Homework 0329

$P_n = \left\{ \sum_{k=0}^{n} a_k t^k \mid a_k \in K \right\}$, $D = \frac{d}{dt}$

$\forall f \in P_n$

$$D(f) = \frac{df}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{n} a_k^f t^k\right) = \sum_{k=0}^{n} \frac{d}{dt}(a_k^f t^k) = \sum_{k=1}^{n} a_k^f t^{k-1} \in P_n$$

$\forall f_1, f_2 \in P_n$, $\forall \lambda \in K$

$$D(f_1+f_2) = \frac{d(f_1+f_2)}{dt} = \frac{df_1}{dt} + \frac{df_2}{dt} = D(f_1) + D(f_2)$$

$$D(\lambda f_1) = \frac{d(\lambda f_1)}{dt} = \lambda \frac{df_1}{dt} + f_1 \frac{d\lambda}{dt} = \lambda \frac{df_1}{dt} = \lambda D(f_1)$$

Hence $D: P_n \longrightarrow P_n$ is linear

10.

$\forall v \in V$

$v = v - P(v) + P(v)$ where $P(v) \in \operatorname{Im} P$

Since P is linear

$P(v - P(v)) = P(v) - P^2(v) \xlongequal{P^2=P} 0$

$\Rightarrow v - P(v) \in \operatorname{Ker} P$

Hence $V = \operatorname{Ker} P + \operatorname{Im} P$

for some $v \in \operatorname{Ker} P \cap \operatorname{Im} P$

$$\begin{cases} P(v) = 0 \\ \exists u \in V \text{ s.t. } P(u) = v \end{cases}$$

$v = P(u) = P^2(u) = P(v) = 0$

i.e. $\operatorname{Ker} P \cap \operatorname{Im} P = \{0\}$

11.

$\forall v \in V$

$v = I(v) = (P+Q)(v) = P(v) + Q(v)$

obviously $P(v) \in \operatorname{Im} P$, $Q(v) \in \operatorname{Im} Q$

Hence $V = \mathrm{Im}P + \mathrm{Im}Q$

for some $u \in \mathrm{Im}P \cap \mathrm{Im}Q$

$\exists u_1 \in V$ s.t. $P(u_1) = u$

$\exists u_2 \in V$ s.t. $Q(u_2) = u$

$$u = P(u) + Q(u) = P(P(u_1)) + PQ(u_1) + Q(Q(u_2)) + QP(u_2)$$
$$= P(u_1) + Q(u_2)$$
$$= 2u \qquad \text{by (b) and (c)}$$

$\Rightarrow u = 0$

i.e. $\mathrm{Im}P \cap \mathrm{Im}Q = \{0\}$

Hence $V = \mathrm{Im}P \oplus \mathrm{Im}Q$

12. $\forall u \in \mathrm{Ker}\,Q$

$$\begin{cases} Q(u) = 0 \\ P(u) + Q(u) = u \end{cases}$$

$\Rightarrow u = P(u)$

i.e. $u \in \mathrm{Im}P$

$\Rightarrow \mathrm{Ker}\,Q \subseteq \mathrm{Im}P$

$\forall u \in \mathrm{Im}P$

$\exists u_1 \in V$ s.t. $P(u_1) = u$

$Q(u) = Q \circ P(u_1) = 0$

i.e. $u \in \mathrm{Ker}\,Q$

$\Rightarrow \mathrm{Ker}\,Q \supseteq \mathrm{Im}P$

$\Rightarrow \mathrm{Ker}\,Q = \mathrm{Im}P$

13.

$$P + Q = \frac{1}{2}(I+T) + \frac{1}{2}(I-T)$$
$$= I$$

$$P^2 = \left[\frac{1}{2}(I+T)\right]^2 = \frac{1}{4}(I + 2T + T^2) = \frac{1}{4}(2I + 2T) = \frac{1}{2}(I+T) = P$$

$$Q^2 = \frac{1}{4}(I-T)^2 = \frac{1}{4}(I - 2T + T^2) = \frac{1}{2}(I-T) = Q$$

$$PQ = \frac{1}{4}(I+T)(I-T) = \frac{1}{4}(I - T + T - T^2) = 0$$
$$QP = \frac{1}{4}(I-T)(I+T) = \frac{1}{4}(I + T - T - T^2) = 0$$

15.

Setting $v_1 \neq v_2 \in V$

$F(v_1) = w_1 \in W$

$F(v_2) = w_2 \in W$

Since F is an isomorphism of V and W over K

F is bijective $\Rightarrow$ F is injective.

it follow that $w_1 \neq w_2$

Similarly

$G(w_1) \neq G(w_2)$

i.e. $G \circ F(v_1) \neq G \circ F(v_2)$

Hence $G \circ F$ is injective

$\forall u \in U$. since G is an isomorphism

G is sujective

$\exists\, w \in W$ s.t. $G(w) = u$

Similarly $\exists\, v \in V$ s.t. $F(v) = w$

It follows that.

$G \circ F(v) = u$

i.e. $G \circ F$ is surjective.

$\Rightarrow$ $G \circ F : V \to U$ is bijective

i.e. it is an isomorphism of V and U over K