

Given player 1's optimal strategy, the corresponding player 2's belief is seen to be

$$(p_H, p_M, p_L) = Anything, \qquad (q_H, q_M, q_L) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

However, it implies that $2q_H + q_M - 4q_L = -\frac{1}{3} < 0$ and the player 2 should not play B if player 1 plays N. Thus, there is no PBE in this case.

**Case 2: If $s_2^* = (NB, B)$**

Then the corresponding player 1's payoff under different actions are given by

| Action | **Type H** | **Type M** | **Type L** |
|---|---|---|---|
| A | $-c$ | $-c$ | $-c$ |
| N | $R_H$ | $R_M$ | $R_L$ |
| Best response | N | N | N |

Given player 1's optimal strategy, the corresponding player 2's belief is seen to be

$$(p_H, p_M, p_L) = Anything, \qquad (q_H, q_M, q_L) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

However, it implies that $2q_H + q_M - 4q_L = -\frac{1}{3} < 0$ and the player 2 should not play B if player 1 plays N. Thus, there is no PBE again in this case.

**Case 3: If $s_2^* = (B, NB)$**

Then the corresponding player 1's payoff under different actions are given by

| Action | **Type H** | **Type M** | **Type L** |
|---|---|---|---|
| A | $R_H - c > 0$ | $R_M - c > 0$ | $R_L - c < 0$ |
| N | $0$ | $0$ | $0$ |
| Best response | A | A | N |

Given player 1's optimal strategy, the corresponding player 2's belief is seen to be

$$(p_H, p_M, p_L) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \ (why?), \qquad (q_H, q_M, q_L) = (0,0,1)$$

One can verify that $2p_H + p_M - 4p_L = \frac{3}{2} > 0$ (so player 2 plays B when player 1 plays A) and $2q_H + q_M - 4q_L = -4 < 0$ (so player 2 plays $NB$ when player 1 plays N). As it is consistent to player 2's optimal strategy, we conclude that the PBE (semi-pooling equilibrium) in this case is

$$s_1^* = (A, A, N), \qquad s_2^* = (B, NB).$$

**Case 4: If $s_2^* = (NB, NB)$**

Then the corresponding player 1's payoff under different actions are given by

| Action | **Type H** | **Type M** | **Type L** |
|---|---|---|---|
| A | $-c$ | $-c$ | $-c$ |
| N | $0$ | $0$ | $0$ |
| Best response | N | N | N |

Given player 1's optimal strategy, the corresponding player 2's belief is seen to be

$$(p_H, p_M, p_L) = Anything, \qquad (q_H, q_M, q_L) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Since $2q_H + q_M - 4q_L = -\frac{1}{3} < 0$ (so player 2 will play NB if player 1 plays N), one can choose $p_H, p_M, p_L$ such that $2p_H + p_M - 4p_L < 0$ (so player 2 will play NB if player plays A too). Thus, there is another PBE (pooling equilibrium) which

$$s_1^* = (N, N, N), \qquad s_2^* = (NB, NB).$$

*Remark of Example 8*

We observe that there exists a semi-pooling equilibrium which allows the firm (of type H and type M) to signal its good type to the buyer so that the buyer will buy its smartphone and the adverse selection issue can be partially resolved (the buyer can rule out the firm is of type L if he observes the firm plays A).

**Equilibrium refinement in signaling games: Intuitive Criterion**

As we have seen in earlier examples that there could be multiple PBEs (separating equilibrium or pooling equilibrium) in a signaling games, one would like to ask which equilibrium is likely to happen.

As an example, we consider the education signaling games considered in Example 6. Recall that there are two PBEs in this games:

- The first equilibrium is separating equilibrium which $s_1^* = (s_{1H}^*, s_{1L}^*) = (Q, N)$ and $s_2^* = \left(s_2^*(Q), s_2^*(N)\right) = (M, T)$. The players' payoff is $(v_1, v_2) = (8,6)$.
- The second equilibrium is pooling equilibrium which $s_1^* = (N, N)$ and $s_2^* = (T, T)$. The player's payoff under this equilibrium will be $(v_1, v_2) = (6,6)$.

Let's investigate the pooling equilibrium which both types of player 1 chooses $N$.

- According to the calculation earlier, player 1 (of any type) never deviate and play $Q$ since the player 2's belief is set to be $(p_2(H|A), p_2(L|A)) = (q, 1-q)$ where $q < 3/8$.
- However, this belief is off-the-equilibrium belief and can be anything (no restriction) in the definition of PBE. We choose a off-the-equilibrium belief just to ensure that the player has no incentive to deviate.

- On the other hand, we find that there is a separating equilibrium which $s_1^* = (Q, N)$ and $s_2^* = (M, T)$. It suggests that player 1 of type H can play $Q$ and signal its type to the player 2. It appears that such proposed off-the-equilibrium may not be reasonable.
- Furthermore, the player 1 of type $H$ can receive a payoff of 8 if he plays $A$ and the signaling is successful. So it appears that he has incentive to deviate and the pooling equilibrium is not a reasonable outcome.

It is clear that player 2 can update his/her belief using Bayes' rule and player 1's strategic profile along on-the-equilibrium path. On the other hand if player 1 chooses a strategy which is on off-the-equilibrium path, player 2 will still try to guess which type of players will *deviate* and choose this strategy.

- ♦ Player 1 of type $H$ has incentive to deviative since he will get a payoff of $8 > 6$ if player 2 believe player 1 is of type $H$ and offer him the manager (play $M$).
- ♦ Player 1 of type $L$ has no incentive to do so since he will get a payoff of $5 < 6$ even if player 2 believe he/she is of type $H$ and offer him the manager position (play $M$)

Combining these results, there is evidence that the player 1 should be of type $H$ if player 2 finds that player 1 plays $A$. And player 1 of type $H$ has incentive to deviate from the pooling strategy. Therefore, the pooling equilibrium is not credible equilibrium.

This equilibrium selection criterion is known as *intuitive criterion* which aims to examine if some types of players have incentive to deviate from a given equilibrium strategy and adopt the strategy in off-the-equilibrium path.

To state the formal statement of intuitive criterion, we need to introduce some additional notations:

For any non-empty subset $T \subseteq \Theta_1$, we let $BR(T, a_1)$ be the set of all pure-strategy best response for player 2 to player 1's action $a_1$ for all beliefs $\mu = \left(p_2(\theta_{11}|a_1), p_2(\theta_{12}|a_1), \ldots, p_2\left(\theta_{1n_1}\middle|a_1\right)\right)$ such that $\sum_{\theta_1 \in T} p_2(\theta_1|a_1) = 1$. That is,

$$BR(T, a_1) = \bigcup_{\mu: \sum_{\theta_1 \in T} p_2(\theta_1|a_1)=1} BR(\mu, a_1)$$

where

$$BR(\mu, a_1) = \underset{a_2}{\operatorname{argmax}} \underbrace{\sum_{\theta_1 \in \Theta_1} p_2(\theta_1|a_1) v_2(a_2; a_1, \theta_1)}_{\substack{expected\ payoff\ of\ player\ 2 \\ (updated\ belief)}}$$

**Definition (Intuitive Criterion)**

We let $s_1^* = \left(s_1^*(\theta_{11}), s_1^*(\theta_{12}), \dots, s_1^*\left(\theta_{1n_1}\right)\right)$ and $s_2^*$ be the equilibrium strategies of player 1 and player 2 in a signaling games.

We say the equilibrium fails the intuitive criterion if and only if there exists a strategy $a_1 \in S_1$ and a set of types $J(a_1) \subseteq \Theta_1$ such that

1. The player 1 of type $\theta_1 \in J(a_1)$ has no incentive to deviate from the equilibrium and play $a_1$, regardless of the player 2's updated belief when he observes player 1 plays $a_1$:
$$v_1(s_1^*(\theta_1); s_2^*, \theta_1) > \max_{a_2 \in BR(\Theta_1, a_1)} v_1(a_1; a_2, \theta_1) \quad for\ all\ \ \theta_1 \in J(a_1)$$
2. There exists a type $\theta_1' \in \Theta_1$ such that the player 1 of type $\theta_1'$ has incentive to deviate to $a_1$ given that the player 2 believes that player 1's type $\theta_1 \in \Theta_1 \backslash J(a_1)$ if player 1 chooses $a_1$. That is,
$$v_1(s_1^*(\theta_1'); s_2^*, \theta_1') < \underbrace{\min_{a_2 \in BR(\Theta_1 \backslash J(a_1), a_1)} v_1(a_1; a_2, \theta_1')}_{\substack{minimum\ payoff \\ (deviate\ and\ play\ a_1)}}$$