Chapter 5 Forecasting

5.1. **Introduction**

Objective of this Chapter:

Given you a sequence of data: $Z_1, Z_2, \cdots, Z_n$, ARMA or ARIMA model,

you can forecast $Z_{t+1}, \cdots, Z_{n+l}$ and give its forecasting interval.

5.2. **Minimum Mean Square Error Forecasts for ARMA models**

Let Z_t be a stationary and invertible ARMA model,

$$\phi(B)Z_t = \theta(B)a_t\,.$$

Given the observations: $Z_n, Z_{n-1}, \cdots,$

how to forecast $Z_{n+1}, \cdots, Z_{n+l}, \cdots$?

Notation:

$\widehat{Z}_n(l)$ denotes the forecast value of Z_{n+l} and is called the l-step ahead of the forecast of Z_{n+l} at the forecast origin n.

Simply say, l-step forecasting.

Forecasting function:

$$\widehat{Z}_n(l) = g(Z_n, Z_{n-1}, \cdots).$$
$$\widehat{Z}_n(l) = g(a_n, a_{n-1}, \cdots).$$

Linear Predictors (LP):

$$\widehat{Z}_n(l) = \psi_l^* a_n + \psi_{l+1}^* a_{n-1} + \psi_{l+2}^* a_{n-2} + \cdots,$$

where ψ_j^* are to be determined.

Criterion of the best LP (BLP):

$\widehat{Z}_n(l)$ is said to be a BLP

if $E[Z_{n+l} - \widehat{Z}_n(l)]^2$ is the smallest among all the LP.

What is the BLP of Z_{n+l}?

Note that

$$Z_{n+l} = \frac{\theta(B)}{\phi(B)} a_{n+l} = \sum_{j=0}^{\infty} \psi_j a_{n+l-j}.$$

According to the above criterion, the BLP is that

$$\widehat{Z}_n(l) = \psi_l a_n + \psi_{l+1} a_{n-1} + \psi_{l+2} a_{n-2} + \cdots .(?)$$

General Result:

$$\mu_n = \widehat{Z}_n(l) = E(Z_{n+l}|Z_n, Z_{n-1}, \cdots).$$

Let $F_n = \{Z_n, Z_{n-1}, \cdots\}$ and g_n be any function of F_n.

Math proof:

$$\begin{aligned}
E(Z_{n+l} - g_n)^2 &= E[Z_{n+l} - \mu_n + \mu_n - g_n)]^2 \\
&= E(Z_{n+l} - \mu_n)^2 + E(\mu_n - g_n)^2 \\
&\quad -2E[(Z_{n+l} - \mu_n)(\mu_n - g_n)] \\
&= E(Z_{n+l} - \mu_n)^2 + E(\mu_n - g_n)^2 \\
&> E(Z_{n+l} - \mu_n)^2 \; if \; g_n \neq \mu_n.
\end{aligned}$$

Forecasting error:

$$e_n(l) = Z_{n+l} - \widehat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}.$$

Forecasting variance:

$$\mathsf{Var}[e_n(l)] = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2.$$

Forecast interval (limit) (FI):

$$\left[\widehat{Z}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1} \psi_j^2},\ \widehat{Z}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1} \psi_j^2}\right]$$

where $N_{\frac{\alpha}{2}}$ is the $\alpha/2$-quantile of the standard normal distribution,

i.e., $P(N > N_{\frac{\alpha}{2}}) = \alpha/2$.

When $\alpha = 0.05$, $N_{\frac{\alpha}{2}} = 1.96$.

Example 5.1. Consider AR(1) model

$$Z_t = 0.8Z_{t-1} + a_t.$$

where $a_t \sim N(0,1)$. Given $Z_{100} = 0.75$, find the $\widehat{Z}_{100}(3)$ and 95% FI.

Example 5.2. Consider AR(1) model

$$Z_t = 0.4 + 0.8Z_{t-1} + a_t.$$

where $a_t \sim N(0,1)$. Given $Z_{100} = 2.75$, find the $\widehat{Z}_{100}(3)$ and 95% FI.

Example 5.3. Consider AR(2) model

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t,$$

where the roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ lie outside the unit circle and $a_t \sim N(0,1)$. Given $Z_1, Z_2, \cdots, Z_n$, find the $\widehat{Z}_n(1)$, $\widehat{Z}_n(2)$ and $\widehat{Z}_n(3)$.

Example 5.4. Consider MA(1) model

$$Z_t = -0.5a_{t-1} + a_t.$$

where $a_t \sim N(0,1)$. Given $Z_{100} = 0.3$ and $Z_{99} = -0.2$, find $\widehat{Z}_{100}(1)$, $\widehat{Z}_{100}(2)$, and 95% FI.

Example 5.5. Consider ARMA(1, 1) model

$$(1 - \phi B)Z_t = (1 - \theta B)a_t,$$

where $|\phi| < 1$, $|\theta| < 1$ and $a_t \sim N(0,1)$. Given Z_n, $Z_{n-1}, \cdots$, find $\widehat{Z}_n(1)$, $\widehat{Z}_n(2)$, $\widehat{Z}_n(l)$, $\mathrm{Var}[e_n(l)]$ and 95% FI.

The Formulas of Computation of Forecasts For ARMA Model:

$$\begin{aligned}\widehat{Z}_n(l) &= \phi_1 \widehat{Z}_n(l-1) + \phi_2 \widehat{Z}_n(l-2) + \\ &\quad \cdots + \phi_p \widehat{Z}_n(l-p) \\ &\quad + \widehat{a}_n(l) - \theta_1 \widehat{a}_n(l-1) - \cdots - \theta_q \widehat{a}_n(l-q).\end{aligned}$$

where

$$\widehat{Z}_n(j) = \begin{cases} E(Z_{n+j}|Z_n, \cdots) & \text{if} \quad j = 1, 2, \cdots, l. \\ Z_{n+j} & \text{if} \quad j = 0, -1, -2, \cdots. \end{cases}$$

$$\widehat{a}_n(j) = \begin{cases} 0 & \text{if} \quad j = 1, 2, \cdots, l. \\ a_{n+j} & \text{if} \quad j = 0, -1, -2, \cdots. \end{cases}$$

5.3 **Minimum Mean Square Error Forecasts for ARIMA models**

A. Model:

Let Z_t be ARIMA(p, d, q) model with $d \neq 0$,

$$\phi(B)(1-B)^d Z_t = \theta(B) a_t.$$

where all the roots of $\phi(z) = 0$ and $\theta(z) = 0$ lie outside the unit circle.

Given the observations: $Z_n, Z_{n-1}, \cdots,$

how to forecast $Z_{n+1}, \cdots, Z_{n+l}, \cdots$?

B. **Minimum Mean Square Error Forecasts**:

$$\widehat{Z}_n(l) = E(Z_{n+l} | Z_n, Z_{n-1}, \cdots).$$

C. **Computation of forecast:**

Denote

$$\begin{aligned}
\Psi(B) &= \phi(B)(1-B)^d \\
&= 1 - \Psi_1 B - \Psi_2 B^2 - \cdots - \Psi_{p+d} B^{p+d}. \\
Z_t &= \Psi_1 Z_{t-1} + \Psi_2 Z_{t-2} + \cdots + \Psi_{p+d} Z_{t-p-d} \\
&\quad + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}.
\end{aligned}$$

Formulas:

$$\begin{aligned}\widehat{Z}_n(l) &= \Psi_1 \widehat{Z}_n(l-1) + \cdots + \Psi_{p+d}\widehat{Z}_n(l-p-d) \\ &\quad + \widehat{a}_n(l) - \theta_1 \widehat{a}_n(l-1) - \cdots - \theta_q \widehat{a}_n(l-q).\end{aligned}$$

where

$$\widehat{Z}_n(j) = \begin{cases} E(Z_{n+j}|Z_n, \cdots) & \text{if} \quad j = 1, 2, \cdots, l. \\ Z_{n+j} & \text{if} \quad j = 0, -1, -2, \cdots. \end{cases}$$

$$\widehat{a}_n(j) = \begin{cases} 0 & \text{if } j = 1, 2, \cdots, l. \\ a_{n+j} & \text{if } j = 0, -1, -2, \cdots. \end{cases}$$

D. Forecast error:

$$e_n(l) = Z_{n+l} - \widehat{Z}_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}\,,$$

where ψ_i can be calculated, recursively:

$$\psi_j = \sum_{i=0}^{j-1} \pi_{j-i}\psi_i\,, \qquad j = 1, 2, \cdots, l-1.$$

π_j is the coefficients of the expansion:

$$\pi(B) = \frac{\phi(B)(1-B)^d}{\theta(B)} = 1 - \sum_{j=1}^{\infty} \pi_j B^j.$$

$$Z_{t+l} = \sum_{j=1}^{\infty} \pi_j Z_{t+l-j} + a_{t+l}.$$

E. Forecast variance:

$$\text{Var}[e_n(l)] = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2.$$

F. Forecast interval (limit) (FI):

$$\left[\widehat{Z}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2},\ \widehat{Z}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right]$$

where $N_{\frac{\alpha}{2}}$ is the $\alpha/2$-quantile of the standard normal distribution,

Example 5.7. Consider ARIMA(0, 1, 1) model:

$$(1 - B)Z_t = (1 - \theta B)a_t.$$

Example 5.8. Consider ARIMA(1, 1, 1) model:

$$(1 - \phi B)(1 - B)Z_t = (1 - \theta B)a_t.$$

5.4 Omitted

5.5 Updating Forecasts

$$Z_n, Z_{n-1}, \cdots.$$

ARMA or **ARIMA**model.

$\widehat{Z}_n(l)$ ——— the forecast value of Z_{n+l}.

Question:

If Z_{n+1} is available, how to forecast Z_{n+l}?

Method I:

$$Z_{n+1}, Z_n, Z_{n-1}, \cdots.$$

the forecast value of Z_{n+l} is $\widehat{Z}_{n+1}(l-1)$, now.

Method II:

Updating the forecast. How to do that?

Relationship of $\widehat{Z}_{n+1}(l-1)$ and $\widehat{Z}_n(l)$:

$$\widehat{Z}_{n+1}(l-1) = \widehat{Z}_n(l) + \psi_{l-1}[Z_{n+1} - \widehat{Z}_n(1)].$$

5.6 Omitted

5.7 A Numerical Example.

Consider AR(1) model:

$$(1-0.6B)(Z_t-9)=a_t,$$

where $a_t \sim N(0, 0.1)$.

Given $Z_{100}=8.9$, $Z_{99}=9$, $Z_{98}=9$, $Z_{97}=9.6$.

(1). Find $Z_{101}, Z_{102}, Z_{103}$ and Z_{104} with their 95% FI.

(2). Given $Z_{101}=8.8$, update the forecasts for Z_{102}, Z_{103} and Z_{104}.

5.8 Some practical forecasts

Assume we have real data: $y_n, y_{n-1}, \cdots$.

Let $Z_t = \ln y_t$, we have data: $Z_n, Z_{n-1}, \cdots$.

If Z_t ia an **ARMA** or **ARIMA** model, then we can forecast Z_{n+l}.

Denote the forecast value of Z_{n+l} by $\widehat{Z}_n(l)$ and the forecast FI by

$$\left[\widehat{Z}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2},\ \widehat{Z}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right]$$

Then the forecast value of y_{n+l} is

$$e^{\widehat{Z}_n(l)+var(e_n(l))/2}$$

and FI is

$$\left[e^{\left\{\widehat{Z}_n(l)-N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right\}},\ e^{\left\{\widehat{Z}_n(l)+N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right\}}\right]$$

Similarly, if $Z_t = \sqrt{y_t}$, then the forecast value and FI of y_{n+l} are, respectively,

$$\widehat{Z}_n^2(l) + var(e_n(l))$$

and

$$\Big[\{\widehat{Z}_n(l) - N_{\frac{\alpha}{2}}\,\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\}^2, \{\widehat{Z}_n(l) + N_{\frac{\alpha}{2}}\,\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\}^2\Big].$$