Chapter 9 GARCH Models

9.1 Introduction

We have know the ARMA(p, q) model:

$$Z_t = \sum_{i=1}^{p} \phi_i Z_{t-i} - \sum_{j=1}^{q} \theta_j a_{t-j} + a_t,$$

where a_t is white noise with a constant variance σ_a^2.

Usually, the conditional variance of a_t is also σ_a^2.

Question: Why is the conditional variance σ_a^2 constant?

Conjecture:

If the conditional variance σ_a^2 includes some past information, then it should be able to improve the statistical inference and forecasting.

How to include the past information?

$$\begin{aligned} \sigma_a^2 &= h_t = f(Z_{t-1}, Z_{t-2}, \cdots) \text{ or} \\ \sigma_a^2 &= h_t = f(a_{t-1}, a_{t-2}, \cdots). \end{aligned}$$

ARCH model (Engle 1982):

$$a_t = \eta_t \sqrt{h_t}\,,$$
$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2.$$

where η_t ~i.i.d. $\mathcal{N}(0,1)$.

a_t is called autoregressive conditional heteroscedasticity [ARCH(r)] model.

GARCH model (Bollerslev 1986):

$$a_t = \eta_t \sqrt{h_t}\,,$$
$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}.$$

a_t is called the general autoregressive conditional heteroscedasticity [GARCH(r, s)] model.

Nelson (1989): Exp-GARCH model.

Duan (1997): Argument-GARCH model.

In finance, h_t is called volatility at time t.

ARMA-GARCH model:

$$\begin{aligned} Z_t &= \sum_{i=1}^{p} \phi_i Z_{t-i} + a_t - \sum_{j=1}^{q} \theta_j a_{t-j}, \\ a_t &= \eta_t \sqrt{h_t}, \\ h_t &= \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}. \end{aligned}$$

Z_t is called ARMA$(p,q)-$GARCH(r,s) model.

ARIMA-GARCH model:

$$\begin{aligned}
\phi_p(B)(1-B)^d Z_t &= \theta_q(B)a_t, \\
a_t &= \eta_t\sqrt{h_t}, \\
h_t &= \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}.
\end{aligned}$$

Z_t is called ARIMA$(p,d,q)-$GARCH(r,s) model.

In finance, many people use the following model to do option:

$$(1-B)Z_t = a_t, \qquad a_t \sim \text{GARCH}(1,1),$$

where $Z_t = \log P_t$.

9.2. Basic Properties of GARCH model

$$\begin{aligned} a_t &= \eta_t \sqrt{h_t}, \\ h_t &= \alpha_0 + \sum_{i=1}^{r} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}. \end{aligned}$$

where $\alpha_0 > 0, \alpha_i \geq 0, \beta_i \geq 0$.

Focus on a GARCH(1,1) model:

$$h_t = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 h_{t-1}, \qquad (1)$$

1. $E(a_t) = 0$,

2. Model (1) has the following expansion (or solution):

$$h_t = \alpha_0 \left[1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \left(\alpha_1 \eta_{t-i}^2 + \beta_1 \right) \right]$$

if and only if

$$E \ln(\alpha_1 \eta_t^2 + \beta_1) < 0, \qquad (2)$$

and the solution is unique and stationary and ergodic.

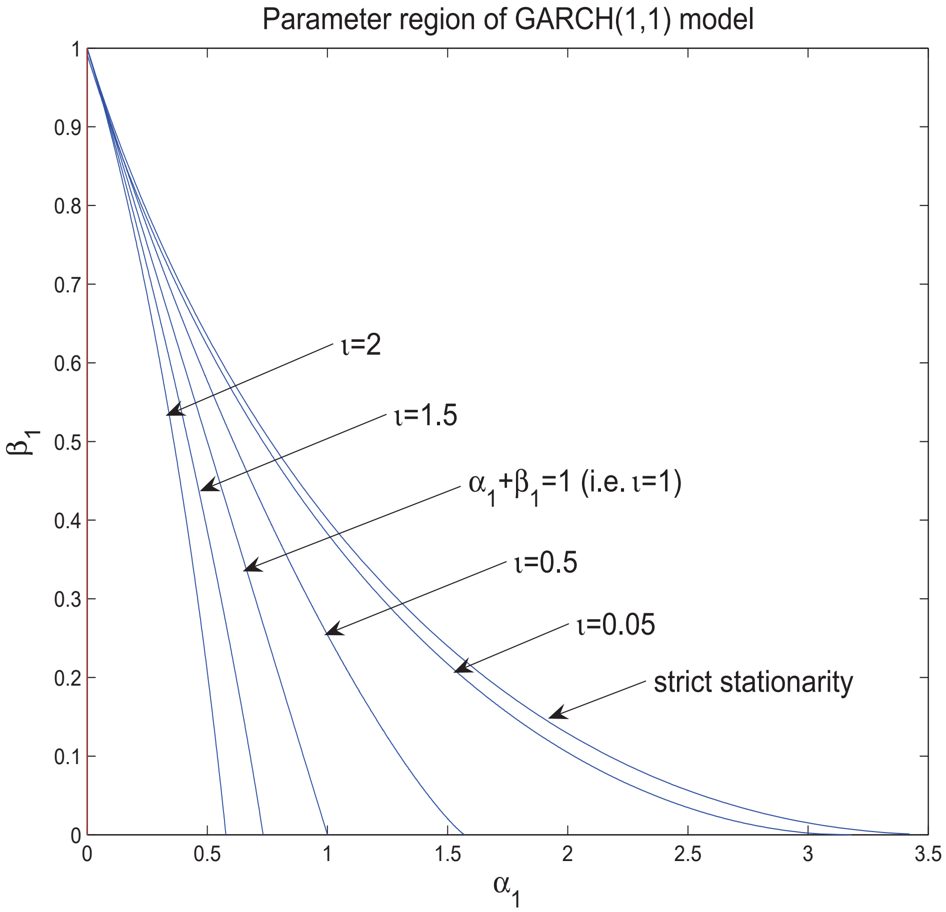
3. $Var(a_t) = \alpha_0/(1 - \alpha_1 - \beta_1)$ if $0 < \alpha_1 + \beta_1 < 1$.

Bollerslev 1986:

The necessary and sufficient condition for the stationarity:

$$\sum_{i=1}^{r} \alpha_i + \sum_{j=1}^{s} \beta_j < 1.$$

The conditions for the strict stationarity, the existence of the moments are given in Ling and Li(1997), Ling(1999), and Ling and McAleer (Econometric Theory 2001).



Regions

$$D_\iota = \{(\alpha_1, \beta_1) : \; E|a_t|^{2\iota} < \infty\}$$
$$D = \{(\alpha_1, \beta_1) : \; E \ln |\alpha_1 \eta_t^2 + \beta_1| < 0\}$$

9.3. Testing whether or not h_t = a constant

Testing for the ARCH effect

When $\mu_t = \mu$, let $\xi_t = (Z_t - \mu)^2$.

If there is not ARCH effect, then the ACF of ξ_t are all zero.

McLeod and Li (1983): use the Ljung-Box to test the null H_0: the ACF ρ_k of ξ_t are all zero, i.e,

$$H_0 : \rho_1 = \cdots = \rho_m = 0.$$

Let $\widehat{\rho}_k$ be the sample ACF of $\{\xi_t\}$. We use the Ljung-Box:

$$Q(m) = n(n-1) \sum_{k=1}^{m} \frac{\widehat{\rho}_k^2}{n-k} \sim \chi^2(m).$$

LM statistic for testing:

$$\begin{aligned} &H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0, \\ &H_1 : H_0 \quad \text{does not hold.} \end{aligned}$$

where k is large. Under H_0,

$$LM \sim \chi^2(k).$$

Maximum Likelihood Estimation

We consider the case with $p = 1$ and $q = 0$ and $s = m = 1$. Assume that random sample $\{Z_1, \cdots, Z_n\}$ is from the AR(1)-GARCH(1,1) model:

$$
\begin{aligned}
Z_t &= \phi_{10} Z_{t-1} + a_t, \\
a_t &= \eta_t \sqrt{h_t}, \\
h_t &= \alpha_{00} + \alpha_{10} a_{t-1}^2 + \beta_{10} h_{t-1},
\end{aligned}
$$

where $\lambda_0 = (\phi_{10}, \alpha_{00}, \alpha_{10}, \beta_{10})'$ is called the true parameters.

Denote $\tilde{Z}_t = (Z_t, Z_{t-1}, \cdots)$. Given $\tilde{Z}_{t-1}$, the conditional density function of Z_t is

$$f(Z_t | \tilde{Z}_{t-1}) = \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{(Z_t - \phi_{10} Z_{t-1})^2}{h_t}\right).$$

Given $\tilde{Z}_0$, the conditional joint density function of $(Z_n, Z_{n-1}, \cdots, Z_1)$:

$$
\begin{aligned}
& f(Z_t, \cdots, Z_1 | \tilde{Z}_0) \\
& = \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{(Z_t - \phi_{10} Z_{t-1})^2}{2h_t}\right) \right\},
\end{aligned}
$$

where

$$h_t = \alpha_{00} + \alpha_{10}(Z_{t-1} - \phi_{10} Z_{t-2})^2 + \beta_{10} h_{t-1}.$$

Replaced λ_0 by its unknown parameter $\lambda = (\phi, \alpha_0,\ \alpha_1, \beta_1)'$, we get

$$\begin{aligned} a_t(\phi) &= Z_t - \phi Z_{t-1}, \\ h_t(\lambda) &= \alpha_0 + \alpha_1 (Z_{t-1} - \phi Z_{t-2})^2 + \beta_1 h_{t-1}(\lambda). \end{aligned}$$

The conditional likelihood function of $(Z_n, Z_{n-1}, \cdots, Z_1)$:

$$f(Z_t, \cdots, Z_1 | \tilde{Z}_0) = \prod_{t=1}^{n} \left\{ \frac{1}{\sqrt{2\pi h_t(\lambda)}} \exp\left(-\frac{a_t^2(\phi)}{2h_t(\lambda)} \right) \right\}.$$

Log -conditional likelihood function of $(Z_n, Z_{n-1}, \cdots, Z_1)$:

$$\begin{aligned} L(\lambda) &\equiv \ln f(Z_t, \cdots, Z_1 | \tilde{Z}_0) \\ &= -\frac{n}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{n} \left\{ \ln h_t(\lambda) + \frac{a_t^2(\phi)}{h_t(\lambda)} \right\}. \end{aligned}$$

The MLE of λ is the maximizer of $L(\lambda)$, denote by $\widehat{\lambda}$. If $Ea_t^4 < \infty$, then

$$\widehat{\lambda} \longrightarrow \lambda_0 \text{ as } n \to \infty,$$
$$\sqrt{n}(\widehat{\lambda} - \lambda_0) \sim N(0, \widehat{\Omega}),$$

where

$$\widehat{\Omega} = E\left[\frac{\partial^2 L(\widehat{\lambda})}{\partial \lambda \partial \lambda'} \right]^{-1} E\left[\frac{\partial L(\widehat{\lambda})}{\partial \lambda} \frac{\partial L(\widehat{\lambda})}{\partial \lambda'} \right] E\left[\frac{\partial^2 L(\widehat{\lambda})}{\partial \lambda \partial \lambda'} \right]^{-1}.$$

9.4. Diagnostic Checking and Model selection

The formal method is not provided in SAS. AIC is a main tool for model selection.

We can use Ljung-Box test for squared standardized residuals:

$$\widehat{\eta}_t = \frac{a_t(\widehat{\phi})}{\sqrt{h_t(\widehat{\lambda})}} \; and \; \widehat{\eta}_t^2 = \frac{a_t^2(\widehat{\phi})}{h_t(\widehat{\lambda})}.$$

9.5. Forecasting

The forecast value $\widehat{Z}_t(l)$ of Z_{t+l} is calculated by

$$\widehat{Z}_t(l) = E(Z_{t+l}|Z_t, Z_{t-1}, \cdots).$$

The formulas is the same as the one in the ARMA model with a constant variance.

The one-step forecast interval for the ARIMA-GARCH model:

$$\left[\widehat{Z}_t(1) - \mathcal{N}_{\frac{\alpha}{2}}\sqrt{\widehat{h}_t(\widehat{\delta})},\ \widehat{Z}_t(1) + \mathcal{N}_{\frac{\alpha}{2}}\sqrt{\widehat{h}_t(\widehat{\delta})}\right]$$

where $\mathcal{N}_{\frac{\alpha}{2}}$ is the $\alpha/2-$ quantile of $\mathcal{N}(0,1)$. SAS only provides the forecasting value and forecast interval for AR-GARCH model. In R, these is code for this purpose.