Complex Analysis Notes

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Abstract

This is KCL undergraduate module 5CCM212A, instructed by Professor Simon Scott. The formal name for this class is "Complex Analysis".

Contents

1 Preliminaries

1.1 The complex plane

The set $\mathbb{C}$ can be identified with the vector space $\mathbb{R}^2$ with addition and multiplication defined as

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda x + a \\ \lambda y + b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} xa - yb \\ xb + ya \end{pmatrix}.$$

The set $\mathbb{R}^2$ with addition and multiplication rules as defined above is a field, as such it has a **unit**, which is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This is because

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Notice that we can express the vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

On the vector space $\mathbb{R}^2$ the vectors $\begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ y \end{pmatrix}$ lie on the vector space $\mathbb{R}^1$, as such we can create a map from

$$\mathbb{R}^2 \to \mathbb{R}^1$$

$$x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} = \text{``}x\text{''}.$$

Note that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Denote $i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $i^2 = \text{``} - 1\text{''}$. Therefore, we can denote the vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{``}x\text{''} + i\text{``}y\text{''}$$

$$= x + iy$$

$$= z.$$

Definition 1.1. The set $\mathbb{C}$ is the set of ordered pairs $z = (x, y)$ with $x, y \in \mathbb{R}$ and the algebraic operations

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$z_1 \times z_2 = (x_1, y_1) \times (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

1.2 Complex numbers

Definition 1.2. Suppose $z \in \mathbb{C}$ and $z = x + iy$. The numbers x and y are said to be the **real** and **imaginary** parts of z and are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Definition 1.3. The **modulus**, $|z|$, of $z = x + iy$ is defined to be distance of z from the origin so,
$$|z| = \sqrt{x^2 + y^2}.$$

Remark 1.4. Notice that
$$|x + iy|_{\mathbb{C}} = |x + iy|$$
$$= \sqrt{x^2 + y^2}$$
$$= \left| \begin{pmatrix} x \\ y \end{pmatrix} \right|_{\mathbb{R}^2}.$$

Definition 1.5. In the polar coordinates, every non-zero $z \in \mathbb{C}$ is defined by a pair (r, θ) where
$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan \theta = \frac{y}{x}.$$

The numbers r and θ are called **modulus** and the **argument** of the complex number z and are denoted by $|z|$ and $\arg(z)$, respectively.

Theorem 1.6. The argument is not uniquely defined.

Proof. One can take $\theta + 2\pi k$ with any $k \in \mathbb{Z}$ instead of θ. $\qquad \square$

Definition 1.7. The number $\theta + 2\pi k$ such that $-\pi < \theta + 2\pi k \leq \pi$ for $k \in \mathbb{Z}$ is said to be the **principal value** of the argument and is denoted by $\mathrm{Arg}(z)$.

Theorem 1.8 (Euler's formula)**.** Define $e^{i\theta} = \cos \theta + i \sin \theta$. Then every non-zero complex number z can be written in the form $|z| \, e^{i \arg(z)}$.

Theorem 1.9 (De Moivre's formula)

For all $n \in \mathbb{N}$ we have
$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Corollary 1.10. In particular $|z_1 z_2| = |z_1| \, |z_2|$.

Definition 1.11. The expression $|z_1 - z_2|$ is the Euclidean distance between z_1 and z_2.

Corollary 1.12. The value $|z|$ is the distance from z to the origin.

Remark 1.13. The triangle inequality, $|z_1 + z_2| \leq |z_1| + |z_2|$, holds.

Definition 1.14. The complex number $\overline{z} = x - iy$ is called the **complex conjugate**.

Theorem 1.15. Results of the complex conjugate:

- $\overline{z} = |z| \, e^{-i \arg(z)}$;

- $z\overline{z} = |z|^2$;

- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$;

- $\overline{(z^{-1})} = (\overline{z})^{-1}$;

- $\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}$;

- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ if $w \neq 0$;

- $z + \overline{z} = 2\operatorname{Re}(z)$;

- $z - \overline{z} = 2i\operatorname{Im}(z)$.

1.2.1   The exponential function

Definition 1.16. For $z = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$ we define the **complex exponential**, denoted by e^z, as follows

$$
\begin{aligned}
e^z &= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x(\cos x + i\sin y).
\end{aligned}
$$

Proposition 1.17. Some properties of the exponential function:

1. $e^0 = 1$;

2. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$,

3. $e^z \neq 0$ for all $z \in \mathbb{C}$,

4. for $z \in \mathbb{C}$, $e^{z+2\pi i} = e^z$,

5. for $z \in \mathbb{C}$, $|e^z| = e^{\operatorname{Re}(z)}$,

6. $\overline{\exp(z)} = \exp(\overline{z})$.

Remark 1.18. In general, if φ is a holomorphic function with φ and $\varphi(\overline{z})$ are defined then

$$
\varphi(\overline{z}) = \overline{\varphi(z)}.
$$

1.2.2   Trigonometric and hyperbolic functions

Definition 1.19. One can define the hyperbolic functions as

$$
\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)
$$
$$
\cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right).
$$

Definition 1.20. We define the trigonometric functions

$$
\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)
$$
$$
\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).
$$

> **Theorem 1.21** (Osborn's rule)
> A key relation between the trigonometric and hyperbolic functions:
> $$\cos(iz) = \cosh(z) \quad \text{and} \quad \sin(iz) = i\sinh(z).$$

Corollary 1.22. We have

$$\cos(z) = \cosh(iz) \quad \text{and} \quad i\sin(z) = \sinh(iz).$$

Proof. Using Osborn's rule and $z = iz$ the identities follow. $\qquad\square$

Theorem 1.23. The addition/subtraction rules for both trigonometric and hyperbolic functions hold in $\mathbb{C}$

1.3 Geometry of the complex plane

We can identify $\mathbb{C}$ as a set with the points in the plane $\mathbb{R}^2$.

Definition 1.24. The complex plane is (sometimes) called the **Argand plane**.
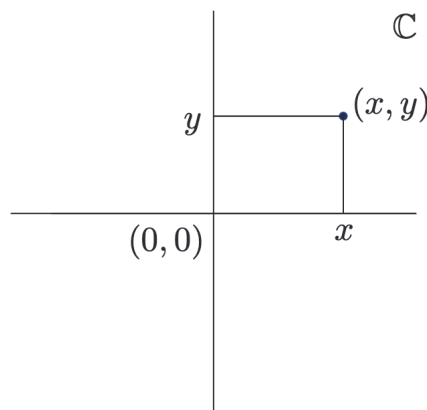
Figure 1: The complex number $x + iy$ in the complex plane.

Geometric meaning of complex addition

With this identification of $\mathbb{C}$ it is clear that addition of complex numbers is just addition of vectors in $\mathbb{R}^2$. By addition of vectors, we mean completing the parallelogram formed by the line segment joining $(0,0)$ to each of the two complex numbers as sides, and then taking the endpoint of the diagonal from $(0,0)$ as the sum of the two given complex numbers. We illustrate this below.
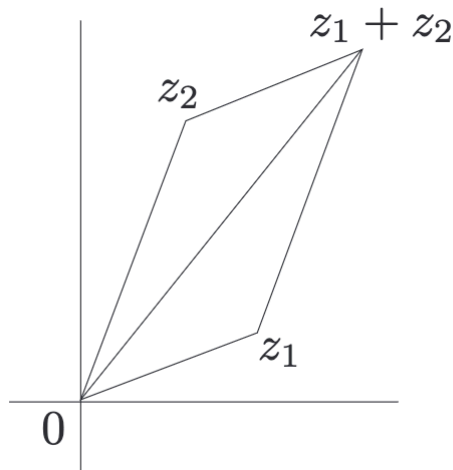
Figure 2: Geometric meaning of complex addition.

Geometric meaning of complex multiplication

For two complex numbers expressed in polar coordinates as

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

we have that

$$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Thus, $z_1 \cdot z_2$ has polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$. That is, the angles z_1 and z_2 make with the positive $\operatorname{Re}(z)$ axis are *added* in order to get the angle $z_1 \cdot z_2$ makes with the positive $\operatorname{Re}(z)$ axis, and the distance to the origin are *multiplied* to get the distance $z_1 \cdot z_2$ has to the origin.
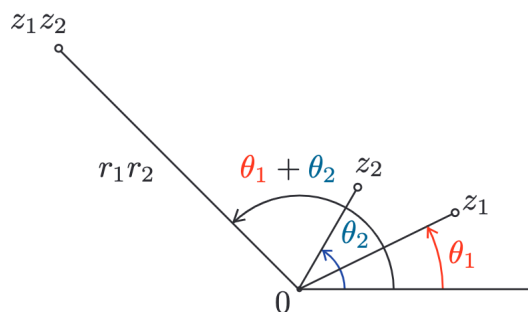
Figure 3: Geometric meaning of complex multiplication: angles get added and distances to the origin get multiplied.

Theorem 1.25

Multiplication of z by i results in an anticlockwise rotation of $\frac{\pi}{2}$ rad.

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1.4   Topological properties
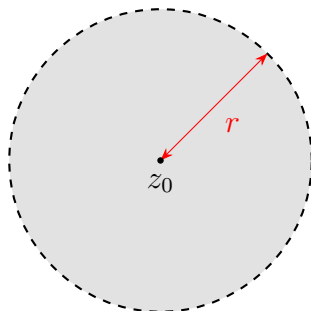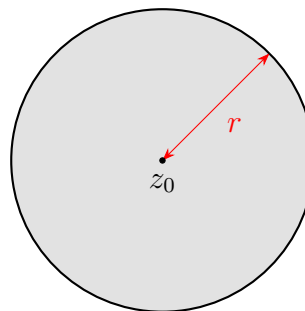
(a) Open disc   (b) Closed disc

Definition 1.26. Let z_0 be a fixed point of $\mathbb{C}$ and $r > 0$. The set of points $z \in \mathbb{C}$ such that $|z - z_0| < r$ is said to be the **open disc** of radius r centred at z_0. We denote this open disc by $D(z_0, r)$, that is $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Definition 1.27. A closed disc is the set $\overline{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$.

Definition 1.28. A set $\Omega \subseteq \mathbb{C}$ is said to be **bounded** if it is a subset of disc $D(z_0, r)$.

Definition 1.29. A set $\Omega \subseteq \mathbb{C}$ is said to be **bounded** if it is a subset of a disc $D(z_0, r)$.

Definition 1.30. A set $\Omega \subseteq \mathbb{C}$ is said to be **open** if for each $z_0 \in A$ there exists $r > 0$ such that $D(z_0, r) \subseteq \Omega$.

> **Note 1.31.** A set $\Omega \subset \mathbb{C}$ being open implies that Ω is two-dimensional and has no boundary points.

Definition 1.32. A set $\Omega \subseteq \mathbb{C}$ is said to be **closed** if its complement in $\mathbb{C}$ is open.

> **Remark 1.33.** An open disc is an open set and similarly, a closed disc is a closed set.

Theorem 1.34. A set $\Omega \subseteq \mathbb{C}$ is closed if and only if the limit of every convergent sequence of points $z_n \in \Omega$ also belongs to Ω.

Definition 1.35. A set $A \subseteq \mathbb{C}$ is said to be **disconnected** if there exists a pair of disjoint open sets $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ such that $A \subseteq \Omega_1 \cup \Omega_2$ and each of these sets contains at least one element of A.

Definition 1.36. A set is **connected** if it is not disconnected.

Definition 1.37. Let $z_0, z_1 \in \mathbb{C}$. A path from z_0 to z_1 is a continuous map $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$.

Definition 1.38. A set $A \subseteq \mathbb{C}$ is said to be path-connected if every two points $z_0, z_1 \in A$ can be joined by a continuous path with values in A.

Theorem 1.39. Every path-connected set is connected. An open set is a path connected. A general (not open) connected set need not be path-connected.

1.5 Convergence

Definition 1.40. The sequence $\{z_n\}$ **converges** to a limit $L \in \mathbb{C}$ if

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \Rightarrow |z_n - L| < \varepsilon.$$

Theorem 1.41

Let $\{z_n\} = \{x_n + iy_n\}$ be a complex sequence. Then $\{z_n\}$ converges in $\mathbb{C}$ $\iff$ $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ $\iff$ the *real* sequences $\{x_n\} = \{\mathrm{Re}(z)\}$ and $\{y_n\} = \{\mathrm{Im}(z)\}$ both converge in $\mathbb{R}^1$.

Proposition 1.42. If $z_n \to L$ then $|z_n| \to |L|$ and $\overline{z_n} \to \overline{L}$.

Example 1.43. Let $z_n = \frac{1}{n} e^{i\frac{\pi}{4}n}$. We have that $z_n \to 0$ as $n \to \infty$ because

$$\begin{aligned} |z_n - 0|_{\mathbb{C}} &= \left| \frac{1}{n} e^{i\frac{\pi}{4}n} \right| \\ &= \frac{1}{n} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

In $\mathbb{R}^2$, we have that $z_n = \frac{1}{n} \begin{pmatrix} \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(\frac{n\pi}{4}\right) \end{pmatrix}$. As before $z_n \to 0$ because

$$\begin{aligned} |z_n - 0|_{\mathbb{C}} &= \left| \frac{1}{n} \begin{pmatrix} \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(\frac{n\pi}{4}\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right|_{\mathbb{R}^2} \\ &= \frac{1}{n} \left| \begin{pmatrix} \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(\frac{n\pi}{4}\right) \end{pmatrix} \right| \\ &= \frac{1}{n} \left(\cos^2\left(\frac{n\pi}{4}\right) + \sin^2\left(\frac{n\pi}{4}\right) \right) \\ &= \frac{1}{n} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Note 1.44. Therefore, the concept of convergence in $\mathbb{C}$ can be thought as the concept of convergence in $\mathbb{R}$.

1.6 Continuity

Definition 1.45. Let $f : S \to \mathbb{C}$ be a function. Then f is **continuous** at $\alpha \in S$ if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \text{ such that } z \in S \text{ and } |z - \alpha| < \delta \Rightarrow |f(z) - f(\alpha)| < \varepsilon.$$

That is, $\lim_{z \to \alpha} f(z)$ exists and equals $f(\alpha)$.

Proposition 1.46

The function $f : \mathbb{C} \to \mathbb{C}$ where $z \mapsto f(z) = u(x, y) + iv(x, y)$, is continuous at $z_0 = x_0 + iy_0 \in \mathbb{C}$ if and only if it is continuous at $(x_0, y_0) \in \mathbb{R}^2$ viewed as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $(x, y) \mapsto (u[x, y], v[x, y])$. Furthermore, this is so if and only if $u, v : \mathbb{R}^2 \to \mathbb{R}^1$ are both continuous at (x_0, y_0).

Proposition 1.47. The usual rules continuity hold in $\mathbb{C}$.

Proposition 1.48. The following two statements are equivalent:

1. $f : V \to W$ is continuous at $y \in V$.

2. For each convergent sequence $v_n \to y$, one has $f(v_n) \to f(y)$ in W.

Corollary 1.49. The function $f : \mathbb{C} \to \mathbb{C}$ is continuous at $w \in \mathbb{C}$ if and only if for each $z_n \to w$ one has $f(z_n) \to f(w)$ in $\mathbb{C} \iff u(x_n, y_n) \to u(x, y)$ and $v(x_n, y_n) \to v(x, y)$, where $z_n = x_n + iy_n$, $w = x + iy$ and $f(z) = u(x, y) + iv(x, y)$.

2 Complex differentiability

Definition 2.1. Let $\Omega \subset \mathbb{C}$ be an open set and $f : \Omega \to \mathbb{C}$ then, f is said to be differentiable at α if $f'(\alpha)$ exists, where it is defined as

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} := f'(\alpha).$$

Remark 2.2. Since $\mathbb{C}$ can be identified with $\mathbb{R}^2$ the limit must exist in all directions thus, we require the function to defined on an open set so that the limit is well-defined. Furthermore, as the limit exists in all directions it must particularly exist for the horizontal and vertical direction.

Theorem 2.3. The statement of differentiability can be equivalently restated as follows:

- Let $z = +h$ then
$$\lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h} = f'(\alpha).$$

- There exists $f'(\alpha) \in \mathbb{C}$ such that
$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } 0 < |z - \alpha| < \delta_\varepsilon \Rightarrow \left| \frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) \right| < \varepsilon.$$

- There exists $f'(\alpha) \in \mathbb{C}$ such that
$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } 0 < |h| < \delta_\varepsilon \Rightarrow \left| \frac{f(\alpha + h) - f(\alpha)}{h} - \xi \right| < \varepsilon,$$
where $z = \alpha + h$.

- There exists a complex number $f'(\alpha)$ such that
$$f(\alpha + h) = f(\alpha) + hf'(\alpha) + o(h),$$
i.e. $f'(\alpha)$ gives 1$^{\text{st}}$ order Taylor approximation to f near α.

Remark 2.4. The "function" $o(h)$ means "a function which goes to zero faster than h". That is, there exists a function $g(\alpha, h)$ (which we generically denote by $o(h)$) such that

$$\lim_{h \to 0} \frac{g(\alpha, h)}{h} = 0,$$

so the limit exists and is zero. Equivalently,

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 \text{ such that } 0 < |h| < \delta_\varepsilon \Rightarrow \left| \frac{g(\alpha, h)}{h} \right| < \varepsilon.$$

Proposition 2.5

If f is differentiable at α then, f is continuous at α.

Example 2.6. The function $f(z) = z$ is differentiable.

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{z + h - z}{h} \\
&= \frac{h}{h} \\
&= 1 \to 1 \quad \text{as } h \to 0.
\end{aligned}$$

Therefore, f is differentiable at all $z \in \mathbb{C}$ with $f'(z) = 1$.

Example 2.7. Let $f(z) = \operatorname{Re}(z) \in \mathbb{C}$. We show that f is not differentiable at any point $z \in \mathbb{C}$.

$$\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{\operatorname{Re}(z+h) - \operatorname{Re}(z)}{h} \\
&= \frac{\operatorname{Re}(h)}{h} \to \begin{cases} 1 & \text{as } h \to 0 \text{ with } h = s \in \mathbb{R} \\ 0 & \text{as } h \to 0 \text{ with } h = it \in \mathbb{C}. \end{cases}
\end{aligned}$$

Example 2.8

The function $f(z) = \overline{z} = x - iy$ is not differentiable.

$$
\begin{aligned}
\frac{f(z+h) - f(z)}{h} &= \frac{\overline{z+h} - \overline{z}}{h} \\
&= \frac{\overline{z} + \overline{h} - \overline{z}}{h} \\
&= \frac{\overline{h}}{h}.
\end{aligned}
$$

We note that $h = s + it$ for $s, t \in \mathbb{R}^1$. So, if $t = 0$ then we have $h = s \neq 0$ thus,

$$
\frac{\overline{h}}{h} = \frac{s}{s} = 1 \to 1 \quad \text{as } h = s \to 0.
$$

However, if $s = 0$ then, $h = it$ hence,

$$
\frac{\overline{h}}{h} = \frac{\overline{it}}{it} = \frac{-it}{it} = -1 \to -1 \quad \text{as } h = it \to 0.
$$

Therefore, the limit of $\frac{\overline{h}}{h}$ does not exist as $h \to 0$ which means $f(z) = \overline{z}$ is not complex differentiable for any $z \in \mathbb{C}$.

Proposition 2.9. Let $f, g : \Omega \to \mathbb{C}$ be functions defined on an open set $\Omega \subset \mathbb{C}$. Then if f and g are both differentiable at $\alpha \in \mathbb{C}$:

1. The sum $af + bg$, for $a, b \in \mathbb{C}$, is differentiable at α with

$$
(af + bg)'(\alpha) = af'(\alpha) + bg'(\alpha).
$$

2. The product function fg is differentiable at α with

$$
(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha).
$$

3. If $g(\alpha) \neq 0$ then the quotient function $\frac{f}{g}$ is differentiable at α with derivative

$$
\left(\frac{f}{g}\right)'(\alpha) = \frac{f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)}{g^2(\alpha)}.
$$

Proposition 2.10. Let f and g be differentiable functions such that f is defined on an open set Ω and differentiable at $\alpha \in \Omega$ while g is defined on an open set containing the image $f(\Omega)$ and differentiable at $f(\alpha) \in f(\Omega)$. Then the composition $g \circ f(z) = g(f(z))$ is also differentiable at α with derivative

$$
(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).
$$

Note 2.11. That is, the usual rules of differentiation hold in $\mathbb{C}$.

2.1   Cauchy-Riemann equations

Definition 2.12. Let f be a complex-valued function on an open set $\Omega \subseteq \mathbb{C}$. Writing

$$f(z) = u(x, y) + iv(x, y) \qquad \text{for } u, v : \mathbb{R}^2 \to \mathbb{R}^1, z = z + iy.$$

We may define f_x and f_y by

$$\frac{\partial f}{\partial x} := \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or } f_x := u_x + iv_x,$$

$$\frac{\partial f}{\partial y} := \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \quad \text{or } f_y := u_y + iv_y,$$

when the partial derivatives u_x, u_y, v_x, v_y exist.

Theorem 2.13. Using the identities $z = x + iy$ and $\overline{z} = x - iy$ the chain rule gives us

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{or } f_z := \frac{1}{2}\left(f_x - if_y\right),$$

$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \quad \text{or } f_{\overline{z}} := \frac{1}{2}\left(f_x + if_y\right).$$

Proof. Let $f(z) = f(x + iy) = f(x, y) = u(x, y) + iv(x, y)$. By the chain rule we have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}.$$

Note that $z = x + iy$ and $\overline{z} = x - iy$ imply

$$x = \frac{1}{2}(z + \overline{z})$$

$$y = \frac{1}{2}(z - \overline{z}),$$

thus,

$$\frac{\partial x}{\partial z} = \frac{1}{2}$$

$$\frac{\partial y}{\partial z} = \frac{1}{2i} = -\frac{i}{2}.$$

Substituting these into f_z and we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

$$= \frac{1}{2}\left(f_x - if_y\right).$$

$\square$

Theorem 2.14. As such we can write

$$f_z = \frac{1}{2}[(u_x + v_y) + i(v_x - u_y)]$$

$$f_{\overline{z}} = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)].$$

Proof. Recall that if $f(z) = u(x,y) + iv(x,y)$ then

$$f_x = u_x + iv_x \quad \text{and} \quad f_y = u_y + iv_y.$$

We have that

$$
\begin{aligned}
f_z &= \frac{1}{2}(f_x - if_y) \\
&= \frac{1}{2}[(u_x + iv_x) - i(u_y + iv_y)] \\
&= \frac{1}{2}[(u_x + v_y) + i(v_x - v_y)].
\end{aligned}
$$

$\square$

Theorem 2.15 (Cauchy-Riemann equations)

Let $f : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{C}$ being open.

(a). Write $f(z) = u(x,y) + iv(x,y)$ if f is differentiable at $z = x + iy \in \Omega$ then u and v have first-order partial derivatives at (x,y) and these satisfy the **Cauchy-Riemann equations**

$$u_x = v_y \quad u_y = -v_x.$$

(b). Equivalently, the Cauchy-Riemann equations can be written as any of the following evaluated at (x,y):

- $f_{\bar{z}} = 0$ or,
- $f_z = f_x$ or,
- $f_z = -if_y$ or,
- $f_y = if_x$.

Note 2.16. It turns out that, provided we impose continuity conditions on the partial derivatives, we do obtain a converse to the theorem above.

Theorem 2.17

If f_x and f_y exists and are continuous at $(x,y) \in \Omega$ (that is, if u_x, u_y, v_x, v_y) exist and are continuous at (x,y), and if they satisfy the Cauchy-Riemann equations at that point then f is complex differentiable at $z = x + iy$ and the equations above hold.

Note 2.18. With the continuity condition the Cauchy-Riemann equation theorem becomes an 'if and only if' statement. That is, suppose f_x and f_y exists and are continuous, f is differentiable on its domain if and only if the Cauchy-Riemann equation hold.

Example 2.19. Suppose $f(z) = \bar{z} = x + i(-y)$. We have that $u(x,y) = x$ and $v(x,y) = -y$. So,

$$
\begin{aligned}
u_x &= 1 \quad \text{and} \quad v_x = 0 \\
u_y &= 0 \quad \text{and} \quad v_y = -1.
\end{aligned}
$$

We have that $u_x \neq v_y$ for all $x, y \in \mathbb{R}$ so, the Cauchy-Riemann equations do not hold hence, $\overline{z}$ is never complex differentiable (as previously shown).

Example 2.20

Suppose $f(z) = |z|^2$. For $z = x + iy$ we can write

$$
\begin{aligned}
f(z) &= |z|^2 \\
&= z\overline{z} \\
&= (x + iy)(x - iy) \\
&= \underbrace{(x^2 + y^2)}_{u(x,y)} + i \underbrace{0}_{v(x,y)}.
\end{aligned}
$$

We have that

$$
\begin{aligned}
u_x &= 2x \quad \text{and} \quad v_x = 0 \\
u_y &= 2y \quad \text{and} \quad v_y = 0.
\end{aligned}
$$

The Cauchy-Riemann equations hold when

$$
\begin{aligned}
u_x = v_y &\Rightarrow x = 0 \\
u_y = -v_x &\Rightarrow y = 0.
\end{aligned}
$$

The CR equations only hold when $z = 0$ thus, $|z|^2$ is complex differentiable **only** at $z = 0$.

Corollary 2.21

If the Cauchy-Riemann equations hold then one has

$$
\begin{aligned}
f'(z) &= f_z(x, y) \\
&= f_x(x, y) \\
&= -i f_y(x, y).
\end{aligned}
$$

Where

$$
\begin{aligned}
f_x(x, y) &= u_x(x, y) + i v_x(x, y) \\
-i f_y(x, y) &= v_y(x, y) - i u_y(x, y).
\end{aligned}
$$

2.2 Holomorphic functions

Definition 2.22. A complex-valued function f which is differentiable at every point on an open set $\Omega \subset \mathbb{C}$ is said to be **holomorphic** in Ω. The set of functions holomorphic in Ω is denoted by $H(\Omega)$.

Definition 2.23. Functions which are holomorphic everywhere, any $f \in H(\mathbb{C})$, are called **entire**.

Proposition 2.24. The set $H(\Omega)$ is a vector space. Furthermore, it is an 'algebra' i.e. the product $f \cdot g \in H(\Omega)$ is defined and if $g(z) \neq 0$ for all $z \in \Omega$, then also $\frac{f}{g} \in H(\Omega)$ is defined.

> **Example 2.25**
>
> Examples of holomorphic functions:
>
> - Any polynomial of the form $p(z) = \sum_{n=0}^{N} a_n z^n$, for complex constants $a_n \in \mathbb{C}$ is an entire function.
>
> - Any polynomial containing non-zero powers of $\overline{z}$ are **NOT** holomorphic.
>
> - Rational functions $\frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomial are holomorphic on the open set in which $q(z)$ is never zero.
>
> - The exponential function.
>
> - The trigonometric and hyperbolic functions since $e^z \in H(\mathbb{C})$.

Theorem 2.26. The usual rules of differentiation hold (these are illustrated in the Complex differentiability section)

> **Theorem 2.27**
>
> The function f is holomorphic if and only if f is continuously differentiable (i.e. the derivative is continuous) and it satisfies the Cauchy-Riemann equations at each point in its domain. Mathematically,
>
> $$f \in H(\Omega) \iff f \in C^1(\Omega)$$
>
> and the C-R equations are satisfied for each $z \in \Omega$.

Theorem 2.28. Holomorphy implies continuity.

> **Proposition 2.29** (Constancy in a region)
>
> Suppose $f \in H(\Omega)$. Then any of the following conditions forces f to be constant in Ω:
>
> 1. $f'(z) = 0$ for all $z \in \Omega$;
>
> 2. $f(z)$ is real for all $z \in \Omega$;
>
> 3. $|f| = $ constant in Ω.

Proof. We split the proof into parts.

1. We have that $f'(z) = u_x + iv_x$. If $f'(z) = 0$ then

$$u_x = -iv_x \Rightarrow u_x = v_x = 0.$$

   By the Cauchy-Riemann equations we have, $u_y = v_y = 0$ hence, f must constant.

2. If f is real-valued, then $v = 0$ which implies $v_x = v_y = 0$. By the Cauchy-Riemann equations, $u_x = u_y = 0$ too. Hence, by (1) f must be constant.

3. If $|f| = 0$ then, $f = 0$. Suppose $|f| = a \neq 0$ then, $|f|^2 = f \cdot \overline{f} = a^2$; consequently, the function

$$\overline{f} = \frac{a^2}{f} \in H(\Omega).$$

This implies that the real-valued functions

$$\mathrm{Re}(f) = \frac{1}{2}\left(f + \overline{f}\right) \quad \text{and} \quad \mathrm{Im}(f) = \frac{1}{2i}\left(f - \overline{f}\right)$$

are also holomorphic. By property (2), both of these functions are constant.

$\square$

3   Complex logarithm

Remark 3.1. When writing log in this course we mean ln.

We define the logarithm on $(0, \infty) \subset \mathbb{R}$ as the function which is inverse to the exponential function: for each positive real number x, there exists a unique real solution $t = \log(x) = \ln(x)$ to the equation $e^t = x$. In the complex case we seek solutions to the equation $e^w = z$.

Suppose $z \in \mathbb{C}$ and $z \neq 0$. Let $z = e^w = e^{u+iv}$ for $u, v \in \mathbb{R}$. Then,

$$|z| = \left|e^u e^{iv}\right| = e^u \quad \text{and} \quad \arg(z) = \{v + 2\pi k : k \in \mathbb{Z}\}.$$

We have derived the important relation

$$e^w = z \iff w = \log|z| + i \arg(z).$$

A problem arises: by this definition the logarithm is not holomorphic! We provide a new definition.

Definition 3.2. Let $\Omega \subset \mathbb{C}$ be an open simply connected set. A **branch of the logarithm** on Ω is the logarithm defined by

$$\log_\theta = \log_{\mathbb{R}}|z| + i \arg_\theta(z) \quad \text{for } \Omega = \mathbb{C}\backslash R_\theta,$$

where

$$\theta - \pi < \arg_\theta(z) < \theta + \pi.$$

Remark 3.3. Usually the argument of a complex number, $\arg(z)$, is defined on $-\pi < \arg(z) \leq \pi$. Whereas, for the argument in the logarithm is defined with strict inequalities.

Remark 3.4. Simply connected set means it has no holes, since in $\mathbb{R}$ the logarithm is not defined for 0 we must delete that point along with a line to make the new 'complex plane' simply connected. Thus, we delete a half-line and the origin from $\mathbb{C}$ to make such a set. This line R_θ is the line opposite the ray emanating at angle θ. Below we illustrate the line R_θ.
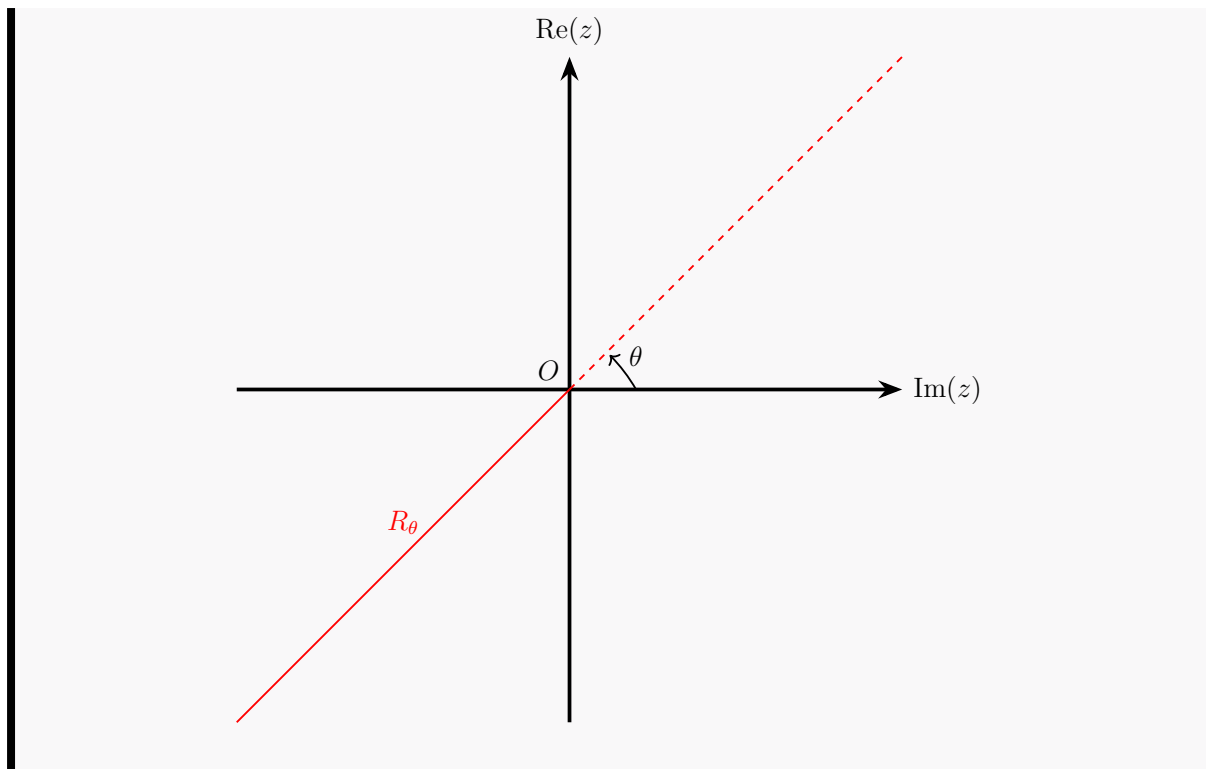
Definition 3.5. Taking $\theta = 0$ gives what is known as the **principal value logarithm**

$$\text{Log}(z) := \log_0(z) = \log_{\mathbb{R}} |z| + i \arg_0(z)$$

with

$$-\pi < \arg_0(z) < \pi.$$

Remark 3.6. Usually the argument of a complex number, $\arg(z)$, is defined on $-\pi < \arg(z) \leq \pi$. Whereas, for the argument in the logarithm is defined with strict inequalities.

Proposition 3.7. For $z, w \in \mathbb{C} \backslash R_\theta$

1. $e^{\log_\theta(z)} = z$.

2. $\log_\theta(e^z) = z + i(2\pi k)$ for some $k \in \mathbb{Z}$.

3. $\log_\theta(zw) = \log_\theta(z) + \log_\theta(w) + i(2\pi k)$ for some $k \in \mathbb{Z}$.

4. $\log_\theta\left(\frac{1}{z}\right) = \log_\theta(z^{-1}) = -\log_\theta(z) + i(2\pi k)$ for some $k \in \mathbb{Z}$.

Example 3.8

Evaluate (i) $\mathrm{Log}(-1+i\sqrt{3})$, (ii) $\log_{\frac{\pi}{4}}(-1+i\sqrt{3})$, (iii) $\log_{\frac{-3\pi}{2}}(-1+i\sqrt{3})$.

Solution. We present the solution to each problem.

(i) $\mathrm{Log}(-1+i\sqrt{3}) = \log_0(-1+i\sqrt{3}) = \log_{\mathbb{R}}(2) + i\arg_0(-1+i\sqrt{3})$. This is the 'usual' arg but is defined on $-\pi < \arg(z) < \pi$. By sketching the point on an Argand diagram it follows that $\arg_0(-1+i\sqrt{3}) = \pi - \arctan\left(\sqrt{3}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. We conclude that

$$\mathrm{Log}(-1+i\sqrt{3}) = \log(2) + i\left(\frac{2\pi}{3}\right).$$

(ii) $\log_{\frac{\pi}{4}}(-1+i\sqrt{3}) = \log_{\mathbb{R}}(2) + i\arg_{\frac{\pi}{4}}(-1+i\sqrt{3})$. The argument is now defined on $-\frac{3\pi}{4} < \arg_{\frac{\pi}{4}}(z) < \frac{5\pi}{4}$. It follows that the argument is the same as before and so,

$$\log_{\frac{\pi}{4}}(-1+i\sqrt{3}) = \log(2) + i\left(\frac{2\pi}{3}\right).$$

(iii) $\log_{\frac{-3\pi}{2}}(-1+i\sqrt{3}) = \log_{\mathbb{R}}(2) + i\arg_{-\frac{3\pi}{2}}(-1+i\sqrt{3})$. The argument is now defined on $-\frac{5\pi}{2} < \arg_{\frac{-3\pi}{2}}(z) < -\frac{\pi}{2}$. Therefore, the branch $\Im(z) \geq 0$ has been cut out from the plane. We conclude,

$$\log_{\frac{-3\pi}{2}}(-1+i\sqrt{3}) = \log(2) + i\left(-\frac{4\pi}{3}\right).$$

Theorem 3.9

The function $\log_\theta$ is holomorphic on $\mathbb{C}\backslash R_\theta$.

Corollary 3.10. We have that

- $\dfrac{\mathrm{d}\log(z)}{\mathrm{d}z} = z^{-1}$ and,

- $\dfrac{\mathrm{d}\log(f(z))}{\mathrm{d}z} = \dfrac{f'(z)}{f(z)}$.

Proposition 3.11. Properties of the log.

1. $\mathrm{Re}[\log f(z)] = \log_{\mathbb{R}}|f(z)|$, where $\log_{\mathbb{R}}: (0,\infty) \to \mathbb{R}$ is the classical logarithm on the positive reals.

2. We have that
$$\log_1(f) - \log_2(f) = i(2\pi k) \quad \text{for some } k \in \mathbb{Z},$$
for $\log_1, \log_2$ any two branches of the logarithm of f on Ω. That is, the difference of any two logarithm functions is a constant function.

3. We have that

$$\log_1(f \cdot g) - \log_2(f) - \log_3(g) = i(2\pi k) \quad \text{for some } k \in \mathbb{Z},$$

for $\log_1, \log_2, \log_3$ respectively of the logarithm $f \cdot g$, f and g on Ω (where $(f \cdot g)(z) = f(z)g(z)$).

4   Complex power series

4.1   Complex series

Definition 4.1. Let $\{w_n\}_{n \geq 0}$ be a sequence of complex numbers. We say that the infinite series $\sum_{n=0}^{\infty} w_n$ **converges** if the sequence of its partial sums $S_m = \sum_{n=0}^{m} w_n$ converges to a limit $w \in \mathbb{C}$ as $m \to \infty$.

> **Remark 4.2.** In this course we will use $\sum w_k < \infty$ to indicate a convergent series and $\sum w_k = \infty$ to indicate a divergent series.

Proposition 4.3

Properties of convergent series.

1. The complex series $\sum_{n=0}^{\infty}$ converges if and only if the **real** series $\sum_{n=0}^{\infty} \text{Re}(w_n)$ and $\sum_{n=0}^{\infty} \text{Im}(w_n)$ converge.

2. Let $\sum_{n=0}^{\infty} w_n$ and $\sum_{n=0}^{\infty} v_n$ be convergent series. Then, for all $\alpha, \beta \in \mathbb{C}$ the series
$$\sum_{n=0}^{\infty}(\alpha w_n + \beta b_n) = \alpha \sum_{n=0}^{\infty} w_n + \beta \sum_{n=0}^{\infty} v_n,$$
is convergent.

3. The series $\sum_{n=0}^{\infty}$ is convergent if and only if the partial sum S_m is a Cauchy sequence.

4. If there exists $z_n \in (0, \infty)$ positive real numbers with $|w_n| < z_n$ for $k > M \in \mathbb{N}$, and if $\sum_{n=0}^{\infty} < \infty$ then $\sum_{n=0}^{\infty} w_n < \infty$.

Definition 4.4. If $\sum_{n=0}^{\infty} |w_n| < \infty$, the series $\sum_{n=0}^{\infty} w_n$ is said to be **absolutely convergent**.

Corollary 4.5. Every absolutely convergent series converges.

4.1.1   Testing for convergence

Proposition 4.6 (The nth root test)**.** Let

$$\limsup_{n \to \infty} |w_n|^{\frac{1}{n}} = L.$$

Then the series $\sum w_n$ converges absolutely if $L < 1$ and diverges if $L > 1$.

Proposition 4.7 (Ratio test)**.** Assume $w_n \neq 0$ and that sequence

$$\frac{|w_{n+1}|}{|w_n|}$$

converges. Let

$$\lim_{n \to \infty} \frac{|w_{n+1}|}{|w_n|} = L,$$

then the series $\sum w_n$ converges absolutely if $L < 1$ and diverges if $L > 1$.

4.2 Power series

Definition 4.8. A **power series** is defined to be a series of the form

$$\sum_{n=0}^{\infty} c_n(z - a)^n,$$

where $a \in \mathbb{C}$ and $c_n \in \mathbb{C}$ (and $n \geq 0$).

Proposition 4.9

For any complex power series $\sum c_n(z - a)^n$ there is a value

$$0 \leq R \leq \infty$$

such that if $0 < R < \infty$ then the series converges

- absolutely for those z with $|z - a| < R$ i.e. $z \in D(a, R)$,

- diverges for those z with $|z - a| > R$ i.e. $z \notin D(a, R)$.

However,

- If $R = \infty$ then the series converges absolutely for all $z \in \mathbb{C}$.

- If $R = 0$ then the series is divergent for all $z \neq a$.

Definition 4.10. The number R from above is called the **radius of convergence** of the power series $\sum_{n=0}^{\infty} c_n(z - a)^n$.

Proposition 4.11

For a complex series $\sum_{n=0}^{\infty} c_n(z - a)^n$ the radius of convergence is given by:

- $R = \left(\limsup_{n \to \infty} |c_n|^{\frac{1}{n}} \right)^{-1} = \frac{1}{\limsup_{n \to \infty} |c_n|^{\frac{1}{n}}}$;

- assuming $c_n \neq 0$, if the sequence $\frac{|c_n|}{|c_{n+1}|}$ converges then

$$R = \lim_{n \to \infty} \frac{|c_n|}{|c_{n+1}|}.$$

Remark 4.12. We can consider $\limsup$ the same as $\lim$.

Note 4.13. An easy way to determine the radius of convergence is to find the largest open disc for which the function is valid in thus, find the poles (singularities) of the function (only if possible).

4.3   Properties of power series

Theorem 4.14

Let $R > 0$ be the radius of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n.$$

Then, f is **uniformly convergent** on the open disc $D(a, r)$ for any $r < R$, and one has

$$f \in H(D(a, R)).$$

Corollary 4.15. For a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n,$$

with radius of convergence R has derivative is

$$f'(z) = \sum_{n=0}^{\infty} n c_n (z - a)^{n-1} = \sum_{n=0}^{\infty} (n + 1) c_{n+1} (z - a)^n$$

with the same radius of convergence, R.

Theorem 4.16 (Fundamental Theorem of Holomorphic Funcions (Discs))

A function, $f \in H(\Omega)$ if and only if f is $\mathbb{C}$ analytic on Ω.

Remark 4.17. By analytic, we mean the function can be written as a power series.

Note 4.18. This is a powerful theorem as it implies that a function which is holomorphic in an open set $\Omega \subset \mathbb{C}$ is such that $f^{(n)}(z)$ exists for $z \in \Omega$ for all n.

Corollary 4.19

The function $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ is infinitely differentiable (i.e. $f^{(m)}(z)$ exists and is holomorphic for all $m \in \mathbb{N}$) on $D(a, R)$ with

$$f^{(m)}(z) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-(m-1))c_n(z-a)^{n-m}.$$

Thus, Taylor's formula holds in $\mathbb{C}$ i.e.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

4.3.1 Multiplying power series

Theorem 4.20. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

are complex power series with radii of convergence R_1 and R_2 respectively. Let

$$h(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \sum_{r=0}^{n} a_r b_{n-r}.$$

Then $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence at least $R := \min\{R_1, R_2\}$ and

$$h(z) = f(z)g(z)$$

for $|z| < R$.

4.4 Geometric series

Definition 4.21. The **geometric series** is defined to be the sum $\sum_{n=0}^{\infty} z^n$.

Theorem 4.22

We have that the geometric series

$$\sum_{n=0}^{\infty} z^n \begin{cases} \text{converges} & \text{if } |z| < 1 \\ \text{diverges} & \text{if } |z| \geq 1. \end{cases}$$

Furthermore, for $|z| < 1$ we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Proof. Let the partial sum

$$S_k = \sum_{n=0}^{k} z^n$$
$$= 1 + z + z^2 + \cdots + z^n.$$

Multiplying both sides by z we have

$$zS_k = z + z^2 + \cdots + z^{n+1}.$$

Subtracting the two equations

$$S_k - zS_k = (1 + z + z^2 + \cdots + z^n) - (z + z^2 \cdots + z^{n+1}),$$

which becomes

$$(1 - z)S_k = 1 - z^{n+1}.$$

Therefore,

$$S_k = \sum_{n=0}^{k} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

For $|z| < 1$ the term $z^{n+1} \to 0$ as $n \to \infty$ hence,

$$S_\infty = \sum_{n=0}^{\infty} z^n$$
$$= \frac{1}{1 - z}.$$
$$= (1 - z)^{-1}$$

$\square$

Definition 4.23. The **geometric identity** is given by

$$(1 - z)(1 + z + \cdots + z^n) = 1 - z^{n+1}.$$

4.4.1 Expansion derived from the geometric series

The geometric series can be viewed in two ways: either as summing an infinite series or as expanding $(1 - z)^{-1}$ as a series when $|z| < 1$. Taking the second viewpoint we may derive many related expansions.

Example 4.24. Examples of power series.

- Suppose $f(z) = \frac{1}{1+z}$. Then we may write

$$\frac{1}{1 + z} = \frac{1}{1 - (-z)}$$
$$= (1 - (-z))^{-1}$$
$$= 1 + (-z) + (-z)^2 + \cdots$$
$$= 1 - z + z^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n z^n$$

for $|z| < 1$.

- Suppose $f(z) = \frac{1}{1-z^2}$. Then we may write

$$\frac{1}{1-z^2} = (1 - (z)^2)^{-1}$$
$$= 1 + z^2 + z^4 + \cdots$$
$$= \sum_{n=0}^{\infty} z^{2n}$$

for $|z| < 1$.

- Suppose $f(z) = \frac{1}{z-4}$. Then we may write

$$\frac{1}{z-4} = -\frac{1}{4} \cdot \frac{1}{\left(1 - \frac{z}{4}\right)}$$
$$= -\frac{1}{4} \left(1 - \frac{z}{4}\right)^{-1}$$
$$= -\frac{1}{4} \left(1 + \frac{z}{4} + \frac{z^2}{16} + \cdots\right)$$
$$= -\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n$$
$$= -\sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}}$$

for $\left|\frac{z}{4}\right| < 1$ i.e. $|z| < 4$.

- Suppose $f(z) = \frac{1}{1+z+z^2}$. Then we may write it as

$$\frac{1}{1+z+z^2} = \frac{1-z}{1-z} \cdot \frac{1}{1+z+z^2}$$
$$= (1-z)\frac{1}{1-z^3}.$$

Now we express $\frac{1}{1-z^3}$ as power series:

$$\frac{1}{1-z^3} = \sum_{n=0}^{\infty} z^{3n}$$

for $|z| < 1$. Therefore,

$$\frac{1}{1+z+z^2} = (1-z) \sum_{n=0}^{\infty} z^{3n}$$
$$= \sum_{n=0}^{\infty} (z^{3n} - z^{3n+1})$$

for $|z| < 1$.

Theorem 4.25. For $a, b \neq 0$ we can write

$$f(z) = \frac{1}{az+b} = \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{b^{n+1}} z^n,$$

for $|z| < \frac{|b|}{|a|}$.

Proof. Let $f(z) = \frac{1}{az+b}$ then we can write

$$\begin{aligned}
\frac{1}{b+az} &= \frac{1}{b} \cdot \frac{1}{\left(1 + \left(\frac{a}{b}\right)z\right)} \\
&= \frac{1}{b} \left(1 + \left(\frac{a}{b}\right)z\right)^{-1} \\
&= \frac{1}{b} \left(1 + \left(\frac{a}{b}\right)z + \left(\frac{a^2}{b^2}\right)z^2 + \cdots\right) \\
&= \frac{1}{b} \sum_{n=0}^{\infty} \left(\frac{a}{b}z\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{a^n}{b^{n+1}}\right) z^n,
\end{aligned}$$

for $\left|\left(\frac{a}{b}\right)z\right| < 1$ i.e. $|z| < \frac{|a|}{|b|}$. $\qquad\qquad \square$

Theorem 4.26. For $a \neq b$ we have

$$f(z) = \frac{1}{(z-a)(z-b)} = \frac{1}{a-b} \sum_{n=0}^{\infty} \left(\frac{1}{b^{n+1}} - \frac{1}{a^{n+1}}\right) z^n$$

for $|z| < \min\{|a|, |b|\}$.

Proof. Suppose $f(z) = \frac{1}{(z-a)(z-b)}$, then we may write using partial fractions

$$\begin{aligned}
f(z) &= \frac{1}{(z-a)(z-b)} \\
&= \frac{1}{a-b} \cdot \frac{1}{z-a} + \frac{1}{b-a} \cdot \frac{1}{z-b}.
\end{aligned}$$

Firstly, we express $\frac{1}{z-a}$ as a power series:

$$\begin{aligned}
\frac{1}{z-a} &= -\frac{1}{a-z} \\
&= -\frac{1}{a} \cdot \frac{1}{1 - \frac{z}{a}} \\
&= -\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n \\
&= \sum_{n=0}^{\infty} -\frac{1}{a^{n+1}} z^n,
\end{aligned}$$

which is valid for $|z| < |a|$. Similarly, we have

$$\frac{1}{z-b} = \sum_{n=0}^{\infty} -\frac{1}{b^{n+1}} z^n,$$

valid for $|z| < |b|$. Substituting the respective power series into $f(z)$, and we have

$$
\begin{aligned}
f(z) &= \frac{1}{a-b} \sum_{n=0}^{\infty} -\frac{1}{a^{n+1}} z^n - \frac{1}{b-a} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} z^n \\
&= \frac{1}{a-b} \sum_{n=0}^{\infty} -\frac{1}{a^{n+1}} z^n + \frac{1}{a-b} \sum_{n=0}^{\infty} \frac{1}{b^{n+1}} z^n \\
&= \frac{1}{a-b} \sum_{n=0}^{\infty} \left(\frac{1}{b^{n+1}} - \frac{1}{a^{n+1}} \right) z^n.
\end{aligned}
$$

Therefore, this power series is valid for $|z| < \min\{|a|, |b|\}$. $\square$

Theorem 4.27. We have that

$$
\begin{aligned}
f(z) = \frac{1}{b-z} &= \frac{1}{(b-a)-(z-a)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(b-a)^{n+1}} (z-a)^n
\end{aligned}
$$

for $|z-a| < |b-a|$.

Proof. Suppose $f(z) = \frac{1}{b-z}$, and we want to express $f(z)$ as a power series centred at $a \in \mathbb{C}$. Then we may write

$$
\begin{aligned}
\frac{1}{b-z} &= \frac{1}{(b-a)-(z-a)} \\
&= \frac{1}{b-a} \cdot \frac{1}{1 - \frac{z-a}{b-a}} \\
&= \frac{1}{b-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a} \right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(b-a)^{n+1}} (z-a)^n.
\end{aligned}
$$

This power series is valid for

$$
\begin{aligned}
\left| \frac{z-a}{b-a} \right| &< 1 \\
\Rightarrow |z-a| &< |b-a|.
\end{aligned}
$$

$\square$

Example 4.28. Suppose $f(z) = \frac{1}{1-z}$ obtain a power series expansion of $f(z)$ which is valid in a disc centre -3. We can write

$$\frac{1}{1-z} = \frac{1}{4-(z+3)}$$

$$= \frac{1}{1 - \left(\frac{z+3}{4}\right)}$$

$$= \left(1 - \frac{z+3}{4}\right)^{-1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{z+3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^{n+1}}(z+3)^n.$$

for

$$\left|\frac{z+3}{4}\right| < 1$$

$$|z+3| < 4.$$

4.4.2 Common power series

Definition 4.29. We define for $z \in \mathbb{C}$,

$$e^z := 1 + z + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos(z) := 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sin(z) := z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

$$\cosh(z) := 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\sinh(z) := z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$

These series are valid for all $z \in \mathbb{C}$ i.e. their radius of convergence is infinite.

Proposition 4.30. One has

$$\log_0(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

valid for $z \in D(0,1)$.

4.5 Laurent series

Definition 4.31. A **Laurent series** is a power series of the form

$$f(z) := \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

Theorem 4.32. A Laurent series means

$$\sum_{n=-\infty}^{\infty} w_n = \sum_{n=-\infty}^{-1} w_n + \sum_{n=0}^{\infty} w_n,$$

or equivalently,

$$\sum_{n=-\infty}^{\infty} w_n = \sum_{m=1}^{\infty} w_{-m} + \sum_{n=0}^{\infty} w_n,$$

where $m = -n$. If these two series converge respectively to $L_-, L_+ \in \mathbb{C}$ then $\sum_{n=\infty}^{\infty} w_n$ is said to converge to $L = L_- + L_+$.

Theorem 4.33

A Laurent power series, $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is convergent on an annulus

$$A(a, R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}$$

for real $R_1, R_2 \in [0, \infty)$; if $R_2 \leq R_1$ then this is the empty set. We have that

$$R_1 = \limsup_{n \to \infty} |c_{-n}|^{\frac{1}{n}}$$

$$R_2 = \frac{1}{\limsup_{n \to \infty} |c_n|^{\frac{1}{n}}} = \left(\limsup_{n \to \infty} |c_n|^{\frac{1}{n}} \right)^{-1}.$$

Remark 4.34. The punctured disc is a degenerate case of an open annulus $R_1 < |z-a| < R_2$, where $R_1 = 0$ i.e. a punctured disc centred at a is of the form

$$0 < |z-a| < R.$$

Note 4.35. We can use Laurent series to find a power series which is valid for $|z| > \alpha$.

Theorem 4.36. If $R_1 \geq R_2$ then the series is not convergent anywhere.

Theorem 4.37

A Laurent series is holomorphic and absolutely convergent inside the annulus $A(a, R_1, R_2)$.

Note 4.38. Suppose that f is holomorphic in $D(a, r)$. It has a Taylor expansion there and also has a Laurent expansion in $D'(a, r)$. The uniqueness of the Laurent coefficients forces these expansions to coincide in $D'(a, r)$ (with $a_k = 0$ for all $n < 0$).

Example 4.39. Determine the power series expansion about $0 \in \mathbb{C}$ of the function

$$f(z) = \frac{1}{(z - i)(z - 2)}$$

in the annuli

(i) $1 < |z| < 2$ and

(ii) $|z| > 2$.

Solution. Notice that $f(z)$ has singularities at i and 2 thus, for each annuli we need to figure out which type of power series we need. We use the method of partial fractions to write

$$f(z) = \frac{1}{(z - i)(z - 2)}$$
$$= \frac{1}{2 - i}\left(\frac{1}{z - 2} - \frac{1}{z - i}\right).$$

(i) We want an expansion which is valid for the blue shaded region illustrated below.
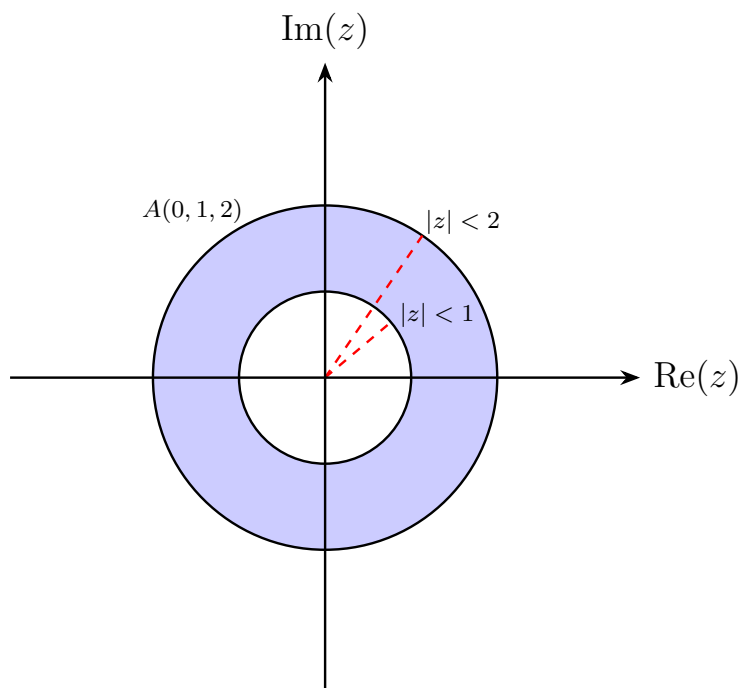
For the term $\frac{1}{z-2}$ we have a singularity at 2, but we want a power series expansion for $|z| < 2$ thus we carry out a Taylor expansion for that term;

$$\frac{1}{z - 2} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{z}{2}}$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

valid for $\left|\frac{z}{2}\right| < 1$ i.e. $|z| < 2$.

For the term $\frac{1}{z-i}$ there is a singularity at $z = i$ therefore, there exists a Taylor expansion valid for $|z| < 1$, but we need an expansion for $|z| > 1$ so, we must find a Laurent series. To obtain a Laurent series we note that we can write $|z| > 1$ as

$$\frac{|z|}{|z|} > \frac{1}{|z|}$$

$$\Rightarrow 1 > \left|\frac{1}{z}\right|$$

We have that $\left|\frac{1}{z}\right| < 1$. We want to find a power with such property so, we can write

$$\frac{1}{z-i} = \frac{1}{z} \cdot \frac{1}{1 - \frac{i}{z}}$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n$$

which is valid for $\left|\frac{i}{z}\right| < 1$ i.e. $|z| > 1$. Note that

$$\frac{1}{z-i} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{i}{z^{n+1}}$$

$$= \sum_{m=1}^{\infty} \frac{i^{m-1}}{z^m} \quad (\text{let } m = n+1)$$

$$= \sum_{m=1}^{\infty} i^{m-1} z^{-m}$$

$$= \sum_{n=-\infty}^{-1} i^{-n-1} z^n \quad (\text{let } m = -n).$$

Hence,

$$f(z) = \frac{1}{2-i} \left(-\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{i}{z}\right)^n \right)$$

$$= \frac{1}{2-i} \left(\sum_{n=-\infty}^{-1} i^{-n-1} z^n - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \right).$$

(ii) If $|z| > 2$ then $\frac{1}{z-2} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$ and so since the expansion for $\frac{1}{z-i}$ is the same as before as it is valid for $|z| > 1$. Therefore,

$$f(z) = \frac{1}{2-i} \left(\sum_{n=-\infty}^{-1} i^{-n-1} z^n - \sum_{n=-\infty}^{-1} \frac{1}{2^{n+1}} z^n \right).$$

5 Integration

5.1 Paths and contours

Definition 5.1. A **path** is a continuous function $\gamma : [a, b] \to \mathbb{C}$.

> **Note 5.2.** In the lecture note we emphasise $[a, b]$ is a non-degenerate interval which means the interval has more than one point.

> **Remark 5.3.** In some texts a 'path' is called a 'curve'.

Theorem 5.4. The convention is to take the anticlockwise direction as positive (because the polar angle is increasing).

Theorem 5.5

A path $\gamma : [a, b] \to \mathbb{C}$ is to give two real functions

$$x : [a, b] \to \mathbb{R} \quad \text{and} \quad y : [a, b] \to \mathbb{C},$$

which determine and are determined by γ via

$$\gamma(t) = x(t) + iy(t)$$

for each $t \in [a, b]$.

Definition 5.6. The geometric image of the map γ is said to be the **curve** traced out by γ or, simply the **trace** of γ and is denoted by C_γ.

> **Remark 5.7.** Whilst γ determines C_γ, the converse does not hold. For example, $\gamma_1(t) = e^{i2\pi t}$ and $\gamma_2(t) = e^{-i4\pi t}$ with $t \in [0, 1]$ have the same trace: the unit circle around the origin. However, γ_1 runs once anticlockwise around the origin and γ_2 goes round twice clockwise.

Definition 5.8. A path is said to be **simple** if it does not cross itself i.e. $\gamma(t_1) \neq \gamma_2(t_2)$ for distinct values $t_1, t_2(a, b)$.

Definition 5.9. A path is said to be **closed** if $\gamma(a) = \gamma(b)$.

Definition 5.10. If $\gamma : [a, b] \to \mathbb{C}$ is a path, then the map $\widetilde{\gamma} : [a, b] \to \mathbb{C}$ given by $\widetilde{\gamma}(t) = \gamma(a + b - t)$ is called the **reverse path**.

> **Note 5.11.** We can think of $\widetilde{\gamma}$ as γ traversed in the opposite direction.

Definition 5.12. A path is said to be **smooth** (or continuously differentiable) if the derivative

$$\gamma'(t) := x'(t) + iy'(t)$$

exists and is continuous. (At the point a we use the right derivative and the point b the left derivative).

Definition 5.13. A **contour** is a piecewise smooth path i.e. the join of finitely many smooth paths.

Proposition 5.14

Key parametrisation:

- A straight line which begins at $a \in \mathbb{C}$ and ends at $b \in \mathbb{C}$:

$$\gamma : [0,1] \to \mathbb{C}, \quad \gamma(t) = a(1-t) + tb = a + t(b-a).$$

- A circle, $C(a,r)$, centred at $a \in \mathbb{C}$ and of radius $r > 0$:

$$\gamma : [0, 2\pi] \to \mathbb{C} \quad \gamma = a + re^{it}.$$

Remark 5.15. Sometimes we can write $\gamma : [0,1] \to [a,b]$.

5.2   Integration in the complex plane

Definition 5.16. The integral of a path $\phi : [a,b] \to \mathbb{C}$ is defined by

$$\int_a^b \phi(t)\, dt := \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt.$$

Remark 5.17. The integrals denoted here are the usual $\mathbb{R}$ integrals.

Lemma 5.18. The integral of a path is linear i.e. for paths γ, ϕ:

$$\int_a^b \alpha\gamma(t) + \beta\phi(t)\, dt = \alpha \int_a^b \gamma(t)\, dt + \beta \int_a^b \phi(t)\, dt \quad \forall \alpha, \beta \in \mathbb{C}.$$

Lemma 5.19. If ϕ is a smooth path then the fundamental theorem of calculus holds:

$$\int_a^b \phi'(t)\, dt = \phi(b) - \phi(a).$$

Definition 5.20. The **length** of a contour γ is given by

$$\int_a^b |\gamma'(t)|\, dt$$

where γ' is the derivative.

Definition 5.21. Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path and f be a continuous complex-valued function in a neighbourhood of C_γ. Then,

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt.$$

Definition 5.22. Let $\gamma : [a_k, a_{k+1}] \to \mathbb{C}$ be a smooth path and f be a continuous function on a neighbourhood of C_γ then

$$\int_\gamma f(z)\, dz := \sum_k \int_{a_k}^{a_{k+1}} f(\gamma(t)) \cdot \gamma_k'(t)\, dt.$$

5.3   Properties of complex integration

Proposition 5.23. Properties of contour integration. Suppose that $\gamma : [a,b] \to \mathbb{C}$ and that $f : C_\gamma \to \mathbb{C}$ is continuous.

1. **Reversal.** If $\widetilde{\gamma}$ is the reverse path

$$\int_{\widetilde{\gamma}} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

   Equivalently,

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz.$$

2. **Reparametrisation.** The contour integral does not depend on the choice of parametrisation of the contour. More precisely, if $\gamma : [a,b] \to \mathbb{C}$ and $\phi : [\alpha,\beta] \to [a,b]$ with $\phi(\alpha) = a$ and $\phi(\beta) = b$ then

$$\int_{\gamma_\phi} f(z)\,dz = \int_{\gamma} f(z)\,dz,$$

   where $\gamma_\phi(t) = \gamma(\phi(t))$.

Proof. We split the proof into parts.

1. We have

$$\begin{aligned}
\int_{\widetilde{\gamma}} f(z)\,dz &= \int_a^b f(\widetilde{\gamma}(t)) \cdot \widetilde{\gamma}'(t)\,dt \\
&= \int_a^b f(\gamma(a+b-t)) \cdot \gamma'(a+b-t) \cdot (-1)\,dt \\
&= -\int_a^b f(\gamma(\tau)) \cdot \gamma'(\tau)\,d\tau \\
&= -\int_{\gamma} f(z)\,dz.
\end{aligned}$$

   We used $\tau = a+b-t$.

2. We have

$$\begin{aligned}
\int_{\gamma_\phi} f(z)\,dz &= \int_a^b f(\gamma_\phi(t)) \cdot \gamma_\phi'(t)\,dt \\
&= \int_a^b f(\gamma(\phi(t))) \cdot \gamma'(\phi(t))\,dt \\
&= \int_\alpha^\beta f(\gamma(\tau)) \cdot \gamma'(\tau)\,d\tau \\
&= \int_{\gamma} f(z)\,dz,
\end{aligned}$$

   where $\tau = \phi(t)$.

$$\square$$

> **Theorem 5.24** (Estimation theorem)
> One has
> $$\left| \int_\gamma f(z)\, dz \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt.$$
> Hence,
> $$\left| \int_\gamma f(z)\, dz \right| \leq |\gamma| \sup_{z \in C_\gamma} |f(z)|,$$
> where $|\gamma|$ is the length of the contour γ.

Proof. We use the following fact. For $\phi : [a, b] \to \mathbb{C}$ we have

$$\left| \int_a^b \phi(t)\, dt \right| \leq \int_a^b |\phi(t)|\, dt.$$

We prove this. Let $\int_a^b \phi(t)\, dt = re^{i\theta}$, where $r \geq 0$ and $\theta \in (-\pi, \pi]$. Then

$$
\begin{aligned}
\left| \int_a^b \phi(t)\, dt \right| &= r = re^{-i\theta}e^{i\theta} \\
&= e^{-i\theta} \int_a^b \phi(t)\, dt \\
&= \int_a^b e^{-i\theta} \cdot \phi(t)\, dt \\
&= \int_a^b \operatorname{Re}(e^{-i\theta} \cdot \phi(t))\, dt + i \int_a^b \operatorname{Im}(e^{-i\theta} \cdot \phi(t))\, dt.
\end{aligned}
$$

The LHS is real so, the integral of the imaginary part on the RHS must be zero. Consequently,

$$
\begin{aligned}
\left| \int_a^b \phi(t)\, dt \right| &= \int_a^b \operatorname{Re}(e^{-i\theta} \cdot \phi(t))\, dt \\
&\leq \left| \operatorname{Re}(e^{-i\theta} \cdot \phi(t)) \right|\, dt \\
&\leq \int_a^b \left| e^{-i\theta} \cdot \phi(t) \right|\, dt \\
&= \int_a^b |\phi(t)|\, dt.
\end{aligned}
$$

Now the proof of the theorem is clear, one has

$$
\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b \underbrace{f(\gamma(t)) \cdot \gamma'(t)}_{\phi(t)}\, dt \right| \leq \int_a^b |f(\gamma(t)) \cdot \gamma'(t)|\, dt
$$

$$
\leq \sup_{z \in C_\gamma} |f(z)| \int_a^b |\gamma'(t)|\, dt.
$$

$\square$

Example 5.25

Let $f(z) = (z^4 + 1)^{-1}$ and let $\gamma = Re^{it}$ for $t \in [0, 2\pi]$. Then by definition,

$$\int_\gamma f(z)\, dz = \int_0^{2\pi} \frac{Rie^{it}}{R^4 e^{i4t} + 1}\, dt,$$

the value of which is not obvious. However, by the estimation theorem we have

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_0^{2\pi} \left| \frac{Rie^{it}}{R^4 e^{i4t} + 1} \right| dt$$

$$\leq \frac{R\pi}{|R^4 + 1|}.$$

Proposition 5.26. If $f_m \to f$ uniformly on Γ and if $C_\gamma \subset \Gamma$ then

$$\lim_m \int_\gamma f_m(z)\, dz = \int_\gamma \underbrace{\lim_m f_m(z)}_{=f(z)}\, dz.$$

Proof. By the estimation theorem we have

$$\left| \int_\gamma f(z)\, dz - \int_\gamma f_m(z)\, dz \right| = \left| \int_\gamma f(z) - f_m(z)\, dz \right|$$

$$\leq |\gamma| \sup_{z \in \gamma} |f(z) - f_m(z)|$$

$$\to 0$$

as $f_m \to f$ uniformly on Γ. $\qquad\square$

Corollary 5.27. If a series $\sum_{n=0}^\infty g_n(z)$ is uniformly convergent on Γ then we integrate the series term by term over any contour $\gamma \in \Gamma$; that is

$$\int_\gamma \sum_{n=0}^\infty g_n(z)\, dz = \sum_{n=0}^\infty \int_\gamma g_n(z)\, dz.$$

Proof. Using the proposition above, take $f_m = \sum_{n=0}^m g_n(z)$. $\qquad\square$

5.4   The fundamental theorem of calculus

Definition 5.28. Let f be a complex function defined on an open set $\Omega \subset \mathbb{C}$. A function F defined on the same set Ω is said to be a **primitive** of f if F is holomorphic in Ω and $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 5.29 (FTC)

Suppose that $\gamma : [a, b] \to \mathbb{C}$ and F is defined on the open set containing the trace of γ, C_γ, and that $F'(z)$ exists and is continuous at each point of C_γ. Then

$$\int_\gamma F'(z)\, dz = \begin{cases} F[\gamma(b)] - F[\gamma(a)] & \text{in general,} \\ 0 & \text{if } \gamma \text{ is closed.} \end{cases}$$

Proof. First assume γ is smooth. The assumptions on F imply that $F \circ \gamma$ is differentiable on $[a, b]$ with $(F \circ \gamma)'(t) = F'[\gamma(t)]\gamma'(t)$. Then

$$
\begin{aligned}
\int_\gamma F'(z)\,dz &= \int_a^b F'[\gamma(t)] \cdot \gamma'(t)\,dt \\
&= \int_a^b (F \circ \gamma)'(t)\,dt \\
&= \int_a^b \mathrm{Re}[(F \circ \gamma)'(t)]\,dt + i \int_a^b \mathrm{Im}[(F \circ \gamma)'(t)]\,dt \\
&= [\mathrm{Re}[(F \circ \gamma)'(t)]]_a^b + i\,[\mathrm{Im}[(F \circ \gamma)'(t)]]_a^b \\
&= F[\gamma(b)] - F[\gamma(a)].
\end{aligned}
$$

Clearly, if the path is closed i.e. $\gamma(a) = \gamma(b)$ then $\int_\gamma F'(z)\,dz = 0$. $\qquad\square$

Corollary 5.30

If f has a primitive, $F \in H(\Omega)$ i.e. $F' = f$, then for any contour $\gamma : [a, b] \to \Omega$ we have

$$
\int_\gamma f(z)\,dz = \begin{cases} F[\gamma(b)] - F[\gamma(a)] & \text{in general,} \\ 0 & \text{if } \gamma \text{ is closed.} \end{cases}
$$

Proof. Clearly, if $F' = f$ then

$$
\begin{aligned}
\int_\gamma f(z)\,dz &= \int_\gamma F'(z)\,dz \\
&= \int_a^b F'[\gamma(t)] \cdot \gamma'(t)\,dt \\
&= F[\gamma(b)] - F[\gamma(a)].
\end{aligned}
$$

If γ is closed then proof is clear. $\qquad\square$

5.5 A fundamental integral

Theorem 5.31 (The fundamental integral)

For $a \in \mathbb{C}$ and $r > 0$,

$$
\int_{\gamma(a,r)} (z - a)^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1, \end{cases}
$$

where $\gamma(a, r)$ denoted the circular contour having centre a and radius r.

Proof. Let $f(z) = (z - a)^n$, since $\gamma(a, r) = a + re^{it}$ for $t \in [0, 2\pi]$ then $\gamma' = rie^{it}$ thus, we

can write

$$
\begin{aligned}
\int_{\gamma(a,r)} f(z)\, dz &= \int_0^{2\pi} f(a + re^{it})(a + re^{it})'\, dt \\
&= \int_0^{2\pi} (re^{it})^n rie^{it}\, dt \\
&= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt \\
&= ir^{n+1} \int_0^{2\pi} (t\cos(n+1) + it\sin(n+1))\, dt \\
&= \begin{cases} ir^{n+1}\left(\left[\frac{t\sin(n+1)}{n+1}\right]_0^{2\pi} - i\left[\frac{t\cos(n+1)}{n+1}\right]_0^{2\pi}\right) & \text{if } n \neq -1 \\ i[1]_0^{2\pi} & \text{if } n = -1. \end{cases}
\end{aligned}
$$

$\square$

5.6   Logarithms

The existence of a primitive depends not only on f but also on the set Ω. For instance, if Ω is an open disc which does not contain the origin then a continuous branch of the logarithm Ω, $\log(z)$ is a primitive for $f(z) = \frac{1}{z}$. However, the function $\frac{1}{z}$ does not have a primitive on any open set containing the unit circle, S^1 about the origin because $\int_{S^1} \frac{1}{z}\, dz \neq 0$ which by the corollary above acts an obstruction to the existence of a primitive.

> **Theorem 5.32**
>
> Suppose $\gamma(0,1)$ is the path of positively oriented unit circle in the plane $\mathbb{C}\backslash\{0\}$ then
>
> $$
> \int_{\gamma(0,1)} \frac{1}{z}\, dz = \begin{cases} 0 & \text{if } z \text{ is outside } \gamma(0,1) \\ 2\pi i & \text{if } z \text{ is inside } \gamma(0,1). \end{cases}
> $$

Proof. The proof follows from the 'fundamental integral proof'. $\square$

Corollary 5.33. Suppose that Ω is a region not containing 0. Then there exists a function $f = \log \in H(\Omega)$ such that $e^{f(z)} = z$ for all $z \in \Omega$ and

$$
f(z) - f(a) = \int_\gamma \frac{1}{w}\, dw
$$

for all $a, z \in \Omega$ where γ is any path in Ω with endpoints a and z.

6   Cauchy integral formula for a disc

> **Theorem 6.1** (CIF for a disc)
> Let $f \in H(D(a, R))$ and let $\gamma(a, r)$ be a simple positively oriented circle (i.e. it is oriented anticlockwise) about a with radius $r < R$. Then if z is inside γ
>
> $$f(z) = \begin{cases} \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{w-z} \, dw & \forall z \in D(a, r) \\ 0 & \forall z \notin D(a, r). \end{cases}$$

> **Note 6.2.** That is, if the singularity is inside the disc the integral is as given above otherwise, if the singularity is outside the disc the function is 0.

Lemma 6.3. Let $\Omega \subset \mathbb{C}$ be open and let $h : \Omega \times [\alpha, \beta] \to \mathbb{C}$ with $(w, s) \mapsto h(w, s)$ be a continuous function (in both w and s). If the derivative $\dfrac{\partial h}{\partial s}$ exists and is continuous and if $\gamma : [a, b] \to \Omega$ is a contour, then

$$\int_\gamma h(w, s) \, dw : \int_a^b h(\gamma(t), s) \cdot \gamma'(t) \, dt$$

is a differentiable function of the variable s with continuous derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_\gamma h(w, s) \, dw = \int_\gamma \frac{\partial}{\partial s} h(w, s) \, dw.$$

> **Example 6.4**
> The CIF is useful for $\mathbb{R}$ integration. Suppose $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ i.e. a circle centred at the origin of radius 1. By the CIF we have
>
> $$\int_\gamma \frac{e^z}{z} \, dz = 2\pi i e^0 = 2\pi i.$$
>
> By direct evaluation we have
>
> $$\int_\gamma \frac{e^z}{z} \, dz = - \int_0^{2\pi} e^{\cos t} \sin(\sin t) \, dt + i \int_0^{2\pi} e^{\cos t} \cos(\sin t) \, dt.$$
>
> Since we know the result is $0 + i(2\pi)$ we can compare the real and imaginary parts of the integral, and we conclude that
>
> $$\int_0^{2\pi} e^{\cos t} \sin(\sin t) \, dt = 0$$
>
> $$\int_0^{2\pi} e^{\cos t} \cos(\sin t) \, dt = 2\pi.$$

6.1   Taylor series coefficients

Remark 6.5. Emphasis that the centre of the discs described in this section coincide with the singularity of the integral.

Theorem 6.6. Let $f \in H(D(a, R))$. Furthermore, let $r < R$ and $\gamma(a, r)$ be the positively oriented circle centred at a of radius r. Then

$$f(z) = \sum_{n=0}^{\infty} c_n(a)(z - a)^n \quad \forall z \in D(a, r)$$

where

$$c_n(a) = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} \, dw$$

and the series is absolutely and uniformly convergent in the disc $D(a, r)$.

Corollary 6.7

Let $f \in H(D(a, R))$. Furthermore, let $r < R$ and $\gamma(a, r)$ be the positively oriented circle centred at a of radius r. The nth derivative, $f^{(n)}$, satisfies

$$f^{(n)}(a) = n! c_n(a)$$
$$= \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} \, dw \quad \forall r < R$$

Proof. By induction. $\qquad \square$

Corollary 6.8 (Cauchy's estimate)

Let $f \in H(D(a, R))$. Furthermore, let $r < R$ and $\gamma(a, r)$ be the positively oriented circle centred at a of radius r. We have

$$\left| f^{(n)}(a) \right| \leq \frac{n!}{r^n} \sup_{w \in C_{\gamma(a,r)}} |f(w)| \quad \forall r < R.$$

Proof. By above

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} \, dw,$$

by the estimation theorem we have

$$\left| f^{(n)}(a) \right| = \left| \frac{n!}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} \, dw \right|$$
$$= \left| \frac{n!}{2\pi i} \right| \left| \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} \, dw \right|$$
$$\leq \frac{n!}{2\pi} \int_{\gamma(a,r)} \left| \frac{f(w)}{(w - a)^{n+1}} \right| \, dw.$$

Notice the contour $\gamma(a, r)$ is a circle so, it can be expressed as $|w - a| < r$, and the length of the contour is $2\pi r$; as such by applying the estimation theorem again we have

$$\frac{n!}{2\pi} \int_{\gamma(a,r)} \left| \frac{f(w)}{(w-a)^{n+1}} \right| dw \leq \frac{n!}{2\pi} \frac{2\pi r}{r^{n+1}} \sup_{z \in C_{\gamma(a,r)}} |f(w)|$$

$$= \frac{n!}{r^n} \sup_{w \in C_{\gamma(a,r)}} |f(w)|.$$

$\square$

6.2   Stocktaking

Theorem 6.9

Let $f \in H(D(z_0, R))$ and let $\gamma(z_0, r)$ be a simple positively oriented circle (i.e. it is oriented anticlockwise) about a with radius $r < R$. Then $f^{(n)}(a)$ exists for $n \in \mathbb{N}$ and $n = 0$, and we have that

$$\int_{\gamma(z_0,r)} \frac{f(z)}{(z-a)^{n+1}} \, dz = \begin{cases} 2\pi i f(a) & \text{if } n = 0 \text{ and } a \in \gamma(z_0, r), \\ \frac{2\pi i}{n!} f^{(n)}(a) & \text{if } n \geq 1 \text{ and } a \in \gamma(z_0, r), \\ 0 & \text{if } a \notin \gamma(z_0, r). \end{cases}$$

---

7   Homotopy of paths

In this section we define the notion of homotopy. Suppose we have two closed paths in Ω. Imagine, we have placed a rubber band along γ_0. For γ_0 to be homotopic to γ_1 we should be able to transform the rubber band to get to γ_1, with the condition that each intermediate position of the rubber band lies in Ω.

Remark 7.1. Clearly this is not always possible: consider $\Omega = \mathbb{C} \backslash \{0\}$ with γ_0 a contour which does not contain the puncture and γ_1 a bigger contour containing both γ_0 and the puncture. There is no way for the rubber band to transform smoothly due to the obstruction of the puncture at $\{0\}$.

Definition 7.2. Let $\gamma_0, \gamma_1 : [0, 1] \to \Omega$ be two **closed** paths in an open set $\Omega \subset \mathbb{C}$. We say that the paths are **homotopic** in Ω if there exists a continuous function

$$H : [0, 1] \times [0, 1] \to \Omega$$

such that the following hold:

(H1) For all $t \in [0, 1]$, $H(t, 0) = \gamma_0(t)$.

(H2) For all $t \in [0, 1]$, $H(t, 1) = \gamma_1(t)$.

(H3) For all $s \in [0, 1]$, $H(0, s) = H(1, s)$.

If two paths, γ_0 and γ_1 are homotopic in Ω we write

$$\gamma_0 \sim \gamma_1$$

or possibly

$$\gamma_0 \sim_\Omega \gamma_1.$$

> **Note 7.3.** We can think of H as a family of closed paths from $[0,1] \to \Omega$ parametrised
> by "time", the s-variable. We can think of the closed path at time s as the position of the
> rubber band at time s: initially, when $s = 0$, $H(\cdot, 0)$ is the path γ_0, while finally, when the
> time $s = 1$, we end up with $H(\cdot, 1)$ which is the path γ_1. This is what (H1) and (H2) say.
> The requirement (H3) says that at each point of time s, the intermediate path $\gamma_s := H(\cdot, s)$
> is closed too. Furthermore, the continuity of H means that the rubber band never break,
> and the transformation takes place smoothly.

Example 7.4. The paths illustrated below are homotopic because they have the same
orientation and γ_1 can 'move' through the plane to deform into γ_2.
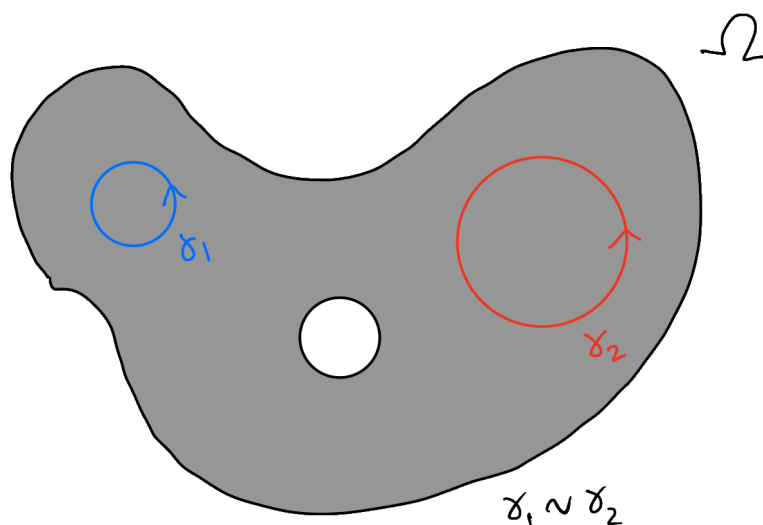
Example 7.5. These paths are NOT homotopic as they have different orientation.
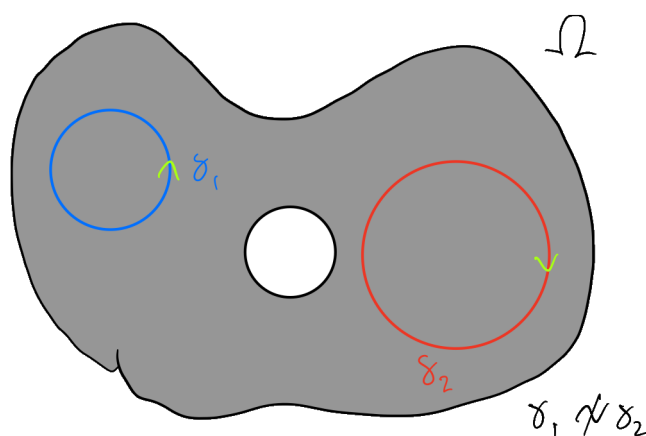
Example 7.6. Suppose we have two contours, γ_1 and γ_2, with a fixed point inside both
contours, $z = a$. If the winding number $n(\gamma_1, a) \neq n(\gamma_2, a)$ then $\gamma_1 \not\sim \gamma_2$.

Definition 7.7. In case a path γ is homotopic to a *point path* (i.e. $\gamma_p(t)$ is constant for all t) we write

$$\gamma \sim 0.$$

> **Note 7.8.** Imagine placing a rubber band along γ and then shrinking it to a point such that each intermediate position of the rubber band is in Ω and is closed. If this is possible we say γ is **contractible**. Indeed, a domain in which *every* closed path is contractible is called *simply connected* i.e. the domain has no holes.

Example 7.9.
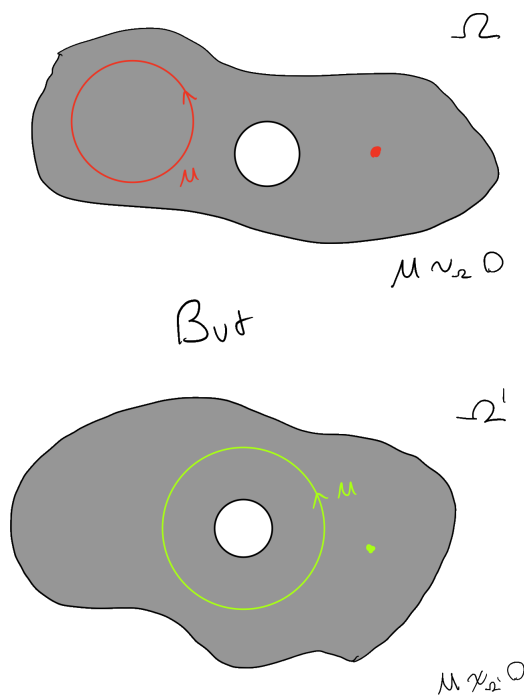
> **Theorem 7.10**
>
> In $\mathbb{C}$ all loops are homotopic to a point i.e. in $\mathbb{C}$ we have that $\gamma_k \sim 0$ for all k.

Proof. $\mathbb{C}$ has no holes so, any loop can be shrunk into a point. $\qquad \square$

> **Example 7.11**
>
> A loop defined on an annulus cannot be shrunk to a point thus, any loop in an annulus is not homotopic to a point.

> **Theorem 7.12** (Deformation theorem)
> Suppose $f \in H(\Omega)$ and that $\gamma_0 \sim \gamma_1$ in Ω. Then
> $$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Proof. To simplify the proof we assume the homotopy H is twice continuously differentiable which allows us to invoke the property to switch the order of partial differentiation:

$$\frac{\partial^2 H}{\partial s\,\partial t}(t,s) = \frac{\partial^2 H}{\partial t\,\partial s}(t,s).$$

Let $\gamma_s := H(\cdot, s)$ be the intermediate curve at time s. Define

$$I(s) := \int_{\gamma_s} f(z)\,dz \quad \text{for } s \in [0,1].$$

We show that $\dfrac{d}{ds}I(s) = 0$ and in particular $I(0) = I(1)$. We have

$$\frac{d}{ds}I(s) = \frac{d}{ds}\int_{\gamma_s} f(z)\,dz = \frac{d}{ds}\int_0^1 f(H(t,s))\frac{\partial H}{\partial t}(t,s)\,dt$$

$$= \int_0^1 \frac{\partial}{\partial s}\left(f(H(t,s))\frac{\partial H}{\partial t}(t,s) \right)\,dt$$

$$= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial s}(t,s)\frac{\partial H}{\partial t}(t,s) + f(H(t,s))\frac{\partial^2 H}{\partial s\,\partial t}(t,s) \right)$$

$$= \int_0^1 \left(f'(H(t,s))\frac{\partial H}{\partial s}(t,s)\frac{\partial H}{\partial t}(t,s) + f(H(t,s))\frac{\partial^2 H}{\partial t\,\partial s}(t,s) \right)$$

$$= \int_0^1 \frac{d}{dt}\left(f(H(t,s))\frac{\partial H}{\partial s}(t,s) \right),$$

and so by FTC,

$$\frac{d}{ds}I(s) = \int_0^1 \frac{d}{dt}\left(f(H(t,s))\frac{\partial H}{\partial s}(t,s) \right)$$
$$= 0,$$

since the paths are closed. Clearly, $I(0) = I(1)$. $\qquad\square$

7.1   Winding numbers

> **Remark 7.13.** In the following it is, in general, more convenient to parametrise loops using the unit interval $[0,1]$, but any non-degenerate interval $[a,b]$ is also fine.

Definition 7.14. Let $\gamma : [0,1] \to \mathbb{C}\backslash\{0\}$ be a piecewise smooth closed curve in $\mathbb{C}$ which does not pass through the origin. Then the **winding number** of γ about 0 is given by

$$w_\gamma = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz$$

Corollary 7.15. The winding number is given by

$$w_\gamma = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} \, dt.$$

Proof. Clear. □

Proposition 7.16

The winding number is an integer i.e. $w_\gamma \in \mathbb{Z}$.

Proof. This follows from direct evaluation. Let $\gamma(t) = \rho(t)e^{i\phi(t)}$, where both ρ and ϕ closed paths and are piecewise differentiable. Then

$$\begin{aligned}
w_\gamma &= \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, dz \\
&= \frac{1}{2\pi i} \left(\int_0^1 \frac{1}{\rho(t)e^{i\phi(t)}} \left(\rho'(t)e^{i\phi(t)} + i\rho(t)\phi'(t)e^{i\phi(t)} \right) dt \right) \\
&= \frac{1}{2\pi i} \left(\int_0^1 \frac{\rho'(t)}{\rho(t)} \, dt + i \int_0^1 \phi'(t) \, dt \right) \\
&= \frac{1}{2\pi i} \left([\log \rho(t)]_0^1 + i \, [\phi(t)]_0^1 \right) \\
&= \frac{1}{2\pi i} \cdot 2\pi i k \quad \text{for } k \in \mathbb{Z} \\
&= k \in \mathbb{Z}.
\end{aligned}$$

□

Definition 7.17. Let $\Omega \subset \mathbb{C}$ be an open set. To each holomorphic function $f : \Omega \to \mathbb{C}\backslash\{0\}$ and each closed contour $\gamma : [0,1] \to \Omega$ consider the contour

$$f \circ \gamma : [0,1] \to \mathbb{C}\backslash\{0\}, \quad \text{where } t \mapsto f(\gamma(t)).$$

Define

$$w_\gamma(f) := w_{f \circ \gamma} \in \mathbb{Z}.$$

Corollary 7.18

Assuming that $f' \in C(\Omega)$ we have

$$w_\gamma(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz.$$

Proof. By the chain rule $f(\gamma(t))$ has derivative $f'(\gamma(t)) \cdot \gamma'(t)$ hence,

$$\begin{aligned}
w_\gamma(f) &= \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z} \, dz \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t)) \cdot \gamma'(t)}{f(\gamma(t))} \, dt \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz.
\end{aligned}$$

□

Proposition 7.19

Let $\gamma : [0,1] \to \Omega$ be a closed contour and let $f, g : \Omega \to \mathbb{C}\backslash\{0\}$ with $f, g \in H(\Omega)$ (with continuous derivatives). Then

$$w_\gamma(f \cdot g) = w_\gamma(f) + w_\gamma(g).$$

Proof. Clear with use of this identity:

$$\frac{(f \cdot g)'(z)}{f(z)g(z)} = \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}.$$

$\square$

Definition 7.20. The **index** of γ around $a \in \mathbb{C}$ (or the **winding number** of γ around $a \in \mathbb{C}$) is defined to be the integer

$$n(\gamma; a) := w_\gamma(z - a).$$

Note 7.21. This allows us to define the winding number for any point $a \in \mathbb{C}$ by a translation of a from 0.

Corollary 7.22

We have that
$$n(\gamma; a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - a}.$$

7.1.1 Intuitive meaning of the winding number

The **winding number**, w_γ of a closed loop γ about the origin is the *net* number of revolutions of the direction of z as it traces out γ once in its given orientation. The notion of a winding number can be extended to any a point in the plane. We add $+1$ to w_γ after each positive (anticlockwise) revolution of the loop around the point and (-1) for each negative (clockwise) revolution.

7.1.2 Finding winding numbers quickly

Remark 7.23. Refer to Visual Complex Analysis book for more explanation (NeedhamVCA.pdf).

Theorem 7.24

If a contour γ is moving from our left to our right [our right to our left] as a particle (a moving point) crosses it, its winding number around that point increases [decreases] by one.

An immediate consequence of this result is a connection between, w_γ and the number of intersection points of γ with a ray emanating from a point p. We illustrate this property with the figure below, where the $w_\gamma = 2$ for p where each intersection point being marked with $\oplus$ or $\ominus$ according as the winding number increases or decreases as it is crossed.

7.1.3   "Inside" a loop

$$v_1 = 2$$
$$v_2 = 0$$
$$v_3 = 1$$
$$v_4 = 0$$
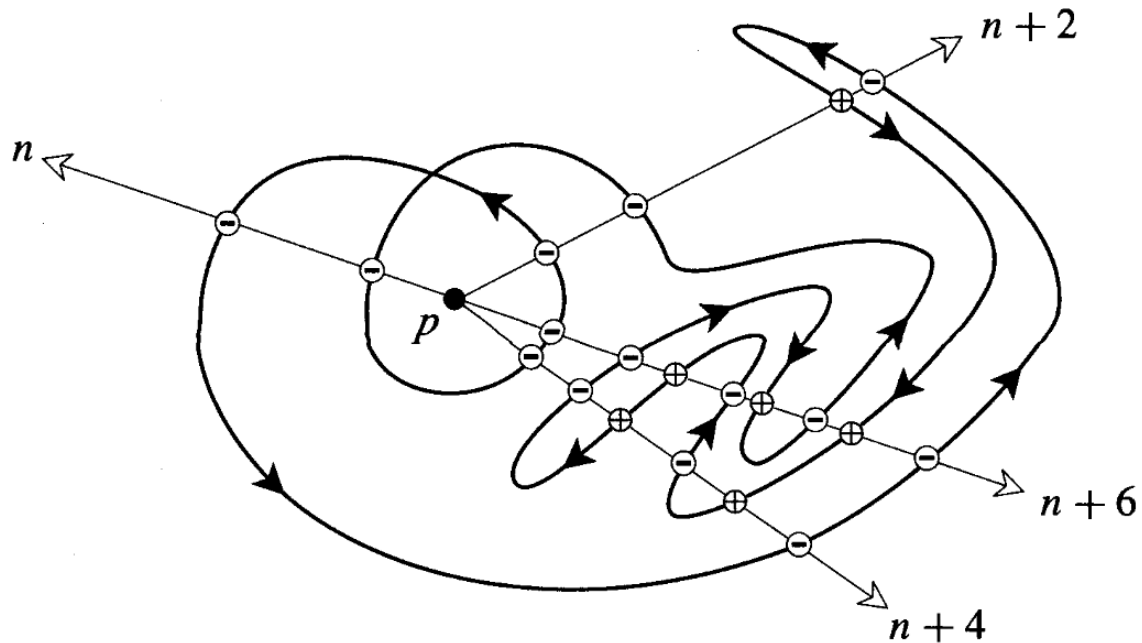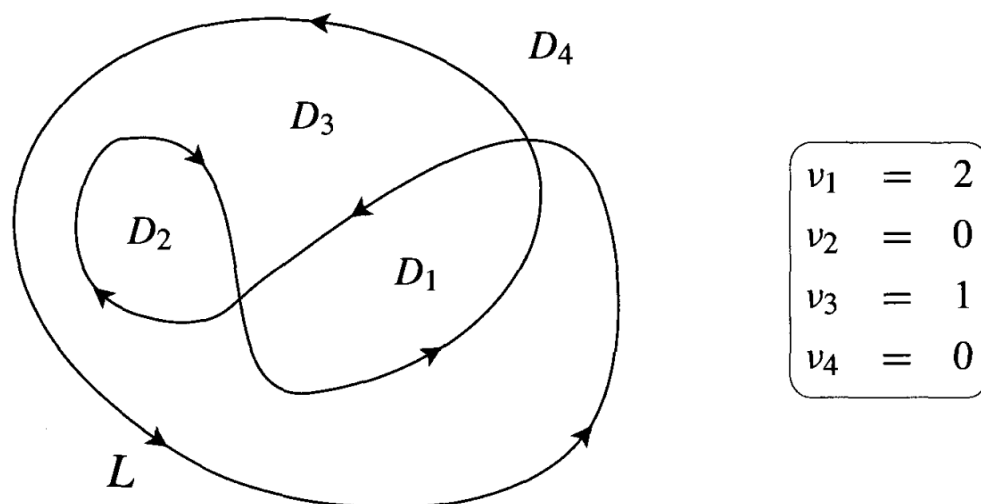
A typical loop such as L (as shown above) will partition the plane into a number of sets D_j.

Definition 7.25. The **"inside"** of a loop can be defined to be those D_j for which the winding number is not 0 i.e. $w \neq 0$.

Definition 7.26. The **"outside"** of a loop can be defined to be those D_j for which the winding number is 0 i.e. $w = 0$.

7.1.4   Homotopic to a point(extra)

> **Theorem 7.27**
>
> If there exists a singularity "outside" the loop then the contour can be deformed into a point. That is, if a loop has a singularity which is "outside" then the loop is homotopic to a point.

7.2   Generalised Cauchy theorem

Definition 7.28. An open set $\Omega \subset \mathbb{C}$ is said to be **simply connected** if every closed contour in Ω is homotopic to a point:

$$\forall \gamma : [0,1] \to \Omega \quad \text{we have} \quad \gamma \sim 0.$$

> **Theorem 7.29** (Cauchy's (integral) theorem)
>
> If f is holomorphic on an open simply connected set $\Omega \subset \mathbb{C}$ we have
>
> $$\int_\gamma f(z) = 0$$
>
> for every closed contour, γ in Ω.

> **Note 7.30.** This theorem extends the CIF to general contours which are not circles. Thus, simply put if
> $$\begin{aligned} & f \in H(\Omega) \\ & \text{and} \qquad \Rightarrow \int_\gamma f(z)\,dz = 0. \\ & \gamma \sim_\Omega 0 \end{aligned}$$

> **Theorem 7.31** (G-CIF)
>
> Let $f \in H(\Omega)$ and let $\gamma \sim_\Omega 0$. Then
>
> $$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$
>
> for each $a \in \Omega$ such that $a \notin \gamma$.

> **Note 7.32.** By $a \notin \gamma$ we mean that a does not lie on the path γ but in the area inside the path.

Proof. Let

$$g(z) = \frac{f(z) - f(a)}{z-a}.$$

By assumption $f(z) \in H(\Omega)$ so, we know that $g(z) \in H(\Omega)$. Since the derivative of f exists at a we know that

$$\lim_{z \to a} g(z) = f'(a).$$

That is

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f(z) & z = a, \end{cases}$$

by Cauchy's integral theorem from above we have

$$\int_\gamma g(z)\,dz = 0 \quad \text{i.e.} \quad \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0.$$

Thus,

$$0 = \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz$$
$$= \int_\gamma \frac{f(z)}{z-a}\,dz - \int_\gamma \frac{f(a)}{z-a}\,dz,$$

that is

$$\int_\gamma \frac{f(a)}{z-a}\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz$$
$$f(a)\int_\gamma \frac{1}{z-a}\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz$$
$$f(a)(2\pi i) = \int_\gamma \frac{f(z)}{z-a}\,dz.$$

We conclude that

$$n(\gamma; a)f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$

$\square$

7.3   The Cauchy Residue Theorem

Definition 7.33. Suppose that $f : D\backslash\{a\} \to \mathbb{C}$ be a holomorphic function which has a singularity at a and let

$$f(z) = \sum_{n \in \mathbb{Z}} c_n(z-a) \quad \text{for } 0 < |z-a| < R.$$

Then, we call the coefficient c_{-1} the residue of f at a, and denote it by

$$\mathrm{Res}(f, a).$$

Remark 7.34. The coefficient c_{-1} is a special term: if we integrate the power series of $f(z)$ the term with the coefficient c_{-1} is the only one that remains.

Theorem 7.35 (Cauchy Residue Theorem)

Let f be a complex function in $\Omega \subseteq \mathbb{C}$ with poles at the points $a_1, a_2, \dots$ then, for every closed contour γ, which is homotopic to a point Ω and does not pass through any a_k, we have

$$\frac{1}{2\pi i} \int_\gamma f(z) \, dz = \sum_k n(\gamma, a_k) \operatorname{Res}(f, a_k).$$

Theorem 7.36

Depending on the form of the function the residue is computed as follows.

1. If
$$f(z) = \frac{\phi(z)}{z - a} \quad \text{where } \phi \text{ is holomorphic at } a$$
then
$$\operatorname{Res}(f, a) = \phi(a).$$

2. If
$$f(z) = \frac{\phi(z)}{(z - a)^m} \quad \text{where } \phi \text{ is holomorphic at } a$$
then
$$\operatorname{Res}(f, a) = \frac{\phi^{(m-1)}(a)}{(m - 1)!},$$
where $\phi^{(m-1)}(a)$ means the $(m - 1)$ derivative of ϕ at a.

3. If
$$f(z) = \frac{\phi(z)}{\psi(z)}$$
with ϕ and ψ holomorphic at $a \in \mathbb{C}$, and that
$$\psi(a) = 0 \quad \text{and} \quad \psi'(a) \neq 0,$$
then f has a simple pole at $z = a$ and
$$\operatorname{Res}(f, a) = \frac{\phi(a)}{\psi'(a)}.$$

Remark 7.37. By ϕ being holomorphic at a we mean that ϕ is differentiable in a (small) disc centred at a.

7.3.1 Computing contour integrals with the CRT

Example 7.38. Using the CRT evaluate the integral

$$\int_{\gamma(0,1)} \frac{e^z}{z} \, dz.$$

Solution. Let $f(z) = \frac{e^z}{z}$, both e^z and z are holomorphic everywhere thus, the only pole of $f(z)$ is at $z = 0$. This implies,

$$\mathrm{Res}(f, 0) = e^0 = 1;$$

since the contour is only traversed once the winding number is 1 hence, by the CRT we have

$$\int_{\gamma(0,1)} \frac{e^z}{z}\, dz = 2\pi i.$$

Example 7.39. Evaluate the integral

$$\int_{\gamma} \frac{e^z}{(z-1)(z-3)}$$

where γ is given by $|z| = 2$ oriented anticlockwise.

Solution. The poles of the integrand are at $z = 1$ and $z = 3$ however, the pole at $z = 3$ lies outside the contour thus, the winding number of this pole is 0 which means the associated residue does not contribute the value of the integral. As such, let $f(z) = \frac{\frac{e^z}{z-3}}{z-1}$ then, $\mathrm{Res}(f, 1) = \phi(1) = \frac{e}{-2}$. By the CRT we have

$$\int_{\gamma} \frac{e^z}{(z-1)(z-3)} = -\pi i e.$$

Example 7.40. The integral as above with contour $|z| = \frac{1}{2}$ will be 0 since, the integrand is holomorphic inside the contour.

Example 7.41. Evaluate the integral

$$\int_{\gamma\left(a, \frac{5}{2}\right)} \cot(\pi z)\, dz$$

traversed anticlockwise once.

Solution. First note that

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)},$$

and $\sin(\pi z) = 0$ for $z \in \mathbb{Z}$ thus, the poles of $\cot(\pi z)$ are $z = n \in \mathbb{Z}$. Let $f(z) = \cot(\pi z)$ then $\phi(z) = \cos(\pi z)$ and $\psi(z) = \sin(\pi z)$, as such the residues are given by

$$\begin{aligned}
\mathrm{Res}(f, n) &= \frac{\phi(n)}{\psi'(n)} \\
&= \frac{\cos(\pi z)}{\pi \cos(\pi z)} \\
&= \frac{1}{\pi}.
\end{aligned}$$

The contour $\gamma(0, \frac{5}{2})$ contains within it only the integers $n = -2, -1, 0, 1, 2$ hence, these are the only poles of $\cot(\pi z)$ which contribute to the integral. Thus, by the CRT we have

$$\int_{\gamma\left(a, \frac{5}{2}\right)} \cot(\pi z)\, dz = 2\pi i \left(\frac{5}{\pi}\right)$$

$$= 10i.$$

8   Evaluating integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$ via the CRT

To evaluate an integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$ over the real line we consider the contour integral $\int_{C_R} f(z)\,dz$ over a simple closed contour

$$C_R = [-R, R] \cup A_R$$

where f is holomorphic on and insider C_R except at the poles inside C_R. We can then split the integral as follows

$$\int_{C_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{A_R} f(z)\,dz$$
$$= \int_{-R}^{R} f(x)\,dx + \int_{A_R} f(z)\,dz/$$

As $R \to \infty$ the aim is to choose A_R such that

$$\lim_{R \to \infty} \int_{A_R} f(z)\,dz = 0$$

hence, we can apply the Cauchy Residue Theorem to infer that

$$\int_{C_R} f(z)\,dz = \lim_{R \to \infty} 2\pi i \sum_{k=1}^{m_R} \operatorname{Res}(f, a_k)$$
$$= \int_{-\infty}^{\infty} f(x)\,dx.$$

> **Note 8.1.** Sometimes we may be interested in computing the integral
>
> $$\int_{0}^{\infty} f(x)\,dx,$$
>
> if $f(x)$ is an even function then
>
> $$\int_{-R}^{R} f(x)\,dx = 2 \int_{0}^{R} f(x)\,dx.$$

8.1   Technical toolkit for contour Integration

Proposition 8.2. Useful limits:

1. for any constant $k > 0$
$$\lim_{x \to \infty} x^k e^{-x} = 0 \quad \text{for } x \in \mathbb{R};$$

2. for any constant $k > 0$

$$\lim_{x \to \infty} x^{-k} \ln(x) = 0 \quad \text{for } x \in \mathbb{R}, x > 0;$$

3. for any constant $k > 0$

$$\lim_{x \to 0} x^k \ln(x) = 0 \quad \text{for } x \in \mathbb{R}, x > 0.$$

Proposition 8.3. Let S_N be a square with vertices at $(\pm 1 \pm i)(N + \frac{1}{2})$ for $N \in \mathbb{N}$ then

- there exists a constant C such that $|\cot(\pi z)| \leq C$ for all $z \in S_N$;

- there exists a constant K such that $|\csc(\pi z)| \leq K$ for all $z \in S_N$;

Proposition 8.4

Inequalities to remember.

1. Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$.

2. Reverse triangle inequality: $|z_1 + z_2| \geq ||z_1| - |z_2||$.

3. $|z_1| \leq |z_2| \iff \frac{1}{|z_1|} \geq \frac{1}{|z_2|}$ for $z_1, z_2 \neq 0$.

4. $\frac{1}{|z_1 + z_2|} \leq \frac{1}{||z_1| - |z_2||}$.

5. Repeated triangle inequality: $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.

6. Repeated reverse triangle inequality: $|z_1 + z_2 + \cdots + z_n| \geq |z_1| - |z_2| - \cdots - |z_n|$ if $|z_1| \geq |z_2| + \cdots + |z_n|$.

7. Estimation theorem:

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_a^b |f(\gamma(t)) \cdot \gamma'(t)|\, dt \leq \sup_{z \in \gamma} |f(z)| \cdot |\gamma'(t)|.$$

8. If the denominator f is at least more than two the degree of the numerator it follows that

$$\left| \int_{A_R} f(z)\, dz \right| \leq \frac{M}{R^2} \cdot \pi R = \frac{M\pi}{R},$$

where A_R is the parametrisation of the semicircle.

9. **Jordan's inequality**:

$$\frac{2}{\pi} \leq \frac{\sin \theta}{\theta} \leq 1 \quad \text{for } 0 < \theta \leq \frac{\pi}{2}.$$

10. Suppose $\gamma : [a, b] \to C \subset \mathbb{C}$ is a parametrisation of C. If $|f(z)| \leq M$ for all $z \in X$ (i.e. f is bounded by M along C) we have

$$\left| \int_C f(z)\, dz \right| \leq ML_C,$$

where L_C is the length of C.

Remark 8.5. Joining (1) and (2) together we have

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Wolfram alpha says:

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Example 8.6. By considering the contour integral

$$\int_{C_R} \frac{e^{iz}}{z - ia} \, dz$$

where $C_R = [-R, R] \cup A_R$ and A_R is the semicircle centre at 0 and radius $R > a > 0$ in the upper-half plane, show that

$$\int_{-\infty}^{\infty} \frac{x \sin x + a \cos x}{x^2 + a^2} \, dx = 2\pi e^{-a}$$

and

$$\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2} \, dx = 0.$$

Solution. Let $f(z) = \frac{e^{iz}}{z - ia}$; note that it has a pole at $z = ia$ with residue $\text{Res}(f, ia) = e^{-a}$. Since, the contour is only traversed once by the CRT we have

$$\int_{C_R} \frac{e^{iz}}{z - ia} \, dz = 2\pi i e^{-a}.$$

Splitting the contour integral over the interval $[-R, R]$, with the parametrisation $\gamma(x) = x$, and the semicircle A_R we have

$$\int_{-R}^{R} \frac{e^{ix}}{x - ia} \, dx + \int_{A_R} \frac{e^{iz}}{z - ia} \, dz = 2\pi i e^{-a}.$$

We now prove

$$\lim_{R \to \infty} \int_{A_R} \frac{e^{iz}}{z - ia} \, dz = 0.$$

Consider

$$\left| \int_{A_R} \frac{e^{iz}}{z - ia} \, dz \right| = \left| \int_0^{\pi} \frac{e^{ia(Re^{i\theta})}}{Re^{i\theta} - ia} \cdot Rie^{i\theta} \, d\theta \right|$$

$$= \left| \int_0^{\pi} \frac{e^{ia[R \cos\theta + iR \sin\theta]}}{Re^{i\theta} - ia} \cdot Rie^{i\theta} \, d\theta \right|$$

$$\le \int_0^{\pi} \left| \frac{e^{ia[R \cos\theta + iR \sin\theta]}}{Re^{i\theta} - ia} \cdot Rie^{i\theta} \right| d\theta.$$

Clearly,

$$\left| Rie^{i\theta} \cdot e^{ia[R \cos\theta + iR \sin\theta]} \right| = \left| Rie^{i\theta} \cdot e^{iaR \cos\theta} \cdot e^{-aR \sin\theta} \right|$$

$$= \left| Rie^{i\theta} \right| \cdot \left| e^{iaR \cos\theta} \right| \cdot \left| e^{-aR \sin\theta} \right|$$

$$= Re^{-aR \sin\theta}.$$

Furthermore, by the reverse triangle inequality we have

$$\left|Re^{i\theta} + (-ia)\right| \geq \left|\left|Re^{i\theta}\right| - |-ia|\right|$$
$$\geq |R - a|,$$

this implies

$$\frac{1}{|Re^{i\theta} - ia|} \leq \frac{1}{|R - a|}.$$

Therefore,

$$\left|\int_{A_R} \frac{e^{iz}}{z - ia}\,dz\right| \leq \int_0^\pi \frac{R}{|R - a|}e^{-aR\sin\theta}\,d\theta.$$

Now, since $R > a$ we can drop the absolute value, and we note that

$$\int_0^\pi e^{-aR\sin\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta.$$

We conclude that the integral

$$\left|\int_{A_R} \frac{e^{iz}}{z - ia}\,dz\right| \leq \frac{2R}{R - a}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta,$$

by Jordan's inequality we can write

$$\frac{2\theta}{\pi} \leq \sin\theta$$

as such we have

$$\left|\int_{A_R} \frac{e^{iz}}{z - ia}\,dz\right| \leq \frac{2R}{R - a}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta$$

$$\leq \frac{2R}{R - a}\int_0^{\frac{\pi}{2}} e^{-aR\frac{2\theta}{\pi}}\,d\theta$$

$$= \frac{2R}{R - a}\left[-\frac{\pi}{2aR}e^{-aR\frac{2\theta}{\pi}}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{Ra - a^2}(-1)\left(e^{-aR} - 1\right)$$

$$= \frac{\pi}{Ra - a^2}\left(1 - e^{-aR}\right).$$

Clearly, as $R \to \infty$ the integral $\int_{A_R} f(z)\,dz \to 0$ (provided $a > 0$). Whereas, as $R \to \infty$ the integral $\int_{-R}^R f(x)\,dx$ becomes

$$\int_{-\infty}^\infty \frac{e^{ix}}{x - ia}\,dx = 2\pi i e^{-a}.$$

To obtain the required integrals we note that

$$\int_{-\infty}^\infty \frac{e^{ix}}{x - ia}\,dx = \int_{-\infty}^\infty \frac{(\cos x + i\sin x)(x + ia)}{(x - ia)(x + ia)}\,dx$$

$$= \int_{-\infty}^\infty \frac{(x\cos x - a\sin x) + i(a\cos x + x\sin x)}{x^2 + a^2}\,dx$$

$$= \int_{-\infty}^\infty \frac{x\cos x - a\sin x}{x^2 + a^2}\,dx + i\int_{-\infty}^\infty \frac{x\sin x + a\cos x}{x^2 + a^2}\,dx$$

$$= 0 + i(2\pi e^{-a}).$$

By comparing the real and imaginary part we have:

- real part:
$$\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2 + a^2} \, dx = 0;$$

- imaginary part:
$$\int_{-\infty}^{\infty} \frac{x \sin x + a \cos x}{x^2 + a^2} \, dx = 2\pi e^{-a}.$$

Example 8.7

By considering

$$\int_{\gamma_N} \frac{\pi \cot(\pi z)}{z^2} \, dz$$

where γ_N is the square with vertices at $(\pm 1 \pm i)(N + \frac{1}{2})$ oriented anticlockwise and traversed once prove

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. The function $f(z) = \frac{\pi \cot(\pi z)}{z^2}$ is holomorphic except at the poles $z = 0$ and $z = n \in \mathbb{Z} \backslash \{0\}$ (we write it in such way as the residue differs for these poles). We can write

$$f(z) = \frac{\frac{\pi \cos(\pi z)}{z^2}}{\sin(\pi z)}$$

thus, letting $\phi(z) = \frac{\pi \cos(\pi z)}{z^2}$ and $\psi(z) = \sin(\pi z)$ the residues

$$\mathrm{Res}(f, n) = \frac{\frac{\pi \cos(\pi n)}{n^2}}{\pi \cos(\pi n)}$$
$$= \frac{1}{n^2}.$$

Whereas, (by computing the relevant Laurent expansion of $f(z)$) the residue at $z = 0$ is given by

$$\mathrm{Res}(f, 0) = -\frac{\pi^2}{3}$$

By the CRT we have

$$\int_{\gamma_N} f(z) \, dz = 2\pi i \left(\sum_{n=-N}^{N} \frac{1}{n^2} - \frac{\pi^2}{3} \right).$$

Note that we can write

$$\sum_{n=-N}^{N} \frac{1}{n^2} = 2 \sum_{n=1}^{N} \frac{1}{n^2}.$$

It is now enough to show that $\int_{\gamma_N} f(z) \, dz \to 0$ as $N \to \infty$. We have

$$\left| \int_{\gamma_N} f(z) \, dz \right| \leq \sup \left| \frac{\pi \cot(\pi z)}{z^2} \right| \cdot \mathrm{length}(\gamma_N)$$
$$\leq \sup |\cot(\pi z)| \frac{4(2N+1)\pi}{\left(N + \frac{1}{2} \right)^2}$$
$$\to 0 \quad \text{as } N \to \infty.$$

By comparing imaginary part of the integral, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

9 Existence of primitives

Theorem 9.1

Let $\Omega \subseteq \mathbb{C}$ be open. Then

$$f \in H(\Omega) \iff f \text{ is complex analytic on } \Omega.$$

Note 9.2. This theorem is saying that f is holomorphic on Ω if and only if f has power series expansion on Ω.

Theorem 9.3.
$$f \in H(\Omega) \Rightarrow f' \in C(\Omega).$$

Lemma 9.4

If $f \in C(\Omega)$ has a primitive on an open set Ω (i.e. there exists $F \in H(\Omega)$ such that $F' = f$) then, $f \in H(\Omega)$ and $f' \in C(\Omega)$. Furthermore, $f^{(k)} \in H(\Omega)$ for all $k \in \mathbb{N}$.

Proof. By assumption the primitive F is a holomorphic function, and it has a continuous derivative $F' = f$. Therefore, the CIF holds for F and F has a power series on every disc $D \subset \Omega$. Since, we can differentiate power series term by term, the function $f'(z)$ is also given by a power series on D. This implies that f' is holomorphic and consequently f' is continuous. Iterating this, we obtain the final statement. $\square$

9.1 Morera's and Goursat's theorems

Theorem 9.5 (Morera's theorem)

Let f be a continuous function on an open simply-connected set $\Omega \subset \mathbb{C}$. If $\int_T f(z)\,dz = 0$ for any triangular contour T in Ω then, f has a primitive on Ω, i.e. $F \in H(\Omega)$ with $F' = f$.

Theorem 9.6 (Goursat's theorem)**.** If Ω is simply connected then,

$$\int_T f(z)\,dz = 0$$

for any arbitrary triangle T in Ω.

Remark 9.7. If we only assume $\int_T f(z)\,dz = 0$ for all triangles $T \subset \Omega$ inside open discs $D \subset \Omega$, Morera and Goursat's theorems are not necessarily true for Ω which are NOT simply connected.

Example 9.8. The function z^{-1} is holomorphic on an open annulus Ω about the origin and its integrals over all such triangular contours in Ω are equal to zero. However, it does not have a primitive on Ω.

> **Corollary 9.9**
>
> Every holomorphic function f on an open simply-connected set Ω has a primitive.

Corollary 9.10. For any holomorphic function $f \in H(\Omega)$ for an open set $\Omega \subset \mathbb{C}$ one has $f^{(k)} \in H(\Omega)$ for all $k \in \mathbb{N}$.

Corollary 9.11. A continuous function f on a simply connected domain is holomorphic if and only if $\int_{\gamma} f(z)\, dz = 0$ for every closed contour γ in Ω.

> **Theorem 9.12**
>
> If f is continuous on an open set $\Omega \in \mathbb{C}$ and $\int_{\gamma} f(z)\, dz = 0$ for every closed contour γ in Ω then, f has a primitive F on Ω and is holomorphic on Ω.

10 Singularities of holomorphic function

Definition 10.1. Let f be a complex-valued function which is not defined at a point $z_0 \in \mathbb{C}$, and suppose that it is holomorphic in some punctured disc $D(z_0, R)\backslash\{z_0\}$ so, it is not holomorphic at z_0. Then, we call z_0 an **isolated singularity** of f.

Example 10.2. Each of the functions

$$\frac{\sin(z)}{z}, \quad \frac{1}{z^3}, \quad e^{\frac{1}{z}}$$

have an isolated singularity at 0.

Definition 10.3. An isolated singularity a of f is called:

1. a **removable singularity** of f if the limit $\lim_{z \to a} |f(z)|$ exists as a finite real number;

2. a **pole** of f if $\lim_{z \to a} |f(z)| = \infty$;

3. an **essential singularity** of f if a is neither removable nor a pole.

> **Note 10.4.** We use the name 'removable singularity' as we can define a new function F such that
> $$F(z) = \begin{cases} \lim_{z \to z_0} f(z) & \text{for } z = z_0 \\ f(z) & \text{otherwise.} \end{cases}$$
> Hence, we are effectively removing the singularity.

> **Example 10.5**
>
> The function $e^{\frac{1}{z}}$ has an essential singularity at 0 as the limit of $\frac{1}{z}$ as $z \to 0$ does not exist (because the left and right limits are not equal) thus, by extension the limit of $e^{\frac{1}{z}}$ as $z \to 0$ does not exist.

Definition 10.6. The point $z = a$ is a zero of a holomorphic function f if $f(a) = 0$.

Lemma 10.7. If $f \in H(\Omega)$ that is not identically to zero and $z = a$ is a zero of f then, f can be represented in the form

$$f(z) = (z - a)^m g(z)$$

in some neighbourhood of $z = a$, where $m \geq 1$ and $g \in H(\Omega)$.

Proof. Since $f \in H(\Omega)$ we can write it as the power series

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} c_n (z - a)^n \\
&= \sum_{n=m}^{\infty} c_n (z - a)^n \\
&= (z - a)^m \sum_{n=0}^{\infty} c_{n+m} (z - a)^n \\
&:= (z - a)^m g(z),
\end{aligned}
$$

where m is the smallest index greater than 0 such that $a_m \neq 0$. $\qquad \square$

Definition 10.8. Suppose $f(a) = 0$ and

$$f(z) = (z - a)^m g(z)$$

then, the number m is called the **order of the zero $z = a$**.

Proposition 10.9

The function f has a zero of order m at $z = a$ if and only if

$$f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$$

but

$$f^{(m)}(a) \neq 0.$$

Proposition 10.10. If $f \in H(\Omega)$ has distinct zeroes, $a_1, \ldots, a_n \in \Omega$ of order $m_1, \ldots, m_n \in \mathbb{N}$ respectively then,

$$f(z) = \prod_{k=1}^{n} (z - a_k)^{m_k} h(z)$$

where $h \in H(\Omega)$ and $h(z) \neq 0$ in each $D(a_k, R_k)$ for $R_k > 0$.

Proposition 10.11 (Compound zeroes)

If f and g have zeroes of order $m \geq 0$ and $n \geq 0$, respectively, at $z = a$ then fg has a zero of order $m + n$ at a.

Definition 10.12. If f is defined and holomorphic in a punctured disc $D(a,R)\backslash\{a\}$, we say that it has a **pole of order** m at $z = a$ if the function $f(z) = \frac{1}{h(z)}$ (defined to be 0 at $z = a$) has a zero of order m at $z = a$. A pole of order 1 is called a **simple pole**.

Lemma 10.13. The function f has a pole of order m at $z = a$ if and only if it can be represented in the form

$$f(z) = (z-a)^{-m}g(z)$$

in a punctured disc centred at $z = a$, where g is holomorphic in this disc and $g(a) \neq 0$.

Proposition 10.14. If f has poles at $b_1, \ldots, b_n \in \Omega$ then,

$$f(z) = \prod_{k=1}^{n}(z-b)^{-m_k}q(z)$$

where $q \neq 0$ and is holomorphic around each pole.

> **Theorem 10.15** (Classification of removable singularity via limits)
>
> A function $f \in H(D(a,r)\backslash\{a\})$ has a removable singularity at $z = a$ if and only if
>
> $$\lim_{z\to a}(z-a)f(z) = 0.$$

> **Proposition 10.16** (Riemann's theorem on removable singularities)
>
> Let $f(z)$ be a holomorphic function which has an isolated singularity at a. Then $f(z)$ has a removable singularity at a if and only if $f(z)$ is bounded for some open disc $D(a,r)$.

Corollary 10.17. If f has a pole at $z = a$ then, the function $\frac{1}{f(z)}$ has a removable singularity at $z = a$.

10.1 Laurent series

Theorem 10.18 (Laurent expansion). Let $0 \leq R_1 < R_2 \leq \infty$ and $A(0, R_1, R_2)$ be the open annulus $R_1 < |z| < R_2$. If $f \in H[A(0, R_1, R_2)]$ then, there exist numbers $c_n \in \mathbb{C}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad \forall z \in A(0, R_1, R_2).$$

This series is **unique** and the coefficients

$$c_m = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{m+1}}\,dz$$

where γ is an arbitrary anticlockwise oriented circle in $A(0, R_1, R_2)$. Furthermore, the Laurent series is absolutely and uniformly convergent on the annulus $A(0, r_1, r_2)$ where $R_1 < r_1 < r_2 < R_2$.

Proposition 10.19 (Classification of singularities via Laurent coefficients)

If f has an isolated singularity at a then, $f \in H(D(a,r)\backslash\{a\})$ and has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n.$$

Then, the point $z = a$ is said to be

- a **removable singularity** if $c_n = 0$ for all $n < 0$;

- a **pole of order** $m \in \mathbb{N}$ $(m \geq 1)$ if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$;

- an **essential singularity** there are infinitely many negative indices n such that $c_n \neq 0$.

Note 10.20. We can interpret the theorem above as follows: given

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n,$$

if the Laurent series of f:

- has no negative terms then, f has removable singularity;

- has finitely many negative terms we can write

$$f(z) = \frac{b_m}{(z-a)^m} + \cdots + \frac{b_1}{z-a} + \sum_{n=0}^{\infty} c_n(z-a)^n$$

  where $b_m \neq 0$ then, f has a pole of order m at $z = a$;

- has infinitely many negative terms then, f has an essential singularity.

10.2 Meromorphic functions

Definition 10.21. A complex function defined on an open set Ω is called **meromorphic** if it is *holomorphic* in Ω except for a set of poles. The set of all meromorphic functions on Ω is denoted by $M(\Omega)$.

Proposition 10.22 (Identity theorem)

The following are equivalent for $f \in H(\Omega)$:

1. $f \equiv 0$ in Ω;

2. there is a point $a \in \Omega$ and a sequence $z_n \in \Omega$ with $z_n \to a$ and $f(z_n) = 0$;

3. there is a point $a \in \Omega$ with $f^{(m)}(a) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$.

Remark 10.23. In literature the Identity theorem is stated as follows.

Theorem 10.24. Let $\Omega \subset \mathbb{C}$ be a connected open set and let $f \in H(\Omega)$. The following statements are equivalent:

- $f \equiv 0$ i.e. $f(z) = 0$ for all $z \in \Omega$;

- the set $\{z \in \Omega : f(z) = 0\}$ has a limit point in Ω i.e. there exists a sequence of <u>distinct</u> points $z_k \in \Omega$ such that $f(z_k) = 0$ and $\lim_{k \to \infty} z_k$ exists and <u>belongs</u> to Ω;

- there exists a point $a \in \Omega$ such that $f^{(n)}(a) = 0$ for every $n \in \mathbb{Z}_{\geq 0}$.

Note 10.25. Thus, the zeroes (i.e. the roots) of a holomorphic function f, on Ω are isolated - meaning there exists $\delta_a > 0$ such that $z = a$ is the <u>unique</u> zero in $D(a, \delta_a)$ so, there are no other zeroes 'nearby'.

Corollary 10.26 (Uniqueness theorem). Suppose that $f, g \in \Omega$ and let $(z_n)_{n \in \mathbb{N}} \subset \Omega$ be a sequence of distinct points which converges to $a \in \Omega$ and such that for all $n \in \mathbb{N}$, $f(z_n) = g(z_n)$. Then, $f(z) = g(z)$ for all $z \in \Omega$.

Note 10.27. The uniqueness follows by the Identity Theorem: from the fact that two analytic functions that agree on a convergent sequence of point are identical.

Example 10.28

Let $f \in H(D(0, 2))$. If $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$ show that $f(z) = z^2$ for all $z \in D(0, 2)$.

Solution. Let $g(z) = z^2$ then,

$$f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \quad \forall n \in \mathbb{N}.$$

Now, define $h(z) = f(z) - g(z)$. Clearly, $z_n = \frac{1}{n} \to 0$ as $n \to \infty$ thus, there exists a point $0 \in D(0, 2)$ and sequence $z_n \in D(0, 2)$ with $z_n \to 0$ and $h(z_n) = 0$ which, by the Identity theorem, is equivalent to the statement $h \equiv 0$ in $D(0, 2)$. Hence,

$$f(z) = g(z) = z^2.$$

Example 10.29. We provide some examples of applications of the identity theorem. Let f be analytic on the open disc $D(0, 2)$.

- If $f\left(\frac{1}{n}\right) = \frac{1}{n^3}$ for all $n \in \mathbb{N}$; define $h(z) = f(z) - g(z)$ where $g(z) = z^3$. Clearly, $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Since $z_n = \frac{1}{n} \to 0$ as $n \to \infty$, we have that $h(z_n) = 0$ therefore, by the Identity theorem, we have the equivalent statement

$$f(z) = g(z) = z^3.$$

- If $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$ then $f(z) = z^3$ for all $z \in D(0, 2)$.

- If $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$ for all $n \in \mathbb{N}$. Consider $f\left(\frac{1}{n}\right) = \frac{1}{n^3}$ then, $f(z) = z^3$ for all $z \in D(0,2)$. However, considering $f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$ then, $f(z) = -z^3$ for all $z \in D(0,2)$. Hence, no such $f \in H(D(0,2))$ exists.

Proposition 10.30. The function $f \in M(\Omega)$ has a pole of order $m > 0$ at $z = a$ if and only if $\frac{1}{f} \in H(\Omega)$ has a zero of order m at $z = a$.

Proposition 10.31

Let $\Omega \subset \mathbb{C}$ be bounded then, $f \in M(\Omega)$ has only finitely many zeroes and poles in Ω. Let $a_1, \ldots, a_p \in \Omega$ denote the zeroes of f with respective multiplicities $m_1, \ldots, m_p \in \mathbb{N}$. Let $b_1, \ldots, b_q \in \Omega$ denote the poles of f with respective multiplicities $r_1, \ldots, r_q \in \mathbb{N}$. Then, f can be written

$$f(z) = \frac{\prod_{i=1}^{p}(z - a_i)^{m_i}}{\prod_{j=1}^{q}(z - b_j)^{r_j}} h(z)$$

where $h \in H(\Omega)$ has no zeroes in Ω.

10.2.1   Counting zeroes of meromorphic functions

Definition 10.32. Let $\gamma \sim 0$ and let $f \in M(\Omega)$ with poles $b_1, b_2, \ldots, b_n$ of multiplicities $r_1, r_2, \ldots, r_n$. Then the **number of poles** of the function f counted with multiplicity is defined to be

$$P_\gamma(f) = \sum_{k=1}^{n} r_k n(\gamma, b_k) \in \mathbb{Z}.$$

Remark 10.33. Above we have assumed that $b_i \notin \gamma$.

Definition 10.34. We define the **index** of $f \in M(\Omega)$ relative to such a closed contour γ by

$$\mathrm{Ind}_\gamma(f) = Z_\gamma(f) - P_\gamma(f).$$

Remark 10.35. Recall that the winding number is also called the index, we can think of this version of the index as one defined for <u>meromorphic</u> functions.

Theorem 10.36

For $f \in H(\Omega)$ one has

$$\mathrm{Ind}_\gamma(f) = w_\gamma(f).$$

Example 10.37

Given the function
$$f(z) = \frac{(z-4)(z-1)^2 \sin z}{z^2+1}$$

evaluate the integral
$$\int_C \frac{f'(z)}{f(z)} \, dz$$

where C is the positively oriented contour with $|z| = 2$.

Solution. Recall that the index of a function over a closed contour is equal to its winding number over the same contour. Recall the winding number is given by

$$w_\gamma(f) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz,$$

and the index of a function is

$$\mathrm{Ind}_\gamma(f) = Z_\gamma(f) - P_\gamma(f).$$

Therefore,

$$\int_C \frac{f'(z)}{f(z)} \, dz = 2\pi i \left(Z_C(f) - P_C(f) \right).$$

The function $f(z)$ has zeros at $4, 1$ (double zero), and $k\pi$ with $k \in \mathbb{Z}$. The only zeros inside C are 1 (double) and 0. Therefore, $Z_C(f) = 3$. The poles are at $\pm i$ and they both lie inside C. Therefore,

$$\int_C \frac{f'(z)}{f(z)} \, dz = 2\pi i \, (3 - 2)$$
$$= 2\pi i.$$

Corollary 10.38. Let $f, g \in M(\Omega)$ such that f, g and $f \cdot g$ have no zeroes or poles in γ then,
$$\mathrm{Ind}_\gamma(f \cdot g) = \mathrm{Ind}_\gamma(f) + \mathrm{Ind}_\gamma(g).$$

Corollary 10.39. One has
$$\mathrm{Ind}_\gamma\left(\frac{1}{f}\right) = -\mathrm{Ind}_\gamma(f).$$

Theorem 10.40

If $\gamma_1 \sim \gamma_2$ and the homotopy does not pass through any zero or pole of $f \in M(\Omega)$ then,
$$\mathrm{Ind}_{\gamma_1}(f) = \mathrm{Ind}_{\gamma_2}(f).$$

Theorem 10.41. If $f_1, f_2 \in M(\Omega)$ are homotopic via a homotopy $F : [0, 1] \times \Omega \to \mathbb{C}$ with $z \mapsto F(t, z) \in M(\Omega)$ and satisfies

$$F(0, z) = f_1(z) \quad \text{and} \quad F(1, z) = f_2(z)$$

such that no zero or pole of $F(t, z)$ lies in γ for any t then,

$$\text{Ind}_\gamma(f_1) = \text{Ind}_\gamma(f_2).$$

10.3   Counting zeroes of holomorphic functions

Definition 10.42. Let $\gamma \sim 0$ and let $f \in H(\Omega)$ with zeroes $a_1, a_2, \ldots, a_n$ of multiplicities $m_1, m_2, \ldots, m_n$. Then the **number of zeroes** of the function f counted with multiplicity is defined to be

$$Z_\gamma(f) = \sum_{k=1}^{n} m_k n(\gamma, a_k) \in \mathbb{Z}.$$

Theorem 10.43 (Zero counting theorem)

Let $\gamma \sim_\Omega 0$ and let $f \in H(\Omega)$ with zeroes $a_1, a_2, \ldots, a_n$ of multiplicities $m_1, m_2, \ldots, m_n$. Then,
$$Z_\gamma(f) = w_\gamma(f).$$

Corollary 10.44. Let $f, g \in H(\Omega)$ then

$$Z_\gamma(f \cdot g) = Z_\gamma(f) + Z_\gamma(g).$$

Theorem 10.45 (Rouché's theorem)

Let Ω be a simply connected set and let γ be a simple closed contour in Ω. If $f, g \in H(\Omega)$ and $|g(z)| < |f(z)|$ for all $z \in \gamma$ then, f and $f + g$ have the same number of zeroes inside γ.

Example 10.46

Determine the number of zeroes for $h(z) = z^7 - 2z^3 + 7$ inside the disc $|z| \leq 2$.

Solution. Let $h(z) = f(z) + g(z)$, we choose $f(z)$ such that $|g(z)| < |f(z)|$ so, we write a table to determine the dominant terms of $h(z)$.

Term	$|\text{term}|$ when $|z| = 2$
z^7	$2^7 = 128$
$-2z^3$	$2^4 = 16$
7	7

Therefore, we choose $f(z) = z^7$ and $g(z) = 7 - 2z^3$. Clearly, $|g(z)| < |f(z)|$ and both functions are holomorphic inside and on the disc. By Rouché's theorem, $f(z) = z^7$ and $f(z) + g(z) = h(z)$ have the same number of zeroes on the disc $|z| \leq 2$. In conclusion, since $f(z)$ has 7 zeroes on the disc then so does $h(z)$.

10.4   Essential singularities

> **Theorem 10.47** (Casorati-Weierstrass)
>
> If $f \in H[D(a,r)\backslash\{a\}]$ and has an essential singularity at $z = a$ then, the image $f(D(a,R)\backslash\{a\})$ of the punctured disc is dense in $\mathbb{C}$.

11   Liouville's theorem

> **Theorem 11.1** (Liouville's theorem)
>
> Let $f \in H(\mathbb{C})$ and let f be a bounded function. Then, f is constant.

Proof. Since f is entire we can write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\gamma(0,R)} \frac{f(w)}{w^{n+1}} \, dw,$$

where $\gamma(0,R)$ is a circle of radius $R > 0$ of centre 0. Suppose f is bounded by M i.e. $|f(z)| \le M$ for all $z \in \mathbb{C}$. We can estimate

$$
\begin{aligned}
|c_n| &= \left| \frac{1}{2\pi i} \int_{\gamma(0,R)} \frac{f(w)}{w^{n+1}} \, dw \right| \\
&\le \frac{1}{2\pi} \cdot \mathrm{Length}(\gamma(0,R)) \cdot \max_{z \in \gamma(0,R)} \left| \frac{f(w)}{w^{n+1}} \right| \\
&= \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M}{R^{n+1}} \\
&= \frac{M}{R^n}.
\end{aligned}
$$

Where we have used the fact $|z| = R$ on the circle $\gamma(0,R)$. Since, the choice of R was an arbitrary positive number by letting R tend to infinity (we let R tend to infinity since f is analytic on the entire plane) gives $c_n = 0$ for all $n \ge 1$. Thus, $f(z) = c_0$ i.e. f the constant function. $\qquad\square$

> **Corollary 11.2**
>
> If there exists $M > 0$ such that
>
> $$|f(z)| < M\,|z|^m \quad \text{for } m \in \mathbb{Z}_{\ge 0} \text{ and } \forall z \in \mathbb{C}$$
>
> then, $f(z)$ is a polynomial of order at most m.

Theorem 11.3 (Fundamental theorem of algebra)**.** Let $p(z)$ be a non-constant polynomial with complex coefficients. Then, there exists $\zeta \in \mathbb{C}$ such that $p(\zeta) = 0$. Consequently, a complex polynomial of degree $n > 1$ has n roots (not necessarily distinct) in $\mathbb{C}$.

Proof. For the sake of contradiction, suppose that $p(z) \neq 0$ for all z. Since $|p(z)| \to \infty$ as $|z| \to \infty$, there exists R such that $\left|\frac{1}{p(z)}\right| < 1$ for $|z| > R$. On the closed disc $\overline{D}(0, R)$ $\frac{1}{p(z)}$ is continuous which implies it is bounded. Hence, $\frac{1}{p(z)}$ is bounded on $\mathbb{C}$. It is also holomorphic on $\mathbb{C}$ so, by Liouville's theorem $\frac{1}{p(z)}$ is constant, which is a contradiction. $\square$

Corollary 11.4

If $P \in H(\mathbb{C})$ such that
$$|P(z)| \leq C\,(|z| + 1)^n$$
for some $C > 0$ and $n \in \mathbb{N}$ then, $P(z)$ is a polynomial.

Corollary 11.5. A polynomial $P(z)$ of degree n can be written as the product
$$P(z) = c(z - a_1)^{m_1}(z - a_2)^{m_2} \cdots (z - a_k)^{m_k},$$
where a_j are the roots of P of multiplicity m_j and c is a constant.

12   The maximum modulus theorem

Theorem 12.1 (Maximum Modulus Theorem)

Let $\Omega \subset \mathbb{C}$ be an open and connected set and let $f \in H(\Omega)$ with $f' \in C(\Omega)$. If $|f|$ obtains its maximum at some point $z_0 \in \Omega$ then, f is constant.

Example 12.2

Does there exist a holomorphic function on the disc $D(0, 1)$ with
$$\left| f\left(\frac{z}{2}\right) \right| > |f(z)|?$$

Solution. The condition above implies that $|f|$ attains its maximum at $z = 0$. By the MMT then, $f(z)$ is constant which contradicts the above assumption. Therefore, no such f exists.

Theorem 12.3 (MMT 2)

Let $\Omega \subset \mathbb{C}$ be an open and connected set and let $f \in H(\Omega)$ with $f' \in C(\Omega)$. If $|f|$ obtains its maximum at some point $z_0 \in \Omega$ then, $|f|$ attains its maximum on the boundary $\partial\overline{\Omega} = \overline{\Omega}\backslash\Omega$.

Remark 12.4. The notation $\overline{\Omega}$ indicates the closure of Ω.

Example 12.5

Let $\Omega = D(0,1)$ then, $\overline{\Omega} = \overline{D}(0,1) = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\partial\overline{\Omega} = |z|$. Consider the function $f(z) = e^z$, this is clearly holomorphic on Ω and $\partial\Omega$. With $z = x + iy$ we have

$$|f(z)| = |e^z| = e^x$$

so, its maximum is the maximum of the real valued function e^x which occurs at $x = 1$.

Appendix

A   Roots of unity

Suppose we want to solve the equation

$$z^n = 1 \quad \text{for } n \in \mathbb{N},$$

since $1 \in \mathbb{C}$ we can write it in exponential polar form:

$$\begin{aligned}
1 &= e^{2\pi i k} \quad \text{for } k \in \mathbb{Z} \\
&= \cos(2\pi k) + i \sin(2\pi k).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
z^n &= 1 \\
z &= 1^{\frac{1}{n}} \\
&= (\cos(2\pi k) + i \sin(2\pi k))^{\frac{1}{n}} \\
&= \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)
\end{aligned}$$

We conclude with the following observation:

- the roots of unity lie on the unit circle;

- they are equally separated by an angle of $\frac{2\pi}{n}$.

Definition A.1. Let $\zeta \in \mathbb{C}$ and n a positive integer. The number ζ is called an n-th **root of unity** if $\zeta^n = 1$.

Theorem A.2

Let n be a positive integer. Define

$$\begin{aligned}
\omega_n &= \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right) \\
&= e^{i\frac{2\pi}{n}}.
\end{aligned}$$

We have:

- ω_n is an n-th root of unity;

- the n-th roots of unity are the n complex numbers of modulus 1, given by $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$.

Corollary A.3. Let n be a positive integer and ω_n an n-th root of unity. Then, the polynomial

$$z^n - 1 = (z-1)(z - \omega_n)(z - \omega_n^2)\cdots(z - \omega_n^{n-1}).$$