hw0424

5. $A(v) = \langle v_0, v \rangle w_0$

$\forall v, w \in V$

$\langle Av, w \rangle = \langle \langle v_0 v \rangle w_0, w \rangle = \langle v_0, v \rangle \langle w_0, w \rangle$

$= \langle \langle w_0, w \rangle v_0, v \rangle$

$= \langle B(w), v \rangle$

$= \langle v, Bw \rangle$

where $B(w) = \langle w_0, w \rangle v_0$

Hence $B = A^T$

i.e. $A^T(w) = \langle w_0, w \rangle v_0$

7. $\forall v \in \operatorname{Ker} A$

$Av = 0$

it follows that, for all $w \in V$

$0 = \langle Av, w \rangle = \langle v, A^T w \rangle$

where $A^T w \in \operatorname{Im} A^T$

Hence $\operatorname{Ker} A \subset (\operatorname{Im} A^T)^\perp$

$\forall v \in (\operatorname{Im} A^T)^\perp$

for all $w \in V$, $\exists w' = A^T w \in \operatorname{Im} A^T$

for all $w' \in \operatorname{Im} A^T$

$0 = \langle v, w' \rangle = \langle v, A^T w \rangle = \langle Av, w \rangle$

Since the product is non-degenerate

we can obtain $Av = 0$

i.e. $v \in \operatorname{Ker} A$

$\Rightarrow \operatorname{Ker} A \supset (\operatorname{Im} A^T)^\perp$

$\Rightarrow \operatorname{Ker} A = (\operatorname{Im} A^T)^\perp$

12. $A^2 = I + P$

$\forall v \in V \quad \exists\ v_1 \in W,\ v_2 \in W^\perp \quad s.t.\ v = v_1 + v_2$
$\forall w \in V \quad \exists\ w_1 \in W,\ w_2 \in W^\perp \quad s.t.\ w = w_1 + w_2$

$$\begin{aligned}\langle A^T v, Aw \rangle &= \langle v, A^2 w \rangle \\ &= \langle v, Iw + Pw \rangle \\ &= \langle v, w \rangle + \langle v, w_1 \rangle \\ &= 2\langle v_1, w_1 \rangle + 2\langle v_2, w_2 \rangle \\ &= 2\langle v, w \rangle\end{aligned}$$

$\langle Av, A^T w \rangle = 2\langle v, w \rangle$

$\langle (A-A^T)v, (A-A^T)w \rangle = 0$

Suppose $N = A - A^T \neq 0$

i.e. $\exists\ v \in V$ s.t. $Nv \neq 0$

$\Rightarrow \langle Nv, Nv \rangle > 0$ (since the product is positive definite

a contradiction

Hence $A = A^T$

i.e A is symmertic.

2. count example.

$V = R^n$

$A: V \longrightarrow V$

$(x_1, x_2, x_3 \cdots x_n) \longmapsto (x_2, -x_1, 0, 0, 0 \cdots 0)$

for all $v \in V$ ①

$\langle Av, v \rangle = 0$

$AE_1 = -E_2 \neq 0$ ②

3. If A is symmertic

i.e. $A = A^T \Rightarrow$

$\forall v, w \in V$

$\langle Av, w \rangle + \langle Aw, v \rangle = 0$

$\Rightarrow \langle Av, w \rangle = 0$

Since the product is non-degenerate

$Av = 0$ for all $v \in V$

It follows that

$A = 0$

8. $I - iA$ is invertible

Since A is hermition

$A = A^*$

$\forall v, w \in V$

$\langle Av, w \rangle = \langle v, Aw \rangle$

$$\begin{aligned}\langle (I - iA)v, w \rangle &= \langle v, w \rangle - i\langle Av, w \rangle \\ &= \langle v, w \rangle + \langle Av, iw \rangle \\ &= \langle v, w \rangle + \langle v, iAw \rangle \\ &= \langle v, (I + iA)w \rangle\end{aligned}$$

Then $I - iA = (I + iA)^*$

$\langle I - iAv_1, (I - iA)v_2 \rangle$

$= \langle (I + iA)(I - iA)v_1, v_2 \rangle$

$= \langle (I + A^2)v_1, v_2 \rangle$

$= \langle v_1, v_2 \rangle + \langle Av_1, Av_2 \rangle$

If $v_1 = v_2$

$\langle (I - iA)v_1, (I - iA)v_1 \rangle = \langle v_1, v_1 \rangle + \langle Av_1, Av_1 \rangle$

Since the product is positive defined

$\forall\, v \in \operatorname{Ker}(I - iA)$

$0 = \langle (I - iA)v, (I - iA)v \rangle = \langle v, v \rangle + \langle Av, Av \rangle \geq 0$

Only when $v = 0$, $\langle v, v \rangle = 0 \wedge \langle Av, Av \rangle = 0$

i.e. $\operatorname{Ker}(I - iA) = \{0\}$

Consider $I - iA$ is linear

$\dim V = \dim V$

$\Rightarrow$ $I - iA$ is bijective.

Hence $I - iA$ is in invertible

Similerly $I + iA$ is also invertible