Hw 0426

5. (a) Let $M = M_{\beta'}^{\beta}(id)$

$\forall v = x_1 v_1 + x_2 v_2 \cdots + x_n v_n = y_1 w_1 + y_2 w_2 \cdots + y_n w_n$

$X = (x_1, x_2, x_3 \cdots x_n)^T$

$Y = (y_1, y_2, y_3 \cdots y_n)^T$

$MX = Y$

Since the product is positive defined and these basis is orthonormal

$$\langle v, v' \rangle = \langle x_1 v_1 + x_2 v_2 \cdots x_n v_n, x_1' v_1 + x_2' v_2 \cdots x_n' v_n \rangle$$
$$= \sum_{i=1}^{n} x_i x_i' \langle v_i, v_i \rangle + \sum_{1 \le i \ne j \le n} x_i x_j' \langle v_i, v_j \rangle$$
$$= \sum_{i=1}^{n} x_i x_i' \langle v_i, v_i \rangle$$
$$= X^T X'$$

Similarly $\langle v, v \rangle = Y^T Y'$

$X^T X' = \langle v, v' \rangle = Y^T Y' = (MX)^T MX' = X^T M^T M X' \quad \forall X, X' \in \mathbb{R}^n$

$\Rightarrow M^T M = I \quad (M^T M = (m_{ij})_{n \times n} \Rightarrow m_{ij} = E_i^T M^T M E_j)$

i.e. M is unitary

(b) $X^T X = \langle v, v \rangle$

$\langle F(v), F(v) \rangle = \langle x_1 w_1 + x_2 w_2 \cdots x_n w_n, x_1 w_1 + x_2 w_2 \cdots x_n w_n \rangle$
$= X^T X$

$\langle F(v), F(v) \rangle = \langle v, v \rangle \quad \Rightarrow F$ is unitary

$M_{\beta'}^{\beta}(F) X = X$ for all X

Hence $M_{\beta'}^{\beta}(F) = I$

$(IX)^T IX = X^T X$

Hence $M_{\beta'}^{\beta}(F)$ is unitary

13. $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \det U = 1 \quad U^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = (\bar{U})^T$

Hence U is unitary

$$U^{-1}BU = A \iff UAU^{-1} = B$$

14 $\mathbb{R}$- Linear

① $\forall z, w \in \mathbb{C}$ $\quad L_\alpha(z+w) = \alpha(z+w) = \alpha z + \alpha w = L_\alpha(z) + L_\alpha(w)$

② $\forall c \in \mathbb{R}\ z \in \mathbb{C}$ $\quad L_\alpha(cz) = \alpha(cz) = c(\alpha z) = c L_\alpha(z)$

$\langle L_\alpha z , L_\alpha w \rangle$

$= \langle \alpha z , \alpha w \rangle$

$= Re[\alpha z \overline{(\alpha w)}]$

$= Re(\alpha \bar{\alpha} z \bar{w})$

$= |\alpha|^2 Re(z\bar{w})$

$= |\alpha|^2 \langle z, w \rangle$

if $|\alpha| = 1$ $\quad$ i.e. $\alpha = e^{i\theta}$ $\quad \theta \in [-\pi, \pi]$

L_α is unitary

Setting $\alpha = e^{i\theta} = \cos\theta + i\sin\theta$

$L_\alpha(1) = \alpha = \cos\theta + i\sin\theta$

$L_\alpha(i) = \alpha i = -\sin\theta + i\cos\theta$

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$