hw 0429

4. $\mathrm{Det}(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} \sum_{j=1}^{n} a_{1j} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \ddots \\ \sum_{j=1}^{n} a_{nj} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

$$= \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

$\Rightarrow |A| = |A - 0I| = 0$

It follows that 0 is a root of $|A-\lambda I| = 0$

i.e. 0 is a eigenvalue of A

5 (a)

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$|A| = -\cos^2\theta - \sin^2\theta = -1$

$\Rightarrow A^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

$0 = |\lambda I - A| = \lambda^2 - \cos^2\theta - \sin^2\theta$

$= \lambda^2 - 1$

$\Rightarrow \lambda_1 = 1$

$\lambda_2 = -1$

$A\gamma_1 = \lambda_1\gamma_1 = \gamma_1$

Setting $\gamma_1 = (x, y)^T$

w.l.o.g, let $x = 1$, we can obtain,

$\cos\theta + y\sin\theta = 1 \Rightarrow y = \dfrac{1-\cos\theta}{\sin\theta}$ $\quad \theta \neq k\pi$ for all $k \in \mathbb{Z}$

$\sin\theta - y\cos\theta = y \qquad = \dfrac{\sin\theta}{1+\cos\theta}$

if $\theta = 2k\pi$, $k \in \mathbb{Z}$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

if $\theta = \pi + 2k\pi \quad k \in \mathbb{Z}$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence for all $\theta \in \mathbb{R}$

1 is eigenvalue of A

$\begin{pmatrix} \sin\theta \\ 1-\cos\theta \end{pmatrix}$ is eigenvector of A $(\theta \neq 2k\pi)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is eigenvector of A $(\theta = 2k\pi)$

(b) w.l.o.g let $x = 1$

$\cos\theta + y\sin\theta = -1 \implies y = -\frac{1+\cos\theta}{\sin\theta}$

Let $v_2 = (-\sin\theta, 1+\cos\theta)^T$

$$Av_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\theta \\ 1+\cos\theta \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -1-\cos\theta \end{pmatrix} = -v_2$$

If $-\sin\theta \neq 0 \wedge 1+\cos\theta \neq 0$,

i.e. $\theta \neq \pi + 2k\pi \quad k \in \mathbb{Z}$

$v_2 \neq 0$

If $\theta = \pi + 2k\pi$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \implies v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Case $\theta \neq k\pi$

$\langle v_1, v_2 \rangle = \sin\theta \cdot (-\sin\theta) + (1-\cos\theta)(1+\cos\theta) = 0$

Case $\theta = 2k\pi$ $\quad \sin\theta = 0 \quad \cos\theta = 1$

$v_2 = (0, 2)^T$

$$\langle v_1, v_2 \rangle = (1, 0)\begin{pmatrix}0\\2\end{pmatrix} = 0$$

Case $\theta = \pi + 2k\pi$ $\quad \sin\theta = 0 \quad \cos\theta = -1$

$v_1 = (0, -2)^T$

$$\langle v_1, v_2 \rangle = (0, -2)\begin{pmatrix}1\\0\end{pmatrix} = 0$$

Therefore $v_1 \perp v_2$, for all θ

14. $v = x_1 v_1 + x_2 v_2 \cdots x_n v_n$

$$\begin{aligned} Av &= A(x_1 v_1 + x_2 v_2 \cdots x_n v_n) \\ &= x_1 A v_1 + x_2 A v_2 \cdots x_n A v_n \\ &= x_1 c_1 v_1 + x_2 c_2 v_2 \cdots x_n c_n v_n \end{aligned}$$

Since v is a eigenvector of A, $\exists \lambda \in K$ s.t.

$$Av = \lambda v = x_1 \lambda v_1 + x_2 \lambda v_2 \cdots x_n \lambda v_n$$

It follows that

$$x_1(c_1 - \lambda)v_1 + x_2(c_2 - \lambda)v_2 \cdots x_n(c_n - \lambda)v_n = 0$$

Since $\{v_1, v_2 \cdots v_n\}$ is L.I.

$\forall i = 1, 2, \cdots n$

$x_i(c_i - \lambda) = 0$

Case $\lambda = c_k$

$\forall i = 1, 2 \cdots k-1, k+1 \cdots n)$

$c_i - \lambda \neq 0$

$\Rightarrow x_i = 0$

i.e. $v = x_k v_k$

Case. $\forall i = 1, 2 \cdots n$

$c_i - \lambda \neq 0$

$\Rightarrow v = 0 \Rightarrow v$ is not eigenvector

A contradiction

Hence

v is some salar multiple of some v_i

15. Suppouse $v \neq 0$ is an eigenvector of AB

λ is the eigenvalue belong to v

i.e. $ABv = \lambda v$

It follows that

$$BA(Bv) = B(ABv) = B(\lambda v) = \lambda(Bv)$$

$\Rightarrow$ Bv is the eigenvector of BA

as long as $Bv \neq 0$

and λ is the eigenvalue belong to Bv

if $Bv = 0$

$\lambda v = ABv = A0 = 0$

Since $v \neq 0$, we can obtain

$\lambda = 0$

Since $|BA| = |B||A| = |A||B| = |AB|$

0 is an eigenvalue of AB $\Rightarrow$ 0 is an eigenvalue of BA

Hence λ is an eigenvalue of AB $\Rightarrow$ λ is an eigenvalue of BA

Similarly,

λ is an eigenvalue of BA $\Rightarrow$ λ is an eigenvalue of AB

Hence they are same.