MATH4425 (T1A) – Tutorial 11

Kazovskaia Anastasiia

May, 2

Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

GARCH model

The model **GARCH(r, s) (Generalized Autoregressive Conditional Heteroscedasticity)** is defined as

$$a_t = \eta_t \sqrt{h_t},$$

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j},$$

where $\eta_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

Testing whether conditional variance is constant. Testing for ARCH effect

Let's consider the model

$$Z_t = \mu + a_t,$$

$$a_t = \eta_t \sqrt{h_t}$$

The null hypothesis $H_0 : h_t = \sigma_a^2$. Notice that

$$\gamma_k = \mathbb{E}(Z_t - \mu)(Z_{t+k} - \mu) = \mathbb{E}a_t a_{t+k} =$$

$$\mathbb{E}\eta_t \sqrt{h_t}\eta_{t+k}\sqrt{h_{t+k}} = \mathbb{E}\eta_t \mathbb{E}\sqrt{h_t}\eta_{t+k}\sqrt{h_{t+k}} = 0 \quad \forall k > 0 \Rightarrow$$

$$\rho_k = 0 \quad \forall k > 0$$

So, we need another statistic. Let's consider $\xi_t := (Z_t - \mu)^2 = a_t^2 = \eta_t^2 h_t$. First notice that under H_0

$$\mathbb{E}\xi_t = \mathbb{E}\eta_t^2 h_t = \mathbb{E}\eta_t^2 \mathbb{E}h_t = \mathbb{E}h_t = h_t = \sigma_a^2$$

$$\gamma_k = \mathbb{E}(\xi_t - \sigma_a^2)(\xi_{t+k} - \sigma_a^2) = \mathbb{E}(\eta_t^2 h_t - \sigma_a^2)(\eta_{t+k}^2 h_{t+k} - \sigma_a^2) =$$

$$\mathbb{E}(\eta_t^2 - 1)\sigma_a^2(\eta_{t+k}^2 - 1)\sigma_a^2 = \mathbb{E}(\eta_t^2 - 1)\mathbb{E}(\eta_{t+k}^2 - 1)\sigma_a^4 = 0$$

Under H_a : *There is ARCH effect*

$$\mathbb{E}\xi_t = \mathbb{E}h_t$$

$$\gamma_k = \mathbb{E}(\xi_t - \mathbb{E}h_t)(\xi_{t+k} - \mathbb{E}h_{t+k}) = \mathbb{E}(\eta_t^2 h_t - \mathbb{E}h_t)(\eta_{t+k}^2 h_{t+k} - \mathbb{E}h_{t+k}) =$$

$$\mathbb{E}\eta_t^2 h_t \eta_{t+k}^2 h_{t+k} - \mathbb{E}\eta_t^2 h_t \mathbb{E}h_{t+k} - \mathbb{E}h_t \mathbb{E}\eta_{t+k}^2 h_{t+k} + \mathbb{E}h_t \mathbb{E}h_{t+k} =$$
$$\mathbb{E}h_t h_{t+k} - \mathbb{E}h_t \mathbb{E}h_{t+k} - \mathbb{E}h_t \mathbb{E}h_{t+k} + \mathbb{E}h_t \mathbb{E}h_{t+k} = \operatorname{cov}(h_t, h_{t+k}) \not\equiv 0$$

To test ACFs of ξ_t we can use **Ljung-Box test**:

$$H_0 : \rho_1 = \cdots = \rho_m = 0$$

with the test statistic

$$Q = n(n+2) \sum_{k=1}^{m} \frac{1}{n-k} \hat{\rho}_k^2 \sim \chi^2(m)$$

Another test to check the parameters of ARCH model is **Score test** with LM statistic:

$$H_0 : \alpha_0 = \cdots = \alpha_k = 0$$

for some big k. Under H_0, LM $\sim \chi^2(k)$.

Maximum Likelihood Estimation

Let's consider AR(1)-GARCH(1, 1) model

$$Z_t = \phi_{1_0} Z_{t-1} + a_t,$$
$$a_t = \eta_t \sqrt{h_t},$$
$$h_t = \alpha_{0_0} + \alpha_{1_0} a_{t-1}^2 + \beta_{1_0} h_{t-1}$$

Let's denote $\lambda_0 := (\phi_{1_0}, \alpha_{0_0}, \alpha_{1_0}, \beta_{1_0})^T$ — **true parameters**, $\tilde{Z}_t := (Z_t, Z_{t-1}, \dots)$.

Then, $a_t \mid \tilde{Z}_{t-1} = \eta_t \sqrt{h_t} \mid \tilde{Z}_{t-1} \sim \mathcal{N}(0, h_t)$. So, $Z_t \mid \tilde{Z}_{t-1} = \phi_{1_0} Z_{t-1} + a_t \mid \tilde{Z}_{t-1} \sim \mathcal{N}(\phi_{1_0} Z_{t-1}, h_t)$.

Therefore, the conditional density function (conditioned on $\tilde{Z}_{t-1}$) of Z_t is

$$f(Z_t \mid \tilde{Z}_{t-1}) = \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{(Z_t - \phi_{1_0} Z_{t-1})^2}{2h_t}\right),$$

where

$$h_t = \alpha_{0_0} + \alpha_{1_0} a_{t-1}^2 + \beta_{1_0} h_{t-1} = \alpha_{0_0} + \alpha_{1_0}(Z_{t-1} - \phi_{1_0} Z_{t-2})^2 + \beta_{1_0} h_{t-1},$$

which can be calculated iteratively.

Thus, given initial values $\tilde{Z}_0^*$, the conditional joint density function of $(Z_n, \dots, Z_1)$ is

$$f(Z_n, \dots, Z_t \mid \tilde{Z}_0^*) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{(Z_t - \phi_{1_0} Z_{t-1})^2}{2h_t}\right)$$

Once we replace λ_0 by unknown parameters $\lambda = (\phi_1, \alpha_0, \alpha_1, \beta_1)^T$, we get

$$a_t(\phi_1) = Z_t - \phi_1 Z_{t-1}$$
$$h_t(\lambda) = \alpha_0 + \alpha_1 (Z_t - \phi_1 Z_{t-2})^2 + \beta_1 h_{t-1}(\lambda)$$

So, **the conditional likelihood function** of $(Z_n, \dots, Z_1)$ is

$$f(Z_n, \dots, Z_t \mid \tilde{Z}_0^*, \lambda) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi h_t(\lambda)}} \exp\left(-\frac{a_t(\phi_1)^2}{2h_t(\lambda)}\right)$$

The conditional log-likelihood function of $(Z_n, \dots, Z_1)$ is

$$L(\lambda) := -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left(\ln h_t(\lambda) + \frac{a_t(\phi_1)^2}{h_t(\lambda)}\right)$$

The MLE of λ_0 denoted by $\hat{\boldsymbol{\lambda}}$ is the *maximizer* of $L(\lambda)$. Moreover, if $\mathbb{E}a_t^4 < \infty$, $\hat{\lambda}$ is **strongly consistent** and **asymptotically normal**:

$$\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda_0$$

$$\sqrt{n}(\hat{\lambda} - \lambda_0) \xrightarrow{\text{d}} \mathcal{N}(0, \hat{\Omega}),$$

where

$$\hat{\Omega} := \mathbb{E}\left[\frac{\partial^2 L}{\partial\lambda\partial\lambda^T}(\hat{\lambda})\right]^{-1}\mathbb{E}\left[\frac{\partial L}{\partial\lambda}(\hat{\lambda})\frac{\partial L}{\partial\lambda^T}(\hat{\lambda})\right]\mathbb{E}\left[\frac{\partial^2 L}{\partial\lambda\partial\lambda^T}(\hat{\lambda})\right]^{-1}$$

Diagnostic Checking and Model Selection

Testing Model Assumptions

There are some **assumptions** to be checked:

1) standardized residuals $\hat{\eta}_t := \frac{\hat{a}_t(\hat{\phi})}{\sqrt{h_t(\hat{\lambda})}}$ are normally distributed (*histogram* + χ^2*-goodness-of-fit test* or *Normality test*)
2) $\hat{\eta}_t^2$ are uncorrelated (*Ljung-Box test*)

Model Selection

We have already discussed some information criteria for ARMA model. Similar techniques can be applied here. The main tool for GARCH model comparison is **AIC**.

Forecasting

As before, the forecast $\hat{Z}_t(l)$ of Z_{t+l} is calculated by

$$\hat{Z}_t(l) = \mathbb{E}(Z_{t+l} \mid Z_t, Z_{t-1}, \dots)$$

So, we get the same formulae as for ARIMA model:

$$\hat{Z}_n(l) = \Psi_1\hat{Z}_n(l-1) + \Psi_2\hat{Z}_n(l-2) + \dots + \Psi_{p+d}\hat{Z}_n(l-p-d) +$$
$$\hat{a}_n(l) - \theta_1\hat{a}_n(l-1) - \theta_2\hat{a}_n(l-2) - \dots - \theta_q\hat{a}_n(l-q),$$

where

$$\hat{Z}_n(j) = \begin{cases} \mathbb{E}(Z_{n+j} \mid Z_n, Z_{n-1}, \dots) & \text{if } j = 1, 2, \dots, l \\ Z_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

$$\hat{a}_n(j) = \begin{cases} 0 & \text{if } j = 1, 2, \dots, l \\ a_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

However, the FI is calculated another way due to non-constant conditional variance of a_t. **One-step FI** for ARIMA-GARCH is simple:

$$\left[\hat{Z}_t(1) - \mathcal{N}_{\frac{\alpha}{2}}\sqrt{\hat{h}_t(\lambda)}, \hat{Z}_t(1) + \mathcal{N}_{\frac{\alpha}{2}}\sqrt{\hat{h}_t(\lambda)}\right],$$

where $\mathcal{N}_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$-quantile of the standard normal distribution, i.e. $\mathbb{P}(\mathcal{N}(0,1) > \mathcal{N}_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$.

Note: However, l-step FI is way more complicated (a_t are not i.i.d. anymore). One way to obtain FI is to model the distribution of $e_t(l)$ using sampling.

Multivariate Time Series Models

We use multivariate time series to model multidimensional data that evolves over time:

$$Z_t = \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \\ \vdots \\ Z_{k,t} \end{bmatrix}$$

Z_t is called **a $\boldsymbol{k}$-dimensional vector time series**.

Covariance and Correlation Matrix Functions

Expected value

$$\mathbb{E}Z_t = \begin{bmatrix} \mathbb{E}Z_{1,t} \\ \mathbb{E}Z_{2,t} \\ \vdots \\ \mathbb{E}Z_{k,t} \end{bmatrix} = \begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \vdots \\ \mu_{k,t} \end{bmatrix} =: \mu_t$$

Cross-covariance matrix

Let's assume $\mu_t = \mu \quad \forall t$. Then **cross-covariance matrix** is defined as

$$\Gamma(l) := \mathrm{Cov}(Z_t, Z_{t+l}) = \mathbb{E}(Z_t - \mu)(Z_{t-l} - \mu)^T = \begin{bmatrix} \gamma_{1,1}(l) & \gamma_{1,2}(l) & \cdots & \gamma_{1,k}(l) \\ \gamma_{2,1}(l) & \gamma_{2,2}(l) & \cdots & \gamma_{2,k}(l) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k,1}(l) & \gamma_{k,2}(l) & \cdots & \gamma_{k,k}(l) \end{bmatrix},$$

where $\gamma_{i,j}(l) := \mathbb{E}(Z_{i,t} - \mu_i)(Z_{j,t-l} - \mu_j)$.

Cross-correlation matrix

Let $D = diag\left(\sqrt{\gamma_{1,1}(0)}, \sqrt{\gamma_{2,2}(0)}, \ldots, \sqrt{\gamma_{k,k}(0)}\right)$.

The **lag-$\boldsymbol{l}$ cross-correlation matrix (CCM)** is defined as

$$\rho(l) := D^{-1}\Gamma(l)D^{-1} = \begin{bmatrix} \rho_{1,1}(l) & \rho_{1,2}(l) & \cdots & \rho_1, k(l) \\ \rho_{2,1}(l) & \rho_{2,2}(l) & \cdots & \rho_{2,k}(l) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}(l) & \rho_{k,2}(l) & \cdots & \rho_{k,k}(l) \end{bmatrix},$$

where

$$\rho_{i,j}(l) := \frac{\gamma_{i,j}(l)}{\sqrt{\gamma_{i,i}(0)\gamma_{j,j}(0)}}$$

Note: When $l = 0$, lag-zero CCM is also called **concurrent CCM**. Moreover,

$$-1 \le \rho_{i,j}(0) = \frac{\mathbb{E}(Z_{i,t} - \mu_i)(Z_{j,t} - \mu_j)}{\sqrt{\mathbb{E}(Z_{i,t} - \mu_i)^2 \mathbb{E}(Z_{j,t} - \mu_j)^2}} \le 1$$

$$\rho_{i,i}(0) = 1$$

Thus, $\rho(0)$ is a symmetric matrix with unit diagonal elements.

Properties

The properties are somewhat similar to the ones we have seen in the 1-dimensional case:

$$\Gamma(l) = \Gamma^T(-l) \geq 0 \text{ (positive semidefinite)}$$

$$\rho(l) = \rho^T(-l) \geq 0 \text{ (positive semidefinite)}$$

Note: When $\mathbb{E}Z_t = 0$ and $\rho(l) = 0 \ \forall l > 0$, Z_t is called **the *l*-dimensional white noise** denoted by a_t:

$$\Gamma(l) = \mathbb{E}a_t a_{t-l}^T = \begin{cases} \Sigma > 0 \text{ (positive definite)}, & \text{if } l = 0 \\ 0, & \text{otherwise} \end{cases}$$

Sample Cross-Covariance and Cross-Correlation Matrices

Given the data $\{Z_t\}_{t=1}^T$, *the cross-covariance matrix* $\Gamma(l)$ can be estimated by

$$\hat{\Gamma}(l) := \frac{1}{T}\sum_{t=1}^{T-l}(Z_t - \bar{Z})(Z_{t-l} - \bar{Z})^T, \quad l \geq 0,$$

where $\bar{Z}$ is the vector of sample means given by

$$\bar{Z} := \frac{1}{T}\sum_{t=1}^{T} Z_t$$

The cross-correlation matrix $\rho(l)$ can be estimated by

$$\hat{\rho}(l) := \hat{D}^{-1}\hat{\Gamma}(l)\hat{D}^{-1}, \quad l \geq 0,$$

where

$$\hat{D} := diag\left(\sqrt{\hat{\gamma}_{1,1}(0)}, \sqrt{\hat{\gamma}_{2,2}(0)}, \ldots, \sqrt{\hat{\gamma}_{k,k}(0)}\right)$$

is the $k \times k$ diagonal matrix of *the sample standard deviations* of the component series.

Multivariate Portmanteau Tests

The univariate Ljung-Box statistic $Q(m)$ has been generalized to the multivariate case. The null hypothesis now is

$$H_0 : \rho(1) = \rho(2) = \cdots = \rho(m) = 0$$

Thus, the statistic is used to test if there are *no auto- and cross-correlations* in the vector series:

$$Q_k(m) = T^2\sum_{l=1}^{m}\frac{1}{T-l}tr\left(\hat{\Gamma}^T(l)\hat{\Gamma}^{-1}(0)\hat{\Gamma}(l)\hat{\Gamma}^{-1}(0)\right) \sim \chi^2(k^2 m),$$

where T is a sample size, k is the dimension of Z_t, $tr(A)$ is the trace of matrix A.