MATH4425 (T1A) – Tutorial 1

Kazovskaia Anastasiia

February, 8

Important information

- Today's tutorial is conducted by **HO Ho Yi Alexis**. Next tutorials will be conducted by **Kazovskaia Anastasiia (Ana)**
- T1A: **Thursday 19:00 - 19:50** (Rm 2610, Lift 31-32)
- Office hours: TBA. **Please vote**, today's the last chance to do that
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

1 Overview

Definition: A time series **(TS)** is a sequence of **random variables** labeled by time t:

$$\{Z_1, Z_2, \ldots, Z_t, \ldots\}$$

or

$$\{\ldots, Z_{-1}, Z_0, Z_1, Z_2, \ldots, Z_t, \ldots\}$$

Denote them by $\{Z_t\}_{t\in\mathbb{N}}$ or $\{Z_t\}_{t\in\mathbb{Z}}$, respectively. For a shorter notation – $\{Z_t\}$.

In case of continuous time (i.e. $t \in \mathbb{R}$), TS is not actually a *series*. Instead, this more complicated concept is usually referred to as a **stochastic process**.

Time series **data** are observations of TS $\{Z_t\}$.

One of the most native examples from the lecture:

Example: Let Z_t =weather on the tth day.

Weather=	29°	30°	9°	$\cdots$
Date t=	1	2	3	$\cdots$
Notation	$Z_1 = 29^\circ$	$Z_2 = 30^\circ$	$Z_3 = 9^\circ$	$\cdots$

Another example to compare:

1) Daily temperature measurements of the *same* patient (Time series)
2) Temperature measurements of *all* patients at the same time (Independent samples)

Main objective of TS analysis is to be able to *predict* (in some sense) possible future of TS according to its observable past:

Past data (available TS observations) $\Longrightarrow$ Appropriate model for TS r.v. Z_t $\Longrightarrow$ Future of TS:

- $\mathbb{E}(Z_{n+l}|y_1, \ldots, y_n)$
- $\mathbb{P}(a \leq Z_{n+l} \leq b|y_1, \ldots, y_n)$ for some $a < b$

2 Fundamental Concepts

2.1 Strict stationarity and weak stationarity

Definition: Let Z_t be a TS

1) When $t = t_1$, we have:
$$Z_{t_1} \to \mathbb{P}(Z_{t_1} \leq z).$$
When $t = t_1 + k$, we have:
$$Z_{t_1+k} \to \mathbb{P}(Z_{t_1+k} \leq z).$$
If $\mathbb{P}(Z_{t_1} \leq z) = \mathbb{P}(Z_{t_1+k} \leq z)$ for $\forall t_1, k, z$, we say: $\{Z_t\}$ is **first order stationary in distribution**.

2) When $t = (t_1, t_2)$, we have:
$$(Z_{t_1}, Z_{t_2}) \to \mathbb{P}(Z_{t_1} \leq z_1, Z_{t_2} \leq z_2).$$
When $t = (t_1 + k, t_2 + k)$, we have:
$$(Z_{t_1+k}, Z_{t_2+k}) \to \mathbb{P}(Z_{t_1+k} \leq z_1, Z_{t_2+k} \leq z_2).$$
If $\mathbb{P}(Z_{t_1} \leq z_1, Z_{t_2} \leq z_2) = \mathbb{P}(Z_{t_1+k} \leq z_1, Z_{t_2+k} \leq z_2)$ for $\forall t_1, t_2, k$ and (z_1, z_2) , we say: $\{Z_t\}$ is **second order stationary in distribution**.

Please, pay attention that distribution $\mathbb{P}$ still depends on t_1, t_2. WLOG $t_1 \leq t_2$ and $t_2 = t_1 + (t_2 - t_1) = t_1 + \Delta$. Then
$$\mathbb{P}(Z_{t_1} \leq z_1, Z_{t_2} \leq z_2) = \mathbb{P}(Z_{t_1} \leq z_1, Z_{t_1+\Delta} \leq z_2) = \mathbb{P}(Z_0 \leq z_1, Z_\Delta \leq z_2)$$
Basically, it means that $\mathbb{P}$ only depends on time difference Δ.

3) When $t = t_1, \ldots, t_n$, we have:
$$(Z_{t_1}, \ldots, Z_{t_n}) \to \mathbb{P}(Z_{t_1} \leq z_1, \ldots, Z_{t_n} \leq z_n).$$
When $t = t_1 + k, \ldots, t_n + k$, we have:
$$(Z_{t_1+k}, \ldots, Z_{t_n+k}) \to \mathbb{P}(Z_{t_1+k} \leq z_1, \ldots, Z_{t_n+k} \leq z_n)$$
If $\mathbb{P}(Z_{t_1} \leq z_1, \ldots, Z_{t_n} \leq z_n) = \mathbb{P}(Z_{t_1+k} \leq z_1, \ldots, Z_{t_n+k} \leq z_n)$ for $\forall t_1, \ldots, t_n, k$ and $(z_1, \ldots, z_n)$ and n, we say: $\{Z_t\}$ is **strictly stationary**.

Definition: Let Z_t be a TS

- Mean function of $\{Z_t\}$: $\mu_t = \mathbb{E}Z_t$
- Variance function of $\{Z_t\}$: $\sigma_t^2 = \mathbb{E}(Z_t - \mu_t)^2$
- Covariance function between Z_{t_1} and Z_{t_2}:
$$\gamma(t_1, t_2) = \mathbb{E}[(Z_{t_1} - \mu_{t_1})(Z_{t_2} - \mu_{t_2})]$$

- Correlation function between Z_{t_1} and Z_{t_2}:

$$\rho(t_1, t_2) = \frac{\gamma(t_1, t_2)}{\sqrt{\sigma^2_{t_1}}\sqrt{\sigma^2_{t_2}}}$$

Definition: Let Z_t be a TS. If

$$\mu_t = \mu < \infty,$$
$$\sigma^2_t = \sigma^2 < \infty,$$
$$\gamma(t, t+k) = \gamma_k$$

for any t, then $\{Z_t\}$ is said **(second-order) weakly stationary**. Usually, we call it simply a **stationary** TS.

Property: Assume $\{Z_t\}$ is strictly stationary

- If $\mathbb{E}|Z_t| < \infty$, then $\mu_t = \mu < \infty$
- If $\mathbb{E}Z^2_t < \infty$, then $\sigma^2_t = \sigma^2 < \infty$
- Furthermore $\gamma(t, t+k) = \gamma_k$ and $\rho(t, t+k) = \rho_k$

Thus, if $\mathbb{E}Z^2_t < \infty$, then the property holds. So, *strict stationarity* $\Longrightarrow$ *second-order weakly stationarity.*

The opposite is **not** necessarily true: Let

$$Z_{2t} \sim^{iid} U[-\sqrt{3}; \sqrt{3}]$$

$$Z_{2t+1} \sim^{iid} \mathcal{N}(0, 1)$$

be independent r.v.s. Then

$$\mu_t = \mathbb{E}Z_t = 0$$
$$\sigma^2_t = \text{var}(Z_t) = 1$$
$$\gamma_k = 0, \text{ if } k > 0$$

So, $\{Z_t\}$ is *second-order weakly stationary* but **not** *strictly stationary.*

Example 2.2: Consider the following time sequence

$$Z_t = A\sin(\omega t + \theta),$$

where A is a r.v. with:

- $\mathbb{E}A = 0$
- $\mathbb{E}A^2 = 1$,

$\theta \sim U[-\pi, \pi]$ is independent of A, and ω is a constant.

Then we have:

$$\mu_t = \mathbb{E}(Z_t) = [\text{due to independence}] = \mathbb{E}A \times \mathbb{E}[\sin(\omega t + \theta)] = 0 = \mu < \infty$$

$$\gamma(t, t+k) = \mathbb{E}(Z_t Z_{t+k}) = \mathbb{E}[A^2 \sin(\omega t + \theta)\sin[\omega(t+k) + \theta] = [\text{independence}] =$$

$$\mathbb{E}A^2 \times \mathbb{E}\left[\frac{1}{2}[\cos(\omega k) - \cos(\omega(2t+k) + 2\theta)]\right] = [\mathbb{E}A^2 = 1, \omega \text{ is a constant}] =$$

$$\frac{1}{2}\cos(\omega k) - \frac{1}{2}\mathbb{E}\cos(\omega(2t+k)+2\theta) = [\theta \sim U[-\pi,\pi]] =$$

$$\frac{1}{2}\cos(\omega k) - \frac{1}{2}\int_{-\pi}^{\pi}\cos(\omega(2t+k)+2\theta)\frac{1}{2\pi}d\theta = [y := \omega(2t+k)+2\theta] =$$

$$\frac{1}{2}\cos(\omega k) - \frac{1}{8\pi}\int_{\omega(2t+k)-2\pi}^{\omega(2t+k)+2\pi}\cos y\ dy =$$

$$\frac{1}{2}\cos(\omega k) - \frac{1}{8\pi}\sin y\Big|_{\omega(2t+k)-2\pi}^{\omega(2t+k)+2\pi} = \frac{1}{2}\cos(\omega k) - \frac{1}{8\pi}[\sin(\omega(2t+k)) - \sin(\omega(2t+k))] =$$

$$\frac{1}{2}\cos(\omega k) = \gamma_k$$

Also, notice that

$$\sigma_t^2 = \gamma(t,t) = \gamma_0 = \frac{1}{2}\cos(\omega \times 0) = \frac{1}{2} = \sigma^2 < \infty$$

Hence, the process is *second-order weakly stationary.*

2.2 Autocovariance and autocorrelation functions

Let $\{Z_t\}$ be a sequence of stationary TS r.v.s. Then $\mathbb{E}Z_t = \mu$, a constant. And

$$\gamma_k = \text{ cov}(Z_t, Z_{t+k}) = \mathbb{E}[(Z_t - \mu)(Z_{t+k} - \mu)]$$

only depends on k.

Definition: γ_k is called **autocovariance (ACV)** of $\{Z_t\}$.

Let

$$\rho_k = \frac{\text{cov}(Z_t, Z_{t+k})}{\sqrt{\text{var}(Z_t)}\sqrt{\text{var}(Z_{t+k})}} = \frac{\gamma_k}{\gamma_0}$$

only depends on k, too.

Definition: ρ_k is called **autocorrelation function (ACF)** of $\{Z_t\}$.

Properties of γ_k and ρ_k:

- $\gamma_0 = \sigma^2, \rho_0 = 1$
- $\gamma_k = \gamma_{-k}, \rho_k = \rho_{-k}$
- $\gamma_k \leq |\gamma_k| \leq \gamma_0, \rho_k \leq |\rho_k| \leq 1$

Important point: The smaller ρ_k, the less (linear) dependency between Z_t and Z_{t+k}.

Intuitively, as $k \to \infty$, $\rho_k \to 0$, generally.

In general, $\rho_k \neq 0$. This is an important feature of TS r.v.s.

2.3 Partial Autocorrelation function (PACF)

Definition: Let $\{Z_t\}$ be a stationary (*reminder: (second order) weakly stationary*) TS

The *conditional* correlation

$$\phi_{kk} := \operatorname{corr}(Z_t, Z_{t+k} | Z_{t+1}, \ldots, Z_{t+k-1}) = \frac{\mathbb{E}[(Z_t - \hat{Z}_t)(Z_{t+k} - \hat{Z}_{t+k})]}{\sqrt{\operatorname{var}(Z_t - \hat{Z}_t)}\sqrt{\operatorname{var}(Z_{t+k} - \hat{Z}_{t+k})}},$$

where $\hat{Z}_t = \mathbb{E}(Z_t | Z_{t+1}, \ldots, Z_{t+k-1})$, is called the **partial autocorrelation function PACF** of Z_t and Z_{t+k}.

Formula: $\phi_{11} = \rho_1$,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

2.4 White noise process

Definition: A process $\{a_t\}$ is called a **white noise process** if

1) $\mathbb{E}a_t = 0$
2) $\text{var}(a_t) = \mathbb{E}a_t^2 = \sigma_a^2$
3) $\gamma_k = \text{cov}(a_t, a_{t+k}) = 0, \text{ if } k \neq 0$

Properties:

1) (ACV)
$$\gamma_k = \begin{cases} \sigma_a^2, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

2) (ACF)
$$\rho_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

3) (PACF)
$$\phi_{kk} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

2.5 Estimation of ACV and ACF

Given $Z_1, Z_2, \ldots, Z_n$, how to estimate μ, σ^2, γ_k and ρ_k?

2.5.1 Sample mean

Definition:
$$\bar{Z} = \frac{1}{n}\sum_{t=1}^{n} Z_t,$$
is called the **sample mean** of Z_t.

$\bar{Z}$ is the estimator of the mean μ. Is this estimator valid?

1) $\bar{Z}$ is an unbiased estimator of μ, i.e.
$$\mathbb{E}\bar{Z} = \mu$$

2) $\bar{Z}$ is an L_2-consistent estimator of μ, i.e.
$$\frac{1}{n}\sum_{t=1}^{n} Z_t \xrightarrow[n\to\infty]{L^2} \mu,$$
if $\rho_k \to 0$ as $k \to \infty$ (**Ergodic property**)

2.5.2 Sample ACV

Definition:

$$\hat{\gamma}_k = \frac{1}{n-k}\sum_{t=1}^{n-k}(Z_t - \bar{Z})(Z_{t+k} - \bar{Z})$$

is called the **sample ACV** of Z_t.

$\hat{\gamma}_k$ are the estimators of γ_k. Are these estimators valid?

1) $\hat{\gamma}_k$ is biased estimator of γ_k, i.e.

$$\mathbb{E}\hat{\gamma}_k \neq \gamma_k$$

2) $\hat{\gamma}_k$ is an L_2-consistent estimator of γ_k, i.e.

$$\hat{\gamma}_k \xrightarrow[n\to\infty]{L^2} \gamma_k,$$

if $\rho_k \to 0$ as $k \to \infty$

In particular,

$$\hat{\sigma}_n^2 = \hat{\gamma}_0 = \frac{1}{n}\sum_{t=1}^{n}(Z_t - \bar{Z})^2 = \frac{1}{n}\sum_{t=1}^{n} Z_t^2 - \bar{Z}^2,$$

is called the **sample variance** of Z_t.

$\hat{\sigma}_n^2$ is an L_2-consistent estimator of σ^2, if $\rho_k \to 0$ as $k \to \infty$.

2.5.3 Sample ACF

Definition:

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0}$$

is called the **sample ACF** of Z_t.

$\hat{\rho}_k$ is the L_2-consistent estimator of ρ_k, if $\rho_k \to 0$ as $k \to \infty$.

Bartlett (1946) showed that

$$\mathrm{var}(\hat{\rho}_k) \approx \frac{1}{n}\sum_{i=-\infty}^{\infty}(\rho_i^2 + \rho_{i+k}\rho_{i-k} - 4\rho_k\rho_i\rho_{i-k} + 2\rho_k^2\rho_i^2)$$

For processes in which $\rho_k = 0$ for $k > m$, Bartlett's approximation becomes

$$\mathrm{var}(\hat{\rho}_k) \approx \frac{1}{n}(1 + 2\rho_1^2 + 2\rho_2^2 + \cdots + 2\rho_m^2)$$

In particular, when $Z_t = a_t$ is a white noise, we have

$$\mathrm{var}(\hat{\rho}_k) \approx \frac{1}{n}$$

How to check whether Z_t is a white noise or not?
Let

$$S_{\hat{\rho}_k} = \sqrt{\frac{1}{n}(1 + 2\hat{\rho}_1^2 + \cdots + 2\hat{\rho}_m^2)},$$

where m is a fixed integer.
If Z_t is a white noise, $S_{\hat{\rho}_k} \approx \sqrt{\frac{1}{n}}$.

2.5.4 Sample PACF

Definition:

$$\hat{\phi}_{11} = \hat{\rho}_1,$$

$$\widehat{\phi}_{kk} = \frac{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & \widehat{\rho}_k \end{vmatrix}}{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-1} \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & 1 \end{vmatrix}}$$

is called the **sample PACF** of Z_t.

$\hat{\phi}_{kk}$ is an L_2-consistent estimator of ϕ_{kk}.

2.6 Moving average and autoregressive representations of time series processes

Definition: Moving average representation (MA) of Z_t is

$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j},$$

where $\psi_0 = 1$, a_t is a white noise, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ (called **Wold's representation** or linear process).

Notation: Backshift operator: $B^j x_t = x_{t-j}$.

Thus, Z_t can be written as

$$\begin{aligned} Z_t &= \mu + B^0 a_t + \psi_1 B^1 a_t + \psi_2 B^2 a_t + \dots \\ &= \mu + \sum_{j=0}^{\infty} \psi_j B^j a_t \\ &= \mu + \Big(\sum_{j=0}^{\infty} \psi_j B^j \Big) a_t \end{aligned}$$

Denote $\dot{Z}_t = Z_t - \mu$ and $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$. Then $\dot{Z}_t = \psi(B) a_t$.

Some properties:

$$\mathbb{E} Z_t = \mu$$

$$\mathrm{var}(Z_t) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2$$

$$\mathbb{E}(a_t Z_{t-j}) = \begin{cases} \sigma_a^2, & \text{if } j = 0 \\ 0, & \text{if } j > 0 \end{cases}$$

$$\gamma_k = \mathbb{E}(\dot{Z}_t \dot{Z}_{t-k}) = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_k \psi_{i+k}}{\sum_{j=0}^{\infty} \psi_j^2}.$$

Definition: Autoregressive representation (AR) of Z_t is

$$\dot{Z}_t = \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \cdots + a_t = \sum_{j=1}^{\infty} \pi_j \dot{Z}_{t-j} + a_t,$$

where $\dot{Z}_t = Z_t - \mu$, $1 + \sum_{j=1}^{\infty} |\pi_j| < \infty$.

Notation: $\pi(B) = 1 - \sum_{j=1}^{\infty} \pi_j B^j$.

Then $\pi(B)\dot{Z}_t = a_t$.

Relationship of MA and AR representations:

1) If all roots of $\pi(z) = 0$ lie outside the unit circle, then

$$\pi(B)\dot{Z}_t = a_t \Longrightarrow \dot{Z}_t = (\pi(B))^{-1} a_t = \psi(B) a_t$$

2) If all roots of $\psi(z) = 0$ lie outside the unit circle, then

$$\dot{Z}_t = \psi(B) a_t \Longrightarrow a_t = (\psi(B))^{-1} \dot{Z}_t = \pi(B)\dot{Z}_t$$

2.7 Time Series Models

Let $\dots, Z_{-t}, \dots, Z_{-1}, Z_0, Z_1, \dots, Z_t, \dots$ be a sequence of TS r.v.

How to describe the relationship between Z_t and the past data $Z_{t-1}, Z_{t-2}, \dots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \dots) + a_t$$

It is called the **time series model**.

1. Autoregressive (AR(1)) model:
$$Z_t = \phi Z_{t-1} + a_t,$$
where ϕ is a constant and called the **parameter**

2. AR(p) model:
$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$
where ϕ_p is a constant and called the **parameter**, and p is called the **order** of the AR(p) model

3. $AR(\infty)$ model:
$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t$$

4. Moving-average (MA) model:
$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots.$$

5. ARMA model

6. Threshold AR model *(Tong 1977)*
7. Long memory model *(Granger (1980)* and *Hosking (1981))*
8. GARCH model *(Engle, 1982)* and *(Bolleslev, 1986)*
9. ARMA-GARCH model
10. Vector ARMA-GARCH model

...

3 Problems

Problem 1

Let

$$Z_t := U_t \sin(2\pi t) + V_t \cos(2\pi t),$$

where $U_t, V_t \sim^{iid} \mathcal{N}(0; 1)$.

(a) Is Z_t strictly stationary?

(b) Is Z_t stationary?

Solution

1) Since $t \in \mathbb{Z}$, actually, we simply have

$$\forall t \; Z_t = V_t$$

2) Note that

$$\mathbb{P}(Z_{t_1} \le z_1, \dots, Z_{t_n} \le z_n) = [\text{independence}] =$$
$$\mathbb{P}(V_{t_1} \le z_1) \times \dots \times \mathbb{P}(V_{t_n} \le z_n) = \Phi(z_1) \times \dots \times \Phi(z_n)$$

Similarly,

$$\mathbb{P}(Z_{t_1+k} \le z_1, \dots, Z_{t_n+k} \le z_n) = \mathbb{P}(V_{t_1+k} \le z_1) \times \dots \times \mathbb{P}(V_{t_n+k} \le z_n) =$$
$$\Phi(z_1) \times \dots \times \Phi(z_n)$$

Since

$$\mathbb{P}(Z_{t_1} \le z_1, \dots, Z_{t_n} \le z_n) = \mathbb{P}(Z_{t_1+k} \le z_1, \dots, Z_{t_n+k} \le z_n)$$

for $\forall t_1, \dots, t_n, k$ and $(z_1, \dots, z_n)$ and n, $\{Z_t\}$ is **strictly stationary**.

3) Moreover,

$$\mathbb{E}Z_t^2 = \mathbb{E}V_t^2 = 1 < \infty$$

By property of strictly stationary TS, $\{Z_t\}$ is **stationary**.

Problem 2

Is the following a valid autocorrelation function for some real-valued stationary TS? Why?

$$\rho_k = \begin{cases} 1, & \text{if } k = 0 \\ \phi, & \text{if } |k| = 1 \\ 0, & \text{if } |k| \ge 2 \end{cases},$$

where $\dfrac{1}{2} < |\phi| < 1$.

Solution

1) $\rho_0 = 1$ by the definition of ρ_0

2) a) $\rho_1 = \rho_{-1} = \phi$ by the definition of $\rho_{\pm 1}$

   b) If $|k| \geq 2$, then $\rho_k = \rho_{-k} = 0$ by the definition of function ρ_k

3) $|\rho_k| \leq 1$ by the definition of function ρ_k

Thus, ρ_k is a **valid ACF**.

4 Supplementary materials

Problem

Given a sequence of iid $\{\epsilon_t\}_{t\in\mathbb{N}_0}$[1] with $\mathbb{E}\epsilon_t = 0$ and distribution P_ϵ which is absolutely continuous with respect to Lebesgue measure λ on $\mathbb{R}$ (i.e. $\mathbb{P}(\epsilon_t \in A) = P_\epsilon(A) = \int_A f_\epsilon(x)\ d\lambda(x)$ for any «reasonable»[2] $A \subset \mathbb{R}$), let $\{Z_t\}_{t\in\mathbb{N}_0}$ be the AR(1) process defined by:

$$Z_t := \theta Z_{t-1} + \epsilon_t, \ \ t \geq 0,$$

where $\theta \in \mathbb{R}$.

Show that if P_ϵ has a positive dense over $\mathbb{R}$ (i.e. $f_\epsilon(x) > 0\ \forall x \in \mathbb{R}$), then

$$\forall B : \lambda(B) > 0\ \forall x \in \mathbb{R} \quad \mathbb{P}(Z_{t+1} \in B \mid Z_t = x) > 0$$

Asymptotic independence and Mixing

Sometimes there is no linear dependence between r.v. Z_t and Z_{t+k}. Although, there might be a more complicated dependence. To characterise the behaviour (in dependence) when *past* and *future* become *far apart*, mixing coefficients are introduced. Therefore, mixing assumptions *(Rosenblatt, 1956)* are used to convey different ideas of asymptotic independence between the *past* and the *future* of a TS. Most popular mixing coefficients are **$\boldsymbol{\alpha}$-mixing** and **$\boldsymbol{\beta}$-mixing** coefficients.

[1] $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

[2] Rigorously, $A \in \mathcal{B}(\mathbb{R})$ – Borel sigma-field of $\mathbb{R}$.