Algebra IV Assessed Coursework 1

This first page is the assessed coursework. Upload your solutions to blackboard before **1pm on 17th February 2026**. You will be marked by the clarity as well as the correctness of your arguments. The total is 100.

1. Let
$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$
be a complex of R-modules. For another R-module M we can apply the the functor $\mathrm{Hom}_R(-, M)$ and obtain a complex
$$\mathrm{Hom}_R(A, M) \leftarrow \mathrm{Hom}_R(B, M) \leftarrow \mathrm{Hom}_R(C, M).$$
Assume this complex is exact for all M. Show that the original complex is exact. (20)

2. Let R be a (commutative) integral domain with fraction field K. Prove the following statements or give counter-examples:
   (a) K is an injective R-module. (5)
   (b) K is a projective R-module. (5)
   (c) The short exact sequence of R-modules
   $$0 \to R \to K \to K/R \to 0$$
   is split. (5)
   (d) A module M is torsion if and only if $M \otimes_R K = 0$. (10)

3. Calculate the following abelian groups:
   (a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/6, \mathbb{Z}/9)$ (5)
   (b) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Q}/\mathbb{Z})$ (5)
   (c) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/3)$ (5)
   (d) $\mathbb{Z}/4 \otimes_{\mathbb{Z}} \mathbb{Z}/16$ (5)
   (e) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ (5)

4. Consider the interval $[0, 1)$. The operation addition modulo 1 makes this into an abelian group A.
   (a) Determine the torsion subgroup A_{tors} of A. (10)
   (b) Show that A_{tors} is a direct summand of A. (10)
   (c) Show that A/A_{tors} is an uncountable $\mathbb{Q}$-vector space. (10)

The rest of this problem sheet is unassessed, just for practice. **Do not upload solutions to these problems**

1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a pair of functors. Suppose that we have natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that $G\epsilon \circ \eta G = \mathrm{id}_G$ and $\epsilon F \circ F\eta = \mathrm{id}_F$. Show that F and G are adjoint, with unit η and co-unit ϵ.

2. Construct infinitely many pairwise non-isomorphic projective $\mathbb{Z}/12$-modules none of which are free.

3. Show that the coproduct $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is a torsion $\mathbb{Z}$-module, but the product $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is not.

4. Let R be a commutative ring. Then the abelian group $\mathrm{Hom}_R(A, B)$ has the structure of an R-module via $f \mapsto rf$ for all $r \in R,\ f \in \mathrm{Hom}_R(A, B)$. Indeed, this R-action really does take R-module homomorphisms to R-module homomorphisms:
$$rf(sa) = rsf(a) = srf(a)$$
so $rf \in Hom_R(A, B)$.
   (a) Prove that there is an isomorphism of R-modules
   $$\mathrm{Bilin}_R(A \times B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_R(A, C))$$
   where $\mathrm{Bilin}_R(A\times B, C)$ denotes the set of R-bilinear maps $f : A\times B \to C$ with the structure of an R-module given by additions of maps $(f, g) \to f + g$ and the ring action $(r, f) \mapsto rf$.
   (b) Prove that there is an isomorphism of R-modules
   $$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Bilin}_R(A \times B, C)\,.$$

   Note, you have just shown that
   $$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_R(A, C))\,.$$
   Everything you did was natural in B and C, so in particular you have shown that the functors $A \otimes_R -$ and $\mathrm{Hom}_R(A, -)$ are adjoint (you only needed a bijection of sets but showed that moreover the bijections are R-module isomorphisms). Tensor is left adjoint to Hom, and Hom is right adjoint to tensor.

5. (a) Give an example of a submodule of a free module which is not free.
   (b) In class we showed that if R is a PID then every submodule of a free R-module is free. The proof was a little long. Prove the following much easier statement: Let R be a PID and let M be a R-module which is free of finite rank. Then every submodule of M is free (of finite rank $\leq n$). Do this without using Zorn's lemma or anything like that.

6. Let M be an R-module. Prove that there is a canonical isomorphism of abelian groups

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})\,.$$

   Here we consider $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ with the left R-module structure given in the lectures.

7. Let R be an integral domain and let M be a finite generated torsion-free R-module. Show that M is isomorphic to a submodule of a free R-module of finite rank.

8. (a) Let R be a ring. Prove the following statement:

   P is a projective R-module if and only if there is a set I and elements $\{e_i\}_{i\in I}$ of P and elements $\{f_i\}_{i\in I}$ of $\mathrm{Hom}_R(P, R)$ such that $p = \sum_i e_i f_i(p)$ for all $p \in P$ (and the sum is always finite).

   This is called the dual basis criterion for projectivity.

   (b) Let $R = C[0,1]$ be the ring of continuous real-valued functions on $[0,1]$ (this is a horribly non-noetherian ring).

   Let $P = C_{(0)}[0,1]$ be the ideal in R of those functions f such that $f[0,\epsilon] = 0$ for some $\epsilon > 0$. Show that P is a projective R-module (hint: use an infinite partition of unity and the previous part). Do the same thing for smooth functions if you like.