Homework 0406

6. (a) $\nu_1 = (1, i, 0)$

$\nu_2 = (1, 1, 1)$

$\nu_1' = \nu_1$

$\nu_2' = \nu_2 - \frac{\langle \nu_2, \nu_1' \rangle}{\langle \nu_1', \nu_1' \rangle} \nu_1'$

$\langle \nu_2, \nu_1' \rangle = (1, 1, 1) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = 1 - i$

$\langle \nu_1', \nu_1' \rangle = (1, i, 0) \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = 2$

$\nu_2' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1-i}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

$= \left(\frac{1+i}{2}, \frac{1-i}{2}, 1 \right)^T$

$\nu_1'' = \frac{\nu_1'}{\|\nu_1'\|} = \frac{1}{\sqrt{2}} (1, i, 0)^T$

$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} i, 0 \right)^T$

$\nu_2'' = \frac{\nu_2'}{\|\nu_2'\|} = \frac{1}{\sqrt{2}} \left(\frac{1+i}{2}, \frac{1-i}{2}, 1 \right)^T$

$= \left(\frac{\sqrt{2} + i\sqrt{2}}{4}, \frac{\sqrt{2} - i\sqrt{2}}{4}, \frac{\sqrt{2}}{2} \right)^T$

$\{ \nu_1'', \nu_2'' \}$ is a othonormal basis

(b) $v_1 = (1, -1\ -i)^T \quad v_2 = (i, 1, 2)^T$

$$v_1' = v_1 = (1, -1, -i)^T$$

$$v_2' = v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1'$$

$$= (i, 1, 2)^T - \frac{3i-1}{3}(1, -1, -i)^T$$

$$= \frac{1}{3}\begin{pmatrix} 1 \\ 2+3i \\ 3-i \end{pmatrix}$$

$$\langle v_1', v_2' \rangle = \frac{1}{3} \cdot (1, -1, -i)\begin{pmatrix} 1 \\ 2-3i \\ 3+i \end{pmatrix}$$

$$= \frac{1}{3}(1 - 2 + 3i - 3i + 1)$$

$$= 0$$

$$v_1'' = \frac{v_1'}{\|v_1'\|} = \frac{\sqrt{3}}{3}(1, -1, -i)^T = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}i\right)$$

$$v_2'' = \frac{v_2'}{\|v_2'\|} = \frac{1}{2\sqrt{6}}(1, 2+3i, 3-i)^T$$

$$= \left(\frac{\sqrt{6}}{12}, \frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{4}i, \frac{\sqrt{6}}{4} - \frac{\sqrt{6}}{12}i\right)^T$$

$\{v_1'', v_2''\}$ is a othonormal basis of span$\{v_1, v_2\}$

7. (a) $\langle A, B\rangle = tr(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}b_{ij}$

Suppose $\forall B \in \mathbb{R}^{n\times n}$ $\langle A, B\rangle = 0$

$\forall\ i, j \in \mathbb{N}^*$ $i, j \le n$

$0 = \langle A, E_{ij}\rangle = a_{ij}$.

i.e. $A = 0$

Hence it is non-degenerate.

(b) Let A be a real symmetric matrix

$tr(AA) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}a_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}^2 \geq 0$

if $A \neq 0$, $\exists\ a_{ji} = a_{ij} \neq 0$

$\Rightarrow a_{ji}^2 > 0$

it follows that $tr AA > 0$

(c) Since $\{E_{ji}+E_{ij} \mid 1\le i\le j \le n\}$ is a basis of V

$\dim V = \frac{n^2+n}{2}$

$trA = 0 \Rightarrow \langle A, I\rangle = 0$

$W = \{A\in V \mid \langle A, I\rangle = 0\}$

It follows that $W^\perp = span\{I\}$

$\dim W^\perp = 1 \Rightarrow \dim W = \frac{n^2+n}{2} - 1$

8. $W = \{A \in \mathbb{R}^{n\times n} \mid a_{ij} = 0 \text{, for all } i \neq j\}$

$tr A = \sum_{i=1}^{n} a_{ii}$ $\quad A \in W,\ B \in V$

$tr(AB) = \sum_{i=1}^{n} a_{ii} b_{ii}$

$B \in W^{\perp} \Rightarrow \forall A \in W,\ \langle A, B \rangle = \sum_{i=1}^{n} a_{ii} b_{ii} = 0$

$\dim V = \frac{n^2+n}{2}$

$\dim W = n$

$\Rightarrow \dim W^{\perp} = \frac{n^2-n}{2}$

9. $\|v\|^2 = \sum_{i=1}^{m} \langle v, v_i \rangle^2$

$\forall\ \alpha_1, \alpha_2 \cdots \alpha_m \in \mathbb{R}$ are not all zero

$\langle \alpha_1 v_1 + \alpha_2 v_2 \cdots \alpha_m v_m, v_i \rangle = \alpha_i \langle v_i, v_i \rangle$

for all $i = 1, 2 \cdots m$, since $\langle v_i, v_j \rangle = 0 (i \neq j)$

it follows that

$\|\alpha_1 v_1 + \alpha_2 v_2 \cdots \alpha_n v_m\|^2 = \sum_{i=1}^{n} \alpha_i^2 \|v_i\|^2 = \sum_{i=1}^{n} \alpha_i^2 \geq 0$

if $\|\sum_{i=1}^{m} \alpha_i v_i\| = 0$

then $\alpha_i = 0$ for all $i = 1, 2 \cdots m$

a contradiction

Hence $\{v_1, v_2 \cdots v_m\}$ are L.I.

$\forall v \in span\{v_1, v_2 \cdots v_m\}$

$v = \alpha_1 v_1 + \alpha_2 v_2 \cdots \alpha_m v_m$

obviously $0 \in span\{v_1, v_2 \cdots v_m\}$

$$\|0\|^2 = 0 = \sum_{i=1}^{m} \langle 0, v_i \rangle^2 \Rightarrow 0 \in V$$

$$\|v\|^2 = \sum_{i=1}^{m} \langle v, a_i v_i \rangle = \sum_{i=1}^{m} \sum_{j=1}^{m} \langle a_j v_j, a_i v_i \rangle$$

since $\langle v_i, v_j \rangle = 0 \quad (i \neq j)$

$$\begin{cases} \|v\|^2 = \sum_{i=1}^{m} a_i^2 \langle v_i, v_i \rangle = \sum_{i=1}^{m} a_i^2 \\ \|v\|^2 = \sum_{i=1}^{m} \langle v, a_i v_i \rangle = \sum_{i=1}^{m} a_i \langle v, v_i \rangle \end{cases}$$

$$\Rightarrow \|v_i\|^2 = \sum_{i=1}^{m} \langle v, v_i \rangle^2$$

i.e. $\operatorname{span}\{v_1, v_2 \dots v_n\} \subseteq V$

$\forall \, v \in V$

$$\|v\|^2 = \sum_{i=1}^{m} \langle v, v_i \rangle^2$$

$$= \sum_{i=1}^{m} \langle v, \langle v, v_i \rangle v_i \rangle$$

$$= \langle v, \sum_{i=1}^{m} \langle v, v_i \rangle v_i \rangle$$

$$\Rightarrow \langle v, v - \sum_{i=1}^{m} \langle v, v_i \rangle v_i \rangle = 0$$

Since $0 \in V \cap \operatorname{span}\{v_1, v_2 \dots v_n\}$

Let $v \neq 0$

$$v = \sum_{i=1}^{m} \langle v, v_i \rangle v_i$$

i.e. $V \subseteq \operatorname{span}\{v_1, v_2 \dots v_m\}$

Hence $V = \operatorname{span}\{v_1, v_2 \dots v_m\}$

$$\begin{cases} \{v_1, v_2 \dots v_m\} \text{ are L.I} \\ V = \operatorname{span}\{v_1, v_2 \dots v_m\} \end{cases}$$

$\Rightarrow \{v_1, v_2 \dots v_m\}$ is a basis of V