Algebraic Geometry 2026
Assessed Coursework 1

As always, we let k be an algebraically closed field. Some of your answers may depend on the characteristic of k; be sure to indicate when this is the case. Each question is worth ten marks, for a total of fifty.

1. Consider the following polynomials in $k[x, y, z]$:

$$f = y^2 - x^2, \qquad g = x^4 - yz, \qquad h = z^2 - x^3 y$$

   Find and describe the irreducible components of the varieties

$$V(f, g), \quad V(f, h), \quad V(f, g, h)$$

2. Let $f, g \in k[x, y]$ be two irreducible polynomials which are not multiples of each other.
   (a) Suppose that at least one of f and g contains a nonzero term in y (i.e. is not an element of $k[x]$). Use Gauss's Lemma to show that f, g have no common factors in the ring $k(x)[y]$.
   (b) Show that there exist nonzero polynomials $h \in k[x]$ and $p, q \in k[x, y]$ such that $h = fp + gq$.
   (c) Show that the set $\{x : (x, y) \in V(f, g)\}$ of first coordinates of points of $V(f, g)$ is finite.
   (d) Show that the set $V(f, g)$ is finite.

3. Let $n, m \geq 1$ be integers and consider the regular map $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ which sends $t \mapsto (t^n, t^m)$. Show that the image of ϕ is an affine subvariety of $\mathbb{A}^2$. Give the condition for which ϕ is bijective to its image, and, in this case, give a birational inverse to ϕ when k has characteristic zero.

4. (a) Let $S = V(x^2 + y^2 - 1)$ be the circle and $H = V(wz - 1)$ be the hyperbola. Show that either $S \cong H$ or $S \cong \mathbb{A}^1$.
   (b) Let k have characteristic p. Show that the map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $t \mapsto t^p$ is a bijection. Show that ϕ is not a birational equivalence.

5. Consider the cubic curve $C := V(y^2 - x^3 - x) \subseteq \mathbb{A}^2$.
   (a) Prove that C is irreducible.
   (b) Find the domain of definition of the rational map $\phi : C \to \mathbb{A}^1$ given by $\phi(x, y) = x/y$.
   (c) Now consider the cubic curve $C' := V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. The same formula defines a rational map $\phi : C' \to \mathbb{A}^1$. Find a dominant rational map $\psi : \mathbb{A}^1 \to C'$ with $\phi\psi = \mathrm{id}_{\mathbb{A}^1}$.