# Sequences and Series Notes

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## Contents

# Notation

The notation $\#(A)$ for a set $A$ denotes the cardinality i.e. the size of the set.

# 1 Set Theory

## 1.1 Operation with sets

**Definition 1.1.** The **union** of two sets $A$ and $B$, denotes $A \cup B$, is the set of all elements which are either in A **or** B (or both).

**Definition 1.2.** The **intersection,** $A \cap B$, is the set of those elements that are both in $A$ and in $B$.

**Definition 1.3.** The notation $A \backslash B$ stands for the **complement** of $B$ in $A$, i.e. the set of all elements of $A$ which are not in $B$.

**Example.** Within the natural numbers $\mathbb{N}$ suppose $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 5\}$. Then

$$A \backslash B = \{3, 4\}.$$

## 1.2 Intervals

**Definition 1.4.** The following subsets of $\mathbb{R}$ are called **intervals:**

(i) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

(ii) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.

(iii) $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

(iv) $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

(v) $(a, \infty) = \{x \in \mathbb{R} : a < x\}$.

(vi) $(-\infty, \infty) = \mathbb{R}$.

**Remark.** *To make operations with intervals easier draw the intervals on the number line to visualise the operation.*

> Note that the symbol $\infty$ is only used as part of the notation. $\infty$ **is not a real number! There is no such real number as** $\infty$.

### 1.3 Multiple and infinite unions and intersections

## 2 Functions

# 2 Functions

*This section will be considered only briefly in lectures, with the foundational material for functions being considered more carefully in 4CCM111a (Calculus I).*

## 2.1 Basics

The concept of function is very important in every branch of mathematics. Here is the formal definition:

**Definition 2.1.** *Let $A$ and $B$ be sets. Then a* function

$$f : A \to B$$

*is a rule which assigns exactly one element of $B$ to each element of $A$. The set $A$ is called the* domain *of the function, the set $B$ is called the* codomain *or the* target set*.*

If $x$ is an element of $A$ then the element of $B$ assigned to $x$ by the function $f$ is usually denoted $f(x)$. In this context, $x \in A$ is usually called the *argument* of the function $f$ and $f(x) \in B$ is called its *value*. Depending on the situation, various synonyms to the term "function" are often used: *map, mapping, morphism.*

In order to set up a function, you have to specify three objects: (i) the set $A$; (ii) the set $B$; (iii) the rule $f$.

Some functions have standard names, others have names (usually a single letter) defined for a particular piece of work. You should be familiar with the following standard functions from $\mathbb{R}$ to $\mathbb{R}$: sin, cos, exp. Here are some other standard functions:

$$\begin{aligned}
\log \colon (0,\infty) &\to \mathbb{R}, & x &\mapsto \log(x);\\
\tan \colon \mathbb{R} \setminus \{\tfrac{\pi}{2} + \pi n \mid n \in \mathbb{Z}\} &\to \mathbb{R}, & x &\mapsto \tan(x);\\
\text{the ceiling function } \lceil \cdot \rceil \colon \mathbb{R} &\to \mathbb{Z}, & x &\mapsto \lceil x \rceil;\\
\text{the square root } \sqrt{\cdot} \colon [0,\infty) &\to [0,\infty), & x &\mapsto \sqrt{x};\\
\text{the square} \colon \mathbb{R} &\to \mathbb{R}, & x &\mapsto x^2;\\
\text{a linear function } \colon \mathbb{R} &\to \mathbb{R}, & x &\mapsto 2x+3.
\end{aligned}$$

Of course, in the last example, 2 and 3 can be replaced by any real numbers.

In setting up a function, there is sometimes a certain freedom in choosing the sets $A$ and $B$. For example, the square root can be defined either as a function from $[0, \infty)$ to $[0, \infty)$ or as a function from $[0, \infty)$ to $\mathbb{R}$. We will discuss this later. For now, remember that you have to be very clear about your choices of the sets $A$, $B$; you can make any suitable choice, but you have to stick to this choice throughout your piece of work.

For functions from $\mathbb{R}$ to $\mathbb{R}$, plotting their graph is a very useful tool in analysing them. In this course we will mostly discuss functions $f : \mathbb{R} \to \mathbb{R}$, but in general, the domain and the target sets can be fairly arbitrary:

**Example.** Let $A$ be the set of students in this class, let $B = \{1, 2, \dots, 100\}$ and for a student $x \in A$ let $f(x)$ be the age of $x$ in years. This sets up a function $f : A \to B$.

**Example.** Let $P$ be the set of all polygons on the plane. We can define two functions, $f$ and $g$, on the set $P$. For $p \in P$, let $f(p)$ be the number of edges of $p$ and let $g(p)$ be the area of $p$. Thus, we have defined two functions $f : P \to \mathbb{N}$ and $g : P \to \mathbb{R}$.

## 2.2 Setting up a function

A function can be set up, or defined, in various ways. The most common way to define a function of real numbers is to write down an analytic expression for it. For example,

$$f(x) = 2x + 3, \qquad g(x) = \sqrt{x-3}, \qquad \text{and} \qquad h(x) = \log(2x+4)$$

define functions of a real variable $x$. In such simple cases, there is usually no need to specify the domain of a function explicitly: the domain is assumed to include all values of $x$ for which the corresponding analytic expression makes sense, i.e. can be computed. For example, one immediately sees that the domain of $f$ is $\mathbb{R}$, the domain of $g$ is $[3, \infty)$ and the domain of $h$ is $(-2, \infty)$.

In more complex cases, a function can be defined by a different analytic expression for different values of its argument. For example, the sign function is defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The *modulus*, or *absolute value* function is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

One can define more complex functions for a particular piece of work; for example,

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ \sqrt{1-x^2} & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 \leq x. \end{cases}$$

In mathematics, one often has to consider very complex and "weird" functions. A classic example is Thomae's function:

$$f(x) = \begin{cases} 1/n & \text{if } x \text{ is rational, } x = m/n,\ n \in \mathbb{N},\ m \text{ and } n \text{ have no common factors;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

One cannot even attempt to construct a graph of this function! However, this is a perfectly well defined function which can be mathematically analysed. In particular, $f$ turns out to be continuous at all irrational points and discontinuous at all rational points.

Another example is the Weierstrass' function:

$$g(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R}, \tag{2.1}$$

where $0 < a < 1$, $b > 1$, $ab \geq 1$ are real parameters. This definition uses the notion of *series*, which we will only briefly mention at the end of this course. Weierstrass proved (with a later improvement by G. H. Hardy) that this function is continuous but nowhere differentiable.

## 2.3 Injective and surjective functions

There is some further terminology for functions which is useful.

**Definition 2.2.** *A function $f$ is said to be* injective *or* one-to-one *if $f(x) = f(y) \Rightarrow x = y$ .*

In words, a function is *injective* if distinct elements of the domain are always mapped to distinct elements of the codomain. (If two different elements of the domain are mapped to the same element of the codomain, then the function is not injective.)

**Example.** Consider the following functions:

$$\begin{array}{rccccl} \sin: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \sin x \\ \exp: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & \exp x = e^x \\ \log: & (0,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \log x \\ f: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^2 \\ g: & (3,\infty) & \to & \mathbb{R}\,, & x & \mapsto & \frac{1}{x-3} \\ h: & \mathbb{R} & \to & \mathbb{R}\,, & x & \mapsto & x^3 - x \end{array}$$

Which functions in the above examples are injective?

**Definition 2.3.** *Let $f : A \to B$ be a function. Then $f(A)$, the* image *(or* range*) of $A$ under the function $f$, is the set*

$$\{y \in B \,|\, y = f(x) \text{ for some } x \in A\} \subset B\,.$$

**Example.** What are the images of the functions in the above examples?

It is sometimes the case that the image of a function is equal to the entire codomain. In this case the function is said to be *surjective.*

**Definition 2.4.** *A function $f : A \to B$ is said to be* surjective *(or* onto*) if*

$$f(A) = B\,.$$

**Example.** Which functions in the above examples are surjective?

It is of course possible for a function to be both injective and surjective. This situation is sufficiently important to have a name of its own.

**Definition 2.5.** *A function which is both injective and surjective is said to be* bijective*.*

**Example.** Which functions in the above examples are bijective?

In principle, you can always change the definition of your function to make it surjective. For example, if you consider sin not as a function from $\mathbb{R}$ to $\mathbb{R}$, but as a function from $\mathbb{R}$ to $[-1, 1]$, then it becomes surjective. This is not always convenient though, because often the range of your function is unknown (or is complicated).

## 2.4 Composition and inverse

**Definition 2.6.** *Let $A$, $B$, $C$, be sets and $f : A \to B$ and $g : B \to C$ be functions. Then the* composition *$g \circ f$ is a function defined by*

$$(g \circ f)(x) = g(f(x)), \qquad x \in A.$$

**Warning 1:** in order to define the composition $g \circ f$, one has to make sure that the image of $f$ is a subset of the domain of $g$. For example, consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, $g : [0, \infty) \to \mathbb{R}$, $g(x) = \log x$. Then $g \circ f$ is NOT defined. Indeed, $g(f(x)) = \log(\sin x)$, and if $\sin x < 0$, this expression makes no sense!

**Warning 2:** in general $g \circ f \neq f \circ g$, even if both expressions are well defined. For example, if $f : \mathbb{R} \to \mathbb{R}$ is $f(x) = x + 1$ and $g : \mathbb{R} \to \mathbb{R}$ is $g(x) = x^2$, then $f \circ g(x) = x^2 + 1$ and $g \circ f(x) = (x + 1)^2 = x^2 + 2x + 1$.

An important feature of a bijective function is that it has a (unique) *inverse.*

**Definition 2.7.** *Let $f : A \to B$ be a bijective function. Then a function $g : B \to A$ is the inverse of the function $f$ if and only if it satisfies*

$$(g \circ f)(x) = x, \quad \forall x \in A \quad \text{and} \quad (f \circ g)(y) = y, \quad \forall y \in B.$$

*The function $g$ with these properties is denoted $f^{-1}$.*

Every bijective function has an inverse. On the other hand, if a function is not bijective then it does not have an inverse.

At a practical level, in order to find the inverse of a function $f$, you need to solve the equation $f(x) = y$ for $x$.

**Example.**

Find the inverse of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.

Find the inverse of $\exp : \mathbb{R} \to (0, \infty)$.

# 3 Logic

## 3.1 Propositions

**Definition 3.1.** For our purposes it will be sufficient to define a **proposition** to be ‘a statement which is either **true or false**’. The word statement is often used as a synonym to proposition.

If $A$ and $B$ are propositions, one can form new propositions as follows:

- $A$ AND $B$ - is true if and only if both $A$ and $B$ are true;
- $A$ OR $B$ - is true if and only if either ($A$ is true) or ($B$ is true) or (both $A$ and $B$ are true);
- NOT $A$ - is true if and only if $A$ is false.

## 3.2 Quantifiers

From propositions with variables one can form new propositions by using the quantifiers $\exists$ ‘there exists’ and $\forall$ ‘for all’.

**If a proposition contains a variable, then without a quantifier one cannot establish whether it is true or false!**

## 3.3 Negating propositions

The process to negate propositions is as follows:

1. Negate the quantifiers, i.e. the negation of $\forall$ is $\exists$ and the negation of $\exists$ is $\forall$.
2. Negate the statement. For example:

$$\text{"}\forall x \in \mathbb{R},\ \text{one has } x > 0\text{"}$$

the negation is

$$\text{"}\exists x \in \mathbb{R},\ \text{one has } x \leq 0\text{"}.$$

**Remark.** *When negating propositions containing operations AND, OR, one should replace AND by OR and vice versa.*

**Example.** Proposition: $\forall a > 0$ one has $(a \geq 1)$ OR $(1/a \geq 1)$. Negation: $\exists a > 0$ such that $(a < 1)$ AND $(1/a < 1)$.

## 3.4 Converse and contrapositive

**Definition 3.2.** Let $A$ and $B$ be propositions depending on a variable. For a statement $A \Rightarrow B$, the **contrapositive** is the statement (NOT $B$) $\Rightarrow$ (NOT $A$). A contrapositive is true if and only if the original statement is true.

**Definition 3.3.** For a statement $A \Rightarrow B$, the **converse** is the statement $B \Rightarrow A$. A converse may or may not be true regardless of whether the original statement is true or not.

# 4 Proofs

## 4.1 Proving two sets are equal

**Theorem 4.1.** Let $A$ and $B$ be two sets. Then $A = B$ if and only if both $A \subset B$ and $B \subset A$.

**Theorem 4.2.** For any sets $A, B$ and $C$, we have:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

## 4.2 Proof by contradictions

In this type of proof instead of proving $A \Rightarrow B$ we prove the contrapositive (which may be easier to prove) NOT $A \Rightarrow$ NOT $B$.

## 4.3 Proof by counterexample

It is enough to find an example in which the proposition proposed does not hold true.

# 5 Boundedness; supremum and infimum

## 5.1 Boundedness

**Definition 5.1.** A subset $S$ of $\mathbb{R}$ is said to be **bounded above** if there is a real number $M$ such that $x \leq M$ for all $x \in S$.

Alternatively: $S$ is bounded above if $S \subset (-\infty, M]$ for some $M \in \mathbb{R}$. Such a number $M$ is called an **upper bound** for $S$.

**Definition 5.2.** A subset $S$ of $\mathbb{R}$ is said to be **bounded below** if there is a real number m such that $x \geq m$ for all $x \in S$.

Alternatively: $S$ is bounded below if $S \subset [m, \infty)$ for some $m \in \mathbb{R}$. Such a number $m$ is called a **lower bound** for $S$.

**Definition 5.3.** A subset $S$ of $\mathbb{R}$ is said to be **bounded** if it is both bounded above and bounded below.

Alternatively: $S$ is bounded if $S \subset [m, M]$ for some real numbers $m, M$.

If $S$ is not bounded, it is called **unbounded**.

## 5.2 Maximum and minimum

**Definition 5.4.** Let $S$ be bounded above and suppose that there exists an upper bound $M$ of $S$ such that $M \in S$. Then $M$ is called the **maximum** of $S$ (or the maximal element of $S$): $M = \max(S)$.

**Definition 5.5.** Let $S$ be bounded below and suppose that there exists a lower bound $m$ of $S$ such that $m \in S$. Then $m$ is called the minimum of $S$ (or the minimal element of $S$): $m = \min(S)$.

**Remark.** *The set $S = (0, 1)$ is bounded. However, neither maximum nor minimum exist for this set.*

**Theorem 5.1.**

(i) Let $S$ be bounded above. If $\max(S)$ exists, then it is unique.

(ii) Let $S$ be bounded below. If $\min(S)$exists, then it is unique.

## 5.3 Supremum and infimum

**Definition 5.6.** Let $S$ be bounded below. Suppose that there exists the largest number $m$ such that $S \subset [m, \infty)$. Then $m$ is called the greatest lower bound for $S$, or the **infimum** of $S$, denoted $\inf(S)$.

**Theorem 5.2.** Let $S$ be bounded from below. If $\min(S)$ exists, then $\inf(S)$ also exists and coincides with $\min(S)$.

**Theorem 5.3.** Let $S \subset \mathbb{R}$ be a set bounded below and let $m = \inf S$. Then for any $\varepsilon > 0$ there exists $x \in S$ such that $x < m + \varepsilon$.

**Definition 5.7.** Let $S$ be bounded above. Suppose that there exists the smallest number $M$ such that $S \subset (-\infty, M]$. Then $M$ is called the least upper bound for $S$, or the **supremum** of $S$, denoted $\sup(S)$.

**Theorem 5.4.** Let $S$ be bounded from above. If $\max(S)$ exists, then $\sup(S)$ also exists and coincides with $\max(S)$.

**Theorem 5.5.** Let $S \subset R$ be a set bounded above and let $M = \sup(S)$. Then for any $\varepsilon > 0$ there exists $x \in S$ such that $M - \varepsilon < x$.

**Note.** *What the theorem above is saying is that number formed by taking a small number, $\varepsilon$, away from the supremum of a set is no longer an upper bound of the set; therefore it cannot be the supremum of the set.*

**Remark.** *The supremum and infimum of a set by definition do not need to be in the set itself.*

## 5.4 Completeness

**Definition 5.8. (Axiom of completeness).** Every non-empty set of real numbers which is bounded above has a supremum.

Every non-empty set of real numbers which is bounded from below has an infimum.

# 6 Sequences: Convergence

## 6.1 Sequences

We say that a list of numbers $s_1, s_2, s_3, \ldots$ generated by a general formula such as,

$$s_n = \frac{1}{n^2} \quad n \in \mathbb{N},$$

is a sequence.

**Remark.** *Some types of notations for sequences are $\{s_n\}_{n=n_0}^{\infty}$, alternatively $(s_n)_{n \geq n_0}$.*

## 6.2 Convergence

**Definition 6.1.** The sequence $s_n$ is said to **converge** to the limit $L$ if for all $\varepsilon > 0$ there exists a natural number $n_0$ such that for all $n \geq n_0$ we have

$$|s_n - L| < \varepsilon.$$

In symbols:

$$\forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \text{ we have } |s_n - L| < \varepsilon.$$

**Remark.** *The defintion does not change if $n > n_0$ or if $\varepsilon = f(\varepsilon)$.*

**Theorem 6.1.** Every convergent sequence has one and only one limit.

## 6.3 Divergence

**Definition 6.2.** If the sequence $s_n$ does not converge to any limit it is said to **diverge**.

**Definition 6.3.** The sequence $s_n$ is said to diverge to $+\infty$, for which we write $s_n \to +\infty$, if, for every positive real number $H$, there exists $n_0$ such that for all $n \geq n_0$ we have

$$s_n > H.$$

i.e.

$$\forall H > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ s_n > H.$$

**Definition 6.4.** The sequence $s_n$ is said to diverge to $-\infty$, for which we write $s_n \to -\infty$ if, for any negative real number $H$, there exists $n_0$ such that for all $n \geq n_0$ we have

$$s_N < H.$$

i.e.

$$\forall H < 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ s_n < H.$$

**Example.** TO DO !!

**Example.** TO DO !!

## 6.4 Boundedness

**Definition 6.5.** The sequence $s_n$ is said to be **bounded** if its terms form a bounded set. i.e., if there exists a real number $M$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$.

**Theorem 6.2.** Every convergent sequence is bounded.

# 7 The algebra of limits

**Theorem 7.1.** Let $s_n \to L$ and $t_n \to M$ as $n \to \infty$. Then

1. for any number $\alpha \in \mathbb{R}$, one has $\alpha s_n \to \alpha L$ as $n \to \infty$;
2. $s_n + t_n \to L + M$ as $n \to \infty$;
3. $s_n t_n \to LM$ as $n \to \infty$;
4. $s_n/t_n \to L/M$ as $n \to \infty$, provided $m \neq 0$ and $t_n \neq 0$ for all $n$.

**Theorem 7.2.** Let $s_n \to \infty$ or $s_n \to -\infty$ as $n \to \infty$ and $s_n \neq 0$ for all $n$. Then $1/s_n \to 0$ as $n \to \infty$.

**Theorem 7.3.** Let $s_n \to L > 0$ and let $t_n \to \pm\infty$ as $n \to \infty$. Then $s_n t_n \to \pm\infty$ as $n \to \infty$.

### 7.1 Sandwich theorem

**Theorem 7.4.** Let $r_n \to L$ and $t_n \to L$ as $n \to \infty$ and suppose that $r_n \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. Then $s_n \to L$ as $n \to \infty$.

**Example.** TO DO !!

### 7.2 Limits with inequalities

**Theorem 7.5.** Let $s_n, t_n$ be convergent sequences such that $s_n \leq t_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.

**Corollary 7.5.1.** *Let $s_n, t_n$ be convergent sequences such that $s_n < t_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.*

**Remark.** *This is because $s_n < t_n \Rightarrow s_n \leq t_n$. The use of $<$ in the inequality for limits may be FALSE sometimes.*

## 8 Standard sequences

The following are 'standard' sequences that converge to zero:

- Exponentials: $a^n \to 0$ as $n \to \infty$, $\forall a \in (-1, 1)$;
- Powers: $n^{-\gamma} \to 0$ as $n \to \infty$, $\forall \gamma > 0$;
- Logarithms: $\frac{1}{\log(n)} \to 0$ as $n \to \infty$.

Alternately we also have the following sequences which diverge to $\infty$ :

- Exponentials: $A^n \to \infty$ as $n \to \infty$, $\forall A > 1$;
- Powers: $n^{\gamma} \to \infty$ as $n \to \infty$, $\forall \gamma > 0$;
- Logarithms: $\log(n) \to \infty$ as $n \to \infty$.

## 8.1 Rates of convergence

Rates of growth:

$n^n \gg n! \gg$ exponentials $(a^n)$ $\gg$ polynomials $(n^k)$ $\gg$ logarithms.

### 8.1.1 Bernoulli inequality

**Lemma 8.1.** ***(Bernoulli inequality).*** *For every $k \in \mathbb{N}$ and $x \geq -1$ one has*

$$(1+k)^k \geq 1 + kx.$$

### 8.1.2 The sequence $a^{1/n}$

**Theorem 8.2.** For any $a > 0$, we have

$$a^{1/n} \to 1 \text{ as } n \to \infty.$$

# 9 Monotone sequences

**Definition 9.1.** A sequence $s_n$ is said to be:

- **increasing**, if $s_{n+1} > s_n$ for all $n \in \mathbb{N}$;
- **decreasing**, if $s_{n+1} < s_n$ for all $n \in \mathbb{N}$;
- **non-decreasing**, if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$;
- **non-increasing**, if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$;
- **monotone** if it is either non-decreasing or non-increasing.

**Theorem 9.1.**

1. Every non-decreasing sequence which is bounded above is convergent. Moreover, the limit of the sequence is the supremum of its terms.
2. Every non-increasing sequence which is bounded below is convergent. Moreover, the limit of the sequence is the infimum of its terms.

# 10 Series

A series is just a special type of sequence.

**Definition 10.1.** Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers. We say that the **series** $\sum_{k=1}^{\infty} a_k$ converges if the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

exists. The limit is called the sum of the series $\sum_{k=1}^{\infty} a_k$. So by definition,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

if the limit exists.

**Remark.** *Essentially if a sequence is composed of $a_1, a_2, a_3, \ldots$ then its corresponding series is $S_n = a_1 + a_2 + a_3 + \ldots$*

**Theorem 10.1.** If the series $\sum_{k=1}^{\infty} a_k$ is convergent then $\lim_{k=\infty} a_k = 0$.

## 10.1 Geometric series

Let $|a| < 1$. Then the series

$$\sum_{k=0}^{\infty} a_k = \frac{1 - a^{n+1}}{1 - a}.$$

Furthermore

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k = \frac{1}{1 - a}.$$

Clearly if $|a| > 1$ the series diverges.

## 10.2 Series with positive terms

**Corollary 10.1.1.** *Let $a_k \geq 0$ be a sequence of non-negative real numbers. Suppose that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$*

$$\sum_{k=1}^{\infty} a_k \leq C.$$

*Then the series $\sum_{k=1}^{\infty} a_k$ converges and the sum, $s$, of this series satisfies $s \leq C$.*

## 10.3 Special series

## 11.5 The series $\sum_{k=1}^{\infty} \frac{1}{k^p}$

Here we consider the important series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for various values of $p > 0$. We denote by $s_n$ the partial sum: $s_n = \sum_{k=1}^{n} \frac{1}{k^p}$.

Case $0 < p < 1$. Let us prove that for $0 < p < 1$, the series diverges. For the partial sum of the series, we have, replacing all terms by the smallest one,

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} > \frac{n}{n^p} = n^{1-p} \to \infty,$$

as $n \to \infty$. It follows that the series diverges.

Case $p = 1$. This is called the *Harmonic Series*. Let us prove that the harmonic series diverges. Let us split the harmonic series into the groups of 2, 4, 8, ... terms:

$$\underbrace{\frac{1}{2} + \frac{1}{3}}_{2}; \quad \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{2^2}; \quad \underbrace{\frac{1}{8} + \cdots + \frac{1}{15}}_{2^3}; \ldots;$$
$$\underbrace{\frac{1}{2^k} + \cdots + \frac{1}{2^{k+1} - 1}}_{2^k}; \ldots$$

Using the estimate

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} > n\frac{1}{2n} = \frac{1}{2},$$

we see that the sum of each group above is at least $1/2$. It follows that for the partial sum of the harmonic series we have:

$$s_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} > \frac{n-1}{2}.$$

So the sequence of partial sums is unbounded, hence the harmonic series diverges.

Case $p > 1$. Let us prove that the series converges for $p > 1$. Again, we split our series into groups of 2, 4, 8, ... terms:

$$\underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{2}; \quad \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}}_{2^2}; \quad \underbrace{\frac{1}{8^p} + \cdots + \frac{1}{15^p}}_{2^3}; \ldots;$$
$$\underbrace{\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1} - 1)^p}}_{2^k}; \ldots$$

Using the simple inequality

$$\frac{1}{n^p} + \frac{1}{(n+1)^p} + \cdots + \frac{1}{(2n-1)^p} < n\frac{1}{n^p} = n^{1-p},$$

we see that the sums of these terms can be estimated above as follows:

$$\frac{1}{(2^k)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} < 2^k \frac{1}{(2^k)^p} = 2^{k(1-p)} = a^k,$$

where $a = 2^{1-p} < 1$. Thus, for the partial sums we have the estimate

$$s_n < a + a^2 + \cdots + a^{k-1} < \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad \text{if } n \leq 2^k - 1.$$

So all partial sums are bounded above by a constant, and the series converges.

**Remark.** The sum of the series considered above is called *Riemann's zeta function:*

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p}, \quad p > 1.$$

Although the definition of zeta function looks simple, some deepest fundamental unsolved mathematical questions are related to this function. The most famous of them is the Riemann hypothesis; the statement of this hypothesis is beyond the scope of this course and is related to the zeros of $\zeta(p)$ for complex values of $p$. A much more "elementary" question relates to the values of $\zeta(p)$ for natural values of $p$. The values $\zeta(2n)$ are known explicitly (they have the form $q_n \pi^{2n}$, where $q_n$ is an explicit rational number). However, very little is known about the values $\zeta(2n+1)$. It is conjectured that these values are irrational, but this is only known for $\zeta(3)$.

## 11.6 The sequence $(1 + \frac{1}{n})^n$ and the number $e$

**Theorem 11.8.** *The sequence $t_n = (1+\frac{1}{n})^n$ is increasing and bounded above, and therefore is convergent.*

*Proof.* 1. Let us prove that the sequence $t_n$ is increasing. It suffices to consider the ratio $t_{n+1}/t_n$ and to prove that this ratio is greater than one for all $n$. We have:

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \Big/ \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{n+1}\right)\left(\frac{n+2}{n+1}\right)^n \frac{n^n}{(n+1)^n}$$
$$= \left(1 + \frac{1}{n+1}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^n = \left(1 + \frac{1}{n+1}\right)\left(1 - \frac{1}{n^2+2n+1}\right)^n.$$

Using Bernoulli's inequality (Lemma 9.4), we have

$$\left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{n^2+2n+1}\right)^n \geq \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{n^2+2n+1}\right)$$
$$> \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{n^2+2n}\right) = \frac{n+2}{n+1}\left(1-\frac{1}{n+2}\right) = \frac{n+2}{n+1}\frac{n+1}{n+2} = 1.$$

Thus, we have proven that $t_n$ is increasing.

2. Let us prove that $t_n$ is bounded above. By the binomial expansion,

$$\begin{aligned} t_n = \left(1+\frac{1}{n}\right)^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n \\ &= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right). \end{aligned} \tag{11.2}$$

Estimating all the expressions in brackets from above by 1, we get

$$t_n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 3.$$

Thus, $t_n$ is bounded above. By Theorem 11.3, it follows that the sequence $t_n$ converges. ∎

The limit of this sequence is denoted by $e$. This is the same number $e$ as the one that serves as a base of natural logarithms, $e \approx 2.71828$:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

In fact, it is more convenient to calculate $e$ according to a different formula:

**Theorem 11.9.** *One has*

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e. \tag{11.3}$$

*Proof.* We use the notation $s_n$ (see (11.1)) for the partial sum of the series in the l.h.s. of (11.3). We also denote $t_n = (1+\frac{1}{n})^n$. We have already seen in Example 11.6 that the series in the l.h.s. of (11.3) converges. We need to prove that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n. \tag{11.4}$$

1. Recall formula (11.2). Consider the $k$'th term in the r.h.s.:

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \leq \frac{1}{k!}.$$

It follows that $t_n \le s_n$ for all $n$. Passing to the limit in this inequality as $n \to \infty$, we get

$$\lim_{n\to\infty} t_n \le \lim_{n\to\infty} s_n. \tag{11.5}$$

2. Again, consider formula (11.2). Fix an integer $k$, $1 \le k \le n$, and let us keep the first $k$ terms in the r.h.s. of (11.2) and erase the rest of the terms. Since we have erased positive numbers, we obtain an inequality

$$t_n > 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\ + \dots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right).$$

Now for fixed $k$ let us pass to the limit $n \to \infty$ in this inequality. We get

$$\lim_{n\to\infty} t_n \ge 2 + \frac{1}{2!} + \dots + \frac{1}{k!} = s_k.$$

The inequality is true for all $k$, and so we can pass to the limit in this inequality. Bearing in mind that the l.h.s. is independent of $k$, we get

$$\lim_{n\to\infty} t_n \ge \lim_{k\to\infty} s_k = \lim_{n\to\infty} s_n. \tag{11.6}$$

Combining (11.5) and (11.6), we obtain (11.4). ∎

The last theorem gives a very efficient way of computing $e$. Indeed, taking the sum of the terms up to $1/10!$, we already obtain 7 correct digits of $e$ after the decimal point.

**Example.**

1. Consider the sequence $t_n = \left(1 + \frac{1}{2n}\right)^{2n}$. Clearly, $t_n = s_{2n}$, where $s_n$ is as in the above theorem. Thus, $t_n$ is a *subsequence* of $s_n$ (we will discuss subsequences in more detail shortly) and therefore it converges to the same limit: $t_n \to e$ as $n \to \infty$. In the same way, one can prove that

$$\lim_{n\to\infty} \left(1 + \frac{1}{kn}\right)^{kn} = e \tag{11.7}$$

   for any natural number $k$. In fact, $k$ doesn't have to be natural; one can prove that (11.7) holds true for any $k \in \mathbb{R}$.

2. Consider the sequence $r_n = \left(1 + \frac{1}{n}\right)^{2n}$. Clearly, $r_n = s_n^2$, where $s_n$ is as in the above theorem. Thus, by the Algebra of limits, $r_n \to e^2$ as $n \to \infty$. In the same way, one can prove that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{kn} = e^k \tag{11.8}$$

   for any natural number $k$. In fact, $k$ doesn't have to be natural here; one can prove that (11.8) holds true for any $k \in \mathbb{R}$.

# 11 Subsequences and limit points

## 11.1 Subsequences

**Definition 11.1.** Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $n_k$ be a strictly increasing sequence of natural numbers (i.e. $1 \leq n_1 < n_2 < n_3 < \ldots$). Then $\{s_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of $\{s_n\}_{n=1}^{\infty}$.

**Example.** Let $s_n = \frac{1}{n}$ and $n_k = k^2$. Then $s_{n_k} = s_{k^2} = \frac{1}{k^2}$.

**Theorem 11.1.** Let $s_n$ be a convergent sequence with $s_n \to L$ as $n \to \infty$. Then every subsequence of $s_n$ converges to $L$.

**Theorem 11.2. (The Bolzano-Weierstrass Theorem).** Every bounded sequence has a convergent subsequence.

More precisely, if $s_n$ is a sequence of real numbers such that $a \leq s_n \leq b$ for all $n$, then there exists a subsequence of $s_n$ which converges to a limit $L \in [a, b]$.

## 11.2 Limit points

**Definition 11.2.** A real number a is called a **limit point** of a sequence $s_n, n \in \mathbb{N}$, if there exists a subsequence $s_{n_k}$ such that $a = \lim_{k\to\infty} s_{n_k}$.

**Theorem 11.3.** A convergent sequence has one and only one limit point; this point is the limit of the sequence.

**Theorem 11.4.** Let $s_n, n \in \mathbb{N}$, be a bounded sequence which has only one limit point $L$. Then $s_n \to L$ as $n \to \infty$.

## 11.3 Cauchy sequences

**Definition 11.3.** The sequence $s_n$ is called a **Cauchy Sequence** if for all $\varepsilon > 0$ there exists a natural number $n_0$ such tha for all $m, n \geq n_0$ we have $|s_m - s_n| < \varepsilon$.

**Theorem 11.5.** Every convergent sequence is a Cauchy sequence.

**Theorem 11.6.** Every Cauchy sequence is bounded.

**Theorem 11.7. (Cauchy's convergent Criterion)** Every Cauchy sequence is a convergent sequence.

**Theorem 11.8.** A sequence of real numbers is a Cauchy sequence if and only if it is convergent.

# 12 Absolute and conditional convergence of series

**Definition 12.1.** The series $\sum_{k=1}^{\infty} a_k$ is said to **converge absolutely** if the series $\sum_{k=1}^{\infty} |a_k|$ is convergent.

**Theorem 12.1.** Every absolutely convergent series is convergent. I.e. if $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ converges as well.

## 12.1 Conditional convergence

**Definition 12.2.** If the series $\sum_{k=1}^{\infty} a_k$ converges, but does not converge absolutely, it is said to **converge conditionally.**

## 12.2 Tests for convergence

**Theorem 12.2. (The Comparison Test).** Let $\sum_{k=1}^{\infty} b_k$ be a convergent series of non-negative numbers and suppose that for some constant $M > 0$ we have $|a_k| \leq M b_k$ for all $k$. Then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

**Theorem 12.3. (The Alternating Series Test).** Let $a_k$ be a non-increasing sequence of positive numbers such that $a_k \to 0$ as $k \to \infty$. Then the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \dots$$

converges. Moreover, its sum lies between $a_1 - a_2$ and $a_1$.

# 13 Appendix

## 13.1 The $O$ symbol

**Definition 13.1.** Let $s_n$ and $t_n$ be sequences. Suppose that there exist $C > 0$ such that for all $n \in N$, one has $|s_n| \leq C|t_n|$. Then one writes

$$s_n = O(t_n) \text{ as } n \to \infty.$$

**Definition 13.2.** Let $s_n$ and $t_n$ be two sequences, such that $t_n \neq 0$for all $n$. Suppose that $\lim_{n\to\infty} \frac{s_n}{t_n} = 0$ Then we write $s_n = o(t_n), n \to \infty$.

## 13.2 Modulus

**Definition 13.3.** For $x \in \mathbb{R}$ the **modulus** of $x$, $|x|$, is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

**Theorem 13.1. (Properties of the modulus)**

1. $\forall x, y \in \mathbb{R} : |xy| = |x||y|$; in particular, $|ax| = a|x|$ if $a > 0$;
2. The triangle inequality: $\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y|$;
3. $\forall x, y \in \mathbb{R} : |x - y| \leq |x| + |y|$.

## 13.3 Floor and Ceiling functions

Property:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1.$$