Algebraic Topology Notes

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Abstract

This is the Imperial College London postgraduate module MATH70034 Algebraic Topology, instructed by Dr. Sara Veneziale. The formal name for this class is "Algebraic Topology".

Contents

Conventions

Definition 0.1. We will use the following notation and conventions throughout the course:

- By "map" we mean a **continuous** map, unless stated otherwise.
- $I = [0, 1]$.
- S^n is the n-sphere which resides in $\mathbb{R}^{n+1}$.
- D^n is the n-ball which resides in $\mathbb{R}^n$.
- T^n is the n-torus i.e. $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$

Equivalently, we may use the boldface notation of these space i.e. $\mathbb{S}, \mathbb{D}, \mathbb{T}$.

1 Introduction

1.1 Point-set topology

The fundamental problem in topology is to classify spaces up to equivalence (that is, homeomorphism). In general, to show that two topological spaces are homeomorphic, one just needs to give a valid homeomorphism. On the other hand, to show that two spaces are not homeomorphic can be harder. It usually amounts to finding a property which is invariant under homeomorphism, and it is satisfied by one space but not the other. In this course, we will build some of these algebraic invariants: the fundamental group, and the homology groups.

> **Remark 1.1.** We will use *map* to mean a *continuous map* between topological spaces. All maps are assumed to be continuous unless stated otherwise.

Let us recall some fundamental definitions from point-set topology.

Definition 1.2. A **topological space** is a set X with a collection Ω of subsets of X such that:

- $\varnothing \in \Omega$ and $X \in \Omega$;
- if $\{U_i\}_{i\in I} \subset \Omega$ then $\bigcup_{i\in I} U_i \in \Omega$;
- if $U, V \in \Omega$ then $U \cap V \in \Omega$.

We call the subsets in Ω the **open sets**. Their complements in X are called **closed sets**.

Example 1.3

Some examples.

The 'smallest' topology on a set X is the **trivial topology**, i.e. $\Omega = \{\emptyset, X\}$.

The 'largest' topology on a set X is the **discrete topology**, i.e. $\Omega = P(X)$, the power set of X.

Let us recall a few important constructions of topological spaces, namely the *product topology*, the *subspace topology*, and the *quotient topology*.

Definition 1.4. Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology generated by sets of the form $U \times V$, for $U \subset X$ open and $V \subset Y$ open.

Definition 1.5. Given a set X, a **basis** is a collection $\mathcal{B} \subset \mathcal{P}(X)$ such that:

- The basis elements cover X.
- For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Then $\mathcal{B}$ will define a topology on X.

Definition 1.6. For a topological space (X, Ω), we say that $\mathcal{B} \subset \Omega$ generates the topology if it is a **basis** of the topology, i.e., for all $U \in \Omega$, there exists $\{B_i\}_{i\in I} \subset \mathcal{B}$ such that

$$U = \bigcup_{i \in I} U_i.$$

> **Remark 1.7.** Every open subset in the product topology is a (possibly infinite) union of **rectangles** $U \times V$ (but not all opens are necessarily rectangles).

Definition 1.8. Let X, Y be topological spaces. The **disjoint union** is defined as

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\}).$$

> **Remark 1.9.** Topologically, X is the same as $X \times \{0\}$ and Y is the same as $Y \times \{1\}$, but we make an arbitrary choice of $0, 1$ to ensure they do not overlap in the union. In fact, $X \cup X = X$, while $X \sqcup X \neq X$.

Corollary 1.10. The topology of the disjoint union is generated by the basis:

$$\{U \times \{0\} \mid U \subset X \text{ open}\} \cup \{V \times \{1\} \mid V \subset Y \text{ open}\}.$$

Corollary 1.11

$$S^0 = \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}.$$

Definition 1.12. Let X be a topological space and let $A \subset X$. The **subspace topology** on A is defined as

$$\{U \cap A \mid U \in \Omega\}$$

where Ω is the set of opens in X.

Example 1.13. We can consider $[0, 1] \subset \mathbb{R}$ with the subspace topology (where $\mathbb{R}$ is taken with the topology induced by the standard metric). Then $(0, 1) \subset \mathbb{R}$ is not open, but $[0, 1] = (0, 2) \cap [0, 1]$ is open in the subspace topology.

Definition 1.14. Let X be a topological space with an equivalence relation $\sim$, and define $Y = X/\sim$. Then we can consider Y with the **quotient topology** defined as

$$\{U \subset Y \mid \pi^{-1}(U) \in \Omega\}$$

where Ω is the set of opens in X and $\pi : X \to Y$ is the **quotient map**.

Note 1.15. The quotient topology collapses parts of X together according to the equivalence relation $\sim$, forming a new space Y. Open sets in Y correspond to those in X that respect the quotient structure.

Corollary 1.16. Often we will consider the quotient space defined by a subset $A \subset X$. This is defined using the equivalence relation

$$x \sim y \text{ if and only if } x = y \text{ or } x, y \in A.$$

By abuse of notation, we denote this quotient space as X/A.

Definition 1.17. A map $f : X \to Y$ between topological spaces is called an **identification map** if

- it is a surjective map,
- such that $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Example 1.18. The quotient map $\pi : X \to X/\sim$ is an example of a identification map.

Proposition 1.19. A surjective map $f : X \to Y$ is an identification map if and only if for any space Z and every function $g : Y \to Z$, $g \circ f$ is continuous if and only if g is continuous. We illustrate this in the following commutative diagram.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Proof. Exercise.

To do

□

1.2 Examples of Topological Spaces

Example 1.20. Let us list some examples of topological spaces which will recur in the course.

- **The unit sphere**

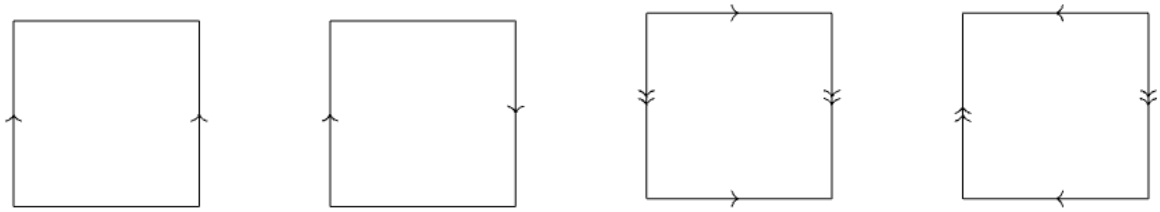
$$S^n = \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

Note that S^0 is the union of two points.

- **The unit ball**
$$D^n = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$$
Note that $D^n / S^{n-1} \cong S^n$.

- **The topological torus**
$$T^n = S^1 \times \cdots \times S^1 \quad (n \text{ times}).$$

Example 1.21. During the course we will often refer to spaces built as quotients of the square $[0,1]^2$. Here are some recurring examples.



(A) Cylinder. (B) Möbius strip. (C) Torus. (D) Klein bottle.

Definition 1.22. Let X, Y be topological spaces. A map $f : X \to Y$ is called a **homeomorphism** if there exists a map $g : Y \to X$ such that
$$f \circ g = \mathrm{id}_Y \quad \text{and} \quad g \circ f = \mathrm{id}_X.$$
If there exists a homeomorphism between two spaces, they are said to be **homeomorphic**.

Example 1.23. We want to show that the topological torus $S^1 \times S^1 \subset \mathbb{R}^4$ is homeomorphic to the torus embedded in $\mathbb{T}^2 \subset \mathbb{R}^3$, parametrised by
$$\gamma(\theta, \phi) = ((b + a\cos\theta)\sin\phi, (b + a\sin\theta)\cos\phi, a\sin\theta)$$
for $\phi, \theta \in [0, 2\pi]$ and fixed $0 < a < b$. It is clear we can construct the homeomorphism $f : S^1 \times S^1 \to \mathbb{T}^2$ as
$$(x_1, y_1, x_2, y_2) \mapsto ((b + ax_1)y_2, (b + ax_2)y_1, ay_1).$$

Example 1.24

Consider $X = [0,1]$ with subspace $A = \{0,1\} \subset X$. The quotient X/A is obtained by glueing the endpoints of the interval, and we obtain $X/A \cong S^1$. We can see this as follows. Consider the continuous map

$$f : [0,1] \to S^1, \quad t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$

This is clearly surjective and injective everywhere except at the endpoints. If we consider $g : X/A \to S^1$ that makes the diagram commute:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & S^1 \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

Note that g is continuous by Proposition 1.1, and it is bijective since the only points at which f was not injective were the endpoints (which we have now identified). Hence, g is a homeomorphism.

We say that f is *factoring over the quotient*, i.e., we are eliminating all points that cause a problem in the map $f : X \to Y$ in order to make $\widetilde{f} : X/\sim \to Y$ a homeomorphism.

Note 1.25. The quotient space X/A identifies 0 and 1, essentially "closing" the interval into a loop, which is exactly the structure of S^1.

2 Homotopies

Intuitively, two spaces are homeomorphic if they are the same up to shrinking or stretching, but not crushing or cutting. It is a rather fine equivalence, and we will now consider a coarser equivalence: *homotopy*.

Definition 2.1. A **pair of spaces** is a pair (X, A) of a topological space X and a subspace $A \subset X$. If $A = \{\text{pt}\}$, then $(X, \{\text{pt}\})$ is a **pointed space**.

Definition 2.2. Let (X, A) be a pair of spaces. A **retraction** of X onto A is a continuous map $r : X \to A$ such that $r(X) = A$ and $r|_A = \text{id}_A$.

Example 2.3

Some examp

- Let $X \neq \emptyset$ be a topological space. The projection $r : X \to \{p\}$ is a retraction for any $p \in X$.
- The set $\mathbb{R}^2 \setminus \{0\}$ retracts to S^1 via the map $x \mapsto x/|x|$. However, $\mathbb{R}^2$ does not retract to S^1.
- The interval $I = [0,1]$ does not retract to $\{0,1\}$ (such a retraction cannot exist by the Intermediate Value Theorem).

We can look at a related concept to the one of retraction that includes the idea of a continuous movement that crushes the bigger space to the smaller space. This is the idea of *deformation retract*. Before we can define this, we need to introduce the concept of *homotopy*. We first define homotopy for two maps between topological spaces.

Definition 2.4. Let X, Y be topological spaces, and $f, g : X \to Y$ be continuous maps. A **homotopy** between f and g is a continuous map $F : X \times I \to Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = g(x).$$

If there exists a homotopy between two functions f, g, we say they are **homotopic** and we denote them by $f \simeq g$.

Note 2.5. A homotopy provides a continuous deformation from f to g, meaning that we can smoothly transition from one function to the other without introducing discontinuities.

Lemma 2.6 (Pasting Lemma)**.** Let $X = A \cup B$, for $A, B \subset X$ closed subsets (taken with the subspace topology). Let $f : X \to Y$ be a function such that $f|_A$ and $f|_B$ are both continuous. Then f is continuous.

Proof.

To do

□

Proposition 2.7

Homotopy equivalence of maps is an equivalence relation.

Proof. Showing that it is reflexive and symmetric is trivial. Let us show transitivity. Assume we have $f, g, h : X \to Y$ related by homotopies

$$F : X \times I \to Y, \quad G : X \times I \to Y$$

such that $F(x,0) = f(x), F(x,1) = g(x)$ and $G(x,0) = g(x), G(x,1) = h(x)$. We can construct a new homotopy $H : X \times I \to Y$,

$$H(x,t) = \begin{cases} F(x,2t) & t \leq 1/2, \\ G(x,2t-1) & t \geq 1/2. \end{cases}$$

Note that we know that H is continuous by the pasting lemma. □

Definition 2.8. Let X, Y be topological spaces. Then a continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that

$$f \circ g \simeq \mathrm{id}_Y$$

$$g \circ f \simeq \mathrm{id}_X$$

If there exists a homotopy equivalence between X and Y, we say that X and Y are **homotopy equivalent** or that they have the same **homotopy type**. We write it as $X \simeq Y$.

Note 2.9. Homotopy equivalence is a weaker notion than homeomorphism. Instead of requiring a bijection, it allows spaces to be continuously deformed into each other while preserving their essential topological structure. Thus, spaces that look different geometrically may still be homotopy equivalent.

Exam Questions 2.10 (Exercise)

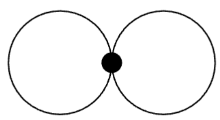
Show that having homotopy equivalence for topological spaces is an equivalence relation.

To do

Corollary 2.11. Note that homotopy equivalence allows for squeezing spaces, but not tearing. A homeomorphism always implies a homotopy equivalence, but the converse is not true.

Example 2.12. Some examples.

- $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ unless $n = m$, but $\mathbb{R}^n \simeq \mathbb{R}^m$ for all n, m (since they are all homotopy equivalent to a point). In general, for all topological spaces X, we have $X \times \mathbb{R}^n \simeq X$.
- $\mathbb{R}^m \backslash \{0\}$ is homotopy equivalent to S^{m-1} using the inclusion map and the retraction $x \mapsto x/|x|$.
- The disc with two holes is homotopy equivalent to the following space:



2.1 Deformation retraction

Definition 2.13. Let (X, A) be a pair of spaces. A **deformation retraction** of X onto A is a retraction that is homotopic to the identity map, i.e., there exists a continuous map

$$F : X \times I \to X, \quad (x, t) \mapsto f_t(x)$$

such that $f_0 = \mathrm{id}_X$ and $f_1 : X \to A$ is the retract, with the property that $f_t|_A = \mathrm{id}_A$ for all $t \in [0, 1]$.

Note 2.14. A deformation retract allows a space to be continuously deformed into a subspace while keeping the subspace fixed. This provides a strong form of homotopy equivalence, as it guarantees that the larger space can be "collapsed" onto the smaller space while preserving its essential topology.

Remark 2.15. In the literature, this is sometimes known as a **strong deformation retract**. We discuss the difference between strong and weak deformation retracts a little more in the example regarding the *comb space* (below).

Example 2.16

Some examples.

- $\mathbb{R}^n$ deformation retracts to a point, by $f_t(x) = tx$.
- The closed n-disc
$$D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$
deformation retracts to the origin of $\mathbb{R}^n$. This is via the map $f_t(x) = tx$. This is because $t = 1$ implies that $f_1 = \mathrm{id}_{D^n}$ and $t = 0$ implies that $f_0 : D^n \to (0, \dots, 0)$. This illustrates how a cylinder retracts onto one of its boundary spheres.
- Let S^n be the n-sphere. The cylinder $S^n \times I$ deformation retracts to $S^n \times \{0\} \cong S^n$, by defining $f_t(x, r) = (x, tr)$.
- $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1} via
$$f_t(x) = (1 - t)x + t\frac{x}{|x|}.$$
This shows that punctured space retracts onto a sphere surrounding the removed point.
- Let $X = \{0, 1\}$, then $\{0, 1\} \mapsto \{0\}$ is a retraction, but it is not a deformation retract since X is not path-connected. This highlights that retraction does not always imply deformation retraction, as connectivity plays a crucial role.

Corollary 2.17. An observation is that if X is a topological space, and $f : X \to \{p\}$ for $p \in X$ is a deformation retraction of X to p, then X is path-connected. Indeed, if $F : X \times I \to X$ is a homotopy from id_X to f and $x \in X$ is a point, then this gives a path

$$I \to X, \quad t \mapsto F(x, t)$$

that connects x to p. This implies that not all retractions are deformation retractions (since we can always build a retraction to a point, but not all spaces are path-connected).

2.2 Contractibility

Definition 2.18. A topological space X is **contractible** if it is homotopy equivalent to a point.

Note 2.19. A space is contractible if it can be continuously shrunk to a single point. This is a stronger property than homotopy equivalence alone, as it ensures that all maps from any space into it are homotopic to a constant map.

Definition 2.20. A continuous map is **nullhomotopic** if it is homotopic to the constant map.

Example 2.21. Some examples.

- $\mathbb{R}^n$ is contractible, and any convex subset of $\mathbb{R}^n$ is contractible.
- S^n is not contractible. An easy way to see this is using the fundamental group.

Exam Questions 2.22 (Exercise)
Prove it for S^0.
To do

Corollary 2.23. Note that if a space deformation retracts to a point, then it is contractible. However, the other way around is not always true.

Example 2.24

Consider the **comb space**, which is given by:

$$\{0\} \times [0,1] \cup (K \times [0,1]) \cup [0,1] \times \{0\}$$

taken with the subspace topology of the standard topology on $\mathbb{R}^2$, where

$$K = \{1/n \mid n \in \mathbb{N}\}.$$

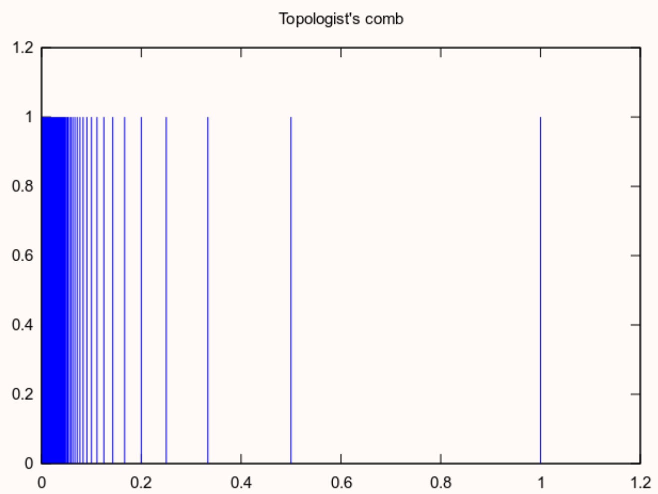


Figure 1: Figure of the comb space.

This space is contractible and path connected, however, it does not deformation retract to any base point on $\{0\} \times (0,1]$.

This is a space that is often used in topology to construct counterexamples. In fact, for this space, we can construct a **weak deformation retract** to $\{0\} \times [0,1]$ (where **weak** means that we are not imposing the condition of being the identity on A, and we just require a retraction that is homotopy equivalent to the identity). However, it does not **strongly** deformation retract to $\{0\} \times [0,1]$, as we just discussed.

Example 2.25. There are spaces that are contractible but do not deformation retract to any of their points.

3 The Fundamental Group

In this section, we define the fundamental group, one of the algebraic invariants that will allow us to tell apart topological spaces.

3.1 Paths

We want to understand the topology of spaces by studying paths and loops on the space.

Definition 3.1. Let X be a topological space. A **path** is a continuous map $f : I \to X$, where $I = [0,1]$.

Definition 3.2. Let $f, g : I \to X$ be two paths such that $f(1) = g(0)$. We define the **product path** $f \cdot g$ as the path

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2}, \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

which is the concatenation of the two paths.

> **Remark 3.3.** A path is not the same as its image.
>
> **Example 3.4.** For example, the paths $I \to S^1, t \mapsto e^{2\pi i t}$ and $I \to S^1, t \mapsto e^{4\pi i t}$ have the same image (the circle), but they are different paths. Note also that a path need not be injective.

Definition 3.5. A topological space X is **path-connected** if for any $x, y \in X$ there exists a path $p : I \to X$ such that

$$p(0) = x, \quad p(1) = y.$$

Definition 3.6. A topological space X is **connected** if it cannot be written as $U \cup V$ where U, V are closed disjoint subsets.

Example 3.7

Not every connected space is path-connected. For example, consider the **topologist sine curve**, defined as

$$T = \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0,1] \right\} \cup \{(0,0)\}.$$

It is connected (when considering the topology induced from the Euclidean plane) but it is not path-connected.

Exam Questions 3.8 (Exercise)

Show that if X is path-connected then every path is homotopic to the constant path.

To do

Definition 3.9. Two paths $f, g : I \to X$ with the same end points, $f(0) = g(0) = x$ and $f(1) = g(1) = y$, are **homotopic with respect to their endpoints** if there exists a homotopy between f and g preserving endpoints, i.e., a continuous map

$$\begin{aligned} F : I \times I &\to X \\ (s,t) &\mapsto f_t(s) \end{aligned}$$

such that

$$\begin{aligned} F(s,0) &= f_0(s) = f(s), \\ F(s,1) &= f_1(s) = g(s), \\ F(0,t) &= x, \\ F(1,t) &= y. \end{aligned}$$

If f and g are homotopic with respect to their endpoints, we denote it by $f \simeq g$ (just as for homotopy equivalence).

Lemma 3.10. Path homotopy is an equivalence relation.

Proof. Exercise.

To do

□

Example 3.11

Path homotopy and homotopy equivalence are not the same thing. Consider, for mdexample, the circle S^1 with two paths

$$f(t) = e^{\pi i t}, \quad g(t) = e^{-\pi i t}.$$

These are homotopy equivalent, but they are not homotopy equivalent with respect to the end points in S^1.

Lemma 3.12

Let $f, g, f', g' : I \to X$ be paths such that

- $f(0) = g(0) = x$,
- $f(1) = g(1) = f'(0) = g'(0) = y$,
- $f'(1) = g'(1) = z$.

If $f \simeq f'$ and $g \simeq g$ we have $f \cdot g \simeq f' \cdot g'$.

Proof. Consider the path homotopies $F, G : I \times I \to X$ between $f_0 = f, f_1 = f'$, $g_0 = g, g_1 = g'$. Define $H : I \times I \to X$,

$$H(s,t) = \begin{cases} F(2s,t) & s \leq 1/2, \\ G(2s-1,t) & s > 1/2. \end{cases}$$

This is a path homotopy between $f \cdot g$ and $f' \cdot g'$.

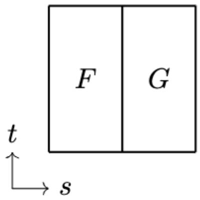


Figure 2: Diagram for the homotopy H.

$\square$

Example 3.13

Let $\phi : I \to I$ with $\phi(0) = 0, \phi(1) = 1$. Let $f : I \to X$, then the composition $\phi \circ f \simeq f$. Here, ϕ is called a **reparametrisation** of f. We can define the homotopy explicitly by

$$f((1-t)\phi(s) + ts).$$

3.2 Constructing the fundamental group

3.3 The Fundamental Group

A big part of this course is constructing an algebraic invariant of topological spaces called the *fundamental group*. Intuitively, this helps us understand the structure of a topological space by examining how *loops* behave in it. We will see that loops on a topological space have a natural group structure, which gives us exactly the algebraic invariant we want to define.

Definition 3.14. A **loop** with basepoint x_0 is a path $f : I \to X$ such that $f(0) = f(1) = x_0$.

Remark 3.15. Note that the composition of two loops with the same basepoint is again a loop.

Definition 3.16. Given a loop f with basepoint x_0, we can define its **homotopy equivalence** class (with respect to the endpoints), since homotopy on paths is an equivalence relation:

$$[f] = \{g : I \to X \mid g(0) = g(1) = x_0, g \simeq f\}.$$

Proposition 3.17. The set $\pi_1(X, x_0)$ with the product defined by concatenation is a group, i.e.

$$[f] \bullet [g] = [f \cdot g].$$

Its unit element is the equivalence class of the constant loop, $[e]$. For every element $[f]$, its inverse is

$$[f]^{-1} = [\overline{f}],$$

where $\overline{f}(t) = f(1-t)$.

Proof. We prove the axioms to be a group.

- We first show that $[f] \bullet [e] = [e] \bullet [f] = [f]$, i.e. we want to show that $f \cdot e \simeq f \simeq e \cdot f$. The first homotopy is defined as

$$F_1(s,t) = \begin{cases} x_0 & t \geq 2s \text{ or } t \leq 2s-1, \\ f(2s-t) & \text{otherwise,} \end{cases}$$

while the second homotopy is defined as

$$F_2(s,t) = \begin{cases} x_0 & t \geq 2s, \\ f\left(\frac{2s-t}{2-t}\right) & \text{otherwise.} \end{cases}$$

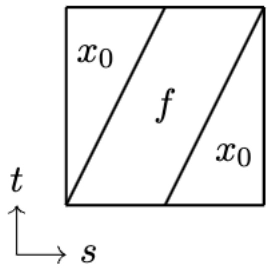


Figure 3: Homotopy F_1.

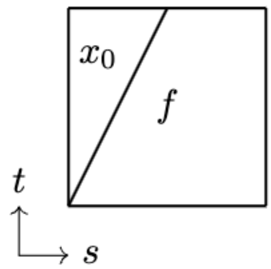


Figure 4: Homotopy F_2.

- Now, we can prove the existence of the inverse for each element, so we need to show that $[f] \bullet [f^{-1}] = [f^{-1}] \bullet [f] = [e]$, i.e. $f \cdot \overline{f} \simeq e$. We can define the homotopy $f \cdot \overline{f} \simeq e$ by

$$F_3(s,t) = \begin{cases} x_0 & t \geq 2s \text{ or } t \geq 2-2s, \\ f(2s-t) & 1 \geq 2s \text{ or } t \leq 2s, \\ \overline{f}(2s+t-1) & \text{otherwise.} \end{cases}$$

Inverting the roles of f and $\overline{f}$, we obtain the homotopy $\overline{f} \cdot f \simeq e$.

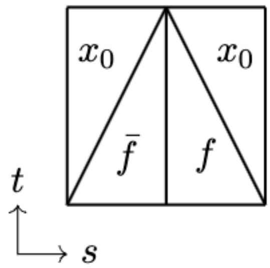


Figure 5: Homotopy F_3.

- We verify associativity by showing

$$(f \cdot g) \cdot h \simeq f \cdot (g \cdot h).$$

The homotopy is defined by

$$F_4(s,t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & \text{if } s \leq \frac{1}{4}(1+t), \\ g(4s-1-t) & \text{if } \frac{1}{4}(1+t) \leq s \leq \frac{1}{4}(2+t), \\ h\left(\frac{4s-2-t}{2-t}\right) & \text{if } \frac{1}{4}(2+t) \leq s \leq 1. \end{cases}$$

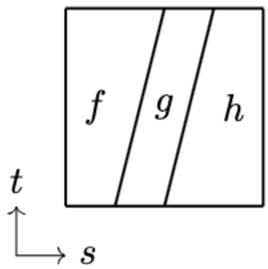


Figure 6: Homotopy F_4.

□

Example 3.18

Some examples.

- $\pi_1(\mathbb{R}^2, \{0\}) = 0$ since every loop is homotopic to the constant loop.
- Let $A \subseteq \mathbb{R}^n$ be a convex set and $x_0 \in X$. Then $\pi_1(A, x_0) = 0$.
- Any space with the trivial topology has trivial fundamental group (since any map $f : I \to X$ is continuous, so X will be path-connected and any two paths are homotopic).
- Any space with the discrete topology has trivial fundamental group (since any continuous map $f : I \to X$ has to be constant).

For certain spaces $\pi_1(X, x_0)$ is a topological invariant of the space itself, i.e. it does not depend on the choice of basepoint.

Proposition 3.19

Let X be a path-connected space and $x_0, x_1 \in X$ be two distinct points. Then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. Therefore, when X is path-connected, we can write $\pi_1(X)$ instead of $\pi_1(X, x_0)$.

Proof. Let $x_0, x_1 \in X$. Let $h : I \to X$ be a path between $x_0 = h(0)$ and $x_1 = h(1)$. Define

$$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$$
$$[f] \mapsto [h \cdot f \cdot \overline{h}]$$

where $\overline{h}(t) = h(1-t)$. This is well-defined since we know that path homotopy behaves well with respect to concatenation. This is a group homomorphism since

$$\beta_h([f] \bullet [g]) = [h \cdot f \cdot g \cdot \overline{h}] = [h \cdot f \cdot \overline{h}] \cdot [h \cdot g \cdot \overline{h}] = \beta_h([f]) \bullet \beta_h([g]),$$

and $\beta_h([e]) = [e]$. It is bijective with $\beta_h^{-1} = \beta_{\overline{h}}$. □

3.4 Simply Connected Spaces

A fundamental property of spaces in topology is **simple connectedness**, which tells us whether all loops in the space can be continuously contracted to a point.

Definition 3.20. A topological space X is **simply connected** if it is path-connected and its fundamental group is trivial, i.e.,

$$\pi_1(X) = 0.$$

Note 3.21. A simply connected space has no "holes" that prevent loops from being contracted to a point. This means that any two paths with the same endpoints can be continuously deformed into each other.

Example 3.22. $\mathbb{R}^n$ is simply connected for any n.

Proposition 3.23

A space X is simply connected if and only if there exists a *unique homotopy class of paths* between any two points of X.

Proof. We prove each direction in turn.

- Proof of $(\Rightarrow)$.
  Since X is path-connected, there exists a path between any two points. Let $f, g : I \to X$ be two distinct paths between $x_0, x_1 \in X$. Define the reverse path $\bar{g}$ as:

$$\bar{g}(t) = g(1-t).$$

Then, we can concatenate the paths as follows:

$$f \cdot \bar{g} \simeq g \cdot \bar{g}.$$

Since $g \cdot \bar{g}$ is homotopic to the constant loop, we obtain:

$$f \simeq f \cdot \bar{g} \cdot g \simeq g \cdot \bar{g} \cdot g \simeq g.$$

Thus, all paths between any two points are homotopic.

- Proof of ($\Leftarrow$).
  Conversely, assume that there exists a unique homotopy class of paths between any two points of X. Then X is path-connected. Moreover, if $x_0 = x_1$, all loops are homotopic to each other, so:
  $$\pi_1(X) = 0.$$

$\square$

3.5 Covering Spaces

Covering spaces play a crucial role in algebraic topology as they allow us to "unfold" a topological space into a simpler, more manageable space. They are particularly useful in computing fundamental groups.

Definition 3.24. A **covering space** (or covering) is a map $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ of X satisfying

$$p^{-1}(U_\alpha) = \bigcup_\beta V_\alpha^\beta$$

where the restrictions $p|_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ are homeomorphisms, and $V_\alpha^\beta \cap V_\alpha^\gamma = \emptyset$ for all $\beta \neq \gamma$.

Definition 3.25. The sets V_α^β are called the **sheets** of the covering.

Proposition 3.26. If X is connected, the number of sheets over each point is constant. This is called the **cardinality** of the covering.

Example 3.27

Some examples.

- Given a topological space X and a discrete space A, the map $p : X \times A \to X$ is the trivial covering.
- For $k \in \mathbb{Z}$, the maps $p_k : S^1 \to S^1, z \mapsto z^k$ are covering maps. The preimage $p_k^{-1}(z)$ of any point $z = e^{2\pi i t} \in S^1$ consists of precisely k distinct points (the kth complex roots of unity).
- The map $p_\infty : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ is a covering map. The preimage $p_\infty^{-1}(z)$ consists of infinitely many points.

Definition 3.28. A covering $p : \widetilde{X} \to X$ is called an n**-fold covering** if for all $x \in X$, $p^{-1}(x)$ consists of precisely n points.

Definition 3.29. Given two coverings $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$, we say that they are **isomorphic** if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $p_1 = p_2 \circ h$. This means the following diagram commutes:

$$\begin{tikzcd} Y_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Example 3.30. Some examples.

- The coverings $p_k : S^1 \to S^1$ and $p_{-k} : S^1 \to S^1$ are isomorphic (via $h : z \mapsto 1/z$).
- The coverings $p_2 : S^1 \to S^1$ and $p_3 : S^1 \to S^1$ are **not** isomorphic.
- The map $p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1, (t, s) \mapsto (e^{2\pi it}, e^{2\pi is})$ is a covering of the torus.

3.6 Deck Transformations and the Automorphism Group

Definition 3.31. Let $p : \widetilde{X} \to X$ be a covering. A **deck transformation** is a homeomorphism $\tau : \widetilde{X} \to \widetilde{X}$ such that $p \circ \tau = p$. That is, τ is an automorphism of the covering space. We denote by $\mathrm{Deck}(p)$ the set of all deck transformations.

Lemma 3.32. $\mathrm{Deck}(p)$ is a group where the action is composition of maps.

Proof. Exercise.

To do

□

Example 3.33

Some examples.

- $\tau : S^1 \to S^1, z \mapsto -z$ is a deck transformation of $p_2 : S^1 \to S^1$.
- For $m \in \mathbb{Z}$, the map $\tau_m : \mathbb{R} \to \mathbb{R}, z \mapsto z + m$ is a deck transformation of $p_\infty : \mathbb{R} \to S^1$.
- Therefore, $\mathrm{Deck}(p_\infty) \cong \mathbb{Z}$.

Note 3.34. Deck transformations describe symmetries of a covering space. For example, in the infinite covering p_∞, shifting by an integer preserves the covering structure.

3.7 Lifting Property

Definition 3.35. A **lift** of a continuous map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$, making the following diagram commute:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X. \end{tikzcd}$$

Example 3.36. Consider $f : I \to S^1, t \mapsto e^{2\pi i n t}$ for $n \in \mathbb{N}$ and the covering $p_\infty : \mathbb{R} \to S^1$. Then $\widetilde{f} : I \to \mathbb{R}, t \mapsto nt$ is a lift of f.

Lemma 3.37

Let $p : \widetilde{X} \to X$ be a covering of X, and let $\widetilde{f}, \widetilde{g} : Y \to \widetilde{X}$ be maps. Then:

1. $\widetilde{f}$ is a lift of $p \circ \widetilde{f}$.
2. If $\widetilde{f} \simeq \widetilde{g}$, then $p \circ \widetilde{f} \simeq p \circ \widetilde{g}$ (homotopies descend).
3. If $\alpha, \beta : I \to \widetilde{X}$ are paths with $\alpha(1) = \beta(0)$, then $p \circ (\alpha \cdot \beta) = (p \circ \alpha) \cdot (p \circ \beta)$ (paths descend).

Proof. We prove each statement in turn.

1. Follows from the definition of lift.
2. Any homotopy $\widetilde{f}_t$ from $\widetilde{f}$ to $\widetilde{g}$ gives rise to a homotopy $p \circ \widetilde{f}_t$ from $p \circ \widetilde{f}$ to $p \circ \widetilde{g}$.
3. Clearly,
$$p \circ (\alpha \cdot \beta)(t) = \begin{cases} p(\alpha(2t)) & \text{if } t \leq \frac{1}{2} \\ p(\beta(2t-1)) & \text{if } t > \frac{1}{2} \end{cases}$$
is equal to $(p \circ \alpha) \cdot (p \circ \beta)$.

□

3.8 Computing $\pi_1(S^1, 1)$ using covering spaces

A fundamental result in algebraic topology is that the fundamental group of the circle is isomorphic to $\mathbb{Z}$. We will prove this by studying loops in S^1 via its covering space $p_\infty : \mathbb{R} \to S^1$.

Our goal is to compute $\pi_1(S^1, 1)$, so we focus on loops in S^1. Consider the loop:

$$\omega_n : I \to S^1, \quad t \mapsto e^{2\pi i t n}.$$

The covering space $p_\infty : \mathbb{R} \to S^1$ is given by:

$$p_\infty(t) = e^{2\pi i t}.$$

We define the lift $\widetilde{\omega}_n : I \to \mathbb{R}$ by:

$$\widetilde{\omega}_n(t) = tn.$$

This satisfies $\omega_n = p_\infty \circ \widetilde{\omega}_n$.

Note 3.38. The covering space $\mathbb{R}$ "unwraps" the circle into an infinite line. A loop in S^1 corresponds to a path in $\mathbb{R}$, where $\widetilde{\omega}_n(1)$ determines how many times the loop winds around S^1.

3.8.1 Deck Transformations and Homotopy

The covering space has deck transformations given by:

$$\tau_m : \mathbb{R} \to \mathbb{R}, \quad t \mapsto t + m.$$

For $m \in \mathbb{Z}$, we observe that $\widetilde{\omega}_m \cdot (\tau_m \circ \widetilde{\omega}_n) \simeq \widetilde{\omega}_{m+n}$ since both are paths in $\mathbb{R}$ from 0 to $n + m$. We construct the homotopy:

$$f_t(s) = (1 - t)\widetilde{\omega}_{m+n}(s) + t(\widetilde{\omega}_m \cdot (\tau_m \circ \widetilde{\omega}_n))(s).$$

Proposition 3.39. The map

$$\Phi : \mathbb{Z} \to \pi_1(S^1, 1), \quad n \mapsto [\omega_n]$$

is a group homomorphism.

Proof. We check that $\Phi(n + m) = \Phi(n) \cdot \Phi(m)$:

$$\Phi(n + m) = [\omega_{n+m}] \overset{(1)}{=} [p_\infty \circ \widetilde{\omega}_{n+m}] \overset{(2)}{=} [p_\infty \circ (\widetilde{\omega}_m \cdot (\tau_m \circ \widetilde{\omega}_n))].$$

Using:

1. $p_\infty \circ \widetilde{\omega}_n = \omega_n$,
2. Homotopies descend, so $\widetilde{\omega}_{m+n} \simeq \widetilde{\omega}_m \cdot (\tau_m \circ \widetilde{\omega}_n)$,
3. Paths descend: $p \circ (\alpha \cdot \beta) = (p \circ \alpha) \cdot (p \circ \beta)$,
4. Associativity of the fundamental group operation,
5. $p \circ \tau_m = p$,
6. Definition of lifting.

We conclude:

$$[p \circ \widetilde{\omega}_m] \cdot [p \circ \widetilde{\omega}_n] = [\omega_m] \cdot [\omega_n] = \Phi(m) \cdot \Phi(n).$$

□

3.8.2 Homotopy Lifting Property

Definition 3.40. The **homotopy lifting property** (HLP) states that if we have a homotopy $F : Y \times I \to X$ and a lift $g : Y \times \{0\} \to Z$ of F_0, then there exists a unique homotopy $\widetilde{F} : Y \times I \to Z$ such that:

1. $\widetilde{F}_0 = g$,
2. $p \circ \widetilde{F} = F$.

Remark 3.41. The homotopy lifting property is summarized by the commutative diagram:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, "\iota"'] & Z \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists! \widetilde{F}"] & X \end{tikzcd}$$

A special case is the **path lifting property**, where $Y = \{\text{pt}\}$.

A crucial tool for studying covering spaces is the ability to **lift** paths and homotopies from a base space X to its covering space Z. This property helps establish isomorphisms between fundamental groups.

Definition 3.42. (Path Lifting Property, PLP). Let $p : Z \to X$ be a map. It has the **path lifting property** if for every path $f : I \to X$ with $f(0) = x_0$ and for every $\widetilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\widetilde{f} : I \to Z$ such that:

$$\widetilde{f}(0) = \widetilde{x}_0 \quad \text{and} \quad p \circ \widetilde{f} = f.$$

Lemma 3.43 (Local Homotopy Lifting Property)**.** . Let $p : \widetilde{X} \to X$ be a covering and $F : Y \times I \to X$ a homotopy. Let $g : Y \times \{0\} \to \widetilde{X}$ be a lift such that $p \circ g = F_0$. Then for each $y_0 \in Y$, there exists an open neighbourhood $N \subset Y$ containing y_0 and a unique homotopy $\widetilde{F}_N : N \times I \to \widetilde{X}$ such that:

$$p \circ \widetilde{F}_N = F|_{N \times I}, \quad \text{and} \quad (\widetilde{F}_N)_0 = g|_{N \times \{0\}}.$$

Moreover, if M is another such neighbourhood, then

$$\widetilde{F}_N|_{(M \cap N) \times I} = \widetilde{F}_M|_{(M \cap N) \times I}.$$

Note 3.44. This lemma guarantees that a homotopy can be lifted **locally**, ensuring that homotopy information is preserved when moving to the covering space.

Proof. Let $U = \{U_\alpha\}$ be the open cover of X from the definition of covering spaces, so that:

$$p^{-1}(U_\alpha) = \bigcup_\beta V_\alpha^\beta,$$

where each $p|_{V_\alpha^\beta}$ is a homeomorphism.

For each β, let $q_{\alpha\beta}$ be the inverse homeomorphism. By continuity of $F : Y \times I \to X$, for any $(y_0, t_0) \in Y \times I$, there exists an open neighbourhood $N \times (a, b)$ such that:

$$F(y, t) \in U_\alpha \quad \text{for all } (y, t) \in N \times (a, b).$$

Since I is compact, we can cover I with finitely many such neighbourhoods, say $N_i \times (a_i, b_i)$. Set $N = \bigcap_i N_i$, ensuring N is still open, and choose a partition:

$$0 = t_0 < t_1 < \cdots < t_n = 1,$$

so that for each i, we have $F(N \times [t_i, t_{i+1}]) \subset U_\alpha$ (using the Lebesgue covering lemma).

We construct a sequence of maps $\widetilde{F}_N^k$ such that:

- $\widetilde{F}_N^k : N \times [0, t_k] \to \widetilde{X}$ is a lift of $F|_{N\times[0,t_k]}$;
- $(\widetilde{F}_N^k)_0 = g|_N$;
- $\widetilde{F}_N^{k+1}|_{N\times[0,t_k]} = \widetilde{F}_N^k$.

These properties uniquely define $\widetilde{F}_N = \lim_{k\to\infty} \widetilde{F}_N^k$. We constrcut inductively:

- For $k = 0$, we set $\widetilde{F}_N^0 = g|_N$.
- Assume $\widetilde{F}_N^k$ is defined up to t_k. Since $F(N \times [t_k, t_{k+1}]) \subset U_\alpha$, we can assume:
$$\widetilde{F}_N^k|_{N\times\{t_k\}} \subset V_\alpha^\beta.$$
- Define:
$$\widetilde{E} = q_{\alpha\beta} \circ F|_{N\times[t_k,t_{k+1}]}.$$
Then $\widetilde{E}|_{N\times\{t_k\}} = \widetilde{F}_N^k|_{N\times\{t_k\}}$.
- Finally, extend:
$$\widetilde{F}_N^{k+1}(z,t) = \begin{cases} \widetilde{F}_N^k(z,t), & t \le t_k, \\ \widetilde{E}(z,t), & t \in [t_k, t_{k+1}]. \end{cases}$$
- By the Pasting Lemma, $\widetilde{F}_N$ is continuous.

We conclude by proving uniqueness. Suppose we have two such lifts $\widetilde{F}_N$ and $\widetilde{F}'_N$. It suffices to show that they must coincide for all $z \in N$ and $t \in I$.

We use induction on the partition $0 < t_1 < \cdots < t_m = 1$.

- The base case $t = 0$ is immediate since $\widetilde{F}_N$ and $\widetilde{F}'_N$ must both equal g on $N \times \{0\}$.
- Assume $\widetilde{F}_N = \widetilde{F}'_N$ on $[0, t_k]$. Since $[t_k, t_{k+1}]$ is connected, there exists a unique β such that:
$$\widetilde{F}_N(\{z\} \times [t_k, t_{k+1}]) \subset V_\alpha^\beta.$$
Similarly, there exists a unique β' for $\widetilde{F}'_N$. Since both maps are lifts of F and must agree at t_k, we conclude $\beta = \beta'$, ensuring:
$$\widetilde{F}_N = \widetilde{F}'_N \text{ on } \{z\} \times [0, t_{k+1}].$$

Thus, by induction, $\widetilde{F}_N$ and $\widetilde{F}'_N$ must be identical. $\square$

Proposition 3.45

A covering map satisfies the **homotopy lifting property** (HLP).

Proof. We assume the previous lemma. Take an open cover $Y = \bigcup_\alpha N_\alpha$ such that each N_α satisfies the local homotopy lifting property. Then, we construct a family of homotopy lifts $F_{N_\alpha} : N_\alpha \times I \to \widetilde{X}$, which coincide in their intersection due to the next lemma. By the Pasting Lemma, we obtain a well-defined homotopy $F : Y \times I \to \widetilde{X}$, which satisfies:
$$F(y,t) = F_{N_\alpha}(y,t) \quad \text{if } y \in N_\alpha.$$
Since the result is continuous by construction, the proof is complete. $\square$

Example 3.46

Consider $p_\infty : \mathbb{R} \to S^1$, with $z_0 = e^{2\pi i t_0} \in S^1$. The preimage under p_∞ is:

$$p_\infty^{-1}(z_0) = \{t_0 + m \mid m \in \mathbb{Z}\}.$$

For the path $w : I \to S^1$, defined by $w(t) = e^{2\pi i (t+t_0)}$, there exists a unique lift:

$$\widetilde{w} : I \to \mathbb{R}, \quad t \mapsto t + m + t_0$$

such that $p_\infty \circ \widetilde{w} = w$.

Note 3.47. This shows that the covering space $\mathbb{R}$ "unwraps" the circle, so each loop in S^1 corresponds to an integer shift in $\mathbb{R}$, encoding winding numbers.

Theorem 3.48

The map

$$\Phi : \mathbb{Z} \to \pi_1(S^1, 1), \quad n \mapsto [\omega_n]$$

is a group isomorphism.

Proof. We prove bijectivity.

- **Surjectivity:** Given any $[\alpha] \in \pi_1(S^1, 1)$, we must find n such that $\alpha \simeq \omega_n$. By the path lifting property (PLP), there exists a unique lift $\widetilde{\alpha} : I \to \mathbb{R}$ such that:

  $$\widetilde{\alpha}(0) = 0, \quad p_\infty \circ \widetilde{\alpha} = \alpha.$$

  Since $\widetilde{\alpha}(1) \in p^{-1}(1)$, it must be of the form n for some $n \in \mathbb{Z}$. Since in $\mathbb{R}$, all paths from 0 to n are homotopic, it follows that $\alpha \simeq \omega_n$.

- **Injectivity:** If $\Phi(n) = [\omega_n] = [e]$, then $\omega_n \simeq e$, meaning there exists a homotopy F_t with:

  $$F_0 = \omega_n, \quad F_1 = e, \quad F_t(0) = F_t(1) = 1 \text{ for all } t.$$

  Define $g : I \times \{0\} \to \mathbb{R}$ by $g(s, 0) = \widetilde{\omega}_n(s) = sn$. Since covering maps satisfy the homotopy lifting property (HLP), we lift F_t to:

  $$\widetilde{F} : I \times I \to \mathbb{R}$$

  with:

  $$\widetilde{F}_0 = g, \quad p_\infty \circ \widetilde{F}_1 = e, \quad p_\infty \circ \widetilde{F} = F.$$

  Since $\widetilde{F}_1(s) \in p^{-1}(1) = \mathbb{Z}$, we must have $\widetilde{F}_1(s) = m$ for some $m \in \mathbb{Z}$. On the other hand, $p_\infty \circ \widetilde{F}_0 = F_0 = 1$ and $p_\infty \circ \widetilde{F}_1 = F_1 = 1$, so:

  $$\widetilde{F}_1(0) = \widetilde{F}_0(0) = 0, \quad \widetilde{F}_1(1) = \widetilde{F}_0(1) = n.$$

  Thus, $m = n = 0$, proving injectivity.

$\square$

4 Induced Maps and Functors

Definition 4.1. Given two pairs of topological spaces (X, A) and (Y, B), a **map of pairs** is a function:

$$f : (X, A) \to (Y, B)$$

such that $f(A) \subset B$.

Definition 4.2. The **induced homomorphism** of a map of pointed spaces $f : (X, x_0) \to (Y, y_0)$ is:

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha].$$

Remark 4.3. An induced homomorphism is sometimes called a **pushforward**.

Lemma 4.4

f_* is a group homomorphism.

Proof. We first verify that f_* is well-defined. If $\alpha \simeq \beta$ are homotopic loops in X, then there exists a homotopy $F : I \times I \to X$ such that:

$$F_0 = \alpha, \quad F_1 = \beta.$$

Define $G = f \circ F$, which is clearly continuous, and observe:

$$G_0 = f \circ \alpha, \quad G_1 = f \circ \beta.$$

Thus, $f \circ \alpha \simeq f \circ \beta$, proving that $f_*([\alpha]) = f_*([\beta])$.

Now we show that f_* respects the group operation:

$$f_*([\alpha] \cdot [\beta]) = [f \circ (\alpha \cdot \beta)].$$

By associativity of concatenation:

$$[f \circ (\alpha \cdot \beta)] = [(f \circ \alpha) \cdot (f \circ \beta)].$$

Thus:

$$f_*([\alpha]) \cdot f_*([\beta]) = [f \circ \alpha] \cdot [f \circ \beta] = f_*([\alpha] \cdot [\beta]).$$

$\square$

4.1 Properties of Induced Homomorphisms

The fundamental group functor π_1 is not just a set-theoretic construction but has important algebraic properties. In particular, it respects composition of maps and identity maps.

Lemma 4.5

The induced homomorphism satisfies the following properties:

1. $(\mathrm{id}_{(X,x_0)})_* = \mathrm{id}_{\pi_1(X,x_0)}$.
2. If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$, then:

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. We prove each statement in turn.

1. Follows immediately since the identity map does not change loops.
2. We compute:

$$(g \circ f)_*([\alpha]) = [g \circ f \circ \alpha] = g_*([f \circ \alpha]) = (g_* \circ f_*)([\alpha]).$$

   Thus, $(g \circ f)_* = g_* \circ f_*$.

□

Corollary 4.6
If $f : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then f_* is an isomorphism.

Proof. Using the above lemma with $g = f^{-1}$, we get:

$$f_*^{-1} \circ f_* = \mathrm{id}_{\pi_1(X,x_0)}, \quad f_* \circ f_*^{-1} = \mathrm{id}_{\pi_1(Y,y_0)}.$$

Thus, f_* is an isomorphism. □

Note 4.7. A homeomorphism preserves all topological structures, so it should also preserve the fundamental group.

Proposition 4.8
If $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then f_* is an isomorphism.

Proof. Since f is a homotopy equivalence, there exists $g : Y \to X$ such that:

$$g \circ f \simeq \mathrm{id}_X, \quad f \circ g \simeq \mathrm{id}_Y.$$

We need to show that f_* is bijective.

- **Injectivity:** Consider the path $h : I \to X$ such that $h(0) = x_0$ and $h(1) = x_1 = g(y_0)$. This path induces an isomorphism of fundamental groups:

$$\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [\gamma] \mapsto [\bar{h} \cdot \gamma \cdot h].$$

  We claim that $(g \circ f)_* = \beta_h$, which implies f_* is injective.
- **Surjectivity:** Using a similar argument for $(f \circ g)_*$, we conclude that f_* is an isomorphism.

□

Example 4.9

Consider the covering map $p_m : S^1 \to S^1$ given by:

$$p_m(e^{2\pi is}) = e^{2\pi ims}.$$

This is an m-fold covering of the circle. Since $\pi_1(S^1, 1) \cong \mathbb{Z}$, the induced homomorphism is:

$$(p_m)_* : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto mn.$$

That is, the map corresponds to multiplication by m.

Note 4.10. The covering map p_m winds the circle around itself m times. Thus, each loop in the base lifts to m loops in the covering space, scaling fundamental group elements by m.

5 Retractions

A retraction is a continuous map that "pulls back" a space onto a subspace while keeping the points of the subspace fixed.

Proposition 5.1. Let (X, A) be a pair of spaces such that there exists a retraction $r : X \to A$. Let $\iota : A \to X$ be the inclusion map. Let $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ be the induced homomorphisms between the fundamental groups. Then:

1. r_* is surjective;
2. ι_* is injective;
3. If $r \circ \iota \simeq \mathrm{id}_A$ (i.e., r is a **weak deformation retract**), then r_* and ι_* are isomorphisms.

Proof. We prove each statement in turn.

1. We prove (1) and (2) together. Since $\mathrm{id}_{\pi_1(A,x_0)} = r_* \circ \iota_*$, it follows that ι_* is injective and r_* must be surjective.
2. To show that r_* is an isomorphism when r is a weak deformation retract, we need to show r_* is injective. Let $[\gamma] \in \pi_1(X, x_0)$ and assume $r_*([\gamma]) = [e]$, i.e., $r \circ \gamma \simeq e$. Then $\iota \circ r \circ \gamma$ is a loop in X that is homotopic to the constant path via a homotopy F, meaning:

$$\iota \circ r \circ \gamma \simeq e.$$

Since $r \circ \iota \simeq \mathrm{id}_X$, we obtain:

$$\gamma \simeq \iota \circ r \circ \gamma \simeq e.$$

Thus, γ is also homotopic to the constant path.

□

Proposition 5.2. There is no retraction $D^2 \to S^1$.

Note 5.3. If a retraction $r : D^2 \to S^1$ existed, it would force the disk to "inherit" the fundamental group structure of S^1, contradicting the fact that D^2 is simply connected. This is a fundamental result in algebraic topology.

Proof. Since $\pi_1(D^2, 0) = 0$ and $\pi_1(S^1, 1) \cong \mathbb{Z}$, there cannot be a surjection $\pi_1(D^2, 0) \to \pi_1(S^1, 1)$. Thus, no retraction can exist. □

5.1 Brouwer Fixed Point Theorem

The Brouwer Fixed Point Theorem states that any continuous function from the closed disk to itself must have a fixed point.

Theorem 5.4 (Brouwer Fixed Point Theorem)
Every continuous map $f : D^2 \to D^2$ has a fixed point.

Remark 5.5. This holds in arbitrary finite dimensions, but we will not prove this here.

Proof. Suppose, for contradiction, that f has no fixed points, i.e., $f(x) \neq x$ for all $x \in D^2$. We define a retraction $r : D^2 \to S^1$ as follows: consider the unique line passing through x and $f(x)$, which can be parametrized as:

$$L_x(t) = tx + (1 - t)f(x).$$

Since $f(x) \neq x$, this line will intersect ∂D^2 in exactly one point. Define:

$$r(x) = L_x(\mathbb{R}_{\geq 0}) \cap \partial D^2.$$

This defines a continuous map $r : D^2 \to S^1$ that is a retraction, contradicting the previous result that no such retraction can exist. Thus, f must have a fixed point. □

5.2 Product Spaces and Fundamental Groups

When dealing with topological spaces, it is useful to consider their product spaces and how their fundamental groups interact under this operation.

Proposition 5.6
Let (X, x_0), (Y, y_0) be pointed spaces. Then,

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof. A map $f : Z \to X \times Y$ is continuous if and only if its components $p_X \circ f : Z \to X$ and $p_Y \circ f : Z \to Y$ are continuous (where p_X and p_Y are the natural projection maps). Considering $Z = I$, this gives a bijection between loops in $X \times Y$ and pairs of loops in X and Y. That is, given a loop $\gamma : I \to X \times Y$, we can decompose it as:

$$\gamma(s) = (\gamma_X(s), \gamma_Y(s)),$$

where γ_X and γ_Y are loops in X and Y, respectively.

The constant loop in $X \times Y$ is mapped to the pair of constant loops in X and Y, and the group operation corresponds to pointwise multiplication in each component. Hence, we obtain the isomorphism:

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

□

Example 5.7
A useful application is computing the fundamental group of the torus:

$$\pi_1(T^2, (1,1)) = \pi_1(S^1 \times S^1, (1,1)) \cong \mathbb{Z} \times \mathbb{Z}.$$

This follows since $\pi_1(S^1, 1) \cong \mathbb{Z}$, and the product rule gives a fundamental group generated by two independent loops.

5.3 An interesting application: The Fundamental Theorem of Algebra

A remarkable application of algebraic topology is an elegant proof of the fundamental theorem of algebra using homotopy theory.

Theorem 5.8 (Fundamental Theorem of Algebra)**.** Any non-constant polynomial

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in \mathbb{C}$$

has a complex root.

Proof. Assume for contradiction that there exists a polynomial $p(z)$ such that $p(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. We define a loop in S^1 as:

$$g_r(s) = \frac{p(re^{2\pi i s})}{|p(re^{2\pi i s})|}$$

for every $r \in \mathbb{R}$, with base point $g_r(0) = g_r(1) = 1$. We claim that g_r is homotopic to ω_n, the standard winding loop, via the homotopy:

$$\widetilde{f}_t(s) = \frac{p_t(re^{2\pi i s})}{|p_t(re^{2\pi i s})|}$$

where

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n).$$

Since $\widetilde{f}_1 = g_r$ and $\widetilde{f}_0 = \omega_n$, this would imply $\omega_n \simeq e$, contradicting the fact that $\pi_1(S^1) \cong \mathbb{Z}$.

To ensure that $\widetilde{f}_t$ is well-defined, we check that the denominator never vanishes. For sufficiently large r, we estimate:

$$|z|^n > (|a_1| + \cdots + |a_n|)|z|^{n-1} > |a_1||z|^{n-1} + \cdots + |a_n|.$$

Thus, the denominator is never zero, and we conclude that $\omega_n \simeq e$ is only true when $n = 0$, which contradicts our assumption. Hence, $p(z)$ must have a root. □

Note 5.9. We summarise the proof.

- The proof translates a question about polynomial roots into a statement about homotopy classes of loops.
- The assumption that $p(z)$ has no roots allows us to construct a loop in S^1 given by normalizing $p(z)$ on large circles in $\mathbb{C}$.
- We then show that this loop is homotopic to ω_n, the standard winding map, which is topologically non-trivial.
- This leads to a contradiction since it would force a null-homotopic loop in S^1, which is impossible unless the polynomial is constant.

6 Galois correspondence

The Galois correspondence examines the relationship between subgroups of the fundamental group and the isomorphism classes of covering spaces. The analogy to Galois theory arises because coverings correspond to field extensions, and the fundamental group plays the role of the Galois group.

Proposition 6.1. Let $p : \widetilde{X} \to X$ be a covering space. Let $x_0 \in X$ and $\widetilde{x}_0 \in p^{-1}(x_0)$. Then:

1. The induced map $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.
2. If $[\alpha] \in \pi_1(X, x_0)$ and $\widetilde{\alpha}$ is a lift of α with $\widetilde{\alpha}(0) = \widetilde{x}_0$, then $\widetilde{\alpha}$ is a loop if and only if $[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

Proof. We prove each statement in turn.

1. We show that if $p_*([\widetilde{\alpha}]) = [e_{x_0}]$, then $\widetilde{\alpha}$ is homotopic to the constant loop. By the *path lifting property*, we have $\alpha = p \circ \widetilde{\alpha}$ and that $\widetilde{\alpha}$ is the unique lift such that $\widetilde{\alpha}(0) = \widetilde{x}_0$. Thus, α is homotopic to the constant loop e_{x_0}, and so $\widetilde{\alpha}$ is homotopic to the constant loop $e_{\widetilde{x}_0}$, proving injectivity.
2. If $\widetilde{\alpha}$ is a loop in $\widetilde{X}$, then $[\alpha] = [p \circ \widetilde{\alpha}] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Conversely, assume $[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Then there exists a loop $\widetilde{\gamma} \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ such that $[\alpha] = [p \circ \widetilde{\gamma}]$. Lifting α, we obtain a path $\widetilde{\alpha}$ such that $\widetilde{\alpha}(0) = \widetilde{x}_0$. By the homotopy lifting property, the homotopy between α and $p \circ \widetilde{\gamma}$ lifts to a homotopy between $\widetilde{\alpha}$ and $\widetilde{\gamma}$, which shows that $\widetilde{\alpha}$ is a loop.

□

Example 6.2
Consider the d-fold covering of S^1, given by

$$p_d : S^1 \to S^1, \quad z \mapsto z^d.$$

This induces an injective map

$$(p_d)_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1),$$

where $(p_d)_*(\mathbb{Z}) \cong d\mathbb{Z} \leq \mathbb{Z}$, showing that the induced map corresponds to multiplication by d.

Definition 6.3. A covering map $p : \widetilde{X} \to X$ is said to have **degree** d if $|p^{-1}(x)| = d$ for all $x \in X$.

Proposition 6.4. Let $p : \widetilde{X} \to X$ be a covering space, and suppose $X, \widetilde{X}$ are path-connected. Then

$$\deg(p) = [\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))],$$

where the right-hand side is the index of the subgroup.

Proof. Let $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Define a map

$$\Phi : G/H \to p^{-1}(x_0),$$

by sending the coset $H[\alpha]$ to the endpoint of the unique lift of α that starts at $\widetilde{x}_0$. The previous proposition ensures that Φ is well-defined.

- **Injectivity:** Assume $\Phi(H[\beta]) = \Phi(H[\alpha])$, meaning that the unique lifts of α and β have the same endpoint. Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ form a loop in $\widetilde{X}$ at $\widetilde{x}_0$, so:

  $$p_*([\widetilde{\alpha} \cdot \widetilde{\beta}^{-1}]) = [p \circ \widetilde{\alpha} \cdot (p \circ \widetilde{\beta})^{-1}] = [\alpha \cdot \beta^{-1}] \in H.$$

  Thus, $H[\alpha] = H[\beta]$, proving injectivity.

- **Surjectivity:** Since $\widetilde{X}$ is path-connected, every $\widetilde{x}_1 \in p^{-1}(x_0)$ is connected to $\widetilde{x}_0$ by a path that projects onto a loop in X, implying surjectivity.

□

6.1 Local Properties and Universal Coverings

A useful concept in topology is understanding when properties hold *locally* at each point. This leads to fundamental notions such as local path-connectedness and local simple connectedness.

Definition 6.5. A topological space X is said to have a property P **locally** if, for each point $x \in X$ and each neighbourhood U of x, there exists an open neighbourhood $V \subset U$ such that V has property P.
For example, we can talk about spaces being *locally path-connected* or *locally simply connected.*

6.2 Lifting Criterion for Covering Spaces

Proposition 6.6. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map, where Y is path-connected and locally path-connected. Then f has a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  If f lifts to $\widetilde{f}$, then clearly $f_* = p_* \circ \widetilde{f}_*$, which implies $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.

- Proof of ($\Leftarrow$).
  Assume $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. For each $y \in Y$, choose a path γ from y_0 to y. Then $f \circ \gamma$ is a path from x_0 to $f(y)$, and by the path lifting property (PLP), we can lift $f \circ \gamma$ to a path in $\widetilde{X}$ starting at $\widetilde{x}_0$. Define:

  $$\widetilde{f}(y) = \text{endpoint of the lift of } f \circ \gamma.$$

  This is well-defined since different choices of γ result in homotopic paths, and lifted homotopies are unique. The function $\widetilde{f}$ satisfies $p \circ \widetilde{f} = f$, so it is a lift.

To show continuity, let $U \subset X$ be an evenly covered neighbourhood of $f(y)$, and let $\widetilde{U}$ be a sheet over U containing $\widetilde{f}(y)$. There exists a path-connected neighbourhood $V \subset Y$ such that $f(V) \subset U$. The map $\widetilde{f}$ restricted to V lifts $f|_V$, so $\widetilde{f}$ is continuous. □

6.3 Universal Coverings

Definition 6.7. A covering space $p : \widetilde{X} \to X$ is a **universal covering** if $\widetilde{X}$ is simply connected.

Note 6.8. Some notes.

- A universal cover is the "largest" possible covering space, where all loops in X lift to paths in $\widetilde{X}$.
- Since $\widetilde{X}$ is simply connected, all loops in X completely unwind in $\widetilde{X}$.
- The fundamental group of X acts as a deck transformation group on the universal covering.

Example 6.9

The map $p_\infty : \mathbb{R} \to S^1$, given by $t \mapsto e^{2\pi i t}$, is a universal cover, since $\mathbb{R}$ is simply connected.

On the other hand, the d-fold covering $p_d : S^1 \to S^1$, given by $z \mapsto z^d$, is *not* universal because S^1 is not simply connected.

Definition 6.10. A topological space X is **semilocally simply connected** if each $x \in X$ has a neighbourhood U such that the inclusion-induced map:

$$\iota_* : \pi_1(U, x) \to \pi_1(X, x)$$

is trivial (i.e., every loop in U is null-homotopic in X).

> **Note 6.11.** A space being *simply connected* means that all loops contract globally, but being *semilocally simply connected* means that small loops around each point contract locally.

Example 6.12 (The Hawaiian Earring)

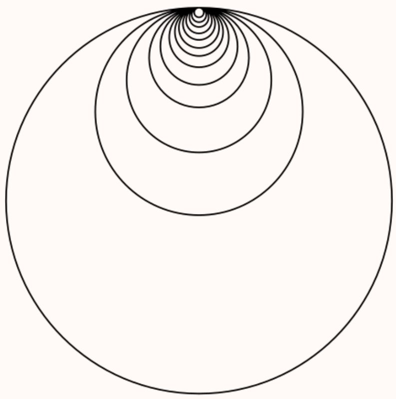
The **Hawaiian Earring** is defined as the union of infinitely many circles C_n:

$$E = \bigcup_{n \geq 1} C_n$$

where

$$C_n = \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n} \right)^2 + y^2 = \frac{1}{n^2} \right\}.$$

The figure below illustrates it.



Proposition 6.13

If $p : \widetilde{X} \to X$ is a universal cover, then X is semilocally simply connected.

Proof. Let $U \subset X$ be an evenly covered neighbourhood of x_0, and let $\widetilde{U}$ be a sheet over U. Let γ be a loop in U based at x_0, which lifts to a loop $\widetilde{\gamma} \subset \widetilde{U}$ based at $\widetilde{x}_0$. Then $\widetilde{\gamma}$ is homotopic to the constant loop at $\widetilde{x}_0$. Composing this homotopy with p implies γ is homotopic to the constant loop at x_0. Thus, $\pi_1(U, x_0)$ is trivial in $\pi_1(X, x_0)$. $\square$

Theorem 6.14

Let X be path-connected, locally path-connected, and semilocally simply connected. Then there exists a universal cover $p : \widetilde{X} \to X$.

Proof. Not examinable. □

Proposition 6.15. Let X be path-connected, locally path-connected, and semilocally simply-connected, and let $x_0 \in X$. Then for every subgroup $H \leq \pi_1(X, x_0)$, there is a covering space

$$p : (X_H, \widetilde{x}_0) \to (X, x_0)$$

such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$ for some basepoint x_0.

Note 6.16. This result tells us that subgroups of the fundamental group correspond to covering spaces. If we take a normal subgroup, the corresponding covering space is regular (Galois), and if the subgroup is trivial, we get the universal cover.

Proof. Define $X_H = \widetilde{X}/\sim$ where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H$. This is an equivalence relation (exercise).

$$\begin{tikzcd}
\widetilde{X} \arrow[r] \arrow[dr, "p"'] & \widetilde{X}/\sim = X_H \arrow[d] \\
& X
\end{tikzcd}$$

Let $U_{[\gamma]}$ and $U_{[\gamma']}$ be basis neighbourhoods. If $[\gamma] \sim [\gamma']$, then $[\gamma \cdot \nu] \sim [\gamma' \cdot \nu]$, so p is a covering space and $p^{-1}(U) = \bigcup_\gamma U_{[\gamma]}$. Let $\widetilde{x}_0 \in X_H$ be the equivalence class of the constant path c_{x_0} at x_0. Let γ be a loop in X based at x_0 such that $[\gamma] \in p_*(\pi_1(X_H, \widetilde{x}_0))$. Again, define the lift $t \mapsto [\gamma_t]$ at $\widetilde{x}_0$ where

$$\gamma_t(s) = \begin{cases} \gamma(s) & s \in [0, t] \\ \gamma(t) & s \in [t, 1] \end{cases}$$

Then $t \mapsto [\gamma_t]$ is a loop in X_H if and only if $[\gamma_1] = [\gamma] = [c_{x_0}]$ in X_H, which happens if and only if $[\gamma] \sim [c_{x_0}]$, i.e., if $\gamma \in H$. □

Proposition 6.17 (Unique Homotopy Lifting Property)

Let $p : \widetilde{X} \to X$ be a covering and $f : Y \to X$ be a continuous map. If there are two lifts $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ of f such that $\widetilde{f}_1(y) = \widetilde{f}_2(y)$ for some $y \in Y$, and Y is connected, then $\widetilde{f}_1 = \widetilde{f}_2$.

Note 6.18. This result tells us that if we lift a function to the covering space and the lift agrees at a single point, then it must agree everywhere. This is important because it ensures uniqueness in homotopy lifting arguments.

Proof. Exercise.

To do

□

Remark 6.19. Any deck transformation with a fixed point of a connected cover (i.e., $p : \widetilde{X} \to X$ with $\widetilde{X}$ connected) is the identity.

Definition 6.20. Given two coverings $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$, a **basepoint-preserving isomorphism** is an isomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ such that $f(\widetilde{x}_1) = \widetilde{x}_2$.

Proposition 6.21. Let X be path-connected and locally path-connected, and let $x_0 \in X$. Two path-connected covering spaces $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ are isomorphic via an isomorphism $h : \widetilde{X}_1 \to \widetilde{X}_2$ mapping $\widetilde{x}_1 \in p_1^{-1}(x_0)$ to $\widetilde{x}_2 \in p_2^{-1}(x_0)$ if and only if

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$$

Note 6.22. This means that two covering spaces are isomorphic if and only if their lifted fundamental groups (the images of the fundamental groups under p_*) are the same. This gives a purely algebraic way to classify covering spaces.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  Since $p_1 = p_2 \circ h$, we have
  $$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) \subseteq (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$$
  and since $p_2 = p_1 \circ h^{-1}$, then
  $$(p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)) \subseteq (p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$$
  so equality follows.

- Proof of ($\Leftarrow$).
  Assume
  $$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$
  We can lift p_1 to a continuous map
  $$\widetilde{p}_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$$
  and p_2 to a continuous map
  $$\widetilde{p}_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$$
  so that $p_1 \circ \widetilde{p}_2 = p_2$ and $p_2 \circ \widetilde{p}_1 = p_1$. Then $\widetilde{p}_1 \circ \widetilde{p}_2$ fixes $\widetilde{x}_2 \in \widetilde{X}_2$. By the unique lifting property, $\widetilde{p}_1 \circ \widetilde{p}_2 = \mathrm{id}_{\widetilde{X}_2}$ and similarly $\widetilde{p}_2 \circ \widetilde{p}_1 = \mathrm{id}_{\widetilde{X}_1}$. Hence $\widetilde{p}_1$ is an isomorphism.

$\square$

Theorem 6.23 (Galois Correspondence)
Let X be path-connected, locally path-connected, and semilocally simply connected, and $x_0 \in X$. Then:

- There is a bijection between subgroups of $\pi_1(X, x_0)$ and path-connected covering spaces $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, up to basepoint-preserving isomorphisms.
- If we ignore basepoints, this correspondence gives a bijection between conjugacy classes of subgroups of $\pi_1(X, x_0)$ and path-connected covering spaces

$$p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0),$$

up to isomorphisms.

Note 6.24. The Galois Correspondence tells us that the topology of covering spaces is deeply linked to algebraic properties of the fundamental group. Just as in Galois Theory where normal subgroups correspond to normal field extensions, here subgroups of $\pi_1(X)$ correspond to covering spaces. When the subgroup is normal, the covering space is a regular (or normal) covering.

Proof. We prove each statement in turn.

- To each covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, we associate the subgroup

$$p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subset \pi_1(X, x_0).$$

  Using previous results, we know that this is well-defined on isomorphism classes and is bijective.

- Conversely, given a subgroup $H \leq \pi_1(X, x_0)$, we construct the covering space associated with H. Define $X_H = \widetilde{X}/\sim$ where $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot (\gamma')^{-1}] \in H$. This defines an equivalence relation. We can then show that $p : X_H \to X$ is a covering space with fundamental group corresponding to H.

□

7 Seifert-Van Kampen theorem

7.1 The free product

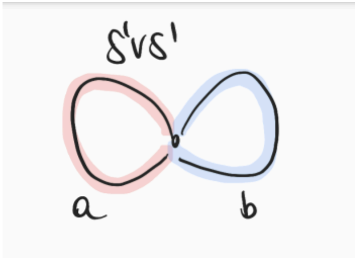
Definition 7.1. Let $\{(X_\alpha, x_\alpha)\}_{\alpha \in A}$ be a collection of pointed topological spaces. The **wedge product** of the collection is defined as

$$\bigvee_{\alpha \in A} X_\alpha = \bigsqcup_{\alpha \in A} X_\alpha / x_\alpha \sim x_\beta.$$

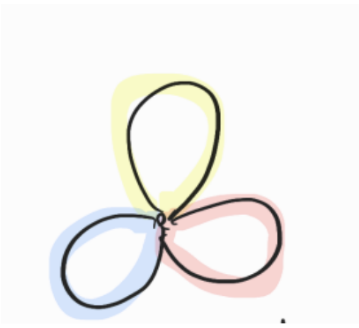
> **Note 7.2.** The wedge product allows us to glue together a collection of topological spaces at a single basepoint.

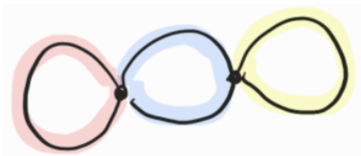
Example 7.3. Some examples.

- $S^1 \vee S^1$ is the figure-eight.



- $S^1 \vee S^1 \vee S^1$ (on the left) versus $S^1 \vee (S^1 \vee S^1)$ (on the right).





Example 7.4. Consider the figure-eight space $S^1 \vee S^1$. We want to compute its fundamental group. Let $A = S^1$ and $B = S^1$ be the two circles. Then,

$$\pi_1(A) \cong \langle a \rangle \cong \mathbb{Z}, \quad \pi_1(B) \cong \langle b \rangle \cong \mathbb{Z}$$

where a and b are loops around A and B, respectively. The fundamental group of the wedge sum allows us to concatenate loops from A and B, forming words such as

$$aa\bar{b}\bar{b}\bar{b}a\bar{a}$$

where $\bar{a}, \bar{b}$ are the inverses of a, b, respectively. This introduces the notion of the **free product** of groups.

Definition 7.5. Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of groups. A **word** in this collection is a finite sequence $g_1 \cdots g_m$ where $g_i \in G_{\alpha_i}$ for some α_i. The **length** of the word is m, and the empty word is denoted by ε.

Intuition: Just like words in a language are made up of letters, words in a free product are made up of elements from different groups. The structure ensures that each step remains well-defined.

Definition 7.6. A word $g = g_1 \cdots g_m$ is **reduced** if:

1. $g_i \neq e_{\alpha_i}$, where e_{α_i} is the identity element of G_{α_i}.
2. Two consecutive letters come from different groups, i.e., $\alpha_i \neq \alpha_{i+1}$.

Definition 7.7. We denote the **set of reduced words** on $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ by $*_\alpha G_\alpha$.

Definition 7.8. The **free product** of groups $*_\alpha G_\alpha$ is defined using reduced words. The product of two words is given by:

- If the last letter of g and the first letter of h are from different groups, concatenate the words.
- If they are from the same group but not inverses, multiply them in G_α.
- If they multiply to the identity, remove them.

This operation is denoted by $g \bullet h$.

Theorem 7.9. The operation $\bullet$ on $*_\alpha G_\alpha$ makes it a group.

Proof. Not examinable. □

Example 7.10. A natural conjecture is:

$$\pi_1((X, x_0) \vee (Y, y_0)) \cong \pi_1(X, x_0) * \pi_1(Y, y_0).$$

However, this is not always true. A counterexample is the Hawaiian earring, where

$$X = \{C_n : n = 1, 2, \dots\}$$

with C_n being a circle of radius $1/n$. The fundamental group is uncountable, whereas $*_n\pi_1(C_n)$ is countable.

Remark 7.11. Given a collection of groups $\{G_\alpha\}_\alpha$, then each G_α is a subgroup of $*_\alpha G_\alpha$ via the map

$$\iota_\alpha : G_\alpha \to *_\alpha G_\alpha$$

sending a group element $g \neq e_\alpha$ to a word made up only of g and e_α to the empty word. In other words, Each G_α embeds into $*_\alpha G_\alpha$ via a map sending g to a word starting with g.

Lemma 7.12

Let $\phi_\alpha : G_\alpha \to G$ be a collection of group homomorphisms. Then there exists a unique group homomorphism

$$*_\alpha\phi_\alpha : *_\alpha G_\alpha \to G$$

such that $(*_\alpha\phi_\alpha) \circ \iota_\beta = \phi_\beta$ for all β.

Proof. Define $*_\alpha\phi_\alpha : *_\alpha G_\alpha \to G$ by

$$g_1 \cdots g_m \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_m}(g_m)$$

where $g_i \in G_{\alpha_i}$. Clearly, $*_\alpha\phi_\alpha \circ \iota_\beta = \phi_\beta$ (note that this works for any words, not only reduced ones).

$$\begin{tikzcd}
& G_\beta \arrow[d, "\iota_\beta"'] \arrow[ddr, "\phi_\beta"] & \\
G_\alpha \arrow[r, "\iota_\alpha"] \arrow[drr, "\phi_\alpha"'] & *_\alpha G_\alpha \arrow[dr] & \\
& & G
\end{tikzcd}$$

Since all ϕ_α are group homomorphisms, we get

$$*_\alpha\phi_\alpha(g_i g_{i+1}) = *_\alpha\phi_\alpha(g_i) *_\alpha \phi_\alpha(g_{i+1})$$

and

$$*_\alpha\phi(e_\alpha) = e.$$

So $*_\alpha\phi_\alpha$ is compatible with the operations that reduce words in the free product.

Now, let $g = g_1 \cdots g_m$ and $h = h_1 \cdots h_n$. Then

$$(*_\alpha\phi_\alpha)(g \cdot h) = \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_m}(g_m) \cdot \phi_{\beta_1}(h_1) \cdots \phi_{\beta_n}(h_n).$$

If $\alpha_m = \beta_1$, then this simplifies to

$$\phi_{\alpha_1}(g_1) \cdots (\phi_{\alpha_m}(g_m) \cdot h_1) \cdots \phi_{\beta_n}(h_n),$$

which respects the structure of the free product.

Hence, we conclude that

$$(*_\alpha\phi_\alpha)(g \cdot h) = (*_\alpha\phi_\alpha)(g)(*_\alpha\phi_\alpha)(h),$$

showing that $*_\alpha\phi_\alpha$ is indeed a group homomorphism.

Finally, the requirement that the restrictions satisfy

$$(*_\alpha\phi_\alpha) \circ \iota_\alpha = \phi_\alpha$$

uniquely determines the map, proving uniqueness. □

7.2 The theorem

We want to apply the free product to topology to simplify the computations of the fundamental group. We want to reduce the computation of the fundamental group of an open cover to the fundamental group of each individual set in the cover.

Consider $X = \bigcup_\alpha X_\alpha$, an open cover of X, and denote the inclusions $\iota_\alpha : X_\alpha \to X$. Assume we have a base point $x_0 \in \bigcap_\alpha X_\alpha$. The inclusion maps induce a map of fundamental groups:

$$(\iota_\alpha)_* : \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0).$$

By the previous lemma, we have a unique map

$$\Phi = *_\alpha(\iota_\alpha)_* : *_\alpha \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0),$$

which is compatible with the inclusions $i_\alpha : \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0)$, meaning that we have

$$*_\alpha(\iota_\alpha)_* \circ i_\alpha = (\iota_\alpha)_*.$$

Note 7.13. The fundamental group of X can be described in terms of the fundamental groups of the open sets X_α. This theorem allows us to compute fundamental groups of complicated spaces by breaking them into simpler pieces and using their known fundamental groups.

Define $X_{\alpha\beta} = X_\alpha \cap X_\beta$ and the inclusions $\iota_{\alpha\beta} : X_{\alpha\beta} \to X_\alpha$ and $\iota_{\beta\alpha} : X_{\alpha\beta} \to X_\beta$. We have corresponding induced maps

$$(\iota_{\alpha\beta})_* : \pi_1(X_{\alpha\beta}, x_0) \to \pi_1(X_\alpha, x_0) \quad \text{and} \quad (\iota_{\beta\alpha})_* : \pi_1(X_{\alpha\beta}, x_0) \to \pi_1(X_\beta, x_0).$$

This set-up can be summarized in the following commutative diagram:

$$\begin{tikzcd}
 & \pi_1(X_\alpha, x_0) \arrow[dr, "i_\alpha"] \arrow[drr, "(\iota_\alpha)_*"] & & \\
\pi_1(X_{\alpha\beta}, x_0) \arrow[ur, "(\iota_{\alpha\beta})_*"] \arrow[dr, "(\iota_{\beta\alpha})_*"'] & & *_\alpha \pi_1(X_\alpha, x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X, x_0) \\
 & \pi_1(X_\beta, x_0) \arrow[ur, "i_\beta"] \arrow[urr] & &
\end{tikzcd}$$

Theorem 7.14 (Seifert-van Kampen)
Let $X = \bigcup_\alpha X_\alpha$ be an open cover and assume $x_0 \in \bigcap_\alpha X_\alpha$. Then,

1. If for all α, β, $X_\alpha \cap X_\beta$ is path-connected, then the map

$$\Phi = *_\alpha(\iota_\alpha)_* : *_\alpha \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0)$$

is surjective.

2. If for all α, β, γ, $X_\alpha \cap X_\beta \cap X_\gamma$ is path-connected, then

$$\ker(\Phi) = N, \quad \text{and} \quad \pi_1(X, x_0) \cong *_\alpha \pi_1(X_\alpha, x_0)/N$$

where N is the normal closure of

$$U = \{(\iota_{\alpha\beta})_*(w)(\iota_{\beta\alpha})_*(w)^{-1} : w \in \pi_1(X_{\alpha\beta}, x_0)\} \subset *_\alpha \pi_1(X_\alpha, x_0).$$

Note 7.15. The theorem essentially tells us that the fundamental group of X can be expressed as the free product of the fundamental groups of the open sets in the cover, subject to identifications dictated by their intersections. The first condition ensures that loops in X can be expressed as loops from the open sets, ensuring surjectivity. The second condition ensures that equivalent representations of loops are identified, making the result a quotient.

Remark 7.16. A subgroup $H \subset G$ is called *normal* if $gHg^{-1} = H$ for any $g \in G$. The normal closure of $U \subset G$ is the smallest normal subgroup of G containing U.

Example 7.17. We would like to have $\pi_1(X) \cong \pi_1(A) * \pi_1(B)$, but $\mathbb{Z}*\mathbb{Z}*\mathbb{Z} \ncong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}$. This is because taking the free product of the fundamental groups makes us have loops that are counted twice, because they get recognised as different words (since they come from different groups). Hence we have an inclusion/exclusion-type problem.

Note that for any $\omega = [\gamma] \in \pi_1(X_\alpha \cap X_\beta, x_0)$ it is represented in $*_\alpha \pi_1(X_\alpha, x_0)$ both as

$$(\iota_{\alpha\beta})_*(\omega) \in \pi_1(X_\alpha, x_0)$$

and

$$(\iota_{\beta\alpha})_*(\omega) \in \pi_1(X_\beta, x_0)$$

Hence we need a way to quotient out these loops in order not to count them twice in the free group.

Let us think of the meaning of this theorem. For $[\gamma] \in \pi_1(X, x_0)$ we have $\gamma \simeq \gamma_1 \cdots \gamma_m$ for $\gamma_i \in X_{\alpha_i}$ for a collection of loops $\{\gamma_i\}_i$. So the induced map from the free product looks something like

$$\Phi : *_\alpha \pi_1(X_\alpha, x_0) \to \pi_1(X, x_0)$$

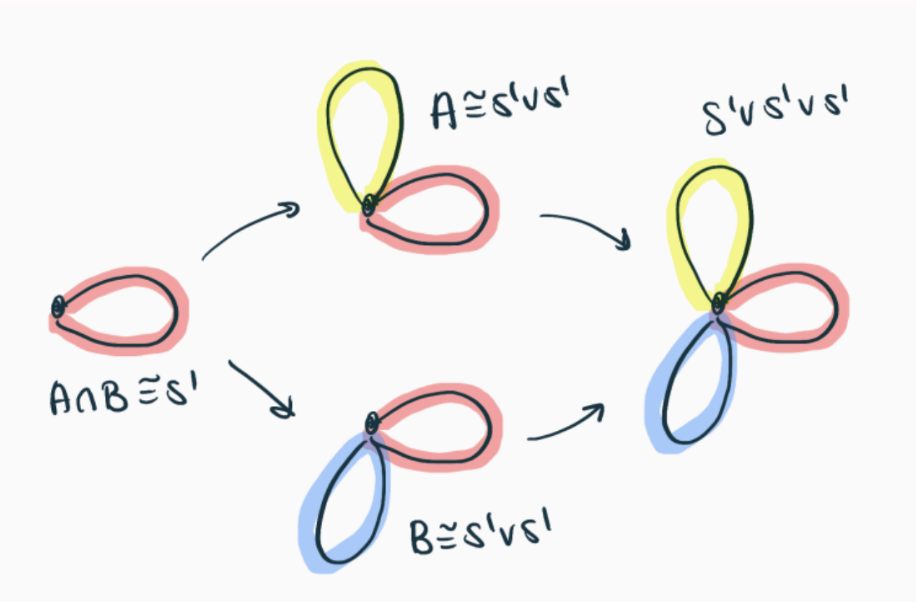
$$[\gamma_1] \cdots [\gamma_m] \mapsto (\iota_{\alpha_1})_*([\gamma_1]) \cdots (\iota_{\alpha_m})_*([\gamma_m]) = [\gamma_1 \cdots \gamma_m] = [\gamma]$$

To show that this map is surjective one needs to **factor out** every loop in X based at x_0 as a concatenation of loops $\gamma_1 \cdots \gamma_m$, where each loop is contained in an open from the cover. The map is not injective because such factorisation is not unique, for examples think of loops in an intersection $X_\alpha \cap X_\beta$. By factoring out its kernel we get an isomorphism.

Note 7.18. The theorem states that if a space X is covered by sets X_α, then the fundamental group of X can be built from the fundamental groups of X_α, but we must be careful about loops that exist in the intersections $X_\alpha \cap X_\beta$. These loops appear in both groups $\pi_1(X_\alpha)$ and $\pi_1(X_\beta)$, so we must quotient by their identification to avoid counting them twice. This provides a way to compute fundamental groups of complicated spaces by breaking them into simpler pieces.

Example 7.19

Consider the n-sphere $S^n = U_1 \cup U_2$, where $U_1 = S^n \setminus \{N\}$ and $U_2 = S^n \setminus \{S\}$, for the north pole $N = (0, \ldots, 0, 1)$ and the south pole $S = (0, \ldots, 0, -1)$. This is an open cover of the n-sphere. We have that $U_1 \cap U_2 = \mathbb{R}^n \setminus \{0\}$, which is path-connected if $n \geq 2$. Since both U_1 and U_2 are homeomorphic to $\mathbb{R}^n$, they are contractible so $\pi_1(U_1) \cong \pi_1(U_2) \cong 0$. Therefore, by Seifert-van Kampen, $\pi_1(S^n) \cong 0$. Note that this argument does not extend to S^1, since $U_1 \cap U_2 = \mathbb{R} \setminus \{0\}$ is not path-connected.



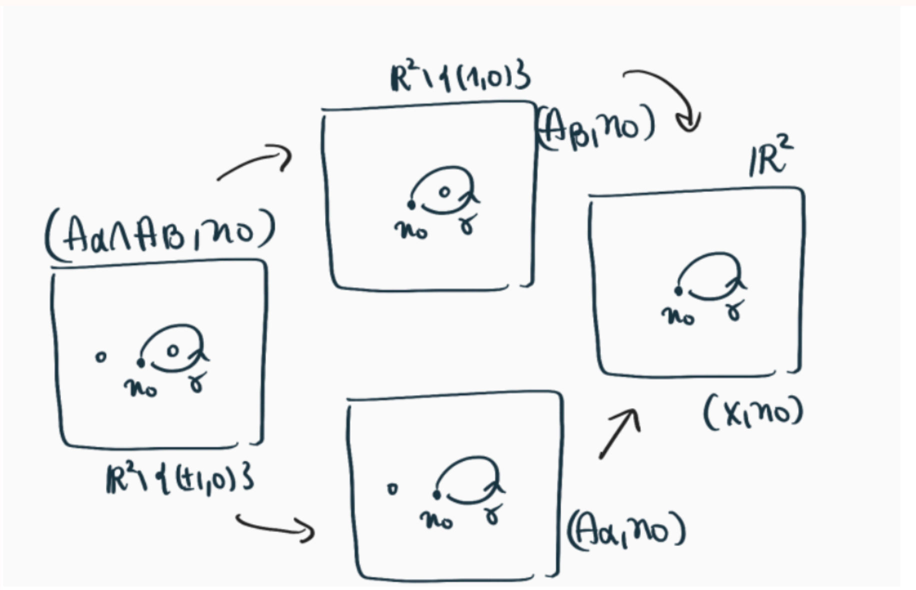
Example 7.20

Let $X = \mathbb{R}^2$, $A_1 = \mathbb{R}^1 \setminus \{(1,0)\}$, $A_2 = \mathbb{R}^2 \setminus \{(-1,0)\}$, and $x_0 = (0,0) \in A_1 \cap A_2$. Consider the loop $\omega = [\gamma] \in \pi_1(A_1 \cap A_2, x_0)$, as in the drawing. We will show that γ gives rise to different elements depending on the space in which it is considered.

- $A_1 \cap A_2 \cong S^1 \vee S^1$, so it has fundamental group $\mathbb{Z} * \mathbb{Z}$ generated by a, b. Then $[\gamma] = a$ (it is one of the two generators).
- $A_1 \cong S^1$, and $(\iota_{1,2})_*([\gamma])$ is the generator of $\pi_1(A_1, x_0) \cong \mathbb{Z}$.
- $A_2 \cong S^1$, and $(\iota_{2,1})_*([\gamma])$ is trivial in $\pi_1(A_2, x_0) \cong \mathbb{Z}$.
- We know that $\mathbb{R}^2$ is contractible, so $[\gamma]$ should be trivial in $\pi_1(\mathbb{R}^2, x_0) = 0$, but if we look at $\pi_1(A_1, x_0) * \pi_1(A_2, x_0)$, $[\gamma]$ will not be trivial there, since when constructing the free product these two groups only have the empty word in common. Then we have

$$(\iota_{1,2})_*([\gamma])(\iota_{2,1})_*([\gamma])^{-1} = (\iota_{1,2})_*([\gamma]),$$

which will give us one of the identifications we make in N in the statement of Seifert-van Kampen (which is what we need in order to be able to construct an isomorphism from $\pi_1(X, x)$ to a free product of some sorts).



7.3 CW-Complexes

We consider a class of topological spaces now called **CW-complexes**. The reasoning behind this is that we want to be able to compare and compute with common topological spaces more effectively. Intuitively, CW-complexes are topological spaces that can be assembled from simpler spaces by *gluing* cells together. Many (but not all!) interesting topological spaces have the structure of a CW-complex.

Remark 7.21. Every smooth manifold can be equipped with a CW structure. Every topological manifold in dimension greater than four can be equipped with a CW structure: this is unknown in dimension four, and is trivial in lower dimensions (since topological is equivalent to smooth in dimensions one, two, and three).

Definition 7.22. A **CW-complex** is a topological space which is built inductively as follows:

- The **zero skeleton** X^0 is a discrete set;
- Given X^{n-1}, a collection of disks $\{D^n_\alpha\}$ with $D^n_\alpha \cong \mathbb{D}^n$ and $S^{n-1}_\alpha = \partial D^n_\alpha$ with **attaching maps**
$$\phi_\alpha : S^{n-1}_\alpha \to X^{n-1}$$
we define
$$X^n = \left[X^{n-1} \sqcup \left(\bigsqcup_\alpha D^n_\alpha \right) \right] / \sim$$
where we are quotienting with respect to the equivalence relation $x \sim \phi_\alpha(x)$ for all $x \in S^{n-1}_\alpha$;
- We define $X = \bigcup_n X^n$ equipped with the **weak topology**, i.e. $U \subset X$ is open if and only if $U \cap X^n$ is open for all n.

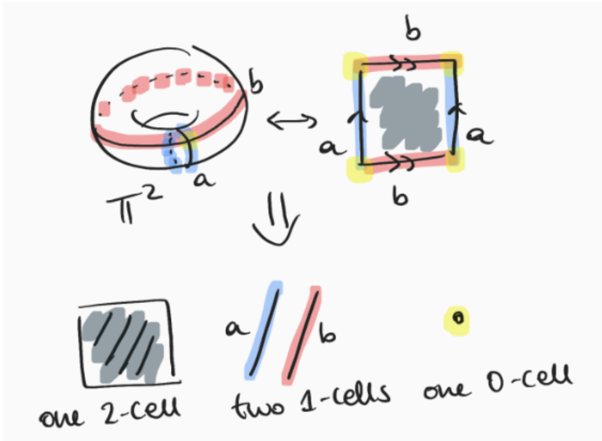
Note 7.23. CW-complexes provide a way to construct complicated spaces from simple building blocks. Starting from discrete points, we inductively attach higher-dimensional disks along their boundaries. The weak topology ensures compatibility of these constructions. Many familiar spaces, such as spheres and tori, can be given CW-complex structures, making them easier to analyze using algebraic topology tools like homotopy and homology.

Remark 7.24. X^n is the n-**skeleton** of the CW-complex X. The disks D^n_α are called the n-**cells**, and their interior are called the **open** n-**cells** denoted by $e^n_\alpha = D^n_\alpha \setminus S^{n-1}_\alpha$.

Remark 7.25. We can have multiple CW-complex structures for the same topological space. There is a notion of **minimal** CW-complex structure, where minimal refers to a CW-complex structure with the minimal number of cells. We will not deal with this, but details can be found in Hatcher section 4.

Example 7.26. Some examples.

- A CW-complex does not need to be made up of a finite number of cells. For example, we could take $X^0 = \mathbb{Z}$ and the attaching maps defined by $\phi_n(0) = n$ and $\phi_n(1) = n + 1$. The resulting CW complex is homeomorphic to $\mathbb{R}$.
- The torus is an example of a CW-complex in two dimensions.



- $\mathbb{RP}^n = S^n/x \sim -x$. We have that $\mathbb{RP}^0 = \{\text{pt}\}$, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup D^n/(x \sim \phi(x))$, i.e. we are gluing the boundary of D^n to the S^{n-1} of $\mathbb{RP}^{n-1}$. Inductively, we get that
$$\mathbb{RP}^n = \{\text{pt}\} \cup e^1 \cup \cdots \cup e^n$$
i.e. it is a CW-complex with an open cell in each dimension.

7.4 Properties of CW-complexes

Definition 7.27. A CW-complex is **finite-dimensional** if $X = X^n$ for some n. The largest n for which there are cells in the complex is the **dimension** of the complex. A CW-complex is **finite** if it has only finitely many cells.

Definition 7.28. A **sub-complex** of a CW-complex X is the closure in X of a collection of open cells in X.

Example 7.29. Not all topological spaces are CW-complexes. For example, the topologist's (or Warsaw) sine curve is not a CW-complex (since it is not locally contractible and is not locally path-connected).

Definition 7.30. A topological space is called **normal** if any two closed subsets have disjoint open neighbourhoods.

Definition 7.31. A topological space is called **Hausdorff** if any two distinct points have disjoint open neighbourhoods.

Proposition 7.32. A CW-complex is **normal (hence Hausdorff)**.

Definition 7.33. A topological space X is called **locally contractible** if for all $x \in X$ and for every open neighbourhood U, there exists an open $V \subset U$ containing x such that V is contractible.

Proposition 7.34. CW-complexes are locally contractible.

Proposition 7.35. Let X be a path-connected topological space and for a fixed n let $\phi_\alpha^n : S_\alpha^{n-1} \to X$ be a collection of attaching maps. Define
$$Y = X \sqcup \left(\bigsqcup_\alpha D_\alpha^n \right) /(x \sim \phi_\alpha^n(x))$$
Let $x_0 \in X$, then:

- if $n = 2$, $\pi_1(X, x_0)/N \cong \pi_1(Y, x_0)$,
- if $n > 2$, $\pi_1(Y, x_0) \cong \pi_1(X, x_0)$.

Remark 7.36. This says that attaching 2-cells could change the topology, for example, it could cover holes, while attaching higher-dimensional cells does not change anything.

As such, when attaching 2-cells, the fundamental group can change (e.g., holes can be filled in). However, when attaching higher-dimensional cells ($n > 2$), the fundamental group remains unchanged because higher-dimensional cells do not affect 1-dimensional loops.

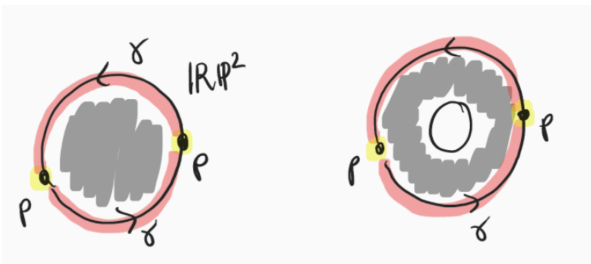
Theorem 7.37

For a path-connected CW-complex X with $x_0 \in X^2$, the inclusion map $\iota : X^2 \to X$ induces an isomorphism $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$.

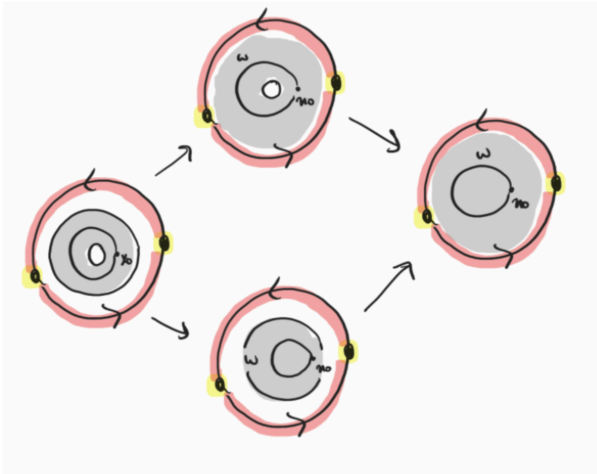
Note 7.38. The fundamental group is entirely determined by the 2-skeleton. This means that the loops in a CW-complex can be studied using only the 2-dimensional structure, making computations significantly easier.

7.5 Example of the Seifert-Van Kampen for CW complexes

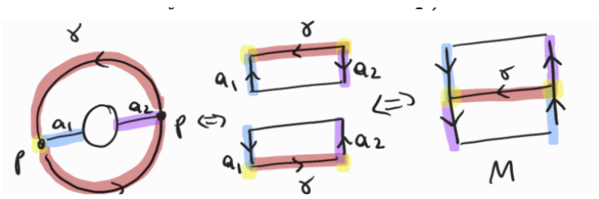
Let us work through an example of using Seifert-van Kampen Theorem to calculate the fundamental group of a CW-complex, namely of $\mathbb{RP}^2$. Consider the CW-complex structure on $\mathbb{RP}^2$ in the left-hand side of the image and take out a closed disk from its interior (right-hand side of the image). Note that we are considering all fundamental groups with respect to the basepoint x_0 in the image.



Now consider the cover given by the following inclusion diagram:



where A is the subset of $\mathbb{RP}^2$ without a closed disk from its interior, and B is $\mathbb{RP}^2$ without its boundary. Clearly, B is contractible, so it has trivial fundamental group, and $A \cap B$ deformation retracts to a circle, so it has fundamental group $\mathbb{Z}$ generated by $[\omega]$. We note that A is the Möbius strip, see the following figure that makes it clear (another way to say this is that we can obtain $\mathbb{RP}^2$ by gluing a 2-cell to the boundary of the Möbius strip).



Note 7.39. The idea is to apply Seifert-van Kampen Theorem to decompose $\mathbb{RP}^2$ into two simpler regions A and B, such that their intersection $A \cap B$ has a known fundamental group (which is $\mathbb{Z}$ since it retracts to a circle). By understanding how loops in A and B interact in X, we can compute $\pi_1(\mathbb{RP}^2, x_0)$.

Therefore, to compute the fundamental group of $\mathbb{RP}^2$ we first take the following free product:

$$\pi_1(A, x_0) * \pi_1(B, x_0) \cong \mathbb{Z}$$

and we need to compute the normal subgroup N:

$$N = \langle \{(\iota_{A,B})_*[\omega](\iota_{B,A})_*[\omega]^{-1}\} \rangle = \langle [\omega] \rangle.$$

Since ω corresponds to the outer-circle of the Möbius strip and γ corresponds to its inner circle, we have that $\langle [\omega] \rangle = \langle [\gamma]^2 \rangle$. Hence,

$$\pi_1(\mathbb{RP}^2, x_0) \cong \langle [\gamma] \rangle / \langle [\gamma]^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

Note 7.40. The relation $[\omega] = [\gamma]^2$ tells us that each loop γ must be considered equivalent to its inverse. This reflects the fact that in $\mathbb{RP}^2$, a loop can be continuously transformed into its reverse. The quotient by $\mathbb{Z}/2\mathbb{Z}$ means that the fundamental group detects only whether a loop winds an even or odd number of times, leading to the group $\mathbb{Z}/2\mathbb{Z}$.

7.6 Higher Homotopy Groups

We can define **higher homotopy groups** similarly to the fundamental group.

Definition 7.41. The **homotopy groups** $\pi_n(X, x_0)$ $(n \geq 0)$ are groups of basepoint-preserving homotopies of continuous maps $\phi : I^n \to X$, such that $\phi(\partial I^n) = x_0$.

Note 7.42. The homotopy groups generalise the fundamental group to higher dimensions. While $\pi_1(X, x_0)$ captures loops up to homotopy, $\pi_n(X, x_0)$ for $n \geq 2$ captures higher-dimensional "holes" in the space.

Remark 7.43. The zeroth homotopy group $\pi_0(X, x_0)$ is just the set of path components of X (it is not a group).

For $n \geq 1$, $\pi_n(X, x_0)$ are groups, and for $n \geq 2$ they are actually **abelian groups**. Note that despite their simple definition, homotopy groups are notoriously difficult to compute for $n \geq 2$. This is because they are not directly computable from a cell structure as π_1 is.

Example 7.44. For S^2, we know:

$$\pi_1(S^2) = 0, \quad \pi_2(S^2) = \mathbb{Z}, \quad \pi_3(S^2) = \mathbb{Z}, \quad \pi_4(S^2) = \mathbb{Z}_2, \quad \pi_5(S^2) = \mathbb{Z}_2, \quad \pi_6(S^2) = \mathbb{Z}_{12}.$$

Note that S^2 has no cells in dimensions greater than 2, but actually $\pi_n(S^2)$ is nonzero for infinitely many n.

7.7 Presentation of groups

We introduce a way of describing groups using generators and relations and discuss how to interpret these for the fundamental groups of topological spaces.

Definition 7.45. A group G has a **group presentation** $\langle S \mid R \rangle$ for some sets S and R (generators and relations respectively) if $G = \langle S \rangle / \langle R \rangle$.

Definition 7.46. A group $G = \langle S \mid R \rangle$ is **finitely generated** if $|S| < \infty$ and it is **finitely presented** if $|S|, |R| < \infty$.

Example 7.47. $\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 \rangle$

7.8 Computing the fundamental group of CW complex

We now describe a way to compute the fundamental group of a CW-complex in terms of generators and relations. Assume we have a path-connected $X = X^2$ CW-complex (recall that $\pi_1(X, x_0) = \pi_1(X^2, x_0)$, so we can restrict to this case without loss of generality). Let $x_0 \in X$. Without loss of generality, we can assume $x_0 \in X^1$ (since this does not change the fundamental group and we can move x_0 by path-connectedness). To compute $\pi_1(X, x_0)$, we proceed as follows:

1. Find a spanning tree of X^1 (e.g. by Dijkstra's algorithm), call it T. Let $\mathcal{A} = \{z \in X^1 \setminus T\}$.

   So every edge not in T gives a loop when adding it to T, and conversely every loop in X^1 based at x_0 is homotopic to a combination of such edge-cycles.

2. Let $e^2_\alpha \subset X^2$ be a 2-cell, and let $\phi_\alpha : S^1 \to X^1$ be the attaching map. We can define a loop by reparametrising
$$\gamma_\alpha(t) = \phi_\alpha(e^{2\pi it})$$
   which must be homotopic to an edge-loop. This might or might not have x_0 as a basepoint.

   If $x_1 = \gamma_\alpha(0)$, let $g_\alpha : I \to X^1$ be a path with $g_\alpha(0) = x_0$ and $g_\alpha(1) = x_1$, and define
$$\omega_\alpha = [g_\alpha \cdot \gamma_\alpha \cdot \bar{g_\alpha}] \in \pi_1(X, x_0)$$
   and therefore it will correspond to a reduced word u_α in $\mathcal{A}$. Set $\{u_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of all reduced words.

3. Now, we can apply Seifert-van Kampen. Consider $A = (\bigcup_\alpha e^2_\alpha) \cup \{\text{path from } x_0 \text{ to each cell}\}$, which is contractible, so it has trivial fundamental group. Choose $y_\alpha \in e^2_\alpha$ (a point from each cell) and define $B = X^2 \setminus \bigcup_\alpha \{y_\alpha\}$ (note that if I take a point from each 2-cell, I get something that deformation retracts to S^1). Then B deformation retracts to X^1, so
$$\pi_1(X^1, x_0) \cong \pi_1(B, x_0).$$
   Applying Seifert-van Kampen,
$$\pi_1(X, x_0) \cong \pi_1(A, x_0) * \pi_1(B, x_0)/\langle\langle U \rangle\rangle.$$

Note 7.48. This describes a systematic way to compute the fundamental group of a CW-complex by first finding a generating set (coming from the edges of the 1-skeleton) and then imposing relations (coming from the attaching maps of the 2-cells). Essentially, every 1-cycle in the space corresponds to a generator, and the 2-cells contribute relations that dictate how these cycles interact. Seifert-van Kampen allows us to compute the fundamental group by splitting the space into contractible and retractive parts.

8 Simplicial and Singular homology

Homology is a fundamental tool in algebraic topology that provides an algebraic means of measuring the shape and structure of topological spaces. It assigns to each space a sequence of algebraic objects, typically abelian groups or modules, which encode information about its connectivity properties.

The key idea of homology is to study a space by analysing chains of simplices and the relationships between them. By defining boundary maps that capture how simplices are assembled, we construct homology groups that classify topological features such as connected components, holes, and higher-dimensional voids.

8.1 Simplicial homology

The basic building blocks of simplicial homology are simplices, which generalise the concept of a triangle to higher dimensions.

Definition 8.1. The **standard n-simplex** is defined as the subset of $\mathbb{R}^{n+1}$ given by

$$\Delta^k = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1, \quad x_i \geq 0 \text{ for all } i = 0, \ldots, n \right\}.$$

Note 8.2. This represents an n-dimensional generalisation of a triangle, where the conditions ensure that the simplex is confined to an affine hyperplane and lies entirely in the positive orthant.

Example 8.3

Geometrically:

- The 0-simplex is a single point.
- The 1-simplex is a closed line segment (edge).
- The 2-simplex is a filled triangle.
- The 3-simplex is a solid tetrahedron.
- Higher-dimensional simplices follow the same pattern.

Definition 8.4. If $\{v_0, v_1, \ldots, v_n\}$ denotes the standard basis of $\mathbb{R}^{n+1}$, then the simplex Δ^n can alternatively be described as the **convex hull** of these points:

$$\Delta^n = [v_0, v_1, \ldots, v_n] = \left\{ \sum_{i=0}^{n} x_i v_i : \sum_{i=0}^{n} x_i = 1, x_0 \geq 0 \right\}.$$

This notation expresses Δ^n as the set of all convex combinations of the vertices $v_0, \ldots, v_n$.

Remark 8.5. The order in which we write the vertices of a simplex matters, as it determines a canonical linear isomorphism between the standard simplex and the convex hull representation. Specifically, we define the mapping

$$\begin{aligned} \sigma : \Delta^n &\to [v_0, \ldots, v_n] \\ (x_0, \ldots, x_n) &\mapsto \sum_{i=0}^{n} x_i v_i. \end{aligned}$$

In this context, the x_i's are called the **barycentric coordinates**. We use the notation $[v_0, \ldots, v_n]$ to refer to both the simplex and the map $\sigma : \Delta^k \to [v_0, \ldots, v_n]$.

In other words, the ordering of vertices does not affect the convex hull itself, as any reordering still spans the same geometric object. However, it is crucial in defining the map σ because we must specify where each standard basis element of Δ^n is mapped.

Definition 8.6. The points $v_0, \dots, v_n$ are said to be in **general position** if they do not lie in any $(n-1)$-dimensional affine subspace.

Equivalently, this means that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent.

> **Remark 8.7.** To see why these conditions are equivalent, note that $v_n - v_{n-1}, \dots, v_1 - v_0$ are linearly dependent if and only if $v_n - v_0, \dots, v_1 - v_0$ are linearly dependent. This, in turn, holds if and only if $v_n - v_0, \dots, v_1 - v_0$ lie in some $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$. This means that $v_0, \dots, v_n$ lie within an affine subspace of lower dimension, violating the definition of general position.

Example 8.8

Some examples

- v_0 is in general position if it is not the same point as any other vertex.
- v_0, v_1 are in general position if they are not collinear.
- v_0, v_1, v_2 are in general position if they are not coplanar.
- More generally, $n+1$ points are in general position if they do not lie in a lower-dimensional affine subspace.

Proposition 8.9. $v_0, \dots, v_n$ are in general position if and only if the map

$$\begin{aligned} \sigma : \Delta^n &\to [v_0, \dots, v_n] \\ (x_0, \dots, x_n) &\mapsto \sum_{i=0}^{n} x_i v_i. \end{aligned}$$

is a homeomorphism.

Proof. We prove each direction in turn.

- Proof of $(\Rightarrow)$.
  We need to show that σ is a homeomorphism. Since σ is already surjective and continuous, it suffices to show that σ is injective. This follows because Δ^n is compact, $[v_0, \dots, v_n]$ is a Hausdorff space, and a bijective continuous function between a compact and a Hausdorff space is a homeomorphism.

  Suppose $(x_0, \dots, x_n) \neq (x'_0, \dots, x'_n)$ such that

  $$\sum_{i=0}^{n} x_i v_i = \sum_{i=0}^{n} x'_i v_i.$$

  Define $y_i = x_i - x'_i$, so that

  $$\sum_{i=0}^{n} y_i v_i = 0.$$

Since the barycentric condition implies $\sum_{i=0}^n x_i = \sum_{i=0}^n x'_i = 1$, we obtain

$$y_0 = -(y_1 + \cdots + y_n).$$

Thus, rewriting the sum, we get

$$0 = \sum_{i=0}^{n} y_i v_i = y_1(v_1 - v_0) + \cdots + y_n(v_n - v_0).$$

Since $v_1 - v_0, \ldots, v_n - v_0$ are assumed to be linearly independent, this forces all y_i to be zero, contradicting our assumption that $(x_0, \ldots, x_n) \neq (x'_0, \ldots, x'_n)$. Thus, σ is injective, and hence a homeomorphism.

- Proof of ($\Leftarrow$).
  Suppose that $v_0, \ldots, v_n$ are not in general position. Then the vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly dependent. This means that there exist scalars $\alpha_0, \ldots, \alpha_{n-1}$, not all zero, such that

$$0 = \alpha_0(v_1 - v_0) + \cdots + \alpha_{n-1}(v_n - v_{n-1}) = -\alpha_0 v_0 + \sum_{i=0}^{n-2}(\alpha_i - \alpha_{i+1})v_{i+1} + \alpha_{n-1}v_n.$$

  Applying σ, we obtain

$$\sigma(-\alpha_0, \ldots, -\alpha_n) = 0.$$

  Since σ is assumed to be a homeomorphism, it must be injective, implying that all α_i must be zero, contradicting our assumption. Therefore, $v_0, \ldots, v_n$ must be in general position. $\square$

Definition 8.10. Given an n-simplex $[v_0, \ldots, v_n]$, if we remove one of the $(n+1)$-vertices and consider the convex hull of the remaining n-vertices, we obtain a **face** of $[v_0, \ldots, v_n]$, which we write as $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$. The vertices of a face of any subsimplex spanned by a subset of the vertices will always be ordered according to their order in the original simplex.

Note 8.11. A face of a simplex is obtained by removing one of its vertices while preserving the convex hull structure. This means that an n-dimensional simplex has $(n+1)$ faces, each of dimension $(n-1)$.

Example 8.12. For example:

- A triangle (2-simplex) has 3 edges (1-faces).
- A tetrahedron (3-simplex) has 4 triangular faces (2-faces).

The order of vertices matters because it maintains consistency in the orientation of the simplex.

Proposition 8.13

We define the inclusion map

$$\begin{aligned} \iota_j : \Delta^n &\to \Delta^{n+1} \\ \sum_{\alpha=0}^{n} t_\alpha e_\alpha &\mapsto \sum_{\alpha=0}^{j-1} t_\alpha e_\alpha + \sum_{\alpha=j+1}^{n+1} t_{\alpha-1} e_\alpha. \end{aligned}$$

Then we can write the face of the simplex $\sigma : \Delta^{n+1} \to [v_0, \dots, v_n]$ opposite to the vertex e_j as $\sigma \circ \iota_j : \Delta^n$.

Definition 8.14. We define the following.

- The union of the $(n-1)$-dimensional faces of $[v_0, \dots, v_n]$ as a set is the **boundary** of the simplex, denoted by $\partial([v_0, \dots, v_n])$.
- We define the **interior** of a simplex as

$$[v_0, \dots, v_n] \setminus \partial[v_0, \dots, v_n].$$

Note 8.15. In other words, the boundary of a simplex consists of all its lower-dimensional faces. Whereas, the interior of a simplex consists of all points that are strictly inside it, meaning they are not part of any of its faces.

Example 8.16

For example:

- the boundary of a line segment (1-simplex) consists of its two endpoints (the 0-simplices);
- the boundary of a triangle (2-simplex) consists of its three edges (1-simplices);
- the boundary of a tetrahedron (3-simplex) consists of its four triangular faces (2-simplices).

Definition 8.17. A **Δ-complex structure** on a space X is a collection of the maps $\sigma_\alpha : \Delta^n \to X$ satisfying the following conditions:

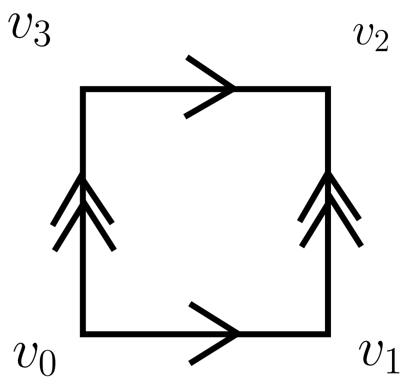
- $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)} : \mathrm{Int}(\Delta^n) \to X$ must be injective, and each point of X is in the image under σ_α of exactly one of these interiors.
- If σ_α is in the collection, then $\sigma_\alpha \circ \iota_j$ is also in the collection (where we identify a face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism that preserves the ordering of the vertices).
- If $U \subset X$ is open, then U is open if and only if $\sigma_\alpha(U)$ is open in Σ^n for all σ_α.

Note 8.18. A Δ-complex structure provides a way to construct topological spaces by gluing simplices together while preserving their combinatorial structure. The conditions ensure:

- Each point in X is uniquely represented inside the interior of a simplex.
- The structure is closed under taking faces, meaning that if a simplex is included, its faces are also included.
- The topology of X is compatible with the topology of simplices.

This concept generalises triangulations and is useful in homology, where we study how simplices fit together to form larger spaces.

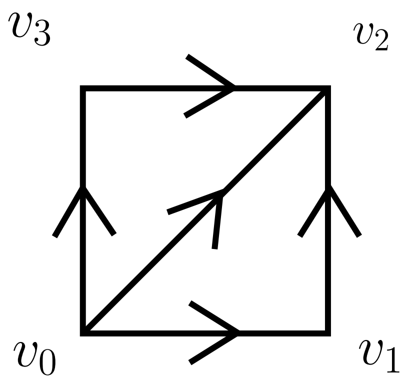
Example 8.19. The torus $S^1 \times S^1$ can be given a Δ-complex structure as follows. Recall that the torus can be represented as a square with opposite sides identified:



We define a Δ-complex structure on the unit square and compose it with the quotient map

$$\begin{aligned} q : [0,1]^2 &\to S^1 \times S^1 \\ (t_1, t_2) &\mapsto (e^{2\pi i t_1}, e^{2\pi i t_2}). \end{aligned}$$

Since a Δ-complex structure consists of simplices glued together, we must decompose the square into triangles. We do this by introducing a diagonal, splitting the square into two 2-simplices:



We define:

- 2-simplices: $[v_0, v_1, v_2]$ and $[v_0, v_3, v_2]$, which together form the entire square.
- 1-simplices: The edges $[v_0, v_1], [v_1, v_2], [v_2, v_3], [v_3, v_0], [v_0, v_2]$.

- 0-simplices: A single vertex, since all vertices are identified under the quotient.

> **Remark 8.20.** The order of $[\cdots]$ matters.

The quotient map q identifies edges according to the torus identification:

$$v_0 \sim v_3, \quad v_1 \sim v_2.$$

This means that when applying the quotient, we identify equivalent simplices.

Since there is no Δ-complex structure for $n \geq 3$ (as the torus is a 2-dimensional space), we limit our analysis to $n \leq 2$:

- For $n = 2$, we take the two 2-simplices:

$$q \circ [v_0, v_1, v_2], \quad q \circ [v_0, v_3, v_2].$$

- For $n = 1$, we consider the boundary edges:

$$q \circ [v_0, v_1], \quad q \circ [v_1, v_2], \quad q \circ [v_0, v_2].$$

- For $n = 0$, since all vertices are identified under the quotient, we have:

$$q \circ [v_0].$$

Using the definition of a Δ-complex, the face maps satisfy:

$$q \circ [v_0, v_1, v_2] \circ i_0 = q \circ [v_1, v_2], \quad q \circ [v_0, v_3, v_2] \circ i_0 = q \circ [v_3, v_2].$$

$$q \circ [v_0, v_1, v_2] \circ i_1 = q \circ [v_0, v_2], \quad q \circ [v_0, v_3, v_2] \circ i_1 = q \circ [v_0, v_2].$$

$$q \circ [v_0, v_1, v_2] \circ i_2 = q \circ [v_0, v_1], \quad q \circ [v_0, v_3, v_2] \circ i_2 = q \circ [v_0, v_3].$$

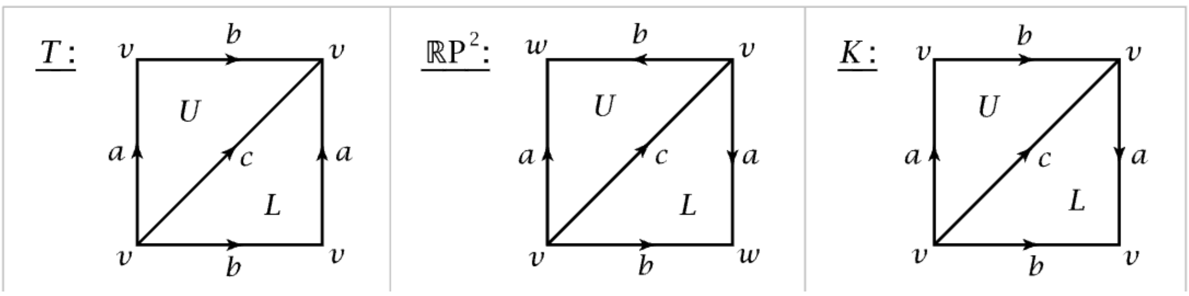
Since under the quotient we have $v_0 \sim v_3$ and $v_1 \sim v_2$, we obtain:

$$q \circ [v_0, v_1], \quad q \circ [v_0, v_2], \quad q \circ [v_1, v_2].$$

For $k = 0$, we only have $q \circ [v_0]$, since all vertices collapse into a single point under the quotient map.

This Δ-complex structure provides a way to study the torus using simplices. However, this decomposition is not unique, and different triangulations can be used to achieve the same result.

Example 8.21. Some more Δ-complexes.



Definition 8.22. Let X be a topological space with a Δ-complex structure. We define $\Delta_n(X)$ to be the free abelian group with basis the set of n-simplices in the Δ-complex structure of X, i.e.

$$\Delta_n(X) = \{\textstyle\sum_\alpha m_\alpha \sigma_\alpha^n : m_\alpha \in \mathbb{Z},\ \text{finitely many non-zero}, \sigma_\alpha^n \text{ is an } n\text{-simplex in the } \Delta\text{-complex structure}\}.$$

Note 8.23. The group $\Delta_n(X)$ is the collection of formal integer linear combinations of n-simplices in X.

Definition 8.25. A **free abelian group** on a set S is the group of all finite integer linear combinations of elements of S, written as

$$\mathbb{Z}[S] = \left\{\sum m_i s_i \mid m_i \in \mathbb{Z},\ s_i \in S,\ \text{finitely many } m_i \neq 0\right\}.$$

There are no relations between elements except commutativity and associativity.

A free abelian group is like a vector space over $\mathbb{Z}$, where elements are sums of basis elements with integer coefficients. For example, if $S = \{a, b\}$, then elements are expressions like $3a - 2b$. The group $\mathbb{Z}^n$ is a free abelian group with basis $\{e_1, \ldots, e_n\}$, and in homology, the chain group $\Delta_n(X)$ is a free abelian group where simplices form the basis.

Definition 8.26. We define the **boundary operator** on $\Delta_n(X)$ as $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by setting it on the generators as

$$\partial_n(\sigma_\alpha^n) = \sum_{j=0}^{n} (-1)^j (\sigma_\alpha^n \circ \iota_j) = \sum_{j=0}^{n} (-1)^j \sigma_\alpha^n|_{[e_0,\ldots,\widehat{e}_j,\ldots,e_n]}.$$

Note 8.27. The boundary of an n-simplex is the formal sum of its $(n-1)$-dimensional faces.

Example 8.28

(Note that in this example we are using $[v_0, \ldots, v_k]$ to indicate the map $\sigma : \Delta^k \to [v_0, \ldots, v_k]$.)

- $\partial_1([v_0, v_1]) = [v_1] - [v_0]$,
- $\partial_1([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$,
- $\partial_1([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$.

Lemma 8.29

$\partial_{n-1} \circ \partial_n = 0$.

Proof. We use the definition of ∂_n.

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i\sigma_{[v_0,\dots,\widehat{v}_i,\dots,v_n]}\right)\\ &= \sum_{j<i}(-1)^i(-1)^j\sigma_{[v_0,\dots,\widehat{v}_j,\dots,\widehat{v}_i,\dots,v_n]} + \sum_{j>i}(-1)^i(-1)^{j-1}\sigma_{[v_0,\dots,\widehat{v}_i,\dots,\widehat{v}_j,\dots,v_n]}\\ &= 0.\end{aligned}$$

□

Definition 8.30. A **chain complex** of abelian groups is a sequence (or equivalently a diagram $(C_\bullet, \partial_\bullet)$ of the form)

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where C_i are abelian groups and $\partial_n : C_n \to C_{n-1}$ are homomorphisms such that $\partial_{n-1} \circ \partial_n = 0$.

> **Remark 8.31.** In this case the abelian groups C_n are Δ_n and the homomorphisms are the boundary operators.

Definition 8.32. We call ∂_n the **boundary homomorphism**, and the elements of C_n are called n**-chains**.

Definition 8.33. The n**-th simplicial homology group** of X is defined as the abelian quotient group:

$$H_n^{\Delta}(X) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}.$$

Remark 8.34. We can take this quotient since $\partial_n \circ \partial_{n+1} = 0$, so $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$.

Example 8.35. Let us consider the Δ-complex structure of the torus from the previous example. The simplices in each dimension are:

- For $n = 2$ (2-simplices):

$$q \circ [v_0, v_1, v_2], \quad q \circ [v_0, v_3, v_2].$$

- For $n = 1$ (1-simplices):

$$q \circ [v_1, v_2], \quad q \circ [v_0, v_2], \quad q \circ [v_0, v_1].$$

- For $n = 0$ (0-simplices):

$$q \circ [v_0].$$

Thus, the chain complex takes the form:

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0.$$

Applying the definition of the boundary operator:

$$\partial_2(q \circ [v_0, v_1, v_2]) = q \circ [v_1, v_2] - q \circ [v_0, v_2] + q \circ [v_0, v_1].$$

$$\partial_2(q \circ [v_0, v_3, v_2]) = q \circ [v_3, v_2] - q \circ [v_0, v_2] + q \circ [v_0, v_3].$$

Thus, the boundary map ∂_2 can be written in matrix form as:

$$\partial_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Similarly, for the 1-simplices:

$$\partial_1(q \circ [v_0, v_2]) = q \circ [v_2] - q \circ [v_0] = 0.$$

$$\partial_1(q \circ [v_1, v_2]) = q \circ [v_2] - q \circ [v_0] = 0.$$

$$\partial_1(q \circ [v_0, v_1]) = q \circ [v_1] - q \circ [v_0] = 0.$$

Thus, $\partial_1 = 0$.

Now we compute homology groups H_n^Δ by finding the kernels and images of the boundary maps.

- For $H_2^\Delta(T^2) = \ker(\partial_2)/\operatorname{im}(\partial_3)$:
  The kernel of $\partial_2 : \mathbb{Z}^2 \to \mathbb{Z}^3$ consists of vectors satisfying:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

  Solving, we get $x = -y$, so the kernel is generated by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $\operatorname{im}(\partial_3) = 0$, we conclude $H_2^\Delta(T^2) \cong \mathbb{Z}$.

- For $H_1^\Delta(T^2) = \ker(\partial_1)/\operatorname{im}(\partial_2)$:
  Since $\partial_1 = 0$, the kernel is all of $\mathbb{Z}^3$, and the image of ∂_2 is generated by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

> **Remark 8.36.** $\operatorname{im}(\partial_2)$ consists of vectors of the form $\begin{bmatrix} x+y \\ -x-y \\ x+y \end{bmatrix}$ from $\partial_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, showing that setting $x = 1, y = 0$ or $x = 0, y = 1$ both yield $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, proving $\operatorname{im}(\partial_2)$ is generated by $\langle (1, -1, 1) \rangle$.

  The quotient:

$$H_1^\Delta(T^2) \cong \mathbb{Z}^3 / \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle \cong \mathbb{Z}^2.$$

  (Since $\mathbb{Z}^3$ has three independent generators, but the relation $(1, -1, 1) = 0$ removes one degree of freedom, the quotient is a free abelian group of with 2 generators).

- For $H_0^\Delta(T^2) = \ker(\partial_0)/\operatorname{im}(\partial_1)$:
  Since ∂_0 is trivial, and $\operatorname{im}(\partial_1) = 0$, we get:

$$H_0^\Delta(T^2) \cong \mathbb{Z}.$$

In conclusion,

$$H_n^{\Delta}(T^2) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n > 3. \end{cases}$$

Remark 8.37. To compute the quotient it is sometimes best to use the structure theorem of finitely generated abelian groups via the Smith normal form (see Appendix).

Exam Questions 8.38 (Exercise 1.1.)

Compute the simplicial homology for the cylinder, circle, Möbius strip, and the Klein bottle.

To do

8.2 Singular Homology

Simplicial homology provides a useful algebraic structure to study topological spaces, but it has some limitations:

- The group $\Delta_n(X)$ depends on the choice of Δ-complex. Even though $H_n^{\Delta}(X)$ does not ultimately depend on this choice, this is not immediately obvious.
- Simplicial homology is not obviously *functorial*, meaning that it is unclear how a continuous map between topological spaces induces a map on homology groups. A homeomorphism of spaces does not necessarily map the simplices of one Δ-complex to simplices in another.

To address these issues, we define *singular homology*, which does not rely on a chosen Δ-complex and allows us to define homology in a more general setting. We will later prove that simplicial and singular homology are equivalent.

Definition 8.39. A **singular n-simplex** in a space X is a continuous map

$$\sigma : \Delta^n \to X.$$

Note 8.40. Unlike simplicial homology, where we use a predefined triangulation, singular homology allows any continuous map from an n-simplex into X.

Remark 8.41. We only require the map $\sigma : \Delta^n \to X$ to be continuous, so we are not assuming that the image is nicely embedded in X. For example, the image of σ may self-intersect or be stretched and bent.

Definition 8.42. Given a topological space X, we define $C_n(X)$ to be the *free abelian group* with basis given by the set of singular n-simplices in X, i.e.

$$C_n(X) = \left\{ \sum_{\alpha} m_\alpha \sigma_\alpha^n \mid m_\alpha \in \mathbb{Z}, \text{ finitely many nonzero}, \sigma_\alpha^n \text{ singular } n\text{-simplex in } X \right\}.$$

The elements of $C_n(X)$ are called n-**chains.**

Note 8.43. The group $C_n(X)$ consists of formal sums of singular simplices with integer coefficients. This allows us to define an algebraic structure that captures the topology of X. Unlike simplicial homology, which relies on a fixed triangulation, singular homology considers *all* possible continuous simplices in X, making $C_n(X)$ 'larger' than $\Delta_n(X)$.

Definition 8.44. We define the **boundary operator** on $C_n(X)$ as $\partial_n : C_n(X) \to C_{n-1}(X)$, defined on the generators by

$$\partial_n(\sigma_\alpha^n) = \sum_{j=0}^{n}(-1)^j(\sigma_\alpha^n \circ \iota_j) = \sum_{j=0}^{n}(-1)^j \sigma_\alpha^n\big|_{[e_0,\ldots,\widehat{e}_j,\ldots,e_n]}.$$

Note 8.45. The boundary operator ∂_n in singular homology is defined similarly to simplicial homology.

Lemma 8.46

$\partial_{n-1} \circ \partial_n = 0$.

Proof. We must show that applying the boundary operator twice always results in zero.

Expanding the boundary operator:

$$\partial_n(\sigma_\alpha^n) = \sum_{i=0}^{n}(-1)^i \sigma_\alpha^n \circ \iota_i.$$

Applying ∂_{n-1} to both sides,

$$\partial_{n-1}(\partial_n(\sigma_\alpha^n)) = \sum_{i=0}^{n}(-1)^i \partial_{n-1}(\sigma_\alpha^n \circ \iota_i).$$

Expanding ∂_{n-1},

$$\partial_{n-1}(\sigma_\alpha^n \circ \iota_i) = \sum_{j=0}^{n-1}(-1)^j(\sigma_\alpha^n \circ \iota_i) \circ \iota_j.$$

Substituting this,

$$\sum_{i=0}^{n}(-1)^i \sum_{j=0}^{n-1}(-1)^j(\sigma_\alpha^n \circ \iota_i) \circ \iota_j.$$

Rearranging the sum,

$$\sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}(\sigma_\alpha^n \circ \iota_i) \circ \iota_j.$$

Each face appears twice with opposite signs, canceling out, proving that $\partial_{n-1} \circ \partial_n = 0$. □

Shorten proof

Definition 8.47. The n**-th singular homology group** of X is the quotient group:

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}.$$

Definition 8.48. We call elements of $Z_n(X) = \ker(\partial_n)$ **cycles** and elements of $B_n(X) = \operatorname{im}(\partial_{n+1})$ **boundaries**.

Note 8.49. The homology group $H_n(X)$ classifies n-cycles (chains with no boundary) up to boundaries of higher-dimensional simplices. This measures the true topological features of X, such as:

- $H_0(X)$ counts connected components.
- $H_1(X)$ detects loops or holes.
- $H_2(X)$ captures enclosed voids, like cavities inside a torus.

8.3 Functoriality of Singular Homology

One of the key advantages of singular homology over simplicial homology is that it is *functorial*, meaning that continuous maps between spaces induce homomorphisms on homology groups. This property ensures that homology is an intrinsic invariant of a topological space, independent of any specific triangulation.

Definition 8.50. Let $f : X \to Y$ be a continuous map between topological spaces. The **induced map on chains** is the group homomorphism

$$f_\# : C_n(X) \to C_n(Y)$$

defined on singular simplices by

$$f_\# \left(\sum_\alpha m_\alpha \sigma_\alpha \right) = \sum_\alpha m_\alpha (f \circ \sigma_\alpha).$$

Note 8.51. This definition simply says that if we have a singular simplex $\sigma : \Delta^n \to X$, then applying f gives another singular simplex $f \circ \sigma : \Delta^n \to Y$. The map $f_\#$ extends linearly to chains, ensuring that every cycle in X is mapped to a corresponding cycle in Y.

Lemma 8.52

$$\partial f_\# = f_\# \partial.$$

Proof.

To do

□

Corollary 8.53. We have the following.

- Since $f_\#$ commutes with the boundary operator, it follows that:

  $$f_\#(\ker(\partial_n^X)) \subseteq \ker(\partial_n^Y),$$

  meaning cycles in X map to cycles in Y.

- Similarly,
$$f_\#(\mathrm{im}(\partial_{n+1}^X)) \subseteq \mathrm{im}(\partial_{n+1}^Y),$$
meaning boundaries in X map to boundaries in Y.

Corollary 8.54

Thus, $f_\#$ descends to a well-defined map on homology groups:
$$\begin{aligned} f_* : H_n(X) &\to H_n(Y) \\ c + B_n(X) &\mapsto f_\#(c) + B_n(Y). \end{aligned}$$

Lemma 8.55. f_* is well-defined.

Proof. Let $c_1, c_2 \in \ker(\partial_n)$ such that $c_1 + B_n(X) = c_2 + B_n(X)$, meaning $c_1 - c_2 \in B_n(X)$. Since $f_\#$ respects boundaries, we have:
$$B_n(Y) = f_\#(B_n(X)) = f_\#(c_1 - c_2) + B_n(Y).$$
Thus,
$$f_\#(c_1) - f_\#(c_2) \in B_n(Y),$$
so
$$f_\#(c_1) + B_n(Y) = f_\#(c_2) + B_n(Y),$$
proving that f_* is well-defined. □

Proposition 8.56

f_* is **functorial**, meaning that for any continuous maps $f : X \to Y$ and $g : Y \to Z$, we have:
$$(f \circ g)_* = f_* \circ g_*.$$

Proof. To show $(f \circ g)_* = f_* \circ g_*$, take any $c + B_n(Z) \in H_n(Z)$. Then:
$$(f \circ g)_*(c + B_n(Z)) = (f \circ g)_\#(c) + B_n(X),$$
$$(f_* \circ g_*)(c + B_n(Z)) = (f_\# \circ g_\#)(c) + B_n(Y).$$
Thus, it suffices to show that $(f \circ g)_\# = f_\# \circ g_\#$. For any chain $\sum_\alpha m_\alpha \sigma_\alpha \in C_n(Z)$,
$$(f \circ g)_\# \left(\sum_\alpha m_\alpha \sigma_\alpha \right) = \sum_\alpha m_\alpha (f \circ g)_\#(\sigma_\alpha)$$
$$= \sum_\alpha m_\alpha (f \circ g \circ \sigma_\alpha) = \sum_\alpha m_\alpha f_\#(g_\#(\sigma_\alpha))$$
$$= (f_\# \circ g_\#) \left(\sum_\alpha m_\alpha \sigma_\alpha \right),$$
proving $(f \circ g)_\# = f_\# \circ g_\#$.

Next, we show that the identity map is preserved, i.e., $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$. For any $c + B_n(X) \in H_n(X)$,

$$(\mathrm{id}_X)_*(c + B_n(X)) = (\mathrm{id}_\#)(c) + B_n(X) = c + B_n(X),$$

since for any chain,

$$(\mathrm{id}_\#)\left(\sum_\alpha m_\alpha \sigma_\alpha\right) = \sum_\alpha m_\alpha (\mathrm{id} \circ \sigma_\alpha) = \sum_\alpha m_\alpha \sigma_\alpha.$$

Thus, $(\mathrm{id}_X)_*$ acts as the identity on homology. □

Corollary 8.57. If X and Y are homeomorphic, then:

$$H_n(X) \cong H_n(Y) \text{ for all } n.$$

Note 8.58. Homology groups are topological invariants, meaning they do not change under homeomorphisms.

Proof. Let $f : X \to Y$ be a homeomorphism with inverse $g : Y \to X$. Then:

$$\begin{aligned} g_* \circ f_* &= (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)} \\ f_* \circ g_* &= (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}_{H_n(Y)} . \end{aligned}$$

Thus, f_* and g_* are inverses, proving $H_n(X) \cong H_n(Y)$. □

Proposition 8.59

$$H_n(\text{point}) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n > 0. \end{cases}$$

Note 8.60. A point has one connected component, so $H_0 = \mathbb{Z}$, but has no higher-dimensional features, so $H_n = 0$ for $n > 0$.

Proof. For a single point $\{pt\}$, there is a unique singular simplex $\sigma^n : \Delta^n \to \{pt\}$, so $C_n(\{pt\}) \cong \mathbb{Z}$ for all n. Moreover, the boundary operator acts as:

$$\partial_n(m\sigma^n) = \sum_{j=0}^{n} m(-1)^j \sigma^{n-1}.$$

Since there is only one simplex σ^{n-1}, this sum is:

$$\partial_n(m\sigma^n) = m \sum_{j=0}^{n} (-1)^j \sigma^{n-1}.$$

For $n > 0$, this sum is zero if n is odd and identity if n is even, showing that ∂_n is either trivial or surjective. This leads to:

$$H_n(\{pt\}) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n > 0. \end{cases}$$

□

8.4 Path connectedness and zeroth homology

Proposition 8.61. Let $X = \bigcup_i X_i$ where each X_i is a path-connected component of X. Since simplices are path-connected, their images must lie entirely within a single X_i. This ensures that the chain complex decomposes as $C_n(X) = \bigoplus_i C_n(X_i)$, and the boundary maps respect this decomposition. Consequently,

$$H_n(X) \cong \bigoplus_i H_n(X_i).$$

Note 8.62. Homology respects disjoint unions, so the homology of X is the direct sum of the homology of its connected components.

Proposition 8.63

X is path-connected if and only if $H_0(X) \cong \mathbb{Z}$.

Note 8.64. $H_0(X)$ measures the number of connected components. If X is path-connected, all points are related by a path, meaning all 0-chains are equivalent modulo boundaries, leaving a single generator.

Proof. We prove each direction in turn.

- Proof of ($\Rightarrow$).
  A map from the 0-simplex to X is simply a choice of a point in X, denoted by $[x]$. If X is path-connected, for any $x, x' \in X$, there exists a singular path $\sigma : \Delta^1 \to X$ with $\partial\sigma = [x'] - [x]$, treating Δ^1 as $[0, 1]$.

  Any 0-chain in X is of the form $\sum_\alpha m_\alpha[x_\alpha]$ with $m_\alpha \in \mathbb{Z}$. Choosing any $x_0 \in X$, path-connectedness ensures the existence of paths $\sigma_\alpha : \Delta^1 \to X$ such that $\partial\sigma_\alpha = [x_\alpha] - [x_0]$, giving:

$$\sum_\alpha m_\alpha[x_\alpha] - \sum_\alpha m_\alpha[x_0] = \sum_\alpha([x_\alpha] - [x_0]) = \sum_\alpha m_\alpha\partial\sigma^1_\alpha = \partial\left(\sum_\alpha m_\alpha\sigma^1_\alpha\right).$$

  Thus, modulo $B_1(X)$, every 0-chain is a multiple of $[x_0]$, yielding the isomorphism:

$$H_0(X) = Z_0(X)/B_1(X) \cong \mathbb{Z}, \quad m[x_0] \mapsto m.$$

  Since $Z_0(X) = C_0(X)$, we obtain $H_0(X) \cong \mathbb{Z}$.

- Proof of ($\Leftarrow$).
  If X is not path-connected, it decomposes as $X = \bigcup_i X_i$, where each X_i is a path component. By the earlier remark,

$$H_0(X) \cong \bigoplus_i H_0(X_i).$$

  Since each $H_0(X_i) \cong \mathbb{Z}$, we conclude $H_0(X) \ncong \mathbb{Z}$, proving the other direction by contrapositive.

□

8.5 Reduced Homology Groups

It is often convenient to define a variation of homology so that a point has a trivial homology group in all dimensions, including zero.

Definition 8.65. The **reduced homology groups**, denoted $\widetilde{H}_n(X)$, are defined as the homology groups of the **augmented chain complex**:

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where the **augmentation map** is given by

$$\varepsilon : C_0(X) \to \mathbb{Z}$$
$$\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i.$$

The **reduced homology groups** are then defined as:

$$\widetilde{H}_n(X) = \begin{cases} H_n(X), & n > 0, \\ \ker(\varepsilon)/\operatorname{im}(\partial_1), & n = 0. \end{cases}$$

Note 8.66. Ordinary homology assigns $H_0(X) \cong \mathbb{Z}$ to any connected space, regardless of its topology. Reduced homology removes this redundancy, ensuring that a single point has trivial homology. The augmentation map sums up all 0-chains, treating them as a single entity, and reduced homology captures the remaining structure.

Corollary 8.67

$$H_n(X) \cong \widetilde{H}_n(X)$$

for $n \geq 1$.

Remark 8.68. Reduced homology is well-defined because $\operatorname{im}(\partial_1) \subseteq \ker(\varepsilon)$, ensuring that the quotient $\ker(\varepsilon)/\operatorname{im}(\partial_1)$ makes sense.

Proof. To see this, consider any 1-chain $c = \sum_\alpha m_\alpha\sigma_\alpha$. Applying ε to its boundary,

$$\varepsilon(\partial_1 c) = \varepsilon\left(\sum_\alpha m_\alpha\sigma_\alpha|_{e_1} - \sum_\alpha m_\alpha\sigma_\alpha|_{e_0}\right),$$

we get:

$$\varepsilon\left(\sum_\alpha m_\alpha\sigma_\alpha|_{e_1}\right) - \varepsilon\left(\sum_\alpha m_\alpha\sigma_\alpha|_{e_0}\right) = \sum_\alpha m_\alpha - \sum_\alpha m_\alpha = 0.$$

Thus, $\operatorname{im}(\partial_1) \subseteq \ker(\varepsilon)$, making reduced homology well-defined. □

Definition 8.69. A sequence of abelian groups and homomorphisms:

$$\cdots \xrightarrow{p_{n+1}} A_n \xrightarrow{p_n} A_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} A_0 \to 0$$

is *exact* if:

$$\ker(p_n) = \operatorname{im}(p_{n+1}) \quad \text{for all } n.$$

Corollary 8.70

Some important consequences:

- Exactness implies $\ker(p_n)/\operatorname{im}(p_{n+1}) = 0$ for all n.
- If $p_{n+1} = 0$, then $\operatorname{im}(p_{n+1}) = \ker(p_n)$, meaning p_n is injective.
- The converse is also true: injectivity of p_n means $\operatorname{im}(p_{n+1}) = \ker(p_n)$.

Proposition 8.71. There is a short exact sequence relating ordinary and reduced homology:

$$0 \to \widetilde{H}_0(X) \to H_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Proof. Since $\ker(\varepsilon) = \widetilde{H}_0(X)$, the short exact sequence follows directly from the definition. □

Corollary 8.72

$$H_0(X) \cong \mathbb{Z} \oplus \widetilde{H}_0(X).$$

Proof. We define the explicit isomorphism:

$$\begin{aligned} f : \widetilde{H}_0(X) \oplus \mathbb{Z} &\to H_0(X) \\ (\widetilde{c} + B_1(X), n) &\mapsto \widetilde{c} + nc + B_1(x) \end{aligned}$$

where $\varepsilon(c) = 1$. Then the map is injective since

$$f(B_1(X), 0) = B_1(X)$$

and it is surjective, sicne for any $c_0 \in C_0(X)$, we have that $\varepsilon(c_1) = m$ for some $m \in \mathbb{Z}$. Then $\varepsilon(c_1 - mc) = m - m = 0$ therefore, $c_1 - mc + B_1(X) \in \widetilde{H}_0(X)$. Hence, for any $[c_1] \in H_0(X)$ we have $f([c_1 - mc], m) = [c_1]$ for $m = \varepsilon(c_1)$. □

Example 8.73. For a single point, we have:

$$\widetilde{H}_n(\{pt\}) = 0 \quad \text{for all } n.$$

8.6 Short exact sequences

Definition 8.74. A sequence of abelian groups:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

where α is injective and β is surjective, is called a **short exact sequence**.

> **Remark 8.75.** In every short exact sequence, the map β induces an isomorphism:
>
> $$C \cong B/\operatorname{im}(\alpha).$$
>
> Alternatively, a short exact sequence can be defined as an exact sequence where only three terms are nonzero.

Definition 8.76. A short exact sequence of abelian groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

splits if there exists a homomorphism $s : C \to B$ such that $\beta \circ s = \mathrm{id}_C$, meaning s is a right inverse of β. Equivalently, the sequence splits if $B \cong A \oplus C$ as abelian groups.

Example 8.77. The sequence:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is a short exact sequence, but it does not split.

8.7 Good pairs

We aim to compute homology groups of quotient spaces X/A, where (X, A) has special properties ensuring homology behaves predictably.

Definition 8.78. Let X be a topological space and $A \subset X$ such that A is a deformation retract of some neighbourhood in X. Then, (X, A) is called a **good pair**.

Note 8.79. A good pair ensures that A is "homotopically simple" within X, meaning we can continuously shrink X onto A without disrupting homology computations.

Example 8.80

For example

- $(\mathbb{D}^n, \partial\mathbb{D}^n)$ is a good pair.
- If X is a CW complex and A is a subcomplex, then (X, A) is a good pair.
- If $X = \mathbb{R}$ and A is the Cantor set, then (X, A) is **not** a good pair. The Cantor set is compact but not locally compact, meaning it cannot be a retract of any open neighbourhood (since open subsets of $\mathbb{R}$ are locally connected, and retractions preserve local connectedness).

Some more to do, it is lecture 17 and 18

Theorem 8.81
Given a short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "i"] & A_n \arrow[r] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "j"] & B_n \arrow[r] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

there exists a long exact sequence of homology

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A_\bullet) \arrow[r, "i_*"] & H_n(B_\bullet) \arrow[r, "j_*"] & H_n(C_\bullet) \arrow[dll, "\partial"] & & \\
 & H_{n-1}(A_\bullet) \arrow[r, "i_*"] & H_{n-1}(B_\bullet) \arrow[r, "j_*"] & H_{n-1}(C_\bullet) \arrow[dll, dashed, "\partial"] & & \\
 & H_0(A_\bullet) \arrow[r] & H_0(B_\bullet) \arrow[r] & H_0(C_\bullet) \arrow[d] & & \\
 & & & 0 & &
\end{tikzcd}$$

where ∂ is called the **connecting homomorphism / boundary homomorphism** (defined in the proof).

Remark 8.82. We often write a short exact sequence of complexes as

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$$

Proof. We will prove this by a technique called **diagram chasing**.

$$\begin{tikzcd}
& & 0 \arrow[d] & \\
& \cdots \arrow[r] & a_{n-1} \arrow[r, "i"] \arrow[d] & \cdots \\
\cdots \arrow[r] & b_n \arrow[r] \arrow[d, "j"] & \partial b_n \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & c_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
& 0 & 0 &
\end{tikzcd}$$

We define the **boundary homomorphism**.

$$\begin{aligned} \partial : H_n(C_\bullet) &\to H_{n-1}(A_\bullet) \\ [c_n] &\mapsto \partial[c_n]. \end{aligned}$$

There are six things to verify (which altogether show that it is a long exact sequence):

$$\begin{gathered} \operatorname{im}(i_*) \subset \ker(j_*), \quad \operatorname{im}(j_*) \subset \ker(i_*), \quad \operatorname{im}(\partial) \subset \ker(i_*), \\ \ker(j_*) \subset \operatorname{im}(i_*), \quad \ker(\partial) \subset \operatorname{im}(j_*), \quad \ker(i_*) \subset \operatorname{im}(\partial). \end{gathered}$$

These six statements altogether prove the exactness of the long exact sequence.

Firstly, we need to show that the boundary homomorphism is well-defined (i.e. we need to show that it does not depend on any choices).

Take any $[c_n] \in H_n(C_*)$ and a representative c_n of the homology class ($c_n \in C_n$). Since it represents a homology class, $\partial c_n = 0$. Since the vertical columns are exact (j is surjective), there exists $b_n \in B_n$ such that $jb_n = c_n$. Also, by commutativity, we have

$$j\partial b_n = \partial(jb_n) = \partial c_n = 0.$$

Therefore, $\partial b_n \in B_{n-1}$ is the kernel of j, hence by exactness b_n must be in the image of i, i.e., there exists $a_{n-1} \in A_{n-1}$ such that $ia_{n-1} = \partial b_n$. Again, by commutativity, we have that

$$i\partial a_{n-1} = \partial i a_{n-1} = \partial\partial b_n = 0.$$

Hence, by injectivity of i, we have that $i\partial a_{n-1}$ implies $\partial a_{n-1} = 0$. In conclusion, we define the connecting homomorphism $\partial([c_n]) = [a_{n-1}]$. We have made two choices in this construction:

- We have chosen b_n such that $jb_n = c_n$.
- We have chosen a representative of $[c_n] \in H_n(C_*)$.

Let us justify why the first choice does not matter. Pick a different $b_n' \in B_n$ such that $jb_n' = c_n$. Following through with the same reasoning, we would obtain $a_{n-1}' \in A_{n-1}$ such that $ia_{n-1}' = \partial b_n'$. Since i is injective, there is a unique preimage of ∂b_n under i, so we have

$$jb_n' = jb_n = c_n.$$

hence $j(b_n - b'_n) = jb_n - jb'_n = c_n - c_n = 0$. So $b_n - b'_n$ is in the kernel of j, so it is in the image of i. Then there exists $a_n \in A_n$ such that $ia_n = b_n - b'_n$. We then note that

$$i(\partial a_n) = \partial(ia_n) = \partial(b_n - b'_n) = ia_{n-1} - ia'_{n-1} = i(a_{n-1} - a'_{n-1}).$$

By injectivity of i, we have that $\partial a_n = a_{n-1} - a'_{n-1}$, meaning that they must belong to the same homology class (since their difference is a boundary).

Next, we show exactness:

- $\ker(i_*) \subset \operatorname{im}(\partial)$. Assume $[a_n] \in \ker(i_*)$, then ia_n is the boundary of something, i.e. there exists $b_{n+1} \in B_{n+1}$ such that $\partial b_{n+1} = ia_n$. By exactness of the columns, we have that $ia_n \in \ker(j)$, so by commutativity,

  $$\partial jb_{n+1} = j\partial b_{n+1} = jia_n = 0.$$

  Hence, $[jb_{n+1}] \in H_{n+1}(C_*)$, therefore $[a_n] = \partial([jb_{n+1}])$.

- $\operatorname{im}(\partial) \subset \ker(i_*)$. Let $[a_n] \in \operatorname{im}(\partial)$, so there exists $[c_{n+1}] \in H_{n+1}(C_*)$ such that $\partial([c_{n+1}]) = [a_n]$. We want to show that $i_*([a_n]) = [ia_n] = 0$. It is obvious that there exists $b_{n+1} \in B_{n+1}$ such that $jb_{n+1} = c_{n+1}$ (by surjectivity of j), and therefore $ia_n = \partial b_{n+1}$.

- $\ker(j_*) \subset \operatorname{im}(i_*)$. Let $[b_n] \in \ker(j_*)$ then $j_*([b_n]) = [jb_n] = 0$, so it must equal a boundary. There exists $c_{n+1} \in C_{n+1}$ such that $\partial c_{n+1} = jb_n$. Since j is surjective, there exists $b_{n+1} \in B_{n+1}$ such that $\partial c_{n+1} = jb_n$.

- $\ker(\partial) \subset \operatorname{im}(j_*)$. Using the same notation, if $[c_n]$ represents a homology class in $\ker(\partial)$, then $a_{n-1} = \partial a_n$ for $a_n \in A_n$. The element $b_n - ia_n$ is a cycle since

  $$\partial(b_n - ia_n) = \partial b_n - \partial ia_n = \partial b_n - i\partial a_n = 0.$$

  and

  $$j(b_n - ia_n) = jb_n - jia_n = jb_n = c_n.$$

  So j_* maps $[b_n - ia_n]$ to $[c_n]$, i.e. $j_*([b_n - ia_n]) = [c_n]$.

□

Example 8.83. We want to use the above result to calculate the homology of S^n. To do so, we use the fact that homology and reduced homology are homotopy invariant: we have not proved this yet, but we will. Recall, $\mathbb{D}^n$ is contractible, then $\widetilde{H}_n(\mathbb{D}^n) \cong \widetilde{H}_n(\text{pt}) \cong 0$ and

$$H_n(\mathbb{D}^n) \cong H_n(\{\text{pt}\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We consider a long exact sequence as above with $X = \mathbb{D}^n$, $A = \partial\mathbb{D}^n \cong S^{n-1}$, and

$X/A = \mathbb{D}^n/\partial\mathbb{D}^n \cong S^n$.

$$\begin{tikzcd}
 & & 0 \arrow[d, dash, "\cong"'] & & & \\
\cdots \arrow[r] & \widetilde{H}_k\left(S^{n-1}\right) \arrow[r, "\iota_*"] & \widetilde{H}_k(D^n) \arrow[r, "q_*"] & \widetilde{H}_k\left(S^n\right) \arrow[dl, "\partial"] & 0 \arrow[dl, dash, "\cong"'] & \\
 & & \widetilde{H}_{k-1}\left(S^{n-1}\right) \arrow[r] & \widetilde{H}_{k-1}(D^n) \arrow[r] & \widetilde{H}_{k-1}\left(S^n\right) \arrow[dl, dashed, "\partial"] & \\
 & & & \widetilde{H}_0\left(S^{n-1}\right) \arrow[r] & \widetilde{H}_0\left(D^n\right) \arrow[r] \arrow[d, dash, "\cong"] & \widetilde{H}_0\left(S^n\right) \arrow[d] \\
 & & & & 0 & 0
\end{tikzcd}$$

Therefore, $\operatorname{im}(q_*) = \ker(\partial) = 0$, meaning that ∂ is injective, which implies that:

$$\operatorname{im}(\partial) \cong \widetilde{H}_k(S^n).$$

Moreover, by exactness:

$$\operatorname{im}(\partial) = \ker(i_*) = \widetilde{H}_k(S^{n-1}),$$

so ∂ is surjective, and therefore it is an isomorphism. Thus, we know that for all $k > 0$,

$$\widetilde{H}_k(S^n) \cong \widetilde{H}_k(S^{n-1}),$$

and $\widetilde{H}_0(S^n) = 0$ for $n > 0$. For $S^0 = \{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$, we have

$$H_k(S^0) \cong \begin{cases} 0 & \text{if } k > 0, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = 0. \end{cases}$$

and

$$\widetilde{H}_k(S^0) \cong \begin{cases} 0 & \text{if } k > 0, \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$

since the short exact sequence:

$$0 \to \widetilde{H}_0(S^0) \to H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z} \to 0$$

splits.

> **Note 8.84.** The sphere S^0 consists of two disconnected points, which gives the direct sum of two copies of $\mathbb{Z}$ in degree 0. However, the reduced homology removes one $\mathbb{Z}$ component, leaving only a single $\mathbb{Z}$.

Thus, we can inductively deduce:

$$\mathbb{Z} \cong \widetilde{H}_0(S^0) \cong \ldots \cong \widetilde{H}_k(S^k) \cong \ldots$$

and $\widetilde{H}_k(S^n) \cong 0$ if $k \neq n$ (also using the fact that $\widetilde{H}_0(S^k) = 0$ for all $k > 0$ by path-connectedness). Hence, we conclude:

$$\widetilde{H}_k(S^n) \cong \begin{cases} 0 & \text{if } k \neq n, \\ \mathbb{Z} & \text{if } k = n. \end{cases}$$

From this, we can clearly get the regular homology for $n > 0$,

$$H_k(S^n) \cong \begin{cases} 0 & \text{if } k \neq n, 0, \\ \mathbb{Z} & \text{if } k = n, 0. \end{cases}$$

8.8 Relative Homology Groups

Let us return to topological spaces. With the machinery of exact sequences, we can now introduce a crucial extension of homology, called *relative homology*, which helps in studying spaces with a subspace.

Definition 8.85. Given a topological space X and a subspace $A \subset X$, we define the **relative chains** as the quotient chain groups:

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}.$$

Note 8.86. Relative chains measure how X differs from A. Elements of $C_n(X)$ that belong to $C_n(A)$ are considered trivial, meaning we only focus on the parts of X that are "not i"n A.

Proposition 8.87

Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ sends $C_n(A)$ to $C_{n-1}(A)$, it induces a well-defined map on the quotient:

$$\begin{aligned} \bar{\partial}_n : C_n(X, A) &\to C_{n-1}(X, A), \\ c + C_n(A) &\mapsto \partial_n(c) + C_{n-1}(A). \end{aligned}$$

Exam Questions 8.88 (Exercise)

Show that $\bar{\partial}_n$ is well-defined and satisfies $\bar{\partial}_{n-1} \circ \bar{\partial}_n = 0$.

To do

Definition 8.89. Thus, we obtain a new chain complex:

$$\cdots \xrightarrow{\bar{\partial}_{n+1}} C_n(X, A) \xrightarrow{\bar{\partial}_n} C_{n-1}(X, A) \xrightarrow{\bar{\partial}_{n-1}} \cdots \xrightarrow{\bar{\partial}_1} C_0(X, A) \to 0.$$

Elements of $C_n(X, A)$ are called **relative chains**.

Definition 8.90. For a pair (X, A), the **relative homology groups** are defined as:

$$H_n(X, A) = \frac{\ker(\bar{\partial}_n)}{\operatorname{im}(\bar{\partial}_{n+1})}.$$

Elements of $H_n(X, A)$ are called **relative cycles**.

Note 8.91. Relative homology measures how homology changes when passing from A to X. Instead of tracking cycles in X, we track cycles in X that may not be in A, modulo those that bound within X.

Proposition 8.92

A relative cycle is trivial in $H_n(X, A)$ if and only if it is a **relative boundary**, meaning it can be written as:

$$\alpha = \partial\beta + \gamma, \quad \text{for } \beta \in C_{n+1}(X), \quad \gamma \in C_n(A).$$

Corollary 8.93

Consider the short exact sequence of chain complexes:

$$0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \xrightarrow{j_\#} C_\bullet(X, A) \to 0$$

where:

- $C_\bullet(A)$ is the chain complex of A,
- $i_\#$ is induced by the inclusion $A \to X$,
- $j_\#$ is the quotient map.

Since homology is functorial, this induces a long exact sequence on homology:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X, A) \arrow[dll, "\partial"'] & & \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r, "j_*"] & H_{n-1}(X, A) \arrow[dll, dashed] & & \\
& H_0(A) \arrow[r, "i_*"] & H_0(X) \arrow[r, "j_*"] & H_0(X, A) \arrow[d] & & \\
& & & 0 & &
\end{tikzcd}$$

9 Homotopy invariance

We have already shown that homeomorphic topological spaces have isomorphic homology groups. Now, we extend this result to homotopy equivalent spaces.

Theorem 9.1

Let $f, g : X \to Y$ be two maps. If $f \simeq g$ (i.e., f and g are homotopic), then their induced maps on homology are equal:

$$f_* = g_*.$$

Note 9.2. Homology does not change under homotopy because we can "continuously deform" one chain map into another without altering cycles and boundaries. The key step is breaking the prism $\Delta^n \times [0,1]$ into simplices so that homology computations remain valid. This ensures that $f_* = g_*$, making homology a homotopy-invariant concept.

Proof. The idea of the proof is to construct a homotopy between the images of simplices under f and g using a family of simplices obtained from a triangulation of Y. The essential ingredient is that we can subdivide $\Delta^n \times [0,1]$ into $(n+1)$ simplices.

Label the vertices of $\Delta^n \times \{0\}$ as $v_0, \dots, v_n$ and the corresponding vertices of $\Delta^n \times \{1\}$ as $w_0, \dots, w_n$. To interpolate from $\Delta^n \times \{0\}$ to $\Delta^n \times \{1\}$, we construct a sequence of n-simplices, each obtained by moving one vertex at a time.

- First, move $[v_0, \dots, v_n]$ to $[v_0, \dots, v_{n-1}, w_n]$.
- Then move this up to $[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$, and so on.

The region between two consecutive n-simplices forms an $(n+1)$-simplex:

$$[v_0, \dots, v_i, w_i, \dots, w_n],$$

where

- The lower face is $[v_0, \dots, v_{i-1}, v_i, w_{i+1}, \dots, w_n]$.
- The upper face is $[v_0, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$.

These $n+1$ simplices cover $\Delta^n \times [0,1]$ and only meet along common n-faces.

Now, given the homotopy $F : X \times I \to Y$ from f to g and a singular simplex $\sigma : \Delta^n \to X$, we can define the composition:

$$F \circ (\sigma \times \mathrm{id}) : \Delta^n \times I \to X \times I \to Y.$$

This shows that $f_* = g_*$, proving that homotopy equivalent spaces have the same homology. □

9.1 The Prism Operator

Given the homotopy $F : X \times I \to Y$ from f to g and a singular simplex $\sigma : \Delta^n \to X$, we can define the composition:

$$F \circ (\sigma \times \mathrm{id}) : \Delta^n \times I \to X \times I \to Y.$$

Using this subdivision, we define the *prism operator*, which serves as a homotopy tool in chain complexes:

Definition 9.3. The *prism operator* is a chain map:

$$P : C_n(X) \to C_{n+1}(Y)$$

defined by:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})\big|_{[v_0,\dots,v_i,w_i,\dots,w_n]}.$$

Note 9.4. The prism operator P tracks the deformation of a simplex σ under the homotopy F, effectively interpolating between f and g. The sum encodes how each vertex v_i moves to its corresponding w_i, decomposing $\Delta^n \times I$ into a sequence of simplices.

Proposition 9.5

$$\partial P(\sigma) = g_\# \sigma - f_\# \sigma - P(\partial \sigma).$$

Note 9.6. This equation states that applying ∂ to $P(\sigma)$ decomposes into three parts:

- $g_\# \sigma$, corresponding to the top face.
- $f_\# \sigma$, corresponding to the bottom face.
- $P(\partial \sigma)$, corresponding to the side faces.

Proof. By definition of P, we expand:

$$\partial P(c) = \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})\big|_{[v_0,\dots,v_i,w_i,\dots,w_n]} \right).$$

Applying ∂ to each term, we have :

$$\partial P(c) = \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})\big|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]} + \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathrm{id})\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}.$$

Now we cancel some terms.

- If both v_i and w_i remain, we get a triangle on a vertical face.
- If one is omitted, we get a sloping face (an interior face).
- Terms where $i = j$ cancel except:

$$F \circ (\sigma \times \mathrm{id})\big|_{[\hat{v}_0,w_0,\dots,w_n]} \quad \text{and} \quad -F \circ (\sigma \times \mathrm{id})\big|_{[v_0,w_0,\dots,\hat{w}_n]}.$$

- The face $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ appears with coefficient $+1$ in the first sum and -1 in the second sum, hence they cancel.

Thus, the only remaining terms are:

$$F \circ (\sigma \times \mathrm{id})\big|_{[\hat{v}_0,w_1,\dots,w_n]} = g_\#(\sigma),$$

$$F \circ (\sigma \times \mathrm{id})\big|_{[v_0,w_1,\dots,\hat{w}_n]} = f_\#(\sigma).$$

For $i \neq j$, the remaining terms are exactly $P(\partial\sigma)$, leading to:

$$P(\partial \sigma) = \sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{id})\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

$$+\sum_{j<i}(-1)^{i-1}(-1)^j F\circ(\sigma\times\mathrm{id})\big|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}.$$

Thus, we conclude:

$$\partial P(\sigma)=g_\#(\sigma)-f_\#(\sigma)-P(\partial\sigma),$$

proving the claim. □

9.2 Chain homotopy

Definition 9.7. Suppose p, q are morphisms of chain complexes $A_\bullet \to B_\bullet$. We say that they are **chain homotopic** if there exists an operator $P : A_n \to B_{n+1}$ (for all $n \in \mathbb{N}$) such that

$$p-q=\partial P+P\partial.$$

Note 9.8. Chain homotopy expresses the idea that two chain maps differ by a "boundary adjustment". This ensures that they induce the same homology maps, meaning they yield identical results at the homology level.

Proposition 9.9

Chain-homotopic chain maps induce the same homomorphisms on homology.

Corollary 9.10

If $h : X \to Y$ is a *homotopy equivalence*, then the induced map on homology

$$h_* : H_n(X)\to H_n(Y)$$

is an isomorphism.

Remark 9.11. In particular, all contractible spaces (i.e., spaces homotopy equivalent to a point) have the same homology as a point, meaning they have trivial reduced homology.

Proof. Let $j : Y \to X$ be the homotopy inverse of h. By previous results, we have:

$$(j\circ h)_*=(\mathrm{id}_X)_*,\quad (h\circ j)_*=(\mathrm{id}_Y)_*.$$

By functoriality of homology, this implies:

$$j_*\circ h_*=\mathrm{id}_{H_n(X)},\quad h_*\circ j_*=\mathrm{id}_{H_n(Y)}.$$

Thus, h_* is a bijection, completing the proof. □

9.3 Excision Theorem

We now work towards proving the equivalence between simplicial and singular homology. The first step in this direction is the excision theorem.

We begin by stating some results that will help us prove the *excision theorem*.

Proposition 9.12. Given any collection $\mathcal{U} = \{U_\alpha\}$ of subsets of X such that

$$X = \bigcup_\alpha \operatorname{Int}(U_\alpha),$$

we define

$$C_\bullet^{\mathcal{U}}(X) = \{n\text{-chains in } X \text{ where each simplex lies in some } U_\alpha\}.$$

Then the inclusion $C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ is a chain homotopy equivalence.

Note 9.13. This result states that the chain complex generated by chains lying in the open sets U_α is homotopy equivalent to the full chain complex of X. This equivalence allows us to replace a space with a collection of open subsets in homology computations.

Remark 9.14. For i to be a chain homotopy equivalence, there must exist $s : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$ such that $i \circ s$ and $s \circ i$ are both chain homotopic to the identity.

Proof. Omitted. □

Theorem 9.15 (Excision Theorem)
Given $Z \subset A \subset X$ such that $\bar{Z} \subset \operatorname{Int}(A)$, the inclusion of pairs

$$(X \setminus Z, A \setminus Z) \to (X, A)$$

induces an isomorphism on relative homology:

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A).$$

Note 9.16. This theorem states that removing a sufficiently 'small' subset Z from both X and A does not change the relative homology. This allows us to simplify homology calculations by "excising" parts of the space that do not contribute new topological information.

Remark 9.17. This statement is equivalent to the following reformulation: If X is a topological space with subspaces $A, B \subset X$ such that $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$, then the inclusion of pairs

$$(B, A \cap B) \to (X, A)$$

induces an isomorphism on relative homology:

$$H_n(B, A \cap B) \to H_n(X, A).$$

This reformulation follows by setting $B = X \setminus Z$ and $Z = X \setminus B$. The condition $\bar{Z} \subset \operatorname{Int}(A)$ ensures that $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$, making the two versions equivalent.

Remark 9.18. We are comparing the following chain complexes:

$$\cdots \to \frac{C_{n+1}(X \setminus Z)}{C_{n+1}(A \setminus Z)} \to \frac{C_n(X \setminus Z)}{C_n(A \setminus Z)} \to \frac{C_{n-1}(X \setminus Z)}{C_{n-1}(A \setminus Z)} \to \cdots$$

with the corresponding chain complex for (X, A):

$$\cdots \to \frac{C_{n+1}(X)}{C_{n+1}(A)} \to \frac{C_n(X)}{C_n(A)} \to \frac{C_{n-1}(X)}{C_{n-1}(A)} \to \cdots$$

The excision theorem guarantees that these two chain complexes induce isomorphic homology groups.

Sketch of proof. Consider subspaces A, B such that $\operatorname{Int}(A) \cup \operatorname{Int}(B) = X$. Define the cover $\mathcal{U} = \{A, B\}$, then we set:

$$C_n^{\mathcal{U}}(X) = \{n\text{-chains that lie either in } A \text{ or in } B\} = C_n(A) \oplus C_n(B) \subset C_n(X).$$

The inclusion $C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ induces the map

$$C_\bullet^{\mathcal{U}}(X, A) = \frac{C_\bullet^{\mathcal{U}}(X)}{C_\bullet^{\mathcal{U}}(A)} = \frac{C_\bullet^{\mathcal{U}}(X)}{C_\bullet(A)} \to \frac{C_\bullet(X)}{C_\bullet(A)} = C_\bullet(X, A).$$

Since this inclusion is a chain homotopy equivalence, by the *second isomorphism theorem for groups*, we obtain

$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \cong \frac{C_n(A) \oplus C_n(B)}{C_n(A)} = \frac{C_n^{\mathcal{U}}}{C_n(A)} = C_n^{\mathcal{U}}(X, A)$$

for all n. Since $C_n(A) \cap C_n(B) = C_n(A \cap B)$, this yields two isomorphisms of chain complexes:

$$C_\bullet(B, A \cap B) \cong C_\bullet^{\mathcal{U}}(X, A) \cong C_\bullet(X, A),$$

which in turn induce isomorphisms on homology:

$$H_n(B, A \cap B) \cong H_n^{\mathcal{U}}(X, A) \cong H_n(X, A).$$

□

9.3.1 Homology of Quotients

The Excision Theorem allows us to prove results about long exact sequences and quotients: for this, we need to show that the relative homology groups are isomorphic to the reduced homology groups of the quotient.

Firstly, let us prove the lemma that will be used in the proof.

Lemma 9.19. Let X be a topological space and $x_0 \in X$, then

$$H_n(X, \{x_0\}) \cong \widetilde{H}_n(X)$$

for all $n \in \mathbb{N}$.

> **Remark 9.20.** We are not requiring $(X, \{x_0\})$ to be a good pair.

Proof. Exercise.

To do

□

Proposition 9.21

For (X, A) a **good pair**, the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms

$$H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A).$$

Note 9.22. This result tells us that computing homology relative to a subspace A is equivalent to computing the homology of the quotient space X/A. In simple terms, collapsing A to a point does not change the essential homological structure when (X, A) is a good pair.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "1"] \arrow[d, dashed, "4"] & H_n(X,V) \arrow[r, "5"] \arrow[d] & H_n(X \setminus A, V \setminus A) \arrow[d, dashed, "3"] \\
H_n(X/A, A/A) \arrow[r, dashed, "2"] & H_n(X/A, V/A) \arrow[r, dashed, "6"] & H_n((X/A) \setminus (A/A), (V/A) \setminus (A/A))
\end{tikzcd}$$

where V is a neighbourhood of A in X that retracts to A (which exists because (X, A) is a good pair). By applying the Excision Theorem, we obtain:

- 5 is an isomorphism by the Excision Theorem.
- 6 is an isomorphism by the Excision Theorem.

To prove the theorem, we want to show that 4 (induced by the quotient map) is an isomorphism. To do so, we show that $1, 2, 3$ are isomorphisms and the rest follows by commutativity of the diagram (note that the diagram is commutative since it is induced by a commutative diagram of inclusions and quotient maps).

Consider the long exact sequence of the triple (X, V, A) induced by the following short exact sequence of chain complexes:

$$0 \to C_\bullet(V, A) \xrightarrow{\phi} C_\bullet(X, A) \xrightarrow{\psi} C_\bullet(X, V) \to 0.$$

We have that ϕ is injective since $C_\bullet(V) \to C_\bullet(X)$ is injective, and we are quotienting by the same equivalence relation (so it is still an inclusion). On the other hand, ψ is surjective because we are quotienting by the same set $C_\bullet(X)$ using a stronger equivalence relation. This induces a long exact sequence on homology:

$$\begin{tikzcd}
 & 0 \arrow[d, dash, dashed, "\cong"] & & \\
\cdots \arrow[r] & H_n(V,A) \arrow[r, "\iota_*"] & H_n(X,A) \arrow[r, "q_*"] & H_n(X,V) \arrow[dll, "\partial"'] \\
0 \arrow[r, dash, "\cong"] & H_{n-1}(V,A) \arrow[r, dashed, "\iota_*"] & H_{n-1}(X,A) \arrow[r, dashed, "q_*"] & H_{n-1}(X,V) \arrow[dll, dashed, "\partial"'] \\
0 \arrow[r, dash, "\cong"] & H_0(V,A) \arrow[r, "\iota_*"] & H_0(X,A) \arrow[r, "q_*"] & H_0(X,V) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

Since $i : A \to V$ is a homotopy equivalence, we know that $H_n(V, A) \cong 0$ (this can be seen by considering the long exact sequence on homology of the pair (V, A)). Hence, 1 is an isomorphism.

By the same argument, we obtain $H_n(V/A, A/A) \cong 0$, so by looking at the long exact sequence for the triple $(X/A, V/A, A/A)$:

$$
\begin{tikzcd}
 & 0 \arrow[d, dash, "\cong"] & & & & \\
\cdots \arrow[r] & H_n(V,A) \arrow[r, "\iota_*"] & H_n(X,A) \arrow[r, "q_*"] & H_n(X,V) \arrow[dl, "\partial"] & & \\
 & 0 \arrow[r, dash, "\cong"] & H_{n-1}(V/A,A/A) \arrow[r, "\iota_*"] & H_{n-1}(X/A,A/A) \arrow[r, "q_*"] & H_{n-1}(X/A,V/A) \arrow[dl, dashed, "\partial"] & \\
 & & 0 \arrow[r, dash, "\cong"] & H_0(V/A,A/A) \arrow[r, "\iota_*"] & H_0(X/A,A/A) \arrow[r, "q_*"] & H_0(X/A,V/A) \arrow[d] \\
 & & & & & 0
\end{tikzcd}
$$

we obtain that 2 is also an isomorphism.

The quotient map $q : X \to X/A$ restricted to $X \setminus A$ and $V \setminus A$ is a homeomorphism (since no identification happens outside of A), so we have a homeomorphism of pairs:

$$(X \setminus A, V \setminus A) \to (X/A \setminus A/A, V/A \setminus A/A),$$

which induces an isomorphism on homology, which is 3. $\square$

Corollary 9.23

Given a collection of spaces $\{X_\alpha\}$ with points $x_\alpha \in X_\alpha$, such that (X_α, x_α) is a good pair, then the inclusions $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ induce an isomorphism on reduced homology:

$$i_* = \bigoplus_\alpha (i_\alpha)_* : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

Note 9.24. This result tells us that the reduced homology of a wedge sum of spaces is just the direct sum of their individual reduced homology groups.

Proof. Take

$$(X, A) = (\bigcup_\alpha X_\alpha, \bigcup_\alpha \{x_\alpha\}).$$

Then we know that

$$H_n(X, A) \cong \widetilde{H}_n(X/A) \cong \widetilde{H}_n\left(\bigvee_\alpha X_\alpha\right).$$

By definition of the wedge sum, we also have

$$H_n(X, A) = H_n\left(\bigcup_\alpha X_\alpha, \bigcup_\alpha \{x_\alpha\}\right) \cong \bigoplus_\alpha H_n(X_\alpha, \{x_\alpha\}) \cong \bigoplus_\alpha \widetilde{H}_n(X_\alpha).$$

Thus, putting everything together, we obtain the desired result. $\square$

Theorem 9.25. Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$. If U and V are homeomorphic, then $n = m$.

Proof. Exercise.

To xo

□

9.3.2 The Five Lemma

Theorem 9.26 (The Five Lemma)**.** In a commutative diagram of abelian groups as shown below, if the two rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is also an isomorphism.

$$\begin{tikzcd} A \arrow[r] \arrow[d, dashed, "\alpha"] & B \arrow[r] \arrow[d, dashed, "\beta"] & C \arrow[r] \arrow[d, dashed, "\gamma"] & D \arrow[r] \arrow[d, dashed, "\delta"] & E \arrow[d, dashed, "\varepsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Proof. Exercise.

To do

□

9.3.3 Chain Complexes

Theorem 9.27

Let (X, A) be a pair of a Δ-complex and a sub-complex. The inclusion of chain complexes $\Delta_*(X, A) \subset C_*(X, A)$ induces an isomorphism on all homology groups, i.e.

$$H_n^\Delta(X, A) \cong H_n(X, A)$$

Remark 9.28. Note that $\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A)$ is a quotient of free abelian groups of countable rank, and $C_*(X, A) = C_*(X)/C_*(A)$ is a quotient of free abelian groups of uncountable rank.

Proof. □

Proof. First, consider the case where X is finite-dimensional and $A = \emptyset$. Let X^k be the collection of all l-simplices with $l \leq k$. Let us do induction on k.

- $k = 0$. Then X^0 is just a collection of points. We know that the simplicial and singular homology of a point coincide, so they must coincide also for a disjoint union of points.
- Assume the statement is true for $k - 1$. Consider the long exact sequence of the pair (X^k, X^{k-1}), for both simplicial and singular homology. We have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^\Delta_{n+1}(X^k, X^{k-1}) \arrow[r] \arrow[d, dashed, "1"] & H^\Delta_n(X^{k-1}) \arrow[r] \arrow[d, dashed, "2"] & H^\Delta_n(X^k) \arrow[r] \arrow[d, dashed, "3"] & H^\Delta_n(X^k, X^{k-1}) \arrow[r] \arrow[d, dashed, "4"] & H^\Delta_n(X^{k-1}) \arrow[r] \arrow[d, dashed, "5"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \arrow[r] & \cdots
\end{tikzcd}$$

Using the induction assumption, we know that maps 2 and 5 are isomorphisms. We just need to show that 1 and 4 are isomorphisms, then 3 is an isomorphism by the Five Lemma.

Note that we have

$$\Delta_n(X^k, X^{k-1}) = \begin{cases} 0, & k \neq n \\ \text{free abelian group of basis the } k\text{-simplices in } X, & k = n \end{cases}$$

Note 9.29. This identifies the homology of the relative pair (X^k, X^{k-1}) with the space of k-simplices, confirming that homology is capturing the k-dimensional holes.

Now, let us look at the singular homology. By looking at the good pair (X^k, X^{k-1}), we have

$$H_n(X^k, X^{k-1}) \cong \widetilde{H}_n(X^k/X^{k-1}) \cong \bigoplus_\alpha \widetilde{H}_n(\Delta^k_\alpha/\partial\Delta^k_\alpha)$$

where Δ^k_α are the k-simplices of X. The second isomorphism is justified as follows. We examine X^k/X^{k-1}, and we have the following commutative diagram

$$\begin{tikzcd}
\bigsqcup_\alpha \Delta^k_\alpha \arrow[r, "\bigoplus_\alpha \sigma^k_\alpha"] \arrow[d, "q_1"'] & X^k \arrow[d, "q_2"] \\
\bigsqcup_\alpha \Delta^k_\alpha / \bigsqcup_\alpha \partial\Delta^k_\alpha \arrow[r, "\cong"'] & X^k/X^{k-1}
\end{tikzcd}$$

Note 9.30. This step shows that quotienting by lower-dimensional simplices leaves only the top-dimensional simplices, making it easier to compute homology.

Thus, we obtain the following induced isomorphisms on homology:

$$\widetilde{H}_n(X^k/X^{k-1}) \cong \widetilde{H}_n\left(\bigvee_\alpha \Delta^k_\alpha/\partial\Delta^k_\alpha\right) \cong \bigoplus_\alpha \widetilde{H}_n(S^k)$$

In particular, since $\Delta_k(X^k, X^{k-1})$ is the free abelian group generated by the k-simplices, it is isomorphic to a direct sum of $\mathbb{Z}$ copies indexed by these simplices.

Thus, we conclude

$$H_n(X^k, X^{k-1}) \cong \widetilde{H}_n(X^k/X^{k-1}) \cong \begin{cases} 0, & k \neq n \\ \bigoplus_\alpha \mathbb{Z}, & k = n \end{cases}$$

Note 9.31. This result tells us that the relative homology $H_n(X^k, X^{k-1})$ precisely captures the new k-dimensional cycles added at each stage.

Generalisation: The above generalises to arbitrary X since $H_n(X)$ only depends on the local structure of the chain complex.

Finally, for a non-empty A, we consider the long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n^{\Delta}(A) \arrow[r] \arrow[d, dashed, "1"] & H_n^{\Delta}(A) \arrow[r] \arrow[d, dashed, "2"] & H_n^{\Delta}(X,A) \arrow[r] \arrow[d, dashed, "3"] & H_{n-1}^{\Delta}(X) \arrow[r] \arrow[d, dashed, "4"] & H_{n-1}^{\Delta}(A) \arrow[r] \arrow[d, dashed, "5"] & \cdots \\
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

Since maps $1, 2, 4, 5$ are isomorphisms, map 3 must also be an isomorphism by the Five Lemma.

9.4 Excision examples

Example 9.32. We now know that

$$H_n(\mathbb{D}^n, \partial\mathbb{D}^n) \cong \widetilde{H}_n(S^n)$$

In this example, we want to find explicit generators for $H_n(\mathbb{D}^n, \partial\mathbb{D}^n)$ and $\widetilde{H}_n(S^n)$ (they are both isomorphic to $\mathbb{Z}$, so they will be generated by one element).

Let us first set up some notation. Using the obvious isomorphism

$$(\Delta^n, \partial\Delta^n) \cong (\mathbb{D}^n, \partial\mathbb{D}^n)$$

where Δ^n is the standard simplex. Let $i_n : \Delta^n \to \Delta^n$ be the identity map, then $i_n \in C_n(\Delta^n)$ and $i_n + C_n(\partial\Delta^n) \in C_n(\Delta^n, \partial\Delta^n)$. We claim that i_n is a cycle generating $H_n(\Delta^n, \partial\Delta^n)$. It is trivially a relative cycle since

$$\partial(i_n + C_n(\partial\Delta^n)) = \partial i_n + C_{n-1}(\partial\Delta^n) = C_{n-1}(\partial\Delta^n)$$

since ∂i_n is a chain in $C_{n-1}(\partial\Delta^n)$. Let us write $[i_n]$ as the homology class of i_n in $H_n(\Delta^n, \partial\Delta^n)$.

Now, let us prove that it generates the homology group by induction.

- $\Delta^0 = \{\text{pt}\}$ and $\partial\Delta^0 = \emptyset$, hence

  $$H_0(\Delta^0, \partial\Delta^0) = H_0(\{\text{pt}\}) = \mathbb{Z}$$

  and it is obviously generated by $i_0 : \{\text{pt}\} \to \{\text{pt}\}$ (since the identity map and its multiples are the only possible maps).

- Let Λ be the union of all $(n-1)$-dimensional faces of Δ^n except one. We claim that there are isomorphisms

  $$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong}_1 H_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{\cong}_2 H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

Here 1 is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ induced by the short exact sequence of chain complexes

$$0 \to C_\bullet(\partial\Delta^n, \Lambda) \to C_\bullet(\Delta^n, \Lambda) \to C_\bullet(\Delta^n, \partial\Delta^n) \to 0$$

We now want to find a cycle generating $\widetilde{H}_n(S^n)$. Consider S^n as two simplices Δ_1^n and Δ_2^n glued together along their boundaries. We claim that $\Delta_1^n - \Delta_2^n$ generates $\widetilde{H}_n(S^n)$ (we consider $\Delta_1^n - \Delta_2^n$ as the map).

Consider the long exact sequence of reduced homology of the pair $(\Delta_1^n \cup \Delta_2^n, \sim, \Delta_2^n)$ where $\sim$ is the identification.

$$\widetilde{H}_n(\Delta_1^n \cup \Delta_2^n / \sim) \cong H_n(\Delta_1^n \cup \Delta_2^n / \sim, \Delta_2^n)$$

Moreover, we claim that

$$H_n(\Delta_1^n \cup \Delta_2^n / \sim, \Delta_2^n) \cong H_n(\Delta_1^n, \partial\Delta_1^n)$$

since it is induced by the inclusion $(\Delta_1^n \cup \Delta_2^n / \sim, \Delta_2^n) \to (\Delta_1^n, \partial\Delta_1^n)$, which is a homeomorphism by similar reasoning as previously. Hence the cycle $\Delta_1^n - \Delta_2^n$ in $\widetilde{H}_n(\Delta_1^n \cup \Delta_2^n / \sim)$ corresponds to Δ_1^n in $H_n(\Delta_1^n \cup \Delta_2^n / \sim, \Delta_2^n)$ and is mapped to i_n in $H_n(\Delta_1^n, \partial\Delta_1^n)$. Since they are all isomorphism, we can conclude that $\Delta_1^n - \Delta_2^n$ is a generator of the homology groups.

Corollary 9.33

Given a CW-complex X, if $X = A \cup B$ for subcomplexes A, B, then the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism on reduced homology groups:

$$H_n(B, A \cap B) \to H_n(X, A)$$

Proof. If $A \cap B \neq \emptyset$, then since CW-pairs are good pairs, we can use a similar argument to show that $B/A \cap B$ is homeomorphic to $A \cup B/A$, inducing an isomorphism. If $A \cap B = \emptyset$, then $X = A \sqcup B$, so:

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)} = \frac{C_n(A) \oplus C_n(B)}{C_n(A)} \cong C_n(B)$$

□

10 The degree of a map

Definition 10.1. Given $f : S^n \to S^n$, it induces a homomorphism on homology:

$$f_* : H_n(S^n) \to H_n(S^n).$$

Since $H_n(S^n) \cong \mathbb{Z}$, we have $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$, which is defined to be the **degree** of f (denoted $\deg(f)$).

10.1 Properties of the degree

- **Identity Map:** $\deg(\mathrm{id}_{S^n}) = 1$.
  (The identity does nothing to the fundamental cycle, so it acts as the identity in homology.)

- **Non-surjective Maps:** If f is not surjective, then $\deg(f) = 0$.
  (Since the image of f avoids a point, the inclusion $S^n \setminus \{x\} \to S^n$ forces f_* to be trivial.) This follows from the commutative diagram:

$$\begin{tikzcd} \widetilde{H}_n(S^n \setminus \{x\}) \arrow[r, "0"] \arrow[d, "f_*"] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\ \widetilde{H}_n(S^n) \arrow[r, "i_*"] & \widetilde{H}_n(S^n) \end{tikzcd}$$

- **Homotopy Invariance:** If $f \simeq g$, then $\deg(f) = \deg(g)$.
  (Homotopic maps induce the same homomorphism on homology.)

- **Composition Rule:** $\deg(f \circ g) = \deg(f)\deg(g)$.
  (Homology is functorial: the effect of f on cycles is passed to g.)

- **Homotopy Equivalences:** If f is a homotopy equivalence, then $\deg(f) = \pm 1$.
  (Since f has a homotopy inverse g with $f \circ g \simeq \mathrm{id}$, we must have $\deg(f \circ g) = 1$, *implying* $\deg(f) = \pm 1$*.)*

- **Reflection Across a Hyperplane:** If $f : S^n \to S^n$ is a reflection through the origin, then $\deg(f) = -1$.
  (The reflection swaps hemispheres, flipping orientation, so it negates the generator of homology.) Using the Δ-complex structure:

$$f_*(\Delta_1^n - \Delta_2^n) = \Delta_2^n - \Delta_1^n.$$

  Since $\Delta_1^n - \Delta_2^n$ is a generator, this flips the sign, giving $\deg(f) = -1$.

- **Antipodal Map:** If $a : S^n \to S^n$ is the antipodal map, then $\deg(a) = (-1)^{n+1}$.
  (This is because the antipodal map consists of $n+1$ reflections, each flipping orientation.)
  Example: The antipodal map on S^2 has $\deg(a) = -1$, since it negates any cycle.

- **Maps Without Fixed Points:** If $f : S^n \to S^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.
  (Such a map is homotopic to the antipodal map, meaning it must have the same degree.) We show this by constructing a homotopy:

$$F(x,t) = f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

  which deforms f to the antipodal map.

11 Cellular homology

Appendix

A Smith normal form and structure theorem

Any integer matrix A can be reduced via elementary integer row and column operations to a diagonal form (SNF):

$$A \sim \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}, \quad \text{with } d_1 \mid d_2 \mid \cdots \mid d_r.$$

These integers d_i (called invariant factors) uniquely determine the quotient structure.

Proposition A.1. The SNF algorithm.

1. Choose smallest nonzero pivot; move it top-left.
2. Clear pivot row and column (integer ops).
3. Ensure pivot divides other entries (use gcd).
4. Repeat on remaining submatrix; arrange so $d_1|d_2|\ldots$.

Theorem A.2 (Structure Theorem for Finitely Generated Abelian Groups)
Every finitely generated abelian group G has the form:

$$G \cong \mathbb{Z}^n \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r\mathbb{Z}$$

with integers $d_1 \mid d_2 \mid \cdots \mid d_r$. In particular: given an integer matrix A, we have:

$$\mathbb{Z}^n/A\mathbb{Z}^m \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r\mathbb{Z} \oplus \mathbb{Z}^{n-r}$$

where d_i are the diagonal elements of SNF of A.

Example A.3
An example

$$A = \begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Thus:

$$\mathbb{Z}^2/A\mathbb{Z}^2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$