hw0414

3. $v_1 = (1, i, 0)^T$

$W = \text{span}\{v_1\}$

If $\mathbb{C}^3$ over $\mathbb{C}$

$v_2 = (0, 0, 1)^T \quad \langle v_1, v_2 \rangle = 0$

$W^\perp = \{X \in \mathbb{C}^3 \mid X \cdot Y = 0,\ \text{for all } Y \in W\}$

obviously v_1, v_2 are L.I.

$\forall\, \alpha v_1 \in W$, where $\alpha \in \mathbb{C}$

$\forall\, av_1 + bv_2 \in \text{span}\{v_1, v_2\}$

$\langle av_1 + bv_2, \alpha v_1 \rangle = a\alpha \langle v_1, v_1 \rangle + b\alpha \langle v_2, v_1 \rangle$

$= 0$

i.e. $\text{span}\{v_1, v_2\} \subseteq W^\perp$

$\forall\, w \in W^\perp$

obviously $\{E_1, E_1, E_3\}$ is a basis of $\mathbb{C}^3$

$\exists\, \alpha, \beta, \gamma$ s.t.

$w = \alpha E_1 + \beta E_2 + \gamma E_3$

$= \alpha E_1 + i\alpha E_2 + (\beta - i\alpha) E_2 + \gamma E_3$

$= \alpha v_1 + (\beta - i\alpha) E_2 + \gamma v_2$

$0 = \langle w, v_1 \rangle = \alpha \langle v_1, v_1 \rangle + (\beta - i\alpha)\, i + \gamma \langle v_2, v_1 \rangle$

$= (\beta - i\alpha)\, i$

$\Rightarrow \beta - i\alpha = 0$

i.e. $w = \alpha v_1 + \gamma v_2$

Hence $W^\perp \subseteq \text{span}\{v_1, v_2\}$

$\Rightarrow \{v_1, v_2\}$ is a basis of $W^\perp$

4.

$\varphi : V \longmapsto K$

Let $\{v_1, v_2 \cdots v_n\}$ be a basis of V

othonormal

Since $\varphi \in V^*$ $\exists (a_1, a_2 \cdots a_n)$ s.t.

$\varphi = a_1\varphi_1 + a_2\varphi_2 \cdots a_n\varphi_n$

$\forall v \in \operatorname{Ker} \varphi$

$$0 = \varphi(v) = \varphi(x_1 v_1 + x_2 v_2 \cdots x_n v_n)$$

$$= a_1 x_1 + a_2 x_2 \cdots a_n x_n$$

$$= (a_1 \cdots a_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\dim \operatorname{Ker} \varphi = \dim W^\perp$

where $W = \operatorname{span}\{(a_1, a_2 \cdots a_n)\} \subseteq K^n$

Hence $\dim \operatorname{Ker} \varphi = \dim W^\perp = n-1$

7. for all $v \in W$

$\forall w' \in W^\perp$

$\langle v, w' \rangle = 0$

i.e. $v \in W^{\perp\perp}$

i.e. $W \subseteq W^{\perp\perp}$

$\forall w'' \in W^{\perp\perp}$

$\forall w' \in W^{\perp}$

$\langle w'', w' \rangle = 0$

i.e. $W^{\perp\perp} \subseteq W$

Hence $W = W^{\perp\perp}$

1. $g(v, v) = f(2v) - fv - f(v)$

$= 4f(v) - 2f(v)$

$= 2f(v)$

Let $g' = \frac{1}{2}g$

obviously g' is also bilinear

obviouly g' determine quadratic from f

Suppose $g''(v, w)$ determine f

$$\begin{cases} g''(v, v) = f(v) \\ g''(w, w) = f(w) \\ g''(v+w, v+w) = f(v+w) \end{cases}$$

$\Rightarrow g''(v, w) = \frac{1}{2}[f(v+w) - f(v) - f(w)] = g'(v, w)$

Hence. g' is unique.

2. $f(X) = x^2 - 3xy + 4y^2$

$$\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. $2x_1x_2 - x_3x_4 = X^TCX$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

$$g(X,Y) = X^TCY = x_1y_2 + x_2y_1 - \frac{1}{2}x_3y_4 - \frac{1}{2}x_4y_3$$