MATH4425 (T1A) – Tutorial 6

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Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

1 Forecasting. Minimum Mean Square Error Forecasts for ARIMA models

Given a sequence of data $Z_1, Z_2, \cdots, Z_n$ from ARMA or ARIMA model, you can forecast $Z_{n+1}, \cdots, Z_{n+l}$ and give their forecasting intervals.

Here, we usually consider $\boldsymbol{a_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_a^2)}$. Now, let Z_t be ARIMA(p, d, q) model with $d \neq 0$,

$$\phi(B)(1-B)^d Z_t = \theta(B) a_t,$$

where all roots of $\phi(z) = 0$ and $\theta(z) = 0$ lie outside the unit circle.

1.1 Best linear predictor

We already know that **best predictor** (in terms of MSE) is a *conditional expectation* $\mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots)$.

The only difference (from ARMA model) is that now $\mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots)$ cannot be directly expressed in terms of past noises a_t. Although, it is still linear (with respect to past Z_n, a_n).

Compare with ARMA model: If $Z_{n+l} = (\phi(B))^{-1}\theta(B) a_{n+l} = \sum_{j=0}^{\infty} \psi_j a_{n+l-j}$ MA(∞) representation of stationary ARMA model, then

$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots) = \psi_l a_n + \psi_{l+1} a_{n-1} + \psi_{l+2} a_{n-2} + \cdots$$

1.2 Forecasting for ARIMA model

Denote

$$\Psi(B) = \phi(B)(1-B)^d = 1 - \Psi_1 B - \Psi_2 B^2 - \cdots - \Psi_{p+d} B^{p+d}$$

Then we can represent Z_t as **a non-stanionary** but **invertible** ARMA$(p+d, q)$ model:

$$Z_t = \Psi_1 Z_{t-1} + \Psi_2 Z_{t-2} + \cdots + \Psi_{p+d} Z_{t-p-d} +$$

$$a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}$$

Due to invertibility, we can get AR(∞) representation of the model:

$$\pi(B)Z_t = a_t,$$

where

$$\pi(B) = (\theta(B))^{-1}\phi(B)(1-B)^d = 1 - \sum_{j=1}^{\infty} \pi_j B^j$$

meaning

$$Z_t = a_t + \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots$$

And we immediately derive that

$$\hat{Z}_n(l) = \mathbb{E}\left(a_{n+l} + \sum_{j=1}^{\infty} \pi_j Z_{n+l-j} \mid Z_n, Z_{n-1}, \dots\right) =$$

$$0 + \sum_{j=1}^{l-1} \pi_j \hat{Z}_{n+l-j} + \sum_{j=l}^{\infty} \pi_j Z_{n+l-j}$$

Forecasting error can be calculated as follows

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j},$$

where ψ_j can be calculated recursively:

$$\psi_j = \sum_{k=0}^{j-1} \pi_{j-k}\psi_k \quad \forall j = 1, 2, \dots, l-1$$

Note: Here, ψ_j are **different** from ψ_j for ARMA model. In ARMA case, ψ_j are coming from MA(∞) representation. In ARIMA case, ψ_j are coming from AR(∞) representation and recursive formula for $e_n(l)$.

It can be proved by induction:

1) First, notice

$$e_n(1) = Z_{n+1} - \hat{Z}_n(1) = \left(a_{n+1} + \sum_{j=1}^{\infty} \pi_j Z_{n+1-j}\right) - \sum_{j=1}^{\infty} \pi_j Z_{n+1-j} =$$

$$a_{n+1} = \psi_0 a_{n+1}$$

2) Now, assume that for $m \leq l-1$ the statement holds. Then

$$e_n(l) = \left(a_{n+l} + \sum_{j=1}^{\infty} \pi_j Z_{n+l-j}\right) - \left(\sum_{j=1}^{l-1} \pi_j \hat{Z}_{n+l-j} + \sum_{j=l}^{\infty} \pi_j Z_{n+l-j}\right) =$$

$$a_{n+l} + \sum_{j=1}^{l-1} \pi_j (Z_{n+l-j} - \hat{Z}_{n+l-j}) = a_{n+l} + \sum_{j=1}^{l-1} \pi_j e_n(l-j) =$$

$$a_{n+l} + \sum_{j=1}^{l-1}\left(\pi_j \sum_{k=0}^{j-1} \psi_k a_{n+l-j-k}\right) =$$

$$a_{n+l} + \pi_1\psi_0 a_{n+l-1} + (\pi_2\psi_0 + \pi_1\psi_1)a_{n+l-2} + \cdots + \left(\sum_{j=0}^{l-1} \pi_{l-j}\psi_j\right) a_{n+1}$$

Forecasting variance: First, note that $e_n(l) \sim \mathcal{N}(0, \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2)$. So,

$$\mathrm{var}[e_n(l)] = \mathbb{E}e_n^2(l) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2 \nearrow \text{ as } l \uparrow$$

Forecast interval (FI):

$$\left[\hat{Z}_n(l) - \mathcal{N}_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \psi_j^2}\ ,\ \hat{Z}_n(l) + N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \psi_j^2} \right],$$

where $\mathcal{N}_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$-quantile of the standard normal distribution, i.e. $P(\mathcal{N}(0,1) > \mathcal{N}_{\frac{\alpha}{2}}) = \alpha/2$.

Usually, we don't know σ_a, ψ_j but they can be **estimated** given the dataset.

Note: When $\alpha = 0.05$, $\mathcal{N}_{\frac{\alpha}{2}} = 1.96$.

1.2.1 Formulas of computing forecasts

Forecasts can be calculated recursively as follows

$$\hat{Z}_n(l) = \Psi_1 \hat{Z}_n(l-1) + \Psi_2 \hat{Z}_n(l-2) + \cdots + \Psi_{p+d} \hat{Z}_n(l-p-d) +$$

$$\hat{a}_n(l) - \theta_1 \hat{a}_n(l-1) - \theta_2 \hat{a}_n(l-2) - \cdots - \theta_q \hat{a}_n(l-q),$$

where

$$\hat{Z}_n(j) = \begin{cases} \mathbb{E}(Z_{n+j} \mid Z_n, Z_{n-1}, \dots) & \text{if } j = 1, 2, \dots, l \\ Z_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

$$\hat{a}_n(j) = \begin{cases} 0 & \text{if } j = 1, 2, \dots, l \\ a_{n+j} & \text{if } j = 0, -1, \dots \end{cases}$$

Actually, $\hat{a}_n(j)$ for $j \le 0$ can be calculated as follows:

$$\hat{a}_n(j) = a_{n+(j-1)+1} = e_{n+j-1}(1) = Z_{n+j} - \hat{Z}_{n+j-1}(1)$$

Note: In practice, $\hat{a}_n(j)$ $(j \le 0)$ is **not** always considered to be zero. If we are provided some observations from the past, we can calculate at least some of the past noises.

Note: All the conclusions about the past noises also apply to **ARMA** model.

2 Updating forecasts for ARMA and ARIMA

If Z_{n+1} turns out to be available, how to forecast Z_{n+l}?

Method 1

Now we are given

$$Z_{n+1}, Z_n, Z_{n-1}, \cdots$$

So, the forecast value of Z_{n+l} is $\hat{Z}_{n+1}(l-1)$.

Method 2

First, note

$$e_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j} = \sum_{j=0}^{l-1} \psi_j a_{(n+1)+(l-1)-j} =$$

$$\sum_{j=0}^{l-2} \psi_j a_{(n+1)+(l-1)-j} + \psi_{l-1} a_{n+1} = e_{n+1}(l-1) + \psi_{l-1} e_n(1)$$

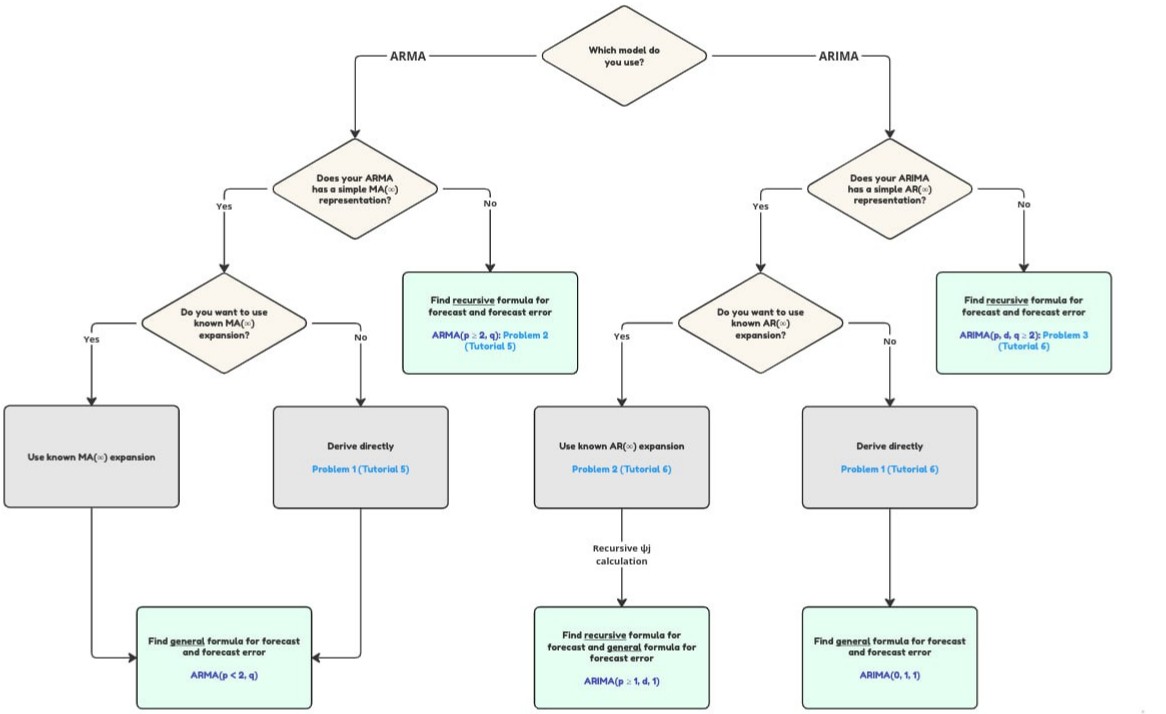
meaning

$$Z_{n+l} - \hat{Z}_n l = Z_{n+l} - \hat{Z}_{n+1}(l-1) + \psi_{l-1}[Z_{n+1} - \hat{Z}_n(1)]$$

So, we can update the forecast using obtained relationship of $\hat{Z}_{n+1}(l-1)$ and $\hat{Z}_n(l)$:

$$\hat{Z}_{n+1}(l-1) = \hat{Z}_n(l) + \psi_{l-1}[Z_{n+1} - \hat{Z}_n(1)]$$

3 Guide to ARMA and ARIMA forecasting



4 Problems

Problem 1

Consider a model

$$(1-B)Z_t = (1-\theta B)a_t,$$

where $|\theta| < 1$.

For arbitrary l:

1) Find the l-step ahead forecast $\hat{Z}_n(l)$ of Z_{n+l}

2) Find the variance of the l-step ahead forecast error

Problem 2

Consider a model
$$(1-\phi B)(1-B)Z_t = (1-\theta B)a_t,$$
where $|\phi| < 1, |\theta| < 1$.

For $l = 1, 2, 3$:

1) Find the l-step ahead forecast $\hat{Z}_n(l)$ of Z_{n+l}

2) Find the variance of the l-step ahead forecast error

Problem 3

Consider a model
$$(1-0.8B)(1-B)Z_t = (1-0.2B-0.4B^2)a_t,$$
given $\sigma_a^2 = 1$.

Suppose that we have the observations from this model:

$$Z_{96} = -0.06$$
$$Z_{97} = 0.88$$
$$Z_{98} = 0.86$$
$$Z_{99} = 0.99$$
$$Z_{100} = 1.90$$

1) Forecast $Z_{101}, Z_{102}, Z_{103}, Z_{104}$

2) Find the 95% forecast interval ($\mathcal{N}_{\frac{0.05}{2}} = 1.96$)

5 Solutions

Solution 1

Here we have the following ARIMA(0, 1, 1) model:
$$Z_t = Z_{t-1} + a_t - \theta a_{t-1}$$

1) Let's find **general** formula for $\hat{Z}_n(l)$:
$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}(Z_{n+l-1} + a_{n+l} - \theta a_{n+l-1} \mid Z_n, Z_{n-1}, \dots) =$$
$$\hat{Z}_n(l-1) + 0 - \theta \hat{a}_n(l-1) = \begin{cases} Z_n - \theta a_n, & l = 1 \\ \hat{Z}_n(l-1), & l \geq 2 \end{cases}$$
meaning
$$\hat{Z}_n(l) = \hat{Z}_n(1) = Z_n - \theta a_n$$

2) For any l we have

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) =$$
$$(Z_{n+l-1} + a_{n+l} - \theta a_{n+l-1}) - \hat{Z}_n(l-1) =$$
$$e_n(l-1) + (a_{n+l} - \theta a_{n+l-1}) =$$
$$e_n(l-2) + (a_{n+l-1} - \theta a_{n+l-2}) + (a_{n+l} - \theta a_{n+l-1}) =$$
$$e_n(l-2) - \theta a_{n+l-2} + (1-\theta) a_{n+l-1} + a_{n+l} = \dots =$$
$$e_n(1) - \theta a_{n+1} + (1-\theta) \sum_{k=1}^{l-2} a_{n+l-k} + a_{n+l} =$$
$$(1-\theta) \sum_{k=1}^{l-1} a_{n+l-k} + a_{n+l}$$

Then

$$\mathrm{var}(e_n(l)) = \sigma_a^2((l-1)(1-\theta)^2 + 1)$$

Solution 2

Here we have the following ARIMA(1, 1, 1) model:

$$Z_t = (\phi + 1) Z_{t-1} - \phi Z_{t-2} + a_t - \theta a_{t-1}$$

1) Let's first find AR(∞) expansion:

$$Z_t - \sum_{i=1}^{\infty} \pi_i Z_{t-i} = a_t \Leftrightarrow$$
$$\pi(B) Z_t = a_t \Leftrightarrow$$
$$(1 - \theta B)^{-1} (1 - (\phi+1)B + \phi B^2) Z_t = a_t \Leftrightarrow$$
$$\sum_{i=0}^{\infty} \theta^i B^i (1 - (\phi+1)B + \phi B^2) Z_t = a_t \Leftrightarrow$$
$$\sum_{i=2}^{\infty} \underbrace{(\theta^{i-2}\phi - \theta^{i-1}(\phi+1) + \theta^i)}_{-\pi_i, i \geq 2} B^i Z_t + \underbrace{(\theta - \phi - 1)}_{-\pi_1} B Z_t + \underbrace{1}_{\pi_0} Z_t = a_t$$

Let's find **recursive** formula for $\hat{Z}_n(l)$:

$$\hat{Z}_n(l) = \mathbb{E}(Z_{n+l} \mid Z_n, Z_{n-1}, \dots) = \mathbb{E}\left(\sum_{j=1}^{\infty} \pi_j Z_{t+l-j} + a_{t+l} \mid Z_n, Z_{n-1}, \dots\right) =$$
$$\sum_{j=1}^{l-1} \pi_j \hat{Z}_{t+l-j} + \sum_{j=l}^{\infty} \pi_j Z_{t+l-j} + 0$$

with initial condition

$$\hat{Z}_n(1) = \sum_{j=1}^{\infty} \pi_j Z_{t+l-j}$$

2) First, let's calculate ψ_j recursively:

$$\psi_0 = 1$$
$$\psi_1 = \pi_1\psi_0 = -(\theta - \phi - 1) = \phi - \theta + 1$$
$$\psi_2 = \pi_2\psi_0 + \pi_1\psi_1 = -(\phi - \theta(\phi + 1) + \theta^2) + (\phi - \theta + 1)^2 =$$
$$-\phi + \theta\phi + \theta - \theta^2 + (\phi - \theta)^2 + 2(\phi - \theta) + 1 =$$
$$\phi - \phi\theta - \theta + \phi^2 + 1$$

For any $l = 1, 2, 3$ we have

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}$$

So, we can calculate $e_n(l)$ using obtained ψ_j:

$$e_n(1) = \psi_0 a_{n+1} = a_{n+1}$$

$$e_n(2) = \psi_0 a_{n+2} + \psi_1 a_{n+1} = a_{n+2} + (\phi - \theta + 1)a_{n+1}$$
$$e_n(3) = \psi_0 a_{n+3} + \psi_1 a_{n+2} + \psi_2 a_{n+1} =$$
$$a_{n+3} + (\phi - \theta + 1)a_{n+2} + (\phi - \phi\theta - \theta + \phi^2 + 1)a_{n+1}$$

Then

$$\text{var}(e_n(1)) = \sigma_a^2$$
$$\text{var}(e_n(2)) = \sigma_a^2[1 + (\phi - \theta + 1)^2]$$
$$\text{var}(e_n(2)) = \sigma_a^2[1 + (\phi - \theta + 1)^2 + (\phi - \phi\theta - \theta + \phi^2 + 1)^2]$$

Solution 3

Here we have ARIMA(1, 1, 2) model, moreover,

$$\begin{cases} 0.4 + 0.2 < 1, \\ 0.4 - 0.2 < 1, \\ -1 < 0.4 < 1 \end{cases}$$

So, the model is invertible. However, it's hard to find $(\theta(B))^{-1}$. We'll use the **recursive** formulas for both $\hat{Z}_n(l)$ and $e_n(l)$.

1) Let's first represent our model as ARMA(1+1, 2):

$$Z_t = 1.8Z_{t-1} - 0.8Z_{t-2} + a_t - 0.2a_{t-1} - 0.4a_{t-2}$$

Then for any l

$$\hat{Z}_n(l) = 1.8\hat{Z}_n(l-1) - 0.8\hat{Z}_n(l-2) + \hat{a}_n(l) - 0.2\hat{a}_n(l-1) - 0.4\hat{a}_n(l-2)$$

First let's calculate a few noise predictions:

1) Since there's no way to calculate *very past* noises (we don't have enough past data), we assume

$$a_{100+j} = 0 \quad \forall j \leq -3,$$

2) For the rest noise predictions:

a) For $j = -2$ we have

$$\hat{a}_{100}(-2) = a_{98} = Z_{98} - \hat{Z}_{97}(1) =$$

$$Z_{98} - (1.8Z_{97} - 0.8Z_{96} + 0 - 0.2a_{97} - 0.4a_{96}) =$$
$$0.86 - (1.8 \times 0.88 - 0.8 \times (-0.06)) \approx -0.77$$

b) For $j = -1$ we have

$$\hat{a}_{100}(-1) = a_{99} = Z_{99} - \hat{Z}_{98}(1) =$$

$$Z_{99} - (1.8Z_{98} - 0.8Z_{97} + 0 - 0.2a_{98} - 0.4a_{97}) =$$
$$0.99 - (1.8 \times 0.86 - 0.8 \times 0.88 - 0.2 \times (-0.77)) \approx -0.01$$

c) For $j = 0$ we have

$$\hat{a}_{100}(0) = a_{100} = Z_{100} - \hat{Z}_{99}(1) =$$
$$Z_{100} - (1.8Z_{99} - 0.8Z_{98} + 0 - 0.2a_{99} - 0.4a_{98}) =$$
$$1.90 - (1.8 \times 0.99 - 0.8 \times 0.86 - 0.2 \times (-0.01) - 0.4 \times (-0.77)) \approx 0.50$$

For $l = 1$ we have

$$\hat{Z}_{101} = \hat{Z}_{100}(1) = 1.8Z_{100} - 0.8Z_{99} + 0 - 0.2a_{100} - 0.4a_{99} =$$

$$1.8 \times 1.90 - 0.8 \times 0.99 + 0 - 0.2 \times 0.50 - 0.4 \times (-0.01) \approx 2.53$$

For $l = 2$ we have

$$\hat{Z}_{102} = \hat{Z}_{100}(2) = 1.8\hat{Z}_{101} - 0.8Z_{100} + 0 - 0 - 0.4a_{100} =$$

$$1.8 \times 2.53 - 0.8 \times 1.9 + 0 - 0 - 0.4 \times 0.50 \approx 2.83$$

Similarly, for $l = 3, 4$.

2) For any l we have

$$e_n(l) = Z_{n+l} - \hat{Z}_n(l) =$$

$$1.8e_n(l-1) - 0.8e_n(l-2) + (a_{n+l} - \hat{a}_n(l)) - 0.2(a_{n+l-1} - \hat{a}_n(l-1)) - 0.4(a_{n+l-2} - \hat{a}_n(l-2))$$

For $l = 1$ we have

$$e_n(1) = 1.8 \times 0 - 0.8 \times 0 + (a_{n+1} - 0) - 0.2(a_n - a_n) - 0.4(a_{n-1} - a_{n-1}) = a_{n+1}$$

And so

$$\text{var}(e_n(1)) = \sigma_a^2 = 1$$

giving FI

$$\left[2.53 - 1.96 \ , \ 2.53 + 1.96 \right]$$

that is,

$$\left[0.57 \ , \ 4.49 \right]$$

Similarly, for $l = 2, 3, 4$.