MATH 4824C: Causal Inference

♠ Solution of HW 2

Q1. Assume the covariates $X_i \in \{1, \ldots, K\}$, and for $z = 0, 1$ and $k = 1, \ldots, K$ define

$$\bar{Y}_{k,z} = \frac{\sum_{X_i = k \wedge Z_i = z} Y_i}{\sum_{i=1}^n \mathbf{1}\{X_i = k \wedge Z_i = z\}}, \quad \bar{Y}_z = \frac{\sum_{Z_i = z} Y_i}{\sum_{i=1}^n \mathbf{1}\{Z_i = z\}}$$

and $n_k = \sum_{i=1}^n \mathbf{1}\{X_i = k\}$ and $n_{k,z} = \sum_{i=1}^n \mathbf{1}\{X_i = k \wedge Z_i = z\}$. Identify the condition that the stratified estimator

$$\widehat{ACE}_{\text{block}} = \sum_{k=1}^{k=K} \frac{n_k}{n} \widehat{ACE}_k$$

is equal to $\widehat{ACE} = \bar{Y}_1 - \bar{Y}_0$ where $\widehat{ACE}_k = \bar{Y}_{k,1} - \bar{Y}_{k,0}$.

Answer
Observe that

$$\begin{aligned}
\widehat{ACE}_{\text{block}} &= \sum_{k=1}^{K} \frac{n_k}{n} \left(\frac{\sum_{X_i = k \wedge Z_i = 1} Y_i}{\sum_{i=1}^n \mathbf{1}\{X_i = k \wedge Z_i = 1\}} - \frac{\sum_{X_i = k \wedge Z_i = 0} Y_i}{\sum_{i=1}^n \mathbf{1}\{X_i = k \wedge Z_i = 0\}} \right) \\
&= \sum_{k=1}^{K} \frac{n_k}{n} \left(\frac{\sum_{X_i = k \wedge Z_i = 1} Y_i}{n_{k,1}} - \frac{\sum_{X_i = k \wedge Z_i = 0} Y_i}{n_{k,0}} \right)
\end{aligned}$$

and

$$\widehat{ACE} = \sum_{k=1}^{K} \left(\frac{\sum_{X_i = k \wedge Z_i = 1} Y_i}{n_1} - \frac{\sum_{X_i = k \wedge Z_i = 0} Y_i}{n_0} \right).$$

Therefore, one sufficient condition that makes $\widehat{ACE}_{\text{block}} = \widehat{ACE}$ is that

$$n_1 = \frac{n \cdot n_{k,1}}{n_k}, \quad n_0 = \frac{n \cdot n_{k,0}}{n_k}$$

holds for all $k = 1, \ldots, K$. In other words, the ratio of treatment and control should be the same across different stratifications.

Q2. Consider the setting in page 16 in Chapter 2 of lecture notes, show that

$$\arg\min_{\gamma} := \text{Var}\left(\frac{n_1}{n} \left(\bar{Y}_1 - \gamma^\top \bar{\mathbf{X}}_1 \right) - \frac{n_0}{n} \left(\bar{Y}_0 - \gamma^\top \bar{\mathbf{X}}_0 \right) \right)$$

is equivalent to $\arg\min_{\alpha,\gamma} E(Y - \alpha - \gamma X)^2$

Answer

Proof. Observa that

$$
\begin{aligned}
&\operatorname{Var}\left(\frac{n_1}{n}\left(\bar{Y}_1-\gamma^{\top}\bar{\mathbf{X}}_1\right)-\frac{n-n_1}{n}\left(\bar{Y}_0-\gamma^{\top}\bar{\mathbf{X}}_0\right)\right)\\
=&E\left(\operatorname{Var}\left(\frac{n_1}{n}\left(\bar{Y}_1-\gamma^{\top}\bar{\mathbf{X}}_1\right)-\frac{n-n_1}{n}\left(\bar{Y}_0-\gamma^{\top}\bar{\mathbf{X}}_0\right)\middle|\sum_{i=1}^{n}Z_i=n_1\right)\right)\\
&+\operatorname{Var}\left(E\left(\frac{n_1}{n}\left(\bar{Y}_1-\gamma^{\top}\bar{\mathbf{X}}_1\right)-\frac{n-n_1}{n}\left(\bar{Y}_0-\gamma^{\top}\bar{\mathbf{X}}_0\right)\middle|\sum_{i=1}^{n}Z_i=n_1\right)\right)\\
=&\frac{1}{n}E\left(\frac{n_1}{n}\operatorname{Var}\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)+\frac{n-n_1}{n}\operatorname{Var}\left(Y_0-\gamma^{\top}\mathbf{X}_0\right)\right)\\
&+\operatorname{Var}\left(\frac{n_1}{n}E(Y_1)-\frac{n-n_1}{n}E(Y_0)\right)\\
=&n^{-1}\color{red}{P(Z=1)\operatorname{Var}\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)+P(Z=0)\operatorname{Var}\left(Y_0-\gamma^{\top}\mathbf{X}_0\right)}\\
&+\operatorname{Var}\left(\frac{n_1}{n}E(Y_1)-\frac{n-n_1}{n}E(Y_0)\right)
\end{aligned}
$$

For $\arg\min_{\alpha,\gamma} E(Y-\alpha-\gamma^{\top}\mathbf{X})^2$, we have

$$
\begin{aligned}
&\min_{\alpha,\gamma} E(Y-\alpha-\gamma^{\top}\mathbf{X})^2\\
=&\min_{\gamma}\left\{\min_{\alpha} E(Y-\alpha-\gamma^{\top}\mathbf{X})^2\right\}\\
=&\min_{\gamma} E\left(Y-\gamma^{\top}\mathbf{X}-E\left(Y-\gamma^{\top}\mathbf{X}\right)\right)^2\\
=&\min_{\gamma}\operatorname{Var}\left(Y-\gamma^{\top}\mathbf{X}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Var}\left(Y-\gamma^{\top}\mathbf{X}\right)\\
=&E\left(\operatorname{Var}\left(Y-\gamma^{\top}\mathbf{X}\middle|Z\right)\right)+\operatorname{Var}\left(E\left(Y-\gamma^{\top}\mathbf{X}\middle|Z\right)\right)\\
=&P(Z=1)\operatorname{Var}(Y_1-\gamma^{\top}\mathbf{X}_1)+P(Z=0)\operatorname{Var}(Y_0-\gamma^{\top}\mathbf{X}_0)\\
&\quad+\operatorname{Var}\left(E\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)\mathbf{1}\{Z=1\}+E\left(Y_0-\gamma^{\top}\mathbf{X}_0\right)\left(1-\mathbf{1}\{Z=1\}\right)\right)\\
=&P(Z=1)\operatorname{Var}(Y_1-\gamma^{\top}\mathbf{X}_1)+P(Z=0)\operatorname{Var}(Y_0-\gamma^{\top}\mathbf{X}_0)\\
&\quad+\operatorname{Var}\left(\left[E\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)-E\left(Y_0-\gamma^{\top}\mathbf{X}_0\right)\right]\mathbf{1}\{Z=1\}\right)\\
=&P(Z=1)\operatorname{Var}(Y_1-\gamma^{\top}\mathbf{X}_1)+P(Z=0)\operatorname{Var}(Y_0-\gamma^{\top}\mathbf{X}_0)\\
&\quad+P(Z=1)P(Z=0)\left[E\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)-E\left(Y_0-\gamma^{\top}\mathbf{X}_0\right)\right]^2\\
=&\color{red}{P(Z=1)\operatorname{Var}(Y_1-\gamma^{\top}\mathbf{X}_1)+P(Z=0)\operatorname{Var}(Y_0-\gamma^{\top}\mathbf{X}_0)}\\
&\quad+P(Z=1)P(Z=0)\left[E\left(Y_1\right)-E\left(Y_0\right)\right]^2
\end{aligned}
$$

Minimizing both objective function is equivalent to minimize the red part,

$$\color{red}{P(Z=1)\operatorname{Var}\left(Y_1-\gamma^{\top}\mathbf{X}_1\right)+P(Z=0)\operatorname{Var}\left(Y_0-\gamma^{\top}\mathbf{X}_0\right).}$$

Hence the γ obtained from two optimization is the same. □

Q3. In observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, if $Y = \alpha + \beta Z + \gamma X + \epsilon$, write down $E(Y(1)|X)$ and ACE;

Answer

Firstly, we have

$$E[Y(1)|X] = \alpha + \beta + \gamma X.$$

Similarly we have

$$E[Y(0)|X] = \alpha + \gamma X.$$

Then,

$$ACE = E[Y(1) - Y(0)] = E[E(Y(1) - Y(0)|X)] = \beta.$$

Q4. In observational studies without unmeasured confounders; i.e. $(Y_i(1), Y_i(0)) \perp Z_i \mid X_i$, if $0 < e(X) = P(Z = 1|X) < 1$, show that $EY_i(0) = E\frac{(1-Z_i)Y_i}{1-e(X_i)}$ and $E\frac{1-Z_i}{1-e(X_i)} = 1$.

Answer

Proof. Observe that

$$\begin{aligned}
&E\frac{(1-Z_i)Y_i}{1-e(X_i)}\\
=&E\left[E\left(\frac{(1-Z_i)Y_i}{1-e(X_i)}\middle|X_i\right)\right]\\
=&E\left[E\left(\frac{(1-Z_i)((1-Z_i)Y_i(0)+Z_iY_i(1))}{1-e(X_i)}\middle|X_i\right)\right]\\
=&E\left[E\left(\frac{(1-Z_i)Y_i(0)}{1-e(X_i)}\middle|X_i\right)\right]\\
=&E\left[\frac{1}{1-e(X_i)}E(1-Z_i|X_i)E[Y_i(0)|X_i]\right]\\
=&E[E[Y_i(0)|X_i]] = E[Y_i(0)].
\end{aligned}$$

and

$$\begin{aligned}
&E\left[E\left(\frac{1-Z_i}{1-e(X_i)}\middle|X_i\right)\right]\\
=&E\left[\frac{1}{1-e(X_i)}E\left(1-Z_i\middle|X_i\right)\right] = 1.
\end{aligned}$$

$\square$