Homework 0331

8. (a) $D(e^t) = e^t = 1e^t + 0e^{2t}$

$D(e^{2t}) = 2e^{2t} = 0e^t + 2e^{2t}$

$\Rightarrow M_\beta^\beta(D) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

(b) $D(1) = 0 = (0,0)\cdot(1,t)$

$D(t) = 1 = (1,0)\cdot(1,t)$

$\Rightarrow M_B^B(D) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(c) $D(e^t) = e^t = (1, 0)\cdot(e^t, te^t)$

$D(te^t) = e^t + te^t = (1, 1)\cdot(e^t, te^t)$

$\Rightarrow M_B^B(D) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(d) $D(1) = 0 = (0, 0, 0)\cdot(1, t, t^2)$

$D(t) = 1 = (1, 0, 0)\cdot(1, t, t^2)$

$D(t^2) = 2t = (0, 2, 0)\cdot(1, t, t^2)$

$\Rightarrow M_\mathcal{B}^\mathcal{B}(D) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(e) $\vec{v} = (1, t, e^t, e^{2t}, te^{2t})$

$D(1) = (0, 0, 0, 0, 0)\cdot\vec{v}$

$D(t) = 1 = (1, 0, 0, 0, 0) \cdot \vec{v}$

$D(e^t) = e^t = (0, 0, 1, 0, 0) \cdot \vec{v}$

$D(e^{2t}) = 2e^{2t} = (0, 0, 0, 2, 0) \cdot \vec{v}$

$D(te^{2t}) = 2te^{2t} + e^{2t} = (0, 0, 0, 1, 2) \cdot \vec{v}$

$$\Rightarrow M_B^B(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

(f) $D(\sin t) = \cos t = 0 \cdot \sin t + 1 \cdot \cos t$

$D(\cos t) = -\sin t = -1 \cdot \sin t + 0 \cdot \cos t$

$$\Rightarrow M_B^B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

9. (a) Since $N$ is nilpotent

$\Rightarrow \exists r \in \mathbb{N}^*$ s.t. $N^r = 0$

Setting $(I + N + N^2 + N^3 \cdots + N^{r-1}) = A$

$(I-N) \cdot A = I - N + N - N^2 + N^2 \cdots - N^{r-1} + N^{r-1} - N^r$

$= I - N^r$ (since $N^r = 0$)

$= I$

$A \cdot (I-N) = I - N^r = I$

Hence $I-N$ is invertible and $(I-N)^{-1} = A$

(b) for a linear mapping $f : V \to V$

Define $f^n := \underbrace{f \circ f \circ f \cdots \circ f}_{n}$

If $\exists r \in \mathbb{N}^*$ s.t. $f^r = 0$

i.e. $\forall x \in V \quad f^r(x) = 0$

Then $(id - f)$ is bijective

proof: suppose $\exists r \in \mathbb{N}^*$, s.t. $f^r = 0$

Setting $g = id + f + f \circ f + f \circ f \circ f \cdots f^r$

$(id - f) \circ g = id \circ g - f \circ g$

$= id + f - f + f^2 - f^2 \cdots - f^r$

$= id - f^n \quad (\text{since } f^r = 0)$

$= id$

Similarly $g \circ (id - f) = id$

Hence $(id - f)$ is invertible and $(id - f)^{-1} = g$

i.e. $id - f$ is bijective.

10. As long as prove that $D$ is nilpotent

As long as prove $D^{n+1} = 0$

Set $B = \{ n!, n!t, \frac{n!}{2!}t^2 \cdots, \frac{n!}{(n-1)!}t^{n-1}, t^n \}$ is a basis of $P_n$

$D(\frac{n!}{i!} t^i) = \frac{n!}{(i-1)!} t^{i-1}$

$$M_B^B(D) = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix} = N$$

$$\Rightarrow M_B^B(D^{n+1}) = N^{n+1}$$

Since Cayley-Hamilto Th

$$\operatorname{Det}(\lambda I - N) = \lambda^{n+1}$$ ($N$ is a strictly upper triangular matrix)

$$\Rightarrow N^{n+1} = 0$$

$$\Rightarrow x \overset{D^{n+1}}{\longmapsto} 0 \quad \text{for all } x \text{ in } P_n$$

$$D^{n+1} = 0 \Rightarrow \begin{cases} (D^2)^{n+1} = 0 \\ (D^m)^{n+1} = 0 \\ \left[\left(\frac{1}{\sqrt[m]{c}} D\right)^m\right]^{n+1} = 0 \quad \left(\frac{1}{c^n} D^{m(n+1)} = 0\right) \end{cases}$$

(a) Since $(D^2)^{n+1} = 0$

$\Rightarrow$ $D^2$ is nilpotent

$\Rightarrow$ $I - D^2$ is invertible

(b) Since $(D^m)^{n+1} = 0$

$\Rightarrow$ $I - D^m$ is invertible

$\Rightarrow$ $D^m - I$ is invertible

(c) Since $\left[\left(\frac{1}{\sqrt[m]{c}} D\right)^m\right]^{n+1} = 0$

$\Rightarrow$ $I - \left(\frac{1}{\sqrt[m]{c}} D\right)^m$ is invertible

$\Rightarrow$ $I - \frac{1}{c} D^m$ is invertible

$\Rightarrow$ $D^m - cI$ is invertible

$D^{n+1}: P_n \to P_n \quad , \quad D^{n+1} = 0$

$n = 0$

$M_{B_0}^{B_0}(D') = 0$

$\Rightarrow D = 0$

if $n = k$ , $D^{k+1} = 0$

then

Since $D^{k+1} = 0$

$D^{k+1}\left(\frac{k!}{i!} t^i\right) = 0$

$D^{k+2}\left(\frac{k+1!}{i!} t^i\right) = D\left(D^{k+1}\left(\frac{k!}{i!} t^i\right)\right) = D(0) = 0 \quad (i \leqslant k$

$D^{k+2}(t^{k+1}) = D^{k+1}(D(t^{k+1}))$

$= D^{k+1}((k+1)t^k) = k+1 \; D^{k+1}(t^k) = 0$

Hence $M_{B_{k+1}}^{B_{k+1}}(D^{k+2}) = 0$

$\Rightarrow D^{k+2}: P_{k+1} \longrightarrow P_{k+1} \quad , \quad D^{k+2} = 0$

By induction

$\forall n \in \mathbb{N}$ , $D: P_n \longrightarrow P_n$

$D^{n+1} = 0$