MATH4425 (T1A) – Tutorial 5

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March, 7

Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

1 Previously on MATH4425

1.1 The autoregressive integrated moving-average ARIMA process

1.1.1 The general ARIMA process

Let Z_t is *a general stochastic trend model*:

$$(1-B)^d Z_t = x_t, \qquad d \geq 1$$

If x_t is a *stationary* and *invertible* ARMA process

$$\phi_p(B)x_t = \theta_q(B)a_t,$$

where $\phi_p(B)$ and $\theta_q(B)$ have *no common* roots. Then:

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t,$$

Z_t is called **ARIMA(p, d, q) process without constant**.

If x_t is the following ARMA process,

$$\phi_p(B)x_t = \theta_0 + \theta_q(B)$$

Then

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t,$$

Z_t is called **ARIMA(p, d, q) process with constant**.

1.1.2 The random walk model

ARIMA(0, 1, 0) process without constant

$$(1-B)Z_t = a_t,$$

or, equivalently,

$$Z_t = Z_{t-1} + a_t$$

is the random walk.

ARIMA(0, 1, 0) process with constant

$$(1 - B)Z_t = \theta_0 + a_t,$$

or, equivalently,

$$Z_t = \theta_0 + Z_{t-1} + a_t$$

is the random walk with drift.

Properties of the random walk (with drift)

Z_0 is usually a deterministic, fixed number. Let's assume this holds. Then

1. $Z_t = \theta_0 + a_t + Z_{t-1} = \theta_0 + a_t + \theta_0 + a_{t-1} + Z_{t-2} = \theta_0 + a_t + \theta_0 + a_{t-1} + \cdots + \theta_0 + a_1 + Z_0 = t\theta_0 + \sum_{i=1}^{t} a_i + Z_0$
2. $\mathbb{E}Z_t = t\theta_0 + Z_0$
3. $\mathrm{var}(Z_t) = t\sigma_a^2$
4. $\mathrm{cov}(Z_t, Z_{t-k}) = (t-k)\sigma_a^2$
5. $\mathrm{corr}(Z_t, Z_{t-k}) = \dfrac{\mathrm{cov}(Z_t, Z_{t-k})}{\sqrt{\mathrm{var}(Z_t)}\sqrt{\mathrm{var}(Z_{t-k})}} \approx 1, \quad$ if t is large enough

2 Forecasting. Minimum mean square error forecasts for ARMA models

Given a sequence of data $Z_1, Z_2, \cdots, Z_n$ from ARMA or ARIMA model, you can forecast $Z_{n+1}, \cdots, Z_{n+l}$ and give their forecasting intervals.

Here, we usually consider $\boldsymbol{a_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_a^2)}$. Now, let Z_t be a **stationary and invertible** ARMA model:

$$\phi(B)Z_t = \theta(B)a_t.$$

Notation: $\hat{Z}_n(l)$ denotes the forecast value of Z_{n+l} and is called the ***l*-step ahead forecast $\hat{Z}_n(l)$ of Z_{n+l}** (or, simply, ***l*-step forecasting**).

Note: In this case, $\hat{Z}_n(l)$ is a function of *past* white noises:

$$\hat{Z}_n(l) = g(a_n, a_{n-1}, \dots)$$

2.1 Criterion of best linear predictors

Definition: Linear predictor (LP)

$$\hat{Z}_n(l) := \sum_{j=0}^{\infty} \beta_j a_{n-j}$$

for some coefficients β_j.

Definition: $\hat{Z}_n(l)$ is said to be **a best LP (BLP)**, if $\mathbb{E}[Z_{n+l} - \hat{Z}_n(l)]^2$ is the smallest *among all LP.*

Statement: BLP is a conditional expectation $\mathbb{E}(Z_{n+l} \mid \sigma(Z_n, Z_{n-1}, \dots))$.

Proof. First, let's note $\boldsymbol{\mathcal{F}_n := \sigma(Z_n, Z_{n-1}, \dots)}$.

$$\mathbb{E}[Z_{n+l} - \hat{Z}_n(l)]^2 = \mathbb{E}[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) + \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)]^2 =$$

$$\mathbb{E}[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)]^2 + \mathbb{E}[\mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)]^2 +$$

$$2\mathbb{E}[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)][\mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)] =$$

$$A + B + 2\mathbb{E}\mathbb{E}\left\{[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)][\mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)] \,\middle|\, \mathcal{F}_n\right\} =$$

$$A + B + 2\mathbb{E}\left[[\mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)]\mathbb{E}\left\{[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)] \,\middle|\, \mathcal{F}_n\right\}\right] = A + B =$$

$$\mathbb{E}[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)]^2 + \mathbb{E}[\mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) - \hat{Z}_n(l)]^2 \geq \mathbb{E}[Z_{n+l} - \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)]^2,$$

and the equality holds iff $\hat{Z}_n(l) \overset{\text{a.s.}}{=} \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n)$. $\square$

Corollary: If $Z_{n+l} = (\phi(B))^{-1}\theta(B)a_{n+l} = \sum_{j=0}^{\infty} \psi_j a_{n+l-j}$, then

$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid \mathcal{F}_n) = \sum_{j=0}^{\infty} \psi_j \mathbb{E}(a_{n+l-j} \mid \mathcal{F}_n) =$$

$$\sum_{j=l}^{\infty} \psi_j \mathbb{E}(a_{n+l-j} \mid \mathcal{F}_n) = \psi_l a_n + \psi_{l+1} a_{n-1} + \psi_{l+2} a_{n-2} + \cdots,$$

since $\mathbb{E}(a_t \mid \mathcal{F}_n) = 0 \ \ \forall t > n \iff a_n$ are independent.

2.2 Forecasting for ARMA model

Definition: Forecasting error is defined as follows

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}$$

Forecasting variance: First, note that $e_n(l) \sim \mathcal{N}(0, \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2)$. So,

$$\text{var}[e_n(l)] = \mathbb{E}e_n^2(l) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2 \nearrow \text{ as } l \uparrow$$

Forecast interval (FI):

$$\left[\hat{Z}_n(l) - \mathcal{N}_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \psi_j^2}\ ,\ \hat{Z}_n(l) + N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \psi_j^2}\right],$$

where $\mathcal{N}_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$-quantile of the standard normal distribution, i.e. $P(\mathcal{N}(0,1) > \mathcal{N}_{\frac{\alpha}{2}}) = \alpha/2$.

Usually, we don't know σ_a, ψ_j but they can be **estimated** given the dataset.

Note: When $\alpha = 0.05$, $\mathcal{N}_{\frac{\alpha}{2}} = 1.96$.

2.2.1 Formulas of computing forecasts

Forecasts can be calculated recursively as follows

$$\hat{Z}_n(l) = \phi_1 \hat{Z}_n(l-1) + \phi_2 \hat{Z}_n(l-2) + \cdots + \phi_p \hat{Z}_n(l-p) +$$

$$\hat{a}_n(l) - \theta_1 \hat{a}_n(l-1) - \cdots - \theta_q \hat{a}_n(l-q),$$

where

$$\hat{Z}_n(j) = \begin{cases} \mathbb{E}(Z_{n+j} \mid Z_n, Z_{n-1}, \dots) & \text{if } j = 1, 2, \dots, l \\ Z_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

$$\hat{a}_n(j) = \begin{cases} 0 & \text{if } j = 1, 2, \dots, l \\ a_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

2.3 Problems

2.3.1 Problem 1

Consider a model

$$(1 - \phi_1 B)(Z_t - \mu) = a_t,$$

where $|\phi_1| < 1$.

For arbitrary l:

1) Find the l-step ahead forecast $\hat{Z}_n(l)$ of Z_{n+l}

2) Find the variance of the l-step ahead forecast error

2.3.2 Problem 2

Consider a model

$$(1 - \phi_1 B - \phi_2 B^2)(Z_t - \mu) = a_t,$$

where roots of $\phi(z) = 0$ lie outside the unit circle.

For $l = 1, 2, 3, 4$:

1) Find the l-step ahead forecast $\hat{Z}_n(l)$ of Z_{n+l}

2) Find the variance of the l-step ahead forecast error

2.3.3 Problem 3

Consider a model

$$Z_t - 1.2Z_{t-1} + 0.6Z_{t-2} = a_t,$$

given $\sigma_a^2 = 1$.

Suppose that we have the observations from this model:

$$Z_{76} = 60.4$$

$$Z_{77} = 58.9$$

$$Z_{78} = 64.7$$

$$Z_{79} = 70.4$$

$$Z_{80} = 62.6$$

1) Forecast $Z_{81}, Z_{82}, Z_{83}, Z_{84}$

2) Find the 99% forecast interval

2.4 Solutions

2.4.1 Solution 1

Here we have the following AR(1) model with drift:

$$Z_t = (1 - \phi_1)\mu + \phi_1 Z_{t-1} + a_t$$

1) First, introduce the notation $\dot{Z}_t := Z_t - \mu$. Then

$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}(\dot{Z}_{n+l} + \mu \mid Z_n, Z_{n-1}, \dots) =$$

$$\mathbb{E}(\dot{Z}_{n+l} \mid Z_n, Z_{n-1}, \dots) + \mu = \hat{\dot{Z}}_n(l) + \mu$$

We can forecast for the AR(1) model **without** drift. Let's find **general** formula for $\hat{\dot{Z}}_n(l)$ (in this case it is possible because the MA(∞) expansion is known and simple):

$$\hat{\dot{Z}}_n(l) = \mathbb{E}(\dot{Z}_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}(\phi_1 \dot{Z}_{n+l-1} + a_{n+l} \mid Z_n, Z_{n-1}, \dots) =$$

$$\phi_1 \hat{\dot{Z}}_n(l-1) + 0 = \phi_1^2 \hat{\dot{Z}}_n(l-2) = \dots = \phi_1^l \hat{\dot{Z}}_n(0) = \phi_1^l \dot{Z}_n$$

Thus, the forecast for the initial AR(1) model with drift:

$$\hat{Z}_n(l) = \hat{\dot{Z}}_n(l) + \mu = \phi_1^l (Z_n - \mu) + \mu$$

2) For any l we have

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = Z_{n+1} - \phi_1^l (Z_n - \mu) - \mu$$

Notice

$$\mathbb{E}e_n(l) = \mathbb{E}Z_{n+l} - \phi_1^l \mathbb{E}(Z_n - \mu) - \mu = 0$$

Then

$$\mathrm{var}(e_n(l)) = \mathbb{E}e_n^2(l) = \mathbb{E}[(Z_{n+l} - \mu) - \phi_1^l (Z_n - \mu)]^2 =$$

$$\mathbb{E}(Z_{n+l} - \mu)^2 + \phi_1^{2l} \mathbb{E}(Z_n - \mu)^2 - 2\phi_1^l \mathbb{E}(Z_{n+l} - \mu)(Z_n - \mu) =$$

$$(1 + \phi_1^{2l}) \mathrm{var}(Z_n) - 2\phi_1^l \mathrm{cov}(Z_{n+1}, Z_n) = (1 + \phi_1^{2l}) \mathrm{var}(\dot{Z}_n) - 2\phi_1^l \mathrm{cov}(\dot{Z}_{n+l}, \dot{Z}_n) =$$

$$(1+\phi_1^{2l})\gamma_0 - 2\phi_1^l\gamma_l = (1+\phi_1^{2l})\frac{\sigma_a^2}{1-\phi_1^2} - 2\phi_1^l\frac{\sigma_a^2\phi_1^l}{1-\phi_1^2} = (1-\phi_1^{2l})\frac{\sigma_a^2}{1-\phi_1^2}$$

2.4.2 Solution 2

Here we have the following AR(2) model with drift:

$$Z_t = (1-\phi_1-\phi_2)\mu + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

1) First, introduce the notation $\dot{Z}_t := Z_t - \mu$. Then

$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}(\dot{Z}_{n+l} + \mu \mid Z_n, Z_{n-1}, \dots) =$$

$$\mathbb{E}(\dot{Z}_{n+l} \mid Z_n, Z_{n-1}, \dots) + \mu = \hat{\dot{Z}}_n(l) + \mu$$

We can forecast for the AR(2) model **without** drift. Let's find **recursive** formula for $\hat{\dot{Z}}_n(l)$ (in this case, it is a better option because the MA(∞) expansion is complicated):

$$\hat{\dot{Z}}_n(l) = \mathbb{E}(\dot{Z}_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}(\phi_1\dot{Z}_{n+l-1} + \phi_2\dot{Z}_{n+l-2} + a_{n+l} \mid Z_n, Z_{n-1}, \dots) =$$

$$\phi_1\hat{\dot{Z}}_n(l-1) + \phi_2\hat{\dot{Z}}_n(l-2) + 0 = \phi_1\hat{\dot{Z}}_n(l-1) + \phi_2\hat{\dot{Z}}_n(l-2)$$

For $l = 1, 2, 3, 4$ we will thus only have to calculate $\hat{\dot{Z}}_n(l)$ recursively having

$$\hat{\dot{Z}}_n(1) = \phi_1\hat{\dot{Z}}_n(0) + \phi_2\hat{\dot{Z}}_n(-1) = \phi_1\dot{Z}_n + \phi_2\dot{Z}_{n-1}$$

$$\hat{\dot{Z}}_n(2) = \phi_1\hat{\dot{Z}}_n(1) + \phi_2\hat{\dot{Z}}_n(0) = \phi_1^2\dot{Z}_n + \phi_1\phi_2\dot{Z}_{n-1} + \phi_2\dot{Z}_n = (\phi_1^2+\phi_2)\dot{Z}_n + \phi_1\phi_2\dot{Z}_{n-1}$$

as initial conditions.

For example, for $l = 3$ we have

$$\hat{\dot{Z}}_n(3) = \phi_1\hat{\dot{Z}}_n(2) + \phi_2\hat{\dot{Z}}_n(1) =$$

$$\phi_1((\phi_1^2+\phi_2)\dot{Z}_n + \phi_1\phi_2\dot{Z}_{n-1}) + \phi_2(\phi_1\dot{Z}_n + \phi_2\dot{Z}_{n-1}) =$$

$$(\phi_1^3 + 2\phi_1\phi^2)\dot{Z}_n + (\phi_1^2\phi_2 + \phi_2^2)Z_{n-1}$$

Finally, for $l = 1, 2, 3, 4$ the forecast for the initial AR(2) model with drift can be obtained.

For example:

$$\hat{Z}_n(1) = \hat{\dot{Z}}_n(1) + \mu = \phi_1 \dot{Z}_n + \phi_2 \dot{Z}_{n-1} + \mu = \phi_1 Z_n + \phi_2 Z_{n-1} + (1 - \phi_1 - \phi_2)\mu$$

2) For $l = 1, 2, 3, 4$ we have

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) =$$

$$(1 - \phi_1 - \phi_2)\mu + \phi_1 Z_{n+l-1} + \phi_2 Z_{n+l-2} + a_{n+l} -$$

$$\phi_1(\hat{Z}_n(l-1) - \mu) - \phi_2(\hat{Z}_n(l-2) - \mu) - \mu =$$

$$\phi_1(Z_{n+l-1} - \hat{Z}_n(l-1)) + \phi_2(Z_{n+l-2} - \hat{Z}_n(l-2)) + a_{n+l} =$$

$$\phi_1 e_n(l-1) + \phi_2 e_n(l-2) + a_{n+l}$$

For $l = 1$ we have

$$e_n(1) = Z_{n+1} - \hat{Z}_n(1) = Z_{n+1} - \phi_1 Z_n - \phi_2 Z_{n-1} - (1 - \phi_1 - \phi_2)\mu = a_{n+1}$$

Then

$$\mathrm{var}(e_n(1)) = \mathrm{var}(a_{n+1}) = \sigma_a^2$$

For $l = 2$ we have

$$e_n(2) = Z_{n+2} - \hat{Z}_n(2) =$$

$$(1 - \phi_1 - \phi_2)\mu + \phi_1 Z_{n+1} + \phi_2 Z_n + a_{n+2} -$$

$$\phi_1(\hat{Z}_n(1) - \mu) - \phi_2(Z_n - \mu) - \mu =$$

$$\phi_1 e_n(1) + a_{n+2} = \phi_1 a_{n+1} + a_{n+2}$$

Then

$$\mathrm{var}(e_n(2)) = \sigma_a^2(\phi_1^2 + 1)$$

The next $e_n(l)$ can be obtained using **recursive** formula. For example, for $l = 3$ we have

$$e_n(3) = \phi_1 e_n(2) + \phi_2 e_n(1) + a_{n+3} =$$

$$\phi_1(\phi_1 a_{n+1} + a_{n+2}) + \phi_2 a_{n+1} + a_{n+3} =$$

$$(\phi_1^2 + \phi_2)a_{n+1} + \phi_1 a_{n+2} + a_{n+3}$$

Then

$$\text{var}(e_n(3)) = \sigma_a^2((\phi_1^2 + \phi_2)^2 + \phi_1^2 + 1) = \sigma_a^2(\phi_1^4 + 2\phi_1^2\phi_2 + \phi_2^2 + \phi_1^2 + 1)$$

2.4.3 Solution 3

Here we have the following AR(2) model without drift:

$$Z_t = 1.2Z_{t-1} - 0.6Z_{t-2} + a_t$$

Here, we only need to calculate forecasts and FI-s using previous solution.

For example, for $t = 81$ we have

$$\hat{Z}_{81} = \hat{Z}_n(1) = 1.2Z_{80} - 0.6Z_{79} = 32.88$$

$$\text{var}(e_{80}(1)) = \sigma_a^2 = 1$$

$$\left[32.88 - 2.58 \times 1\ ,\ 32.88 + 2.58 \times 1\right],$$

that is,

$$\left[30.30\ ,\ 35.46\right]$$