Homework 0324

$$H_n = \{A \in \mathbb{C}^{n\times n} \mid A^* = A\}$$

over $\mathbb{C}$

$a \in \mathbb{C} \setminus \mathbb{R}$, $A \neq 0$

$$(aA)^* = {}^t(\overline{aA}) = \bar{a}\,{}^t(\bar{A}) = \bar{a}A \neq aA$$

i.e. $aA \notin H_n$

$\Rightarrow$ H_n is not a VS over $\mathbb{C}$

over $\mathbb{R}$

$$\{E_{ii} \mid i = 1, 2, \dots n\}$$

$$\{E_{ij} + E_{ji} \mid i \neq j \wedge i,j \in \mathbb{N}^* \wedge i,j \leq n\}$$

$$\{(E_{ij} - E_{ij})i \mid i \neq j \wedge i,j \in \mathbb{N}^* \wedge j,i \leq n\}$$

$$\Rightarrow \dim H_n = n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n^2$$

2.

$\forall\, v = (x, y, z)^T \in K^3 = V$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x - y + z)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (y - z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\vec{e_1} := (1, 0, 0)^T$ $\quad \vec{e_2} := (1, 1, 0)^T$ $\quad \vec{e_3} := (0, 1, 1)^T$

Since $(x - y + z), (y - z), z \in K$

Since $W = \operatorname{span}\{\vec{e_1}\}$

$U = \operatorname{span}\{\vec{e_1}, \vec{e_2}\}$

$w = (x-y+z)\vec{e_1} \in W$

$u = (y-z)\vec{e_2} + z\vec{e_3} \in U$

s.t $v = w + u$

i.e. $V = W + U$

$\forall v_0 \in W \cap U$

$\exists ! \alpha, \beta, \gamma \in K$

$\alpha\vec{e_1} = v_0 = \beta\vec{e_2} + \gamma\vec{e_3}$

$$\Rightarrow \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ \beta+\gamma \\ \gamma \end{pmatrix}$$

$\Rightarrow \alpha = \beta = \gamma = 0$

$\Rightarrow v_0 = 0$

i.e. $W \cap U = \{0\}$

Hence $V = U \oplus W$

36. Lemma 1: Cayley - Hamilto Th

$A \in \mathbb{C}^{n\times n}$, $\phi_A(\lambda) = \det(\lambda I_n - A)$

$\phi_A(A) = 0$

Proof.

Setting B is an Adjugate Matrix of $\lambda I - A$

$$B(\lambda I - A) = (\lambda I - A)B = \phi_A(\lambda) I = F(\lambda)$$

$$B = \lambda^{n-1} B_{n-1} + \lambda^{n-2} B_{n-2} \ \cdots \ B_0$$

where $B_\alpha = \{b_{ij}^{\alpha}\}^{n\times n}$ b_{ij}^{α} is a numbers (not polynomial of λ)

$$\Rightarrow F(\lambda) = \lambda^n a_n + \lambda^{n-1} a_{n-1} \cdots a_0$$

$a_i \in \mathbb{C}^{n \times n}$ $\quad i = 0, 1, 2 \cdots n$

$$F(A) = (AI - A)B = 0$$
$$= \phi_A(A) I$$
$$\Rightarrow \phi_A(A) = 0$$

Since $N = A - I_n$

N is a strictly upper triangular matrix $\underline{n+1 \times n+1}$

$\phi_N(\lambda) = \lambda^{n+1}$

$\Rightarrow \phi_N(N) = N^{n+1} = 0$

Hence $N^{n+1} = 0$

$$(I+N)(I - N + N^2 - N^3 \cdots + (-1)^n N^n)$$
$$= I + (-1)^n N^{n+1} = I$$

Hence $(I+N)^{-1} = I - N + N^2 \cdots + (-1)^n N^n$

39. Since A, B are nilpotend

W.L.G. $\exists r_A, r_B : r_A \le r_B \wedge r_A, r_B \in N^*$

s.t. $A^{r_A} = 0 \quad B^{r_B} = 0$

Since $AB = BA$

$$(AB)^{r_A} = A^{r_A} B^{r_A} = 0 \cdot B^{r_A} = 0$$

Hence AB is nilpotent.

Since A, B, AB are nilpotent.

since $AB = BA$

$$(A+B)^{2r_B} = A^{2r_B} + B^{2r_B} + \sum_{i=1}^{2r_B-1} \binom{2r_B}{i} A^i B^{2r_B-i}$$

① $2r_B > r_A$

$\Rightarrow A^{2r_B} = 0$

② $2r_B > r_B$

$\Rightarrow B^{2r_B} = 0$

③ $i \in (0, r_B) \cap N^*$

$2r_B - i > r_B$

$\Rightarrow A^i B^{2r_B-i} = 0$

④ $i \in [r_B, 2r_B) \cap N^*$

$i \geq r_B > r_A$

$\Rightarrow A^i B^{2r_B-i} = 0$

Hence $(A+B)^{2r_B} = 0$

i.e. $A+B$ is nilpotent.