Chapter 1: Overview

Observe two data sets:

Hang Seng Index	12877	12850	13023	$\cdots$
Date	30.8.04	31.8.04	01.9.04	$\cdots$

Student's Weights	130kg	200kg	45kg	$\cdots$
Students	A	B	C	$\cdots$

What is the difference between these two data sets?

Definition:

A time series (**TS**) is a sequence of random variables labeled by time t:

$$\{Z_1, Z_2, \cdots, Z_t, \cdots\}$$

or

$$\{\cdots, Z_{-1}, Z_0, Z_1, Z_2, \cdots, Z_t, \cdots\}.$$

Denote them by $\{Z_t\}$.

Example: Let $Z =$ weather.
Let $Z_t =$ weather on the tth day. Then $\{Z_1, Z_2, \cdots, Z_t, \cdots\}$ is a TS.

$Z_t =$exchange rate of USAD/HKD at the tth hour.
$Z_t =$daily Hang Seng Index on the tth day.
$Z_t =$average personal consumption in HongKong in the tth month.
$Z_t =$USA beer production at the tth quarter.
$Z_t =$USA tobacco production at the tth year.

All these $\{Z_t\}$ are TS.

Time series data are observations of TS $\{Z_t\}$.

Example: Let Z_t =weather on the tth day.

Weather=	29°	30°	9°	$\cdots$
Date t=	1	2	3	$\cdots$
Notation	$Z_1 = 29°$	$Z_2 = 30°$	$Z_3 = 9°$	$\cdots$

The types of TS data:

$$\text{Discrete time data}\begin{cases}\text{annual data}\\\text{monthly data}\\\text{quarterly data}\\\text{daily data}\\\text{hourly data}\end{cases}$$

Continuous time data

Main objective of TS analysis:

Past data $\Longrightarrow$TS r.v. $Z_t\Longrightarrow$ future of TS.

$$(a) \qquad E\left(Z_{n+l}|y_1,\cdots,y_n\right),$$

$$(b) \qquad P(a \le Z_{n+l} \le b|y_1,\cdots,y_n) \text{ for some } a < b.$$

2.1. **Strict stationarity and weak stationarity**

Definition: Let $\{Z_t\}$ be a TS.

When $t = t_1$, we have:

$$Z_{t_1} \to P(Z_{t_1} \leq z).$$

When $t = t_{t_1+k}$, we have:

$$Z_{t_1+k} \to P(Z_{t_1+k} \leq z).$$

If $P(Z_{t_1} \leq z) = P(Z_{t_1+k} \leq z)$ for $\forall t_1, k, z$, then we say: Z_t is the first order stationary in distribution.

When $t = t_1, t_2$, we have:

$$(Z_{t_1}, Z_{t_2}) \to P(Z_{t_1} \leq z_1, Z_{t_2} \leq z_2)$$

When $t = t_1 + k, t_2 + k$, we have:

$$(Z_{t_1+k}, Z_{t_2+k}) \to P(Z_{t_1+k} \leq z_1, Z_{t_2+k} \leq z_2)$$

If

$$P(Z_{t_1} \leq z_1, Z_{t_2} \leq z_2) = P(Z_{t_1+k} \leq z_1, Z_{t_2+k} \leq z_2),$$

for $\forall t_1, t_2, k$ and (z_1, z_2), then we say: Z_t is the second order stationary in distribution.

When $t = t_1, \cdots, t_n$, we have:

$(Z_{t_1}, \cdots, Z_{t_n}) \to P(Z_{t_1} \le z_1, \cdots, Z_{t_n} \le z_n)$

When $t = t_1 + k, \cdots, t_n + k$, we have:

$(Z_{t_1+k}, \cdots, Z_{t_n+k}) \to P(Z_{t_1+k} \le z_1, \cdots, Z_{t_n+k} \le z_n)$

If

$$P(Z_{t_1} \le z_1, \cdots, Z_{t_n} \le z_n)$$

$$= P(Z_{t_1+k} \le z_1, \cdots, Z_{t_n+k} \le z_n),$$

for $\forall t_1, \cdots, t_n, k$ and $(z_1, \cdots, z_n)$ and n, we say: $\{Z_t\}$ is a **strictly stationary TS**.

Definition: Let $\{Z_t\}$ be a TS.

Mean function of Z_t: $\mu_t = EZ_t$.

Variance function of Z_t: $\sigma_t^2 = E(Z_t - \mu_t)^2$.

Covariance function between Z_{t_1} and Z_{t_2}:

$$\gamma(t_1, t_2) = E[(Z_{t_1} - \mu_{t_1})(Z_{t_2} - \mu_{t_2})]\,,$$

and their correlation function

$$\rho(t_1, t_2) = \frac{\gamma(t_1, t_2)}{\sqrt{\sigma_{t_1}^2}\sqrt{\sigma_{t_2}^2}}$$

Definition: Let Z_t be a TS. If

$$\begin{aligned} \mu_t &= \mu < \infty, \\ \sigma_t^2 &= \sigma^2 < \infty, \\ \gamma(t, t+k) &= \gamma_k\,, \end{aligned}$$

for any t, then $\{Z_t\}$ is said **(second order) weakly stationary**.

Property: Assume $\{Z_t\}$ is strictly stationary.

If $E|Z_t| < \infty$, then $\mu_t = \mu < \infty$.

If $E|Z_t|^2 < \infty$, then $\sigma_t^2 = \sigma^2 < \infty$.

Furthermore

$$\gamma(t, t+k) = \gamma_k, \quad \rho(t, t+k) = \rho_k.$$

Thus, if $EZ_t^2 < \infty$, then strictly stationary $\Longrightarrow$ second-order weakly stationary.

Example 2.2: Consider the following time sequence

$$Z_t = A \sin(\omega t + \theta)\,,$$

where A is a random variable with a zero mean and a unit variance and θ is a r.v. with a uniform distribution on the interval $[-\pi, \pi]$ independent of A. Then

$$E(Z_t) = 0, \quad E(Z_t Z_{t+k}) = \frac{1}{2}\cos(\omega k).$$

Example 2.3: Let $X_t \sim N(0,1)$ be i.i.d. and $Y_t = \{1, -1\}$ be i.i.d so that $P(Y_t = 1) = P(Y_t = -1) = 1/2$ Let

$$Z_t = \begin{cases} X_t & \text{if } t \text{ is odd} \\ Y_t & \text{if } t \text{ is even} \end{cases},$$

where $\{X_t\}$ and $\{Y_t\}$ are independent. Then

$$EZ_t = 0\,, EZ_t^2 = 1,$$

$$E(Z_t Z_s) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases},$$

$$\rho(t,s) = \begin{cases} 0 & \text{if } t \neq s \\ 1 & \text{if } t = s \end{cases}.$$

From now on, the term "stationary" means " second-order weakly stationary".

2.2 Autocovariance and autocorrelation functions

Let Z_t be a sequence of stationary TS r.v.s. Then $EZ_t = \mu$, a constant.

$\gamma_k = \textbf{cov}(Z_t, Z_{t+k}) = E[(Z_t - \mu)(Z_{t+k} - \mu)]$ only depends on k, γ_k is called **autocovariance** (ACV) of Z_t.

Let

$$\rho_k = \frac{\textbf{cov}(Z_t, Z_{t+k})}{\sqrt{\textbf{var}(Z_t)}\sqrt{\textbf{var}(Z_{t+k})}} = \frac{\gamma_k}{\gamma_0}.$$

Then ρ_k only depends on k. ρ_k is called **autocorrelation function** (ACF) of Z_t.

Properties of γ_k and ρ_k:

(1). $\gamma_0 = \sigma^2, \quad \rho_0 = 1.$

(2). $\gamma_k = \gamma_{-k}, \quad \rho_k = \rho_{-k}.$

(3). $\gamma_k \leq \gamma_0, \quad \rho_k \leq \rho_0.$

Important point: The smaller ρ_k, the less dependency between Z_t and Z_{t+k}.

Intuitively, as $k \to \infty, \rho_k \to 0$, generally.

In general, $\rho_k \neq 0$, this is an important feature of TS r.v.s..

2.3 **Partial Autocorrelation function** (PACF).

Definition:

Let Z_t be a stationary TS process. The conditional correlation

$$
\begin{aligned}
&Corr(Z_t, Z_{t+k}|Z_{t+1}, \cdots, Z_{t+k-1}) \\
&\qquad = \frac{Cov[(Z_t - \hat{Z}_t)(Z_{t+k} - \hat{Z}_{t+k})]}{\sqrt{Var(Z_t - \hat{Z}_t)Var(Z_{t+k} - \hat{Z}_{t+k})}}
\end{aligned}
$$

is called the PACF of Z_t and Z_{t+k}, denoted by ϕ_{kk}, where $\hat{Z}_t$ and $\hat{Z}_{t+k}$ is the best linear estimate of Z_{t+k} given $\{Z_{t+1}, \cdots, Z_{t+k-1}\}$ in mean square errors as $k \geq 2$. $\hat{Z}_t = \hat{Z}_{t+k} = EZ_t$ as $k = 1$.

Formula: $\phi_{11} = \rho_1$,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

2.4 White noise processes

A process $\{a_t\}$ is called a white noise process if

$$Ea_t = 0,$$
$$\textbf{var}(a_t) = \sigma_a^2,$$
$$\gamma_k = \textbf{cov}(a_t, a_{t+k}) = 0, \quad \text{if } k \neq 0.$$

Properties of the white noise: if a_t is a white noise, then

$$\text{(1).(ACV)} \quad \gamma_k = \begin{cases} \sigma_a^2 & k = 0 \\ 0 & k \neq 0 \end{cases},$$

$$\text{(2).(ACF)} \quad \rho_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases},$$

$$\text{(3).(PACF)} \quad \phi_{kk} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

2.5 Estimation of ACV and ACF

Given $Z_1, Z_2, \cdots, Z_n$, how to estimate μ, σ^2, γ_k and ρ_k?

2.5.1 Sample mean

$$\overline{Z} = \frac{1}{n}\sum_{t=1}^{n} Z_t$$

is called the sample mean of Z_t. $\overline{Z}$ is the estimator of the mean μ. Is this estimator valid?

(1). $\overline{Z}$ is an unbiased estimator of μ, i.e.

$$E\overline{Z} = \mu.$$

(2). $\overline{Z}$ is a consistent estimator of μ, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} Z_t = \mu\,,$$

almost surely, if $\rho_k \to 0$ as $k \to \infty$. (ergodic property)

2.5.2 Sample ACV

$$\widehat{\gamma}_k = \frac{1}{n-k}\sum_{t=1}^{n-k}(Z_t - \overline{Z})(Z_{t+k} - \overline{Z})$$

is called the sample ACV of Z_t.

$\widehat{\gamma}_k$ is the estimators of γ_k.

Are these estimators valid?

(1). $\widehat{\gamma}_k$ is biased estimator of γ_k, i.e.

$$E\widehat{\gamma}_k \neq \gamma_k,$$

(2). $\widehat{\gamma}_k$ is consistent estimator of γ_k, i.e.

$$\lim_{n\to\infty} \widehat{\gamma}_k = \gamma_k,$$

if $\rho_k \to 0$ as $k \to \infty$.

In particular,

$$\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{t=1}^{n}(Z_t - \overline{Z})^2 = \frac{1}{n}\sum_{t=1}^{n} Z_t^2 - \overline{Z}^2$$

is called the sample variance of Z_t.

$\widehat{\sigma}_n^2$ is an estimator of σ^2, and

$$\lim_{n\to\infty} \widehat{\sigma}_n^2 = \sigma^2 \qquad \text{if} \quad \rho_k \to 0 \quad \text{as} \quad k \to \infty.$$

2.5.3 **Sample ACF**.

$$\widehat{\rho}_k = \frac{\widehat{\gamma}_k}{\widehat{\gamma}_0}$$

is called the sample ACF of Z_t.

$\widehat{\rho}_k$ is the estimator of ρ_k.

$\widehat{\rho}_k$ is the consistent estimator of ρ_k, i.e.

$$\lim_{n\to\infty} \widehat{\rho}_k = \rho_k \qquad \text{if} \quad \rho_k \to 0 \quad \text{as} \quad k \to \infty.$$

Bartlett(1946) showed that

$$\textbf{Var}(\widehat{\rho}_k) \approx \frac{1}{n} \sum_{i=-\infty}^{\infty} (\rho_i^2 + \rho_{i+k}\rho_{i-k} - 4\rho_k\rho_i\rho_{i-k} + 2\rho_k^2\rho_i^2).$$

In particular, when $Z_t = a_t$ is a white noise, we have

$$\textbf{Var}(\widehat{\rho}_k) \approx \frac{1}{n}.$$

How to check whether Z_t is a white noise or not?

Let

$$S_{\widehat{\rho}_k} = \sqrt{\frac{1}{n}(1 + 2\widehat{\rho}_1^2 + \cdots + 2\widehat{\rho}_m^2)}\ ,$$

where m is a fixed integer.

If Z_t is a white noise, $S_{\widehat{\rho}_k} \approx \sqrt{\frac{1}{n}}$.

2.5.4 **Sample PACF**.

$\widehat{\phi}_{11} = \widehat{\rho}_1$,

$$\widehat{\phi}_{kk} = \frac{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & \widehat{\rho}_k \end{vmatrix}}{\begin{vmatrix} 1 & \widehat{\rho}_1 & \widehat{\rho}_2 & \cdots & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-1} \\ \widehat{\rho}_1 & 1 & \widehat{\rho}_1 & \cdots & \widehat{\rho}_{k-3} & \widehat{\rho}_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \widehat{\rho}_{k-1} & \widehat{\rho}_{k-2} & \widehat{\rho}_{k-3} & \cdots & \widehat{\rho}_1 & 1 \end{vmatrix}}$$

is called the sample PACF of Z_t.

$\widehat{\phi}_{kk}$ is the estimator of ϕ_{kk} and is consistent.

2.6 Moving average and autoregressive representations of time series processes

Definition: Moving average representation of Z_t:

$$\begin{aligned} Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \\ &= \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}\,, \end{aligned}$$

where $\psi_0 = 1$, a_t is a white noise, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$. (called Wold's representation or linear process)

Notation Backshift operator : $B^j x_t = x_{t-j}$.

Thus, Z_t can be written as

$$\begin{aligned} Z_t &= \mu + B^0 a_t + \psi_1 B^1 a_t + \psi_2 B^2 a_t + \cdots \\ &= \mu + \sum_{j=0}^{\infty} \psi_j B^j a_t \\ &= \mu + \left(\sum_{j=0}^{\infty} \psi_j B^j \right) a_t. \end{aligned}$$

Denote $\dot{Z}_t = Z_t - \mu$ and $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$.

Then $\dot{Z}_t = \psi(B) a_t$.

Some properties:

$$EZ_t = \mu \,,$$

$$\textbf{Var}(Z_t) = \sigma_a^2 \sum_{j=0}^{\infty} \psi_j^2 \,,$$

$$E(a_t Z_{t-j}) = \begin{cases} \sigma_a^2 & \text{for} \quad j = 0 \\ 0 & \text{for} \quad j > 0 \end{cases} \,,$$

$$\gamma_k = E(\dot{Z}_t \dot{Z}_{t-k}) = \sigma_a^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \,,$$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \psi_i \psi_{i+k}}{\sum_{j=0}^{\infty} \psi_j^2}.$$

Definition Autoregressive representations of Z_t:

$$\begin{aligned}\dot{Z}_t &= \pi_1 \dot{Z}_{t-1} + \pi_2 \dot{Z}_{t-2} + \cdots + a_t \\ &= \sum_{j=0}^{\infty} \pi_j \dot{Z}_{t-j} + a_t\,,\end{aligned}$$

where $\dot{Z}_t = Z_t - \mu$, $1 + \sum_{j=0}^{\infty} |\pi_j| < \infty$.

Let $\pi(B) = 1 - \sum_{j=0}^{\infty} \pi_j B^j$. Then $\pi(B)\dot{Z}_t = a_t$.

Relationship of MA and AR representations:

(1) if the root of $\pi(z) = 0$ all lie outside the unit circle, then

$$\pi(B)\dot{Z}_t = a_t \Longrightarrow \dot{Z}_t = \frac{1}{\pi(B)} a_t = \psi(B) a_t.$$

(2) if the root of $\psi(z) = 0$ all lie outside the unit circle, then

$$\dot{Z}_t = \psi(B) a_t \Longrightarrow a_t = \frac{1}{\psi(B)} \dot{Z}_t = \pi(B)\dot{Z}_t.$$

2.7 Time Series Models

Let $\{\cdots, Z_{-t}, \cdots, Z_1, Z_0, Z_1, \cdots, Z_t, \cdots\}$ be a sequence of TS r.v.

How to describe the relationship between Z_t and the past data $Z_{t-1}, Z_{t-2}, \cdots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \cdots) + a_t$$

$- - --$ is called time series models.

1. Autoregressive (AR(1)) model:

$$Z_t = \phi Z_{t-1} + a_t,$$

where ϕ is a constant and called the parameter.

2. AR(p) model:

$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$

where ϕ_i is a constant and called the parameter and p is called the order of the AR(p) model.

3. AR(∞) model:

$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t.$$

4. Moving-average (MA) model:

$$Z_t = \mu + a_t + \psi a_{t-1} + \psi_2 a_{t2} + \cdots.$$

5. ARMA model.

6. Threshold AR model (Tong 1977).

7. Long memory model (Granger (1980) and Hosking (1981)).

8. GARCH model (Engle, 1982/ Bolleslev 1986).

9 ARMA-GARCH model.

10. Vector ARMA-GARCH model.

...

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many and many models.

Given a sequence of data, none knows its true model.

However, we can find a better model for the given data.

Chapter 3 Stationary Time Series Models

3.1 **Autoregressive Processes**

3.1.1 **The first Order Autoregressive AR(1) process**

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance σ_a^2. $\dot{Z}_t$ satisfies the following equation:

$$\dot{Z}_t = \phi \dot{Z}_{t-1} + a_t.$$

$\dot{Z}_t$ is called the AR(1) model.

$$\begin{aligned} \dot{Z}_{t+1} &= \phi \dot{Z}_t + a_{t+1}\,, \\ \dot{Z}_t &= \phi \dot{Z}_{t-1} + a_t\,, \\ \dot{Z}_{t-1} &= \phi \dot{Z}_{t-2} + a_{t-1}. \end{aligned}$$

A. Expansion of AR(1) model.

$\dot{Z}_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-1} a_1 + \phi^t \dot{Z}_0.$

When $\phi = 1$,

$$\dot{Z}_t = a_t + a_{t-1} + \cdots + a_1 + \dot{Z}_0.$$

$\dot{Z}_t$ is called the random walk or unstable process.

When $|\phi| > 1$, e.g. $\phi = 3$,

$\dot{Z}_t = a_t + 3a_{t-1} + 3^2 a_{t-2} + \cdots + 3^{t-1} a_1 + 3^t \dot{Z}_0.$

$\dot{Z}_t$ is called the explosive process.

When $|\phi| < 1$, e.g. $\phi = 0.5$,

$\dot{Z}_t = a_t + 0.5a_{t-1} + \cdots + 0.5^{t-1}a_1 + 0.5^t\dot{Z}_0$.

$\dot{Z}_t$ is called stable (?).

When the time t goes very far away today, the impact of the past noises and the initial value on the current value $\dot{Z}_t$ almost disappear !!!

$$\begin{aligned} \dot{Z}_0 &= \phi\dot{Z}_{-1} + a_0\,, \\ \dot{Z}_{-1} &= \phi\dot{Z}_{-2} + a_{-1}\,, \\ \dot{Z}_{-2} &= \phi\dot{Z}_{-3} + a_{-2}. \end{aligned}$$

In general, we have the following expansion:

$$\dot{Z}_t = \sum_{i=0}^{m} \phi^i a_{t-i} + \phi^{m+1}\dot{Z}_{t-m-1}.$$

How far can the m go?

Let $S_m = \sum_{i=0}^{m} \phi^i a_{t-i}$.

Whether $\lim_{n\to\infty} S_m$ exists or not?

Definition: If

$$E(\xi_m - \xi)^2 \to 0 \quad \text{as} \quad m \to \infty.$$

we say that the sequence ξ_m of random variables converges to the random variable ξ in mean square.

We can prove that

$$S_m \to \sum_{i=0}^{\infty} \phi^i a_{t-i} \quad \text{in mean square.}$$

if and only if $|\phi| < 1$.

The second term $\phi^{m+1}\dot{Z}_{t-m} \to 0$ (??).

Thus, we have the following result:

If and only if $|\phi| < 1$, $\dot{Z}_t$ in the AR(1) model has the following expansion:

$$\dot{Z}_t = \sum_{i=0}^{\infty} \phi^i a_{t-i}\,,$$

where the infinite sum converges in mean square.

B. ACF of the AR(1) Process.

When $|\phi| < 1$,

$$\mu = E\dot{Z}_t = E\left(\sum_{i=0}^{\infty} \phi^i a_{t-i}\right) = 0,$$

$$\sigma^2 = \textbf{Var}(\dot{Z}_t) = \frac{\sigma_a^2}{1-\phi^2},$$

$$\gamma_k = E[(\dot{Z}_t - \mu)(\dot{Z}_{t+k} - \mu)] = \frac{\sigma_a^2 \phi^k}{1-\phi^2},$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k.$$

Thus, in this case, $\dot{Z}_t$ is stationary.

When $|\phi| \geq 1$, $\dot{Z}_t$ is not stationary.

C. Partial Autocorrelation function (PACF) of AR(1) Process

$$\phi_{kk} = \begin{cases} \rho_1 = \phi, & k = 1, \\ 0, & k \geq 2. \end{cases}$$

The AR(1) model can be written as:

$$(1 - \phi B)\dot{Z}_t = a_t\,,$$

Example 3.1 Simulated 250 values from the model:

$$(1 - \phi B)(Z_t - 10) = a_t,$$

where $\phi = 0.9$ and $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

Example 3.2 Simulated 250 values from the model:

$$(1 - \phi B)(Z_t - 10) = a_t,$$

where $\phi = -0.65$ and $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

3.1.2 **The Second Order Autoregressive AR(2) Model**

A. Model

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + a_t$$
$$\text{or} \quad \phi(B)\dot{Z}_t = a_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$.

B. Condition for stationarity:

the roots of $\phi(z) = 0$ **lie outside the unit circle**, or equivalently,

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$

all the roots of $(1 - \phi_1 z - \phi_2 z^2) = 0$ lie outside the unit circle.

Decompose $1 - \phi_1 z - \phi_2 z^2 = (1 - \alpha_1 z)(1 - \alpha_2 z)$.

Then $|\alpha_1| < 1$ and $|\alpha_2| < 1$.

$$(1 - \alpha_1 B)(1 - \alpha_2 B)\dot{Z}_t = a_t.$$

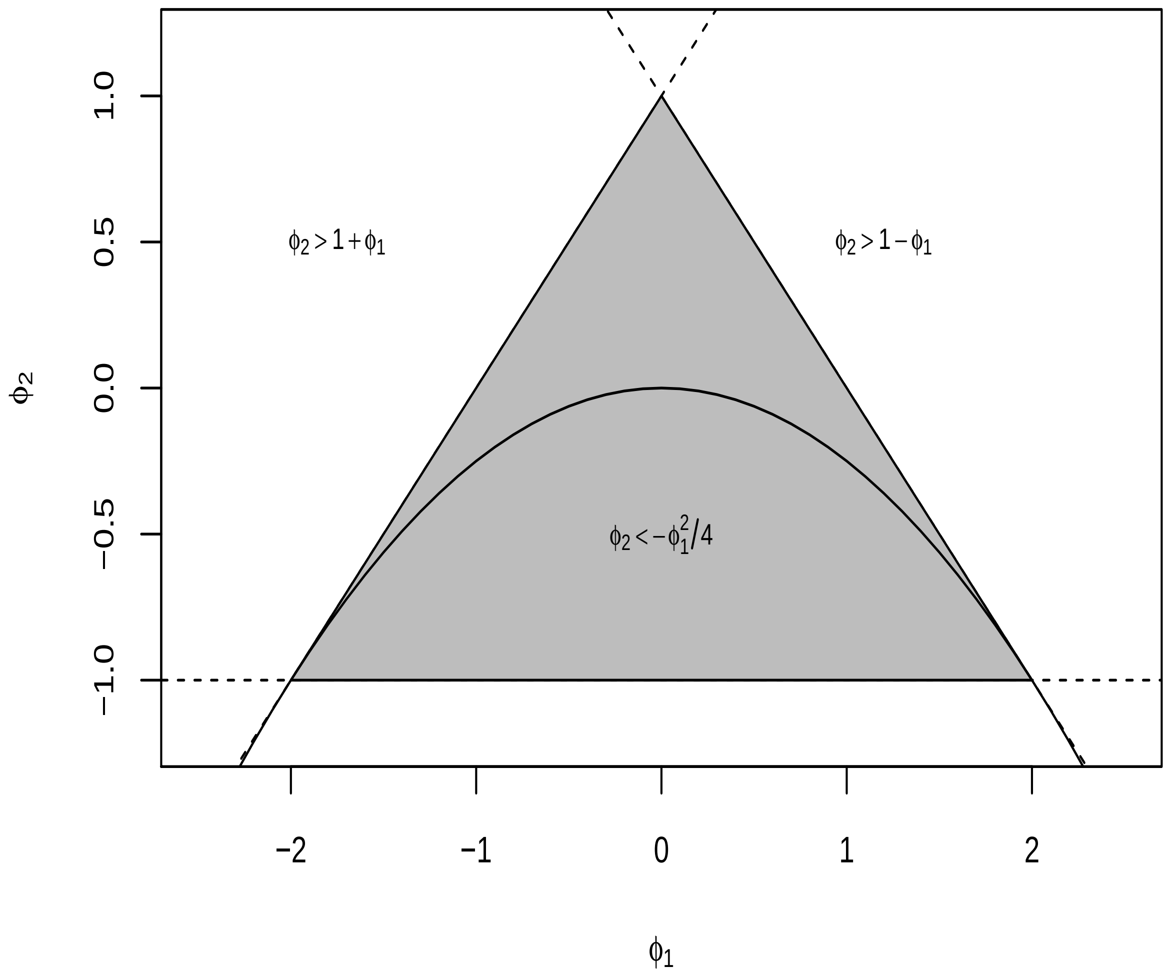
Let $u_t = (1 - \alpha_2 B)\dot{Z}_t$. Then

$$u_t = \alpha_1 u_{t-1} + a_t = a_t + \sum_{i=1}^{\infty} \alpha_1^i a_{t-i}.$$

$$\dot{Z}_t = \alpha_2 \dot{Z}_{t-1} + u_t = u_t + \sum_{j=1}^{\infty} \alpha_2^j u_{t-j}.$$

Stationarity condition is equivalent to

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$



Stationary region for AR(2) model.

C. ACF of the AR(2) model:

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}, \quad k \geq 1.$$
$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \quad k \geq 1.$$

When $k = 1, 2$,

$$\rho_1 = \frac{\phi_1}{1-\phi_2} \quad \text{and} \quad \rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2}.$$

D. PACF of the AR(2) model:

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1-\phi_2},$$
$$\phi_{22} = \phi_2,$$
$$\phi_{kk} = 0, \quad \text{as} \quad k \geq 3.$$

Example 3.3 Simulated 250 values from the AR(2) model:

$$(1 - B + 0.5B^2)Z_t = a_t,$$

where $a_t \sim N(0, 1)$. Show the sample ACF and PACF.

3.1.3. The General pth Order Autoregressive AR(p) Model

A. Model:

Let $\{a_t\}$ be a sequence of white noise with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be an AR(p) model, if

$$\begin{aligned} &\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t \\ &\text{or} \quad \phi_p(B)\dot{Z}_t = a_t, \end{aligned}$$

where p is an positive integer, and $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$.

B. Condition for Stationarity:

the roots of $\phi_p(z) = 0$ lie outside the unit circle, or equivalently,

all the eigenvalues of the following matrix lie outside the unit circle,

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

C. ACF of AR(p) model:

$$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}, \quad k > 0.$$

$$\rho_k = \phi_1 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}, \quad k > 0.$$

$---$ the difference equation of ρ_k.

Solve the following sets of equations:

$$\begin{cases} \rho_1 - \phi_1 \rho_0 - \cdots + \phi_p \rho_{p-1} = 0, \\ \cdots\cdots \\ \rho_p - \phi_1 \rho_{p-1} - \cdots + \phi_p \rho_0 = 0. \end{cases}$$

$---$ find $\rho_1, \cdots, \rho_p$.

When $k \geq p+1$, calculate $\rho_{p+1}, \rho_{p+2}, \cdots$ by:

$$\rho_{p+1} - \phi_1 \rho_p - \cdots - \phi_p \rho_1 = 0,$$
$$\cdots\cdots$$
$$\rho_k - \phi_1 \rho_{k-1} - \cdots - \phi_p \rho_{k-p} = 0.$$

D. PACF of AR(p) Model:

ψ_{kk} can be obtained from $\rho_1, \cdots, \rho_k$.

In particular, $\psi_{kk} = 0$ when $k > p$.