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Assessed Coursework 1

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Exercise 1.

50 points: 10 + 10 + 10 + 10 + 10.
As always, we let k be an algebraically closed field. Some answers may depend on the characteristic of k.

1. Consider the following polynomials in $k[x, y, z]$:

$$f = y^2 - x^2, \qquad g = x^4 - yz, \qquad h = z^2 - x^3 y.$$

Find and describe the irreducible components of the varieties $V(f, g)$, $V(f, h)$, $V(f, g, h)$.

Solution. Throughout, note that $f = (y - x)(y + x)$.

$V(f, g)$**:** On $V(y - x)$: substituting $y = x$ into g gives $x^4 - xz = x(x^3 - z) = 0$, so $x = 0$ or $z = x^3$.

- $x = 0, y = 0$: the z-axis $V(x, y) = \{(0, 0, t) : t \in k\}$.
- $z = x^3, y = x$: the twisted cubic $C_1 = V(y - x,\ z - x^3) = \{(t, t, t^3) : t \in k\}$.

On $V(y + x)$: substituting $y = -x$ gives $x^4 + xz = x(x^3 + z) = 0$, so $x = 0$ (the z-axis again) or $z = -x^3$.

- $z = -x^3, y = -x$: the curve $C_2 = V(y + x,\ z + x^3) = \{(t, -t, -t^3) : t \in k\}$.

Each component is irreducible:

$$k[x, y, z]/(x, y) \cong k[z], \quad k[x, y, z]/(y - x, z - x^3) \cong k[x], \quad k[x, y, z]/(y + x, z + x^3) \cong k[x].$$

These are integral domains. Also, none is contained in the union of the others. For example,

$$(0, 0, 1) \in V(x, y), \qquad (0, 0, 1) \notin C_1 \cup C_2.$$

And when $\operatorname{char} k \neq 2$,

$$(1, 1, 1) \in C_1, \qquad (1, 1, 1) \notin V(x, y) \cup C_2.$$

If $\operatorname{char} k = 2$: then $y + x = y - x$, so $C_1 = C_2$ and

$$V(f,g) = V(x,y) \cup V(y-x,\ z-x^3) \quad \text{(two components)}.$$

If $\operatorname{char} k \neq 2$:

$$V(f,g) = V(x,y) \cup V(y-x,\ z-x^3) \cup V(y+x,\ z+x^3) \quad \text{(three components)}.$$

We are using explicitly:

[Definition, §2.1, p.3, Algebraic Geometry, 2026]: $V(S) = \{x \in \mathbb{A}^n_k : f(x) = 0\, \forall f \in S\}$,

and

[Lemma 2.1, §2.2, p.4, Algebraic Geometry, 2026]: $V(I) \cup V(J) = V(IJ)$.

Hence each piece above is an affine subvariety (a zero-locus), and their finite union is again Zariski closed.

$V(f,h)$**:** On $V(y-x)$: $h = z^2 - x^4 = (z-x^2)(z+x^2)$, giving components $V(y-x, z-x^2)$ and $V(y-x, z+x^2)$.

On $V(y+x)$:

$$h = z^2 + x^4.$$

Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Then

$$z^2 + x^4 = (z - ix^2)(z + ix^2),$$

giving components $V(y+x, z-ix^2)$ and $V(y+x, z+ix^2)$.

Each quotient $k[x,y,z]/(y \pm x, z - \alpha x^2) \cong k[x]$ is a domain, so each component is irreducible.

If $\operatorname{char} k = 2$: $i = 1$, $y + x = y - x$, and $z + x^2 = z - x^2$, so all four collapse to a single component:

$$V(f,h) = V(y-x,\ z-x^2) \quad \text{(one component)}.$$

If $\operatorname{char} k \neq 2$: four distinct irreducible components,

$$V(f,h) = V(y-x, z-x^2) \cup V(y-x, z+x^2) \cup V(y+x, z-ix^2) \cup V(y+x, z+ix^2).$$

$V(f,g,h)$**:** We intersect the components of $V(f,g)$ with the condition $h = 0$.

On $V(x,y)$: $h = z^2 = 0$, so $z = 0$. This gives the origin $(0,0,0)$.

On $C_1 = \{(t,t,t^3)\}$: $h = t^6 - t^4 = t^4(t^2-1) = 0$, so $t = 0$ or $t = \pm 1$. Points: $(0,0,0)$, $(1,1,1)$, $(-1,-1,-1)$.

On $C_2 = \{(t,-t,-t^3)\}$ (when $\operatorname{char} k \neq 2$):

$$h = t^6 + t^4 = t^4(t^2+1) = 0,$$

so $t = 0$ or $t = \pm i$. Hence the points are

$$(0, 0, 0),\ (i, -i, i),\ (-i, i, -i).$$

If $\operatorname{char} k \neq 2$: $V(f, g, h)$ consists of five points,

$$V(f, g, h) = \{(0, 0, 0),\ (1, 1, 1),\ (-1, -1, -1),\ (i, -i, i),\ (-i, i, -i)\}.$$

If $\operatorname{char} k = 2$: $t^2 - 1 = (t + 1)^2$, so $t = 0$ or $t = 1$, giving

$$V(f, g, h) = \{(0, 0, 0),\ (1, 1, 1)\}.$$

In both cases, the irreducible components are the individual points. □

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2. Let $f, g \in k[x, y]$ be two irreducible polynomials which are not multiples of each other.

   (a) Suppose that at least one of f and g contains a nonzero term in y (i.e. is not an element of $k[x]$). Use Gauss's Lemma to show that f, g have no common factors in $k(x)[y]$.

   Solution. View f, g as elements of $R[y]$ where $R = k[x]$ is a UFD with fraction field $K = k(x)$.
   Suppose first that f involves y, i.e. $\deg_y f \geq 1$. Then f is primitive in $R[y]$: if a non-unit $c(x) \in R$ divided every coefficient of f (viewed in y), then

   $$f = c(x)\,\tilde{f}(x, y),$$

   with $\tilde{f}$ still involving y, contradicting irreducibility of f in $k[x, y]$. Hence, by Gauss's Lemma, f is irreducible in $K[y]$.
   If g also involves y, the same argument shows g is irreducible in $K[y]$. Since f and g are not multiples in $k[x, y]$, they are not associates in $K[y]$. Indeed, if

   $$f = (a/b)\,g, \qquad a, b \in R \setminus \{0\},$$

   then $bf = ag$. Since f is irreducible in $R[y]$ with $\deg_y f \geq 1$, no non-unit $a \in R$ divides f. So a is a unit; similarly b is a unit. This would force f, g to be scalar multiples in $k[x, y]$, contradiction. Thus f, g are distinct irreducibles in the PID $K[y]$, hence coprime.
   If $g \in k[x]$: then g is a nonzero element of $R \subset K$, hence a unit in $K[y]$. So $\gcd(f, g) = 1$ trivially. □

   (b) Show that there exist nonzero polynomials $h \in k[x]$ and $p, q \in k[x, y]$ such that $h = fp + gq$.

Solution. By part (a), f and g are coprime in the PID $k(x)[y]$. By the Euclidean algorithm, there exist $P, Q \in k(x)[y]$ with

$$1 = fP + gQ.$$

Write

$$P = p_0/d_1, \qquad Q = q_0/d_2,$$

where $p_0, q_0 \in k[x, y]$ and $d_1, d_2 \in k[x] \setminus \{0\}$. Multiplying through by $d_1 d_2$:

$$d_1 d_2 = f(d_2 p_0) + g(d_1 q_0).$$

Setting $h = d_1 d_2 \in k[x] \setminus \{0\}$, $p = d_2 p_0$, $q = d_1 q_0$ gives the result. □

(c) Show that the set $\{x : (x, y) \in V(f, g)\}$ of first coordinates of points of $V(f, g)$ is finite.

Solution. By part (b), $h(x) = f(x, y)\, p(x, y) + g(x, y)\, q(x, y)$ for some nonzero $h \in k[x]$.
If $(a, b) \in V(f, g)$, then $f(a, b) = g(a, b) = 0$, so

$$h(a) = f(a, b)\, p(a, b) + g(a, b)\, q(a, b) = 0.$$

Since $h \in k[x]$ is nonzero, it has at most $\deg h$ roots. Therefore the set of first coordinates is finite. □

(d) Show that the set $V(f, g)$ is finite.

Solution. By part (c), the set

$$A = \{a \in k : (a, b) \in V(f, g) \text{ for some } b\}$$

is finite. For each $a \in A$, the fiber

$$\{b \in k : (a, b) \in V(f, g)\}$$

is contained in

$$\{b \in k : f(a, b) = 0\}.$$

Since at least one of f, g involves y, say f, write

$$f = \sum_j c_j(x) y^j, \qquad c_m \neq 0,\ m \geq 1.$$

If $f(a, y) \equiv 0$ as a polynomial in y, then $c_j(a) = 0$ for all j. So $(x - a)$ divides every $c_j(x)$, hence $(x - a) \mid f$ in $k[x, y]$. But f is irreducible with $\deg_y f \geq 1$, contradiction. Therefore $f(a, y)$ is nonzero for every a, so each fiber is finite.

A finite union of finite sets is finite, so $V(f, g)$ is finite. $\square$

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3. Let $n, m \geq 1$ and consider

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^2, \qquad t \mapsto (t^n, t^m).$$

Show that $\mathrm{im}(\varphi)$ is an affine subvariety of $\mathbb{A}^2$. Give the condition for which φ is bijective onto its image. In that case, give a birational inverse when $\mathrm{char}\, k = 0$.

Solution. Let $d = \gcd(n, m)$, $a = n/d$, $b = m/d$, so $\gcd(a, b) = 1$.

Image is a subvariety: By [Definition, §2.1, p.3, Algebraic Geometry, 2026],

it is enough to prove $\mathrm{im}(\varphi) = V(S)$ for some $S \subset k[x, y]$.

Since k is algebraically closed, the map $t \mapsto t^d$ is surjective on k, so

$$\mathrm{im}(\varphi) = \{(t^n, t^m) : t \in k\} = \{(s^a, s^b) : s \in k\}.$$

Claim. $\mathrm{im}(\varphi) = V(x^b - y^a)$.

If $(x, y) = (s^a, s^b)$, then $x^b - y^a = s^{ab} - s^{ab} = 0$, so $\mathrm{im}(\varphi) \subseteq V(x^b - y^a)$.

Conversely, let $(x, y) \in V(x^b - y^a)$. If $x = 0$, then $y^a = 0$, so $y = 0 = (0^a, 0^b) \in \mathrm{im}(\varphi)$. If $x \neq 0$, choose $s_0 \in k$ with $s_0^a = x$. Then $s_0^{ab} = x^b = y^a$, so

$$\left(\frac{y}{s_0^b}\right)^a = 1.$$

Let $\zeta := y/s_0^b \in \mu_a$. Since $\gcd(a, b) = 1$, the map

$$\mu_a \to \mu_a, \qquad \eta \mapsto \eta^b$$

is bijective. Hence there exists $\eta \in \mu_a$ with $\eta^b = \zeta$. Set $s := \eta s_0$. Then

$$s^a = \eta^a s_0^a = x, \qquad s^b = \eta^b s_0^b = \zeta s_0^b = y.$$

Thus $(x, y) = (s^a, s^b) \in \mathrm{im}(\varphi)$.

The polynomial $x^b - y^a$ is irreducible. Consider

$$k[x, y] \to k[t], \qquad x \mapsto t^a, \ y \mapsto t^b.$$

Its image is $k[t^a, t^b]$. The monomials $\{x^i y^j : i \geq 0, \ 0 \leq j \leq a - 1\}$ map to $\{t^{ia+jb}\}$, and these exponents are distinct since $\gcd(a, b) = 1$. Hence

$$k[x, y]/(x^b - y^a) \hookrightarrow k[t]$$

is a domain, so $x^b - y^a$ is irreducible.

Therefore

$$\mathrm{im}(\varphi) = V(x^b - y^a),$$

so the image is an affine subvariety in the sense of [Definition, §2.1, p.3, Algebraic Geometry, 2026].

Bijectivity: φ is bijective onto its image iff

$$t^n = s^n \text{ and } t^m = s^m \implies t = s.$$

For $t, s \neq 0$, set $\omega = s/t$. Then

$$\omega^n = \omega^m = 1 \Longrightarrow \omega^d = 1.$$

So we need the only such ω to be 1.

In characteristic zero, $\mu_d(k) = \{1\}$ iff $d = 1$. So φ is bijective to its image if and only if $\gcd(n, m) = 1$.

In characteristic $p > 0$, φ is bijective iff $\gcd(n, m)$ is a power of p, since the only p^e-th root of unity in k is 1.

Birational inverse (char 0, $\gcd(n, m) = 1$): Here we invoke the Chapter 7 framework:

[Definition, §7, p.22, Algebraic Geometry, 2026]: rational maps/functions and $k(V)$,

[Definition, §7, p.25, Algebraic Geometry, 2026]: birational equivalence via rational inverses.

By Bézout, choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha n + \beta m = 1$. Define the rational map

$$\psi \colon V(x^m - y^n) \dashrightarrow \mathbb{A}^1, \qquad \psi(x, y) = x^\alpha y^\beta,$$

(where negative exponents denote division). Then

$$\psi(\varphi(t)) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n + \beta m} = t,$$

On the dense open subset where $x \neq 0$ and $y \neq 0$, this is a composition of regular functions. In function fields,

$$\psi^*(t) = x^\alpha y^\beta, \qquad \varphi^*(x) = t^n,\ \varphi^*(y) = t^m,$$

so

$$(\varphi^* \circ \psi^*)(t) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n + \beta m} = t.$$

Also

$$(\psi^* \circ \varphi^*)(x) = x, \qquad (\psi^* \circ \varphi^*)(y) = y,$$

so

$$\psi^* \circ \varphi^* = \mathrm{id}_{k(V(x^m - y^n))}.$$

Hence ψ and φ are inverse as rational maps ([Definition, §7, p.25, Algebraic Geometry, 2026]). So ψ is a birational inverse of φ. $\square$

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4. (a) Let $S = V(x^2 + y^2 - 1)$ be the circle and $H = V(wz - 1)$ be the hyperbola. Show that either $S \cong H$ or $S \cong \mathbb{A}^1$.

Solution. **Case** $\operatorname{char} k \neq 2$**:** Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Define

$$\Phi\colon S \to H, \qquad (x, y) \mapsto (x + iy,\, x - iy).$$

On S: $(x + iy)(x - iy) = x^2 + y^2 = 1$, so Φ maps into H. The inverse is

$$\Phi^{-1}\colon H \to S, \qquad (w, z) \mapsto \left(\frac{w + z}{2},\, \frac{w - z}{2i}\right),$$

which is well-defined since $\operatorname{char} k \neq 2$. One verifies

$$\Phi^{-1} \circ \Phi = \mathrm{id}_S, \qquad \Phi \circ \Phi^{-1} = \mathrm{id}_H.$$

Both maps are morphisms, so $S \cong H$.
Case $\operatorname{char} k = 2$**:** In characteristic 2,

$$x^2 + y^2 - 1 = (x + y)^2 + 1 = (x + y + 1)^2,$$

since $(x + y + 1)^2 = x^2 + y^2 + 1$ and $-1 = 1$. Therefore

$$S = V((x + y + 1)^2) = V(x + y + 1),$$

which is a line in $\mathbb{A}^2$. The map

$$t \mapsto (t, t + 1)$$

is an isomorphism $\mathbb{A}^1 \xrightarrow{\sim} S$ with inverse $(x, y) \mapsto x$. So $S \cong \mathbb{A}^1$. □

(b) Let k have characteristic p. Show that

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^1, \qquad t \mapsto t^p$$

is a bijection, and show that φ is not a birational equivalence.

Solution. **Bijection:**
Surjectivity: For any $a \in k$, the polynomial $t^p - a$ has a root in k, since k is algebraically closed.
Injectivity: If $s^p = t^p$, then

$$(s - t)^p = s^p - t^p = 0,$$

because $\binom{p}{i} = 0$ in characteristic p for $0 < i < p$. Hence $s = t$.

Not a birational equivalence: By [Corollary 7.3, §7, p.26, Algebraic Geometry, 2026], irreducible affine varieties are birational iff their function fields are k-isomorphic. Here

$$\varphi^* \colon k(t) \to k(t), \qquad f(t) \mapsto f(t^p),$$

and

$$\mathrm{im}(\varphi^*) = k(t^p) \subsetneq k(t).$$

Moreover

$$[k(t) : k(t^p)] = p > 1,$$

because t is algebraic over $k(t^p)$ with equation

$$X^p - t^p = 0.$$

Therefore φ^* is not an isomorphism, hence φ is not birational. □

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5. Consider the cubic curve $C := V(y^2 - x^3 - x) \subseteq \mathbb{A}^2$.

   (a) Prove that C is irreducible.

Solution. View $F = y^2 - x^3 - x$ as an element of $k[x][y]$. If F were reducible in $k[x, y]$, then since $\deg_y F = 2$, it would factor as

$$F = (y - p(x))(y - q(x))$$

for some $p, q \in k[x]$, giving

$$p + q = 0, \qquad pq = -(x^3 + x).$$

Thus $q = -p$ and $p^2 = x^3+x$. But $\deg(p^2) = 2 \deg p$ is even, while $\deg(x^3+x) = 3$ is odd. Contradiction. Therefore F is irreducible. □

   (b) Find the domain of definition of the rational map

$$\varphi \colon C \dashrightarrow \mathbb{A}^1, \qquad \varphi(x, y) = x/y.$$

Solution. On C, $y^2 = x^3 + x = x(x^2 + 1)$, so

$$\frac{x}{y} = \frac{x}{y} \cdot \frac{y}{y} = \frac{xy}{y^2} = \frac{xy}{x(x^2 + 1)} = \frac{y}{x^2 + 1}.$$

The representation x/y is regular where $y \neq 0$. The representation $y/(x^2 + 1)$ is regular where $x^2 + 1 \neq 0$. Together, φ is regular on

$$C \setminus V(y, x^2 + 1).$$

At a point $P \in C$ with $y(P) = 0$ and $x(P)^2 + 1 = 0$:

$$x(P)^2 = -1, \qquad y(P) = 0.$$

Locally, $y^2 = x(x^2+1)$ has a simple zero in $(x - x(P))$ at such P. Set $u = x - x(P)$. Then

$$y^2 \sim x(P) \cdot u \cdot (\text{unit}),$$

so $y \sim \sqrt{u}$. Hence

$$\varphi = \frac{y}{x^2+1} \sim \frac{\sqrt{u}}{u} \to \infty.$$

So φ has a pole at P and cannot be extended.
Therefore the domain of definition is $C \setminus V(y,\, x^2+1)$.
Explicitly, the excluded points satisfy $x^2 = -1$ and $y = 0$:

$$(i, 0),\ (-i, 0) \text{ if } \operatorname{char} k \neq 2; \qquad (1, 0) \text{ if } \operatorname{char} k = 2.$$

□

(c) Now consider $C' := V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. The same formula defines a rational map $\varphi : C' \dashrightarrow \mathbb{A}^1$. Find a dominant rational map $\psi : \mathbb{A}^1 \dashrightarrow C'$ with

$$\varphi\psi = \mathrm{id}_{\mathbb{A}^1}.$$

Solution. We seek $\psi(t) = (a(t), b(t)) \in C'$ with $\varphi(\psi(t)) = a/b = t$, i.e. $a = tb$. Substituting into $b^2 = a^3 + a^2 = (tb)^3 + (tb)^2$:

$$b^2 = t^3b^3 + t^2b^2, \qquad b^2(1 - t^2) = t^3b^3, \qquad b = \frac{1 - t^2}{t^3},$$

and then $a = tb = \dfrac{1 - t^2}{t^2}$.
Define the rational map

$$\psi \colon \mathbb{A}^1 \dashrightarrow C', \qquad t \mapsto \left(\frac{1 - t^2}{t^2}, \frac{1 - t^2}{t^3} \right).$$

Verification: $\varphi(\psi(t)) = \dfrac{(1 - t^2)/t^2}{(1 - t^2)/t^3} = t.$ ✓ $\psi(t) \in C'$:

$$\begin{aligned} y^2 - x^3 - x^2 &= \frac{(1 - t^2)^2}{t^6} - \frac{(1 - t^2)^3}{t^6} - \frac{(1 - t^2)^2}{t^4} \\ &= \frac{(1 - t^2)^2\big(1 - (1 - t^2) - t^2\big)}{t^6} \\ &= 0. \end{aligned}$$

✓

Dominance: ψ is defined for $t \neq 0$. Every $(x, y) \in C'$ with $y \neq 0$ is in the image: set $t = x/y$, then recover $\psi(t) = (x, y)$ on C'. Since $C' \setminus V(y)$ is dense in irreducible C', ψ is dominant. $\square$
