

Assessed Coursework 1

Student name: *Frankie Feng-Cheng WANG*

Email: *maths@frankie.wang*

fw225@ic.ac.uk

Course: *MATH70029-Algebraic Geometry* – Lecturer: *Matt Booth*

Due date: *16th February 2026, 1 PM*

Exercise 1.

50 points: 10 + 10 + 10 + 10 + 10.

As always, we let k be an algebraically closed field. Some answers may depend on the characteristic of k .

1. Consider the following polynomials in $k[x, y, z]$:

$$f = y^2 - x^2, \quad g = x^4 - yz, \quad h = z^2 - x^3y.$$

Find and describe the irreducible components of the varieties $V(f, g)$, $V(f, h)$, $V(g, h)$.

Solution. Throughout, note that $f = (y - x)(y + x)$.

$V(f, g)$: On $V(y - x)$: substituting $y = x$ into g gives $x^4 - xz = x(x^3 - z) = 0$, so $x = 0$ or $z = x^3$.

- $x = 0, y = 0$: the z -axis $V(x, y) = \{(0, 0, t) : t \in k\}$.
- $z = x^3, y = x$: the twisted cubic $C_1 = V(y - x, z - x^3) = \{(t, t, t^3) : t \in k\}$.

On $V(y + x)$: substituting $y = -x$ gives $x^4 + xz = x(x^3 + z) = 0$, so $x = 0$ (the z -axis again) or $z = -x^3$.

- $z = -x^3, y = -x$: the curve $C_2 = V(y + x, z + x^3) = \{(t, -t, -t^3) : t \in k\}$.

Each component is irreducible: $k[x, y, z]/(x, y) \cong k[z]$, $k[x, y, z]/(y - x, z - x^3) \cong k[x]$, and $k[x, y, z]/(y + x, z + x^3) \cong k[x]$ are all integral domains. One checks that none is contained in the union of the others (e.g. $(0, 0, 1) \in V(x, y)$ but not in C_1 or C_2 ; $(1, 1, 1) \in C_1$ but not in $V(x, y)$ or C_2 when $\text{char } k \neq 2$).

If $\text{char } k = 2$: then $y + x = y - x$, so $C_1 = C_2$ and

$$V(f, g) = V(x, y) \cup V(y - x, z - x^3) \quad (\text{two components}).$$

If $\text{char } k \neq 2$:

$$V(f, g) = V(x, y) \cup V(y - x, z - x^3) \cup V(y + x, z + x^3) \quad (\text{three components}).$$

We are using explicitly:

lecturenotes, Section 2.1: $V(S) = \{x \in \mathbb{A}_k^n : f(x) = 0 \ \forall f \in S\}$,

and

lecturenotes, Lemma 2.1: $V(I) \cup V(J) = V(IJ)$.

Hence each piece above is an affine subvariety (a zero-locus), and their finite union is again Zariski closed.

$V(f, h)$: On $V(y - x)$: $h = z^2 - x^4 = (z - x^2)(z + x^2)$, giving components $V(y - x, z - x^2)$ and $V(y - x, z + x^2)$.

On $V(y + x)$: $h = z^2 + x^4$. Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Then $z^2 + x^4 = (z - ix^2)(z + ix^2)$, giving components $V(y + x, z - ix^2)$ and $V(y + x, z + ix^2)$.

Each quotient $k[x, y, z]/(y \pm x, z - \alpha x^2) \cong k[x]$ is a domain, so each component is irreducible.

If $\text{char } k = 2$: $i = 1$, $y + x = y - x$, and $z + x^2 = z - x^2$, so all four collapse to a single component:

$$V(f, h) = V(y - x, z - x^2) \quad (\text{one component}).$$

If $\text{char } k \neq 2$: four distinct irreducible components,

$$V(f, h) = V(y - x, z - x^2) \cup V(y - x, z + x^2) \cup V(y + x, z - ix^2) \cup V(y + x, z + ix^2).$$

$V(f, g, h)$: We intersect the components of $V(f, g)$ with the condition $h = 0$.

On $V(x, y)$: $h = z^2 = 0$, so $z = 0$. This gives the origin $(0, 0, 0)$.

On $C_1 = \{(t, t, t^3)\}$: $h = t^6 - t^4 = t^4(t^2 - 1) = 0$, so $t = 0$ or $t = \pm 1$. Points: $(0, 0, 0)$, $(1, 1, 1)$, $(-1, -1, -1)$.

On $C_2 = \{(t, -t, -t^3)\}$ (when $\text{char } k \neq 2$): $h = t^6 + t^4 = t^4(t^2 + 1) = 0$, so $t = 0$ or $t = \pm i$. Points: $(0, 0, 0)$, $(i, -i, i)$, $(-i, i, -i)$.

If $\text{char } k \neq 2$: $V(f, g, h)$ consists of five points,

$$V(f, g, h) = \{(0, 0, 0), (1, 1, 1), (-1, -1, -1), (i, -i, i), (-i, i, -i)\}.$$

If $\text{char } k = 2$: $t^2 - 1 = (t + 1)^2$, so $t = 0$ or $t = 1$, giving

$$V(f, g, h) = \{(0, 0, 0), (1, 1, 1)\}.$$

In both cases, the irreducible components are the individual points. \square

2. Let $f, g \in k[x, y]$ be two irreducible polynomials which are not multiples of each other.

- (a) Suppose that at least one of f and g contains a nonzero term in y (i.e. is not an element of $k[x]$). Use Gauss's Lemma to show that f, g have no common factors in the ring $k(x)[y]$.

Solution. View f, g as elements of $R[y]$ where $R = k[x]$ is a UFD with fraction field $K = k(x)$.

Suppose first that f involves y , i.e. $\deg_y f \geq 1$. Then f is primitive as an element of $R[y]$: if a non-unit $c(x) \in R$ divided every coefficient of f (viewed in y), then $f = c(x) \cdot \tilde{f}(x, y)$ with \tilde{f} involving y , contradicting the irreducibility of f in $k[x, y]$. Since f is primitive and irreducible in $R[y]$, by Gauss's Lemma f is irreducible in $K[y]$.

If g also involves y , the same argument shows g is irreducible in $K[y]$. Since f and g are not multiples of each other in $k[x, y]$, they are not associates in $K[y]$: if $f = (a/b)g$ for $a, b \in R \setminus \{0\}$, then $bf = ag$; since f is irreducible in $R[y]$ with $\deg_y f \geq 1$, no non-unit $a \in R$ divides f , so a is a unit; similarly b is a unit, making f and g scalar multiples in $k[x, y]$, a contradiction. Thus f and g are distinct irreducibles in the PID $K[y]$, hence coprime.

If $g \in k[x]$: then g is a nonzero element of $R \subset K$, hence a unit in $K[y]$. So $\gcd(f, g) = 1$ trivially. \square

- (b) Show that there exist nonzero polynomials $h \in k[x]$ and $p, q \in k[x, y]$ such that $h = fp + gq$.

Solution. By part (a), f and g are coprime in the PID $k(x)[y]$. By the Euclidean algorithm, there exist $P, Q \in k(x)[y]$ with

$$1 = fP + gQ.$$

Write $P = p_0/d_1$ and $Q = q_0/d_2$ where $p_0, q_0 \in k[x, y]$ and $d_1, d_2 \in k[x] \setminus \{0\}$. Multiplying through by d_1d_2 :

$$d_1d_2 = f(d_2p_0) + g(d_1q_0).$$

Setting $h = d_1d_2 \in k[x] \setminus \{0\}$, $p = d_2p_0$, $q = d_1q_0$ gives the result. \square

- (c) Show that the set $\{x : (x, y) \in V(f, g)\}$ of first coordinates of points of $V(f, g)$ is finite.

Solution. By part (b), $h(x) = f(x, y)p(x, y) + g(x, y)q(x, y)$ for some nonzero $h \in k[x]$.

If $(a, b) \in V(f, g)$, then $f(a, b) = g(a, b) = 0$, so

$$h(a) = f(a, b)p(a, b) + g(a, b)q(a, b) = 0.$$

Since $h \in k[x]$ is nonzero, it has at most $\deg h$ roots. Therefore the set of first coordinates is finite. \square

-
- (d) Show that the set $V(f, g)$ is finite.
-

Solution. By part (c), the set of first coordinates $A = \{a \in k : (a, b) \in V(f, g) \text{ for some } b\}$ is finite. For each $a \in A$, the fiber $\{b \in k : (a, b) \in V(f, g)\} \subseteq \{b \in k : f(a, b) = 0\}$.

Since at least one of f, g involves y , say f . Write $f = \sum_j c_j(x)y^j$ with $c_m \neq 0$ for some $m \geq 1$. If $f(a, y) \equiv 0$ as a polynomial in y , then $c_j(a) = 0$ for all j , so $(x - a)$ divides every $c_j(x)$, hence $(x - a) \mid f$ in $k[x, y]$. But f is irreducible with $\deg_y f \geq 1$, a contradiction. Therefore $f(a, y)$ is a nonzero polynomial in y for every a , so the fiber above each a is finite.

A finite union of finite sets is finite, so $V(f, g)$ is finite. \square

3. Let $n, m \geq 1$ be integers and consider the regular map $\varphi: \mathbb{A}^1 \rightarrow \mathbb{A}^2$ which sends $t \mapsto (t^n, t^m)$. Show that the image of φ is an affine subvariety of \mathbb{A}^2 . Give the condition for which φ is bijective to its image, and, in this case, give a birational inverse to φ when k has characteristic zero.
-

Solution. Let $d = \gcd(n, m)$, $a = n/d$, $b = m/d$, so $\gcd(a, b) = 1$.

Image is a subvariety: By *lecturenotes*, Section 2.1 (Definition of affine subvarieties),

it is enough to prove $\text{im}(\varphi) = V(S)$ for some $S \subset k[x, y]$.

Since k is algebraically closed, the map $t \mapsto t^d$ is surjective on k , so

$$\text{im}(\varphi) = \{(t^n, t^m) : t \in k\} = \{(s^a, s^b) : s \in k\}.$$

Claim. $\text{im}(\varphi) = V(x^b - y^a)$.

If $(x, y) = (s^a, s^b)$, then $x^b - y^a = s^{ab} - s^{ab} = 0$, so $\text{im}(\varphi) \subseteq V(x^b - y^a)$.

Conversely, let $(x, y) \in V(x^b - y^a)$. If $x = 0$ then $y^a = 0$ so $y = 0 = (0^a, 0^b) \in \text{im}(\varphi)$. If $x \neq 0$, choose $s_0 \in k$ with $s_0^a = x$ (possible since k is algebraically closed). Then $s_0^{ab} = x^b = y^a$, so

$$\left(\frac{y}{s_0^b}\right)^a = 1.$$

Let $\zeta := y/s_0^b \in \mu_a$. Since $\gcd(a, b) = 1$, the map $\mu_a \rightarrow \mu_a$, $\eta \mapsto \eta^b$, is bijective. Hence there exists $\eta \in \mu_a$ with $\eta^b = \zeta$. Set $s := \eta s_0$. Then

$$s^a = \eta^a s_0^a = x, \quad s^b = \eta^b s_0^b = \zeta s_0^b = y.$$

Thus $(x, y) = (s^a, s^b) \in \text{im}(\varphi)$.

The polynomial $x^b - y^a$ is irreducible: the map $k[x, y] \rightarrow k[t]$ sending $x \mapsto t^a$, $y \mapsto t^b$ has image $k[t^a, t^b]$. The monomials $\{x^i y^j : i \geq 0, 0 \leq j \leq a-1\}$ map to $\{t^{ia+jb}\}$,

which are all distinct (since $\gcd(a, b) = 1$). So $k[x, y]/(x^b - y^a) \hookrightarrow k[t]$ is an integral domain, hence $x^b - y^a$ is irreducible.

Therefore

$$\text{im}(\varphi) = V(x^b - y^a),$$

so the image is an affine subvariety in the sense of Section 2.1 of *lecturenotes*.

Bijectivity: φ is bijective onto its image iff $t^n = s^n$ and $t^m = s^m$ implies $t = s$. For $t, s \neq 0$, set $\omega = s/t$; then $\omega^n = \omega^m = 1$, so $\omega^d = 1$. We need the only such ω to be 1.

In characteristic zero, $\mu_d(k) = \{1\}$ iff $d = 1$. So φ is bijective to its image if and only if $\gcd(n, m) = 1$.

(In characteristic $p > 0$, φ is bijective iff $\gcd(n, m)$ is a power of p , since the only p^e -th root of unity in k is 1.)

Birational inverse (char 0, $\gcd(n, m) = 1$): Here we invoke two specific Lecture 2 facts from *lecturenotes*:

(i) “rational function on V ” and function field $K(V)$,

(ii) birationality is equivalence via inverse rational maps (Lecture 2 lemma).

By Bézout, choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha n + \beta m = 1$. Define the rational map

$$\psi: V(x^m - y^n) \dashrightarrow \mathbb{A}^1, \quad \psi(x, y) = x^\alpha y^\beta,$$

(where negative exponents denote division). Then

$$\psi(\varphi(t)) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n + \beta m} = t,$$

On the dense open subset where $x \neq 0$ and $y \neq 0$, this is a composition of regular functions. In function fields,

$$\psi^*(t) = x^\alpha y^\beta, \quad \varphi^*(x) = t^n, \quad \varphi^*(y) = t^m,$$

so

$$(\varphi^* \circ \psi^*)(t) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n + \beta m} = t.$$

Also

$$(\psi^* \circ \varphi^*)(x) = x, \quad (\psi^* \circ \varphi^*)(y) = y,$$

so

$$\psi^* \circ \varphi^* = \text{id}_{k(V(x^m - y^n))}.$$

Hence ψ and φ are inverse as rational maps (Lecture 2 birational criterion). So ψ is a birational inverse of φ . \square

4. (a) Let $S = V(x^2 + y^2 - 1)$ be the circle and $H = V(wz - 1)$ be the hyperbola. Show that either $S \cong H$ or $S \cong \mathbb{A}^1$.

Solution. **Case** $\text{char } k \neq 2$: Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Define

$$\Phi: S \rightarrow H, \quad (x, y) \mapsto (x + iy, x - iy).$$

On S : $(x + iy)(x - iy) = x^2 + y^2 = 1$, so Φ maps into H . The inverse is

$$\Phi^{-1}: H \rightarrow S, \quad (w, z) \mapsto \left(\frac{w+z}{2}, \frac{w-z}{2i} \right),$$

which is well-defined since $\text{char } k \neq 2$. One verifies $\Phi^{-1} \circ \Phi = \text{id}_S$ and $\Phi \circ \Phi^{-1} = \text{id}_H$. Both maps are morphisms (polynomial maps), so $S \cong H$.

Case $\text{char } k = 2$: In characteristic 2, $x^2 + y^2 - 1 = (x+y)^2 + 1 = (x+y+1)^2$ (since $(x+y+1)^2 = x^2 + y^2 + 1$ in char 2, and $-1 = 1$). Therefore

$$S = V((x+y+1)^2) = V(x+y+1),$$

which is a line in \mathbb{A}^2 . The map $t \mapsto (t, t+1)$ is an isomorphism $\mathbb{A}^1 \xrightarrow{\sim} S$ with inverse $(x, y) \mapsto x$. So $S \cong \mathbb{A}^1$. \square

- (b) Let k have characteristic p . Show that the map $\varphi: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ defined by $t \mapsto t^p$ is a bijection. Show that φ is not a birational equivalence.

Solution. **Bijection:** *Surjectivity:* For any $a \in k$, the polynomial $t^p - a$ has a root in k since k is algebraically closed. *Injectivity:* If $s^p = t^p$, then $(s-t)^p = s^p - t^p = 0$ (using the Frobenius endomorphism: all binomial coefficients $\binom{p}{i}$ for $0 < i < p$ are divisible by p , hence zero in characteristic p). So $s = t$.

Not a birational equivalence: By Lecture 2 in *lecturenotes*, birational equivalence implies an isomorphism of function fields. Here

$$\varphi^*: k(t) \rightarrow k(t), \quad f(t) \mapsto f(t^p),$$

and

$$\text{im}(\varphi^*) = k(t^p) \subsetneq k(t).$$

Moreover

$$[k(t) : k(t^p)] = p > 1,$$

because t is algebraic over $k(t^p)$ with equation

$$X^p - t^p = 0.$$

Therefore φ^* is not an isomorphism, hence φ is not birational. \square

5. Consider the cubic curve $C := V(y^2 - x^3 - x) \subseteq \mathbb{A}^2$.

-
- (a) Prove that C is irreducible.
-

Solution. View $F = y^2 - x^3 - x$ as an element of $k[x][y]$. If F were reducible in $k[x, y]$, then since $\deg_y F = 2$, it would factor as

$$F = (y - p(x))(y - q(x))$$

for some $p, q \in k[x]$, giving $p + q = 0$ and $pq = -(x^3 + x)$. Thus $q = -p$ and $p^2 = x^3 + x$. But $\deg(p^2) = 2 \deg p$ is even while $\deg(x^3 + x) = 3$ is odd, a contradiction. Therefore F is irreducible. \square

- (b) Find the domain of definition of the rational map $\varphi: C \dashrightarrow \mathbb{A}^1$ given by $\varphi(x, y) = x/y$.
-

Solution. On C , $y^2 = x^3 + x = x(x^2 + 1)$, so

$$\frac{x}{y} = \frac{x}{y} \cdot \frac{y}{y} = \frac{xy}{y^2} = \frac{xy}{x(x^2 + 1)} = \frac{y}{x^2 + 1}.$$

The representation x/y is regular where $y \neq 0$; the representation $y/(x^2 + 1)$ is regular where $x^2 + 1 \neq 0$. Together, φ is regular on $C \setminus V(y, x^2 + 1)$.

At a point $P \in C$ with $y(P) = 0$ and $x(P)^2 + 1 = 0$: we have $x(P)^2 = -1$, $y(P) = 0$, and locally $y^2 = x(x^2 + 1)$ has a simple zero in $(x - x(P))$ at such P . Setting $u = x - x(P)$, we get $y^2 \sim x(P) \cdot u \cdot (\text{unit})$, so $y \sim \sqrt{u}$, and $\varphi = y/(x^2 + 1) \sim \sqrt{u}/u \rightarrow \infty$. Thus φ has a pole at P and cannot be extended.

Therefore the domain of definition is $C \setminus V(y, x^2 + 1)$.

Explicitly: the excluded points are those with $x^2 = -1$ and $y = 0$, which are $(i, 0)$ and $(-i, 0)$ when $\text{char } k \neq 2$, and $(1, 0)$ when $\text{char } k = 2$. \square

- (c) Now consider the cubic curve $C' := V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. The same formula defines a rational map $\varphi: C' \dashrightarrow \mathbb{A}^1$. Find a dominant rational map $\psi: \mathbb{A}^1 \dashrightarrow C'$ with $\varphi\psi = \text{id}_{\mathbb{A}^1}$.
-

Solution. We seek $\psi(t) = (a(t), b(t)) \in C'$ with $\varphi(\psi(t)) = a/b = t$, i.e. $a = tb$. Substituting into $b^2 = a^3 + a^2 = (tb)^3 + (tb)^2$:

$$b^2 = t^3b^3 + t^2b^2, \quad b^2(1 - t^2) = t^3b^3, \quad b = \frac{1 - t^2}{t^3},$$

and then $a = tb = \frac{1 - t^2}{t^2}$.

Define the rational map

$$\psi: \mathbb{A}^1 \dashrightarrow C', \quad t \mapsto \left(\frac{1 - t^2}{t^2}, \frac{1 - t^2}{t^3} \right).$$

Verification: $\varphi(\psi(t)) = \frac{(1-t^2)/t^2}{(1-t^2)/t^3} = t.$ ✓

$$\begin{aligned} \psi(t) \in C': y^2 - x^3 - x^2 &= \frac{(1-t^2)^2}{t^6} - \frac{(1-t^2)^3}{t^6} - \frac{(1-t^2)^2}{t^4} = \\ \frac{(1-t^2)^2[(1) - (1-t^2) - t^2]}{t^6} &= 0. \end{aligned}$$

Dominance: ψ is defined for $t \neq 0$, and every $(x, y) \in C'$ with $y \neq 0$ is in the image (set $t = x/y$ and recover $\psi(t) = (x, y)$ on C'). Since $C' \setminus V(y)$ is dense in the irreducible curve C' , ψ is dominant. □
