MATH4321 -- Game Theory
Problem Set 2

Problem 1

We consider the following two-players dynamic games as follows:

- Firstly, player 1 can choose either strategy A or strategy B. The games end immediately if player 1 choose A and the payoffs of two players is $(2,0)$.
- If player 1 chooses B in the first round, then player 2 can choose either strategy C or strategy D. The games end immediately if player 2 choose C and the payoffs of two players is $(3,1)$.
- If player 2 chooses D in the second round, then player 1 will choose either strategy E or strategy F. The games end after player 1 made the decision. The payoff is $(0,0)$ if player 1 chooses E and is $(1,2)$ if player 1 chooses F.

(a) Express the above games I extensive form using games tree.

(b) Determine all Nash equilibrium of the games (😊Hint: Pay attention to the strategic profiles of two players)

(c) Which of the Nash equilibria obtained in (a) is sequentially rational?

Problem 2 (Market deterrence and price war)

Store A is selling a smartphone to the customers in a district. Another store (Store B) in the same district is considering selling the same smartphone to the customers. Store B must decide whether to sell the smartphone.

- If store B decides to sell the smartphone, the store A then decides whether to lower the selling price of the smartphone to attract more customers or not.
  - ✓ If store A decides to do so, the payoffs to two store are given by $(V_A, V_B) = (50 - L, -10)$, where $L > 0$ can be treated as the loss due to the "price war" (*Note: L is a fixed number)
  - ✓ Otherwise, the payoffs would be $(V_A, V_B) = (30, 20)$.
- If store B decides not to sell the smartphone, the corresponding payoffs is $(V_A, V_B) = (50,0)$.

(a) Express the above games in the extensive form tree.

(b) Determine the sequentially rational Nash equilibrium. (☺Hint: Different equilibria may be resulted for different values of x)

Problem 3

Two students plan to work as a group in a final year project. There are 3 topics (denoted by T1, T2 and T3) and they can choose to work on one of them. The student's payoff (in terms of utility) of working on various topics is given by the following table:

	Topic 1 (T1)	**Topic (T2)**	**Topic 3 (T3)**
Student 1	10	6	2
Student 2	7	10	6

The games goes as follows:

- First, they choose their preferences simultaneously and independently.
- If they choose the same preference, they will work on this topic.

- Otherwise, student 1 will first eliminate one of the topics and student 2 chooses a topic from the remaining 2 topics. Then they will work the topic selected student 2.

(a) Find all possible subgame perfect equilibrium in this games. Show that at least one student chooses T1 in the first round of the games under equilibrium.

(b) Suppose that the roles of student 1 and student 2 are *reversed*, find the corresponding subgame perfect equilibrium.

Problem 4

Two firms cooperate and work on an investment project. Each firm chooses the amount of effort $e_i \in [0, \infty)$ ($i = 1,2$) invested in the project. The contributed cost is $2e_i^2$. Given the efforts e_1, e_2 put by the firms, the Firm i's revenue gained from the project is assume to be

$$R_i = 4e_i + e_ie_j.$$

Here, e_ie_j represents the additional revenue due to synergy effect. Hence, the profit (payoff) of Firm i is found to be

$$V_i(e_i; e_j) = \underbrace{R_i}_{revenue} - \underbrace{2e_i^2}_{cost} = 4e_i + e_ie_j - 2e_i^2.$$

(a) Suppose that two firms choose e_i *simultaneously*, find the Nash equilibrium for this problem.

(b) Suppose that Firm 1 chooses e_1 first. After knowing e_1 chosen by firm 1, firm 2 decides e_2. Find the corresponding sequentially rational Nash equilibrium. Compare the results with that in **(a)** and examine the issue of first-mover advantage (The first player can enjoy a higher payoff then the second player).

Problem 5 (Second price auction games)

In a second price auction games, each bidder submits a single bid for an object. The bidder who submits the highest bid is the winner. Different from first price auction games, the winner just needs to pay the second-highest bid, If there are more than 1 bidder submitting the highest bid, the object will be randomly assigned to one of these bidders with equal probability. We assume that the bid submitted must be an integer.

(a) Suppose that there are only two bidders in the games, Bidder 1 first submits the bid and bidder 2 submits the bid after knowing the bid submitted by Bidder 1. It is given that bidder 1 values the object at $v_1 = 50$ and bidder 2 values the object at $v_2 = 55$. Find the sequentially rational Nash equilibrium for this games.

(b) **(Harder)** Suppose that there are three bidders in the games, Bidder 2 first submits the bid. Bidder 1 and Bidder 3 then submits the bid simultaneously after knowing the bid submitted by Bidder 2. It is given that bidder 1 values the object at $v_1 = 50$, bidder 2 values the object at $v_2 = 55$ and bidder 3 values the object at $v_3 = 60$. Find the subgames perfect equilibrium for the games.
(☺Hint: We let b_2 be the bid submitted by bidder 2. To analyze the equilibrium strategies adopted by bidder 1 and 3, consider the case when $b_2 < 50$ and $b_2 \geq 50$.)

Problem 6 (Multi-stage games)

There are two pots. Pot 1 contains $10 and Pot 2 contains $12. Two players need to decide how to split each pot: They first spilt the Pot 1 and then split the Pot 2 (after knowing the result in the first round).

In each round, each player can choose to either take all money (T) or share the pot equally (S).

- If both players choose to take all money, two players share the money equally.
- If only one player chooses to take all money, this player will take all money in the pot and another player receives nothing.
- If both players chooses to share the money, additional money will be deposited into the pot ($6 for pot 1 and $$B$ for pot 2), then the players will share the money in the pot equally.

The player's payoff is simply the total amount of money received in these two rounds. We let D denote the discounting factor between two games so that the player's payoff in the entire games is

$$V_i = \underbrace{V_i^{(1)}}_{\substack{1st \\ pot}} + D \underbrace{V_i^{(2)}}_{\substack{2nd \\ pot}}$$

(a) Suppose that $B < 12$, show that the only subgames perfect equilibrium is that all players choose T (take all money).

(b) Suppose that $B \geq 12$, show that there is a subgame perfect equilibrium in which two players decide to share the pot in every round of the games, provided that D is sufficiently large.

Problem 7 (Contribution games)

We consider the contribution games in Problem 4(a).

(a) Find a strategic profile $\vec{e}' = (e_1', e_2')$ (not necessarily to be Nash equilibrium) which Pareto-dominates the Nash equilibrium obtained in Problem 4(a).
(☺Hint: You may consider the scenario which two firms cooperates and each firm chooses e (common for both firms) to maximize the sum of their payoff $O_1 + O_2$.)

(b) It is clear that the strategic profile $\vec{e}'$ will not be played in equilibrium if the games is being played once. We now assume that the games is played repeatedly for infinitely many times. We let $D \in [0,1]$ denote the discounted factor over two consecutive games. Construct a subgame perfect equilibrium in which all players play $\vec{e}'$ in the repeated games. Verify your answer by one-stage deviation principle.

Problem 8 (Price competition)

Two competing stores are selling a common product to customers in a district. Each store can decide the selling price s_i of the product. It is given that

- the selling price s_i can be either 20, 25 or 30;
- The customer prefers to buy the good at lower price so that the store who sells the good at lower price can get a larger market share.
- There are 100 customers in a district. We assume that the store which sells the product at lower price can attract 90% of the customers. If two stores sell the product at the same price, we assume that each store will get 50% of the customers.
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(a) Suppose that two stores choose the selling price simultaneously,

(i) find the Nash equilibrium for this games.

(ii) Suppose the games is played repeated for infinitely many times, is it possible to have a subgame perfect equilibrium in which two stores sell the product at $30? Explain your answer. (You may assume that $D = 1$.)

(b) Suppose that store 1 first choose the selling price and store 2 chooses the selling price after knowing the selling price of store 1, find all possible sequentially rational Nash equilibrium.

Problem 9 (Contribution games)

Two players are working on a project. Each player can decide whether to contribute effort to the project. The contribution cost is known to be $c > 0$. The value of the project, which depends on the number of players contribute (denoted by n), is assumed to be $V = 2n$. Hence, the player's payoff can be expressed as

$$Payoff = \begin{cases} \underbrace{V}_{2\ or\ 4} - c & if\ \ player\ i\ contributes \\ \underbrace{V}_{0\ or\ 2} & if\ otherwise \end{cases}.$$

(a) Suppose that two players takes turn to choose his decision (Player 1 first and Player 2 next) and the last player knows the strategy chosen by first player, Find the sequentially rational Nash equilibrium for each value of c.

(b) Suppose that two players make their decision simultaneously in the above games.

(i) Find all possible value of c such that no players is willing to contribute under Nash equilibrium.

(ii) Take $c = 3$ and suppose that the games is played repeated for infinitely many times, find a subgame perfect equilibrium in which all players contribute in every round of the games, provided that the discounting factor D is sufficiently large.

Problem 10

The following matrix shows the players' payoffs in a two-person static games G:

		Player 2	
		A	B
Player 1	A	(3,3)	(0,20)
	B	(20,0)	(1,1)

(a) Find all possible pure strategy Nash equilibrium for this games.

(b) Suppose that the games is played repeatedly in an infinite repeated games (with discounting factor $D \in (0,1)$,

(i) show that when D is sufficiently large, there is a subgame perfect equilibrium such that two players play (A, A) in every round of the games.

(ii) Show that when D is sufficiently large, there is a subgame perfect equilibrium such that two players play (A, B) in odd round of the games and play (B, A) in even round of the games.

(*Remark: In fact, this strategic profile can generate a higher average payoff (around 10 per round if D is close to 1) than that in (a)(i) (3 per round).

Problem 11 (Bargaining games)
We consider the following bargaining games. There are 3 periods in the games. Three players decides on how to share a pie. They take turn (Player 1 plays first, Player 2 plays second and Player 3 plays last) make an proposal $\vec{x} = (x_1, x_2, x_3)$ (where $x_1 + x_2 + x_3 = 1$) to other two players. The proposal is approved when at least two players accept the proposal.

- At period 1, player 1 makes a proposal to other two players. If the proposal is approved, the games is over. Otherwise, the games move to period 2.
- At period 2, player 2 makes a proposal to other two players. If the proposal is approved, the games is over. Otherwise, the games move to period 3.
- At period 3, player 3 makes a proposal to other two players. If the proposal is approved, the games is over. Otherwise, the games ends and all players receive nothing.

Find the outcome of the bargaining games by finding the sequentially rational Nash equilibrium.

Problem 12 (Bargaining games)
Two companies intend to work together in an investment project. The project revenue is known to be 150. It is given that the costs for two companies are $c_1 = 10$ and $c_2 = 20$. Before executing the project, two companies need to comprise how to split the revenue generated from the project. The bargaining games goes as follows:

- At period 1, company 1 proposes an offer $\vec{x} = (x, 1 - x)$ to company 2. The offer specifies that company 1 get a fraction x of the project revenue and the company 2 gets a fraction of $1 - x$ of the project revenue. Knowing $\vec{x}$, company 2 can choose to whether accept the offer or reject the offer. If company 2 accepts the offer, the games is over. Otherwise, the bargaining continues at period 2.
- At period 2, company 2 proposes an offer $\vec{x} = (x, 1 - x)$ to company 1. If company 1 accepts the offer, the game ends. Otherwise, the bargaining continues at period 3.
- In the remaining period, each company takes turn to propose an offer (company 1 proposes at odd period and company 2 offers at period) to another firm until one company accepts the offer made by another company.

We assume that <u>the costs will be *doubled* over period</u> so that both companies need to pay a higher cost to execute the project if the bargaining process takes too long. We assume that there is no further discounting over periods.

Find the bargaining outcome by finding the sequentially rational Nash equilibrium.
(☺Hint: At period 4, the costs are grown to 80 (for company 1) and 160 (for company 2) respectively. Since the total costs override the project revenue, it is impossible that two firms can comprise an agreement in this cases and two firms receive nothing as a result. Thus the bargaining games must end at or before period 3.)

Problem 14 (Harder)

We let G be a two-person static games. It is given that the games G has two Pure strategy Nash equilibrium, denoted by (s_1', s_2') and (s_1'', s_2''), which satisfies

$$V_i(s_i'; s_j') > V_i(s_i'', s_j'')$$

for any player i.

Suppose that there is another strategic profile (s_1^0, s_2^0) (may not be the Nash equilibrium) such that

- $V_i(s_i^0; s_j^0) > V_i(s_i'; s_j')$ for any player i and
- $\left[\max_s V_i(s; s_j^0) - V_i(s_i^0, s_j^0)\right] < V_i(s_i'; s_j') - V_i(s_i''; s_j'') \ldots\ldots (*)$

(a) If the games G is played repeated for finitely many times, show that there is a subgame perfect equilibrium in which two players may play (s_1^0, s_2^0) in some round of the games.
(☺Hint: To prove this, you first construct a candidate of subgame perfect equilibrium. Verify your answer by one-stage deviation principle.

(b) Show, by constructing a counter-example, that the statement in (a) is no longer valid if the inequality $(*)$ does not satisfy.

Problem 15 (Harder)

We consider a two-person multi-stage games in which the games $G_1, G_2, \ldots, G_N$ are played sequentially. Let D denote the usual discounting factor. It is given that each static games G_i has at least one pure strategy Nash equilibrium (denoted by $s^{(i)*} = \left(s_1^{(i)*}, s_2^{(i)*}\right)$).

- In games G_1, it is given that there is another strategic profile $\left(s_1^{(1)}, s_2^{(1)}\right)$ such that $V_1\left(s_1^{(1)}; s_2^{(1)}\right) > V_1\left(s_1^{(1)*}; s_2^{(1)*}\right)$ and $V_2\left(s_2^{(1)}; s_1^{(1)}\right) > V_2\left(s_2^{(1)*}; s_1^{(1)*}\right)$.
- It is given that there is a games G_k (where $k = 2,3, \ldots, N$) such that the games has multiple pure strategy Nash equilibrium.
- We let $\left(s_1^{(k)*}, s_2^{(k)*}\right)$ and $\left(s_1^{(k)**}, s_2^{(k)**}\right)$ be two Nash equilibrium in the games G_k. It is given that these equilibria satisfy

$$V_i^{(k)}\left(s_1^{(k)*}, s_2^{(k)*}\right) > V_i\left(s_1^{(k)**}, s_2^{(k)**}\right)$$

and

$$\left[\max_s V_i^{(1)}\left(s; s_j^{(1)}\right) - V_i^{(1)}\left(s_i^{(1)}, s_j^{(1)}\right)\right] < V_i^{(k)}\left(s_1^{(k)*}, s_2^{(k)*}\right) - V_i\left(s_1^{(k)**}, s_2^{(k)**}\right)$$

for any player i.

For simplicity, we take the discounting factor to be 1. That is, $D = 1$. Show that there is a subgame perfect equilibrium in which players play $\left(s_1^{(1)}, s_2^{(1)}\right)$ in games G_1.
(☺Hint: To construct such equilibrium, you need to develop some "penalty-reward" mechanism at games G_k. For the games other than G_1, G_k, you may assume that all players play the equilibrium strategy in the games. Verify your answer using one-stage deviation principle.)

Problem 16 (Harder)

We consider a two-person multi-stage games in which the games $G_1, G_2, \ldots, G_N$ are played sequentially. It is given that $s_1^{(1)} \in S_1^{(1)}$ is a player 1's dominated strategy in games G_1. It is well-known that player 1 should not choose $s_1^{(1)}$ under equilibrium if G_1 is played alone.

Is it always true that player 1 does not choose $s_1^{(1)}$ under any subgames perfect equilibrium in this multi-stage games? If your answer is yes, provide a proof. If your answer is no, explain your answer by using a counter-example.

(☺Hint: Pick $N = 2$ and consider the following two games:

Games G_1		**Player 2**	
		A	**B**
Player 1	**A**	$(6,6)$	$(8,2)$
	B	$(3,15)$	$(4,10)$

Games G_2		**Player 2**	
		C	**D**
Player 1	**C**	$(10,7)$	$(0,0)$
	D	$(0,0)$	$(1,8)$