Applied Differential Equations Notes

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Abstract

This is KCL undergraduate module 5CCM211A, instructed by Professor Giuseppe Tinaglia. The formal name for this class is "Applied Differential Equations".

Contents

1 The Laplace transform **3**
- 1.1 The step function . . . 7
- 1.2 Properties of the Laplace transform . . . 8
- 1.3 Solving $y''(t) + \lambda y(t) = 0$ for $\lambda > 0$ . . . 10
- 1.4 Solving $y''(t) + \lambda y(t) = 0$ for $\lambda < 0$ . . . 11
- 1.5 Convolution . . . 12
- 1.6 The Dirac delta function . . . 14
- 1.7 2^{nd} order linear ODEs with constant coefficients . . . 14
- 1.8 2^{nd} order linear homogeneous ODEs with constant coefficients . . . 16
- 1.9 "Uniqueness" of the Laplace transform . . . 17

2 Picard's theorem **18**
- 2.1 Proof of the existence . . . 18
- 2.2 PLACEHOLDER TITLES BELOW . . . 18
- 2.3 Picard's iteration method . . . 18

3 SOME NONSENSE TO CHANGE ABOVE **19**

4 Power series methods **19**
- 4.1 Ordinary points . . . 20
- 4.2 Regular singular points . . . 20
- 4.3 Euler equations . . . 21
  - 4.3.1 Real and distinct roots . . . 21
  - 4.3.2 Complex roots . . . 22
  - 4.3.3 Equal roots . . . 23
- 4.4 Frobenius method . . . 23
- 4.5 Power series . . . 24

5 Heat equation **25**
5.1 Homogeneous boundary conditions . . . 26
5.2 Fourier series and the initial condition . . . 27
5.3 Separation of variables . . . 30
5.3.1 Solution of time ODE . . . 31
5.3.2 Solution of 'position' ODE . . . 31
5.3.3 The general solution . . . 33
5.4 Uniqueness of the solution . . . 34
5.5 Insulated ends . . . 35
5.5.1 Solution for 'position' ODE . . . 36
5.6 Constant boundary conditions . . . 38
5.7 Maximum principle . . . 39

6 The wave equation **40**
6.1 Zero initial velocity . . . 42
6.1.1 Separation of variables . . . 43
6.2 Zero initial displacement . . . 44
6.2.1 Separation of variables . . . 45
6.3 Uniqueness of the solution . . . 46
6.4 Cracking the whip . . . 47
6.5 Can you hear the shape of a drum? . . . 48
6.6 The 2-dimensional wave equation . . . 48
6.6.1 Separation of variables . . . 48
6.7 The Dirichlet eigenvalues of a disk . . . 49

Appendix **50**

A Links **50**

B Laplace transform table **50**

C Techniques of integration **50**
C.1 Integration by parts . . . 50

D Tricks **50**
D.1 Step function . . . 50

E Hessian matrix **51**

1 The Laplace transform

Definition 1.1. The **Laplace transform** of a function

$$y(t) : [0, \infty) \to \mathbb{R}$$

is the function

$$\mathcal{L}[y(t)](s) = Y(s) = \int_0^\infty y(t)e^{-st}\,dt$$

for all numbers s for which this integral converges.

Note 1.1. The Laplace transform takes a function of t as an input and outputs a function of s.

Remark 1.1. Not all functions have a Laplace transform.

Example 1.1. Functions which do not have Laplace transform:

- $y(t) = \frac{1}{t}$ grows to fast near zero independently of s:

$$\int_0^\infty \frac{1}{t}e^{-st}\,dt = \infty.$$

- $y(t) = e^{(t^2)}$ grows too fast as $t \to \infty$ independently of s:

$$\int_0^\infty e^{(t^2)}e^{-st}\,dt = \infty.$$

Example 1.1

Find the Laplace transform of

$$f(t) = \begin{cases} 1 & \text{if } t \in [0, 1) \\ k & \text{if } t = 1 \\ 0 & \text{if } t \in (1, \infty). \end{cases}$$

Solution.

$$\begin{aligned} \mathcal{L}[f(t)](s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^1 e^{-st}\,dt + \int_1^1 k\,dt + \int_1^\infty 0\,dt \\ &= \int_0^1 e^{-st}\,dt \\ &= \frac{1 - e^{-s}}{s}. \end{aligned}$$

Example 1.2

Compute the Laplace transform of $y(t) = e^{at}$.

Solution. Applying the definition of Laplace transform we have

$$\begin{aligned}
\mathcal{L}[e^{at}](s) &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{(a-s)t}\,dt \\
&= \lim_{N\to\infty}\int_0^N e^{(a-s)t}\,dt \\
&= \lim_{N\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^N \\
&= \frac{1}{a-s}\lim_{N\to\infty}\left(e^{(a-s)N}-e^0\right) \\
&= \frac{1}{a-s}\lim_{N\to\infty}\left(e^{(a-s)N}-1\right).
\end{aligned}$$

We now consider when the integral converges and diverges thus, we look at the size of s relative to a.

- If $s = a$ then the integral diverges.
- If $s < a$ then the integral diverges.
- If $s > a$ then the integral converges to

$$-\frac{1}{a-s} = \frac{1}{s-a}.$$

Therefore, the Laplace transform of $y(t) = e^{at}$ is

$$\mathcal{L}[e^{at}](s) = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{undefined} & \text{if } s \leq a. \end{cases}$$

Example 1.2. Compute the Laplace transform of $y(t) = 1$.

Solution. By the previous example we have,

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a} \quad \text{if } s > a.$$

If $a = 0$ then, $e^{at} = e^0 = 1$ for all t. Therefore,

$$\mathcal{L}[1](s) = \frac{1}{s} \quad \text{if } s > 0.$$

Theorem 1.1
We have the following property

$$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a).$$

Proof. Consider

$$\begin{aligned}\mathcal{L}[e^{at}f(t)(s)] &= \int_0^\infty f(t)e^{at}e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{(a-s)t}\,dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= \mathcal{L}[f(t)](s-a).\end{aligned}$$

□

Definition 1.2. A function, $y(t)$, is said to be **piecewise continuous** on a finite interval $[a,b]$ if it is continuous at every point in $[a,b]$, except possibly for a finite number of points at which $y(t)$ has a jump discontinuity.

Example 1.3. Consider the function

$$g(t) = \begin{cases} t & \text{if } t \in [0,1) \\ 0 & \text{if } t \in [1,2]. \end{cases}$$

Then $g(t)$ is continuous on $[0,2]$ with a jump discontinuity at $t=1$. Whereas, the function

$$f(t) = \begin{cases} \frac{1}{1-t} & \text{if } t \in [0,1) \\ 0 & \text{if } t \in [1,2] \end{cases}$$

has an infinite discontinuity at $t=1$ so, it does not have a jump discontinuity on $[0,2]$. Therefore, it is not piecewise continuous.

Definition 1.3. A function, $y(t)$, is said to be **piecewise continuous** on $[0,\infty)$ if it is piecewise continuous on $[0,N]$ for any $N>0$.

Definition 1.4. A function, $y(t)$, is said to be of **exponential order** (or of exponential order α) if there exist positive constants, $T, M, \alpha > 0$ such that

$$\forall t \in [T,\infty) \quad \text{we have} \quad |y(t)| \leq Me^{\alpha t}.$$

Theorem 1.2
Let $y(t)$ be a piecewise continuous function on $[0,\infty)$ and of exponential order ($\alpha > 0$). Then, $Y(s) = \mathcal{L}[y](s)$ exists for all $s > \alpha$.

Proof. Fix $s > \alpha$ then we need to show that

$$Y(s) = \int_0^\infty y(t)e^{-st}\,dt$$

is finite i.e.

$$Y(s) = \int_0^\infty y(t)e^{-st}\,dt < \infty.$$

By the triangle inequality we have

$$\left|\int_0^\infty y(t)e^{-st}\,dt\right| \le \left|\int_0^T y(t)e^{-st}\,dt\right| + \left|\int_T^\infty y(t)e^{-st}\,dt\right|.$$

We consider the two integrals separately and prove they are both finite. Note that $y(t)$ is a function of exponential therefore,

$$\forall t \in [T, \infty] \quad \text{we have} \quad |y(t)| \le Me^{\alpha t}.$$

- We prove $\left|\int_0^T y(t)e^{-st}\,dt\right| < \infty$. First, we note that since $y(t)$ is a piecewise continuous function on $[0, T]$, there exists a constant K such that

  $$\max_{t\in[0,T]} |y(t)| \le K.$$

  This implies that

  $$\begin{aligned}\left|\int_0^T y(t)e^{-st}\,dt\right| &\le \int_0^T \left|y(t)e^{-st}\right|\,dt \\ &\le \max_{t\in[0,T]} \left|y(t)e^{-st}\right|\,dt \\ &\le \int_0^T \max_{t\in[0,T]} |y(t)| \max_{t\in[0,T]} \left|e^{-st}\right|\,dt \\ &\le \int_0^T \max_{t\in[0,T]} |y(t)|\,dt \\ &\le \int_0^T K\,dt \\ &= TK \\ &< \infty.\end{aligned}$$

- We prove $\left|\int_0^\infty y(t)e^{-st}\,dt\right| < \infty$.

  $$\begin{aligned}\left|\int_T^\infty y(t)e^{-st}\,dt\right| &\le \int_T^\infty \left|y(t)e^{-st}\right|\,dt \\ &\le \int_T^\infty Me^{\alpha t}e^{-st}\,dt \\ &\le M\lim_{N\to\infty}\int_T^N e^{(\alpha-s)t}\,dt \\ &= M\lim_{N\to\infty}\left[\frac{e^{(\alpha-s)t}}{\alpha-s}\right]_T^N \\ &= -\frac{M}{\alpha-s}e^{(\alpha-s)T} \\ &< \infty.\end{aligned}$$

$\square$

1.1 The step function

From now (unless stated otherwise), assume $y(t)$ will be a piecewise continuous function on $[0, \infty)$, and it is of exponential order.

Definition 1.5. Given $a \in \mathbb{R}$ such that $a \geq 0$ let

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a. \end{cases}$$

We call this the **step function**.

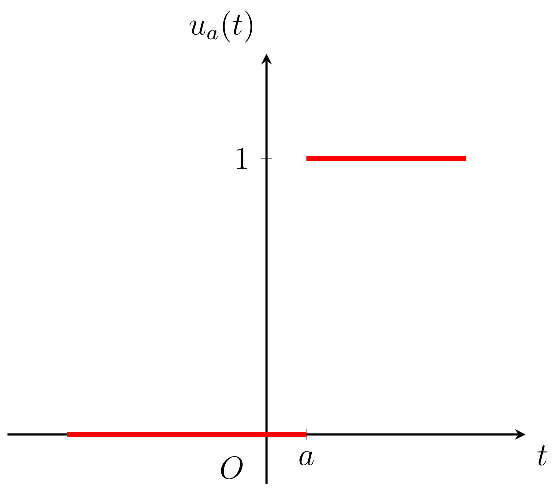


Figure 1: Graph of the step function

Proposition 1.1
Let $f(t)$ be a piecewise continuous function of exponential order. Then

$$\mathcal{L}\left[f(t-a)\, u_a(t)\right](s) = e^{-as}\mathcal{L}[f(t)](s).$$

In particular,

$$\mathcal{L}\left[u_a(t)\right](s) = \frac{e^{-as}}{s}.$$

Proof. Applying the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\left[f(t-a)\, u_a(t)\right](s) &= \int_0^\infty f(t-a)\, u_a(t)e^{-st}\, dt \\ &= \int_0^a f(t-a)e^{-st}\, dt + \int_a^\infty f(t-a)e^{-st}\, dt \\ &= \int_a^\infty f(t-a)e^{-st}\, dt.\end{aligned}$$

When $t < a$, the first integral vanishes by the definition of the step function. Now, we change variable to $z = t - a$ which gives

$$\begin{aligned}\int_a^\infty f(t-a)e^{-st}\,dt &= \int_0^\infty f(z)e^{-as-sz}\,dz \\ &= e^{-as}\int_0^\infty f(z)e^{-sz}\,dz \\ &= e^{-as}\mathcal{L}[f(t)](s).\end{aligned}$$

To prove that

$$\mathcal{L}\left[u_a(t)\right](s) = \frac{e^{-as}}{s}$$

we apply the previous formula with $f(t) = 1$. □

Corollary 1.1. The proposition above is equivalent to the statement

$$\mathcal{L}[u_a(t)f(t)](s) = e^{-as}\mathcal{L}[f(t+a)](s).$$

Proof. Let $g(t) = f(t+a)$ and $f(t) = g(t-a)$ therefore,

$$\mathcal{L}[g(t)](s) = \mathcal{L}[f(t+a)](s).$$

We can write

$$\begin{aligned}\mathcal{L}[u_a(t)f(t)](s) &= \mathcal{L}[u_a(t)g(t-a)](s) \\ &= e^{-as}\mathcal{L}[g(t)](s) \\ &= e^{-as}\mathcal{L}[f(t+a)](s).\end{aligned}$$

□

Example 1.3

Compute the Laplace transform of $u_{2\pi}(t)\cos(t)$.

Solution. First note that by adding 0 we can write

$$u_{2\pi}(t)\cos(t) = u_{2\pi}(t)\cos(t+2\pi-2\pi).$$

Therefore, by using the formula in Proposition 1.1 the Laplace transform is as follows:

$$\begin{aligned}\mathcal{L}[u_{2\pi}(t)\cos(t+2\pi-2\pi)](s) &= e^{-2\pi s}\mathcal{L}[\cos(t+2\pi)](s) \\ &= e^{-2\pi s}\mathcal{L}[\cos(t)](s) \\ &= e^{-2\pi s}\frac{s}{s^2+1}.\end{aligned}$$

1.2 Properties of the Laplace transform

Theorem 1.1. Given functions f and g and a constant $c \in \mathbb{R}$,

$$\begin{aligned}\mathcal{L}[f+g] &= \mathcal{L}[f] + \mathcal{L}[g] \\ \mathcal{L}[cf] &= c\,\mathcal{L}[f].\end{aligned}$$

In other words, the Laplace transform is a **linear operator**.

Theorem 1.2. The inverse Laplace transform is a linear operator.

Proof. We prove the first property of linearity,

$$\begin{aligned}\mathcal{L}^{-1}[f+g] &= \mathcal{L}^{-1}\left(\mathcal{L}\left(\mathcal{L}^{-1}[f]\right)+\mathcal{L}\left(\mathcal{L}^{-1}[g]\right)\right)\\ &= \mathcal{L}^{-1}\left(\mathcal{L}\left(\mathcal{L}^{-1}[f]+\mathcal{L}^{-1}[g]\right)\right)\\ &= \mathcal{L}^{-1}[f]+\mathcal{L}^{-1}[g].\end{aligned}$$

We prove the second property of linearity,

$$\begin{aligned}\mathcal{L}^{-1}[\alpha f] &= \mathcal{L}^{-1}\left(\alpha\mathcal{L}\left(\mathcal{L}^{-1}[f]\right)\right)\\ &= \mathcal{L}^{-1}\left(\mathcal{L}\left(\alpha\mathcal{L}^{-1}[f]\right)\right)\\ &= \alpha\mathcal{L}^{-1}[f].\end{aligned}$$

□

Example 1.4. Compute the Laplace transform of $y(t)=c\in\mathbb{R}$ for $s>0$.

Solution. We previously computed the Laplace transform of 1 which is $\mathcal{L}[1](s)=\frac{1}{s}$ for $s>0$. Using the linearity of the Laplace transform we have that

$$\mathcal{L}[c](s)=c\,\mathcal{L}[1](s)=\frac{c}{s}$$

for $s>0$.

Theorem 1.3

Given a function $y(t)$ with Laplace transform, $Y(s)=\mathcal{L}[y(t)](s)$, the Laplace transform of $\frac{\mathrm{d}y}{\mathrm{d}t}(t)$ is

$$\mathcal{L}\left[\frac{\mathrm{d}^n y}{\mathrm{d}t^n}(t)\right](s)=s^n\mathcal{L}[y(t)](s)-\sum_{i=0}^{n-1}s^{n-i-1}\frac{\mathrm{d}^i y}{\mathrm{d}t^i}(0),$$

for $n\geq 1$.

Corollary 1.1

In particular,

- the Laplace transform of $\frac{\mathrm{d}y}{\mathrm{d}t}(t)$ is

$$\mathcal{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}(t)\right](s)=s\mathcal{L}[y(t)](s)-y(0).$$

- the Laplace transform of $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)$ is

$$\mathcal{L}\left[\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)\right](s)=s^2\mathcal{L}[y(t)](s)-s\,y(0)-\frac{\mathrm{d}y}{\mathrm{d}t}(0).$$

Example 1.4
Let $y(t)$ be a solution to the initial value problem

$$y' = y - 4e^{-t} \quad \text{for } y(0) = 1.$$

Compute the Laplace transform of $y(t)$.

Solution. We apply the Laplace transform to both sides of the equation i.e.

$$\mathcal{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right](s) = \mathcal{L}\left[y - 4e^{-t}\right](s).$$

Using the properties of the Laplace transform we get

$$sY(s) - y(0) = Y(s) - 4\mathcal{L}\left[e^{-t}\right](s).$$

Substituting the initial condition, $y(0) = 1$, the equation above becomes

$$sY(s) - 1 = Y(s) - 4\mathcal{L}\left[e^{-t}\right](s).$$

Recall that $\mathcal{L}\left[e^{at}\right](s) = \frac{1}{s-a}$. Applying this result with $a = -1$ we obtain that

$$sY(s) - 1 = Y(s) - \frac{4}{s+1}.$$

Rearranging for $Y(s)$, we obtain

$$Y(s) = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}.$$

1.3 Solving $y''(t) + \lambda y(t) = 0$ for $\lambda > 0$

Proposition 1.2
Let $\omega \neq 0$ then

$$\mathcal{L}\left[\cos(\omega t)\right](s) = \frac{s}{s^2 + \omega^2}$$

and

$$\mathcal{L}\left[\sin(\omega t)\right](s) = \frac{\omega}{s^2 + \omega^2}.$$

Proposition 1.3
Consider the ODE
$$y''(t) + \lambda y(t) = 0 \quad \text{for } \lambda > 0.$$
Then the general solution of this ODE is given by
$$y(t) = A\sin\left(t\sqrt{\lambda}\right) + B\cos\left(t\sqrt{\lambda}\right)$$
for $A, B \in \mathbb{R}$.

Remark 1.2. By general solution we mean "every solution can be written as".

Proof. Assume $y(t)$ is a solution to the ODE above. Applying Laplace transform to the differential equation we have that

$$\begin{aligned} 0 &= \mathcal{L}\left[y''(t) + \lambda y(t)\right](s) \\ &= \mathcal{L}[y''(t)](s) + \lambda\mathcal{L}[y(t)](s) \\ &= s^2Y(s) - sy(0) - y'(0) + \lambda Y(s). \end{aligned}$$

Rearranging for $\mathcal{L}[y(t)](s) = Y(s)$, we have that

$$\begin{aligned} Y(s) &= \frac{sy(0)}{s^2+\lambda} + \frac{y'(0)}{s^2+\lambda} \\ &= y(0)\frac{s}{s^2+\left(\sqrt{\lambda}\right)^2} + \frac{y'(0)}{\sqrt{\lambda}}\frac{\sqrt{\lambda}}{s^2+\left(\sqrt{\lambda}\right)^2}, \end{aligned}$$

this holds as $\lambda > 0$. Notice that

$$\mathcal{L}\left[y(0)\cos\left(t\sqrt{\lambda}\right)\right](s) = y(0)\frac{s}{s^2+\left(\sqrt{\lambda}^2\right)}$$

and that

$$\mathcal{L}\left[\frac{y'(0)}{\sqrt{\lambda}}\sin\left(t\sqrt{\lambda}\right)\right](s) = \frac{y'(0)}{\sqrt{\lambda}}\frac{\sqrt{\lambda}}{s^2+\left(\sqrt{\lambda}\right)^2}.$$

Therefore, we can write

$$y(t) = y(0)\cos\left(t\sqrt{\lambda}\right) + \frac{y'(0)}{\sqrt{\lambda}}\sin\left(t\sqrt{\lambda}\right).$$

□

1.4 Solving $y''(t) + \lambda y(t) = 0$ for $\lambda < 0$

Proposition 1.4
Consider the ODE
$$y''(t) + \lambda y(t) = 0 \quad \text{for } \lambda < 0.$$
Then the general solution of this ODE is given by
$$y(t) = Ae^{t\sqrt{-\lambda}} + Be^{-t\sqrt{-\lambda}}$$
for $A, B \in \mathbb{R}$.

Proof. Apply Laplace transform. □

1.5 Convolution

Note 1.2. The Laplace transform of the product of two functions is **not** the product of the related Laplace transforms.

Definition 1.6. Let $f, g : [0, \infty) \to \mathbb{R}$ be two integrable functions. Then the **convolution** of f and g, denoted by $f * g$, is the function
$$(f * g)(t) := \int_0^t f(k)g(t-k)\, dk.$$

Theorem 1.3. Properties of the convolution: let c be a constant and f, g and h be functions then

- $f * g = g * f$;
- $(cf) * g = f * (cg) = c(f * g)$;
- $(f * g) * h = f * (g * h)$.

Theorem 1.4
Let f and g be piecewise continuous functions of exponential order, then
$$\mathcal{L}[(f * g)(t)](s) = \mathcal{L}[f(t)](s) \cdot \mathcal{L}[g(t)](s).$$

Remark 1.3. This statement is equivalent to
$$(f * g)(t) = \mathcal{L}^{-1}\{\mathcal{L}[f(t)](s) \cdot \mathcal{L}[g(t)](s)\}(t).$$

Note 1.3. The $\cdot$ is to emphasise the multiplication.

Proof. Let $F(s) = \mathcal{L}[f(t)](s)$ and $G(s) = \mathcal{L}[g(t)](s)$. From the definition of the Laplace transform we have:
$$F(s) = \int_0^\infty f(k)e^{-sk}\, dk \quad \text{and} \quad G(s) = \int_0^\infty g(u)e^{-su}\, du.$$

The product of $F(s)$ and $G(s)$ is given by

$$\left(\int_0^\infty f(k)e^{sk}\,dk\right)\left(\int_0^\infty f(u)e^{-su}\,du\right).$$

Since first integral does not depend on u, we can write the product as a double integral:

$$F(s)G(s) = \int_0^\infty \int_0^\infty f(k)g(u)e^{-s(k+u)}\,dkdu.$$

Changing variable to $t = k + u$ for each fixed u. So, $dt = dk$ and that $k = t - u$. We obtain

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_u^\infty f(t-u)g(u)e^{-st}\,dtdu \\ &= \int_0^\infty \int_0^t f(t-u)g(u)e^{-st}\,dudt. \end{aligned}$$

(Note that the domain of integration changes when switching the order of the integrals). Finally, isolating the terms that contain u, we get

$$\begin{aligned} F(s)G(s) &= \int_0^\infty \int_0^t f(t-u)g(u)\,du\,e^{-st}\,dt \\ &= \int_0^\infty (f*g)(t)e^{-st}\,dt \\ &= \mathcal{L}[(f*g)(t)](s). \end{aligned}$$

□

Example 1.5

Suppose we have the function defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\cdot\frac{1}{s^2}.$$

We recognise the entries as

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t.$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s+1}\cdot\frac{1}{s^2}\right] &= \mathcal{L}^{-1}[\mathcal{L}[e^{-t}]\cdot\mathcal{L}[t]] \\ &= \mathcal{L}^{-1}\left[\mathcal{L}[(e^{-t}*t)(t)]\right] \\ &= (e^{-t}*t)(t) \\ &= \int_0^t e^{-v}(t-v)\,dv \\ &= e^{-t}+t-1. \end{aligned}$$

1.6 The Dirac delta function

Definition 1.7. The **Dirac delta function** is defined by the following properties

$$\delta(t) = 0 \text{ when } t \neq 0,$$

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Given $b > 0$, define

$$g_b(t) = \begin{cases} \frac{1}{2b} & \text{if } -b \leq t \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Then, one can think of the δ-function as

$$\delta(t) = \lim_{b \to 0} g_b(t)$$

and

$$\delta(t-a) = \lim_{b \to 0} g_b(t-a).$$

This limit is zero for all values t except at $t = a$, where it is infinite.

Theorem 1.4.

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt := \lim_{b \to 0} \int_{-\infty}^{\infty} g_b(t-a) f(t) = f(a).$$

Theorem 1.5

The Laplace transform of the Dirac delta function (for $a > 0$) is

$$\begin{aligned} \mathcal{L}[\delta(t-a)](s) &:= \lim_{b \to 0} \mathcal{L}[g_b(t-a)](s) \\ &= e^{-as}. \end{aligned}$$

Corollary 1.2

For $a = 0$ the Laplace transform of $\delta(t)$ is defined as

$$\begin{aligned} \mathcal{L}[\delta(t)](s) &:= \lim_{a \to 0} \mathcal{L}[\delta(t-a)](s) \\ &= 1. \end{aligned}$$

1.7 2^{nd} order linear ODEs with constant coefficients

Definition 1.8. A second order linear differential equation with constant coefficients is one of the form

$$ay'' + by' + cy = g(t)$$

for $a, b, c \in \mathbb{R}$ and $g : I \subset \mathbb{R} \to \mathbb{R}$.

Definition 1.9. The associated equation

$$ay'' + by' + cy = 0$$

is called the **homogeneous equation**.

Definition 1.10. Let $\xi(t)$ be the solution of the initial value problem

$$ay'' + by' + cy = \delta(t) \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 0. \end{aligned}$$

The function $\xi(t)$ is called the **impulse response**.

Corollary 1.2. Let $\xi(t)$ be the impulse response. Then

$$\mathcal{L}[a\xi''(t) + b\xi'(t) + c\xi(t)](s) = \mathcal{L}[\delta(t)](s),$$

and applying the properties of the Laplace transform, and the initial condition gives that

$$\mathcal{L}[\xi(t)](s) = \frac{1}{as^2 + bs + c}.$$

Theorem 1.6

Consider the following initial value problem,

$$ay'' + by' + cy = g(t) \qquad \begin{aligned} y(0) &= 0 \\ y'(0) &= 0. \end{aligned}$$

The unique solution is

$$\begin{aligned} y(t) &= (\xi * g)(t) \\ &= \int_0^t \xi(t-k)g(k)\,dk. \end{aligned}$$

Proof. Applying the Laplace transform on both sides, we have:

$$\begin{aligned} \mathcal{L}[ay'' + by' + cy] &= \mathcal{L}[g(t)] \\ s^2\mathcal{L}[y] - sy(0) - y'(0) + s\mathcal{L}[y] - y(0) + c\mathcal{L}[y] &= \mathcal{L}[g(t)] \\ s^2\mathcal{L}[y] + s\mathcal{L}[y] + c\mathcal{L}[y] &= \mathcal{L}[g(t)] \\ (s^2 + s + c)\mathcal{L}[y] &= \mathcal{L}[g(t)]. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}[y] &= \mathcal{L}[g(t)] \cdot \frac{1}{s^2 + s + c} \\ &= \mathcal{L}[g(t)](s) \cdot \mathcal{L}[\xi(t)](s). \end{aligned}$$

$\square$

Corollary 1.3

Consider the following the initial value problem

$$ay'' + by' + cy = g(t) \qquad \begin{aligned} y(0) &= y_0 \\ y'(0) &= y_0. \end{aligned}$$

The solution is

$$\begin{aligned} y(t) &= (\xi * g)(t) + \widehat{y}(t) \\ &= \int_0^t \xi(t-k)g(k)\,dk + \widehat{y}(t), \end{aligned}$$

where $\widehat{y}(t)$ is the solution of

$$ay'' + by' + cy = 0 \qquad \begin{aligned} y(0) &= y_0 \\ y'(0) &= y_0'. \end{aligned}$$

1.8 2^{nd} order linear homogeneous ODEs with constant coefficients

Definition 1.11. Given a second order linear homogeneous equation with constant i.e.,

$$ay'' + by' + cy = 0$$

for $a, b, c \in \mathbb{R}$, the equation

$$ar^2 + br + c = 0$$

is called the **characteristic equation**.

Theorem 1.7

Let r_1 and r_2 be the roots of the characteristic equation.

1. If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$) then, the characteristic equation has general solution

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where $C_1, C_2 \in \mathbb{R}$.

2. If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then the characteristic equation has the general solution

$$y = (C_1 + C_2 t)e^{r_1 t}$$

where $C_1, C_2 \in \mathbb{R}$.

3. If r_1 and r_2 are complex roots of the form $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution to the characteristic equation is

$$y = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x)$$

where $C_1, C_2 \in \mathbb{R}$.

1.9 "Uniqueness" of the Laplace transform

Theorem 1.8

If $f(t)$ is a **continuous** function with $\mathcal{L}[f(t)](s) = F(s)$, then $f(t)$ is the **only** continuous function whose Laplace transform is $F(s)$.

Theorem 1.5. If h and g are piecewise continuous functions with $\mathcal{L}[h] = \mathcal{L}[g]$, then $h = g$ except possibly at he points of discontinuity.

2 Picard's theorem

Theorem 2.1 (Picard's theorem – existence and uniqueness)
Let $R \subset \mathbb{R}^2$ be a closed rectangle of the form

$$R := \{(t, y) : a \leq t, c \leq y \leq d\},$$

for $a, b, c, d \in \mathbb{R}$ and let

$$f(t, y) : R \to \mathbb{R}$$
$$\frac{\partial}{\partial y} f(t, y) : R \to \mathbb{R}$$

be continuous functions. Let $(t_0, y_0) \in (a, b) \times (c, d)$ be a point in the open rectangle. Then there exists $\varepsilon > 0$ and

$$y(t) : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$$

such that y is the unique solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad \text{for } y(t_0) = y_0$$

in the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Note 2.1. In essence, if $f(t, y)$ and $\dfrac{\partial}{\partial y} f(t, y)$ are continuous functions then the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad \text{for } y(t_0) = y_0$$

has only one unique solution, y.

2.1 Proof of the existence

The proof of the existence part of Picard's theorem is also known as **Picard's Iteration method**, which provides us a method to find a solution.

2.2 PLACEHOLDER TITLES BELOW

2.3 Picard's iteration method

Theorem 2.2 (Picard's iteration)
Suppose we have an initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \quad \text{for } y(t_0) = y_0.$$

Then, the solutions are given by

$$\begin{aligned} y_0(t) &:= y_0 \\ y_1(t) &:= y_0 + \int_{t_0}^{t} f(u, y_0(u))\, du \\ &\vdots \\ y_k(t) &:= y_0 + \int_{t_0}^{t} f(u, y_{k-1}(u))\, du, \end{aligned}$$

for any $k \in \mathbb{N}$.

Definition 2.1. Let $I \subset \mathbb{R}$ and let

$$\phi_n : I \to \mathbb{R} \quad \text{for } n \in \mathbb{N},$$

be a sequence of functions. We say that ϕ_n is **uniformly convergent** on I to $\phi : I \to \mathbb{R}$ if for any M there exists $n_M \in \mathbb{N}$ such that

$$\sup_I |\phi - \phi_n| < M.$$

Lemma 2.1. Let $I \subset \mathbb{R}$ be a bounded interval and let

$$\phi_n : I \to \mathbb{R} \quad \text{for } n \in \mathbb{N},$$

be a sequence of integrable functions uniformly converging on I to $\phi : I \to \mathbb{R}$. Then,

$$\lim_{n\to\infty} \int_I \phi_n = \int_I \phi.$$

3 SOME NONSENSE TO CHANGE ABOVE

4 Power series methods

In this section we illustrate solution to the second order linear homogeneous ODE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $P(x), Q(x)$ and $R(x)$ are analytic functions at $c \in \mathbb{R}$. From now on unless stated otherwise, we will consider ODEs of this form.

Note 4.1. By analytic, we mean there exists a power series expansion of each respective function at the point $x = c \in \mathbb{R}$.

4.1 Ordinary points

Definition 4.1. Consider the ODE from above, if $P(c) \neq 0$ then c is called an **ordinary point**.

Definition 4.2. If $P(c) = 0$ (and either $Q(c)$ or $R(c)$ is different from zero) then c is called a **singular point**.

Definition 4.3. Two solution of an ODE y_1 and y_2 are said to be **linearly independent** if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha y_1(t) + \beta y_2(t) = 0$$

if and only if $\alpha = \beta = 0$.

Theorem 4.1

Suppose $x = c$ is an ordinary point of the ODE then, the ODE has two linearly independent analytic solution of the form:

$$y = \sum_{n=0}^{\infty} a_n (x - c)^n.$$

Furthermore, the radius of convergence is at least as large as the distance from c to the nearest singular point (real or complex-valued) of the ODE.

Remark 4.1. If there are no singular points then the radius of convergence is infinite.

4.2 Regular singular points

Definition 4.4. Let $x = c$ be a singular point of the ODE. If

$$\lim_{x \to c}(x - c)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to c}(x - c)^2\frac{R(x)}{P(x)}$$

are both finite then, $x = c$ is called a **regular singular point**. Otherwise, is called an **irregular singular point**.

Remark 4.2. Since $x = c$ is a singular point, at least one of the functions $\frac{Q(x)}{P(x)}$ or $\frac{R(x)}{P(x)}$ blows up at $x = c$ and in particular they are not analytic.

Example 4.1
Consider the equation

$$(x-2)^2(x-1)^2y'' + (x-1)y' + 5y = 0.$$

We can identify

$$\begin{aligned} P(x) &= (x-2)^2(x-1)^2 \\ Q(x) &= x-1 \\ R(x) &= 5. \end{aligned}$$

Since $P(1) = P(2) = 0$ and R is never zero then $x = 1, 2$ are the only singular points of the ODE. We have that

$$\begin{aligned} \lim_{x\to 2}(x-2)\frac{Q(x)}{P(x)} &= \lim_{x\to 2}(x-2)\frac{(x-1)}{(x-1)^2(x-2)^2} \\ &= \infty \end{aligned}$$

so, $x = 2$ is an irregular singular point. Whereas, at $x = 1$ we have

$$\begin{aligned} \lim_{x\to 1}\frac{Q(x)}{P(x)} &= \lim_{x\to 1}(x-1)\frac{x-1}{(x-1)^2(x-2)^2} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to 1}(x-1)^2\frac{R(x)}{P(x)} &= \lim_{x\to 1}(x-1)^2\frac{5}{(x-1)^2(x-2)^2} \\ &= 5. \end{aligned}$$

Since they are both finite we have that $x = 1$ is a regular singular point.

4.3 Euler equations

Definition 4.5. The **Euler equation** is an ODE of the form

$$x^2y'' + \alpha xy' + \beta y = 0.$$

> **Remark 4.3.** The solutions presented in this section to the Euler equation are only valid for $x > 0$.

Definition 4.6. Given an Euler equation the following equation

$$r(r-1) + \alpha r + \beta = 0,$$

is called the **indicial equation**.

4.3.1 Real and distinct roots

Theorem 4.2

If the indicial equation has real and distinct roots, r_1 and r_2, then the general solution to the Euler equation is given by

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

where $C_1, C_2 \in \mathbb{R}$ and $x > 0$.

Example 4.1. Solve the following ODE

$$2x^2 y'' + 3xy' - y = 0 \quad \text{for } x > 0.$$

Clearly, this is an Euler equation since we can write the ODE as

$$x^2 y'' + \frac{3}{2} xy' - \frac{1}{2} y - 0.$$

The corresponding indicial equation is $r(r-1) + \frac{3}{2}r - \frac{1}{2} = 0$. Equivalently,

$$\begin{aligned} 0 = 2r(r-1) + 3r - 1 &= 2r^2 + r - 1 \\ &= (2r-1)(r+1). \end{aligned}$$

Therefore, the roots of the indicial equation are $r_1 = \frac{1}{2}$ and $r_2 = -1$, we conclude the general solution of the Euler equation is

$$y = C_1 x^{\frac{1}{2}} + C_2 x^{-1} \quad \text{for } x > 0.$$

4.3.2 Complex roots

Theorem 4.3

If the indicial equation has complex roots, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where $\alpha, \beta \in \mathbb{R}$. Then the general solution to the Euler equation is given by

$$y(x) = C_1 e^{\alpha \ln(x)} \cos(\beta \ln x) + C_2 e^{\alpha \ln(x)} \sin(\beta \ln x)$$

where$C_1, C_2 \in \mathbb{R}$ and $x > 0$.

Example 4.2. Consider the Euler equation:

$$x^2 y'' + xy + y = 0 \quad \text{for } x > 0.$$

The corresponding indicial equation is

$$r(r-1) + r + 1 = r^2 + 1 = 0.$$

Therefore, the roots are $r_1 = i$ and $r_2 = -i$ which implies the general solution of the Euler equation is given by

$$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x) \quad \text{for } x > 0.$$

4.3.3 Equal roots

Theorem 4.4

If the indicial equation has a repeated root r_1 then the general solution to the Euler equation is given by

$$y(x) = C_1 x^{r_1} + C_2 x^{r_1} \ln(x)$$

where $C_1, C_2 \in \mathbb{R}$ and $x > 0$.

Example 4.3. Consider the Euler equation

$$x^2 y'' + 5xy' + 4y = 0.$$

The corresponding indicial equation is

$$\begin{aligned} 0 = r(r-1) + 5r + 4 &= r^2 + 4r + 4 \\ &= (r+2)^2. \end{aligned}$$

Therefore, $r_1 = r_2 = -2$ which implies the general solution is

$$y(x) = x^{-2}(C_1 + C_2 \ln x) \quad \text{for } x > 0.$$

4.4 Frobenius method

Definition 4.7. Let $x = c$ be a regular singular point and let

$$p_0 := \lim_{x \to c} (x-c) \frac{Q(x)}{P(x)} \quad \text{and} \quad q_0 := \lim_{x \to c} (x-c)^2 \frac{R(x)}{P(x)}.$$

The equation

$$r(r-1) + p_0 r + q_0 = 0$$

is called the **indicial equation** at $x = c$.

Example 4.4. Consider the equation

$$(x-2)^2 (x-1)^2 y'' + (x-1)y' + 5y = 0.$$

From a previous example we know that

$$\lim_{x \to 1} (x-1) \frac{Q(x)}{P(x)} = 1 \quad \text{and} \quad \lim_{x \to 1} (x-1)^2 \frac{R(x)}{P(x)} = 5.$$

Therefore, the indicial equation at $x = 1$ is

$$r(r-1) + r + 5 = 0.$$

Theorem 4.5

Let $x = 0$ be a regular singular point of the ODE. Suppose that $r_1, r_2 \in \mathbb{R}$ with $r_1 \geq r_2$ are solutions of the indicial equation. Then there exists a solution of the form

$$y(x) = x^{r_1} \sum_{k=0}^{\infty} a_k x^k \quad x > 0,$$

with $a_0 \neq 0$.

4.5 Power series

Definition 4.8. A **power series** is an expression of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where a_n and c are constants.

> **Remark 4.4.** For $x = c$ the series always converge to a_0.

Definition 4.9. We say that a power series **converges absolutely** at x whenever the limit

$$\lim_{N\to\infty} \sum_{k=0}^{N} |a_k|\,|x-c|^k$$

exists. That is the series $\sum_{k=0}^{\infty} |a_k|\,|x-c|^k$ is convergent.

Definition 4.10. Given a power series there exists a number $R \in [0,\infty)$ called the **radius of convergence** if the power series converges absolutely for any $x \in (c-R, c+R)$. Otherwise, it does not converge absolutely.

Theorem 4.6 (Cauchy-Hadamard formula)
The radius of convergence is

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

> **Remark 4.5.** A key property of power series is that they can be differentiated term by term, added and multiplied together, within the radius of convergence.

Definition 4.11. Let $f(x)$ be a smooth function at $x = c$ and let

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

be its Taylor expansion at c. If its Taylor expansion has radius of converge $R > 0$ then, the function is said to be an **analytic function** at $x = c$.

Example 4.2
Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the power series solution about $x = 0$ of the initial value problem

$$2y'' + xy' + y = 0 \quad y(0) = 1$$
$$y'(0) = 0.$$

Find the value of a_0, a_1, a_2 and a_3.

Solution. Since the solution is a power series we know it must be a Taylor series. As such each

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Using this we determine that, $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$. From the ODE we can write

$$y'' = \frac{-xy' - y}{2},$$

which implies $y''(0) = -\frac{1}{2}$ hence, $a_2 = -\frac{1}{4}$. To find a_3 we take a derivative of the ODE and obtain

$$2y''' + y' + xy'' + y' = 0$$

and conclude $a_3 = 0$.

5 Heat equation

Definition 5.1. The **heat equation** for a wire of length $L > 0$ takes the form of

$$\frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t)$$

where $k > 0$ and $u(x,t)$ represents the temperature of the wire at the position $x \in (0, L)$.

Remark 5.1. In $\mathbb{R}^3$ the heat equation becomes of the form:

$$\frac{\partial}{\partial t} u(x,y,z,t) = k \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(x,y,z,t)$$

Theorem 5.1. If $u(x,t)$ and $v(x,t)$ are solutions to the heat equation then

$$\alpha u(x,t) + \beta v(x,t)$$

for $\alpha, \beta \in \mathbb{R}$ is also a solution.

Definition 5.2. Related to the heat equation there are also **boundary conditions** (BC). These take the form of:

- **homogeneous** BC, where

$$u(0,t) = u(L,t) = 0 \quad \forall t \geq 0;$$

- **insulated ends** or **Neumann** BC, where

$$\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(L,t) = 0 \quad \forall t \geq 0.$$

Remark 5.2. We can have other types of boundary conditions which take on a more complicated form, for example:

$$u(0,t) = e^t \quad \text{and} \quad u(L,t) = \sin(t) \quad \forall t \geq 0.$$

Definition 5.3. The **initial condition** of the heat equation is defined as

$$u(x,0) = f(x) \quad \text{for } x \in [0,L]$$

for an $f : [0,L] \to \mathbb{R}$.

5.1 Homogeneous boundary conditions

In this section we show methods to solve the heat equation with the following conditions:

$$\begin{cases} u_t = ku_{xx} & x \in (0,L), \quad t,k > 0 \\ u(0,t) = u(L,t) = 0 & \text{(Homogeneous BC)} \\ u(x,0) = f(x) & \text{(initial condition)}, \end{cases}$$

where $f : [0,L] \to \mathbb{R}$ is a continuous function such that f' is piecewise conditions and $f(0) = f(L) = 0$.

Theorem 5.1 (Unique solution to HE with homogeneous BC)
The **unique** solution to the heat equations with the conditions specified above is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L}x\right) \exp\left(-k\left(\frac{\pi n}{L}\right)^2 t\right) \\ &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{L}x\right) e^{-k\left(\frac{\pi n}{L}\right)^2 t} \end{aligned}$$

where

$$c_n := \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n}{L}x\right)\,dx.$$

Sketch proof. Need to check that we can differentiate the power series term by term in respect to both x and t i.e. the power series converges absolutely and uniformly on $[0,L]$. $\square$

Corollary 5.1. We have that $u(x,t) = 0$ is a solution to the heat equation with homogeneous BC $\iff f(x) = 0$. We call this the **trivial solution**.

Example 5.1
Find a solution to the following heat conduction problem:

$$\begin{cases} u_t = 7u_{xx} & x \in (0,\pi) \quad t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{(homogeneous BC)} \\ u(x,0) = 3\sin(2x) - 6\sin(5x) & \text{(initial condition).} \end{cases}$$

Solution. By the theorem there exists a unique solution to the problem and is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \sin(nx)\exp(-7n^2 t) \\ &= \sum_{n=1}^{\infty} c_n \sin(nx) e^{-7n^2 t}. \end{aligned}$$

To find the coefficients c_n we can evaluate the integral, but it is easier to impose the initial condition; we note that

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} c_n \sin(nx) \\ &= 3\sin(2x) - 6\sin(5x), \end{aligned}$$

by comparing the LHS and RHS we have that $c_2 = 3, c_5 = -6$ and the remaining $c_n = 0$. Therefore, the solution to the problem is given by

$$u(x,t) = 3e^{-28t}\sin(2x) - 6e^{-175t}\sin(5x).$$

5.2 Fourier series and the initial condition

In this section we address the problems related to the convergence of the series

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

to the initial condition.

Proposition 5.1. Let $h(x) : [-L, L] \to \mathbb{R}$ and let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

be a function series converging uniformly to h. Then,

$$a_n = \frac{1}{L}\int_{-L}^{L} h(x)\cos\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N} \cup \{0\}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N}.$$

Lemma 5.1

We have the following results.

- For $n, m \in \mathbb{N} \cup \{0\}$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\, dx = 0.$$

- For $m, n \in \mathbb{N}$,

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m. \end{cases}$$

- For $m, n \in \mathbb{N} \cup \{0\}$,

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m \neq 0, \\ 2 & \text{if } n = m = 0. \end{cases}$$

Lemma 5.2

Some properties of the trigonometric functions:

- $\cos(-x) = \cos(x)$ ($\cos(x)$ is an even function);
- $\sin(-x) = -\sin(x)$ ($\sin(x)$ is an odd function);
- $\cos(n\pi) = (-1)^n$;
- $\sin(n\pi) = 0$.

Remark 5.3. Even functions: $f(x) = f(-x)$.
Odd functions: $-f(x) = f(-x)$.

Example 5.2

Finding Fourier series of a function.

Definition 5.4. Let $h(x) : [-L, L] \to \mathbb{R}$, the infinite sum

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} h(x)\cos\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N} \cup \{0\}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi}{L}x\right) \quad \text{for } n \in \mathbb{N},$$

is called the **Fourier series** of $h(x)$.

Definition 5.5. Let $h(x) : [0, L] \to \mathbb{R}$, the infinite sum

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$$

with

$$a_n = \frac{2}{L}\int_{0}^{L} h(x)\sin\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N},$$

is called the **Fourier sine series** of $h(x)$.

Definition 5.6. Let $h(x) : [0, L] \to \mathbb{R}$, the infinite sum

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

with

$$a_n = \frac{2}{L}\int_{0}^{L} h(x)\cos\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N}\cup\{0\},$$

is called the **Fourier cosine series** of $h(x)$.

Theorem 5.2. Let $h, h' : [-L, L] \to \mathbb{R}$ be piecewise continuous functions and let

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

be the Fourier series of h. Then, for any $x \in (-L, L)$ we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right] = \frac{1}{2}\left[h(x^+) + h(x^-)\right].$$

For $x = \pm L$, the series converges to

$$\frac{1}{2}\left[h((-L)^+) + h(L^-)\right].$$

Here,

$$h(x^+) = \lim_{\delta\to\infty} h\left(x + \frac{1}{\delta}\right) \quad \text{and} \quad h(x^-) = \lim_{\delta\to\infty} h\left(x - \frac{1}{\delta}\right).$$

Furthermore, if h is continuous at x then, $h(x^+) = h(x^-)$.

Corollary 5.2. If h is continuous with $h(0) = h(L)$ and $h'' : [-L, L] \to \mathbb{R}$ is piecewise continuous then, the Fourier series of $h'(x)$ can be obtained by term wise differentiation. Namely, the Fourier series of $f'(x)$ is

$$\sum_{n=1}^{\infty} \frac{\pi n}{L}\left[-a_n \sin\left(\frac{n\pi}{L}x\right) + b_n \cos\left(\frac{n\pi}{L}x\right)\right].$$

Definition 5.7. Given $h : [0, L] \to \mathbb{R}$, let $\widehat{h}_{\text{odd}} : [-L, L] \to \mathbb{R}$ be the **odd extension** of h. That is,

$$\widehat{h}_{\text{odd}}(x) = \begin{cases} h(x) & \text{for } x \in [0, L] \\ -h(-x) & \text{for } x \in [-L, 0). \end{cases}$$

Proposition 5.2. The Fourier series of $\widehat{h}_{\text{odd}}$ is equal to the Fourier Sine series of h. That is, the Fourier series of $\widehat{h}_{\text{odd}}$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

where

$$b_n = \frac{2}{L}\int_0^L \widehat{h}_{\text{odd}}(x) \sin\left(\frac{n\pi}{L}x\right)\, dx \quad \text{for } n \in \mathbb{N}.$$

Corollary 5.3. Let $h : [0, L] \to \mathbb{R}$ be a continuous function such that $h(0) = h(L) = 0$ and such that h' is piecewise continuous. Let $\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$ be the Fourier sine series of h. Then,

$$h(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right).$$

Definition 5.8. Given $h : [0, L] \to \mathbb{R}$, let $\widehat{h}_{\text{even}} : [-L, L] \to \mathbb{R}$ be the **even extension** of h. That is,

$$\widehat{h}_{\text{even}}(x) = \begin{cases} h(x) & \text{for } x \in [0, L] \\ h(-x) & \text{for } x \in [-L, 0). \end{cases}$$

Proposition 5.3. The Fourier series of $\widehat{h}_{\text{even}}$ is equal to the Fourier cosine series of h. That is, the Fourier series of $\widehat{h}_{\text{even}}$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right),$$

where

$$a_n = \frac{2}{L}\int_0^L \widehat{h}_{\text{even}}(x) \cos\left(\frac{n\pi}{L}x\right) \quad \text{for } n \in \mathbb{N} \cup \{0\}.$$

Corollary 5.4. Let $h : [0, L] \to \mathbb{R}$ be a continuous function such that h' is piecewise continuous. Let $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$ be the Fourier cosine series of h. Then,

$$h(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right).$$

5.3 Separation of variables

In this section we solve the following problem

$$\begin{cases} u_t = ku_{xx} & x \in (0, L) \quad t, k > 0 \\ u(0, t) = u(L, t) = 0, \end{cases}$$

with the method of **separation of variables** i.e. when

$$u(x, t) = X(x)T(t).$$

By plugging in the boundary conditions we obtain the trivial solution of the heat equation. We note that

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial t}X(x)T(t)\\ &= X(x)T'(t),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \frac{\partial}{\partial x}X(x)T(t)\\ &= X''(x)T(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= k\frac{\partial^2}{\partial x^2}u(x,t)\\ \iff X(x)T'(t) &= kX''(x)T(t)\\ \iff \frac{T'(t)}{kT(t)} &= \frac{X''(x)}{X(x)}.\end{aligned}$$

The only way this equation holds if both the RHS and LHS are equal to a constant. Let this constant be $\lambda \in \mathbb{R}$ such that

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

Note 5.1. We use $-\lambda$ as a convention.

This can be rephrased as the following: there exists $\lambda \in \mathbb{R}$ such that

$$T'(t) = -\lambda k T(t) \quad \text{and} \quad X''(x) = -\lambda X(x).$$

5.3.1 Solution of time ODE

Consider the time dependent ODE:

$$T'(t) = -\lambda k T(t).$$

With the standard methods of solving first order ODE we obtain the solution

$$T(t) = Ce^{-\lambda kt}$$

for $C \in \mathbb{R}$.

5.3.2 Solution of 'position' ODE

Consider the position dependent ODE:

$$\begin{cases} X''(x) = -\lambda X(x) & x \in [0, L] \\ X(0) = X(L) = 0, \end{cases}$$

clearly, the trivial solution $X(x) = 0$ for all $t \geq 0$ is a valid solution, but we are interested in the non-trivial solutions.

Definition 5.9. The value λ for which a non-trivial solution of the ODE above is called an **eigenvalue** for the Dirichlet problem with homogeneous BC. A non-zero solution related to this value of λ is called an **eigenfunction**.

Theorem 5.2
The ODE above:

- does **NOT** have a non-zero solution when $\lambda < 0$;
- does **NOT** have a non-zero solution when $\lambda = 0$;

Proof. We prove each bullet point in turn.

- Proof when $\lambda < 0$.
  The general solution to the ODE is of the form
  $$X(x) = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}$$
  for $C_1, C_2 \in \mathbb{R}$. Imposing the boundary condition $X(0) = 0$ gives $C_2 = -C_1$; imposing $X(L) = 0$ gives
  $$C_1 \left(e^{L\sqrt{-\lambda}} - e^{-L\sqrt{-\lambda}}\right) = 0,$$
  if $C_1 = 0$ then $X(x) = 0$. Therefore, we can assume $C_1 \neq 0$. In this case we have a non-zero solution when $\lambda < 0$ satisfies
  $$\begin{aligned} 0 &= e^{L\sqrt{-\lambda}} - e^{-L\sqrt{-\lambda}} \\ &= \left(e^{2L\sqrt{-\lambda}} - 1\right) e^{-L\sqrt{-\lambda}} \end{aligned}$$
  This is 0 if and only if $e^{2L\sqrt{-\lambda}} - 1 = 0$ that is, if $2L\sqrt{-\lambda} = 0$. However, since both λ and L are assumed to be non-zero we have that $2L\sqrt{-\lambda} \neq 0$ always.

- Proof when $\lambda = 0$.
  The general solution to the ODE is of the form
  $$X(x) = C_1 x + C_2$$
  for $C_1, C_2 \in \mathbb{R}$. By imposing the boundary conditions we have that $C_1 = C_2 = 0$ hence, $X(x) = 0$.

□

Proposition 5.1
The ODE above a non-zero solution when $\lambda = \left(\frac{\pi n}{L}\right)^2$ for $n \in \mathbb{N}$. The solution is given by the function
$$X_n(x) = \sin\left(\frac{\pi n}{L}\right).$$
Furthermore, any other solution can be obtained by multiplying $X_n(x)$ by a constant.

Proof. Suppose $\lambda > 0$ then

$$X(x) = C_1 \cos(x\sqrt{\lambda}) + C_2 \sin(x\sqrt{\lambda})$$

for $C_1, C_2 \in \mathbb{R}$. Imposing the initial condition $X(0) = 0$ gives that $C_1 = 0$ and imposing $X(L) = 0$ we have

$$X(x) = C_2 \sin(L\sqrt{\lambda}).$$

This is true when

$$\begin{aligned} L\sqrt{\lambda} &= n\pi \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \end{aligned}$$

for $n \in \mathbb{N}$. Furthermore, for $\lambda = \left(\frac{n\pi}{L}\right)^2$ the function

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

is the non-zero solution and, any other solution can be obtained by multiplying $X_n(x)$ by a constant. □

5.3.3 The general solution

We know solutions exists for $\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n \in \mathbb{N}$ so, a solution to

$$T'(t) = -k\left(\frac{n\pi}{L}\right)^2 T(t)$$

is

$$T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

In conclusion, the method of separation of variables gives that the solution to

$$\begin{cases} u_t = ku_{xx} & x \in (0, L)\ t, k > 0 \\ u(0,t) = u(L,t) = 0, \end{cases}$$

is given by

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

Remark 5.4. The set

$$S = \left\{\sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} : n \in \mathbb{N}\right\}$$

is a countable set of solution of the heat equation which satisfies the homogeneous BC. Therefore, any finite linear combination of the elements in S is a solution to the ODE. That is, the function series

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{for } a \in \mathbb{R}$$

is a solution. By the theory of Fourier series we have that the coefficients are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

5.4 Uniqueness of the solution

Proposition 5.2

The function $v(x,t) = 0$ is the **unique** solution to the heat conduction problem:

$$\begin{cases} v_t = kv_{xx} \quad x \in [0,L] \text{ and } t,k > 0 \\ v(L,t)\dfrac{\partial}{\partial x}v(L,t) - v(0,t)\dfrac{\partial}{\partial x}v(0,t) = 0 \quad \text{(BC)} \\ v(x,0) = 0. \end{cases}$$

Proof. Clearly, $v(x,t) = 0$ is a solution the problem. It remains to prove that it is a unique solution. Let

$$I(t) = \frac{1}{2}\int_0^L v(x,t)^2\,dx.$$

Note that $I(t) \geq 0$ for all $t \in [0,\infty)$ and that $I(0) = 0$. It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I(t) &= \int_0^L v(x,t)\frac{\partial}{\partial t}v(x,t) \\ &= k\int_0^L v(x,t)\frac{\partial^2}{\partial x^2}v(x,t), \end{aligned}$$

where the last equality is achieved since the heat conduction problem tells us $v_t = kv_{xx}$. Using integration by parts we have that

$$\begin{aligned} k\int_0^L v(x,t)\frac{\partial^2}{\partial x^2}v(x,t)\,dx &= k\left[v(x,t)\frac{\partial}{\partial x}v(x,t)\right]_0^L - k\int_0^L\left(\frac{\partial}{\partial x}v(x,t)\right)^2\,dx \\ &= -k\int_0^L\left(\frac{\partial}{\partial x}v(x,t)\right)^2\,dx. \end{aligned}$$

Notice that $\int_0^L\left(\frac{\partial}{\partial x}v(x,t)\right)^2\,dx \geq 0$ and $k > 0$ so,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I(t) &= -k\int_0^L\left(\frac{\partial}{\partial x}v(x,t)\right)^2\,dx \\ &\leq 0. \end{aligned}$$

That ism $I(t)$ is a non-increasing function of t and since $I(0) = 0$ and $I(t) \geq 0$ we must have $I(t) = 0$ i.e.

$$\frac{1}{2}\int_0^L v(x,t)^2\,dx = 0.$$

Clearly, this implies that $v(x,t) = 0$. $\square$

Theorem 5.3
The solution to the heat equation with homogeneous BC is **unique**.

Proof. For the sake of contradiction assume the solution is not unique and let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the heat equation with homogeneous BC. We have that

$$v(x,t) = u_1(x,t) - u_2(x,t)$$

is a solution to the heat equation above hence, by applying the proposition above we have that $v(x,t) = 0$ is the unique solution. We conclude,

$$u_1(x,t) = u_2(x,t).$$

□

5.5 Insulated ends

In this section we want to solve the following heat conduction problem:

$$\begin{cases} u_t = k u_{xx} & x \in (0,L), \quad t,k > 0 \\ u_x(0,t) = u_x(L,t) = 0 & \text{(Neumann BC)} \\ u(x,0) = f(x) & \text{(initial condition)}, \end{cases}$$

where $f : [0,L] \to \mathbb{R}$ is a continuous function such that f' is piecewise continuous and $f'(0) = f'(L) = 0$.

Theorem 5.4 (Unique solution to HE with insulated ends)
The **unique** solution to the heat equation with the conditions specified above is

$$\begin{aligned} u(x,t) &= \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{\pi n}{L}x\right) \exp\left(-k\left(\frac{\pi n}{L}\right)^2 t\right) \\ &= \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{\pi n}{L}x\right) e^{-k\left(\frac{\pi n}{L}\right)^2 t}, \end{aligned}$$

where

$$c_n := \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n}{L}x\right)\,dx.$$

Example 5.3

Consider the following heat conduction initial boundary value problem:

$$\begin{cases} u_t = 7u_{xx} & x \in (0, \pi), \quad t > 0, \\ u_x(0,t) = u_x(\pi, t) = 0 & \text{(Neumann BC)} \\ u(x,0) = 5 + \cos(2x) - 2\cos(3x) & \text{(initial condition).} \end{cases}$$

By the theorem there exists a unique solution of the form

$$u(x,t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) e^{-7n^2 t}.$$

To find the coefficients c_n we can evaluate the integral, but it is easier to impose the initial condition; we note that

$$\begin{aligned} u(x,0) &= \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) \\ &= 5 + \cos(2x) - 2\cos(3x). \end{aligned}$$

Therefore, $\frac{1}{2}c_0 = 5, c_1 = 1, c_3 = -2$ and the remaining $c_n = 0$. We conclude, the general solution to this problem is

$$u(x,t) = 5 + e^{-28t}\cos(2x) - 2e^{-63t}\cos(3x).$$

5.5.1 Solution for 'position' ODE

Assume $u(x,t) = X(x)T(t)$ by our previous discussion of separation of variables we now consider the ODE

$$\begin{cases} X''(x) = -\lambda X(x) & x \in [0, L] \\ X'(0) = X'(L) = 0 \end{cases}$$

and when it has solution.

Definition 5.10. The value λ for which a non-trivial solution of the ODE with the specified condition above exists is called an **eigenvalue** for the Dirichlet problem with **Neumann boundary conditions**. A non-zero solution related to this value of λ is called **eigenfunction**.

Theorem 5.5

The ODE with the specified conditions above does **NOT** have a non-zero solution when $\lambda < 0$.

Proof. In this case the solution is of the form

$$X(x) = C_1 e^{x\sqrt{-\lambda}} + C_2 e^{-x\sqrt{-\lambda}}$$

for $C_1, C_2 \in \mathbb{R}$ so,

$$X'(x) = \sqrt{-\lambda}\left(C_1 e^{x\sqrt{-\lambda}} - C_2 e^{-x\sqrt{-\lambda}}\right).$$

Imposing the condition $X'(0) = 0$ gives us $C_1 = C_2$; imposing $X'(L) = 0$ gives

$$\sqrt{-\lambda}\left(C_1 e^{L\sqrt{-\lambda}} - C_2 e^{-L\sqrt{-\lambda}}\right) = 0 \iff C_1\left(e^{L\sqrt{-\lambda}} - e^{-L\sqrt{-\lambda}}\right) = 0.$$

Since we are not interested in the trivial solution we can assume $C_1 \neq 0$ therefore,

$$\begin{aligned} 0 &= e^{L\sqrt{-\lambda}} - e^{-L\sqrt{-\lambda}} \\ &= \left(e^{2L\sqrt{-\lambda}} - 1\right) e^{-L\sqrt{-\lambda}} \end{aligned}$$

This is 0 if and only if $e^{2L\sqrt{-\lambda}} - 1 = 0$ that is, if $2L\sqrt{-\lambda} = 0$. However, since both λ and L are assumed to be non-zero we have that $2L\sqrt{-\lambda} \neq 0$ always. □

Theorem 5.6

The ODE with the specified conditions has a non-zero solution when $\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n \in \mathbb{N} \cup \{0\}$. The solution is given by

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right).$$

Furthermore, any other solution can be obtained by multiplying $X_n(x)$ by a constant.

Proof. There are two special cases.

- If $\lambda > 0$ then,
$$X(x) = C_1\cos(x\sqrt{\lambda}) + C_2\sin(x\sqrt{\lambda})$$
which implies
$$X'(x) = -C_1\sqrt{\lambda}\sin(x\sqrt{\lambda}) + C_2\sqrt{\lambda}\cos(x\sqrt{\lambda}).$$
Imposing the condition that $X'(0) = 0$ gives $C_2 = 0$ so,$X(x) = C_1\cos(x\sqrt{x})$; imposing the condition $X'(L) = 0$ gives
$$-C_1\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0.$$
This is true when
$$L\sqrt{L} = n\pi \quad \text{for } n \in \mathbb{N},$$
that is when,
$$\lambda = \left(\frac{n\pi}{L}\right)^2.$$
Hence, the solution is given by
$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right),$$
and any other solution can be obtained by multiplying $X_n(x)$ by a constant.

- If $\lambda = 0$ then,
$$X_0(x) = \cos(0) = 1$$
which is a non-zero solution.

□

Remark 5.5. The set

$$S = \left\{\cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} : n \in \mathbb{N} \cup \{0\}\right\}$$

is a countable set of solution of the heat equation which satisfies the Neumann BC. Therefore, any finite linear combination of the elements in S is a solution to the ODE. That is, the function series

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \quad \text{for } a \in \mathbb{R}$$

is a solution. By the theory of Fourier series we have that the coefficients are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\, dx.$$

5.6 Constant boundary conditions

In this section we show methods to solve the heat equation with the following conditions:

$$\begin{cases} u_t = ku_{xx} & x \in (0,L), \quad t,k > 0 \\ u(0,t) = U_1 \\ u(L,t) = U_2 \\ u(x,0) = f(x) & \text{(initial condition)}, \end{cases}$$

where $f : [0,L] \to \mathbb{R}$ is a continuous function such that f' is piecewise conditions and $f(0) = U_1$ and $f(L) = U_2$.

Definition 5.11. Given a heat conduction problem, the **equilibrium solution** or the **steady-state solution** is a solution $u(x,t)$ of the problem (not including the initial condition) that does not depend on t. That is,

$$\frac{\partial}{\partial t}u(x,t) = 0 \quad \forall t \geq 0.$$

We denote such solution by $u_e(x)$.

Remark 5.6. Requiring that $\frac{\partial}{\partial t}u_e(x) = 0$ and that it satisfies the heat equation gives that $\frac{\partial^2}{\partial t^2}u_t(x) = 0$. Therefore, $u_e(x)$ must be of the form $Ax + B$. By imposing the initial condition we obtain the values of A and B.

Theorem 5.7 (Equilibrium solution)

The equilibrium solution of the heat conduction problem with the specified conditions above is

$$u_e(x) = U_1 + \frac{(U_2 - U_1)x}{L}.$$

Theorem 5.8 (Unique solution to HE with constant BC)
The **unique** solution to the heat equation with the conditions specified above is

$$u(x,t) := w(x,t) + u_e(x)$$

where $u_e(x)$ is the equilibrium solution and $w(x,t)$ is the solution of the following problem:

$$\begin{cases} u_t = ku_{xx} & x \in (0,L) \quad t,k > 0 \\ u(0,t) = u(L,t) = 0 & \text{(homogeneous BC)} \\ u(x,0) = f(x) - u_e(x). \end{cases}$$

5.7 Maximum principle

Note 5.2. Refer to this video Maximum principle at minute $8:00$.

Note 5.3. Let $u(x,t)$ be a solution to the heat equation $u_t - ku_{xx} \leq 0$ for $k > 0$ then,

$$\max u(x,t) = u(0,0) \text{ or } u(L,0).$$

Theorem 5.9 (Maximum principle)
Suppose that $u(x,t)$ satisfies

$$u_t - ku_{xx} \leq 0 \quad \text{for } k > 0$$

in the spacetime rectangle $\Omega_T = (0,L) \times (0,T]$. Then,

$$\max_{\overline{\Omega}_T = [0,L]\times[0,T]} u = \max_{\overline{\Omega}_T \setminus \Omega_T} .$$

In particular,

$$\sup_{[0,L]\times[0,\infty]} u = \max_{\overline{\Omega}_\infty \setminus \Omega_\infty} .$$

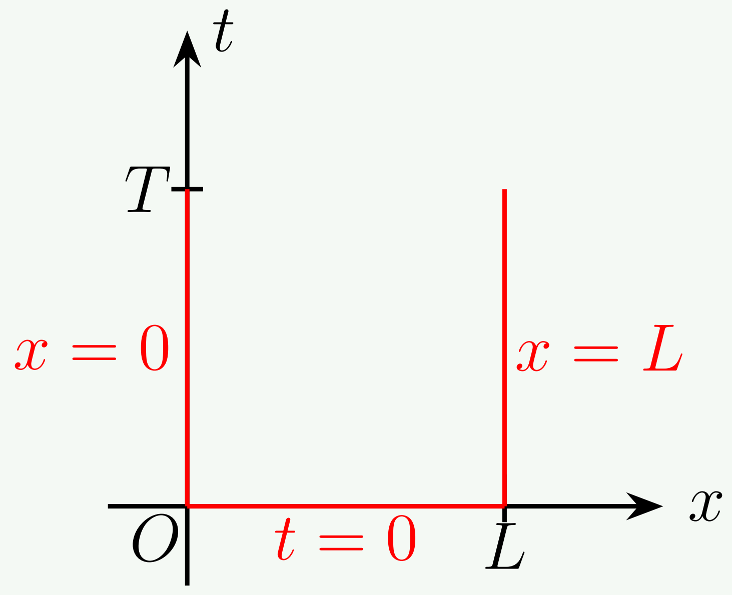
Note 5.4. We can interpret the Maximum principle as follows:
Suppose that $u(x,t)$ satisfies

$$u_t - ku_{xx} \leq 0 \quad \text{for } k > 0$$

in the spacetime rectangle $0 \leq x \leq L$, $0 \leq t \leq T$. Then, the maximum value of u occurs at some point on the boundary lines

$$t = 0, x = 0 \text{ or } x = L$$

Below we have an illustration of the rectangle.



Lemma 5.1. If u attains its maximum over $\overline{\Omega}_T$ at a point $(x_0, t_0) \in \Omega_T$ then,

$$u_t(x_0, t_0) \geq 0 \quad \text{and} \quad u_{xx}(x_0, t_0) \leq 0.$$

In particular,

$$u_t(x_0, t_0) - k u_{xx}(x_0, t_0) \geq 0.$$

Proposition 5.3

Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of

$$\begin{cases} u_t = k u_{xx} \quad x \in (0, L),\, t, k > 0 \\ u(0, t) = u(L, t) = 0 \\ u_1(x, 0) = f_1(x) \\ u_2(x, 0) = f_2(x). \end{cases}$$

Then,

$$\max_{\overline{\Omega}_T} |u_1 - u_2| \leq \max_{[0,L]} |f_1 - f_2| \,.$$

6 The wave equation

Given a length $L > 0$ of a ‘perfectly flexible’ elastic string stretched between two points at distance L, the **wave equation** says that the displacement $u(t, x)$ for $x \in (0, L)$ and time $t > 0$ changes according to the following problem.

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & x \in (0, L),\ t > 0 \\ u(0,t) = u(L,t) = 0 & \text{(Homogeneous BC)} \\ u(x,0) = f(x) & \text{(initial displacement condition)} \\ u_t(x,0) = g(x) & \text{(Initial velocity condition),} \end{cases}$$

where α^2 depends on the properties of the string and $f, g : [0, L] \to \mathbb{R}$ are smooth functions with $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$.

Remark 6.1. This is a second order linear PDE.

Theorem 6.1

The **unique** solution to the ODE above with the specified conditions is

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{\pi n \alpha}{L} t\right) + b_n \sin\left(\frac{\pi n \alpha}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right)$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n}{L} x\right)\, dx,$$

and

$$\frac{n\pi\alpha}{L} b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{\pi n}{L} x\right)\, dx.$$

Example 6.1

Solve the following problem:

$$\begin{cases} u_{tt} = 9u_{xx} \quad x \in (0, \pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 4\sin(3x) \\ u_t(x,0) = 14\sin(7x). \end{cases}$$

By the theorem above, there exists a unique solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \cos(3nt) + b_n \sin(3nt)] \sin(nx).$$

To find the coefficients we impose the initial conditions:

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} a_n \sin(nx) \\ &= 4\sin(3x) \end{aligned}$$

and

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty} 3nb_n \sin(nx) \\ &= 14\sin(7x). \end{aligned}$$

These conditions imply that $a_3 = 4$ and all other $a_n = 0$, it also implies $b_7 = \frac{14}{21} = \frac{2}{3}$ and all other $b_n = 0$. Therefore, the solution is

$$u(x,t) = 4\cos(9t)\sin(3x) + \frac{2}{3}\sin(21t)\sin(7x).$$

6.1 Zero initial velocity

In this section we consider the problem

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & x \in (0, L), t > 0 \\ u(0,t) = u(L,t) = 0 & \\ u(x,0) = f(x) & \\ u_t(x,0) = 0 & \text{(Zero initial velocity).} \end{cases}$$

Theorem 6.2
The **unique** solution to the problem above is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi\alpha}{L}t\right) \sin\left(\frac{n\pi}{L}x\right)$$

with

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)\, dx.$$

6.1.1 Separation of variables

In order to prove the theorem above we will consider a simpler problem. Consider the problem

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} \quad x \in (0,L),\, t > 0 \\ u(0,t) = u(L,t) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Suppose, $u(x,t)$ is separable i.e.

$$u(x,t) = X(x)T(t),$$

once again $u(x,t) = 0$ is a solution, but we are interested in non-trivial solutions. Imposing the boundary conditions, and we have

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0$$

for all $t > 0$. In order to have $X(0)T(t) = 0$ for all $t > 0$ we either have $X(0) = 0$ or $T(t) = 0$ for all $t > 0$ however, the second option leads back to the trivial solution thus, we must have

$$X(0) = 0.$$

By a similar argument $X(L) = 0$. Now imposing the initial condition, $u_t(x,0) = 0$, we have that

$$\frac{\partial}{\partial t}u(x,0) = X(x)T'(0) = 0$$

by a similar reasoning as above this gives that $T'(0) = 0$. Substituting, $u(x,t) = X(x)T(t)$ into the problem implies, there exist $\lambda \in \mathbb{R}$ such that

$$T''(t) = -\lambda\alpha^2 T(t) \quad \text{and} \quad X''(x) = -\lambda X(x).$$

Consider the position dependent ODE,

$$\begin{cases} X''(x) = -\lambda X(x) \\ X(0) = X(L) = 0. \end{cases}$$

We have seen that this ODE has non-zero solutions only when

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } n \in \mathbb{N}$$

and, when that is the case, the solution is given by

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

Any other solution can be obtained by multiplying $X_n(x)$ by a constant.

With this in mind, for a given $\lambda = \left(\frac{n\pi}{L}\right)^2$ the time dependent ODE becomes

$$T''(t) = -\left(\frac{n\pi\alpha}{L}\right)^2 T(t)$$

and, the general solution is

$$T_n(t) = a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right).$$

We have that

$$T_n'(t) = -a_n\left(\frac{n\pi\alpha}{L}\right)\sin\left(\frac{n\pi\alpha}{L}t\right) + b_n\left(\frac{n\pi\alpha}{L}\right)\cos\left(\frac{n\pi\alpha}{L}t\right)$$

and imposing the initial condition gives

$$T_n'(0) = b_n\left(\frac{n\pi\alpha}{L}\right) = 0.$$

This implies that $b_n = 0$ and

$$T_n(t) = a_n \cos\left(\frac{n\pi\alpha}{L}t\right).$$

We have that

$$u(x,t) = a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi\alpha}{L}t\right).$$

Consider the set

$$S = \left\{\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi\alpha}{L}t\right) : n \in \mathbb{N}\right\},$$

is a countable set of solutions and since the wave equation is linear, any finite linear combination of elements in S is a solution. Therefore, the function series

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi\alpha}{L}t\right) \quad \text{for } a_n \in \mathbb{R},$$

is the solution to the problem.

6.2 Zero initial displacement

In this section we consider the problem

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = 0 & \\ u(x,0) = 0 & \text{(Zero initial displacement)} \\ u_t(x,0) = g(x). & \end{cases}$$

Theorem 6.3

The **unique** solution to the problem above is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\alpha}{L}t\right) \sin\left(\frac{n\pi}{L}x\right)$$

with

$$\frac{n\pi\alpha}{L}b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\, dx.$$

6.2.1 Separation of variables

In order to prove the theorem above we will consider a simpler problem. Consider the problem

$$\begin{cases} u_{tt} = \alpha^2 u_{xx} & x \in (0,L),\ t > 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 0. \end{cases}$$

Suppose, the solution $u(x,t)$ is separable i.e. it can be written as

$$u(x,t) = X(x)T(t).$$

Once again $u(x,t) = 0$ is a solution, but we are interested in non-trivial solutions.

The boundary conditions for $X(x)$ are $X(0) = X(L) = 0$ and imposing the condition $u(x,0) = X(x)T(0) = 0$ which implies $T(0) = 0$.

As in a previous section, we obtain that there must exist $\lambda \in \mathbb{R}$ such that

$$T''(t) = -\lambda\alpha^2 T(t) \quad \text{and} \quad X''(x) = -\lambda X(x).$$

Considering the position dependent ODE

$$\begin{cases} X''(x) = -\lambda X(x) x \in [0,L] \\ X(0) = X(L) = 0, \end{cases}$$

as shown previously this has a solution when

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \text{for } n \in \mathbb{N}$$

and the solution is the function

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

Any other solution can be obtained by multiplying $X_n(x)$ by a constant.

With this in mind, for a given $\lambda = \left(\frac{n\pi}{L}\right)^2$ the time dependent ODE becomes

$$T''(t) = -\left(\frac{n\pi\alpha}{L}\right)^2 T(t),$$

where the general solution is then,

$$T_n(t) = a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right).$$

Imposing the initial condition gives

$$T_n(0) = a_n = 0,$$

which implies that $a_n = 0$ and

$$T_n(t) = b_n \sin\left(\frac{n\pi\alpha}{L}t\right).$$

We have that

$$u(x,t) = b_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi\alpha}{L}t\right).$$

Consider the set

$$S = \left\{\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi\alpha}{L}t\right) : n \in \mathbb{N}\right\},$$

which is a countable set of solutions and since the wave equation is linear, any finite linear combinations of elements in S is a solution. Therefore, the function series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

is the solution.

6.3 Uniqueness of the solution

In this section we prove that the solution to the wave equation and its variants is unique.

Theorem 6.4

Consider the following initial-boundary value problem for the wave equation:

$$\begin{cases} v_{tt} = \alpha^2 v_{xx} & x \in (0,L)\, t > 0 \\ v(L,t) = v(0,t) = 0 \\ v(x,0) = 0 \\ v_t(x,0) = 0. \end{cases}$$

The function $v(x,t) = 0$ is the unique solution to this problem.

Proof. Clearly, $v(x,t) = 0$ is a solution to the problem. First note that the conditions $v(L,t) = v(0,t) = 0$ and $v(x,0) = 0$ imply that $v_t(L,t) = v_t(0,t) = 0$ and $v_x(x,0) = 0$. Let

$$E(t) = \frac{1}{2}\int_0^L \alpha^2 v_x^2 + v_t^2 \, dx \quad \text{(the energy)}.$$

We have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &= \int_0^L \frac{\partial}{\partial t}\left(\alpha^2 v_x^2 + v_t^2\right) dx \\ &= \int_0^L \alpha^2 v_x v_{xt} + v_t v_{tt} \, dx \\ &= \int_0^L \alpha^2 v_x v_{xt} \, dx + \int_0^L v_t v_{tt} \, dx. \end{aligned}$$

□

Theorem 6.5
FINISH PROOF

Theorem 6.6
Solution is unique

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the wave equation.

Theorem 6.7
TO FINISH.

□

6.4 Cracking the whip

In this section we consider the problem

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & x \in (0,\infty),\, t > 0 \\ u(0,t) = h(t) & \lim_{x\to\infty} \sup_{t\geq 0} |u(x,t)| = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Theorem 6.8
The **unique** solution to the problem above is given by

$$u(x,t) = g_{\frac{x}{\alpha}}(t) \cdot h\left(t - \frac{x}{\alpha}\right),$$

where the function

$$g_{\frac{x}{\alpha}}(t) = \begin{cases} 1 & t \geq \frac{x}{\alpha} \\ 0 & \text{otherwise,} \end{cases}$$

is a step function.

Example 6.2
Let

$$h(t) = \begin{cases} \sin(t) & t \in [0, \pi] \\ 0 & t \geq \pi. \end{cases}$$

Then the solution to the problem above is given by

$$u(x,t) = g_{\frac{x}{\alpha}}(t) \cdot h\left(t - \frac{x}{\alpha}\right).$$

Note that $h(t) = (1 - g_\pi(t)) \sin\left(t - \frac{x}{\alpha}\right)$, as such we can write

$$\begin{aligned} u(x,t) &= g_{\frac{x}{\alpha}}(t)\left(1 - g_\pi\left(t - \frac{x}{\alpha}\right)\right) \sin\left(t - \frac{x}{\alpha}\right) \\ &= \begin{cases} \sin\left(t - \frac{x}{\alpha}\right) & t \in \left[\frac{x}{\alpha}, \frac{x}{\alpha} + \pi\right] \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

6.5 Can you hear the shape of a drum?

NO

6.6 The 2-dimensional wave equation

Let Ω be a bounded domain in $\mathbb{R}^2$ and given a function $u(x,y,t)$, let $\Delta u = u_{xx} + u_{yy}$. Then the 2-dimensional wave equation

$$\begin{cases} u_{tt}(x,y,t) = \alpha^2 \Delta u(x,y,t) & (x,y) \in \Omega\, t > 0 \\ u(x,y,t) = 0 & (x,y) \in \partial\Omega \quad \text{(Homogeneous BC)} \\ u(x,y,0) = f(x,y) & \text{(Initial displacement)} \\ u_t(x,y,0) = g(x,y) & \text{(Initial velocity).} \end{cases}$$

Remark 6.2. The notation $\partial\Omega$ means the boundary of the set Ω.

6.6.1 Separation of variables

However, in this section we are going to investigate this problem without the initial displacement and velocity conditions, namely the problem:

$$\begin{cases} u_{tt}(x,y,t) = \alpha^2 \Delta u(x,y,t) & (x,y) \in \Omega\, t > 0 \\ u(x,y,t) = 0 & (x,y) \in \partial\Omega \quad \text{(Homogeneous BC).} \end{cases}$$

Assume the separation is separable, assume that $u(x,y,t) = X(x,y)T(t)$. Now, checking the boundary condition

$$u(x,y,t) = X(x,y)T(t) = 0 \quad \text{for } (x,y) \in \partial\Omega.$$

In order to have this we must either have $X(x, y) = 0$ or $T(t) = 0$ for all $t > 0$. The second option leads us to the trivial solution so, if we are interested in the non-trivial solution we assume:

$$X(x, y) = 0 \quad \text{for } (x, y) \in \partial\Omega.$$

Calculating the second derivatives we have that

$$\begin{aligned} u_{tt} &= XT'', \\ u_{xx} &= X_{xx}T, \\ u_{yy} &= X_{yy}T. \end{aligned}$$

Therefore, wave equation

$$u_{tt} = \alpha^2 \Delta u$$

becomes

$$\begin{aligned} XT'' &= \alpha^2(u_{xx} + u_{yy}) \\ &= \alpha^2 (X_{xx} + X_{yy})\, T \\ &= \alpha^2 T \cdot \Delta X, \end{aligned}$$

This is equivalent to

$$\frac{T''}{\alpha^2 T} = \frac{\Delta X}{X},$$

for this to be true there must exist $\lambda \in \mathbb{R}$ such that

$$\frac{T''}{\alpha^2 T} = -\lambda = \frac{\Delta X}{X}.$$

6.7 The Dirichlet eigenvalues of a disk

Appendix

A Links

- Series

B Laplace transform table

Function	L-Transform
$y(t)$	$Y(s) = \mathcal{L}[y(t)](s)$
e^{at}	$\frac{1}{s-a}$ for $s > a$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega}$

C Techniques of integration

C.1 Integration by parts

Theorem C.1 (Integration by parts)
Let $f, g \in C[a, b]$ with $f', g' \in C[a, b]$; then

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Note C.1. The acronym **LIATE** can be used to choose which function to differentiate. (The I stands inverse trigonometric/hyperbolic functions).

D Tricks

D.1 Step function

Example D.1. Let f satisfy $f(T + t) = f(t)$ for all $t \geq 0$ and for some fixed $T > 0$. Show that

$$\mathcal{L}[f(t)](s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}.$$

Solution. Notice that

$$u_T(t) = \begin{cases} 0 & \text{if } t < T \\ 1 & \text{if } t \geq T. \end{cases}$$

Therefore,

$$1 - u_T(t) = \begin{cases} 1 & \text{if } t < T \\ 0 & \text{if } t \geq T. \end{cases}$$

Now, we can write

$$\begin{aligned}\int_0^T y(t)e^{-st}\,dt &= \int_0^T 1\cdot y(t)e^{-st}\,dt + \int_0^\infty 0\cdot y(t)e^{-st}\,dt \\ &= \int_0^\infty (1-u_T(t))y(t)e^{-st}\,dt.\end{aligned}$$

Example D.2. Compute $\mathcal{L}^{-1}\left[\frac{e^{-4s}}{2s-1}\right](t)$.

Solution. Recall

- $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s > a$.
- $\mathcal{L}[u_a(t)y(t-a)](s) = e^{-as}\mathcal{L}[y(t)](s)$.

We know

$$\mathcal{L}\left[e^{\frac{1}{2}t}\right](s) = \frac{1}{s-\frac{1}{2}}.$$

Therefore, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{e^{-4s}}{2s-1}\right] &= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s-\frac{1}{2}}\right] \\ &= \frac{1}{2}\mathcal{L}^{-1}\left[e^{-4s}\mathcal{L}\left[e^{\frac{1}{2}t}\right]\right] \\ &= \frac{1}{2}\mathcal{L}^{-1}\left[\mathcal{L}\left[u_4(t)e^{\frac{1}{2}(t-4)}\right]\right] \\ &= \frac{1}{2}u_4(t)e^{\frac{1}{2}(t-4)}.\end{aligned}$$

E Hessian matrix

Hessian matrix