Repeated games

Repeated games is a kind of multi-stage games in which players play a particular game G repeatedly. There are two types of repeated games: (1) finite repeated games and (2) infinite repeated games.

In this section, we shall explore the players' behavior in repeated games. Do all players play the equilibrium strategy G every time? Will they play differently at different stages?

Finite repeated games

Suppose that the games G is played for n times, we let $V_i^{(j)}$ be the player i's payoff in the j^{th} round. We define the total payoff of player i as follows:

$$V_i(s_i; s_{-i}) = \sum_{j=1}^{n} D^{j-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right),$$

where $D \in (0,1]$ denotes the discounted factor.

Example 24

We consider a two-person games G with the following payoff:

		Player 2		
		X	***Y***	***Z***
Player 1	***X***	$(10,10)$	$(4,11)$	$(2,12)$
	Y	$(11,4)$	$(6,6)$	$(2,4)$
	Z	$(12,2)$	$(4,2)$	$(3,3)$

One can show that there are only two Nash equilibria: (Y, Y) and (Z, Z) in this games. Although two players can enjoy higher payoffs if they choose (X, X), (X, X) is not the equilibrium because each player has incentive to deviate.

We consider the case when the games G is played repeated for 10 times. We let $D \in (0,1]$ be the discount factor. Show that if D is sufficiently large, there is a subgame perfect equilibrium in which the players choose to play (X, X) at earlier round.

☺Solution

Step 1: Construct a candidate of subgame perfect equilibrium

In order to "motivate" the players to choose (X, X) with no deviation, the player has to propose some "punishment" in later rounds if another player does not choose X at earlier stage. Player chooses Z (following equilibrium (Z, Z) if another player does not play X in earlier games.

So the players' strategic profile $s^* = (s_1^*, s_2^*)$ is constructed to be

$$\left(s_1^{(1)*}, s_2^{(1)*}\right) = (X, X),$$

$$\left(s_1^{(k)*}, s_2^{(k)*}\right) = \begin{cases} (X, X) & if \ \left(s_1^{(k-1)*}, s_2^{(k-1)*}\right) = (X, X) \\ (Z, Z) & if \ otherwise \end{cases};$$

for $k = 2,3, \ldots ,9$

$$\left(s_1^{(10)^*}, s_2^{(10)^*}\right) = \begin{cases} (Y, Y) & if \ \left(s_1^{(9)^*}, s_2^{(9)^*}\right) = (X, X) \\ (Z, Z) & if \ otherwise \end{cases}.$$

Step 2: Verify that $\boldsymbol{s}^* = (\boldsymbol{s}_1^*, \boldsymbol{s}_2^*)$ constructed in Step 1 is the subgame perfect equilibrium.

It is obvious that s^* is Nash equilibrium at the last round (10th round).

It remains to show that s^* is always Nash equilibrium at k^{th} round of the repeated games. However, it is too tedious to do the checking directly since there are 9 stages. To simplify the verification, we apply one-stage deviation principle and verify that

$$V_i(s_i^*; s_{-i}^*)|_{h_i^{(k)}} \geq V_i\left(s_i^{*\left(s,h_i^{(k)}\right)}; s_{-i}^*\right)|_{h_i^{(k)}}$$

for any information set $h_i^{(k)}$.

We consider k^{th} round ($k-1$ rounds have been played) of the repeated games. Our objective is to show that none of the players have incentive to adopt "one-shot deviation" strategy. Mathematically, we need to show that

$$V_i\left(\underbrace{s_i^{(1)},\dots,s_i^{(k-1)}}_{past\ strategies},s_i^{(k)*},s_i^{(k+1)*},\dots,s_i^{(N)*};s_{-i}^*\right)$$
$$\geq V_i\left(s_i^{(1)},\dots,s_i^{(k-1)},s_i^{(k)},s_i^{(k+1)*},\dots,s_i^{(N)*};s_{-i}^*\right),\dots(*)$$

for any past strategies $s_i^{(1)},\dots,s_i^{(k-1)}$ (information set).

We consider the following two cases:

Case 1: $\left(s_1^{(k-1)},s_2^{(k-1)}\right)=(X,X)$

Then $s_i^{(k)^*}=X$. The corresponding payoff can be expressed as

$$V_i(X)=\underbrace{V_i^{(1)}+\cdots+D^{k-2}V_i^{(k-1)}}_{denoted\ by\ C}+D^{k-1}(10)+D^k(10)+\cdots+D^{N-2}(10)$$
$$+D^{N-1}(6).$$

Suppose that player i chooses Y at k^{th} round, the payoff is given by

$$V_i(Y)=C+D^{k-1}(11)+D^k(3)+\cdots+D^{N-2}(3)+D^{N-1}(3).$$

If player i chooses Z at k^{th} round, the payoff is given by

$$V_i(Z) = C + D^{k-1}(12) + D^k(3) + \cdots + D^{N-2}(3) + D^{N-1}(3).$$

So the player i has no incentive to deviate only when

$$V_i(X) \geq \max\big(V_i(Y), V_i(Z)\big) = V_i(Z)$$

$$\Leftrightarrow -2D^{k-1} + \underbrace{7D^k + \cdots + 7D^8}_{vanish\ when\ k=9} + 3D^9 \geq 0,$$

$$\Leftrightarrow -2 + \underbrace{7D + \cdots + 7D^{9-k}}_{vanish\ when\ k=9} + 3D^{10-k} \geq 0, \quad for\ all\ k = 1,2,\ldots,9.$$

Since L.H.S. is decreasing with respect to k (L.H.S. is smaller when k is larger), thus the above inequality holds when

$$-2 + 3D^{10-9} \geq 0 \Rightarrow D \geq \frac{2}{3} \ldots\ldots (**)$$

Hence, we conclude that $(*)$ holds when $D \geq 2/3$.

Case 2: $\left(s_1^{(k-1)}, s_2^{(k-1)}\right) \neq (X, X)$

Then $s_i^{(k)*} = Z$. The corresponding payoff can be expressed as

$$V_i(Z) = C + D^{k-1}(3) + D^k(3) + \cdots + D^{N-2}(3) + D^{N-1}(3).$$

Suppose that player i chooses X at k^{th} round, the payoff is given by

$$V_i(X) = C + D^{k-1}(2) + D^k(3) + \cdots + D^{N-2}(3) + D^{N-1}(3).$$

If player i chooses Y at k^{th} round, the payoff is given by

$$V_i(Y) = C + D^{k-1}(2) + D^k(3) + \cdots + D^{N-2}(3) + D^{N-1}(3).$$

It is clear that $V_i(Z) > \max\left(V_i(X), V_i(Y)\right)$ and the ineq. $(*)$ holds unconditionally.

In summary, we conclude that when $D \geq \frac{2}{3}$, there is a subgame perfect equilibrium (constructed in Step 1) such that all players choose X in the first 9 rounds and choose Y in the last round.

Example 25

A game G is played for N times in a repeated games. Suppose that G has a unique pure strategy Nash equilibrium $a^* = (a_1^*, a_2^*, \dots, a_n^*)$, show that the repeated games has a unique subgame perfect equilibrium in which all players play s^* in every round of the games.

☺Solution

We let $s^* = (s_1^*, \dots, s_n^*)$ be a subgame perfect equilibrium. Our goal is to show that $s_i^{(j)*} = a_i^*$ for all $j = 1,2, \dots, N$ and all information set $h_i^{(j)}$. This can be done by backward induction.

For $j = N$, it is clear that all players should play the equilibrium strategy in G. That is, $s_i^{(N)*} = a_i^*$.

Assume that $s_i^{(j)*} = a_i^*$ for all $j = k+1, k+2 \dots, N$, we proceed to show that $s_i^{(k)*} = a_i^*$.

Applying the one-stage deviation principle on s_i^*, we have

$$V_i(s_i^*; s_{-i}^*)|_{h_i^{(k)}} \geq V_i\left(s_i^{*\left(s, h_i^{(k)}\right)}; s_{-i}^*\right)|_{h_i^{(k)}}$$

$$\Leftrightarrow V_i\left(\underbrace{s_i^{(1)}, \dots, s_i^{(k-1)}}_{past\ strategies}, s_i^{(k)*}, s_i^{(k+1)*}, \dots, s_i^{(N)*}; s_{-i}^*\right)$$

$$\geq V_i\left(s_i^{(1)}, \dots, s_i^{(k-1)}, s, s_i^{(k+1)*}, \dots, s_i^{(N)*}; s_{-i}^*\right), \dots (*)$$

for any past strategies s, $s_i^{(1)}, \dots, s_i^{(k-1)}$.

Expanding the above inequality, we get

$$\sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right) + V_i^{(k)}\left(s_i^{(k)*}; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{N} V_i^{(j)}\left(s_i^{(j)*}\left(s_i^{(k)*}\right); s_{-i}^{(j)*}\left(s_i^{(k)*}\right)\right)$$

$$\geq \sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)*}\right) + V_i^{(k)}\left(s; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{N} V_i^{(j)}\left(s_i^{(j)*}(s); s_{-i}^{(j)*}(s)\right).$$

By assumption, we get $s_i^{(j)*}\left(s_i^{(k)}\right) = s_i^{(j)*}(s) = a_i^*$ for $j = k+1, \dots, N$.

Thus, the inequality can be simplified into

$$V_i^{(k)}\left(s_i^{(k)}; s_{-i}^{(k)}\right) + \sum_{j=k+1}^{N} V_i^{(j)}(a_i^*; a_{-i}^*) \geq V_i^{(k)}\left(s; s_{-i}^{(k)}\right) + \sum_{j=k+1}^{N} V_i^{(j)}(a_i^*; a_{-i}^*)$$
$$\Rightarrow V_i^{(k)}\left(s_i^{(k)}; s_{-i}^{(k)}\right) \geq V_i^{(k)}\left(s; s_{-i}^{(k)}\right).$$

The above inequality reveals that $s_i^{(k)}$ is the best response to $s_{-i}^{(k)}$.

Recall that $V_i^{(k)}$ is simply the payoffs in the games G, thus $s^{(k)} = \left(s_1^{(k)}, \dots, s_n^{(k)}\right)$ is the Nash equilibrium of the games G.

Then by the uniqueness of equilibrium in G, we can conclude that

$$s_i^{(k)} = a_i^*.$$

This completes the induction.

Infinite repeated games

Next, we consider infinite repeated games in which the players play a games G for infinite number of times.

Strategies and payoff function in infinite repeated games

We let $s_i^{(j)} = s_i^{(j)}\left(h_i^{(j)}\right)$ be the player i's strategy at j^{th} round. Then the player i's strategy s_i is defined as $s_i = \left(s_i^{(1)}, s_i^{(2)}, \dots\right)$.

We let $O_i^{(j)}$ be the player i's payoff at i^{th} round, then the player i's payoff in this repeated games can be defined as

$$V_i = V_i^{(1)}\left(s_i^{(1)}; s_{-i}^{(1)}\right) + DV_i^{(2)}\left(s_i^{(2)}; s_{-i}^{(2)}\right) + D^2 V_i^{(3)}\left(s_i^{(3)}; s_{-i}^{(3)}\right) + \cdots.$$

For convergence of the above infinite sum, we assume that $D < 1$ and the payoff function $V_i^{(j)}$ is bounded.

Equilibrium strategy in infinite repeated games

Since there is no terminal nodes in infinite repeated games, the full characterization of all subgame perfect equilibria may not be feasible since the standard backward induction cannot be applied. Hence, we can only concentrate on some equilibria that fit our interest.

We let $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ be a pure strategy Nash equilibrium in the games G (that is played repeatedly in the repeated games). The following theorem shows that one can construct a (trivial) subgame perfect equilibrium from this Nash equilibrium a^*.

Theorem 3

Suppose $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ is a pure strategy Nash equilibrium in the games G which is played repeatedly in infinite repeated games, then the strategic profile $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$, defined by $s_i^{(j)*}\left(h_i^{(j)}\right) = a_i^*$ for all $j = 1,2, \ldots$ and all $h_i^{(j)}$, constitutes the subgame perfect equilibrium.

Proof of the theorem 3

One can prove this using one-stage deviation principle. We consider k^{th} stage which $k-1$ rounds have been played. For any past strategies $s_i^{(1)}, \dots, s_i^{(k-1)}$, we have

$$V_i\left(\underbrace{s_i^{(1)}, \dots, s_i^{(k-1)}}_{past\ strategies}, s_i^{(k)*}, s_i^{(k+1)*}, \dots, ; s_{-i}^{*}\right)$$

$$= \sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right) + V_i^{(k)}\left(s_i^{(k)*}; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} V_i^{(j)}\left(s_i^{(j)*}\left(s_i^{(k)*}\right); s_{-i}^{(j)*}\left(s_i^{(k)*}\right)\right)$$

$$\overset{\substack{s_i^{(j)*}=a_i^* \\ for\ all\ h_i^{(j)}}}{\triangleq} \sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right) + V_i^{(k)}(a_i^*; a_{-i}^*) + \sum_{j=k+1}^{\infty} V_i^{(j)}(a_i^*; a_{-i}^*)$$

$$\overset{\substack{a^* \text{ is Nash} \\ \text{equilibrium}}}{\geq} \sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right) + V_i^{(k)}(s; a_{-i}^*) + \sum_{j=k+1}^{\infty} V_i^{(j)}(a_i^*; a_{-i}^*), \quad for\ any\ s \in S_i$$

$$\overset{\substack{s_i^{(j)^*} = a_i^* \\ for\ all\ h_i^{(j)}}}{=} \sum_{j=1}^{k-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right) + V_i^{(k)}\left(s; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} V_i^{(j)}\left(s_i^{(j)^*}(s); s_{-i}^{(j)*}(s)\right)$$

$$= V_i\left(\underbrace{s_i^{(1)}, \dots, s_i^{(k-1)}}_{past\ strategies}, s, s_i^{(k+1)^*}, \dots; s_{-i}^*\right).$$

The above inequality reveals that no players have incentive to adopt one-stage deviation strategy. Thus the given strategic profile s^* constitutes the subgame perfect equilibrium by one stage deviation principle.

Example 26

Two restaurants in the same district compete for the customers in that district. Every day, they need to decide whether to provide some special discount (P) to the customers in order to attract more customers. Although such offer is costly for the restaurant, the restaurant will lose a lot of profit if another restaurant offers the discount and it chooses to do nothing. The daily profits made by two restaurants are summarized by the following matrix: (Here, "N" denotes “does not offer discount”.)

		Restaurant 2	
		P	**N**
Restaurant 1	**P**	$(3,3)$	$(7,1)$
	N	$(1,7)$	$(5,5)$

We let $D < 1$ be the discount factor, then the total profit made by restaurant i is given by $V_i = \sum_{j=1}^{\infty} D^{j-1} V_i^{(j)}$, where $V_i^{(j)}$ is the profit made in day j.

(a) Suppose that two restaurants only concern the profit made in a particular day, show that (P, P) is the only pure strategy Nash equilibrium.

(b) Suppose that two restaurants care about their *total profits*, find a subgame perfect equilibrium in which two restaurants may choose (N, N) in some days, provided that D is sufficiently large.

☺Solution of (a)

We first determine the best responses of each player which are summarized in the following payoff matrix:

		Restaurant 2	
		P	**N**
Restaurant 1	**P**	$(\overline{\mathbf{3}}, \overline{\mathbf{3}})$	$(\overline{\mathbf{7}}, 1)$
	N	$(1, \overline{\mathbf{7}})$	$(5,5)$

So we conclude that there is an unique Nash equilibrium (P, P).

Remark of (a)

One can observe from the above payoff matrix that two firms can earn more profits if both firms choose not to offer discount (N, N). We observe from (a) that this outcome is impossible to happen if two firms only concern their short-term profits. However, this outcome may be possible if both firms concern their total profits.

☺Solution of (b)

We construct the strategic profiles $s^* = (s_1^*, s_2^*)$ as follows:

$$\left(s_1^{(1)*}, s_2^{(1)*}\right) = (N, N) \quad and$$

$$\left(s_1^{(k+1)*}, s_2^{(k+1)*}\right) = \begin{cases} (N, N) & if \ \left(s_1^{(k)*}, s_2^{(k)*}\right) = (N, N) \\ (P, P) & if \ otherwise \end{cases}.$$

Next, we proceed to show the strategic profile $\vec{s}^*$ is the subgame perfect Nash equilibrium using one-stage deviation principle.

We consider k^{th} round of the games. We consider the following two cases:

Case 1: The players chooses (N, N) in the previous round.

Then $s_i^{(j)*} = N$ for all $j \geq k$ and $i = 1,2$. The corresponding payoff is found to be

$$V_i(s_1^{(k)*}) = \underbrace{\sum_{j=1}^{k-1} D^{j-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right)}_{denoted\ by\ C} + \sum_{j=k}^{\infty} D^{j-1} V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right)$$

$$\overset{\substack{\left(s_1^{(j)*}, s_2^{(j)*}\right)=(N,N) \\ for\ all\ j\geq k}}{\widehat{=}} \quad C + \sum_{j=k}^{\infty} D^{j-1}(5) = C + \frac{5D^{k-1}}{1-D}.$$

Suppose that player i chooses to deviate at k^{th} round and choose P, then the corresponding payoff is found to be

$$V_i(P) = \underbrace{\sum_{j=1}^{k-1} D^{j-1} V_i^{(j)}\left(s_i^{(j)}; s_{-i}^{(j)}\right)}_{denoted\ by\ C} + D^{k-1} V_i^{(k)}\left(P; s_{-i}^{(k)*}\right)$$

$$+ \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right)$$

$$\overset{\substack{\left(s_1^{(j)*}, s_2^{(j)*}\right) = (P,P) \\ for\ all\ j \geq k+1}}{\stackrel{\triangle}{=}} \quad C + D^{k-1} V_i^{(k)}(P; N) + \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}(P; P)$$

$$= C + 7D^{k-1} + \sum_{j=k+1}^{\infty} D^{j-1}(3) = C + 7D^{k-1} + \frac{3D^k}{1-D}.$$

If the players have no incentive to deviate, it must be that

$$V_i\left(s_i^{(k)*}\right) \geq V_i(P) \Rightarrow \frac{5D^{k-1}}{1-D} \geq 7D^{k-1} + \frac{3D^k}{1-D} \Rightarrow D \geq \frac{1}{2}.$$

Case 2: The players do not play (N, N) in the previous round.

Then $s_i^{(j)*} = P$ for all $j \geq k$ and $i = 1,2$. The corresponding payoff is found to be

$$V_i\left(s_i^{(k)*}\right) = C + \sum_{j=k}^{\infty} D^{j-1} V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right) = C + \sum_{j=k}^{\infty} D^{j-1} V_i^{(j)}(P; P)$$

$$= C + \sum_{j=k}^{\infty} D^{j-1}(3) = C + \frac{3D^{k-1}}{1-D}.$$

Suppose that player i deviates and plays N in the k^{th} round, the corresponding payoff is found to be

$$V_i(N) = C + D^{k-1} V_i^{(k)}\left(N; s_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}\left(s_i^{(j)*}; s_{-i}^{(j)*}\right)$$

$$= C + D^{k-1} V_i^{(k)}(N; P) + \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}(P; P)$$

$$= C + D^{k-1}(1) + \sum_{j=k+1}^{\infty} D^{j-1}(3)$$

$$= C + D^{k-1} + \frac{3D^k}{1-D}.$$

Since $V_i\left(s_i^{(k)*}\right) - V_i(N) = 2D^{k-1} \geq 0$, so we have $V_i\left(s_i^{(k)*}\right) \geq V_i(N)$ and player i has no incentive to deviate from s^*.

In summary, we conclude that the strategic profile $\vec{s}^*$ constructed is the required subgame perfect equilibrium when $D \geq \frac{1}{2}$.

Remark of (b)

One can observe that both players choose to cooperate and play N in equilibrium (when D is sufficiently large). It is because the long-term benefit (from cooperating) override the short-term benefit (from deviating).

Example 27 (Trust games)

A company (player 1) needs someone to provide dining services for its workers. Every day, the company can choose whether to ask help from a restaurant (player 2). Both players receive nothing if the company choose not to do so (N). If the company choose to ask the restaurant for help (A), then the restaurant can choose to provide either good service (G) or bad service (B) to the company. It is clear that company prefers good service from the restaurant and providing good service is more costly for the restaurant. We assume that

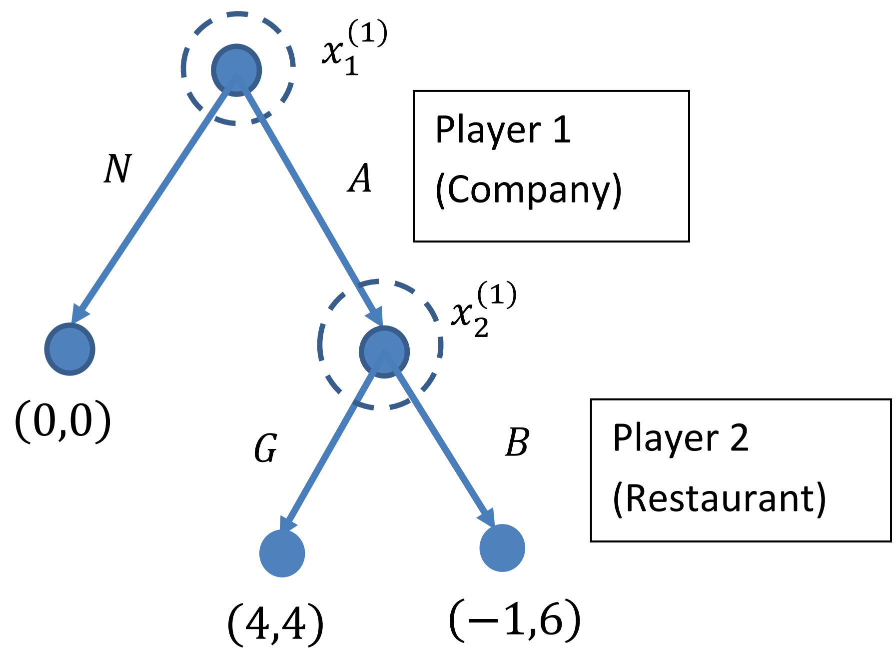
- The players' payoffs are $(V_1, V_2) = (4,4)$ if the restaurant provides good service.
- The players' payoffs are $(V_1, V_2) = (-1,6)$ if the restaurant provides bad service.

(a) Suppose that the company only needs the service from restaurant for 1 day, find the equilibrium strategy for this games.

(b) Suppose that the company needs the dining service every day, find a subgame perfect equilibrium in which the restaurant (player 2) will provide good services to the company.

☺Solution of (a)

Since the games is a dynamic games in which player 1 moves first and player 2 moves next. We first develop the game trees for this games:



Since the games is of perfect information, one can solve for the sequentially rational Nash equilibrium using backward induction.

- We first consider the last stage where the player 2 makes the last move. It is obvious that the player 2 should choose "B".
- We consider the first stage where player 1 makes the move. Knowing that player 2 will choose B in the second stage, the player 1 should choose N.

Thus the sequentially rational Nash equilibrium of this games is $s^* = (s_1^*, s_2^*) = (N, B)$.

☺Solution of **(b)**

We consider the following strategic profile $s^* = (s_1^*, s_2^*)$:

$$\left(s_1^{(1)*}, s_2^{(1)*}\right) = (A, G), \quad \left(s_1^{(k+1)*}, s_2^{(k+1)*}\right) = \begin{cases} (A, G) & if \ \left(s_1^{(k)*}, s_2^{(k)*}\right) = (A, G) \\ (N, B) & if \ otherwise \end{cases},$$

for $k = 1,2, \ldots$.

In other words, player 1 will choose to ask help from the player 2 as long as the player 2 provides good service.

We proceed to verify that s^* is the subgame perfect equilibrium using one-stage deviation principle.

At k^{th} round, we divide our analysis into two cases:

Case 1: If $\left(s_1^{(k-1)}, s_1^{(k-1)}\right) = (A, G)$,

Then $\left(s_1^{(k)*}, s_2^{(k)*}\right) = (A, G)$. We first consider player 1. If player 1 chooses to play $s_1^{(k)*} = A$, the payoff is seen to be

$$V_1(s_i^{(k)*}) = \underbrace{\sum_{j=1}^{k-1} D^{j-1} V_1^{(j)}\left(s_1^{(j)}; s_2^{(j)}\right)}_{denoted\ by\ C} + \sum_{j=k}^{\infty} D^{j-1} V_1^{(j)}\left(s_i^{(j)*}; s_i^{(j)*}\right)$$

$$\overset{\left(s_1^{(j)*}, s_2^{(j)*}\right) = (A,G)\ for\ all\ j \geq k}{\cong} C + \sum_{j=k}^{\infty} D^{j-1}(4) = C + \frac{4D^{k-1}}{1-D}.$$

If player 1 chooses to play N, the corresponding payoff is seen to be

$$V_1(N) = C + D^{k-1}V_1^{(k)}(N;G) + \sum_{j=k+1}^{\infty} D^{j-1}V_1^{(j)}\left(s_1^{(j)^*};s_2^{(j)^*}\right)$$

$$\overset{\left(s_1^{(j)^*},s_2^{(j)^*}\right)=(N,B)\ \ for\ all\ j\geq k+1}{\triangleq} C + D^{k-1}\underbrace{V_1^{(k)}(N;G)}_{=0} + \sum_{j=k+1}^{\infty} D^{j-1}\underbrace{V_1^{(j)}(N,B)}_{=0} = C.$$

Since $V_1\left(s_i^{(k)*}\right) \geq V_1(N)$, so player 1 has no incentive to deviate.

We proceed to consider player 2. If player 2 chooses to play $s_2^{(k)*} = G$, the payoff is seen to be

$$V_2(s_2^{(k)*}) = \underbrace{\sum_{j=1}^{k-1} D^{j-1}V_2^{(j)}\left(s_2^{(j)};s_1^{(j)}\right)}_{denoted\ by\ C_2} + \sum_{j=k}^{\infty} D^{j-1}V_2^{(j)}\left(s_2^{(j)^*};s_1^{(j)^*}\right)$$

$$\overset{\left(s_1^{(j)^*},s_2^{(j)^*}\right)=(A,G)\ \ for\ all\ j\geq k}{\triangleq} C_2 + \sum_{j=k}^{\infty} D^{j-1}(4) = C_2 + \frac{4D^{k-1}}{1-D}.$$

If player 2 deviates and chooses B, the payoff is seen to be

$$V_2(B) = C_2 + D^{k-1}V_2^{(k)}(B;A) + \sum_{j=k+1}^{\infty} D^{j-1}V_2^{(j)}\left(s_2^{(j)*};s_1^{(j)*}\right)$$

$$\overset{\left(s_1^{(j)*},s_2^{(j)*}\right)=(N,B)\ \text{for all}\ j\geq k+1}{\cong} \quad C_2 + D^{k-1}\underbrace{V_2^{(k)}(B;A)}_{6} + \sum_{j=k+1}^{\infty} D^{j-1}\underbrace{V_2^{(j)}(B;N)}_{=0}$$

$$= C_2 + 6D^{k-1}.$$

We note that player 2 has no incentive to deviate only when

$$V_2\left(s_2^{(k)*}\right) \geq V_2(B) \Rightarrow \frac{4D^{k-1}}{1-D} \geq 6D^{k-1} \Rightarrow D \geq \frac{1}{3}.$$

So when $D \geq \frac{1}{3}$, no players have incentive to deviate.

Case 2: If $\left(s_1^{(k-1)}, s_1^{(k-1)}\right) \neq (A, G)$,

Then $\left(s_1^{(k)*}, s_2^{(k)*}\right) = (N, B)$. If player 1 chooses to play $s_1^{(k)*} = N$, the payoff is seen to be

$$V_1\left(s_1^{(k)*}\right) = C + \sum_{j=k}^{\infty} D^{j-1} \underbrace{V_1^{(j)}(N; B)}_{0} = C.$$

If player 1 chooses to play A, the corresponding payoff is

$$V_1(A) = C + D^{k-1} \underbrace{V_1^{(k)}(A; B)}_{-1} + \sum_{j=k+1}^{\infty} D^{j-1} \underbrace{V_1^{(j)}(N; B)}_{0} = C - D^{k-1}.$$

Since $V_1\left(s_1^{(k)^*}\right) \geq V_1(A)$, thus player 1 has no incentive to deviate.

On the other hand, if player 2 chooses to play $s_2^{(k)*} = B$, the payoff is seen to be

$$V_2\left(s_2^{(k)*}\right) = C_2 + \sum_{j=k}^{\infty} D^{j-1} \underbrace{V_2^{(j)}(B;N)}_{0} = C_2.$$

If player 2 chooses to play $s_2^{(k)*} = G$, the payoff is found to be

$$V_2(G) = C_2 + D^{k-1} \underbrace{V_2^{(k)}(G;N)}_{=0} + \sum_{j=k+1}^{\infty} D^{j-1} \underbrace{V_2^{(j)}(B;N)}_{0} = C_2.$$

Since $V_2\left(s_2^{(k)*}\right) = V_2(G)$, so player 2 has no strict incentive to deviate.

Hence, we can conclude from one-stage deviation principle that the strategic profile $\vec{s}^*$ is the subgame perfect equilibrium when $D \geq \frac{1}{3}$.

Example 28 (Tactic collusion)

We revisit the Cournot duopoly games: Two firms (Firm 1 and Firm 2) are producing common goods for a market. Each firm needs to decide the number of goods (denoted by q_i) produced. It is given that

- The production cost for Firm i is $C_i(q_i) = 40q_i$.
- The market price of the goods is $250 - q_1 - q_2$.

Thus the profits made by each firm is

$$V_i^{(k)}(q_i; q_j) = (250 - q_i - q_j)q_i - 40q_i = 210q_i - q_i^2 - q_iq_j.$$

Suppose that the games is played once, one can show that $(q_1^*, q_2^*) = (70,70)$ is the unique Nash equilibrium of the games and the corresponding payoffs is $\left(V_1^{(j)}, V_2^{(j)}\right) = (4900,4900)$.

However, this is not a "best" outcome for both firms. If each firm produces $q_i = 52.5$ units of goods, then the profit made by each firm will be

$$V_i^{(j)} = (250 - 52.5 - 52.5)(52.5) - 40(52.5) = 5512.5,$$

which is higher than that under Nash equilibrium. The corresponding market price will be 145. One would like to ask whether it is possible for the firms to cooperate and achieve a better profits.

(*Note: In fact, $(q_1, q_2) = (52.5,52.5)$ maximizes the total payoffs of two firms. That is, $\max\left(V_1^{(j)} + V_2^{(j)}\right)$.)

We consider the case when the market games is now played repeatedly at periods $1, 2, 3, \ldots$. We let $D \in (0,1)$ be the discounted factor over 1 period and $V_i^{(j)}$ be the Firm i's payoff in the games played at period j. The total profit received by Firm i is seen to be

$$V_i = V_i^{(1)} + DV_i^{(2)} + D^2V_i^{(3)} + D^3V_i^{(4)} + \cdots.$$

Find a subgame perfect Nash equilibrium so that each firm will produce 52.5 units of goods at every period (assuming D is sufficiently large).

☺Solution

Step 1: Construct a candidate of subgame perfect equilibrium.

We let $q_i^* = \left(q_i^{(1)*}, q_i^{(2)*}, q_i^{(3)*}, \dots\right)$ be the strategy chosen by Firm i. We take

$$\left(q_1^{(1)*}, q_2^{(1)*}\right) = (52.5, 52.5) \quad and$$

$$\left(q_1^{(k+1)*}, q_2^{(k+1)*}\right) = \begin{cases} (52.5, 52.5) & if \ \left(q_1^{(k)*}, q_2^{(k)*}\right) = (52.5, 52.5) \\ (70,70) & if \ otherwise \end{cases},$$

for $k = 1,2, \dots$.

Step 2: Verify (q_1^*, q_2^*) is the subgame perfect equilibrium using one-stage deviation principle.

At any period k, we need to show that two firms have no incentive to deviate from (q_1^*, q_2^*) by adopting one-shot deviation strategy. We consider the following two cases:

Case 1: If $\left(\boldsymbol{q}_1^{(\boldsymbol{k}-\mathbf{1})}, \boldsymbol{q}_2^{(\boldsymbol{k}-\mathbf{1})}\right) \neq (\mathbf{52.5}, \mathbf{52.5})$,

then $\left(q_1^{(k)*}, q_2^{(k)*}\right) = (70,70)$ and $\left(q_1^{(j)*}, q_2^{(j)*}\right) = (70,70)$ for all $j \geq k + 1$. Player i has no incentive to deviate at period k if and only if

$$V_i\left(q_1^{(1)}, \dots, q_1^{(k-1)}, q_1^{(k)*}, q_1^{(k+1)*}, \dots; q_{-i}^*\right) \geq \max_{q\neq 70} V_i\left(q_1^{(1)}, \dots, q_1^{(k-1)}, q, q_1^{(k+1)*}, \dots; q_{-i}^*\right)$$

$$\Rightarrow \sum_{j=1}^{k-1} D^{j-1} V_i^{(j)}\left(q_i^{(j)}; q_{-i}^{(j)}\right) + D^{k-1} V_i^{(k)}\left(q_i^{(k)*}; q_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}\left(q_i^{(j)*}; q_{-i}^{(j)*}\right)$$

$$\geq \sum_{j=1}^{k-1} D^{j-1} V_i^{(j)}\left(q_i^{(j)}; q_{-i}^{(j)}\right) + \max_{q\neq 70} D^{k-1} V_i^{(k)}\left(q; q_{-i}^{(k)^*}\right) + \sum_{j=k+1}^{\infty} D^{j-1} V_i^{(j)}\left(q_i^{(j)^*}; q_{-i}^{(j)^*}\right)$$

$$\Rightarrow V_i^{(k)}\left(q_i^{(k)*}; q_{-i}^{(k)*}\right) \geq \max_{q\neq 70} V_i^{(k)}\left(q; q_{-i}^{(k)*}\right)$$

$$\Rightarrow V_i^{(k)}(70; 70) \geq \max_{q\neq 70} V_i^{(k)}(q; 70)$$

$$\Rightarrow 4900 \geq \max_{q \neq 70}\left((250 - 70 - q)q - 40q\right) \overset{q^{max}=70}{\triangleq} 4900.$$

The last inequality holds trivially. Thus no players will deviate at k^{th} round.

Case 2: If $\left(q_1^{(k-1)}, q_2^{(k-1)}\right) = (52.5, 52.5)$,

then $\left(q_1^{(k)*}, q_2^{(k)*}\right) = (52.5,52.5)$. Player i has no incentive to deviate at period k if and only if

$$V_i\left(q_1^{(1)}, \dots, q_1^{(k-1)}, q_1^{(k)*}, q_1^{(k+1)*}, \dots; q_{-i}^*\right)$$
$$\geq \max_{q \neq 52.5} V_i\left(q_1^{(1)}, \dots, q_1^{(k-1)}, q, q_1^{(k+1)*}, \dots; q_{-i}^*\right)$$

$$\Rightarrow D^{k-1}V_i^{(k)}\left(q_i^{(k)*}; q_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} D^{j-1}V_i^{(j)}\left(q_i^{(j)*}; q_{-i}^{(j)*}\right)$$
$$\geq \max_{q \neq 52.5} D^{k-1}V_i^{(k)}\left(q; q_{-i}^{(k)*}\right) + \sum_{j=k+1}^{\infty} D^{j-1}V_i^{(j)}\left(q_i^{(j)*}(q); q_{-i}^{(j)*}(q)\right)$$

$$\Rightarrow D^{k-1}V_i^{(k)}(52.5;52.5) + \sum_{j=k+1}^{\infty} D^{j-1}\underbrace{V_i^{(j)}(52.5;52.5)}_{=5512.5}$$

$$\geq \max_{q\neq 52.5} D^{k-1}V_i^{(k)}(q;52.5) + \sum_{j=k+1}^{\infty} D^{j-1}\underbrace{V_i^{(j)}(70;70)}_{=4900}$$

$$\Rightarrow D^{k-1}(5512.5) + \frac{5512.5D^k}{1-D} \geq \max_{q\neq 52.5} D^{k-1}V_i^{(k)}(q;52.5) + \frac{4900D^k}{1-D}$$

$$\Rightarrow 5512.5 + \frac{5512.5\ D}{1-D} \geq \max_{q\neq 52.5}\big((250-52.5-q)q-40q\big) + \frac{4900D}{1-D}$$

$$\overset{q^{max}=78.75}{\Rightarrow} \quad 5512.5 + \frac{5512.5\ D}{1-D} \geq 6201.5625 + \frac{4900D}{1-D}$$

$$\Rightarrow D \geq 0.529411765.$$

Using one-stage deviation principle, we can conclude that (q_1^*, q_2^*) constructed in Step 1 is the subgame perfect Nash equilibrium when $D \geq 0.529412$.

General Theory – Folk theorem

We observe from the above examples that players may cooperate and choose strategies which is not Nash equilibrium in order to achieve a better long-run return. The following theorem confirms this observation.

Theorem 4 (Folk theorem – Version 1)

A static games G is played repeatedly in n-person infinite repeated games. We let $a^* = (a_1^*, a_2^*, \dots, a_n^*)$ be a pure strategy Nash equilibrium of the games G and $v_i^* = V_i^G(a_i^*; a_{-i}^*)$ be the corresponding payoff of player i in the game G.

Suppose that there exists a strategic profile $a' = (a_1', a_2', \dots, a_n')$ such that $v_i' = V_i^G(a_i'; a_{-i}') > v_i^*$ for all i, there exists $D^* \in (0,1]$ such that for any $D \geq D^*$, there exists a subgame perfect Nash equilibrium of the infinite repeated games which each player (player i) will play a_i' in every stage and achieve an average payoff of v_i'.

Proof of Theorem 4

Step 1: Construct a strategic profile for subgame perfect equilibrium

We define $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ be the strategic profile where

$$s_i^{(1)*} = a_i', \qquad s_i^{(k+1)*} = \begin{cases} a_i' & if \quad s^{(k)} = a' \\ a_i^* & if \quad s^{(k)} \neq a' \end{cases} \quad for \quad k = 1,2,\dots.$$

Step 2: Verify the optimality of s^* using one-stage deviation principle

Suppose that the first k games are played and the players are playing $(k+1)^{th}$ round. We consider the following two cases:

Case 1: If $s^{(k)} = a'$

Then $s^{(j)*} = a'$ for all $j \geq k+1$ and the corresponding player i's payoff is given by

$V_i(s_i^*, s_{-i}^*)_{h_i^{(k)}}$

$$= \underbrace{\sum_{j=1}^{k} D^{j-1} V_i\left(s_i^{(j)}; s_{-i}^{(j)}\right)}_{=C} + D^k \underbrace{V_i^{(k+1)}\left(\overbrace{s_i^{(k+1)*}}^{=a_i'}; \overbrace{s_{-i}^{(k+1)*}}^{=a_{-i}'}\right)}_{=v_i'} + \sum_{j=k+2}^{\infty} D^{j-1} \underbrace{V_i^{(j)}\left(\overbrace{s_i^{(j)*}}^{=a_i'}; \overbrace{s_{-i}^{(j)*}}^{=a_{-i}'}\right)}_{=v_i'}$$

Suppose that player i tries to deviate and adopt another strategy a_i^0 which yield a payoff of $v_i^0 = V_i^{(G)}(a_i^0; a'_{-i})$, then all players will always play a_i^* afterwards. Then the corresponding payoff of player i can be expressed as

$$V_i\left(s_i^{\left(a_i^0, h_i^{(k)}\right)}, s_{-i}^*\right)_{h_i^{(k)}}$$

$$= \underbrace{\sum_{j=1}^{k} D^{j-1} V_i\left(s_i^{(j)}; s_{-i}^{(j)}\right)}_{=C} + D^k \underbrace{V_i^{(k+1)}(a_i^0; a'_{-i})}_{=v_i^0} + \sum_{j=k+2}^{\infty} D^{j-1} \underbrace{V_i^{(j)}(a_i^*; a_{-i}^*)}_{=v_i^*}$$

The player has no incentive to deviate if and only if

$$V_i(s_i^*, s_{-i}^*)_{h_i^{(k)}} \geq V_i\left(s_i^{\left(a_i^0, h_i^{(k)}\right)}, s_{-i}^*\right)_{h_i^{(k)}} \Leftrightarrow D^k v_i' + \sum_{j=k+2}^{\infty} D^{j-1} v_i' \geq D^k v_i^0 + \sum_{j=k+2}^{\infty} D^{j-1} v_i^*$$

$$\Leftrightarrow \frac{D}{1-D} \underbrace{(v_i' - v_i^*)}_{>0} \geq v_i^0 - v_i' \dots \dots (*)$$

If $v_i^0 - v_i' \leq 0$, the above inequality holds trivially.

If $v_i^0 - v_i' > 0$, one can show that $\lim_{D \to 1^-} \frac{D}{1-D}(v_i' - v_i^*) = +\infty$ and $\frac{D}{1-D}$ is increasing with respect to D, then there exists $D^* \in (0,1]$ such that

$$\frac{D^*}{1-D^*}(v_i' - v_i^*) = v_i^0 - v_i' \quad and \quad \frac{D}{1-D}(v_i' - v_i^*) > v_i^0 - v_i' \ \ for \ \ D > D^*.$$

So the player will not deviate when $D \geq D^*$.

Case 2: If $\boldsymbol{s^{(k)} \neq a'}$

Then $s^{(j)*} = a^*$ for all $j \geq k+1$ and the corresponding player i's payoff is given by

$$V_i(s_i^*, s_{-i}^*)_{h_i^{(k)}} = \underbrace{\sum_{j=1}^{k} D^{j-1} V_i\left(s_i^{(j)}; s_{-i}^{(j)}\right)}_{=C} + D^k \underbrace{V_i^{(k+1)}(a_i^*; a_{-i}^*)}_{=v_i^*} + \sum_{j=k+2}^{\infty} D^{j-1} \underbrace{V_i^{(j)}(a_i^*; a_{-i}^*)}_{=v_i^*}$$

If the player deviate and adopt another strategy a_i^0 in $(k+1)^{th}$ game, then $V_i^{(G)}\left(a_i^0; a_{-i}^*\right) \leq V_i^{(G)}(a_i^*; a_{-i}^*)$ since a_i^* is the best response to a_{-i}^* by the definition of Nash equilibrium. Since $s^{(j)*} = a^*$ for all $j \geq k+2$, then the corresponding payoff will be

$$V_i\left(s_i^{\left(a_i^0, h_i^{(k)}\right)}, s_{-i}^*\right)_{h_i^{(k)}} = C + D^k \underbrace{V_i^{(k+1)}\left(a_i^0; a_{-i}^*\right)}_{\leq v_i^*} + \sum_{j=k+2}^{\infty} D^{j-1} \underbrace{V_i^{(j)}(a_i^*; a_{-i}^*)}_{=v_i^*} \leq V_i(s_i^*, s_{-i}^*)_{h_i^{(k)}}.$$

Hence, the player i will not deviate in this case also.

Therefore, the player i will not adopt one-stage deviation and the proposed strategic profile is the subgame perfect equilibrium by one-stage deviation principle.

Extension of Folk theorem

We consider the following infinite repeated games which the following games G is played repeatedly

		Player 2	
		A	**B**
Player 1	**A**	$(6,6)$	$(1,99)$
	B	$(99,1)$	$(3,3)$

One can show that

- (B, B) is the only pure strategy Nash equilibrium of the games G and each player gets a payoff of 3 per games.
- Note that $V_i(A; A) = 6 > 3$ for all players, it follows from Folk theorem that there exists a subgames perfect equilibrium which two players keep playing (A, A) for all games and achieve a payoff of 6 per games if the discounting factor D is sufficiently large.

Question: Can the players "work together" and do even better?

As we observed from the above payoff matrix that one player (i.e. the player who chooses the strategy A) can earn a huge payoff 99 if two players adopt the strategies (B, A) or (A, B). However, two players will not cooperate and play 1 of these strategy since another player is strictly worse payoff and he/she has incentive to deviate.

Question: What can they do?

Suppose that two players agree that they will play (A, B) in 50% of the time and play (B, A) in 50% of the time (i.e. mixing between (A, B) and (B, A) with equal chance) and they will play (B, B) if someone breaks the agreement. Then two players can enjoy an average payoff of $0.5(99) + 0.5(1) = 50 > 3$.

Question 2: How to execute the strategy?

- As it involves "mixed strategy", one can execute the strategy using *public randomization:* A third party (referee) is invited to draw one of the two strategies randomly in *each* round and two players then strictly follow the strategy chosen.

Therefore, one can extend the folk theorem into the case that the players can enhance the payoffs by mixing multiple strategies through public randomization.

To state this version of folk theorem, we need the following notations:

Additional notation 1: Average payoff

We let $V_i^{(k)}\left(s_i^{(k)}; s_{-i}^{(k)}\right)$ be the payoff received from k^{th} round of the games.

We can define the average discounted payoff be the *weighted average of payoffs received from different rounds of the games*. That is,

$$\bar{V}_i = \sum_{k=1}^{N} \underbrace{\frac{D^{k-1}}{\sum_{k=1}^{N} D^{k-1}}}_{w_k} V_i^{(k)}(\cdot) = \sum_{k=1}^{N} \frac{1-D}{1-D^N} D^{k-1} V_i^{(k)}(\cdot)$$

for finite repeated games and

$$\bar{V}_i = \sum_{k=1}^{\infty} \underbrace{\frac{D^{k-1}}{\sum_{k=1}^{\infty} D^{k-1}}}_{w_k} V_i^{(k)}(\cdot) = (1-D) \sum_{k=1}^{\infty} D^{k-1} V_i^{(k)}(\cdot)$$

for infinite games.

In other words, less weight is assigned to payoff which is paid at later times.

Additional notation 2: Convex combination and Convex hull

We let $V^{(1)}, V^{(2)}, \dots, V^{(m)}$ be a finite set of vectors in $\mathbb{R}^n$ where $V^{(k)} = \left(V_1^{(k)}, V_2^{(k)}, \dots, V_n^{(k)}\right)$. Then we say a vector v is the *convex combination* of the vectors $V^{(1)}, V^{(2)}, \dots, V^{(m)}$ if it can be expressed as

$$v = \lambda_1 V^{(1)} + \lambda_2 V^{(2)} + \cdots + \lambda_m V^{(m)},$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$ and $\lambda_i \geq 0$.

The *convex hull* of a set of vectors $W = \{V^{(1)}, V^{(2)}, \dots, V^{(m)}\}$ is defined as the collection of vectors which are convex combinations of these m vectors. That is,

$$C(W) = \left\{v \in \mathbb{R}^n : v = \lambda_1 V^{(1)} + \lambda_2 V^{(2)} + \cdots + \lambda_m V^{(m)},\ \ \lambda_i \geq 0\ \ and\ \ \sum_{i=1}^{m} \lambda_i = 1\right\}.$$

In our case, W can be interpreted as a set of possible payoffs under some pure strategies and the corresponding convex hull can be viewed as a set of possible payoffs (called *feasible payoff*) by mixing the corresponding pure strategies $(s^{(1)}, s^{(2)}, \dots, s^{(m)}$, where $s^{(k)}$ is the pure strategic profile that yields the payoff of $V^{(k)}$) through public randomization and λ_k represents the probability that the strategic profile $s^{(k)}$ will be picked.

Theorem 5 (Folk theorem – Version 2)

A static games G is played repeatedly in n-person infinite repeated games. We let $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ be a pure strategy Nash equilibrium of the games G and $v_i^* = V_i^G(a_i^*; a_{-i}^*)$ be the corresponding payoff of player i in the game G. We let W be the set of possible payoffs that can be achieved using pure strategy.

Suppose that there exists a feasible payoff $v \in C(W)$ such that $v_i' > v_i^*$ for all i, there exists $D^* \in (0,1]$ such that for any $D \geq D^*$, there exists a subgame perfect Nash equilibrium of the infinite repeated games such that each player (player i) can achieve an average payoff arbitrarily close to v_i'.

The proof of this theorem is very similar to that of theorem 4. We just need to replace the strategy a' by a mixed strategic profile (through public randomization) in the proof of the theorem.

Some remarks

According to the theorem 5, any feasible average payoff can be achieved through *public randomization*. However, some may criticize the feasibility since it involves randomization and the actual payoff to each player could be different from the expected payoff v_i. We would like to ask if there is a more realistic way to achieve such payoff.

As an example, we consider the infinite repeated games presented in p.139. We consider the following strategic profiles:

- Two players play (A, B) in *odd* rounds (i.e. Rounds 1, 3, 5, …) and
- Two players play (B, A) in *even* rounds (i.e. Rounds 2, 4, 6, …)
- Two players play (B, B) if one of the players do not follow the above strategies.

Then the average payoff to player 1 is found to be

$$\bar{v}_1 = (1-D)\sum_{k=1}^{\infty} D^{k-1} V_1^{(k)}(\cdot) = (1-D)(1 + 99D + D^2 + 99D^3 + D^4 + \cdots)$$

$$= (1-D)\left(\frac{1}{1-D^2} + \frac{99D}{1-D^2}\right) = \frac{1+99D}{1+D}$$

And the average payoff to player 1 is given by

$$\bar{v}_2 = (1-D)\sum_{k=1}^{\infty} D^{k-1} V_1^{(k)}(\cdot) = (1-D)(99 + D + 99D^2 + D^3 + \cdots)$$

$$= (1-D)\left(\frac{99}{1-D^2} + \frac{D}{1-D^2}\right) = \frac{99+D}{1+D}.$$

One can show that

$$\lim_{D\to 1^-} \bar{v}_1 = \lim_{D\to 1^-} \frac{1+99D}{1+D} = 50, \qquad \lim_{D\to 1^-} \bar{v}_2 = \lim_{D\to 1^-} \frac{99+D}{1+D} = 50.$$

So two players can achieve an average payoff of 50 when the discounting factor is close to 1.

For completeness, it remains to verify that such strategic profile constitute the subgame perfect equilibrium using one-stage deviation principle: Suppose that the players are in k^{th} stage and we shall consider the following three scenarios:

Case 1: If some players did not follow the agreement in previous $(k-1)^{th}$ stage. Then all players will pay (B, B) afterwards. As (B, B) is the Nash equilibrium, it follows from Theorem 1 that this constitutes the subgames perfect equilibrium.

Case 2: If all players follows the agreement in previous games and the players play (A, B) in k^{th} games

- For player 1 (who plays A), then the total payoff received is given by

$$V_1(A) = \underbrace{\sum_{i=1}^{k-1} D^{i-1} V_1^{(i)}}_{C_1} + D^{k-1}(1) + D^k(99) + D^{k+1}(1) + D^{k+2}(99) \ldots$$

$$= C_1 + \frac{D^{k-1}}{1-D^2} + \frac{99D^k}{1-D^2}.$$

If the player chooses B in k^{th} games instead, then all players will play (B, B) afterwards, then the corresponding total payoff is seen to be

$$V_1(B) = C_1 + D^{k-1}(3) + D^k(3) + D^{k+1}(3) + \cdots = C_1 + \frac{3D^{k-1}}{1-D}.$$

One can show that

$$V_1(A) \geq V_1(B) \Leftrightarrow \frac{D^{k-1}}{1-D^2} + \frac{99D^k}{1-D^2} \geq \frac{3D^{k-1}}{1-D} \Leftrightarrow D \geq \frac{1}{48}.$$

Given that $D \in [0,1]$, player 1 has no incentive to deviate.

- For player 2 (who plays B), then the corresponding total payoff is seen to be

$$V_2(B) = \underbrace{\sum_{i=1}^{k-1} D^{i-1} V_2^{(i)}}_{C_2} + D^{k-1}(99) + D^k(1) + D^{k+1}(99) + D^{k+2}(1) \dots$$

$$= C_2 + \frac{99D^{k-1}}{1-D^2} + \frac{D^k}{1-D^2}.$$

If the player chooses A in k^{th} games instead, then all players will play (B, B) afterwards, then the corresponding total payoff is seen to be

$$V_2(A) = C_2 + D^{k-1}(6) + D^k(3) + D^{k+1}(3) + \cdots = C_1 + 6D^{k-1} + \frac{3D^k}{1-D}$$

We can show that

$$V_2(B) \geq V_2(A) \Leftrightarrow \frac{99D^{k-1}}{1-D^2} + \frac{D^k}{1-D^2} \geq 6D^{k-1} + \frac{3D^k}{1-D} \Leftrightarrow 3D^2 - 2D + 93 \geq 0.$$

The above inequality holds for all $D \in [0,1]$ (I omitted the details here). Thus, player 2 has no incentive to deviate.

Hence, the strategic profile also constitutes the subgames perfect equilibrium and each player can achieve an average payoff close to 50 when $D \to 1^-$.

Application 2: Strategic Bargaining

Another nice example of dynamic games is about strategic bargaining in which the players negotiate and comprise on certain proposal.

- Two companies engage in a joint project. They need to negotiate how to share the revenue/profit generated from the project.
- A foreign firm may acquire another firm in China (target firm) to start its business in mainland China. Two firms need to negotiate the amount (cash or shares) to be paid to the target firm.

In a bargaining games, one player will first propose a deal/proposal to the remaining players. Other players decide whether to accept the proposal. If the agreement is made, the games will end. If the players cannot make an agreement, either they move on another round of bargaining or the games end.

One can observe that the bargaining games is a kind of dynamic games.

Simple strategic bargaining games: A general framework

We consider the following bargaining games: There are T ($T \geq 1$) periods in the games. Two players (Player 1 and player 2) need to split a pie which its value is assumed to be 1. The rundown of the bargaining games is as follows:

Period 1 (Current time)

- Player 1 (first-mover) first makes a proposal $\vec{x}^{(1)} = (x_1, x_2) = (x, 1 - x)$, where $x \geq 0$, to player 2 on how to split the pie. Under this proposal, player 1 will receive x of the pie and player 2 will receive $1 - x$ of the pie.
- After receiving the proposal from player 1, player 2 decides whether to accept the proposal or not. If player 2 accepts the proposal, the games is over and the players' payoffs are $(V_1, V_2) = (x, 1 - x)$. If player 2 rejects the proposal, the games will move on to period 2.

Period 2

- Player 2 makes another similar proposal $\vec{x}^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) = (x, 1-x)$ to player 1: player 1 will receive x of the pie and player 2 will receive $1-x$ of the pie under this proposal.
- After receiving the proposal, player 1 now decides whether to accept the proposal or not. If player 1 accepts the proposal, the games is over and the players' payoffs are $(V_1, V_2) = (x, 1-x)$. If player 2 rejects the proposal, the games will move on to period 3.

In the remaining periods, players takes turn to offer a proposal to another player (player 1 makes the proposal in odd period and player 2 makes the proposal in even period) and another player chooses to whether accept or reject the proposal.

Period T (Last period)

- Player i (last mover) makes an offer $\vec{x}^{(T)} = \left(x_1^{(T)}, x_2^{(T)}\right) = (x, 1-x)$ to player j. If player j accepts the offer, the players' payoffs are $(V_1, V_2) = (x, 1-x)$. Otherwise, all players will receive nothing, i.e. $(V_1, V_2) = (0, 0)$.

We assume that the discounter factor over 1-period is $D \in (0,1)$ so that the "present value" of the payoff V_i received at period k is $D^{k-1}V_i$.

Objective

Our objective is to examine the bargaining results of this bargaining games. We also examine the critical factors that affects the "bargaining power" of each player.

- Order of moves: First mover advantage & Last mover advantage.
- Other parameters: Discounting factor D (which affect the player's payoff at later stage) and time horizon T.

Example 29 (One round bargaining games)

We take $T = 1$ in the above games in which player 1 will make a "take-it-or-leave it" offer $\vec{x}^{(1)} = (x, 1 - x)$ to player 2 and player 2 choose to either accept the offer or reject the offer. Determine the optimal proposal offered by player 1 by finding the corresponding Nash equilibrium of the bargaining games.

☺Solution

The above bargaining games can be described by the following games tree:

Since the games is of perfect information, we proceed to solve for sequentially rational Nash equilibrium of the games using backward induction.

- We consider the last stage which the player 2 makes the last move. Given the proposal $\vec{x}^{(1)} = (x, 1-x)$ offered by player 1, the player 2 has two choices:
  - If the player 2 chooses to accept the offer, his payoff will be $1-x$.
  - If the player 2 chooses to reject the offer, his payoff will be 0.

  It is clear that player 2 chooses to accept the offer if and only if $1-x > 0$.
- We consider the first stage which the player 1 makes his offer. Knowing the acceptance criterion of player 2, the player 1's payoff can be described as

$$V_1\left(\vec{x}^{(1)}\right) = \begin{cases} x & if \;\; 1-x > 0 \\ 0 & if \; otherwise \end{cases}, \quad x \in [0,1].$$

We observe that the it is optimal for player 1 to offer a proposal $\vec{x} = (1-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is a very small constant. Player 2 will accept the proposal since $\underbrace{\varepsilon}_{accept} > \underbrace{0}_{reject}$. By taking $\varepsilon \to 0^+$, the sequentially rational Nash equilibrium is seen to be

$$(s_1^*, s_2^*) = \lim_{\varepsilon \to 0^+} \left(\underbrace{(1-\varepsilon, \varepsilon)}_{proposal}, A \right) = \big((1,0), A\big),$$

and the corresponding payoffs are $(V_1, V_2) = (1,0)$.

Remark of Example 29

We observe that the player 1 will get the "whole pie" in this bargaining games. It is because player 2 has no bargaining power. In fact, player 2 does not have any opportunity to offer any proposal in this one-round bargaining games and he is very passive in this games.

In the coming examples, we shall examine the factors that affect the "bargaining power" of each player.

Example 30 (Two round bargaining games: First mover advantage and last mover advantage)

We take $T = 2$ so that each player has chance to offer his own proposal at some period. Determine the final outcome of the games by finding the corresponding sequentially rational Nash equilibrium.

☺Solution

Similar to the previous example, we find the equilibrium using backward induction. We first consider period 2 when player 2 can choose his proposal. Similar to the analysis as in one round case, player 2 should offer $\vec{x}^{(2)} = (0,1)$ to player 1 and player 1 accept the offer. The corresponding payoffs, denoted by $\left(V_1^{(2)}, V_2^{(2)}\right)$, is given by

$$\left(V_1^{(2)}, V_2^{(2)}\right) = (0,1).$$

Next, we consider period 1 when player 1 chooses his proposal. We first determine the optimal strategy of player 2. Given the proposal $\overrightarrow{x_1} = (x, 1-x)$ submitted by player 1, player 2 chooses to accept the proposal if and only if

$$\underbrace{1-x}_{accept} \geq \underbrace{DV_2^{(2)}}_{\substack{reject \\ (delay\ ot\ period\ 2)}} = D(1) \Rightarrow x \leq 1-D.$$

Then player 1's payoff in period 1 can be expressed as

$$V_1^{(1)}(x) = \begin{cases} x & if \;\; x \leq 1-D \\ D\left(V_1^{(2)}\right) & if\ otherwise \end{cases} = \begin{cases} x & if \;\; x \leq 1-D \\ 0 & if\ otherwise \end{cases}.$$

Hence, it is optimal for player 1 to submit a proposal $\vec{x}^{(1)} = (1-D-\varepsilon, D+\varepsilon)$, where $\varepsilon > 0$ is a very small positive constant. Player 2 must accept the offer since $D+\varepsilon > D$.

Hence, the sequentially rational Nash equilibrium is given by

$$\left(s_1^{(1)}, s_2^{(1)}\right) = \big((1-D, D), A\big), \qquad \left(s_1^{(2)}, s_2^{(2)}\right) = \big(A, (0,1)\big).$$

The corresponding payoffs in the games is seen to be

$$(V_1, V_2) = (1-D, D).$$

Remark of Example 30

Comparing with the result in Example 29, player 2 can enjoy some extra surplus in the bargaining games since the player 2 can make the offer in the last round and has "last-mover advantage".

Player 2 can get the whole pie in the second round since player 1 will accept any offer (although he will get zero payoff). Knowing this, player 1 has to offer an attractive deal to player 2 in the first round to "encourage" player 2 to accept his proposal.

Therefore, this "last-mover advantage" is one of the key factor that affects the bargaining power.

On the other hand, the player 1 enjoys first mover advantage since he can enjoy the "largest" pie in the games since player 2 suffers from some "discount" on the payoff. So player 1 is able to get some surplus by taking this advantage although player 2 is expected to get the whole pie in the second round.

One would like to ask which of these two advantages will be a dominant factor in bargaining games. In fact,

- If the discount factor D is large, then the last-mover advantage will dominate the first-mover advantage. In this example, this happens when $1 - D < D \Rightarrow D > \frac{1}{2}$
- If the discounting factor D is small, then the first-mover advantage will dominate the last-mover advantage. In this example, this happens when $D < \frac{1}{2}$.

Example 31 (Three-period bargaining games: Can I say no?)

We now take $T = 3$ so that player 1 has both first mover advantage and last mover advantage. We examine whether the player 1 can "dominate" the bargaining games as in Example 1. Find the sequentially rational equilibrium for this games.

☺Solution

We use backward induction and first find the equilibrium in the last period when player 1 makes his proposal.

Using the result in Example 29, we deduce that player 1 should offer $x^{(3)} = (1,0)$ and player 2 accepts the offer). The corresponding payoffs, denoted by $\left(V_1^{(3)}, V_2^{(3)}\right)$, is given by

$$\left(V_1^{(3)}, V_2^{(3)}\right) = (1,0).$$

We proceed to determine the equilibrium in period 2 when player 2 makes his proposal. We first determine the decision rule of player 1. Give the proposal $\vec{x}^{(2)} = (x, 1-x)$ made by player 2, player 1 accepts the offer immediately if and only if

$$\underbrace{x}_{accept} \geq \underbrace{DV_1^{(3)}}_{reject} = D.$$

So player 2's payoff in period 2 can be expressed as

$$V_2^{(2)} = \begin{cases} 1-x & if \;\; x \geq D \\ DV_2^{(3)} & if \; otherwise \end{cases} = \begin{cases} 1-x & if \;\; x \geq D \\ 0 & if \; otherwise \end{cases}.$$

One can find that it is optimal for player 2 to offer $\vec{x}^{(2)} = (D+\varepsilon, 1-D-\varepsilon)$ and player 1 accepts it immediately. By taking $\varepsilon \to 0^+$, the players' payoff is found to be

$$\left(V_1^{(2)}, V_2^{(2)}\right) = (D, 1-D).$$

Finally, we consider period 1 and determine the corresponding equilibrium strategy. Given player 1's offer $\vec{x}^{(1)} = (x, 1 - x)$ at period 1, player 2 chooses to accept the offer if and only if

$$1 - x \geq DV_2^{(2)} = D(1 - D) \Rightarrow x \leq 1 - D + D^2.$$

So player 1's payoff in period 1 can be expressed as

$$V_1^{(1)} = \begin{cases} x & if \;\; x \leq 1 - D + D^2 \\ DV_1^{(2)} & if \; otherwise \end{cases} = \begin{cases} x & if \;\; x \leq 1 - D + D^2 \\ D^2 & if \; otherwise \end{cases}.$$

By some simple observation, we find that it is optimal for player 1 to offer

$$\vec{x}^{(1)} = (1 - D + D^2 - \varepsilon, D - D^2 + \varepsilon),$$

and player 2 accepts the offer. By taking $\varepsilon \to 0^+$, the players' payoffs are given by

$$(V_1, V_2) = \left(V_1^{(1)}, V_2^{(1)}\right) = (1 - D + D^2, D - D^2).$$

So we conclude that the sequentially rational Nash equilibrium is

$$\left(s_1^{(1)}, s_2^{(1)}\right) = \left((1 - D + D^2, D - D^2), A\right),$$

$$\left(s_1^{(2)}, s_2^{(2)}\right) = \left(A, (D, 1 - D)\right),$$

$$\left(s_1^{(3)}, s_2^{(3)}\right) = \left((1, 0), A\right).$$

Remark of Example 31

Although player 1 has both first-mover advantage and last-mover advantage, his surplus $1 - D + D^2$ is less than that in one round bargaining case in Example 29 (i.e. 1).

This is because the player 2 *has a right to make a proposal* in the intermediate stage. This gives player 2's incentive to "say no" to player 1's offer in the first stage if he finds the offer unattractive.

General case T

One can extend the above results easily into the general case when the players are allowed to negotiate over a finite period of time T. We summarize the result in the following theorem:

Theorem 6

The sequentially rational Nash equilibrium of two person T-period bargaining games can be described as follows:

(1) All players must reach an agreement in the first round.

(2) Player 1 offers $\vec{x} = \left(\underbrace{\sum_{k=0}^{T-1} (-1)^k D^k - \varepsilon}_{player\ 1}, \underbrace{\sum_{k=1}^{T-1} (-1)^{k-1} D^k + \varepsilon}_{player\ 2} \right)$

The corresponding players' payoffs are given by

$$(V_1, V_2) = \begin{cases} \left(\dfrac{1 + D^T}{1 + D}, \dfrac{D - D^T}{1 + D} \right) & if\ \ T\ is\ odd \\ \left(\dfrac{1 - D^T}{1 + D}, \dfrac{D + D^T}{1 + D} \right) & if\ T\ is\ even \end{cases}.$$

Remark of Theorem 6

The result allows us to examine the effect of time horizon on the bargaining power of each player. (We take $D = 0.8$)

T is odd		
T	V_1	V_2
1	1	0
3	0.84	0.16
5	0.7376	0.2624
11	0.6033	0.3967
∞	0.5556	0.4444

T is even		
T	V_1	V_2
2	0.2	0.8
4	0.328	0.672
6	0.4099	0.5901
10	0.4959	0.5041
∞	0.5556	0.4444

- We observe that player 1's bargaining power decreases when T is odd (player 1 has both first mover advantage and last mover advantage).
- We observe that player 1's bargaining power increases when T is even (player 1 has only first mover advantage). One possible explanation is that player 2 owns the last mover advantage and the longer time horizon will become advantageous to player 1.

Proof of Theorem 6

We first consider the last period T. The games is equivalent to one-round bargaining case. According to the result in Example 29, the players' payoffs, depending on which player offers the proposal, are given by

$$\left(V_1^{(T)}, V_2^{(T)}\right) = \begin{cases} (1,0) & if\ \ T\ is\ odd \\ (0,1) & if\ T\ is\ even \end{cases}.$$

Given the player's payoffs $\left(V_1^{(k+1)}, V_2^{(k+1)}\right)$ at $(k+1)^{th}$ round of the bargaining games, we proceed to determine the players' payoff at k^{th} round. We consider the following two cases:

Case 1: If player 1 makes a proposal at k^{th} round

We let $\vec{x}^{(k)} = (x, 1-x)$ be the proposal made by player 1. Player 2 accepts the offer only when

$$1 - x \geq DV_2^{(k+1)} \Rightarrow x \leq 1 - DV_2^{(k+1)}.$$

So player 1's payoff is given by

$$V_1^{(k)} = \begin{cases} x & if \;\; x \le 1 - DV_2^{(k+1)} \\ DV_1^{(k+1)} & if \; otherwise \end{cases}$$

If the agreement is made at the current round, player 1 should offer $\vec{x}^{(k)} = \left(1 - DV_2^{(k+1)} - \varepsilon, DV_2^{(k+1)} + \varepsilon\right)$ to player 2, where ε is a very small positive constant. It remains to check whether it is optimal to do so.

Using the fact that $V_1^{(k+1)} + V_2^{(k+1)} = 1$, we deduce that

$$\underbrace{\lim_{\varepsilon \to 0^+} \left(1 - DV_2^{(k+1)} - \varepsilon\right)}_{payoff\,(offer)} = 1 - D\left(1 - V_1^{(k+1)}\right) = \underbrace{1 - D}_{>0} + DV_1^{(k+1)}$$

$$> \underbrace{DV_1^{(k+1)}}_{payoff\,(delay)} .$$

Then it is optimal for player 1 to offer $\vec{x}$ and player 2 accepts the offer. The players' payoffs are

$$\left(V_1^{(k)}, V_2^{(k)}\right) = \left(1 - DV_2^{(k+1)},\;\; DV_2^{(k+1)}\right) \ldots\ldots (1)$$

Case 2: If player 2 makes a proposal at k^{th} round

One can apply the similar reasoning and deduce that the players' payoff at k^{th} round can be expressed as

$$\left(V_1^{(k)}, V_2^{(k)}\right) = \left(DV_1^{(k+1)},\ 1 - DV_1^{(k+1)}\right) \dots\dots (2)$$

Using the recursive formula (1) and (2), one can deduce that

$$V_1 = V_1^{(1)} \overset{By\ (1)}{\triangleq} 1 - DV_2^{(2)} \overset{By\ (2)}{\triangleq} 1 - D\left(1 - DV_1^{(3)}\right)$$

$$\overset{By\ (1)}{\triangleq} 1 - D + D^2\left(1 - DV_2^{(4)}\right) \overset{By\ (2)}{\triangleq} 1 - D + D^2 - D^3\left(1 - DV_1^{(5)}\right)$$

$$= \cdots = 1 - D + D^2 - D^3 + \cdots + (-1)^{T-1}D^{T-1} = \frac{1 - (-D)^T}{1 + D}.$$

Similarly, we get

$$V_2 = D - D^2 + \cdots - (-1)^{T-1}D^{T-1} = \frac{D - (-D)^T}{1 + D}.$$

Example 32 (Harder)

101 players bargains on how to split a unit of resource. The games goes as follows:

- At each round, one player is randomly selected from these 101 players with equal probability and this player can make a proposal $\vec{x} = (x_1, x_2 \dots, x_{101})$ on how to split the resource among 101 players.
- If the proposal is supported by the majority of players (including proposer), the games end. Otherwise, the bargaining will be started over. The games will be repeated until an agreement is made.

Determine the final outcome of the games by finding the sequential rational Nash equilibrium. Find the payoffs to each player. Take $D = 0.8$.

☺Solution

Since all players are symmetric in the games, we let V^* be the (common) expected payoff of player i in the bargaining games.

Firstly, we determine the optimal strategy made by the player (player i) who is chosen to make a proposal.

- In order that his proposal is accepted, he must get support from at least 50 $(= \frac{100}{2})$ members.
- Player j (who is not proposer) will accept the current proposal if and only if

$$\underbrace{x_j}_{\substack{payoff\\(accept)}} \geq \underbrace{DV^*}_{\substack{payoff\\(delay)}}.$$

Hence, the optimal strategy of player i is to give $DV^* + \varepsilon$ to 50 of the 100 players and remaining 50 players receive nothing. Then 51 players will support the proposal and the proposal is accepted. The corresponding payoff to the player i (proposer) is given by

$$V_i = \lim_{\varepsilon \to 0^+} 1 - 50(DV^* + \varepsilon) = 1 - 50(DV^*).$$

It remains to compute V^*. We observe that

- A player is chosen to be proposer with probability $\frac{1}{101}$ and gets a payoff of $V_i = 1 - 50(DV^*)$.
- If the player is not proposer (with probability $\frac{100}{101}$) , he can receive DV^* with probability $\frac{50}{100} = \frac{1}{2}$ (50 of the players will receive DO^* according to the proposal).

Thus, V^* can be expressed as

$$V^* = \frac{1}{101}[1 - 50(DV^*)] + \frac{1}{2}\left(\frac{100}{101}\right) DV^* \Rightarrow V^* = \frac{1}{101}.$$

Thus the optimal proposal (with $D = 0.8$) is found to be

$$x_{Proposer} = 1 - 50DV^* = \frac{61}{101}, \qquad x_{support\ member} = \frac{4}{505}.$$

(*Remark: One can observe that the strategic profile is sequentially rational Nash equilibrium.)