Homework 0403

1. $\vec{v_1} = (1, 1, -2, 3, 4, 5)^T$

$\vec{v_2} = (0, 0, 1, 1, 0, 7)^T$

Setting $A = \text{span}\{\vec{v_1}, \vec{v_2}\} \subseteq \mathbb{R}^6$

$B = \{\vec{v} \in \mathbb{R}^6 \mid \vec{v} \cdot \vec{v_1} = 0 \wedge \vec{v} \cdot \vec{v_2} = 0\}$

B . A are subspace of $\mathbb{R}^6$ and $B = A^\perp$

Since $\dim A + \dim A^\perp = \dim \mathbb{R}^6 = 6$

$\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow$

$\vec{v_1}$ and $\vec{v_2}$ are L.I. $\Rightarrow \dim A = 2$

$\Rightarrow \dim B = \dim A^\perp = 6 - 2 = 4$

2. (a) $\vec{v_1} = (1, 2, 1, 0)^T$, $\vec{v_2} = (1, 2, 3, 1)^T$

$\vec{v_1}' = \vec{v_1} = (1, 2, 1, 0)^T$

$\vec{v_2}' = \vec{v_2} - \frac{\langle \vec{v_1}', \vec{v_2} \rangle}{\langle \vec{v_1}', \vec{v_1}' \rangle} \vec{v_1}' = (1, 2, 3, 1)^T - \frac{4}{3}(1, 2, 1, 0)$

$= (-\frac{1}{3}, -\frac{2}{3}, \frac{5}{3}, 1)^T$

$\vec{v_1}'' = \frac{\vec{v_1}'}{\|\vec{v_1}'\|} = \frac{\sqrt{6}}{6}(1, 2, 1, 0)^T = (\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, 0)^T$

$\vec{v_2}'' = \frac{\vec{v_2}'}{\|\vec{v_2}'\|} = \frac{\sqrt{39}}{39}(-1, -2, 5, 3)^T = (-\frac{\sqrt{39}}{39}, -\frac{2\sqrt{39}}{39}, \frac{5\sqrt{39}}{39}, \frac{\sqrt{39}}{13})$

$\{\vec{v_1}'', \vec{v_2}''\}$ is an orthonormal basis

(b) $\vec{v_1} = (1, 1, 0, 0)^T$

$\vec{v_2} = (1, -1, 1, 1)^T$

$\vec{v_3} = (-1, 0, 2, 1)^T$

since $\langle \vec{v_1}, \vec{v_2} \rangle = 0$

$\vec{v_1}' = \vec{v_1}$

$\vec{v_2}' = \vec{v_2}$

$\vec{v_3}' = \vec{v_3} - \frac{\langle \vec{v_3}, \vec{v_1}' \rangle}{\langle \vec{v_1}', \vec{v_1}' \rangle} \vec{v_1}' - \frac{\langle \vec{v_3}, \vec{v_2}' \rangle}{\langle \vec{v_2}', \vec{v_2}' \rangle} \vec{v_2}'$

$= \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

$= (-1, 1, \frac{3}{2}, \frac{1}{2})^T$

$\vec{v_1}'' = \frac{\vec{v_1}'}{\|\vec{v_1}'\|} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0)^T$

$\vec{v_2}'' = \frac{\vec{v_2}'}{\|\vec{v_2}'\|} = (\frac{\sqrt{4}}{4}, -\frac{\sqrt{4}}{4}, \frac{\sqrt{4}}{4}, \frac{\sqrt{4}}{4})^T$

$\vec{v_3}'' = \frac{\vec{v_3}'}{\|\vec{v_3}'\|} = (-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{6})^T$

$\{ \vec{v_1}'', \vec{v_2}'', \vec{v_3}'' \}$

3. $\langle f, g \rangle = \int_0^t f(t) g(t)\, dt$

① obviously $\langle f, g \rangle \in \mathbb{R}$

② $\langle f, g \rangle = \int_0^t f(t) g(t)\, dt = \int_0^t g(t) f(t)\, dt = \langle g, f \rangle$

③ $f, h, g \in C[0,1]$ are real-valued

$\langle f+h, g \rangle = \int_0^1 (f(t) + h(t)) g(t)\, dt = \int_0^1 (f(t)g(t) + h(t)g(t))\, dt$

$= \int_0^1 f(t) g(t)\, dt + \int_0^1 h(t) g(t)\, dt = \langle f, g \rangle + \langle h, g \rangle$

④ $\forall c \in \mathbb{R}$

$\langle cf, g \rangle = \int_0^1 (cf)(t) g(t) = \int_0^1 c\, f(t) g(t)\, dt = c \int_0^1 f(t)\, g(t)\, dt = c\langle f, g \rangle$

$\int_0^1 f(t) (cg)(t)\, dt = \langle f, cg \rangle$

$\Rightarrow \langle f, g \rangle$ is a scalar product

5. $\vec{v_1} = 1$

$$\vec{v_2} = t - \frac{\langle 1, t \rangle}{\langle 1, 1 \rangle} 1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt} t = t - \frac{1}{2} 1 = t - \frac{1}{2}$$

$$\vec{v_3} = t^2 - \frac{\langle 1, t^2 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle t - \frac{1}{2}, t^2 \rangle}{\langle t - \frac{1}{2}, t - \frac{1}{2} \rangle} (t - \tfrac{1}{2})$$

$$= t^2 - \frac{\int_0^1 t^2\,dt}{\int_0^1 1\,dt} 1 - \frac{\int_0^1 t^3 - \frac{1}{2} t\,dt}{\int_0^1 t^2 - t + \frac{1}{4}\,dt} (t - \tfrac{1}{2})$$

$$= t^2 - \frac{1}{3}$$

$$\langle t - \tfrac{1}{2}, t - \tfrac{1}{2} \rangle = \int_0^1 t^2 - t + \frac{1}{4} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$$\langle t^2 - \tfrac{1}{3}, t^2 - \tfrac{1}{3} \rangle = \int_0^1 t^4 - \frac{2}{3} t^2 + \frac{1}{9}\,dt = \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{4}{45}$$

$$\vec{v_1}' = \frac{\vec{v_1}}{\|\vec{v_1}\|} = 1$$

$$\vec{v_2}' = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \sqrt{12}(t - \tfrac{1}{2})$$

$$\vec{v_3}' = \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{3\sqrt{5}}{2} (t^2 - \tfrac{1}{3})$$

$$\langle 1, t - \tfrac{1}{2} \rangle = \int_0^1 t - \frac{1}{2}\,dt = 0$$

$$\langle 1, t^2 - \tfrac{1}{3} \rangle = \int_0^1 t^2 - \frac{1}{3}\,dt = 0$$

$$\langle t - \tfrac{1}{2}, t^2 - \tfrac{1}{3} \rangle = \int_0^1 t^3 - \frac{1}{2} t^2 - \frac{1}{3} t + \frac{1}{6}\,dt = \frac{1}{4} - \frac{1}{4} - \frac{1}{6} + \frac{1}{6} = 0$$

Hence $\{\vec{v_1}', \vec{v_2}', \vec{v_3}'\}$ is what we want.