$$D_n = a D_{n-1} - bc D_{n-2}$$

$$x^2 = ax - bc \qquad \Rightarrow \begin{cases} x_1 + x_2 = a \\ x_1 x_2 = bc \end{cases}$$

$$\Delta = a^2 - 4bc$$

Case $\Delta \neq 0$ i.e. $x_1 \neq x_2 \in \mathbb{C}$

$$D_n - x_1 D_{n-1} = x_2 (D_{n-1} - x_1 D_{n-2})$$

$$D_n - x_2 D_{n-1} = x_1 (D_{n-1} - x_2 D_{n-2})$$

Setting $C_n = D_n - x_1 D_{n-1}$

$B_n = D_n - x_2 D_{n-1}$

Redefine $D_0 = 1 \Rightarrow \begin{array}{l} C_1 = D_1 - x_1 D_0 = x_1 + x_2 - x_1 = x_2 \\ B_1 = D_1 - x_2 D_0 = x_1 + x_2 - x_2 = 1 \end{array}$

$C_n = x_2^n$

$B_n = x_1^n$

$$\Rightarrow \begin{cases} D_n - x_1 D_{n-1} = x_2^n \\ D_n - x_2 D_{n-1} = x_1^n \end{cases}$$

$$\Rightarrow (x_2 - x_1) D_n = x_2^{n+1} - x_1^{n+1}$$

$$\Rightarrow D_n = \frac{(x_2^{n+1} - x_1^{n+1})}{x_2 - x_1} \qquad \text{where} \begin{cases} x_1 = \dfrac{a - \sqrt{a^2 - 4bc}}{2} \\ x_2 = \dfrac{a + \sqrt{a^2 + 4bc}}{2} \end{cases}$$

Case $\Delta = 0 \Rightarrow x_1 = x_2 = \frac{a}{2}$

$$D_n - \frac{a}{2} D_{n-1} = \left(\frac{a}{2}\right)^n$$

$$\frac{a}{2} D_{n-1} - \left(\frac{a}{2}\right)^2 D_{n-2} = \left(\frac{a}{2}\right)^n$$

$$\vdots$$

$$\left(\frac{a}{2}\right)^{n-1} D_1 - \frac{a}{2}^n D_0 = \left(\frac{a}{2}\right)^n$$

$$\Rightarrow \quad D_n - \left(\frac{a}{2}\right)^n = n \cdot \left(\frac{a}{2}\right)^n$$

i.e. $D_n = (n+1)\left(\frac{a}{2}\right)^n$

$$= \frac{(n+1) a^n}{2^n}$$

11. suppose $\{e^{a_1 t}, e^{a_2 t}, \cdots e^{a_n t}\}$ is L.D.

it follows that $\exists\ (c_1, c_2 \cdots c_n) \in \mathbb{C}^n$

Such that, $\forall\ t \in \mathbb{C}$

$$F(t) = c_1 e^{a_1 t} + c_2 e^{a_2 t} \cdots c_n e^{a_n t} = 0$$

$$F'(t) = c_1 a_1 e^{a_1 t} + c_2 a_2 e^{a_2 t} \cdots c_n a_n e^{a_n t} = 0$$

$$\cdots$$

$$F^{(n-1)}(t) = c_1 a_1^{n-1} e^{a_1 t} \cdots + c_n a_n^{n-1} e^{a_n t} = 0$$

Setting $t=0$, we can obtain

$$\begin{cases} c_1 + c_2 \cdots c_n = 0 \\ c_1 a_1 + c_2 a_2 \cdots c_n a_n = 0 \\ \vdots \\ c_1 a_1^{n-1} + c_2 a_2^{n-1} \cdots c_n a_n^{n-1} = 0 \end{cases}$$

i.e. $A_n C = 0$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & \cdots & a_n \\ a_1^2 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ a_1^{n-1} & \cdots & \cdots & \cdots & a_n^{n-1} \end{pmatrix}}_{\substack{\parallel \\ A}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\substack{\parallel \\ C}} = 0$$

Since C is not zero

$Det(A)$ must be zero

however, $a_1, a_2 \cdots a_n$ are distinct

$$Det(A_{1,n}, A_{2,n}, A_{3,n} \cdots A_{n,n})$$

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2-a_1 & a_3-a_1 & \cdots & a_n-a_1 \\ a_1^2 & a_2^2-a_1^2 & a_3^2-a_1^2 & \cdots & a_n^2-a_1^2 \\ \vdots & & & & \\ a_1^{n-1} & & \cdots & & a_n^{n-1}-a_1^{n-1} \end{vmatrix} = \begin{vmatrix} a_2-a_1 & a_3-a_1 & \cdots & a_n-a_1 \\ a_2^2-a_1^2 & & \cdots & a_n^2-a_1^2 \\ \vdots & & \ddots & \vdots \\ a_1^{n-1} & & \cdots & a_n^{n-1}-a_1^{n-1} \end{vmatrix}$$

$$= \prod_{i=2}^{n} (a_i - a_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2+a_1 & a_3+a_1 & & \vdots \\ a_2^2+a_1a_2+a_1^2 & & \ddots & \vdots \\ \vdots & & & \vdots \\ \sum_{i=0}^{n-2} a_1^i a_2^{n-2-i} & & & \sum_{i=1}^{n-2} a_1^i a_n^{n-2-i} \end{vmatrix}$$

$$= \prod_{i=2}^{n} (a_i - a_1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & & a_n \\ a_2^2+a_1a_2 & & \cdots & & a_n^2+a_1a_n \\ \vdots & & & & \\ \sum_{i=0}^{n-3} a_1^i a_2^{n-2-i} & & \cdots & & \end{vmatrix}$$

$$= \prod_{i=2}^{n} (\alpha_i - \alpha_1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & & \alpha_n \\ \alpha_2^2 & \alpha_3^2 & \cdots & & \alpha_n^2 \\ \vdots & & & & \\ \alpha_2^{n-2} & \cdots & & & \alpha_n^{n-2} \end{vmatrix}$$

$$= \prod_{i=2}^{n} (\alpha_i - \alpha_1) \prod_{i=3}^{n} (\alpha_i - \alpha_2) \cdots (\alpha_n - \alpha_{n-1})$$

$$= \prod_{1 \le j < k \le n} (\alpha_k - \alpha_j) \neq 0$$

a contradiction

Hence $\{e^{\alpha_1 t}, e^{\alpha_2 t} \cdots e^{\alpha_n t}\}$ are L.I.