hw 0417

1. (a) $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

$$0 = \mathrm{Det}(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 5$$

$\lambda_1 = \sqrt{5}$

$\lambda_2 = -\sqrt{5}$

index of nullity is 0

index of positivity is 1

(b) $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 1 = 0$

$\lambda_1 = 0$

$\lambda_2 = 2$

$\Rightarrow \begin{cases} \text{index of nullity is } 1 \\ \text{index of positivity is } 1 \end{cases}$

(c) $\begin{vmatrix} 1-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 7 = 0$

$\lambda_1 \lambda_2 = -7 < 0$

$\Rightarrow \begin{cases} \text{index of nullity is } 0 \\ \text{index of nullity is } 1 \end{cases}$

2. Let $\{v_1, v_2 \cdots v_n\}$ be the othogonal basis for V s.t.

$$\begin{cases} \langle v_i, v_i \rangle > 0 & i \in [1, r] \cap \mathbb{N}^* \\ \langle v_i, v_i \rangle < 0 & i \in [r+1, s] \cap \mathbb{N}^* \\ \langle v_i, v_i \rangle = 0 & i \in [s+1, n] \cap \mathbb{N}^* \end{cases}$$

$V_0 = \text{span}\{v_{s+1}, v_{s+2} \cdots v_n\}$

$V^{+'} = \text{span}\{v_1, v_2 \cdots v_r\}$

$V^{-'} = \text{span}\{v_{r+1}, v_{r+2} \cdots v_s\}$

obviously $V = V_0 \oplus V^{+'} \oplus V^{-'}$

$\forall\ v = a_1 v_1 + a_2 v_2 + a_3 v_3 \cdots a_r v_r \in V^{+'}$ if $v \neq 0$

$\langle v, v \rangle = \sum_{i=1}^{r} a_i^2 \langle v_i, v_i \rangle$

Since $\langle v_i, v_i \rangle > 0$ for all $i = 1, 2 \cdots r$

$\langle v, v \rangle > 0$

i.e. $v \in V^+$

It follows that $V^{+'} \subseteq V^+$

$\forall\ v \in V^+$ if $v \neq 0$

$v = \sum_{i=1}^{n} a_i v_i$

$\langle v, v \rangle > 0$

$\langle v, v \rangle = \sum_{i=1}^{n} a_i^2 \langle v_i, v_i \rangle$

obviosuly

$v_1 = \sum_{i=1}^{r} a_i v_i \quad \in V^+ \quad$ (since $\langle v_1, v_1 \rangle > 0$

Since V^+ is a VS

$v - v_1 \in V^+$

$0 \leq \langle v - v_1 \quad v - v_1 \rangle = \sum_{i=r+1}^{n} a_i^2 \langle v_i, v_i \rangle \leq 0$

$\Rightarrow \quad v = v_1 \in V^{+'}$

i.e. $V^+ \subseteq V^{+'}$

Hence $V^+ = V^{+'}$

i.e. $V^+ = \text{span}\{v_1, v_2 \cdots v_r\}$

Similarly

$V^- = \text{span}\{v_{r+1}, v_{r+2} \cdots v_s\}$

Then $\dim V^+$ = index of positivity

$\dim V^-$ = index of negtivity