Homework 0329

$$P_n = \left\{ \sum_{k=0}^{n} a_k t^k \,\middle|\, a_k \in K \right\}, \quad D = \frac{d}{dt}$$

$\forall f \in P_n$

$$D(f) = \frac{df}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{n} a_k^f t^k\right) = \sum_{k=0}^{n} \frac{d}{dt}(a_k^f t^k) = \sum_{k=1}^{n} a_k^f t^{k-1} \in P_n$$

$\forall f_1, f_2 \in P_n, \ \forall \lambda \in K$

$$D(f_1+f_2) = \frac{d(f_1+f_2)}{dt} = \frac{df_1}{dt} + \frac{df_2}{dt} = D(f_1) + D(f_2)$$

$$D(\lambda f_1) = \frac{d(\lambda f_1)}{dt} = \lambda \frac{df_1}{dt} + f_1 \frac{d\lambda}{dt} = \lambda \frac{df_1}{dt} = \lambda D(f_1)$$

Hence $D: P_n \to P_n$ is linear

10.

$\forall v \in V$

$v = v - P(v) + P(v)$ where $P(v) \in \operatorname{Im} P$

Since P is linear

$$P(v - P(v)) = P(v) - P^2(v) \xlongequal{P^2=P} 0$$

$\Rightarrow v - P(v) \in \operatorname{Ker} P$

Hence $V = \operatorname{ker} P + \operatorname{Im} P$

for some $v \in \operatorname{Ker} P \cap \operatorname{Im} P$

$$\begin{cases} P(v) = 0 \\ \exists u \in V \text{ s.t. } P(u) = v \end{cases}$$

$v = P(u) = P^2(u) = P(v) = 0$

i.e. $\operatorname{Ker} P \cap \operatorname{Im} P = \{0\}$

11.

$\forall v \in V$

$v = I(v) = (P+Q)(v) = P(v) + Q(v)$

obviously $P(v) \in \operatorname{Im} P$, $Q(v) \in \operatorname{Im} Q$