Homework 23.3.22

1. In a Vector space $V$

Since VS2 $\exists\, 0 \in V$ s.t.

$0 + 0 = 0$

Since VS5 $\forall c \in \mathbb{C}$

$c0 = c(0+0) = c0 + c0$

Since VS3

$\exists\, x \in V$ s.t. $c0 + x = 0$

$0 = c0 + x = (c0 + c0) + x$

Since VS1

$0 = c0 + (c0 + x) = c0 + 0 = c0$

Then

if $c$ is a number, $c0 = 0$

II

1. Closure under addition

$\forall\, x, y \in L$, since $L$ is field,

$x + y \in L$

by the closure of addition in $L$

VS1 — VS4 Since $L$ is a field

$\langle L, + \rangle$ is a Ablian Group

VS1-4 is obviously holds

2. Closure under scalar multiplication

$\forall\, a \in K$, $\forall\, x \in L$

Since $K$ is a subfield of $L$,

$a \in L$

Since $L$ is a field

$ax \in L$

VS5 — VS7

$\forall\, a, b \in K$, $\forall\, u, v \in L$

since $K$ is a subfield of $L$

$a, b \in L$

Since $L$ is a field

$$\begin{cases} \text{VS5} & a(u+v) = au + av \\ \text{VS6} & (a+b)v = av + bv \end{cases}$$

1 VS 7 $(ab)v = a(bv)$

holds

VS 8

Since $K$ and $L$ are fields

$\exists\, 1_K \in K$ s.t. $\forall u \in K$ $\quad 1_K \cdot u = u \cdot 1_K = u$

$\exists\, 1_L \in L$ s.t $\forall u \in L$ $\quad 1_L \cdot u = u\, 1_L = u$

Since $1_K \neq 0$ $\quad \exists\, 1_K^{-1} \in L$ s.t. $1_K \cdot 1_K^{-1} = 1_K^{-1} \cdot 1_K = 1_L$

since $1_K \in L$ and $1_K \in K$

$1_K = 1_K \cdot 1_L = 1_K \cdot 1_K \cdot 1_K^{-1} = 1_K \cdot 1_K^{-1} = 1_L$

Then $\exists\, 1_K \in K$ $\forall u \in L$ s.t. $1_K \cdot u = u$

i.e. VS 8 holds

Hence $L$ is a Vector Space over $K$

14. Let $K = \{a + b\gamma \mid a, b \in \mathbb{Q}\}$

Case 1 $\gamma \in \mathbb{Q}$

$\forall u \in \mathbb{Q}$ $\quad u = u + 0\gamma \in K$

$\forall\, a + b\gamma \in K$

Since $a, b, \gamma \in \mathbb{Q}$

$a + b\gamma \in \mathbb{Q}$

$\mathbb{Q} \subseteq K \wedge K \subseteq \mathbb{Q} \Rightarrow K = \mathbb{Q}$

$\mathbb{Q}$ is a field $\Rightarrow$ $K$ is a field

Case 2 $\gamma \in \mathbb{R} \setminus \mathbb{Q}$

1. Let $\begin{cases} x = a_1 + b_1\gamma \in K \\ y = a_2 + b_2\gamma \in K \end{cases}$

i) $x + y = (a_1 + a_2) + (b_1 + b_2)\gamma$

Since $a_1, a_2, b_1, b_2 \in \mathbb{Q}$

$\Rightarrow a_1 + a_2,\ b_1 + b_2 \in \mathbb{Q}$

Then $x+y \in K$

ii) $xy = a_1a_2 + b_1b_2r^2 + (a_1b_2+a_2b_1)r$

$= (a_1a_2 + b_1b_2c) + (a_1b_2+b_1a_2)r$

Similarly, $a_1a_2+b_1b_2c \in \mathbb{Q}$, $a_1b_2+b_1a_2 \in \mathbb{Q}$

$\Rightarrow xy \in K$

2.

$x = a+br \in K$

define $-x = -a-br \in K$ since $\begin{cases} -a \in \mathbb{Q} \\ -b \in \mathbb{Q} \end{cases}$

s.t. $x+(-x) = 0$

if $x \neq 0$. i.e. $a \neq -br$ and $(a, b) \neq (0, 0)$

Since $r \in \mathbb{R} \setminus \mathbb{Q}$ and $a, b \in \mathbb{Q}$

$\Rightarrow a \neq br$

Hencce

$a^2-b^2c \neq 0$

define $x^{-1} = \dfrac{a-br}{a^2-b^2c} = \dfrac{a}{a^2-b^2c} - \dfrac{b}{a^2-b^2c}r$

$\frac{a}{a^2-b^2r}, \frac{-b}{a^2-b^2r} \in \mathbb{Q}$

$\Rightarrow x^{-1} \in K$

3. $a=b=0 \Rightarrow 0 \in K$

$a=1\ b=0 \Rightarrow 1 \in K$

Hence K is a field