3.4. **Autoregressive Moving-average ARMA(p, q) Model**

3.4.1. The General ARMA(p, q) Model

A. Model:

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be an ARMA(p, q) model, if

$$\begin{aligned}\dot{Z}_t &= \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} \\ &\quad -\theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} + a_t,\end{aligned}$$

where p and q are positive integers.

(p, q) is called the order or lag of the model.

The ARMA(p, q) model can be written as

$$\phi_p(B)\dot{Z}_t = \theta_q(B)a_t$$

where

$$\begin{aligned}\phi_p(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \\ \theta_q(B) &= 1 - \theta_1 B - \cdots - \theta_q B^q.\end{aligned}$$

B. Condition for the Stationarity:

all the roots of $\phi_p(z) = 0$ lie outside the unit circle.

$$\frac{\theta_q(B)}{\phi_p(B)} = 1 + \sum_{i=1}^{\infty} \psi_i B^i$$

$\Longrightarrow$

$$\begin{aligned}\dot{Z}_t &= \left(1 + \sum_{i=1}^{\infty} \psi_i B^i\right) a_t \\ &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots .\end{aligned}$$

where $\psi_i = O(h^i)$ with $|h| < 1$.

C. Condition for Invertibility:

all the roots of $\theta_q(z) = 0$ lie outside the unit circle.

$$\frac{\phi_p(B)}{\theta_q(B)} = 1 - \sum_{i=1}^{\infty} \pi_i B^i$$

$\Longrightarrow$

$$\begin{aligned}a_t &= \left(1 - \sum_{i=1}^{\infty} \pi_i B^i\right) \dot{Z}_t \\ &= \dot{Z}_t - \pi_1 \dot{Z}_{t-1} - \pi_2 \dot{Z}_{t-2} - \cdots .\end{aligned}$$

where $\pi_i = O(h^i)$ with $|h| < 1$.

D. ACF of ARMA(p, q) model:

$$\begin{aligned}\gamma_k &= \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_{k-p}, \quad k \geq q+1,\\ \rho_k &= \phi_1\rho_{k-1} + \cdots + \phi_p\rho_{k-p}, \quad k \geq q+1.\end{aligned}$$

Important feature:

1. $\rho_{q+1}, \rho_{q+2}, \cdots$ do not direct depend on the coefficients in the MA part.

2. $\rho_1, \rho_2, \cdots, \rho_q$ depend on the coefficients in both AR and MA parts.

3. $\rho_k = O(h^k)$, where $|h| < 1$, i.e., ρ_k goes down to zero, exponentially.

E. PACF of ARMA(p, q) model:

ϕ_{kk} can be obtained from $\rho_1, \rho_2, \cdots, \rho_q$.

Important feature:

$\phi_{kk} = O(h^k)$, where $|h| < 1$, i.e., ϕ_{kk} goes down to zero, exponentially.

3.4.2. The ARMA(1, 1) Model

A. Model:

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t - \theta_1 a_{t-1}$$

The ARMA(1, 1) model can be written as

$$(1 - \phi_1 B)\, \dot{Z}_t = (1 - \theta_1 B)\, a_t. \qquad \frac{\phi_1 B - \theta_1 B - 1 + 1}{1 - \phi_1 B} =$$

B. Condition for the Stationarity: $|\phi_1| < 1$.

$$\frac{1 - \theta_1 B}{1 - \phi_1 B} = 1 + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} B^i \qquad \frac{\phi_1 - \theta_1}{\phi_1} \sum_{i=1}^{\infty} (\phi_1 B)^i$$

$\Longrightarrow$

$$\begin{aligned} \dot{Z}_t &= \left[1 + (\phi_1 - \theta_1) \sum_{i=1}^{\infty} \phi_1^{i-1} B^i\right] a_t \qquad = \frac{\cancel{\phi_1 B}}{1 - \phi_1 B} \frac{\phi_1 - \theta_1}{\cancel{\phi_1}} \\ &= a_t + (\phi_1 - \theta_1)\, a_{t-1} + (\phi_1 - \theta_1)\, \phi_1 a_{t-2} + \cdots . \end{aligned}$$

C. Condition for Invertibility: $|\theta_1| < 1$.

$$\frac{1 - \phi_1 B}{1 - \theta_1 B} = 1 + (\theta_1 - \phi_1) \sum_{i=1}^{\infty} \theta_1^{i-1} B^i$$

$\Longrightarrow$

$$\begin{aligned} a_t &= \left[1 + (\theta_1 - \phi_1) \sum_{i=1}^{\infty} \theta_1^{i-1} B^i\right] \dot{Z}_t \\ &= \dot{Z}_t + (\theta_1 - \phi_1)\, \dot{Z}_{t-1} + (\theta_1 - \phi_1)\, \theta_1 \dot{Z}_{t-2} + \cdots . \end{aligned}$$

D. ACF of ARMA(1, 1) model:

$$
\begin{aligned}
\gamma_0 &= \phi_1\gamma_1 + \sigma_a^2 - \theta_1\sigma_a^2(\phi_1 - \theta_1), \\
\gamma_1 &= \phi_1\gamma_0 - \theta_1\sigma_a^2, \\
\gamma_k &= \phi_1\gamma_{k-1}, \quad k \geq 2.
\end{aligned}
$$

$$
\begin{aligned}
\rho_0 &= 1, \quad k = 0, \\
\rho_1 &= \frac{(\phi_1 - \theta_1)(1 - \phi_1\theta_1)}{1 + \theta_1^2 - 2\theta_1\phi_1}, \quad k = 1, \\
\rho_k &= \phi_1\rho_{k-1}, \quad k \geq 2.
\end{aligned}
$$

E. PACF of ARMA(1, 1) model:

ϕ_{kk} can be obtained from $\rho_1, \rho_2, \cdots, \rho_k$.

Example 3.7. Simulated 250 values from

$$(1 - 0.9B)Z_t = (1 - 0.5B)a_t.$$

Show the sample ACF and PACF.

Example 3.8. Simulated 250 values from

$$(1 - 0.6B)Z_t = (1 - 0.5B)a_t.$$

Show the sample ACF and PACF.

3.4.3. AR, MA and ARMA model with drift

A. AR(p) Model with drift:

Let Z_t be an AR model:

$$Z_t = \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$

where all the roots of $1 - \sum_{i=1}^{p} \phi_i z^i = 0$ lie outside the unit circle.

Z_t is called AR(p) Model with drift, or called AR(p) model, and θ_0 is called drift of Z_t.

When $\theta_0 = 0$, then

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t.$$

when $\theta_0 \neq 0$, let $\mu = \theta_0/(1 - \phi_1 - \cdots - \phi_p)$. Then we have

$$(Z_t - \mu) = \phi_1(Z_{t-1} - \mu) + \cdots + \phi_p(Z_{t-p} - \mu) + a_t,$$
$$or\ \phi_p(B)(Z_t - \mu) = a_t,$$

where $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.

If $\dot{Z}_t = Z_t - \mu$, then

$$\phi_p(B)\dot{Z}_t = a_t.$$

Properties of Z_t: $EZ_t = \mu$,

Condition for stationarity, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$.

B. MA(q) Model with drift:

Let Z_t be an MA model:

$$Z_t = \theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t.$$

Z_t is called MA(q) Model with drift, or called MA(q) model.

θ_0 is called drift of Z_t.

When $\theta_0 = 0$, then

$$Z_t = -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t.$$

when $\theta_0 \neq 0$, then

$$Z_t - \theta_0 = -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t.$$

If $\dot{Z}_t = Z_t - \theta_0$, then

$$\dot{Z}_t = -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t.$$

Properties of Z_t:
$EZ_t = \theta_0$,
Condition for invertibility, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$.
C. ARMA(p, q) Model with drift:
Let Z_t be an ARMA(p, q) model:

$$\begin{aligned} Z_t &= \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} \\ &\quad -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t, \end{aligned}$$

where all the root of $1 - \sum_{i=1}^{p} \phi_i z^i = 0$ lie outside the unit circle.

Z_t is called ARMA(p, q) model with drift, or simply called ARMA(p, q) model. θ_0 is called the drift of ARMA model.

When $\theta_0 = 0$, then

$$\begin{aligned} Z_t &= \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} \\ &\quad -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t. \end{aligned}$$

when $\theta_0 \neq 0$, let $\mu = \theta_0/(1 - \phi_1 - \cdots - \phi_p)$. Then we have

$$\begin{aligned} Z_t - \mu &= \phi_1(Z_{t-1} - \mu) + \cdots + \phi_p(Z_{t-p} - \mu) \\ &\quad -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t. \end{aligned}$$

Let $\dot{Z}_t = Z_t - \mu$. Then

$$\begin{aligned} \dot{Z}_t &= \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} \\ &\quad -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t. \end{aligned}$$

Properties of Z_t: $EZ_t = \mu$,

Condition for stationarity, invertibility, variance, ACV, ACF, PACF of Z_t are the same as those of $\dot{Z}_t$.