MATH4425 (T1A) – Tutorial 3

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Important information

- T1A: **Thursday 19:00 - 19:50** (Rm 1033, LSK Bldg)
- Office hours: **Wednesday 14:00 - 14:50** (Math support center, 3rd floor, Lift 3)
- Any questions to be addressed to **akazovskaia@connect.ust.hk**

1 Time series models

Let $\ldots, Z_{-t}, \ldots, Z_{-1}, Z_0, Z_1, \ldots, Z_t, \ldots$ be a sequence of TS r.v.

How to describe the relationship between Z_t and the past data $Z_{t-1}, Z_{t-2}, \ldots$?

$$Z_t = f(Z_{t-1}, Z_{t-2}, \ldots) + a_t$$

It is called the **time series model**.

1. Autoregressive (AR(1)) model:
$$Z_t = \phi Z_{t-1} + a_t,$$
where ϕ is a constant and called the **parameter**

2. AR(p) model:
$$Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t,$$
where ϕ_p is a constant and called the **parameter**, and p is called the **order** of the AR(p) model

3. $AR(\infty)$ model:
$$Z_t = \sum_{i=1}^{\infty} \phi_i Z_{t-i} + a_t$$

4. Moving-average (MA) model:
$$Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots.$$

5. ARMA model

2 Stationary time series models. Moving-average processes

2.1 Special case of AR(∞)

$$\dot{Z}_t = \sum_{i=1}^{\infty} \phi_i \dot{Z}_{t-i} + a_t$$

is called **AR(∞) process**.

A special case: when $\phi_k = -\theta^k$ and $|\theta| < 1$,

$$\dot{Z}_t = \sum_{i=1}^{\infty} (-\theta^i) \dot{Z}_{t-i} + a_t$$

$$\left(1 + \sum_{i=1}^{\infty} \theta^i B^i\right) \dot{Z}_t = a_t$$

$$(1 - \theta B)^{-1} \dot{Z}_t = a_t \Rightarrow$$

$$\dot{Z}_t = (1 - \theta B) a_t = a_t - \theta a_{t-1}$$

2.2 The first order moving-average MA(1) process

2.2.1 Model

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a **MA(1) process**, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1}.$$

Notation: $\dot{Z}_t = (1 - \theta_1 B) a_t$.

2.2.2 Properties

If $|\theta_1| < 1$, then a_t can be written as

$$a_t = \dot{Z}_t + \sum_{i=1}^{\infty} \theta_1^i \dot{Z}_{t-i}.$$

Definition: Given $\dot{Z}_t, \dot{Z}_{t-1}, \dots$, if we can calculate a_t, then the process is said to be **invertible**.

By the definition, AR processes are *invertible*:

$$\dot{Z}_t = \phi_1 Z_{t-1} + \phi_2 \dot{Z}_{t-2} + \cdots + \phi_p \dot{Z}_{t-p} + a_t$$

MA(1) process is *invertible* if $|\theta_1| < 1$.

2.2.3 ACF of MA(1) process

Actually, we already know that MA process is stationary.

Some properties of general MA process $\boldsymbol{Z_t = \mu + \sum_{j=0}^{\infty} \theta_j a_{t-j}}$:

$$\mathbb{E} Z_t = \mu < \infty$$

$$\operatorname{var}(Z_t) = \sigma_a^2 \sum_{j=0}^{\infty} \theta_j^2 < \infty$$

$$\mathbb{E}(a_t Z_{t-j}) = \begin{cases} \sigma_a^2, & \text{if } j = 0 \\ 0, & \text{if } j > 0 \end{cases}$$

$$\gamma_k = \mathbb{E}(\dot{Z}_t \dot{Z}_{t-k}) = \sigma_a^2 \sum_{i=0}^{\infty} \theta_i \theta_{i+k}$$

$$\rho_k = \frac{\sum_{i=0}^{\infty} \theta_k \theta_{i+k}}{\sum_{j=0}^{\infty} \theta_j^2}.$$

Let's calculate those quantities directly and compare the results:

$$\mu_t = \mathbb{E}\dot{Z}_t = \mathbb{E}a_t - \theta_1 \mathbb{E}a_{t-1} = 0 = \mu < \infty$$

$$\sigma_t^2 = \mathbb{E}\dot{Z}_t^2 = \mathbb{E}(a_t^2 - 2\theta_1 a_t a_{t-1} + \theta_1^2 a_{t-1}^2) = \sigma_a^2(1 + \theta_1^2) = \sigma^2 < \infty$$

$$\gamma(t, t+1) = \mathbb{E}\dot{Z}_t \dot{Z}_{t+1} = \mathbb{E}(a_t - \theta_1 a_{t-1})(a_{t+1} - \theta_1 a_t) = -\theta_1 \sigma_a^2 = \gamma_1$$

$$\gamma(t, t+k) = \mathbb{E}\dot{Z}_t \dot{Z}_{t+k} = \mathbb{E}(a_t - \theta_1 a_{t-1})(a_{t+k} - \theta_1 a_{t+k-1}) = 0 = \gamma_k, \quad \text{if} \quad k > 1$$

$$\rho_1 = -\frac{\theta_1}{1 + \theta_1^2}$$

$$\rho_k = 0, \quad \text{if} \quad k > 1$$

MA(1) is **always stationary**.

2.2.4 PACF of MA(1) process

$$\phi_{11} = \rho_1 = \frac{-\theta_1(1 - \theta_1^2)}{1 - \theta_1^4}$$

$$\phi_{22} = \frac{-\theta_1^2(1 - \theta_1^2)}{1 - \theta_1^6}$$

$$\phi_{kk} = \frac{-\theta_1^k(1 - \theta_1^2)}{1 - \theta^{2(k+1)}}, \qquad k > 1$$

ϕ_{kk} goes down to 0, **exponentially**.

Remark: In notes and SAS, MA(1) process is defined as

$$\dot{Z}_t = a_t - \theta_1 a_{t-1},$$

while in R or RStudio

$$\dot{Z}_t = a_t + \theta_1 a_{t-1}$$

2.3 The second order moving-average MA(2) process

2.3.1 Model

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a **MA(2) process**, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}.$$

Notation: $\dot{Z}_t = \theta(B) a_t$, where $\theta(B) = 1 - \theta_1 B - \theta_2 B^2$.

2.3.2 Condition for invertibility

All roots of $\theta(z) = 0$ lie outside the unit circle, or equivalently,

$$\begin{cases} \theta_2 + \theta_1 < 1, \\ \theta_2 - \theta_1 < 1, \\ -1 < \theta_2 < 1 \end{cases}$$

Compare this condition with *stationarity condition* for AR(2) process:

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \phi_2 \dot{Z}_{t-2} + a_t = \phi(B)\dot{Z}_t = a_t,$$

where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$.

Condition for stationarity of AR(2) process: all roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle, or equivalently,

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1 \end{cases}$$

2.3.3 ACF of MA(2) process

Similar to MA(1), we can use direct calculations to get the following results:

$$\mu = 0 < \infty$$

$$\sigma^2 = \mathbb{E}\dot{Z}_t^2 = \sigma_a^2(1 + \theta_1^2 + \theta_2^2) < \infty$$

$$\gamma_1 = -\theta_1(1 - \theta_2)\sigma_a^2$$

$$\gamma_2 = -\theta_2 \sigma_a^2$$

$$\gamma_k = 0, \quad k > 2$$

$$\rho_1 = \frac{-\theta_1(1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_k = 0, \quad k > 2$$

MA(2) process is **always stationary**.

2.3.4 PACF of MA(2) process

$$\phi_{11} = \rho_1$$

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

$$\cdots$$

$\phi_{kk} = O(\rho^k)$, where $|\rho| < 1$, i.e. ϕ_{kk} goes down to zero, **exponentially**.

2.4 The general q-th order moving-average MA(q) process

2.4.1 Model

Let $\{a_t\}$ be a sequence of white noises with mean 0 and variance σ_a^2.

$\dot{Z}_t$ is said to be a **MA(q) process**, if

$$\dot{Z}_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q},$$

where q is a positive integer.

In this case, q is called **the order** or **lag** of the process.

Notation: $\dot{Z}_t = \theta_q(B)a_t$, where $\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$.

2.4.2 Condition for invertibility

All roots of $\theta_q(z) = 0$ lie outside the unit circle,

or equivalently,

all the eigenvalues of the following matrix lie outside the unit circle,

$$\begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_q \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

2.4.3 ACF of MA(q) process

Similar to MA(1) and MA(2), we can use direct calculations to get the following results:

$$\mu = 0 < \infty$$

$$\sigma^2 = \sigma_a^2(1 + \theta_1^2 + \cdots + \theta_q^2) < \infty$$

$$\gamma_k = \sigma_a^2(-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q), \quad k = 1, \ldots, q$$

$$\gamma_k = 0, \quad k > q$$

$$\rho_k = \frac{-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2}, \quad k = 1, \ldots, q$$

$$\rho_k = 0, \quad k > q$$

MA(q) process is **always stationary**.

2.4.4 PACF of MA(q) process

As usual, ϕ_{kk} can be obtained from $\rho_1, \rho_2, \ldots, \rho_k$. Moreover, $\phi_{kk} = O(\rho^k)$, where $|\rho| < 1$, i.e. ϕ_{kk} goes down to zero, **exponentially**.

3 The dual relationship between AR(p) and MA(q) processes

3.1 Basic lemma

Let $f(z) = 1 + b_1 z + b_2 z^2 + \cdots + b_s z^s$. If all roots of $f(z) = 0$ lie outside the unit circle, the $f^{-1}(B)$ has the following expansion

$$f^{-1}(B) = (1 + b_1 B + b_2 B^2 + \cdots + b_s B^s)^{-1} = \sum_{i=0}^{\infty} c_i B^i,$$

where $c_0 = 1$ and $c_i = O(h^i)$ with $|h| < 1$.

3.2 AR(p) process

Let $\phi_p(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$,

$$\phi_p(B)\dot{Z}_t = a_t$$

If all the roots of $\phi_p(z) = 0$ lie outside the unit circle, then

$$\phi_p^{-1}(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i \Longrightarrow$$

$$\dot{Z}_t = \phi_p^{-1}(B)a_t = \left(1 + \sum_{i=1}^{\infty} \psi_i B^i\right) a_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \ldots,$$

where $\psi_i = O(h^i)$ with $|h| < 1$.

Remark: A *stationary* AR(p) process has an MA(∞) expansion. This expansion tells us that, originally, $\dot{Z}_t$ comes from a sequence of white noises.

3.3 MA(q) process

Let $\theta_q(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$,

$$\dot{Z}_t = \theta(B)a_t$$

If all roots of $\theta_q(z) = 0$ lie outside the unit circle, then

$$\theta^{-1}(B) = 1 - \sum_{i=1} \pi_i B^i \Longrightarrow$$

$$a_t = \theta^{-1}(B)\dot{Z}_t = \left(1 - \sum_{i=1}^{\infty} \pi_i B^i\right) \dot{Z}_t = \dot{Z}_t - \pi_1 \dot{Z}_{t-1} - \pi_2 \dot{Z}_{t-2} - \ldots,$$

where $\pi_i = O(h^i)$ with $|h| < 1$, i.e.

$$\dot{Z}_t = \sum_{i=1}^{\infty} \pi_i \dot{Z}_{t-i} + a_t.$$

Remark: An *invertible* MA(q) process is a special AR(∞) process.

4 Problems

4.1 Problem 1

a) Show that the ACF ρ_k for the AR(1) process satisfies the difference equation

$$\rho_k - \phi_1 \rho_{k-1} = 0 \quad \text{for} \quad k \geq 1$$

b) Find the general expression for ρ_k

4.2 Problem 2

a) Find the range of α such that the AR(2) process

$$Z_t = Z_{t-1} + \alpha Z_{t-2} + a_t$$

is stationary

b) Find the PACF for the process in a) with $\alpha = -\frac{1}{2}$

4.3 Problem 3

Consider the MA(2) process $Z_t = a_t - 0.1a_{t-1} + 0.21a_{t-2}$.

a) Is the process stationary? Why?

b) Is the process invertible? Why?

c) Find the ACF for the above process

5 Solutions

5.1 Solution 1

a) Direct calculation leads us to

$$\gamma_k = \mathbb{E}\dot{Z}_t\dot{Z}_{t+k} = \mathbb{E}\dot{Z}_t(\phi_1\dot{Z}_{t+k-1} + a_{t+k}) = \phi_1\gamma_{k-1} + 0 \Longrightarrow$$

$$\rho_k = \phi_1\rho_{k-1}$$

b) Two approaches:

1) Direct calculation using an expansion $\dot{Z}_{t+k} = \sum_{i=0}^{k-1}\phi^i a_{t+k-i} + \phi^{k-1+1}\dot{Z}_{t+k-(k-1)-1}$ (check *ACF of AR(1) process* section in *MATH4425 – Tutorial 2*)
2) Derivation using recurrence obtained in a)

5.2 Solution 2

a) According to our notation

$$\phi(z) = 1 - z - \alpha z^2$$

All roots of $\phi(z) = 0$ lie outside the unit circle is equivalent to

$$\begin{cases} \alpha + 1 < 1, \\ \alpha - 1 < 1, \\ -1 < \alpha < 1 \end{cases}$$

That is, $-1 < \alpha < 0$.

b) First, notice $-1 < -\frac{1}{2} < 0 \implies$ the process is stationary. Then, according to *PACF of AR(2) process* section in *MATH4425 – Tutorial 2*,

$$\phi_{11} = \rho_1 = \frac{\phi_1}{1-\phi_2} = \frac{1}{1-\frac{-1}{2}} = \frac{2}{3}$$

$$\phi_{22} = \phi_2 = -\frac{1}{2}$$

$$\phi_{kk} = 0 \text{ as } \quad k \geq 3$$

5.3 Solution 3

a) MA(2) process is always stationary.

b) According to our notation

$$\theta(z) = 1 - 0.1z + 0.21z^2$$

All roots of $\theta(z) = 0$ lie outside the unit circle is equivalent to

$$\begin{cases} -0.21 + 0.1 = -0.11 < 1, \\ -0.21 - 0.1 = -0.31 < 1, \\ -1 < -0.21 < 1 \end{cases}$$

That is, the process is invertible.

c) According to *ACF of MA(2)* section,

$$\rho_1 = \frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2} = \frac{-0.1(1+0.21)}{1+0.1^2+0.21^2} = \frac{-0.121}{1.0541} = -0.11478986813$$

$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} = \frac{0.21}{1+0.1^2+0.21^2} = \frac{0.21}{1.0541} = 0.19922208519$$

$$\rho_k = 0, \quad k > 2$$