Algebraic Geometry, Jan–Mar 2026

These lecture notes are based mostly on the notes of Martin Orr for the 2018 course, revised by Travis Schedler. They were further revised by Pierre Descombes for the 2025 course, who added material from the lecture notes of David Helm for the 2023 course. Small changes were made by Matt Booth for 2026.

1. Introduction

Practical information about the course.

Lectures will be based on the material below.

The exam will be worth 90 percent of the course; the remaining ten percent will be awarded based on performance in two assessed courseworks, the deadlines to be provided soon. Nonassessed problems sheets will also be provided.

Books:

- "Undergraduate algebraic geometry" by Miles Reid
- Chapter 1 of "Algebraic Geometry" by Robin Hartshorne
- "Basic Algebraic Geometry I" by Igor Shafarevich
- "Algebraic Geometry: A first course" by Joe Harris

During the course we will sometimes *assume* results from commutative algebra. Books which contain these results (and much much more) include

- "Commutative algebra with a viewpoint toward algebraic geometry" by D. Eisenbud.
- "Commutative algebra" by Atiyah and Macdonald;
- "Commutative ring theory" by H. Matsumura.

If you are curious to see more advanced (and thorough) treatments of algebraic geometry, you may look at, e.g., the Stacks project and Ravi Vakil's "The Rising Sea" notes online.

Course outline.

(1) Affine varieties – definition, examples, maps between varieties, translating between geometry and commutative algebra (the Nullstellensatz).
(2) Projective varieties – definition, examples, maps between varieties, rigidity and images of maps.
(3) Dimension – several different definitions (all equivalent, but useful for different purposes), calculating dimensions of examples.
(4) Smoothness and singularities – definition, examples, key theorems

What is not in the course?

(1) Schemes (aside from some nonexaminable words)
(2) Sheaves and cohomology (although you may find this in the module on Complex Manifolds, and I may insert some nonexaminable comments about them)

(3) Curves, divisors and the Riemann–Roch theorem

The base field. Let k be an algebraically closed field.

We are going to be thinking about solutions to polynomials, so everything is much simpler over algebraically closed fields. Number theorists might be interested in other fields, but you generally have to start by understanding the algebraically closed case first. In this course we will stop with the algebraically closed case too.

Apart from being algebraically closed, it usually does not matter much which field we use to do algebraic geometry – except sometimes it matters whether the characteristic is zero or positive. In this course I will take care to mention results which depend on the characteristic, and sometimes we might consider only the characteristic zero case. You will not lose much if you just assume that $k = \mathbb{C}$ throughout the course (except when it will be explicitly something else).

Indeed it is often useful to think about $k = \mathbb{C}$ because then you can use your usual geometric intuition. When I draw pictures on the whiteboard, I am usually only drawing the real solutions because it is hard to draw shapes in $\mathbb{C}^2$. This is cheating but it is often very useful – the real solutions are not the full picture but in many cases we can still see the important features there.

Exercise 1.1. Think of as many examples of algebraically closed fields as you can. In particular, think of at least one example of positive characteristic.

Affine space.

Definition. Algebraic geometers write $\mathbb{A}^n$ to mean k^n, and call it **affine n-space**.

You may think of this as just a funny choice of notation, but there are at least two reasons for it:

(i) When we write k^n, it makes us think of a vector space, equipped with operations of addition and scalar multiplication. But $\mathbb{A}^n$ means just a set of points, described by coordinates $(x_1, \dots, x_n)$ with $x_i \in k$, without the vector space structure.
(ii) Because it usually doesn't matter much what our base field k is (as long as it is algebraically closed), it is convenient to have notation which does not prominently mention k.

On occasions when it *is* important to specify which field k we are using, we write $\mathbb{A}^n_k$ for affine n-space.

2. Affine varieties (reminder from commutative algebra)

2.1. **Definition of affine subvarieties.** Let k be an algebraically closed field. We denote:

$$\mathbb{A}_k^n := \{(c_1, ..., c_n) | c_i \in k \text{ for } i = 1, ..., n\} \tag{1}$$

the n-dimensional affine space over k. For $S \subset k[x_1, ..., x_n]$, we consider the vanishing locus of S:

$$V(S) := \{x \in \mathbb{A}_k^n | f(x) = 0 \text{ for all } f \in S\} \subset \mathbb{A}_k^n \tag{2}$$

it is called an affine variety. Elementary facts:

- If $S_1 \subset S_2$, then $V(S_2) \subset V(S_1)$.
- $V(S) = V((S))$: in particular, one can always suppose that S is an ideal.

If $S = (f_1, ..., f_n)$, then we denote $V(f_1, ..., f_n) := V(\{f_1, ..., f_n\})$. By the Hilbert basis theorem (Noetherianity of $k[x_1, ..., x_n]$), every ideal of $k[x_1, ..., x_n]$ is finitely generated, hence every affine variety can be written in the form $V(f_1, ..., f_n)$.

Examples of affine algebraic varieties.

Exercise 2.1. Think of some examples and non-examples of affine algebraic varieties.

Examples.

(1) The whole space $\mathbb{A}^n$, defined by the polynomial $f_1 = 0$ (or by the empty set of polynomials).
(2) The set $\{2\}$, defined by the polynomial $X - 2$. More generally, any point in $\mathbb{A}^1$.
(3) Any union of finitely many affine algebraic varieties (see proof below). Combining with (2), we deduce that any finite subset of $\mathbb{A}^1$ is an affine algebraic variety.
(4) An algebraic curve in $\mathbb{A}^2$, that is, a set of the form

$$\{(x_1, x_2) \in \mathbb{A}^2 : f(x_1, x_2) = 0\}$$

for some polynomial $f \in k[X_1, X_2]$.

Non-examples.

(1) Any infinite subset of $\mathbb{A}^1$ (other than $\mathbb{A}^1$ itself). This is because a one-variable polynomial with infinitely many roots must be the zero polynomial.

Here are some additional examples of affine algebraic varieties.

Further examples.

(5) Any point in $\mathbb{A}^n$. The single-point set $\{(a_1, \ldots, a_n)\}$ is defined by the equations
$$X_1 - a_1 = 0, \ldots, X_n - a_n = 0.$$
Using (3), we see that any finite subset of $\mathbb{A}^n$ is an affine algebraic variety.

(6) Embeddings of $\mathbb{A}^m$ in $\mathbb{A}^n$ where $m < n$:
$$\{(x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{A}^n\} = \{(x_1, \ldots, x_n) \in \mathbb{A}^n : x_{m+1} = \cdots = x_n = 0\}.$$
More generally, the image of a linear map $\mathbb{A}^m \to \mathbb{A}^n$:
$$\{(x_1, \ldots, x_n) \in \mathbb{A}^n : \text{some linear conditions}\}.$$

Further non-example.

Example (6) does not generalise to images of maps where each coordinate is given by a polynomial. For example, consider the map
$$\phi \colon \mathbb{A}^2 \to \mathbb{A}^2 \text{ where } f(x, y) = (x, xy).$$
The image of ϕ is
$$S = \left(\mathbb{A}^2 \setminus \{(0, y)\}\right) \cup \{(0, 0)\}.$$

To prove that S is not an affine algebraic variety, consider a polynomial $g(X, Y) \in k[X, Y]$ which vanishes on S. For each fixed $y \in k$, the one-variable polynomial $g(X, y)$ vanishes at all $x \neq 0$. This implies that $g(X, y)$ is the zero polynomial. Thus $g(x, y) = 0$ for all $x, y \in k^2$, that is, g is the zero polynomial.

2.2. **Zariski topology.** One of the examples was a union of finitely many affine algebraic varieties. Now we prove that the union of two affine algebraic varieties is an affine algebraic variety.

Lemma 2.1. For two ideals $I, J \subset k[x_1, ..., x_n]$, $V(I) \cup V(J) = V(IJ)$. Here $IJ = \{ij : i \in I, j \in J\}$ is the product of ideals.

Lemma 2.2. For a family of ideals I_k, we have $\bigcap_k V(I_k) = V(\sum_k I_k)$.

We have seen that affine algebraic subvarieties of $\mathbb{A}^n$ satisfy the following conditions:

(i) $\mathbb{A}^n$ and $\emptyset$ are affine algebraic subvarieties of $\mathbb{A}^n$. (The empty set is the vanishing set of a non-zero constant polynomial.)
(ii) A finite union of affine algebraic subvarieties is an affine algebraic subvariety.
(iii) An arbitrary intersection of affine algebraic subvarieties is an affine algebraic subvariety.

These are precisely the conditions satisfied by the *closed* sets in a topological space:

Definition. The Zariski topology on $\mathbb{A}^n_k$ is the topology whose closed subsets are the affine algebraic subvarieties. The Zariski topology on a subvariety $V \subset \mathbb{A}^n_k$ is the induced topology.

Basic facts about the Zariski topology. We defined the Zariski topology on an affine algebraic subvariety $V \subseteq \mathbb{A}^n$ to be the subset topology induced by the Zariski topology on $\mathbb{A}^n$. Thus: a subset of V is Zariski closed in V if and only if it is Zariski closed in $\mathbb{A}^n$, but a Zariski open subset of V need not to be Zariski open in $\mathbb{A}^n$. (For example: let V be the x-axis in $\mathbb{A}^2$. Then $V \setminus \{0\}$ is open in V, but not open in $\mathbb{A}^2$.)

Example. The Zariski topology on $\mathbb{A}^1$ is the same as the cofinite topology.

Thus we see that that Zariski topology on $\mathbb{C}$ has much fewer closed sets (equivalently, much fewer open sets) than the usual Euclidean topology.

Lemma 2.3. Suppose that $k = \mathbb{C}$ (so there is a Euclidean topology on $\mathbb{A}^n_{\mathbb{C}}$). If V is a Zariski closed subset of $\mathbb{A}^n_{\mathbb{C}}$, then V is closed in the Euclidean topology. ("The Euclidean topology is finer than the Zariski topology.")

Proof. Let $f \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial. It is a continuous function $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{C}$ for the Euclidean topology. Since $\{0\}$ is a closed subset of $\mathbb{C}$, $V(f) = f^{-1}(0)$ is a closed subset of $\mathbb{A}^n_{\mathbb{C}}$ in the Euclidean topology. We conclude by noting that intersections of closed sets are closed. $\square$

The open subsets of the Zariski topology are all "very big." This is made precise (for $\mathbb{A}^1$) by the following lemma.

Lemma 2.4. Prove that every pair U_1, U_2 of non-empty open sets in $\mathbb{A}^1$ has a non-empty intersection $U_1 \cap U_2$.

Hence the Zariski topology on $\mathbb{A}^1$ is not Hausdorff.

A subset of $\mathbb{A}^1$ is dense in the Zariski topology if and only if it is infinite.

At the moment, the Zariski topology is likely to seem very strange. It might also seem like: what is the point of such a strange topology? We will not use it in a very deep way, it is just a convenient language to be able to talk about open and closed sets. (It does get used more seriously in the theory of schemes.)

Products. Just a remark on one other way of constructing new affine algebraic varieties from existing ones:

If $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ are affine algebraic subvarieties, then their Cartesian product $V \times W \subseteq \mathbb{A}^{m+n}$ is an affine algebraic subvariety. Write

$$V = \{(x_1, \dots, x_m) \in \mathbb{A}^m : f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\},$$
$$W = \{(y_1, \dots, y_n) \in \mathbb{A}^n : g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

Then

$$V \times W = \{(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{A}^{m+n} : f_1(\underline{x}) = \dots = f_r(\underline{x}) = g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

This looks a bit like the equations defining $V \cap W$, but here the f_i involve different variables from the g_j, while for $V \cap W$ both used the same variables.

Notice that the Zariski topology on $V \times W$ is different from the product topology!

2.3. **Ideals and algebraic subvarieties: back and forth.** We can also go in the other direction: from affine algebraic subvarieties to ideals.

Definition. If A is any subset of $\mathbb{A}^n$ (usually A will be an affine algebraic subvariety), we define

$$I(A) = \{f \in k[X_1, \dots, X_n] : f(\underline{x}) = 0 \text{ for all } \underline{x} \in A\}.$$

Note that $I(A)$ is an ideal in $k[X_1, \dots, X_n]$.

We have now defined two functions

$$V : \{\text{ideals in } k[X_1, \dots, X_n]\} \to \{\text{affine algebraic subvarieties of } \mathbb{A}^n\},$$
$$I : \{\text{affine algebraic subvarieties of } \mathbb{A}^n\} \to \{\text{ideals in } k[X_1, \dots, X_n]\}.$$

These functions are not inverses of each other. For example, for the ideal $(X^2) \subseteq k[X]$:

$$I\Big(V\Big(\,(X^2)\,\Big)\Big) = (X) \neq (X^2).$$

But composing V and I in the other order gives the identity.

Lemma 2.5. If V is an affine algebraic subvariety, then $V(I(V)) = V$.

Proof. It is clear that $V \subseteq V(I(V))$ (and this works when V is any subset of $\mathbb{A}^n$, not necessarily algebraic).

For the reverse inclusion, we have to use the hypothesis that V is an affine algebraic subvariety. By the definition of affine algebraic subvariety, $V = V(J)$ for some ideal $J \subseteq k[X_1, \dots, X_n]$.

Suppose that $\underline{y} \notin V$. We shall show that $\underline{y} \notin V(I(V))$.

Because $\underline{y} \notin V = V(J)$, there exists $f \in J$ such that $f(\underline{y}) \neq 0$. By definition, $J \subseteq I(V)$ and so $f \in I(V)$. Hence $f(\underline{y}) \neq 0$ tells us that $\underline{y} \notin V(I(V))$. $\square$

Statement of the Nullstellensatz. When does $I(V(I)) = I$? It turns out that the only reason that this can fail is where elements of the ideal I have n-th roots which are not in I, just as with the example of $I = (X^2)$ where $X^2 \in I$ has a square root X which is not in I.

Recall the definition of the radical of an ideal from Commutative Algebra:

Definition. Let I be an ideal in a ring R. The **radical** of I is

$$\sqrt{I} = \{f \in R : \exists n > 0 \text{ s.t. } f^n \in I\}.$$

We say that I is a **radical ideal** if $\sqrt{I} = I$.

Theorem 2.6 (Hilbert's Nullstellensatz)**.** Let I be any ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. Then

$$I(V(I)) = \sqrt{I}.$$

This is a substantial theorem, fundamental to algebraic geometry, which was proven in the Commutative algebra lectures.

Note that, to calculate $\sqrt{I}$, we need to add in n-th roots of all elements of I, not just the generators. For example, if $I = (X, Y^2 - X) \subseteq k[X, Y]$, then we can rewrite this as $I = (X, Y^2)$ and so $\sqrt{I} = (X, Y) \neq I$, even though neither of the original generators of I had any non-trivial n-th roots.

3. Irreducible varieties and irreducible components

3.1. **Connected and irreducible sets.**

Question. Consider the following affine algebraic subvarieties of $\mathbb{A}^2$. Do they have 1 or 2 pieces? (I have deliberately not specified what I mean by "pieces." There are multiple sensible interpretations, so there is not always a unique "correct" answer.)

(1) The union of two disjoint lines $V(X(X-1))$.
(2) The union of two intersecting lines $V(XY)$.
(3) The hyperbola $V(XY-1)$.

Answer.

(1) $V(X(X-1))$ unambiguously has 2 pieces: the two lines $X = 0$ and $X = 1$.
Recall that a topological space is **connected** if it is not possible to write it as the union of two disjoint non-empty closed sets. This notion makes sense for the Zariski topology.
$V(X(X-1))$ is *not connected* because it is $V(X) \cup V(X-1)$.

(2) This has more than one answer. The two axes form 2 pieces. However they intersect at the origin, joining them into 1 piece. The set $V(XY)$ is *connected* but *reducible*.

Definition. A topological space S is **reducible** if it is empty, or there exist closed sets $S_1, S_2 \subseteq S$ such that $S = S_1 \cup S_2$, and neither S_1 nor S_2 is equal to S.

The opposite: A topological space S is **irreducible** if it is non-empty and it is not possible to write it as the union $S_1 \cup S_2$ of two closed sets, unless at least one of S_1 and S_2 is equal to S itself. (Change from the definition of *connected*: S_1 and S_2 are not required to be disjoint.)

Irreducibility is not a very useful notion for the topological spaces we consider in analysis. For example, considering the real line with the Euclidean topology, we can write it as a union of proper closed subsets:

$$\mathbb{R} = \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x \geq 0\}$$

These subsets are not disjoint because they intersect at 0.

It can often be convenient to rewrite the definition of irreducible spaces in terms of open sets instead of closed sets:

Lemma 3.1. The following conditions on a topological space S are equivalent to irreducibility:

(i) S is non-empty, and every pair of non-empty open subsets $U_1, U_2 \subseteq S$ have non-empty intersection $U_1 \cap U_2$.

(ii) S is non-empty, and every non-empty open subset of S is dense in S.

Corollary 3.2. Let S be a irreducible topological space and $U \subseteq S$ a non-empty open subset. Then U is irreducible (in the subspace topology).

(Proofs of the lemma and corollary are just manipulations of the topological definitions.)

Corollary 3.2(1) says that "irreducible" is a very long way from "Hausdorff": the Hausdorff condition says that a space has lots of pairs of disjoint non-empty open subsets, while an irreducible space has none. For example, we saw that $\mathbb{R}$ (with the Euclidean topology) is reducible in many ways.

(3) A drawing of $V(XY - 1)$ in $\mathbb{R}^2$ looks like it has two pieces. But (as mentioned before) we are missing a lot by only looking at real solutions. Over $\mathbb{C}$ it unambiguously has one piece.

One way to visualise this is to note that $V(XY - 1)$ "looks like" the set $\mathbb{A}^1 \setminus \{0\}$ (projecting onto the x coordinate is a bijection between these sets). This is not a formal statement – we have not yet defined a notion of isomorphism of affine algebraic varieties, and even if we had, $\mathbb{A}^1 \setminus \{0\}$ is not an affine algebraic variety. In a few weeks we will develop technology to make this into a rigorous statement.

But for now we use it as a heuristic. $\mathbb{R} \setminus \{0\}$ unambiguously has 2 pieces, but $\mathbb{C} \setminus \{0\}$ unambiguously has 1 piece. So the hyperbola (over an algebraically closed field) should have only one piece. We prove below the lecture that $V(XY - 1)$ is *irreducible* (and also *connected*).

Here's a bonus fact about connected sets in the Zariski topology which I didn't mention in the lecture. The proof is surprisingly hard.

Theorem. (Not part of the course.) Over $\mathbb{C}$, an affine algebraic variety is connected in the Zariski topology if and only if it is connected in the Euclidean topology.

3.2. Prime ideals and irreducible varieties.

Definition. (from Commutative Algebra) An ideal I in a ring R is a **prime ideal** if $I \neq R$ and for every $f, g \in R$, if $fg \in I$, then $f \in I$ or $g \in I$ (or both).

Lemma 3.3. An affine algebraic subvariety $V \subseteq \mathbb{A}^n$ is irreducible if and only if $I(V)$ is a prime ideal in $k[X_1, \dots, X_n]$.

Proof. First suppose that V is irreducible. Suppose we have $f, g \in k[X_1, \dots, X_n]$ such that $fg \in I(V)$. Let

$$V_1 = \{\underline{x} \in V : f(\underline{x}) = 0\},\ V_2 = \{\underline{x} \in V : g(\underline{x}) = 0\}.$$

For every $\underline{x} \in V$, $f(\underline{x})g(\underline{x}) = 0$ and hence either $f(\underline{x}) = 0$ or $g(\underline{x}) = 0$. Thus for every $\underline{x} \in V$, either $\underline{x} \in V_1$ or $\underline{x} \in V_2$. In other words, $V = V_1 \cup V_2$. Furthermore V_1 and V_2 are closed subsets of V. Hence as V is irreducible, either $V_1 = V$ or $V_2 = V$. If $V_1 = V$ then $f \in I(V)$ and if $V_2 = V$ then $g \in I(V)$.

Now suppose that V is reducible. Then we can write it as a union $V_1 \cup V_2$ of proper closed subsets. Since V_1 is a proper closed subset of V, there exists some $f \in k[X_1, \dots, X_n]$ vanishing on V_1 but not on all of V. Similarly there exists g vanishing on V_2 but not on all of V. Thus neither f nor g is in $I(V)$, but the product fg vanishes on $V_1 \cup V_2$ and hence we have $fg \in I(V)$. Thus $I(V)$ is not prime.

V is empty if and only if $I(V) = k[X_1, \dots, X_n]$, which is explicitly defined to not be a prime ideal. So it was OK to ignore this case above. □

Just like the definition of connected components, we can define:

Definition. Let S be a topological space. An **irreducible component** of S is a maximal irreducible subset of S.

Using the dictionary between ideals and subvarieties, one finds that an irreducible component of V corresponds to a minimal prime ideal containing the radical ideal $I(V)$. Hence, the primary decomposition theorem (for radical ideals) gives the irreducible decomposition:

Proposition 3.4. Any affine algebraic variety has a finite number of irreducible components, and is the union of its irreducible components.

Unlike connected components, irreducible components need not be disjoint. For example, the irreducible components of $\{(x, y) : xy = 0\}$ are the lines $x = 0$ and $y = 0$, which intersect in $\{(0, 0)\}$.

More generally, the irreducible components of a hypersurface $V(f)$ correspond to the irreducible factors of f: if $f = f_1^{a_1} \cdots f_m^{a_m}$ (where the f_i are distinct irreducible polynomials), then the irreducible components of $V(f)$ are $V(f_1), \dots, V(f_m)$.

4. Regular functions and regular maps

4.1. Regular functions.

So far we have only considered algebraic subvarieties as sets, sitting individually. Now we look at functions between them. Just as one uses continuous functions for topological spaces, holomorphic functions for complex manifolds, homomorphisms for groups, etc., so algebraic geometry has its own type of functions – regular functions. Of course, these are given by polynomials.

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic subvariety. A **regular function** on V is a function $f \colon V \to k$ such that there exists a polynomial $F \in k[X_1, \dots, X_n]$ with $f(\underline{x}) = F(\underline{x})$ for all $x \in V$.

Note that the polynomial F is not uniquely determined by the function f: $F, G \in k[X_1, \dots, X_n]$ determine the same regular function on V if and only if $F - G$ vanishes on V, that is iff $F - G \in I(V)$.

Definition. The regular functions on V form a k-algebra: they can be added and multiplied by each other and multiplied by scalars in k. This is called the **coordinate ring** of V and denoted $k[V]$.

There is a ring homomorphism $k[X_1, \dots, X_n] \to k[V]$ which sends a polynomial F to the function $F_{|V}$ which it defines on V. This homomorphism is surjective and its kernel is $I(V)$, so

$$k[V] \cong k[X_1, \dots, X_n]/I(V).$$

Example. What are the coordinate rings of the following affine algebraic subvarieties?

(i) $\mathbb{A}^n$.
(ii) A point.
(iii) $\{x \in \mathbb{A}^1 : x(x-1) = 0\}$ (two points).
(iv) $\{(x, y) \in \mathbb{A}^2 : xy = 0\}$ (two intersecting lines).
(v) $\{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$ (hyperbola).

Answers.

(i) $k[X_1, \dots, X_n]$.
(ii) k. A regular function on a point is just a single value.
(iii) $k \times k$. A regular function on two points is determined by two scalars, namely its value on each of the two points. For any pair of values $(a, b) \in k \times k$, one can easily write down a polynomial $f \in k[X]$ such that $f(1) = a$ and $f(0) = b$. Alternatively, one can check algebraically that the map

$$(a, b) \mapsto (a-1)X + b \bmod (X(X-1))$$

is a k-algebra isomorphism $k \times k \to k[X_1, \dots, X_n]/(X(X-1))$.

(iv) $\{(f,g) \in k[X] \times k[Y] : f(0) = g(0)\}$.

To prove this, note that

$$k[X,Y]/(XY) \cong \Big\{a_0 + \sum_{r=1}^{m} b_r X^r + \sum_{s=1}^{n} c_s Y^s : a_0, b_1, \dots, b_m, c_1, \dots, c_n \in k, m, n \in \mathbb{N}\Big\}.$$

We can compare these two descriptions by observing that

$$k[X] = \Big\{a_0 + \sum_{r=1}^{m} b_r X^r\Big\}, \quad k[Y] = \Big\{a_0 + \sum_{s=1}^{n} c_s Y^s\Big\},$$

and the condition that $f(0) = g(0)$ is equivalent to insisting that these two polynomials have the same constant coefficient a_0.

(v) The quotient ring $k[X,Y]/(XY-1)$.

To describe this more explicitly, note that any term $a_{i,j}X^rY^s$ of a two-variable polynomial is congruent (mod $XY-1$) to either $a_{r,s}X^{r-s}$ (if $r \geq s$) or $a_{r,s}Y^{s-r}$ (if $s > r$). Thus every coset in $k[X,Y]/(XY-1)$ has a representative of the form

$$\sum_{i=0}^{m} a_i X^i + \sum_{j=1}^{n} a_j Y^j.$$

The polynomials of this form determine different functions on V, so we have written down exactly one representative of each coset.

Since $XY = 1$ in $k[V]$, we may relabel Y as X^{-1}; then the multiplication rule will be what the notation leads us to expect. So we can write

$$k[V] = k[X, X^{-1}] = \Big\{\sum_{j=-n}^{m} a_j X^m : a_{-n}, \dots, a_m \in k, m, n \in \mathbb{N}\Big\}.$$

Example (iii) generalises: if V is a disconnected affine algebraic variety, we can write V as a union $V_1 \cup V_2$ of disjoint Zariski closed subsets, and then

$$k[V] = k[V_1] \times k[V_2].$$

On the other hand, if V is reducible but connected, so that the sets V_1 and V_2 are not disjoint, then $k[V]$ is a proper subset of $k[V_1] \times k[V_2]$ (see example (iv)).

Example (iv) does not generalise to arbitrary reducible algebraic varieties: we may have $V = V_1 \cup V_2$ where V_1 and V_2 are closed subsets, but

$$k[V] \neq \{(f,g) \in k[V_1] \times k[V_2] : f_{|V_1 \cap V_2} = g_{|V_1 \cap V_2}\}.$$

There will be an example of this on problem sheet 2.

Lemma 4.1. An affine algebraic variety V is irreducible if and only if $k[V]$ is an integral domain.

Proof. V is irreducible if and only if $I(V)$ is a prime ideal in $k[X_1, \dots, X_n]$. $\square$

4.2. **Regular maps.** A regular function goes from an algebraic variety V to the field k. We can also define regular maps, which go from one algebraic variety V to another algebraic variety W.

Definition. Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be affine algebraic subvarieties. A **regular map** $\varphi\colon V \to W$ is a function $V \to W$ such that there exist polynomials $F_1, \dots, F_n \in k[X_1, \dots, X_m]$ such that

$$\varphi(\underline{x}) = (F_1(\underline{x}), \dots, F_n(\underline{x}))$$

for all $\underline{x} \in V$.

Regular maps are often called **morphisms**.

In order to check that a given list of polynomials $F_1, \dots, F_n$ defines a regular map $V \to W$, it is necessary to check that $(F_1(\underline{x}), \dots, F_n(\underline{x})) \in W$ for every $\underline{x} \in V$. Equivalently, we need to check that the regular functions $F_{1|V}, \dots, F_{n|V} \in k[V]$ satisfy the equations

$$g(F_{1|V}, \dots, F_{n|V}) = 0$$

in the coordinate ring $k[V]$, for each polynomial $g \in I(W)$.

Examples.

(1) Let $V \subseteq \mathbb{A}^m$ be an affine algebraic subvariety. For any $n < m$, the projection $\pi\colon V \to \mathbb{A}^n$ defined by

$$\pi(x_1, \dots, x_m) = (x_1, \dots, x_n)$$

is a regular map.

(2) A regular function on V is the same thing as a regular map $V \to \mathbb{A}^1$. More generally, the set of regular maps $V \to \mathbb{A}^m$ is in bijection with the product $k[V]^m = k[V] \times \dots \times k[V]$.

(3) Consider SL_n, the set of $n \times n$ matrices with determinant 1. This is an affine algebraic subvariety in $\mathbb{A}^{n^2}$ because the determinant is a polynomial in the entries of a matrix. The map $a \mapsto a^{-1}$ is a regular map $\mathrm{SL}_n \to \mathrm{SL}_n$: Cramer's rule tells us how to write each entry of a^{-1} as a polynomial in the entries of a divided by $\det a$, and because we are only considering $a \in \mathrm{SL}_n$ we can drop the division by $\det a$.

Regular maps and Zariski topology. A regular map $\varphi\colon V \to W$ is a continuous function with respect to the Zariski topology. This is because, if $A \subseteq W$ is a Zariski closed subset defined by polynomials $f_1, \dots, f_r$, then $\varphi^{-1}(A)$ is the zero set of the polynomials $f_1 \circ \varphi, \dots, f_r \circ \varphi$ and therefore $\varphi^{-1}(A)$ is Zariski closed. In complex analysis, "holomorphic" is a much stricter condition than "continuous in the Euclidean topology," and similarly "regular" is much stricter than "continuous in the Zariski topology."

The following fact is very useful:

Lemma 4.2. Let $\varphi, \psi\colon V \to W$ be regular maps. If there exists a Zariski dense subset $A \subseteq V$ such that $\varphi_{|A} = \psi_{|A}$, then $\varphi = \psi$ on all of A.

Note that, if X and Y are Hausdorff topological spaces, then any continuous maps $X \to Y$ which agree on a dense set must agree everywhere. However the lemma does not follow immediately from the fact that regular maps are continuous, because the Zariski topology is not Hausdorff! (And the lemma is definitely false if we try to generalise it to all continuous maps with respect to the Zariski topology.) Thus in order to prove the lemma, we have to use something special about regular maps as opposed to general continuous maps.

Proof. Write $\varphi = (F_1, \dots, F_m), \psi = (G_1, \dots, G_m)$, where $F_1, \dots, F_m, G_1, \dots, G_m$ are polynomials. Then $F_i - G_i$ is also a polynomial for each i, and so

$$V_{\text{eq}} = \{\underline{x} \in V : \varphi(\underline{x}) = \psi(\underline{x})\} = \{\underline{x} \in V : (F_i - G_i)(\underline{x}) = 0 \text{ for all } i\}$$

is a Zariski closed subset of V. But we know that V_{eq} contains A, which is Zariski dense in V. Hence $V_{\text{eq}} = V$. □

5. Dictionary between algebra and geometry

Reduced finitely generated k-algebras.

To fully understand the relationship between affine algebraic varieties and k-algebras, there is one more question to answer: Which k-algebras can occur as $k[V]$ where V is an affine algebraic variety?

We write down some algebraic properties which obviously hold for $A = k[V]$:

(1) A is finitely generated, because if $V \subseteq \mathbb{A}^n$ then A is generated by the coordinate functions $X_1, \dots, X_n$.
(2) A is reduced, meaning that if $f \in A$ and $f^k = 0$ for some $k > 0$, then $f = 0$. This is because A is a ring of functions in the usual set-theoretic sense: if $f^k = 0$ then $f(x)^k = 0$ for all $x \in V$, so $f(x) = 0$ for all $x \in V$.

Using the Nullstellensatz, we can prove that these properties are enough to characterise the k-algebras which are coordinate rings of affine algebraic varieties.

Proposition 5.1. Let A be a finitely generated reduced k-algebra. Then there exists an affine algebraic variety V such that $k[V] \cong A$.

Proof. Pick a finite set $f_1, \dots, f_n \in A$ which generates A as a k-algebra. We can define a homomorphism $\alpha\colon k[X_1, \dots, X_n] \to A$ by $X_1 \mapsto f_1, \dots, X_n \mapsto f_n$.

Let $I = \ker \varphi$ and let $V = V(I) \subseteq \mathbb{A}^n$.

The homomorphism α is surjective because $f_1, \dots, f_n$ generate A, and so

$$A \cong k[X_1, \dots, X_n]/I.$$

Thus $k[X_1, \dots, X_n]/I$ is a reduced k-algebra. It follows that I is a radical ideal.

Hence the Nullstellensatz tells us that $I = I(V)$. Thus

$$k[V] \cong k[X_1, \dots, X_n]/I(V) \cong k[X_1, \dots, X_n]/I \cong A.$$ □

Dictionary between algebraic subsets and ideals.

Can we do something similar with Zariski closed subsets of V, and work them out from the algebra of $k[V]$?

Suppose that $V \subseteq \mathbb{A}^n$.

In $\mathbb{A}^n$: the Nullstellensatz tells us that the functions I and V are bijections

$$\{\text{Zariski closed subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[X_1, \dots, X_n]\}.$$

Since I and V reverse the direction of inclusions, we deduce that they restrict to bijections

$$\{\text{Zariski closed subsets of } V\} \longleftrightarrow \{\text{radical ideals in } k[X_1, \dots, X_n] \text{ containing } I(V)\}.$$

We know that

$$k[V] \cong k[X_1, \dots, X_n]/I(V).$$

It is a basic algebraic fact that

$$\{\text{ideals in } k[X_1, \dots, X_n] \text{ containing } I(V)\} \longleftrightarrow \{\text{ideals in } k[X_1, \dots, X_n]/I(V)\}.$$

Under this correspondence, radical ideals on one side correspond to radical ideals on the other side and similarly for prime ideals.

We conclude that the natural maps are bijections

$$\{\text{Zariski closed subsets of } V\} \longleftrightarrow \{\text{radical ideals in } k[V]\}$$

and

$$\{\text{irreducible Zariski closed subsets of } V\} \longleftrightarrow \{\text{prime ideals in } k[V]\}.$$

Can we describe the points of an affine algebraic variety V in terms of the algebra of $k[V]$? The points of V are the smallest non-empty Zariski closed subsets. Since the bijection between Zariski closed subsets and ideals reverses direction of inclusion, they correspond to maximal ideals:

$$\{\text{points of } V\} \longleftrightarrow \{\text{maximal ideals in } k[V]\}.$$

Isomorphisms.

Definition. A regular map $\varphi\colon V \to W$ is an **isomorphism** if there exists a regular map $\psi\colon W \to V$ such that $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_W$.

Example. If V is the parabola $\{(x, y) : y - x^2 = 0\}$, then the regular map $\varphi\colon V \to \mathbb{A}^1$ given by

$$\varphi(x, y) = x$$

is an isomorphism because it has an inverse $\psi\colon \mathbb{A}^1 \to V$ given by

$$\psi(x) = (x, x^2).$$

Example. On the other hand, if H is the hyperbola $\{(x, y) : xy = 1\}$, then the projection $(x, y) \mapsto x$ is not an isomorphism $H \to \mathbb{A}^1$ because it is not surjective so it cannot possibly have an inverse. This is not enough to prove that H is not isomorphic to $\mathbb{A}^1$, because maybe there is some other regular map $H \to \mathbb{A}^1$ which is an isomorphism. (We will soon prove that H is not isomorphic to $\mathbb{A}^1$.)

Example. Consider the affine algebraic subvariety $W = \{(x, y) : y^2 - x^3 = 0\}$. The regular map $\varphi \colon \mathbb{A}^1 \to W$ given by

$$\varphi(t) = (t^2, t^3)$$

is a bijection but it is not an isomorphism. Note that we should expect W not to be isomorphic to $\mathbb{A}^1$ because it has a singularity at the origin.

To prove that $\varphi \colon \mathbb{A}^1 \to W$ is not an isomorphism: Consider a regular map $\psi \colon W \to \mathbb{A}^1$. It must be given by a polynomial $g(X, Y) \in k[X, Y]$ and so

$$\psi \circ \varphi(t) = \psi(t^2, t^3)$$

is a polynomial in t which can have a constant term and terms of degree 2 or greater, but no term of degree 1. Hence we cannot find ψ such that $\psi \circ \varphi(t) = t$.

5.1. **Regular maps and the coordinate ring.**

Suppose we have a regular map $\varphi \colon V \to W$ between affine algebraic varieties. For each regular function g on W, we get a regular function $\varphi^* g$ on V defined by

$$(\varphi^* g)(x) = g(\varphi(x)).$$

We call $\varphi^* g \in k[V]$ the **pull-back** of $g \in k[W]$.

Thus φ induces a k-algebra homomorphism

$$\varphi^* \colon k[W] \to k[V].$$

Note that φ^* goes in the opposite direction to φ.

If we have two regular maps $\varphi \colon V \to W$ and $\psi \colon W \to Z$, then we can compose them to get $\psi \circ \varphi \colon V \to Z$. One can easily check that the associated pullback maps on coordinate rings satisfy

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^* \colon k[Z] \to k[V]. \tag{*}$$

For those who know category theory, we say that $V \mapsto k[V]$ is a contravariant functor

$$\{\text{affine algebraic varieties}\} \to \{\text{reduced } k\text{-algebras of finite type}\}.$$

The next proposition shows that this functor is an equivalence of categories:

Proposition 5.2. $\varphi \mapsto \varphi^*$ is a bijection

$$\{\text{regular maps } V \to W\} \longrightarrow \{k\text{-algebra homomorphisms } k[W] \to k[V]\}.$$

Proof. Consider $V = V(I) \subset \mathbb{A}^n$ and $W = V(J) \subset \mathbb{A}^m$, such that $k[V] = k[x_1, ..., x_n]/I$ and $k[W] = k[x_1, ..., x_m]/J$. We will build an inverse to $\phi \to \phi^*$. Consider a ring morphism:

$$\Phi : k[W] = k[x_1, ..., x_m]/J \to k[x_1, ..., x_n]/I = k[V] \tag{3}$$

Consider polynomials $F_1, ..., F_m$ such that $F_i + I = \Phi(x_i) \in k[x_1, ..., x_n]/I$. For any polynomial $g \in k[x_1, ..., x_m]$, we have $g(F_1, ..., F_m) = \Phi(g)$. In particular, if $g \in I$ then $g(F_1, ..., F_m) = \Phi(g) \in J$. Hence, for $x \in V$, $x \mapsto (F_1(x), ..., F_m(x))$ defines a regular map $\tilde{\Phi} : V \to W$, such that $\left(\tilde{\Phi}\right)^* = \Phi$. Notice that the value of $\tilde{\Phi}$ on V does not depend on the choice of the F_i. One checks that $\Phi \to \tilde{\Phi}$ is the inverse of $\phi \to \phi^*$. □

Corollary 5.3. Affine algebraic varieties V and W are isomorphic if and only if their coordinate rings $k[V]$ and $k[W]$ are isomorphic as k-algebras.

The moral is: if we only care about affine algebraic varieties up to isomorphism, then coordinate rings contain exactly the same information as algebraic varieties themselves.

Example. Now we can prove that the hyperbola H is not isomorphic to $\mathbb{A}^1$: we know that $k[H] = k[X, X^{-1}]$, and this is not isomorphic to $k[\mathbb{A}^1] = k[X]$ because in $k[X]$ the only invertible elements are the scalars, while $k[X, X^{-1}]$ contains non-scalar invertible elements, such as X.

6. Defining regular maps locally

Distinguished open subsets. We begin by giving complements on open subsets and open coverings of affine algebraic varieties. There is a certain class of open subsets of an affine algebraic variety V, known as distinguished open subsets of V, that are particularly easy to work with.

Definition. Let V be an affine algebraic variety, and let $f \in K[V]$. The distinguished open subset $D_V(f)$ is defined to be the open subset $V - V(f)$ of V.

Note that since $V(fg) = V(f) \cup V(g)$, we have $D_V(f) \cap D_V(g) = D_V(fg)$. Every point of V contains a basis of neighborhoods consisting of distinguished open sets. In other words:

Lemma 6.1. Let U be an open subset of V, and let p be a point of U. Then there exists an $f \in k[V]$ such that $p \in D_V(f) \subseteq U$.

Proof. Since U is open in V, it is a set of the form $V - V(I)$ for some ideal $I \subset K[V]$. Since p is not an element of $V(I)$ (since $p \in U$), there exists an $f \in I$ such that $p \notin V(f)$. Then $p \in D_V(f)$. On the other hand, since $f \in I$, $V(I) \subseteq V(f)$, so $D_V(f)$ (which equals $V - V(f)$) is contained in U. $\square$

Corollary 6.2. Any open set U in V is a union of finitely many distinguished open sets

Proof. Using the above lemma, one can cover U by distinguished open subset, and from Noetherianity a finite union of them cover U. $\square$

The Nullstellensatz gives us a nice criterion for when a collection of distinguished open sets covers V:

Proposition 6.3. Let $f \in k[V]$, and let $g_1, ..., g_r \in k[V]$ such that $D_V(g_i) \subseteq D_V(f)$. Then the $D_V(g_i)$ cover $D_V(f)$ if, and only if, f is in the radical of $(g_1, ..., g_r)$. In particular the $D_V(g_i)$ cover V if, and only if, the g_i generate the unit ideal of $k[V]$.

Proof. The union of the $D_V(g_i)$ is the union of $V - V(g_i)$, which is $V - V(g_1, ..., g_r)$. For this to contain $D_V(f)$ we must have $V(g_1, ..., g_r) \subseteq V(f)$. Taking ideals of both sides, we find that $I(V(f)) \subseteq I(V(g_1, ..., g_r))$. By the Nullstellensatz we then have:

$$\sqrt{(f)} \subseteq \sqrt{(g_1, ..., g_r)} \tag{4}$$

Since f is an element of $\sqrt{(f)}$ the first claim follows. The second claim follows by taking $f = 1$, so that $D_V(f) = V$. Then we have $1 \in (g_1, ..., g_r)$ in $k[V]$. $\square$

Regular functions on open subsets. In the last lectures, we have studied regular functions and maps defined everywhere on an affine variety. In geometry, it is in general also useful to consider functions and maps which are defined only locally, i.e. on an open subset. The basic idea is that we can invert a nonvanishing polynomial on the open locus where it does not vanish, *i.e.* we must consider that, given $f, g \in K[V]$,:

$$\begin{aligned} \phi : & V - V(g) \to \mathbb{A}^1 \\ & x \mapsto \frac{f(x)}{g(x)} \end{aligned} \tag{5}$$

must be regular on the open subset $V - V(g)$ where g does not vanish, hence one can work locally with rational fractions instead of polynomials. Moreover, we want to be able to check regularity locally near each point, i.e. we must not ask for a global presentation of ϕ as global fraction, but only for a local presentation. That is, regularity of a function is a property that, like continuity, differentiability, holomorphy, etc, can be checked in small neighborhoods of every point in the domain. This leads to the definition:

Definition. Let U be an open subset of an affine algebraic variety V. We say that $f : U \to k$ is a regular function on U if there exists an open cover (U_i) of U, together with regular functions g_i, h_i such that h_i is nonvanishing on U_i and such that $f(x) = \frac{g_i(x)}{h_i(x)}$ for all $x \in U_i$, and for all i.

The ring of regular functions on $U \subseteq V$ will be denoted $\mathcal{O}_V(U)$. Note that it is not immediately obvious that this agrees with our definition of a regular function on V in the case $U = V$ (i.e., that $\mathcal{O}_V(V) = k[V]$); we will verify this shortly.

We observe that if f is a regular function on U, then the set $f^{-1}(0)$ of zeros of f on U, is closed in U. Indeed, we can fix an open cover U_i of U on which f is given by $\frac{g_i}{h_i}$, with g_i and hi polynomials such that h_i is nonvanishing on U_i. Then $f^{-1}(0) \cap U_i$ is equal to $V(g_i) \cap U_i$, and the latter is clearly closed in U_i.

A disadvantage of the above definition is that it is typically quite hard to describe the ring of regular open functions on an arbitrary open subset U of V. However, when U is a distinguished open subset, things are easier:

Proposition 6.4. Let V be an affine algebraic variety, $h \in k[V]$, and let $U = D_V(h)$. Then every regular function on U is globally of the form $\frac{g}{h^n}$, where $g \in k[V]$. In particular the natural map:

$$k[V][\frac{1}{h}] \to \mathcal{O}_V(U) \tag{6}$$

is an isomorphism. In particular, taking $h = 1$, one obtains $\mathcal{O}_V(V) = k[V]$.

Proof. By definition, a regular function $f : U \to k$ admits an open cover (U_i) of U, and polynomial functions g_i, h_i on U_i, with h_i nonvanishing on U_i, such that $f(x) = \frac{g_i(x)}{h_i(x)}$ for all x on U_i. Our first step is to replace this cover (and the g_i and h_i) with ones in a more convenient form. Note we can cover U_i with distinguished opens $D_V(h_{ij})$. Then (since $D_V(h_{ij})$ is contained in U_i, and h_i is nonvanishing on

U_i, we have $V(h_i) \subseteq V(h_{ij})$. Taking ideals we find that $h_{ij} \in \sqrt{(}h_i)$, so, for some r, we have write $h_{ij}^r = g'_{ij} h_i$. Then:

$$f(x) = \frac{g_i(x)}{hi(x)} = \frac{g_i(x) g'_{ij}(x)}{h_{ij}^r} \tag{7}$$

Since $D_V(h_{ij}) = D_V(h_{ij}^r)$ we can set $h'_{ij} = h_{ij}^r$, we then have a cover of U_i by $D_V(h'_{ij})$ and an identity $f(x) = \frac{(polynomial)}{h'_{ij}}$ on U_i.

Doing this for all of the U_i's, we end up with an open cover of U by distinguished opens of the form $D_V(h_i)$ (not the same h_i as in the last paragraph!), and expressions for f of the form $f(x) = \frac{g_i(x)}{h_i(x)}$ on U_i for some polynomials $g_i(x)$.

On $D_V(h_i) \cap D_V(h_j)$ we have $g_i(x) h_i(x) = g_j(x) h_j(x)$, which implies that $g_i h_j - g_j h_i$ is identically zero on $D_V(h_i) \cap D_V(h_j)$. Since $h_i h_j$ vanishes identically on the complement of $D_V(h_i) \cap D_V(h_j)$, we deduce that $g_i h_i h_j^2 - g_j h_i^2 h_j$ vanishes identically on all of V, for all i and j.

Now, since the $D_V(h_i)$ cover $U = D_V(h)$, and $D_V(h_i) = D_V(h_i^2)$ for all i, the sets $D_V(h_i^2)$ cover $D_V(h)$. By our criterion for when a collection of distinguished opens covers another distinguished open, there exists an r such that h^r lies in the ideal of $k[V]$ generated by the h_i^2; that is, we can write:

$$h^r = a_1 h_1^2 + ... + a_s h_s^2 \tag{8}$$

where the $a_i \in k[V]$. Thus, for all $x \in V$, we have:

$$h^r(x) = a_1(x) h_1^2(x) + ... + a_s(x) h_s^2(x) \tag{9}$$

I claim that we have:

$$f(x) = \frac{1}{h^r(x)} \sum_{i=1}^{s} a_i(x) g_i(x) h_i(x) \tag{10}$$

for all $x \in D_V(h)$, proving the theorem.

Indeed, if x lies in $D_V(h_i)$, then $f(x) = \frac{g_i(x)}{h_i(x)}$ on $D_V(h_i)$. Thus

$$f(x) h_i(x)^2 h(x)^r = \frac{g_i(x)}{h_i(x)} h_i(x)^2 \sum_{j=1}^{s} a_j(x) h_j(x)^2 \tag{11}$$

$$= \sum_{j=1}^{s} a_j(x) g_i(x) h_i(x) h_j(x)^2 \tag{12}$$

Using that $g_i(x) h_i(x) h_j(x)^2 = g_j(x) h_i(x)^2 h_j(x)$ for all j we find that the right hand sum is equal to:

$$h_i(x)^2 \sum_{j=1}^{s} a_j(x) g_j(x) h_j(x) \tag{13}$$

Dividing both sides by $h_i(x)^2 h(x)^r$, the claim follows. □

Note (Non-examinable material!)**.** We have then for each $U \subseteq V$ a ring $\mathcal{O}_V(U)$, but there is a structure relating these different rings. We can restrict a regular function on an open subset, namely for for $U' \subseteq U \subseteq V$ open, and $f : U \to k$ regular, $f|_{U'}$ is regular too, which gives a restriction morphism $\mathrm{res}_{U,U'} : \mathcal{O}_V(U) \to \mathcal{O}_V(U')$. One can easily check that given another open subset $U'' \subseteq U'$, we have an equality $\mathrm{res}_{U,U''} = \mathrm{res}_{U',U''} \circ \mathrm{res}_{U,U'}$.

Also, because the notion of regularity is local, we can glue regular functions, i.e. given U_i open subsets covering U and $f_i : U_i \to W$ regular such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for each i, j, there is a unique regular function $f : U \to k$ such that $f|_{U_i} = f_i$. It means that the regular functions form a *sheaf*, just like continuous, differentiable, holomorphic functions on topological, differentiable or complex manifolds. The sheaf formalism lies at the core of the foundation of scheme theory, but we will not require any knowledge of sheaf theory in these lectures.

For those that know category theory: a $\mathcal{C}$-valued presheaf on a space X is a functor $\mathcal{O} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$, where we regard the partially ordered set of open subsets of X as a category $\mathrm{Open}(X)$. This data yields the assignments $U \mapsto \mathcal{O}(U)$ as well as the restriction morphisms and their compatibilities. A presheaf is a sheaf if it satisfies existence and uniqueness of gluing (a presheaf is said to be separated if it satisfies just the uniqueness part).

Regular maps on open subsets. As for regular maps on affine varieties, we can define regular maps on open subsets as maps having regular coordinates:

Definition. Let $V \subseteq \mathbb{A}^n_k$ and $W \subseteq \mathbb{A}^m_k$ be affine algebraic subvarieties, and let $U \subsetneq V$, $U' \subsetneq W$ be open subsets. A map $f : U \to U'$ is a regular map if there exist regular functions $f_1, ..., f_m$ on U such that $f(p) = (f_1(p), ..., f_m(p))$ for all $p \in U$.

Note that, since being a regular function is a local property, being a regular map is also local: a map $f : U \to U'$ is regular if, and only if, every point p in U has a neighborhood on which f is regular. Also note that if $g : U' \to k$ is a regular function, and $f : U \to U'$ is a regular map, then the function $g \circ f : U \to k$ is also regular. Indeed, we can check this locally near $p \in U$. We can choose a neighborhood of $f(p)$ on which g is given by a rational function $\frac{t}{u}$, with t and u polynomials and u nonvanishing on this neighborhood. Similarly choose a neighborhood of p on which f is given by $(\frac{f_1}{h_1}, ..., \frac{f_m}{h_m})$, where the f_i and h_i are polynomials and h_i is nonvanishing on this neighborhood. Then $g \circ f$ is given by the expression:

$$\frac{t(\frac{f_1}{h_1}, ..., \frac{f_m}{h_m})}{s(\frac{f_1}{h_1}, ..., \frac{f_m}{h_m})} \tag{14}$$

and this is clearly a regular function near p, since the composition of rational functions is a rational function, whose denominator does not vanish on a neighborhood of p. We thus obtain a "pullback" homomorphism $f^* : \mathcal{O}_W(U') \to \mathcal{O}_V(U)$, defined by $f^*(g) = g \circ f$, which generalize the one for regular functions on affine space.

Corollary 6.5. Any regular map $f : U \to U'$ is continuous in the Zariski topology

Proof. It suffices to check that preimages of closed sets are closed. Note that any closed subset of U' is an intersection of subsets of the form $V(g)$ for g regular on W , and that the preimage of such a set is of the form $(f^*(g))^{-1}(0)$ and hence closed in U. The claim follows. $\square$

In particular, since a regular function on U is the same as a regular map $U \to \mathbb{A}^1_k$, we see that regular functions are also continuous with respect to the Zariski topology on U and $\mathbb{A}^1_k$. Note that there are certainly many continuous functions on U that are not regular!

Proposition 6.6. A map $f : U \to U'$ is regular if, and only if, for all regular functions g on U' , the composition $g \circ f$ is regular on U. In particular, any composition of regular maps is regular.

Proof. Recall that we have $U' \subseteq W \subseteq \mathbb{A}^m_k$ for some affine algebraic subvariety W. In particular we have the coordinate functions $x_1, ..., x_m$ on W . Note that $f(p)$ is then equal to $((x_1 \circ f)(p), ..., (x_m \circ f)(p))$, so that if $x_i \circ f$ is regular for all i, then f is regular. The converse was proven above. $\square$

Definition. Consider $U \subset V$ and $U' \subset W$ two open subsets of affine varieties.

- A regular map $f : U \to U'$ is said to be an isomorphism if there exists a regular map $g : U' \to U$ that is a two-sided inverse to f.
- A regular map $f : U \to U'$ is said to be an open (resp. closed) embedding if its image is open (resp. closed) in U' and the induced map $U \to f(U)$ is an isomorphism.
- A regular map $f : U \to U'$ is said to be dominant if $f(U)$ is dense in U'. In this case the pullback map $\mathcal{O}_W(U') \to \mathcal{O}_V(U)$ is injective: if $g \in \mathcal{O}_W(U')$ is a function such that $f^*(g) = 0$, then $f(U) \subseteq V(g)$ and in particular $\overline{f(U)} \subseteq V(g)$. Since $f(U)$ was dense, we have $V(g) = U'$ and hence $g = 0$.

7. Rational functions and rational maps

Rational maps and domains. Now let V be an open subset of an affine variety, and suppose that V is irreducible. In particular, every nonempty open subset U of V is dense in V, and so the intersection of any two nonempty open subsets is nonempty and dense.

Definition. Let V be an irreducible open in an affine variety (and W an open in an affine variety). A rational function $f : V \dashrightarrow k$ (resp. a rational map $f : V \dashrightarrow W$) is an equivalence class of pairs (U, f), where U is nonempty and open in V and f is a regular function on U (resp. a regular map $U \to W$). We say that (U, f) is equivalent to (U', f') if the restrictions of f and f' to $U \cap U'$ agree.

The only subtle point is the transitivity of the relation. If $(U, f) \sim (U', f') \sim (U'', f'')$, then:

$$f|_{U\cap U'\cap U''} = f|_{U\cap U'}|_{U''} = f'|_{U\cap U'}|_{U''} = f'|_{U'\cap U''}|_U \simeq f''|_{U'\cap U''}|_U = f''|_{U\cap U'\cap U''} \quad (15)$$

but $U\cap U'\cap U''$ is dense in $U\cap U''$ (here one really uses the irreducibility of V), which means that $f|_{U\cap U''}$ and $f''|_{U\cap U''}$ are two regular functions (resp., maps) agreeing on an open dense subset, hence they are equal, which means $(U, f) \sim (U'', f'')$.

We use the broken arrow symbol $\dashrightarrow$ instead of the usual arrow because a rational map is not a function on V in the usual set-theoretic sense. From the definition, a rational function is a rational map $V \dashrightarrow \mathbb{A}^1$. If (U, f) is equivalent to (U', f'), we can in particular glue f and f' to obtain a regular map $g : U \cup U' \to W$, and then a representative $(U \cup U', g)$ of the same rational function. In particular, we can find a maximal representative, defined on the union of all the open subsets on which the representatives are defined. This motivates:

Definition. The domain $\mathrm{dom}(\phi)$ of a rational function ϕ is the maximal open subset on which ϕ is defined, i.e. the set of points $x \in V$ (called regular points) such that there exists an open neighborhood $x \in U \subseteq V$ and a regular map $f : U \to W$ representing ϕ.

We can add and multiply rational functions by restricting to where they are both defined:

$$(U, f) + (U', f') := (U \cap U', f|_{U\cap U'} + f'|_{U\cap U'}) \quad (16)$$

and similarly for multiplication. What is new is that we also have multiplicative inverses of nontrivial rational functions:

$$(U, f) \cdot (U - f^{-1}(0), \frac{1}{f}) \quad (17)$$

so that the rational functions on V have the structure of a field. This field is called the function field of V, and denoted $k(V)$.

Proposition 7.1. Let V be an irreducible affine algebraic variety. Then $k(V)$ is the field of fractions of $k[V]$.

Proof. We certainly have a map from the field of fractions of $k[V]$ to $k(V)$, taking $\frac{f}{g}$ to the equivalence class of $(D_V(g), \frac{f}{g})$. On the other hand, when restricted to a sufficiently small neighborhood, every rational function on V has a representative of this form. Thus this map is a surjection of fields, hence an isomorphism. $\square$

Hence a rational function on V is an equivalence class of fractions $\frac{f}{g}$ where $f, g \in k[V]$, where as usual $\frac{f}{g} \sim \frac{f'}{g'}$ if $fg' = f'g$. The set of regular points $\mathrm{dom}(\phi)$ is then the set of points $x \in V$ such that there exists a representative $\frac{f}{g}$ of ϕ with $g(x) \neq 0$.

For example, consider the algebraic subvariety defined by the equation $XY = ZT$ in $\mathbb{A}^4$. Let

$$\varphi = X/Z \in k(V).$$

The defining equation implies that we also have

$$\varphi = T/Y.$$

Looking at the fraction X/Z shows us that φ is regular wherever $Z \neq 0$, and looking at the fraction T/Y shows us that φ is regular wherever $Y \neq 0$. On the other hand, φ is not regular on the closed subset $Y = Z = 0$. (One can verify that there is no other fraction representing φ which is non-zero on this closed subset.)

Projection from a point onto a hyperplane.

Example. An important example of a rational map is projection from a point onto a hyperplane.

Let H be a hyperplane in $\mathbb{A}^n$ (that is, a set defined by a single *linear* equation). Let p be a point in $\mathbb{A}^n \setminus H$. For simplicity, we shall assume that p is the origin and that

$$H = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 1\}.$$

(We could always reduce to this case by a suitable change of coordinates.)

Let us write H_p for the hyperplane through p parallel to H, that is,

$$H_p = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 0\}.$$

For each point $x \in \mathbb{A}^n \setminus H_p$, let L_x denote the line which passes through p and x. Since $x \notin H_p$, L_x intersects H in exactly one point. Call this point $\varphi(x)$.

We can write this algebraically as

$$\varphi(x_1, \dots, x_n) = (x_1/x_n, \dots, x_{n-1}/x_n, 1)$$

and so φ is a rational map $\mathbb{A}^n \dashrightarrow H$. This map is called **projection from p onto H**.

We have $\mathrm{dom}\,\varphi = \mathbb{A}^n \setminus H_p$. (Note that we have not proved this, because we have not proved that there is no other list of fractions which define the same rational map but have non-zero denominators at points in H_p.)

For any affine algebraic subvariety $V \subseteq \mathbb{A}^n$ such that $V \not\subseteq H_p$, we can restrict φ to get a rational map $V \dashrightarrow H$. (Note that p might be in V, or it might not.)

Example. Let V be the circle $\{(x,y) : x^2 + y^2 = 1\}$. Consider the projection from the point $p = (1,0)$ on to the line $x = 0$. This is a rational map $\pi\colon V \dashrightarrow \mathbb{A}^1$ with the formula

$$\pi(x,y) = y/(1-x).$$

We can see geometrically that this projection induces a bijection between the circle (excluding p) and the line (at least for real points). If we compute the formula for the inverse map, we get

$$\psi(t) = \Big(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\Big),$$

a well-known parameterisation of the circle. Thus we see that the inverse is a rational map $\psi\colon \mathbb{A}^1 \dashrightarrow V$. Note that ψ is not regular at $t = \pm i$ – we don't see this on the picture, which only shows the real points.

Next time we will define formally what it means to say that the rational maps π and ψ are inverse to each other, taking into account that they are not true functions between the sets V and $\mathbb{A}^1$ because they are not regular everywhere.

Composing rational maps.
Last time we defined rational maps $\pi\colon V \dashrightarrow \mathbb{A}^1$ and $\psi\colon \mathbb{A}^1 \dashrightarrow V$ where V is the circle. These maps are inverses in that composing them (either way round) gives the identity, if we ignore the points where the maps are not regular.

In order to rigorously define composition of rational maps, we need to notice that sometimes the set of points where a composite map is undefined is "everywhere" and exclude that situation. For example, consider the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ defined by

$$\xi(x,y) = \frac{1}{1-x^2-y^2}.$$

This map is not regular anywhere on the circle V, and hence it does not make sense to try to define the composite map $\xi \circ \psi\colon \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ (it is not defined anywhere!).

This problem can occur because the image of ψ is not dense in $\mathbb{A}^2$. So to rule it out, we make the following definition.

Definition. The **image** of a rational map $\varphi\colon V \dashrightarrow W$ is the set of points

$$\{\varphi(x) \in W : x \in \operatorname{dom}\varphi\}.$$

A rational map is **dominant** if its image is Zariski dense in W.

For example, ψ from the end of the previous lecture is dominant if we consider it as a rational map $\mathbb{A}^1 \dashrightarrow V$ but it is not dominant if we consider it as a rational map $\mathbb{A}^1 \dashrightarrow \mathbb{A}^2$. (This is like surjectivity: whether a function is surjective or not depends on what codomain you declare it to have.)

Let V, W, T be irreducible affine algebraic varieties. If $\varphi\colon V \dashrightarrow W$ is a *dominant* rational map and $\psi\colon W \dashrightarrow T$ is a rational map (ψ is not required to be dominant), then it makes sense to compose them because we know that $\operatorname{dom}\psi$ is a Zariski open subset of W, while $\operatorname{im}\varphi$ is a Zariski dense subset of W and so

$$\operatorname{dom}\psi \cap \operatorname{im}\varphi \neq \emptyset.$$

Thus there are at least some points where $\psi \circ \varphi$ is defined. One can check (by writing out ψ in terms of fractions of polynomials, then substituting in fractions of polynomials representing φ) that $\psi \circ \varphi$ is a rational map $V \dashrightarrow T$.

Definition. Rational maps $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow V$ are **rational inverses** if both are dominant and $\varphi \circ \psi = \mathrm{id}_W$ and $\psi \circ \varphi = \mathrm{id}_V$, everywhere these composite rational maps are well-defined.

A rational map $\varphi\colon V \dashrightarrow W$ is a **birational equivalence** if it is dominant and has a rational inverse.

We say that irreducible algebraic varieties V and W are **birational** (or **birationally equivalent**) if there exists a birational equivalence $V \dashrightarrow W$.

Our example from the previous lecture showed that the circle is birational to $\mathbb{A}^1$.

Another example is the cuspidal cubic

$$W = \{(x, y) : y^2 = x^3\}.$$

This is also birational to $\mathbb{A}^1$, as shown by the rational maps

$$W \dashrightarrow \mathbb{A}^1 : (x, y) \mapsto y/x,$$
$$\mathbb{A}^1 \dashrightarrow W : t \mapsto (t^2, t^3).$$

Birationally equivalent affine algebraic varieties look the same "almost everywhere." For example, the cuspidal cubic is the same as the affine line everywhere *except* at the origin.

On the other hand, $\mathbb{A}^1$ is not birationally equivalent to $\mathbb{A}^2$ or to an elliptic curve

$$\{(x, y) : y^2 = f(x)\} \text{ where } f \text{ is a cubic polynomial with no repeated roots.}$$

We will prove this later in the course once we have more tools.

Correspondence between algebra and geometry. If $\varphi\colon V \dashrightarrow W$ is a dominant rational map between irreducible affine varieties, then we can use it to pull back rational functions from W to V (just like we earlier used regular maps to pull back regular functions). We get a k-homomorphism of fields

$$\varphi^*\colon k(W) \to k(V)$$

defined by $\varphi^*(g) = g \circ \varphi$. (A k-homomorphism means that φ^* restricts to the identity on the copies of k which are contained in $k(W)$ and $k(V)$, namely the constant functions.)

If φ is a birational equivalence, then φ^* is a k-isomorphism of fields.

One can do the same thing that for regular functions:

Proposition 7.2. $\varphi \mapsto \varphi^*$ is a bijection

$$\{\text{dominant rational maps } V \dashrightarrow W\} \longrightarrow \{k\text{-field homomorphisms } k(W) \to k(V)\}.$$

The proof is similar than the proof of Proposition 5.2, and one can deduce an equivalent of Corollary 5.3:

Corollary 7.3. Irreducible affine algebraic varieties V and W are birationally equivalent if and only if their function fields $k(V)$ and $k(W)$ are k-isomorphic.

Hypersurfaces and birational equivalence. We will now use the algebraic description of birational equivalence to give a geometric interpretation of this algebraic theorem:

Lemma 7.4. Let k be an algebraically closed field and let K be a finitely generated extension field of k. Then there exist $t_1, \dots, t_d, u \in K$ such that

(i) $K = k(t_1, \dots, t_d, u)$;
(ii) $t_1, \dots, t_d$ are algebraically independent over k (that is, there is no non-zero polynomial in d variables with coefficients in k whose value at $(t_1, \dots, t_d)$ is zero); and
(iii) u is algebraic over $k(t_1, \dots, t_d)$ (that is, there exists a non-zero polynomial in one variable with coefficients in the field $k(t_1, \dots, t_d)$ which is zero at u).

Proof. This follows from the primitive element theorem in field theory. For a full proof, see Proposition A.7 in the Appendix of Shafarevich, *Basic Algebraic Geometry 1*. □

Proposition 7.5. Let K be a finitely generated extension of k. Then there exists an irreducible hypersurface $H \subseteq \mathbb{A}^{d+1}$ for some d such that K is isomorphic to the field of functions $k(H)$.

Corollary 7.6. Let $V \subseteq \mathbb{A}^n$ be an irreducible affine algebraic subvariety. Then there exists an irreducible hypersurface $H \subseteq \mathbb{A}^{d+1}$ for some d such that V is birationally equivalent to H.

The corollary is obtained by applying the proposition to $K = k(V)$. It tells us that, even if V is a complicated algebraic variety defined by many equations, provided we only care about properties of V which are preserved by birational equivalence, we can replace V by a simpler set defined by just one equation, that is, a hypersurface. Note that it is *not* true that every irreducible affine algebraic variety is *isomorphic* to a hypersurface.

Proof of Proposition 7.5. Write $K = k(t_1, \dots, t_d, u)$ as in Lemma 7.4, and let $K' = k(t_1, \dots, t_d)$. Let $p(X) \in K'[X]$ be the minimal polynomial of u over K'.

Each coefficient of $p(X)$ is a fraction whose numerator and denominator are polynomials in $t_1, \dots, t_d$. We can multiply up by a suitable element of $k[t_1, \dots, t_d]$

to clear the denominators, and also replace $t_1, \dots, t_d$ by indeterminates $T_1, \dots, T_d$ to get a polynomial $g \in k[T_1, \dots, T_d, X]$ such that

$$g(t_1, \dots, t_d, u) = 0 \text{ in the field } K.$$

Assuming we multiplied up by a lowest common denominator for the coefficients of p, g is irreducible.

Let H be the hypersurface $V(g) \subseteq \mathbb{A}^{d+1}$, which we give coordinates $T_1, \dots, T_d, X$. Because g is irreducible, it generates a radical ideal in $k[T_1, \dots, T_d, X]$ and so the (Strong) Nullstellensatz implies that

$$I(H) = (g).$$

Thus the coordinate ring is given by

$$k[H] = k[T_1, \dots, T_d, X]/(g).$$

There is a k-algebra homomorphism $\alpha\colon k[T_1, \dots, T_d, X] \to K$ defined by

$$T_1 \mapsto t_1,\ \dots,\ T_d \mapsto t_d,\ X \mapsto u.$$

If $\alpha(f) = 0$ then $p(X)$ divides $F(X) = f(t_1, \dots, t_d, X)$ in $K'[X]$, since $p(X)$ was the minimal polynomial of u. Hence $g(X)$ also divides $F(X)$ in $K'[X]$. Gauss's Lemma now implies that g divides f in $k[T_1, \dots, T_d, X]$, and hence the kernel of α is generated by g, so α induces an injection $k[H] \hookrightarrow K$. Furthermore, the image of α generates K as a field, so α induces an isomorphism from the fraction field of $k[H]$ to K.

The fraction field of $k[H]$ is the function field $k(H)$. Thus we have shown that $k(H) \cong k(V)$. By Corollary 7.3, this implies that V is birationally equivalent to H. $\square$

8. Projective varieties

Projective space.

Projective space consists of affine space together with "points at infinity", one for each direction. The purpose for adding extra points is that it avoids special cases where a point "disappears to infinity". For example, a pair of parallel lines do not intersect in affine space but they do intersect at a point at infinity in projective space.

Definition. Projective n-space, $\mathbb{P}^n$, is the quotient of $k^{n+1} \setminus \{(0,\dots,0)\}$ by the equivalence relation

$$(x_0,\dots,x_n) \sim (\lambda x_0,\dots,\lambda x_n) \text{ where } \lambda \in k \setminus \{0\}.$$

We call a representative for an equivalence class the **homogeneous coordinates** of that point in $\mathbb{P}^n$ (and there are many choices for each point, by scaling by λ). To avoid confusion between homogeneous coordinates for $\mathbb{P}^n$ and ordinary coordinates for $\mathbb{A}^n$, we usually write homogeneous coordinates as

$$[x_0 : x_1 : \cdots : x_n].$$

Observe that we can embed $\mathbb{A}^n$ into $\mathbb{P}^n$ by the map

$$(x_1,\dots,x_n) \mapsto [1 : x_1 : \cdots : x_n].$$

Any other homogeneous coordinates where the first coordinate is non-zero can be re-scaled to have first coordinate 1. So we are left with the points with first coordinate equal to 0: these are the "points at infinity." A point $[0 : x_1 : \cdots : x_n]$ can be seen as a point in $\mathbb{P}^{n-1}$, by just dropping the initial zero. Thus

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

Similarly

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\text{a point}\}.$$

Thinking about projective space as affine space plus points at infinity can be useful if we want to make use of our geometric intuition about affine space or the algebraic tools we have developed for working with affine algebraic varieties. On the other hand, thinking about projective space in terms of homogeneous coordinates emphasises that all points of projective space look the same: we can only distinguish points at infinity from points in affine space after choosing a convention for how we embed $\mathbb{A}^n$ into $\mathbb{P}^n$ (for example, we could have used $[x_1 : \cdots : x_n : 1]$ instead); throughout this lecture we will use the convention above.

8.1. **Zariski Topology on $\mathbb{P}^n$, homogeneous ideals.** We define the Zariski topology on $\mathbb{P}^n$ to be the topology induced from the Zariski topology on $\mathbb{A}^{n+1} - \{(0,...,0)\}$ by the quotient map $\pi : \mathbb{A}^{n+1} - \{(0,...,0)\} \to \mathbb{P}^n$. Unwrapping the definition, we hence have:

$$V \subset \mathbb{P}^n \text{ is closed} \iff \pi^{-1}(V) \subset \mathbb{A}^{n+1} - \{(0,...,0)\} \text{ is closed.}$$

We can simplify this condition a little further. If $X \subseteq \mathbb{A}^{n+1} - \{(0, ..., 0)\}$ is nonempty then we put $\hat{X} = X \cup \{(0, ..., 0)\} \subseteq \mathbb{A}^{n+1}$. If X is empty then we put $\hat{X} = X = \emptyset$.

Lemma 8.1. Let $X \subseteq \mathbb{A}^{n+1} - \{(0, ..., 0)\}$. Then X is closed in the subspace topology if and only if $\hat{X} \subseteq \mathbb{A}^{n+1}$ is closed.

Proof. Put $U = \mathbb{A}^{n+1} - \{(0, ..., 0)\} \subseteq \mathbb{A}^{n+1}$. If X is closed in U then we have $X = U \cap V$ for some $V \subseteq \mathbb{A}^{n+1}$ closed. Then either $\hat{X} = V \cup \{(0, ..., 0)\}$ or $\emptyset$, which is closed. Conversely, if $\hat{X}$ is closed then $X = \hat{X} \cap U$ is closed. $\square$

The above Lemma then tells us that we have

$$\begin{aligned} V \subset \mathbb{P}^n \text{ is closed } &\iff \pi^{-1}(V) \subset \mathbb{A}^{n+1} - \{(0, ..., 0)\} \text{ is closed} \\ &\iff \widehat{\pi^{-1}(V)} \subset \mathbb{A}^{n+1} \text{ is closed.} \end{aligned} \tag{18}$$

Take $V \subseteq \mathbb{P}^n$ closed. The closed subset $\widehat{\pi^{-1}(V)} \subset \mathbb{A}^{n+1}$ is called the affine cone on V. It is indeed a cone: if $x \in \widehat{\pi^{-1}(V)}$ then $\lambda x \in \widehat{\pi^{-1}(V)}$ for all $\lambda \in k$. If V is nonempty, this is equivalent to the condition that the origin is in $\widehat{\pi^{-1}(V)}$, and if $x \in \widehat{\pi^{-1}(V)}$ is not the origin, then the line L joining x to the origin is contained in $\widehat{\pi^{-1}(V)}$.

Lemma 8.2. If $V \subseteq \mathbb{P}^n$ is a closed subset then $\widehat{\pi^{-1}(V)}$ is the closure of the set $\pi^{-1}(V) \subseteq \mathbb{A}^{n+1}$.

Proof. The case where V is empty is clear so assume that V is nonempty, so that we may choose $x \in \pi^{-1}(V)$. Hence the punctured line $L' = \{\lambda x : \lambda \in k^\times\}$ is a subset of $\pi^{-1}(V)$. If $L = \hat{L}'$ denotes the closure of L', then $\overline{\pi^{-1}(V)} \cap L$ is a closed subset of L containing L'. It is hence all of L and so the origin must be contained in $\overline{\pi^{-1}(V)}$. Hence we have $\widehat{\pi^{-1}(V)} \subseteq \overline{\pi^{-1}(V)}$, which implies that they are equal. $\square$

In general, $\widehat{\pi^{-1}(X)} \subseteq \overline{\pi^{-1}(X)}$, but if X is not closed then this inclusion is proper.

Again we let $V \subseteq \mathbb{P}^n$ be closed. Since the affine variety $\widehat{\pi^{-1}(V)}$ is stable by the scaling action of k, its vanishing ideal must be homogeneous:

Definition. An ideal $I \subset k[x_0, ..., x_n]$ is said to be homogeneous if, for any $f \in I$ and $\lambda \in k$, $f(\lambda x_0, ..., \lambda x_n)$ is in I.

Proposition 8.3. Let I be an ideal of $k[x_0, ..., x_n]$. The following are equivalent:

i) I is a homogeneous ideal.

ii) For any polynomial $f \in I$, every degree d homogeneous part $f^{(d)}$ of f is also an element of I.

iii) I is generated by homogeneous polynomials.

(Here the homogeneous part $f^{(d)}$ of f is the sum of all terms of degree d in f, and a polynomial is called "homogeneous of degree d" if every term appearing in it has degree d.)

Proof. It is clear that *ii*) implies *iii*), as if $f_1, ..., f_r$ generate I then their homogeneous parts are also elements of I, and these homogeneous parts are a homogeneous generating set for I.

It is also easy to see that *iii*) implies *i*): if I is generated by homogeneous polynomials $f_1, ..., f_r$ of degree d_i, then we have $f_i(\lambda\underline{x}) = \lambda^{d_i} f(\underline{x})$, so that if $f = \sum_i a_i f_i$ is an element of I then we have $f(\lambda\underline{x}) = \sum_i a_i(\lambda\underline{x})\lambda^{d_i} f(\underline{x}) \in I$.

We thus are left with proving that *i*) implies *ii*). Let $f \in I$ and write $f = f^{(0)} + f^{(1)} + ... + f^{(d)}$. Since k is infinite we can find $c_0, ..., c_d \in k^\times$ such that the matrix $(c_i^j)_{0\leq i,j\leq d}$ is invertible (this is known as a Vandermonde matrix, and its determinant is zero precisely when two entries are the same). On the other hand we have:

$$f(c_i x_0, ..., c_i x_n) = \sum_j c_i^j f^{(j)}(x_0, ..., x_n) \tag{19}$$

using *i*), the left hand side is in I, hence, applying the inverse of the matrix $(c_i^j)_{0\leq i,j\leq d}$ we obtain that each $f^{(j)}$ is in I. Notice that we have only used the fact that k is an infinite field here. □

In particular, from the Hilbert basis theorem, a homogeneous ideal has a finite basis of homogeneous polynomials.

8.2. **Dictionary between homogeneous ideals and closed subvarieties of $\mathbb{P}^n$.** One can now define maps:

$$\tilde{V} : \{\text{Homogeneous ideals of } k[x_0, ..., x_n]\} \leftrightarrow \{\text{closed subsets of } \mathbb{P}^n\} : \tilde{I} \tag{20}$$

by:

$$\tilde{V}(I) = \pi(V(I) - \{(0, ..., 0)\}) = \{[x_0 : ... : x_n] \in \mathbb{P}^n | f(x_0, ..., x_n) = 0 \text{ for all } f \in I\}$$
$$\tilde{I}(V) = I(\hat{\pi^{-1}}(V)) = \{f \in k[x_0, ..., x_n] | f(x_0, ..., x_n) = 0 \text{ for all } [x_0 : ... : x_n] \in V\} \tag{21}$$

One can show as in the affine case that $\tilde{V}(\tilde{I}(V)) = V$ (and, if V is not closed $\tilde{V}(\tilde{I}(V))$ is the Zariski closure of V). The converse is given by the projective Nullstellensatz:

Theorem 8.4. (Projective Nullstellensatz) Let I be a homogeneous ideal. Then $\tilde{I}(\tilde{V}(I)) = \sqrt{I}$ unless $\sqrt{I}$ is the irrelevant ideal $(x_0, ..., x_n)$ (in which case $\tilde{V}(I)$ is empty). In particular, the maps $\tilde{I}$ and $\tilde{V}$ give a bijection between closed subsets of $\mathbb{P}^n$ and homogeneous radical ideals of $k[x_0, \ldots, x_n]$ other than the irrelevant ideal.

Proof. If $V(I) \neq \{(0, ..., 0)\}$, i.e. $\sqrt{I} \neq (x_0, ..., x_n)$, one has that $\tilde{I}(\tilde{V}(I) = I(V(I))$, hence it suffice to applies the affine Nullstellensatz. □

Projective varieties.

Definition. A projective subvariety of $\mathbb{P}^n$ is a closed subset $V = \tilde{V}(I) \subset \mathbb{P}^n$, i.e a subset of the form:

$$V = \{[x_0 : ... : x_n] \in \mathbb{P}^n | f_1(x_0, ..., x_n) = ... = f_r(x_0, ..., x_n)\} \tag{22}$$

where $f_1, ..., f_r$ are homogeneous polynomials. The Zariski topology on V is the topology induced from the Zariski topology on $\mathbb{P}^n$.

Example. An example of a projective algebraic subvariety is

$$V = \{[w : x : y] \in \mathbb{P}^2 : wx - y^2 = 0\}.$$

What is $V \cap \mathbb{A}^2$? (Using the embedding $\mathbb{A}^2 \to \mathbb{P}^2$ which we considered before.) To find this, we just substitute $w = 1$ into the equation for V:

$$V \cap \mathbb{A}^2 = \{(x, y) \in \mathbb{A}^2 : x - y^2 = 0\},$$

that is, a parabola.

We can also work out the intersection of V with the "$\mathbb{P}^1$ at infinity:" it is the points where $w = 0$. Substituting that into the equation for V, we get

$$\{[x : y] \in \mathbb{P}^1 : -y^2 = 0\} = \{[1 : 0]\}.$$

Thus V consists of the parabola together with a point at infinity "in the direction $(1, 0)$", i.e. along the x-axis (informally, the two arms of the parabola close up at infinity).

We can draw a rough and heuristic 'real picture' of $\mathbb{P}^2$ as a sphere (more accurately, S^2 is either a picture of $\mathbb{P}^1_{\mathbb{C}}$, the Riemann sphere, or a double cover of $\mathbb{P}^2_{\mathbb{R}}$, which is homeomorphic to S^2 with antipodal points identified), and this parabola hence defines a closed curve on it.

We can also consider the embedding $\mathbb{A}^3 \subset \mathbb{P}^3$ given by $(w, x) \mapsto [w : x : 1]$, with the "$\mathbb{P}^1$ at infinity" given by $[w : x] \mapsto [w : x : 0]$. One has:

$$V \cap \mathbb{A}^2 = \{(w, x) \in \mathbb{A}^2 | wx - 1 = 0\} \tag{23}$$

that is, a hyperbola. The intersection with the "$\mathbb{P}^1$ at infinity" is given by

$$\{[wx] \in \mathbb{P}^1 : wx = 0\} = \{[1 : 0], [0 : 1]\}.$$

Thus V consists of the hyperbola together with a point at infinity "in the direction $(1, 0)$", i.e. along the w-axis, and one "in the direction $(0, 1)$", i.e. along the y-axis, corresponding to the two asymptotes of the hyperbola.

Consider the subvariety $H_i \subseteq \mathbb{P}^n$ defined by the equation $x_i = 0$. This is a "hyperplane at infinity" and intuitively is isomorphic to $\mathbb{P}^{n-1}$. Its open complement U_i is intuitively a copy of $\mathbb{A}^n$. Since every point $[x_0 : \ldots : x_n]$ has at least one nonzero coordinate, it is contained in one of the U_i. We intuitively see that $\mathbb{P}^n$ has an open cover consisting of $n + 1$ copies of $\mathbb{A}^n$. We will be able to make this precise once we consider the notion of quasi-projective algebraic variety.

Homogenisation. Given a projective variety we can obtain an affine variety via the affine cone construction. We would like to reverse this process, and go from an affine algebraic subvariety to a projective algebraic subvariety.

Example. Consider the affine hyperbola $H = \{(w, x) \in \mathbb{A}^2 : wx - 1 = 0\}$. We want to turn the polynomial $WX - 1$ into a homogeneous polynomial, using a new variable Y. To do this, note that the highest degree term in $WX - 1$ has degree 2. We multiply each term by an appropriate power of Y to get all terms of degree 2: thus we get $WX - Y^2 = 0$.

This process generalises:

Definition. For any polynomial $f \in k[X_1, \ldots, X_n]$, we define the **homogenisation** of f to be the polynomial $\bar{f} \in k[X_0, \ldots, X_n]$ obtained by the following procedure: let d be the maximum degree of terms of f. Then multiply each term of f by X_0^{d-e}, where e is the degree of this term in f.

For example: if $f(X_1, X_2, X_3) = X_1^3 + 4X_1X_2X_3 - X_1^2 - X_2^2 + 5X_3 + 8$, then the homogenisation is

$$\bar{f}(X_0, X_1, X_2, X_3) = X_1^3 + 4X_1X_2X_3 - X_1^2X_0 - X_2^2X_0 + 5X_3X_0^2 + 8X_0^3.$$

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic subvariety with $I = I(V)$. Let $\bar{I}$ be the ideal of $k[X_0, \ldots, X_n]$ defined by $\bar{I} = (\bar{f} : f \in I)$. We call $\bar{I}$ the **homogenisation** of I.

It follows that $W = \tilde{V}(\bar{I})$ is the smallest projective algebraic subvariety of $\mathbb{P}^n$ containing V. When we substitute $x_0 = 1$ into the polynomials defining W, we just get back $I(V)$, so we have

$$W \cap \{[1 : x_1 : \cdots : x_n]\} = V.$$

The following example shows that if we homogenise a generating set, instead of all of I, we still get a projective algebraic subvariety W' such that $W' \cap \mathbb{A}^n = V$, but it might not be the smallest such set.

Example. Here's a more complex example (the twisted cubic curve). Let

$$C = \{(t, t^2, t^3) \in \mathbb{A}^3\} = V(Y - X^2,\ Z - XY).$$

Homogenising the polynomials, we get

$$C' = \{[w : x : y : z] \in \mathbb{P}^3 : wy - x^2 = wz - xy = 0\}.$$

It is still true that we can reverse this by just setting $w = 1$, so $C' \cap \mathbb{A}^3 = C$.

But what happens at infinity? Subsituting in $w = 0$, we get

$$\{[0 : x : y : z] \in \mathbb{P}^3 : -x^2 = -xy = 0\} = \{[0 : 0 : y : z] \in \mathbb{P}^3\}.$$

Thus the intersection of C' with the plane at infinity is a copy of $\mathbb{P}^1$. This is not what we should expect, if C' were the smallest possible projective algebraic subvariety containing C: the dimension of the intersection with the plane at infinity should be smaller than the dimension of the initial affine algebraic subvariety (speaking informally).

In fact, the smallest possible projective algebraic subvariety containing C' is

$$C'' = \{[w : x : y : z] \in \mathbb{P}^3 = wy - x^2 = wz - xy = zx - y^2 = 0\}.$$

The extra polynomial involves only x, y, z and is in the ideal generated by $Y - X^2$ and $Z - XY$. You can calculate:

$$C'' = C \cup \{[0 : 0 : 0 : 1]\}.$$

(I am not giving a procedure to find the smallest projective algebraic subvariety containing a given affine algebraic subvariety - I just assert that this happens to work in this case. There is an algorithm but you would not want to have to use it by hand.)

9. Regular maps between projective varieties

Quasi-projective algebraic varieties.

So far, we have defined affine algebraic varieties and projective algebraic varieties, as separate types of object. It is very convenient to have a single notion that unifies both affine and projective algebraic varieties (for example to save us from having to prove a lemma for affine algebraic varieties, then the same lemma for projective algebraic varieties).

Definition. A **quasi-projective algebraic variety** is the intersection between an open subset and a closed subset of $\mathbb{P}^n$ (in the Zariski topology).

A projective algebraic variety is quasi-projective (just take the open subset to be $\mathbb{P}^n$ itself). An affine algebraic variety is quasi-projective: it is the intersection between $\mathbb{A}^n$ (which is open in $\mathbb{P}^n$) and a projective algebraic variety $\overline{V}$. There are other quasi-projective algebraic varieties, for example $\mathbb{A}^1 \setminus \{0\}$ which is an open subset of $\mathbb{P}^1$.

Regular maps between projective varieties.

We want to define regular maps between quasi-projective algebraic subvarieties. Let $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ be quasi-projective algebraic subvarieties. We expect a regular map $\varphi\colon V \to W$ to be a function which can be expressed as polynomials in the homogeneous coordinates:

$$\varphi([x_0 : \dots : x_m]) = [f_0(x_0, \dots, x_m) : \dots : f_n(x_0, \dots, x_m)].$$

In order for this to be a well-defined function, all the f_i must be homogeneous polynomials of the same degree, so that

$$\begin{aligned}&[f_0(\lambda x_0, \dots, \lambda x_m) : \dots : f_n(\lambda x_0, \dots, \lambda x_m)]\\ &= [\lambda^d f_0(x_0, \dots, x_m) : \dots : \lambda^d f_n(x_0, \dots, x_m)].\end{aligned}$$

All the coordinates are multiplied by λ^d, so this is the same point in $\mathbb{P}^n$ as $[f_0(x_0, \dots, x_m) : \dots : f_n(x_0, \dots, x_m)]$.

There is another condition which must be imposed to get a well-defined function $V \to \mathbb{P}^m$: we must never have

$$f_0(x_0, \dots, x_n) = \dots = f_m(x_0, \dots, x_n) = 0$$

because $[0 : \dots : 0]$ is not the homogeneous coordinates of a point in $\mathbb{P}^m$.

This is a very strong condition and there are too few lists of polynomials which satisfy it. However, as for the case of regular maps on open subsets, we can turn the notion of regularity into a local notion, by asking for the existence of local representations by homogeneous polynomials near each point. The homogeneous nature of the coordinates allows us to do this in such a way that the different lists of polynomials define the same map wherever they overlap. This leads us to make the following definition:

Definition. A **regular map** $\varphi\colon V \to W$ between two quasi-projective varieties $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ is a function $V \to W$ such that for every point $x \in V$, there exists a Zariski open set $U \subseteq V$ containing x and polynomials $f_0, \dots, f_n \in k[X_0, \dots, X_m]$ such that:

(i) $f_0, \dots, f_n$ are homogeneous of the same degree;
(ii) for every $y \in U$, $f_0, \dots, f_n$ are not all zero at y;
(iii) for every $y = [y_0 : \dots : y_m] \in U$, $\varphi(y) = [f_0(y_0, \dots, y_m) : \dots : f_n(y_0, \dots, y_m)]$.

When we restrict this definition to affine varieties, viewed as quasi-projective varieties, this recovers our old definition of regular map. Similarly, this gives us a notion of regular map between projective varieties, or between affine and projective varieties.

Definition. A **regular function** on a quasi-projective algebraic variety V is a regular map $V \to \mathbb{A}^1$.

We will later prove that the only regular functions on a *projective* algebraic variety are the constant functions.

In practice, every regular map can be written down by specifying lists of polynomials on just finitely many open sets (this follows ultimately from the Hilbert Basis Theorem). To check that a purported definition really does define a regular map $V \to W$, you have to check:

(1) each set on which an expression is defined is Zariski open;
(2) an expression never gives $[0 : \dots : 0]$ on its associated set;
(3) two expressions agree wherever they are both defined;
(4) the image of the map is contained in W.

Example. Let V be the projective closure of the parabola, i.e.

$$V = \{[w : x : y] \in \mathbb{P}^2 : wy = x^2\}.$$

Let

$$V' = V \cap \{[w : x : y] : w \neq 0\} = \{(x, y) \in \mathbb{A}^2 : y = x^2\}.$$

There is a regular map $\varphi'\colon V' \to \mathbb{A}^1$ given by

$$\varphi'(x, y) = x.$$

Does this extend to a regular map $\varphi\colon V \to \mathbb{P}^1$? (We guess it should send the point at infinity $[0 : 0 : 1] \in V$ to the point at infinity $[0 : 1] \in \mathbb{P}^1$.)

To attempt to construct such a map, write φ' in homogeneous coordinates using the embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$:

$$[1 : x : y] \mapsto [1 : x].$$

Now we homogenise, i.e. multiply by powers of the "extra" coordinate w to make all the polynomials homogeneous of degree 1:

$$[w : x : y] \mapsto [w : x].$$

This is maps $[0 : 0 : 1]$ to $[0 : 0]$ which is not allowed! But we can fix this by expressing the same map differently. Using the homogeneous nature of the coordinates, and the equation $x^2 = wy$ defining V, we have

$$[w : x] = [wx : x^2] = [wx : wy] = [x : y]$$

whenever the values we multiplied/divided by (w and x) are non-zero.

The expression $[x : y]$ is well-defined at $[0 : 0 : 1]$, with value $[0 : 1]$. On the other hand, $[x : y]$ gives $[0 : 0]$ at the point $[1 : 0 : 0] \in V$, so we cannot use $[x : y]$ alone to define a map $V \to \mathbb{P}^1$.

At least one of these two expressions is defined everywhere on V, and they agree where they overlap, so the two expressions together give a well-defined regular map $\varphi \colon V \to \mathbb{P}^1$:

$$\varphi([w : x : y]) = \begin{cases} [w : x] \text{ if } w \neq 0, \\ [x : y] \text{ if } y \neq 0. \end{cases} \tag{*}$$

Note that each expression is defined on a *Zariski open* subset of V. This is important because it is how we ensure that the value of φ at each point is *polynomially* related to its value at nearby points. (Open sets are the natural way to talk about "nearby points" in a topological space. This still applies in the Zariski topology, even though open sets are very big.)

Note that questions 5 and 6 on problem sheet 2 give examples of regular maps defined everywhere except at a single point of an affine algebraic subvariety, where there is an obvious value the map "should" take at the missing point, but the map is not regular at that point because there is no way to extend it to that point using polynomials. This is why we are not allowed just to write down polynomials on arbitrary (non-open) subsets of V and claim they define a regular map.

Example. As another example, let's try to extend the inverse of φ from affine to projective algebraic varieties. On affine algebraic varieties, the inverse of φ' is $\psi' \colon \mathbb{A}^1 \to V'$ given by

$$\psi'(t) = (t, t^2).$$

In projective coordinates, this is

$$[1 : t] \mapsto [1 : t : t^2].$$

Homogenising (inserting powers of s to make all the polynomials on RHS degree 2), we get

$$[s : t] \mapsto [s^2 : st : t^2].$$

Now s^2, st, t^2 are never simultaneously zero for $[s : t] \in \mathbb{P}^1$, so in this case the single expression $[s^2 : st : t^2]$ is enough to define a regular map $\varphi \colon \mathbb{P}^1 \to V$ (note that the image of φ is indeed contained in V).

The two maps $\varphi \colon V \to \mathbb{P}^1$ and $\psi \colon \mathbb{P}^1 \to V$ are inverses, so we conclude that the projective parabola V is isomorphic to $\mathbb{P}^1$.

Note that this homogenisation procedure does not always work. There are regular maps between affine algebraic subvarieties which it is impossible to extend to regular maps between their projective closures (there are points for which it is impossible to avoid sending them to $[0 : \cdots : 0]$).

Regular maps equal on a dense subset.

Lemma 9.1. Let $\varphi, \psi \colon V \to W$ be regular maps between quasi-projective varieties V, W. If there exists a Zariski dense subset $A \subseteq V$ such that $\varphi_{|A} = \psi_{|A}$, then $\varphi = \psi$.

If V was irreducible, then any nonempty Zariski open subset of V is dense, and we obtain the useful:

Corollary 9.2. Let $\varphi, \psi \colon V \to W$ be regular maps between quasi-projective varieties V, W with V irreducible. If there exists a nonempty Zariski open subset $U \subseteq V$ such that $\varphi_{|U} = \psi_{|U}$, then $\varphi = \psi$.

So the corollary tells us that, given a list of polynomials on a Zariski open subset of V, there is *at most* one regular map which is given by that list of polynomials on that set. In other words, to give a regular map on an open subset, we may simply give a list of polynomials; sometimes this may not define a regular map, but when it does, it defines one uniquely. Note however that different lists may give the same map, since e.g. we may multiply the list by any nonzero scalar.

Proof of 9.1. Let $Z = \{x \in V \colon \varphi(x) = \psi(x)\}$. By hypothesis, Z contains a dense subset of V. Hence in order to show that $Z = V$, it suffices to show that Z is closed in V.

We will use the following topological fact:

Fact. Let S be any topological space. Let $\{U_\alpha\}$ be a collection of open subsets of S whose union is all of S. Let Z be any subset of S such that $Z \cap U_\alpha$ is closed in the subspace topology on U_α for every α. Then Z is closed as a subset of S.

From the definition of regular maps, we know that we can cover V by Zariski open sets U_α such that on each U_α, both φ and ψ are defined by sequences of homogeneous polynomials:

$$\varphi_{|U_\alpha} = [f_{\alpha,0} : \cdots : f_{\alpha,m}], \quad \psi_{|U_\alpha} = [g_{\alpha,0} : \cdots : g_{\alpha,m}].$$

By the topological fact, it suffices to show that $Z \cap U_\alpha$ is closed in the subspace topology on U_α for every α.

Now

$$Z \cap U_\alpha = \{x \in U_\alpha : [f_{\alpha,0}(x) : \cdots : f_{\alpha,m}(x)] = [g_{\alpha,0}(x) : \cdots : g_{\alpha,m}(x)]\}.$$

This is the same as the set of $x \in U_\alpha$ where the vectors $(f_{\alpha,0}(x), \ldots, f_{\alpha,m}(x))$ and $(g_{\alpha,0}(x), \ldots, g_{\alpha,m}(x))$ are proportional (for any choice of homogeneous coordinates for x). A little algebra shows that this condition is equivalent to

$$f_{\alpha,i}(x) g_{\alpha,j}(x) - f_{\alpha,j}(x) g_{\alpha,i}(x) = 0 \text{ for all } i, j \in \{0, \ldots, m\}.$$

This last condition is given by homogeneous polynomials, and therefore defines a closed subset in the subspace topology on U_α. $\square$

We remark that the key point of the proof is to show that, if $\varphi, \psi \colon V \to W$ are regular maps between quasi-projective varieties, then the set $Z \subseteq V$ where they agree is Zariski closed.

Isomorphisms of quasi-projective varieties. Since we have a notion of regular map, we have in particular an induced notion of isomorphism for quasi-projective algebraic varieties. We can now make rigorous the claim that "$\mathbb{A}^1 \setminus \{0\}$ is isomorphic to the affine hyperbola $H = \{(x, y) \in \mathbb{A}^2 : xy = 1\}$." The set

$$\mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{[1:0], [0:1]\}$$

is a Zariski open subset of $\mathbb{P}^1$, because its complement is finite. Hence $\mathbb{A}^1 \setminus \{0\}$ is a quasi-projective algebraic variety. The map $\varphi \colon \mathbb{A}^1 \setminus \{0\} \to H$ given by $\varphi(t) = (t, 1/t)$ can be written in homogeneous coordinates as

$$\varphi([1:t]) = [1:t:1/t] = [t:t^2:1]$$

so homogenising, we get

$$\varphi([s:t]) = [st:t^2:s^2].$$

So long as $[s:t] \in \mathbb{A}^1 \setminus \{0\}$, this does give a point in

$$H = \{[w:x:y] \in \mathbb{P}^2 : xy = w^2\} \cap \mathbb{A}^2$$

so φ is a regular map $\mathbb{A}^1 \setminus \{0\} \to H$. The projection $(x, y) \mapsto x$ is a regular inverse to φ. Hence $\mathbb{A}^1 \setminus \{0\}$ and H are isomorphic as quasi-projective algebraic varieties.

10. Rational maps

Rational maps between quasi-projective algebraic varieties.

Definition. Given V, W quasi-projective with V irreducible, we define rational maps $V \dashrightarrow W$ exactly as in the affine case: a **rational map** is an equivalence class of pairs (U, f) with $U \subseteq V$ open and $f : U \to W$ regular, under the equivalence relation $(U, f) \sim (U', f')$ if $f|_{U\cap U'} = f'|_{U\cap U'}$.

Lemma 10.1. Let $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ be quasi-projective algebraic subvarieties with V irreducible.

Let S denote the set of lists $(f_0, \dots, f_n) \in k[X_0, \dots, X_m]^{n+1}$ such that:

(1) $f_0, \dots, f_n$ are homogeneous of the same degree;
(2) $f_0, \dots, f_n$ are not all identically zero on V (note that this looks a little like the $b \neq 0$ condition in defining the field of fractions);
(3) there exists a non-empty Zariski open set $A \subseteq V$ such that, for all $x \in A$, $[f_0(x) : \cdots : f_n(x)] \in W$.

Define an equivalence relation $\sim$ on S by: $(f_0, \dots, f_n) \sim (g_0, \dots, g_n)$ if

$$f_i g_j = f_j g_i \text{ for all } i, j.$$

A **rational map** $\varphi\colon V \dashrightarrow W$ is the same as an equivalence class in S for $\sim$.

Proof. This is an unraveling of the definitions: we already observed that a list of polynomials from S defines a regular map on an open subset; it does not matter which one we choose. The proof of 9.1 shows that two such lists define the same regular map precisely when they are equivalent. Conversely a regular map provides a list of polynomials from S. □

We usually specify rational maps by just giving one representative $[f_0 : \cdots : f_n]$ from S.

We also define similarly the **domain** $\mathrm{dom}(\phi)$, i.e. the open set of regular points, of a regular map. We similarly have:

Lemma 10.2. A rational map $\varphi\colon V \to W$ is **regular** at a point $x \in V$ if there exists at least one list of polynomials $(f_0, \dots, f_n) \in S$ representing φ such that

$$(f_0(x) : \cdots : f_n(x)) \neq (0, ..., 0)$$

.

It follows immediately from the definition of regular maps between quasi-projective algebraic varieties that a rational map is regular at every point exactly when it is a regular map.

Note that the domain of definition of a rational map can change if we change the target set W. For example, consider the map $\mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is regular at every point. We could interpret the same formula as defining a rational map $\mathbb{P}^1 \dashrightarrow W$ where $W \subseteq \mathbb{P}^2$ is the open set

$$W = \{[w : x : y] : w \neq 0\}.$$

As a rational map $\mathbb{P}^1 \dashrightarrow W$, this is not regular at the point $[0 : 1]$ because this point maps to $[0 : 0 : 1] \notin W$.

For a simpler example: the identity map $\mathbb{A}^1 \to \mathbb{A}^1$ defines a rational map that is regular everywhere (and hence a regular map). We can also view it as a rational map $\mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$ which is not regular at the origin.

Example. Let C denote the affine algebraic subvariety

$$C = \{(x, y) \in \mathbb{A}^2 : y = x^3\}.$$

This has projective closure

$$\overline{C} = \{[w : x : y] \in \mathbb{P}^2 : w^2 y = x^3\} = C \cup \{[0 : 0 : 1]\}.$$

Consider the regular map of affine algebraic varieties $\varphi\colon C \to \mathbb{A}^1$ given by

$$\varphi(x, y) = x.$$

If we try to extend this to a map of projective algebraic varieties $\overline{\varphi}\colon \overline{C} \to \mathbb{P}^1$, we must say that for points $[1 : x : y] \in C \subseteq \overline{C}$,

$$\overline{\varphi}([1 : x : y]) = [1 : x]$$

and this homogenises to

$$\overline{\varphi}([w : x : y]) = [w : x].$$

Thus $\overline{\varphi}$ is a rational map $\overline{C} \dashrightarrow \mathbb{P}^1$.

The above expression for $\overline{\varphi}$ is not defined at the point $[0 : 0 : 1] \in \overline{C}$. We can prove that there is no other expression for $\overline{\varphi}$ which is defined at that point, and so $\overline{\varphi}$ is not regular at $[0 : 0 : 1]$ (see problem sheet 3, question 3).

The above example illustrates a general phenomenon: a regular map of affine algebraic varieties extends to a rational map between their projective closures, but the extended map may not be regular at the points at infinity.

Birational maps.

Just as in the affine case, if we have irreducible quasi-projective varieties V, W, T and rational maps $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow T$, if the image of φ is dense in W, then the composite $\psi \circ \varphi$ is a rational map $V \dashrightarrow T$.

The following definitions are the same as the affine case:

Definition. A rational map $\varphi\colon V \dashrightarrow W$ is **dominant** if its image is dense in W.

A rational map $\varphi\colon V \dashrightarrow W$ is a **birational equivalence** if it is dominant and there exists a dominant rational map $\psi\colon W \dashrightarrow V$ such that $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_W$ (where these composite rational maps are defined).

Irreducible algebraic varieties V and W are **birational** if there exists a birational equivalence $V \dashrightarrow W$.

Notice that the codomain of a dominant rational map is automatically irreducible:

Lemma 10.3. Let V, W be quasi-projective varieties, and $f : V \to W$ be a dominant rational map. If V is irreducible, then W is irreducible.

Proof. Suppose that f is given by a regular map $U \to W$ where U is open in V. Since open subsets of quasi-projective varieties are quasi-projective varieties, and open subsets of irreducible spaces are irreducible, it follows that U is an irreducible quasi-projective variety. If we take U large enough then we may assume that $f : U \to W$ is a dominant regular map.

Suppose that W is reducible: consider a decomposition $W = W_1 \cup W_2$, with $W_1; W_2$ proper closed subsets. Then $U = f^{-1}(W_1) \cup f^{-1}(W_2)$. These are both closed subsets of U, so for a contradiction it suffices to show that they are proper. If the image of f is contained in W_1, then since f is dominant we have $W_1 = W$, a contradiction, and similarly for W_2. Hence $f^{-1}(W - W_i) \neq \emptyset$, and hence $f^{-1}(W_i)$ is a proper subset of U, which contradicts the irreducibility of U. □

Example. $\mathbb{A}^n$ is birational to $\mathbb{P}^n$: consider the regular map

$$\varphi\colon \mathbb{A}^n \to \mathbb{P}^n : (x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$$

and the rational map

$$\psi\colon \mathbb{P}^n \dashrightarrow \mathbb{A}^n : [x_0 : \cdots : x_n] \mapsto (x_1/x_0, \ldots, x_n/x_0).$$

Each of these is dominant and composing them in either direction gives the identity, so these are birational equivalences.

Observe that φ is an isomorphism from $\mathbb{A}^n$ to an open subset of $\mathbb{P}^n$.

We can generalise the preceding example to the following stronger result, which makes precise the intuition that varieties are birational if and only if they are the same "almost everywhere."

Lemma 10.4. Two irreducible quasi-projective varieties V and W are birational if and only if there exist non-empty Zariski open subsets $A \subseteq V$ and $B \subseteq W$ such that A is isomorphic to B (as quasi-projective varieties).

Proof. Given a map $f : A \to B$ we can extend it to a rational map (A, f); if f has an inverse g then (A, f) is inverse to (B, g). For the other direction, let $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow V$ be an inverse pair of rational maps. Let $A_1 = \operatorname{dom} \varphi$ and $B_1 = \operatorname{dom} \psi$. B_1 is a non-empty open subset of W.

Since φ induces a continuous map $A_1 \to W$, $A = \varphi_{|A_1}^{-1}(B_1)$ is an open subset of V. Furthermore, since φ is dominant, its image intersects the open set $B_1 \subseteq W$. Therefore A is non-empty.

Similarly $B = \psi_{|B_1}^{-1}(A_1)$ is a non-empty open subset of W.

One can now check that $\varphi_{|A}$ and $\psi_{|B}$ form an inverse pair of isomorphisms between A and B. $\square$

If V is a quasi-projective algebraic variety, we define a **rational function** on V to be a rational map $\varphi\colon V \dashrightarrow \mathbb{A}^1$. By definition, this is the same as a rational map $\varphi'\colon V \dashrightarrow \mathbb{P}^1$ except that we declare φ to be non-regular at points where $\varphi'(x) = \infty = [0:1] \in \mathbb{P}^1$. We can therefore say

$$\varphi(x) = [f(x) : g(x)] = [1 : g(x)/f(x)] = \frac{g(x)}{f(x)} \in \mathbb{A}^1$$

whenever $f(x) \neq 0$, for suitable polynomials f, g. Of course, as always with rational maps, we might need to use different polynomials to evaluate it at different points.

The rational functions on V form a field $k(V)$. Just as in the affine case, V is birational to W if and only if $k(V)$ is k-isomorphic to $k(W)$.

11. Projections from a point to a hyperplane

Linear spaces in $\mathbb{P}^n$.

We want to define a fundamental example of a rational map: projection from a point to a hyperplane. First, we need to make a few other definitions.

Definition. A **hyperplane** in $\mathbb{P}^n$ is a projective algebraic subvariety defined by a single homogeneous linear equation:

$$H = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : h_0x_0 + \cdots + h_nx_n = 0\}$$

for some $h_0, \dots, h_n \in k$, not all zero.

More generally, a **linear subspace** of $\mathbb{P}^n$ is a subset defined by any set of homogeneous linear equations.

Examples of linear subspaces are $\mathbb{P}^n$ itself (empty set of equations), $\emptyset$ (too many equations), and singletons. We can't define the singleton $\{[p_0 : \cdots : p_n]\}$ by the equations $x_0 = p_0, \dots, x_n = p_n$ because these are not homogeneous. Instead, we can write homogeneous equations asserting that the ratios between pairs of coordinates are correct:

$$\{[p_0 : \cdots : p_n]\} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : p_ix_j = p_jx_i \text{ for all } i, j\}.$$

If Λ is a linear subspace of $\mathbb{P}^n$, then the affine cone $C(\Lambda)$ (the set of points in $\mathbb{A}^{n+1}$ satisfying the same equations as Λ) is a vector subspace of k^{n+1}.

As a vector space, we know what is meant by $\dim C(\Lambda)$. We define

$$\dim \Lambda = \dim C(\Lambda) - 1.$$

(We have not yet defined the dimension of an arbitrary algebraic variety; this definition is only for linear subspaces of projective space. The -1 is because $C(\Lambda)$ contains a line for each point in Λ.) For example, $\mathbb{P}^n$ has dimension n, a hyperplane has dimension $n - 1$ and a point has dimension 0.

If Λ is a linear subspace of $\mathbb{P}^n$ of dimension d, then $C(\Lambda) \cong k^{d+1}$ (as a vector space) and

$$\Lambda = \big(C(\Lambda) \setminus \{0\}\big)/(\text{multiplying by scalars})$$

so $\Lambda \cong \mathbb{P}^d$.

Lines in $\mathbb{P}^n$.

Definition. A **line** in $\mathbb{P}^n$ is a linear subspace of dimension 1.

Lemma 11.1. For any two distinct points $p, q \in \mathbb{P}^n$, there exists a unique line L_{pq} through p and q.

Proof. One could prove this by saying: $\mathbb{P}^n$ can be written as a union $\mathbb{A}^n \cup \mathbb{P}^{n-1}$, and going through the cases $p, q \in \mathbb{A}^n$; $p, q \in \mathbb{P}^{n-1}$; $p \in \mathbb{A}^n$ and $q \in \mathbb{P}^{n-1}$. In order to make this into a full proof, we would need to check that a line in $\mathbb{P}^n$, intersected with $\mathbb{A}^n$, is the same as the ordinary definition of a line in $\mathbb{A}^n$ (which is true!)

Instead we shall give a proof using linear algebra. A benefit of this proof is that it gives a description of the homogeneous coordinates of points in the line L_{pq}.

Let $p = [p_0 : \cdots : p_n]$ and $q = [q_0 : \cdots : q_n]$. The affine cones $C(p)$ and $C(q)$ are the one-dimensional vector spaces of generated by $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$ respectively. Since $p \neq q$, these vector spaces are linearly independent so there is a unique 2-dimensional vector subspace $W \subseteq k^{n+1}$ which contains $C(p)$ and $C(q)$. The image of $W \setminus \{0\}$ in $\mathbb{P}^n$ is the unique line through p and q.

Explicitly: W consists of all linear combinations of the vectors $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$. It follows that

$$L_{pq} = \{[p_0 s + q_0 t : \cdots : p_n s + q_n t] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}. \qquad \square$$

Projections.

A fundamental example of a rational map is projection from a point to a hyperplane. Let $p = [p_0 : \cdots : p_n] \in \mathbb{P}^n$ and let $H \subseteq \mathbb{P}^n$ be a hyperplane such that $p \notin H$. To simplify the calculations, we shall assume that

$$H = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_n = 0\}.$$

Any line in $\mathbb{P}^n$ which is not contained in H meets H in exactly one point. This is geometrically clear; one can prove it algebraically via linear algebra using the affine cones, or by the following calculation:

Let $x \in \mathbb{P}^n \setminus \{p\}$. Then

$$L_{px} = \{[p_0 s + x_0 t : \cdots : p_n s + x_n t] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}. \qquad (*)$$

Hence, to find $L_{px} \cap H$, we need to choose $[s : t]$ such that $p_n s + x_n t = 0$: we can choose $[s : t] = [x_n : -p_n]$ (note that $p_n \neq 0$ because $p \notin H$, so we do not get $[0 : 0]$). Substituting in to (*), the unique point of $L_{px} \cap H$ is

$$[p_0 x_n - x_0 p_n : \cdots : p_{n-1} x_n - x_{n-1} p_n : 0].$$

The final $0 = p_n x_n - x_n p_n$ is what we expect for a point in H. Note that, if $p \neq x$, then this is not $[0 : \cdots : 0]$ so it is well-defined.

Thus, for $x \in \mathbb{P}^n \setminus \{p\}$, it makes sense to define $\pi(x)$ to be the unique point of $L_{px} \cap H$. The above calculation shows that π is a rational map $\mathbb{P}^n \dashrightarrow H$, regular on $\mathbb{P}^n \setminus \{p\}$. We show below that π is not regular at p.

This rational map is called **projection from** p **to** H. One could replace this particular fixed H by any hyperplane not containing p, and carry out the same recipe.

Lemma 11.2. Let $n \geq 2$. The projection of $\mathbb{P}^n$ from p to H is not regular at p.

Proof. Intuitively: there are many lines passing through p and p (not a typo!), so the projection would have to map p to "everywhere at once."

We can make this rigorous: Pick a point $s \in H$ and consider the line L_{ps}. For any $x \in L_{ps} \setminus \{p\}$, the geometric description of π shows that $\pi(x) = s$.

If we assume that π is regular at p, then it restricts to a regular map $L_{ps} \to H$. We have just shown that this map is constant on $L_{ps} \setminus \{p\}$ and therefore it is constant on L_{ps}. Hence $\pi(p) = s$.

We could pick another point $t \in H$ and repeat exactly the same argument using L_{pt}, so tha $\pi(p) = t$. This is a contradiction.

(The condition $n \geq 2$ is needed to ensure that $H \cong \mathbb{P}^{n-1}$ has two distinct points s and t. If $n = 1$, then H is just a point and π is a constant map, so it is regular everywhere.) $\square$

12. Products of quasi-projective algebraic varieties

The problem of products. Many sets that we want to work with (for example, the graph of a regular map $V \to W$) are naturally defined as subsets of products $V \times W$ of algebraic varieties. Therefore we would like to be able to say that the product of algebraic varieties are also algebraic varieties.

We saw that this is easy for affine algebraic varieties: $V \times W$ is an affine algebraic subset of $\mathbb{A}^{m+n}$. The key point here is the isomorphism $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$.

For projective algebraic varieties, things are harder because $\mathbb{P}^m \times \mathbb{P}^n \not\cong \mathbb{P}^{m+n}$. To see informally why $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$, recall that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\mathrm{pt}\}$ so

$$\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^1 \times \mathbb{A}^1) \cup (\mathbb{A}^1 \times \{\mathrm{pt}\}) \cup (\{\mathrm{pt}\} \times \mathbb{A}^1) \cup (\{\mathrm{pt}\} \times \{\mathrm{pt}\}) = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \{\mathrm{pt}\}.$$

Meanwhile

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{\mathrm{pt}\}.$$

Thus $\mathbb{P}^1 \times \mathbb{P}^1$ contains an extra copy of $\mathbb{A}^1$ compared to $\mathbb{P}^2$. This is only an informal argument!

We could try giving an ad hoc definition for $\mathbb{P}^m \times \mathbb{P}^n$. It is fairly clear what algebraic subsets of $\mathbb{P}^m \times \mathbb{P}^n$ should mean (sets defined by polynomials in the two sets of homogeneous coordinates $[x_0 : \cdots : x_m], [y_0 : \cdots : y_n]$; in order for the zero set of such a polynomial to be well-defined, it must be **bihomogeneous**, that is, homogeneous in the x variables and homogeneous in the y variables, but potentially of different degrees in x and y degrees can be different). Similarly, we could give a definition of regular maps between subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ involving bihomogeneous polynomials. But it would be annoying to have just defined quasi-projective varieties, unifying affine and projective varieties, and then immediately have to introduce *ad hoc* definitions for another different kind of variety. So we aim to construct the product in a way which makes it a (quasi-)projective variety, and then we can just reuse the definitions from before.

The Segre embedding. To construct the product $\mathbb{P}^m \times \mathbb{P}^n$ as a projective algebraic subvariety, we embed it inside some larger $\mathbb{P}^N$. The homogeneous coordinates of a point in $\mathbb{P}^m \times \mathbb{P}^n$ will be given by an $(m+1) \times (n+1)$ matrix, so we need

$$N = (m+1)(n+1) - 1 = mn + m + n.$$

Label the homogeneous coordinates of a point in $\mathbb{P}^N$ as if they were entries of a matrix:

$$[(z_{ij} : 0 \leq i \leq m, 0 \leq j \leq n)],$$

rather than the usual $[z_0 : \cdots : z_N]$.

Define a map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ by sending $([x_0 : \cdots : x_m], [y_0 : \cdots : y_n])$ to the point in $\mathbb{P}^N$ whose homogeneous coordinates $[(z_{ij})]$ are given by

$$z_{ij} = x_i y_j$$

for each pair of indices i, j. Another way to describe this is to say that the homogeneous coordinates of $\sigma_{m,n}([x_0 : \cdots : x_m], [y_0 : \cdots : y_n])$ are given by the product matrix

$$\begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_0 & \cdots & y_n \end{pmatrix}.$$

This matrix has rank 1. A little thought shows that we may obtain all matrices of rank 1 in this way: to give a rank 1 matrix, it suffices to give a nonzero column, and then all other columns are scalar multiples of the given one. Let

$$\Sigma_{m,n} = \{[z_{00} : \cdots : z_{mn}] \in \mathbb{P}^N : \text{the matrix } (z_{ij}) \text{ has rank } 1\}.$$

By the above discussion, the set $\Sigma_{m,n}$ is the image of the map $\sigma_{m,n}$. In fact, $\Sigma_{m,n}$ is a projective variety (given by an intersection of quadrics). In order to see this we will use the following fact from linear algebra:

Proposition 12.1. Let Z be an $m \times n$ matrix. Then Z has rank r if and only if

(1) At least one $r \times r$ minor of Z is nonzero.
(2) All $(r+1) \times (r+1)$ minors of Z are nonzero.

In particular, a matrix has rank 1 if and only if it has a nonzero entry and all of its 2×2 minors vanish. So we can describe $\Sigma_{m,n}$ as the subset of $\mathbb{P}^N$ where all 2×2 submatrices of the matrix (z_{ij}) have zero determinant. Thus $\Sigma_{m,n}$ is a projective algebraic subvariety, defined by the equations

$$z_{ij} z_{k\ell} = z_{kj} z_{i\ell} \text{ for } 0 \leq i, k \leq m,\ 0 \leq j, \ell \leq n.$$

Lemma 12.2. $\sigma_{m,n}$ is a bijection from $\mathbb{P}^m \times \mathbb{P}^n$ to $\Sigma_{m,n}$.

Proof. (This proof is not part of the course.)

We can define an inverse to $\sigma_{m,n}$ as follows:

Let $a \in \Sigma_{m,n}$, and let A be a matrix giving homogeneous coordinates for a. A is not the zero matrix (because it is a set of homogeneous coordinates), so we can pick j such that the j-th column of A contains a non-zero entry. Define $\pi_1(a) \in \mathbb{P}^m$ to be the point with homogeneous coordinates given by the j-th column of A, that is,

$$\pi_1(a) = [A_{1j} : \cdots : A_{mj}].$$

This is independent of the choice of j because the matrix has rank 1 (every non-zero column is a multiple of every other non-zero column).

Similarly we can pick i such that the i-th row of A contains a non-zero entry, and define $\pi_2(a) \in \mathbb{P}^n$ to be the point with homogeneous coordinates given by the i-th row of A. Again this is independent of the choice of i.

Now $(\pi_1, \pi_2) \colon \Sigma_{m,n} \to \mathbb{P}^m \times \mathbb{P}^n$ is an inverse to $\sigma_{m,n}$. $\square$

This construction shows that the projections $\pi_1 \colon \Sigma_{m,n} \to \mathbb{P}^m$ and $\pi_2 \colon \Sigma_{m,n} \to \mathbb{P}^n$ are regular maps (each column of the matrix is non-zero on a Zariski open subset of $\Sigma_{m,n}$).

The map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ is called the **Segre embedding** and its image $\Sigma_{m,n} \subseteq \mathbb{P}^N$ is called the **Segre variety**.

Closed subsets of the Segre variety.

Example. When $m = n = 1$, $N = 3$. The Segre variety $\Sigma_{m,n} \subseteq \mathbb{P}^3$ is defined by the single equation

$$\det \begin{pmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{pmatrix} = z_{00}z_{11} - z_{10}z_{01} = 0.$$

The Segre embedding is given by

$$\sigma_{m,n}([x_0 : x_1], [y_0 : y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$

We see that $\Sigma_{m,n}$ is an irreducible quadric hypersurface in $\mathbb{P}^3$. Therefore by problem sheet 3, question 4, it is birational to $\mathbb{P}^2$. This is not surprising, because of course $\mathbb{P}^1 \times \mathbb{P}^1$ should have an open subset isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, which in turn is an open subset of $\mathbb{P}^2$.

The Zariski topology on $\mathbb{P}^N$ induces a subspace topology on $\Sigma_{m,n}$. One can check that this topology is the same as what we expect, namely:

Lemma 12.3. Let $V \subseteq \mathbb{P}^m \times \mathbb{P}^n$. Then $\sigma_{m,n}(V) \subseteq \Sigma_{m,n}$ is closed if and only if

$$V = \{([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) : f_i(x_0, \dots, x_m, y_0, \dots, y_n = 0) \text{ for } 1 \leq i \leq s\}$$

where $f_1, \dots, f_s \in k[X_0, \dots, X_m, Y_0, \dots, Y_n]$ are bihomogeneous polynomials.

Proof. (not part of the course)

Suppose that $\sigma_{m,n}(V)$ is Zariski closed in $\mathbb{P}^N$. Then it is defined by some homogeneous polynomials $g_r(z_{00}, \dots, z_{mn})$. Making the substitutions $z_{ij} = x_iy_j$ (as in the definition of $\sigma_{m,n}$), we get a finite set of polynomials which define V. If g_r is homogeneous in z_{ij} of degree d_r, then $g_r \circ \sigma_{m,n}$ is bihomogeneous of degree (d_r, d_r).

It is easy to see that if V is defined by polynomials f_r, where f_r is bihomogeneous of degree (d_r, d_r), then we can reverse this process to get homogeneous polynomials in z_{ij} which define $\sigma_{m,n}(V)$.

But what if the defining polynomials for V include some f which is bihomogeneous of degree (d, e), where $d \neq e$? Without loss of generality, suppose that $d > e$. Then $f = 0$ is equivalent to the system of equations

$$y_0^{d-e} f = 0, \; \dots, \; y_n^{d-e} f = 0$$

and these equations are bihomogeneous of degree (d, d). $\square$

If $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ are projective algebraic subvarieties, then $V \times W \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is Zariski closed: the homogeneous polynomials defining V become bihomogeneous polynomials of bidegree $(d, 0)$ while those defining W become bihomogeneous polyomials of bidegree $(0, e)$.

Similarly, if $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ are quasi-projective algebraic subvarieties, then the product $V \times W$ is also quasi-projective (it is the intersection of an open subset and a closed subset in $\mathbb{P}^m \times \mathbb{P}^n$, and therefore in $\mathbb{P}^N$ via the Segre embedding).

Graphs of regular functions.

Example. One useful example of a subvariety of a product is the graph of a regular function.

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be quasi-projective algebraic subvarieties, and let $\varphi\colon V \to W$ be a regular map. The **graph** of φ is

$$\Gamma = \{(x, y) \in V \times W : y = \varphi(x)\}.$$

To check that this is closed in $V \times W$, observe that Γ is the preimage of the diagonal $\Delta \subseteq \mathbb{P}^m \times \mathbb{P}^m$ under the regular map

$$(\iota \circ \varphi, \iota)\colon V \times W \to \mathbb{P}^m \times \mathbb{P}^m$$

where ι denotes the inclusion map $W \to \mathbb{P}^m$. Since $(\iota \circ \varphi, \iota)$ is a regular map, it is continuous. Therefore it suffices to check that the diagonal is a Zariski closed subset of $\mathbb{P}^m \times \mathbb{P}^m$. This is true because we can describe the diagonal by bihomogeneous equations as follows:

$$\Delta = \{([x_0 : \cdots : x_m], [y_0 : \cdots : y_m]) : x_i y_j = x_j y_i \text{ for all } i, j\}.$$

13. Complete varieties

A remark on compactness.

Over the complex numbers, every projective algebraic variety is compact in the analytic topology. This is because they are closed subsets of $\mathbb{P}^n_{\mathbb{C}}$, which is compact.

In the Zariski topology, the notion of compactness is not very interesting: every algebraic variety is compact in the Zariski topology, even affine algebraic varieties. Affine algebraic varieties do not behave in ways matching our intuition about compactness: this intuition only works in Hausdorff spaces.

There is a converse to this, which tells us that there is a very close relationship between analytic and algebraic geometry in $\mathbb{P}^n_{\mathbb{C}}$:

Theorem 13.1 (Chow's theorem)**.** Let V be an analytic subset of $\mathbb{P}^n_{\mathbb{C}}$ which is closed in the analytic topology. Then V is a projective algebraic subvariety.

I won't define analytic subsets here, but roughly it means a set defined by zeroes of holomorphic functions. This theorem requires too much complex analytic geometry to prove here.

Proper maps and complete varieties. Recall that we have seen that the image of a closed subset under a regular map need not be closed.

Definition. We say that a regular map $f : V \to W$ is closed if $f(Z)$ is closed in W for all closed subsets Z of V.

A stronger condition on maps is that of being universally closed, also known as proper:

Definition. A regular map $f : V \to W$ is universally closed (or proper) if for all algebraic varieties Z, the map $f \times Id : V \times Z \to W \times Z$ defined by $(f \times Id)(x, z) = (f(x), z)$ is closed.

Being proper is much stronger than being closed. Indeed, the "constant" regular map f from $\mathbb{A}^1_k$ to a point p is certainly closed (as every subset of a one-point topological space is closed!) but it is not proper; for example the product $f \times Id : \mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ is not closed. Indeed, let W be the closed subset of $\mathbb{A}^1_k \times \mathbb{A}^1_k$ defined by the equation $XY = 1$; then $(f \times Id)(W)$ is the subset $\{p\} \times (\mathbb{A}^1_k - \{0\})$ of $\mathbb{A}^1_k$, and this subset is not closed.

Having seen this example, one might reasonably wonder whether any map of varieties is universally closed! In fact we will see that this is rather common, and that in fact any map of projective varieties is proper (and thus, in particular, closed!). We first introduce an auxiliary notion that is rather useful:

Definition. A quasi-projective variety V is complete if the constant map $V \to \{p\}$ (where $\{p\}$ is a single point, considered as a variety over k) is proper, i.e. for any quasi-projective variety W, the second projection $p_2 : V \times W \to W$ is closed.

We will see that completeness of varieties is a notion that is (roughly) analogous to compactness for manifolds. (Note again that compactness itself is not a very useful notion for varieties as every variety is compact; thus the usual intuition of a compact object having "no holes" fails in algebraic geometry. Instead, the correct intuition is that a complete variety is one with "no holes".)

We will show below the theorem:

Theorem 13.2. The complete quasi-projective varieties are exactly the projective varieties.

Notice that if we go beyond the world of quasi-projective varieties (we have not defined non-quasi-projective varieties at all in this course) then it is possible to find algebraic varieties which are complete but not projective. We remark that, over the complex numbers, an algebraic variety is complete if and only if it is compact for the analytic topology (this is hard to prove).

Formal properties of proper maps. Before proving completeness of projective varieties, we develop several formal properties associated with the notions of completeness and properness. The claims here all follow more-or less directly from the definitions.

Lemma 13.3. Let $i : V \to W$ be a closed embedding (i.e, the inclusion of a closed subset). Then i is proper.

Proof. If i is a closed embedding then it is clearly closed, as it is a homeomorphism onto its image and its image is closed. But for any variety Z, $i \times Id : V \times Z \to W \times Z$ also a closed embedding, hence is closed. So i is proper. □

It follows directly from the definition that compositions of proper maps are proper (as compositions of closed maps are clearly closed). From this and the previous lemma we deduce:

Corollary 13.4. A closed subvariety of a complete variety is complete.

Proof. Let W be complete and V closed in W. Then the map $V \to \{p\}$ is the composition of the inclusion of V in W (a closed embedding) and the map $V \to \{p\}$, which is proper, hence is proper. □

Lemma 13.5. Let $f : V \to W$ be a proper map, and let y be a point of V. Then $f^{-1}(y)$ is a complete variety.

Proof. Let Z be a quasi-projective variety, and let Y be a closed subset of $f^{-1}(y) \times Z$. We must show that the image of Y under the projection to Z is closed. But this image coincides with the image of Y under the map $V \times Z \to W \times Z$, which is closed since f is proper. □

Lemma 13.6. Let $f : V \to W$ be a proper map. Then for any Z, the map $(f \times Id) : V \times Z \to W \times Z$ is proper.

Proof. For any variety Y, the map $f \times Id : V \times (Z \times Y) \to W \times (Z \times Y)$ is closed. The result follows by "moving the parentheses to the left". $\square$

Corollary 13.7. Any product of complete varieties is complete.

Proof. If V is complete, then $V \to \{p\}$ is proper, hence from the last lemma $V \times W \to W$ is proper. Composing this with the proper map $W \to \{p\}$, we find that $V \times W$ is complete. $\square$

Proposition 13.8. Let V be a complete variety and $f : V \to W$ a surjective regular map. Then W is complete.

Proof. Let Z be a variety, and Y a closed subset of $W \times Z$. We must show that its image under the projection to Z is closed. On the other hand, let Y' be the preimage of Y in $V \times Z$. Then the projection of Y' to Z is closed since V is complete. But since f is surjective, the projection of Y' to Z coincides with the projection of Y to Z, which is hence closed. $\square$

Images of complete varieties. One can prove analytically that every holomorphic function on a connected compact complex manifold is constant (for example, this holds on the Riemann sphere, which is equal to $\mathbb{P}^1_{\mathbb{C}}$). Polynomials are holomorphic, so every regular function on a connected projective algebraic variety over $\mathbb{C}$ is constant. The same is true for over arbitrary algebraically closed fields, which we show below.

Proposition 13.9. If V is complete, and $f : V \to W$ is any regular map of quasiprojective varieties, then f is proper. In particular, the image of V under any regular map is closed.

Proof. Consider the graph $\Gamma_f := \{(x, y : y = f(x)\}$; we have shown that it is a closed subset of $V \times W$, isomorphic to V. The map $V \to V \times W$ defined by $x \to (x, f(x))$ is an isomorphism on its closed image Γ_f (with inverse given by projection onto the first factor), hence it is a closed embedding, so it is proper. On the other hand the projection $V \times W \to W$ is proper by Lemma 13.6. The composition of these two maps is the map $f : V \to W$, so f is proper. $\square$

A Hausdorff topological space is compact if and only if its image under any continuous map is closed, so the above proposition shows that completeness is a good replacement for compactness in the (very non Hausdorff) Zariski topology. We can now prove half of Theorem 13.2:

Corollary 13.10. A complete quasi-projective variety is projective.

Proof. Any quasi-projective variety V comes with an embedding $i : V \to \mathbb{P}^n$ for some n. If V is complete, then the previous Proposition tells us that the image of i is closed, and hence a projective variety. $\square$

Proving that a projective variety is complete is harder and we will return to it later.

Corollary 13.11. Every regular function on a connected complete variety is constant.

Proof. Let V be a connected complete variety and $\varphi \colon V \to \mathbb{A}^1$ a regular function. Let $\iota \colon \mathbb{A}^1 \to \mathbb{P}^1$ be the natural inclusion.

Then $\iota \circ \varphi \colon V \to \mathbb{P}^1$ is a regular map, so by Proposition 13.9, its image is a closed subset of $\mathbb{P}^1$. But the image of $\iota \circ \varphi$ is contained in $\mathbb{A}^1$, so it cannot be all of $\mathbb{P}^1$. Therefore the image of φ is finite.

Since V is connected, its image is also connected and therefore consists of a single point. $\square$

Thus complete (and then, using the theorem, projective) algebraic varieties are essentially "opposite" to affine ones, since an affine algebraic variety is determined by its ring of regular functions while a projective algebraic variety has no regular functions except constants.

Corollary 13.12. The image of a regular map from a connected complete variety to an affine variety is a point.

Proof. Suppose we have a regular map $\varphi \colon V \to W$, where V is projective and connected and W is affine. We can suppose that $W \subseteq \mathbb{A}^m$, and let $X_1, \dots, X_m$ denote the coordinate functions on W. Then $X_1 \circ \varphi, \dots, X_m \circ \varphi$ are all constant by Corollary 13.11, and so φ is constant. $\square$

Lemma 13.13. Let $V \subseteq \mathbb{P}^n$ be an infinite complete variety and let $H \subseteq \mathbb{P}^n$ be a hyperplane. Then the intersection $V \cap H$ is non-empty.

Proof. Suppose for contradiction that $V \cap H = \emptyset$. Then $V \subseteq \mathbb{P}^n \setminus H$, which is isomorphic to $\mathbb{A}^n$. Hence we get an injective regular map $\iota \colon V \to \mathbb{A}^n$.

By Corollary 13.12, ι is constant on each irreducible component of V. Since ι is injective, each irreducible component of V is a point.

But V has only finitely many irreducible components, so this contradicts the hypothesis that V is infinite. $\square$

14. Proof of completeness of projective varieties

Affine open covers.

By definition, every quasi-projective algebraic variety is contained in a projective algebraic variety. We can use this to reduce some proofs for quasi-projective algebraic varieties to the projective case (proving from the outside in). On the other hand, it is often useful to know that we can find affine varieties as open sets *inside* each quasi-projective algebraic variety. This can be used to reduce some proofs to the affine case (proving from the inside out).

Lemma 14.1. Let V be a quasi-projective variety. For every point $x \in V$, there exists an open set $U \subseteq V$ which contains x and is (isomorphic to) an affine variety.

Proof. Write $V = V_0 \cap U_0$ where $V_0 \subseteq \mathbb{P}^n$ is closed and $U_0 \subseteq \mathbb{P}^n$ is open.

Given a point $x \in V$, we may assume that x is in $\mathbb{A}^n \subseteq \mathbb{P}^n$ (embedded by setting $X_0 = 1$ – we can achieve this by changing the coordinate system if necessary).

Since $\mathbb{P}^n \setminus U_0$ is a projective algebraic variety which does not contain x, there is some homogeneous polynomial f which vanishes on $\mathbb{P}^n \setminus U_0$ but not at x. Then x is contained in the set

$$U = V_0 \cap D(f) = V \cap D(f)$$

where $D(f) = \{y \in \mathbb{A}^n : f(1, y_1, \ldots, y_n) \neq 0\}$. (We have $V_0 \cap D(f) = V \cap D(f)$ because $D(f) \subseteq U_0$.)

U is an open subset of V because $D(f)$ is an open subset of $\mathbb{P}^n$.

U is a closed subset of $D(f)$, so in order to show that U is an affine variety, it suffices to show that $D(f)$ is an affine variety. We can prove this using the "hyperbola trick": consider the set

$$E(f) = \{(y_1, \ldots, y_n, z) \in \mathbb{A}^{n+1} : z.f(1, y_1, \ldots, y_n) = 1\}.$$

$E(f)$ is an affine algebraic subvariety in $\mathbb{A}^{n+1}$, and projection onto the first n coordinates gives an isomorphism between $E(f)$ and $D(f)$. □

Proof of completeness.

We will now prove the completeness of projective varieties (Theorem 13.2, which we recall here for convenience:

Theorem 14.2. Let V be a projective variety. For any quasi-projective variety W, the second projection map $p_2 \colon V \times W \to W$ maps closed sets to closed sets.

Let Z be a closed subset of $V \times W$.

By Lemma 14.1, we may cover W by open sets U_α such that each U_α is an affine variety. According to the topological fact from the proof of Lemma 9.1, in order to show that $p_2(Z)$ is closed in W, it suffices to show that $p_2(Z) \cap U_\alpha$ is closed in U_α for every α. In other words (replacing W by U_α), it suffices to prove Theorem 14.2 for the case where W is affine.

Then we can replace $V \subseteq \mathbb{P}^m$ by $\mathbb{P}^m$ and $W \subseteq \mathbb{A}^n$ by $\mathbb{A}^n$ ($Z \subseteq V \times W$ is closed in $\mathbb{P}^m \times \mathbb{A}^n$). The benefit of doing this is that it simplifies the algebra when we change everything into coordinates. Thus it suffices to prove the following special case of Theorem 14.2.

Theorem 14.3. The second projection map $p_2 \colon \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ maps closed sets to closed sets.

Proof. We can describe a Zariski closed subset $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ as the zero set of some polynomials $f_1, \dots, f_r \in k[X_0, \dots, X_m, Y_1, \dots, Y_n]$ which are homogeneous with respect to $X_0, \dots, X_m$. ($Y_1, \dots, Y_n$ are affine coordinages, so there is no homogeneity condition with respect to them.)

For each point $(y_1, \dots, y_n) \in \mathbb{A}^n$, we can substitute the values $(y_1, \dots, y_n)$ into these polynomials and get a projective algebraic subvariety

$$Z_{\underline{y}} = \{[x_0 : \cdots : x_m] \in \mathbb{P}^n : f_i(\underline{x}, \underline{y}) = 0 \text{ for all } i\}.$$

Observe that $\underline{y} \in p_2(Z)$ if and only if $Z_{\underline{y}}$ is non-empty.

Let $I_{\underline{y}}$ denote the ideal in $k[X_0, \dots, X_m]$ generated by the polynomials

$$f_0(X_0, \dots, X_m, \underline{y}), \dots, f_r(X_0, \dots, X_m, \underline{y}).$$

By the Projective Nullstellensatz, $Z_{\underline{y}}$ is non-empty if and only if $\sqrt{I_{\underline{y}}}$ is not equal to either the full ring $k[X_0, \dots, X_m]$ or to the ideal $(X_0, \dots, X_m)$. It is easy to see that this is equivalent to: $I_{\underline{y}}$ does not contain S_d for any positive integer d, where S_d denotes the set of all homogeneous polynomials of degree d in $k[X_0, \dots, X_m]$.

For each positive integer d, write

$$W_d = \{(y_1, \dots, y_n) \in \mathbb{A}^n : I_{\underline{y}} \not\supseteq S_d\}.$$

We have shown that $p_2(Z) = \bigcap_{d \in \mathbb{N}} W_d$.

Let the polynomials $f_0, \dots, f_r$ have degrees $d_0, \dots, d_r$ with respect to the X variables. We shall show that W_d is closed for $d \geq \max(d_0, \dots, d_r)$. Since the W_d are a descending chain of sets, this is sufficient to show that $p_2(Z)$ is closed.

If $g \in S_d$, then $g \in I_{\underline{y}}$ if and only if we can write

$$g(X_0, \dots, X_m) = \sum_{i=1}^{r} f_i(X_0, \dots, X_m, \underline{y})\, h_i(X_0, \dots, X_m)$$

for some homogeneous polynomials $h_1, \dots, h_r$, where $\deg h_i = d - d_i$. Hence $S_d \cap I_{\underline{y}}$ is the image of the linear map $\alpha_{d,\underline{y}} \colon \bigoplus_{i=1}^r S_{d-d_i} \to S_d$ given by

$$\alpha_{d,\underline{y}}(h_1, \dots, h_r) = \sum_{i=1}^{r} f_i(X_0, \dots, X_m, \underline{y})\, h_i(X_0, \dots, X_m).$$

Therefore

$$\begin{aligned} W_d &= \{\underline{y} \in \mathbb{A}^n : \alpha_{d,\underline{y}} \text{ is not surjective}\} \\ &= \{\underline{y} \in \mathbb{A}^n : \operatorname{rk} \alpha_{d,\underline{y}} < \dim S_d\} \\ &= \{\underline{y} \in \mathbb{A}^n : \text{all } (\dim S_d \times \dim S_d) \text{ submatrices of } \alpha_{d,\underline{y}} \text{ have determinant } 0\} \end{aligned}$$

(where we fix bases for S_d and $\bigoplus_i S_{d-d_i}$ and use these to write $\alpha_{d,\underline{y}}$ as a matrix). The determinants of these submatrices are polynomials in $y_1, \ldots, y_n$, proving that W_d is Zariski closed in $\mathbb{A}^n$. $\square$

15. Definition of dimension

Axiomatic approach.

We want to define the dimension of algebraic varieties. There are several different definitions, all equivalent but each being useful in different situations. Nome of these definitions is particularly obvious, so we begin by listing some properties that the "dimension" of an irreducible quasi-projective variety V ought to have. (We only consider irreducible varieties here, because a reducible variety might have components of different dimensions, so it is harder to be confident about what properties the dimension of a reducible variety should have.)

1) $\dim \mathbb{A}^n = n$.
2) If U is an open subset of V, then $\dim U = \dim V$ (note that this holds for manifolds in differential geometry).

In particular, because $\mathbb{A}^n$ is open in $\mathbb{P}^n$, Axioms 1) and 2) implies that $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$, as expected. Moreover, if V and W are birational, then some open subset of V is isomorphic to some open subset of W, then Axiom 2) implies that $\dim V = \dim W$, hence dimension is a birational invariant. In particular, the dimension of V is expected to depend only on $k(V)$.

Not every quasi-projective variety is isomorphic to some $\mathbb{A}^n$, so the two preceding axioms do no suffice to attribute a dimension to each quasi-projective variety. But we can add:

3) If $f : V \to W$ is finite to one (i.e. each fibre of f is finite) and dominant, then $\dim V = \dim W$

This axiom is natural, because, the fibres of f being points means intuitively that f 'doesn't add any dimension' to V: geometrically, f is a kind of ramified covering.

It turns out that these properties are enough to compute the dimension of every irreducible quasi-projective variety, thanks to the following lemma.

Lemma 15.1 ('projective Noether normalisation')**.** Let V be an irreducible projective variety. Then there exists a finite to one surjective regular map $V \to \mathbb{P}^d$ for some d.

Proof. Embed V in $\mathbb{P}^N$ as a closed subset for some sufficiently large N; this is possible because V is projective. If $V = \mathbb{P}^N$ then we are done. Otherwise, choose a point p not in V. Consider a hyperplane $H \simeq \mathbb{P}^{N-1}$ in $\mathbb{P}^N$ not containing p, and the projection $\pi : \mathbb{P}^N - \{p\} \to \mathbb{P}^{N-1}$ from p to H. Because $V \subseteq \mathbb{P}^N - \{p\}$, this restricts to a regular map $\pi : V \to \mathbb{P}^{N-1}$. The fibre of this map at x is $L_{px} \cap V$, which is a closed subset of $L_{px} \simeq \mathbb{P}^1$ (because V is closed) not containing p, hence is a finite set of point, hence π is finite to one. Denote by $V_1 = \pi(V) \subseteq \mathbb{P}^{N-1}$, V_1 is closed because V is projective, hence complete. If $V_1 = \mathbb{P}^{N-1}$ we are done, otherwise we can continue the procedure, which have to finish, and obtain a sequence of finite to one surjective maps:

$$V = V_0 \to V_1 \to ... \to V_r = \mathbb{P}^{N-r} \tag{24}$$

with $V_i \subseteq \mathbb{P}^{N-i}$ closed. The composition of all these arrows is then a finite to one and surjective map $V \to \mathbb{P}^{N-r}$. □

Then, if V is quasi-projective and irreducible, by Lemma 14.1, V has a non-empty affine open subset $U \subseteq V$. Using an embedding $U \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ and taking the projectivization, one obtain an irreducible projective variety $\bar{U}$ birational to V: then Axiom 2) gives $\dim V = \dim \bar{U}$. This lemma gives a finite to one and surjective map $f : \bar{U} \to \mathbb{P}^d$, then by Axiom 3) $\dim \bar{U} = \dim \mathbb{P}^d = d$.

Algebraic approach. There is one problem with the axiomatic definition given above: there were a lot of choices made in the procedure used to attribute a dimension to a quasi-projective variety (choice of open affine subset, of embedding in a $\mathbb{P}^N$, of points and hyperplane), and we are not sure a priori that different choices would have lead to the same dimension. In fact this is the case, and we will prove this by finding an algebraic invariant corresponding to the geometric notion of dimension introduced here.

Theorem 15.2. Let V and W be irreducible varieties over k, and let $f : V \to W$ be a dominant, finite-to-one regular map. Then the induced map $k(W) \to k(V)$ makes $k(V)$ into an algebraic extension of $k(W)$.

We first establish a lemma:

Lemma 15.3. Let W be an irreducible variety over k, and let $V \subseteq \mathbb{A}^1 \times W$ be an irreducible closed subvariety. Then the induced map $k(W) \to k(V)$ is a algebraic extension of fields.

Proof. Let t be the coordinate on $\mathbb{A}^1$. Then the coordinate ring of $\mathbb{A}^1 \times W$ is $k[W][t]$. Let I be the ideal of $k[W][t]$ consisting of functions that vanish on V; since the projection of V onto W is dominant, I contains no nonzero elements of $k[V]$. Suppose that I is the zero ideal. Then $V = \mathbb{A}^1 \times W$ by the Nullstellensatz, and so the projection of V onto W is not finite-to-one.

Thus I contains a nonconstant polynomial P(t) with coefficients in $k[W]$. In particular $k[V]$ is a quotient of $k[W][t]/P(t)$. But then $k(V)$ is generated over $k(W)$ by t, and t is a root of the polynomial $P(t)$ over $k(W)$. So $k(V)$ is algebraic over $k(W)$. □

Proof. (of Theorem 15.2) Replacing V and W by suitable open subsets, we may assume that both V and W are affine. Embed V in $\mathbb{A}^n$, and consider the closed subset $\Gamma_f \subseteq \mathbb{A}^n \times W$ given by $\Gamma_f = \{(x, y) : x \in V, y = f(x))$. Then V is isomorphic to Γ_f, and the composition of this isomorphism with the projection $\Gamma_f \to W$ is precisely the map f. We can thus factor f as a series of projections:

$$V = \Gamma_f = V_0 \to V_1 \to ... \to V_n = W \tag{25}$$

where $V_i \subseteq \mathbb{A}^i \times W$ is the closure of the image of V_{i-1} under the projection $\mathbb{A}^{n-i+1} \times W \to \mathbb{A}^{n-i} \times W$. By the lemma, the induced extension $k(V_{i-1}/V_i$ is algebraic, hence $k(V)/k(W)$ is algebraic. □

This theorem leads us to consider naturally the transcendance degree:

Definition. Let k and K be fields, with $k \subseteq K$. The **transcendence degree** of K over k is the size of a maximal k-algebraically independent set in K. (By a result in field theory, all maximal k-algebraically independent sets in K have the same size, so this is well-defined.)

Definition. The **dimension** of an irreducible quasi-projective variety V is the transcendence degree (over k) of the field of rational functions $k(V)$.

Notice the formal similarity between this definition of dimension and the one for vector spaces: one replace here basis of vector spaces by transcendence basis of fields of rational functions.

We see that this algebraic definition satisfies the three axioms:

1) We have $k(\mathbb{A}^n) = k(X_1, ..., X_n)$, of transcendence degree n.
2) The definition of dimension depends only on $k(V)$, and then it is a birational invariant, as expected.
3 If $f : V \to W$ is finite to one and dominant, from Theorem 15.2, $k(V)$ is algebraic over $k(W)$, hence they have the same transcendence degree.

Basic properties of dimension. We establish here basic properties of dimension. Notice that it is useful to know how to use more algebraic or more geometric proofs, depending on the context.

Lemma 15.4. If V, W are irreducible quasi-projective varieties, and $f : V \to W$ is dominant, then $\dim W \leq \dim V$.

Proof. Because f is dominant, it induces a field extension $f^* : k(W) \to k(V)$, then the transcendence degree of $k(V)$ over k is the transcendence degree of $k(W)$ over k plus the transcendence degree of $k(V)$ over $k(W)$, and hence $\dim W \leq \dim V$. □

Lemma 15.5. Let V, W be irreducible quasi-projective varieties. Then $\dim(V \times W) = \dim V + \dim W$.

Proof. Replacing V and W first with affine open subsets, and then with their projective closures, which does not change the dimension, we can assume that V and W are projective. If $d = \dim V$ and $e = \dim W$, we then have surjective, finite-to-one rational maps $f : V \to \mathbb{P}^d$ and $g : W \to \mathbb{P}^e$. Then $f \times g : V \times W \to \mathbb{P}^d \times \mathbb{P}^e$ is a surjective, finite to one map, hence:

$$\dim(V \times W) = \dim(\mathbb{P}^d \times \mathbb{P}^e) \tag{26}$$

But $\mathbb{P}^d \times \mathbb{P}^e$ has an open subset isomorphic to $\mathbb{A}^d \times \mathbb{A}^e \simeq \mathbb{A}^{d+e}$, and is then of dimension $d + e$. □

Recall that, when $V \subseteq \mathbb{P}^n$ is a projective variety, the projective cone $\tilde{V} \subseteq \mathbb{A}^{n+1}$ is defined by:

$$\tilde{V} = \{0\} \cup \{(x_0, ..., x_n) : [x_0 : ... : x_n] \in V\} \tag{27}$$

It is a closed subset, which is the zero locus of the homogeneous ideal defining V. V is obtained by quotienting $\tilde{V} - \{0\}$ by the scaling, so, intuitively, it must have one dimension less than $\tilde{V}$. This is indeed the case:

Corollary 15.6. Let $V \subseteq \mathbb{P}^n$ be irreducible and projective, and let $\tilde{V} \subseteq \mathbb{A}^{n+1}$ be the projective cone of X. Then $\dim \tilde{V} = \dim V + 1$.

Proof. Choose an i such that X is not contained in the hyperplane $x_i = 0$ $\mathbb{P}^n$ (by permuting the variables we may assume $i = 0$). Then $\dim V = \dim U$ where U is the complement of the hyperplane $x_0 = 0$ in V. The cone $\tilde{U}$ of Uin $\mathbb{A}^{n+1}$ is the set of all points of the form $(t, tx_1, ..., tx_n)$, where $[1 : x_1 : ... : x_n]$ is a point of U. On one hand, $\tilde{U}$ is dense in $\tilde{V}$ so that $dim\tilde{U} = dim\tilde{V}$; on the other hand we have an isomorphism:

$$\begin{aligned} &(\mathbb{A}^1 - \{0\}) \times U \to \tilde{U} - \{(0, ..., 0)\} \\ &(t, [1 : x_1, ..., x_n]) \to (t, tx_1, ..., tx_n) \end{aligned} \tag{28}$$

Thus:

$$\dim \tilde{U} = \dim((\mathbb{A}^1 - \{0\}) \times U) = \dim(\mathbb{A}^1 - \{0\}) + \dim U = \dim U + 1 \tag{29}$$

where we have used the previous lemma, and the fact that $\mathbb{A}^1 - \{0\}$ is open in $\mathbb{A}^1$, hence is of dimension 1. $\square$

16. Dimension and intersections

Intersection of vector spaces. Let's recall two facts about dimensions of vector spaces:

Proposition 16.1. Let $E \subseteq k^n$ a sub-vector space, and $H \subseteq k^n$ a hyperplane not containing E, then $\dim(E \cap H) = \dim E - 1$.

Proposition 16.2. Let $E, F \subseteq k^n$ two sub-vector spaces of dimension d, e, such that $d + e \geq n$. Then $\dim(E \cap F) \geq d + e - n$.

The goal of this section will be to provide analogous results for quasi-projective varieties.

Dimension of intersections.

Theorem 16.3. Let $V \subseteq \mathbb{P}^n$ be irreducible and quasi-projective variety of dimension ≥ 1, and let H be a hyperplane of $\mathbb{P}^n$ not containing V. Then either $V \cap H$ is empty or every irreducible component of $V \cap H$ has dimension $\dim V - 1$. Moreover, if V is projective, $V \cap H$ is never empty.

Proof. Firstly we reduce to the case where V is projective. Let $\bar{V}$ be the closure of V in $\mathbb{P}^n$, then it is projective, and does not contain H. $V \cap H$ is open in $\bar{V} \cap H$: in particular, either $V \cap H$ is empty, or each irreducible component of $V \cap H$ is open in a irreducible component of $\bar{V} \cap H$. The claim for $\bar{V}$ implies then the claim for V.

We assume now that V is projective, and we argue by induction on n. If $n = 1$ the result is clear. Indeed, any irreducible projective subvariety V of $\mathbb{P}^1$ of dimension ≥ 1 is $\mathbb{P}^1$, of dimension 1 and H is a point in $\mathbb{P}^1$ not containing V, such that $V \cap H = H$, of dimension 0.

Assume the result is true for $n-1$. If V is all of $\mathbb{P}^n$, the claim is clear, so assume this is not the case. We first show that V can't contain H. If so, then $I(V) \subseteq I(H)$, so $I(V)$ is a homogeneous prime ideal contained in the ideal generated by the linear polynomial L defining H. Let $P(X)$ be an irreducible homogeneous polynomial contained in $I(V)$; then $L(X)$ divides $P(X)$ so $L(X) = P(X)$. Thus $I(V) = I(H)$, which is impossible since then $V = H$.

Now fix a point $p \in H$ but not in V, and a hyperplane $\mathbb{P}^{n-1}$ not containing p. Consider the projection from p to $\mathbb{P}^{n-1}$ $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$, which gives a regular map $\pi : V \to H$. Because $p \notin V$, this is a finite-to-one map surjective map on its image $V' := \pi(\bar{V})$ which is automatically irreducible, and closed because V is projective, and then complete, and $\dim V = \dim V' \geq 1$. On the other hand, $H' := \pi(H)$ is a hyperplane in $\mathbb{P}^{n-1}$, since H contains p, which does not contain V'. It follows that π is a surjective finite-to-one map from $V \cap H$ to $V' \cap H'$, hence it maps each irreducible component of $V \cap H$ to an irreducible of $V' \cap H'$ of the same dimension. By the induction hypothesis on $V' \subseteq \mathbb{P}^{n-1}$, $V' \cap H'$ in sot empty and each of its irreducible components is of dimension $\dim V' - 1 = \dim V - 1$, hence the same holds for $V \cap H$. □

Corollary 16.4. Let $V, W \subseteq \mathbb{A}^n$ be irreducible and quasi-projective of dimensions d and e, respectively, and suppose that $d + e \geq n$. Then either $V \cap W$ is empty or every irreducible component of $V \cap W$ has dimension at least $d + e - n$.

Proof. We first note that $V \cap W = (V \times W) \cap \Delta$, where $\Delta \subseteq \mathbb{A}^n \times \mathbb{A}^n$ is the diagonal. Note that $\Delta = V(x_1 - y_1, ..., x_n - y_n)$, where x_i are the coordinates on the first factor and the y_i are coordinates on the second factor. If we let H_i be the hyperplane $V(x_i - y_i)$ then intersecting with Δ is the same as successively intersecting with each H_i. By the lemma, when we intersect with one of the H_i, the dimension of each irreducible component drops by at most one (or the whole intersection becomes empty). At the end of this process the dimension of each irreducible component is at least $\dim(X \times Y) - n = d + e - n$. □

Theorem 16.5. Let $V, W \subseteq \mathbb{P}^n$ be irreducible projective varieties of dimensions d and e, respectively, and suppose that $d + e \geq n$. Then $V \cap W$ is nonempty, and every irreducible component of $V \cap W$ has dimension at least $d + e - n$.

Proof. As in the proof of Theorem 16.3, by considering the closure $\bar{V}$ and $\bar{W}$, it suffice to prove the theorem in the projective case. We assume now that V, W are projective.

Let $\tilde{V}$ and $\tilde{W}$ be the projective cones of V and W respectively: they are respectively of dimension $d+1$ and $e+1$ from corollary 15.6. Their intersection is nonempty since both contain the origin; thus by the previous Corollary every irreducible component of $\tilde{V} \cap \tilde{W}$ has dimension at least $(d+1) + (e+1) - (n+1) = d + e - n + 1$. In particular this is at least one, so this intersection contains a point other than the origin. Since $\tilde{V} \cap \tilde{W}$ is the projective cone of $V \cap W$, it follows that $V \cap W$ is nonempty!

Now if Z is an irreducible component of $V \cap W$ then its projective cone $\tilde{Z}$ is an irreducible component of $\tilde{V} \cap \tilde{W}$. It follows from Corollary 15.6 that Z has dimension at east $d + e - n$. □

Veronese embedding.

In order to generalise Theorem 16.3 from intersections with hyperplanes to intersections with hypersurfaces, we use the Veronese embedding. This is defined as follows.

Let d and n be positive integers and let $N = \binom{n+d}{d} - 1$. There are $N+\ 1$ monomials of degree d in variables $X_0, \ldots, X_n$ (expressions of the form $X_0^{a_0} X_1^{a_1} \cdots X_n^{a_n}$ where $a_0, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ with $a_0 + \cdots + a_n = d$). We define a regular map $\nu_{n,d} \colon \mathbb{P}^n \to \mathbb{P}^N$ by writing down all these monomials of degree d (in some order). For example, for $n = d = 2$ we get $N = 5$ and

$$\nu_{2,2}([X_0 : X_1 : X_2]) = [X_0^2 : X_1^2 : X_2^2 : X_0X_1 : X_1X_2 : X_0X_2].$$

This is called the **degree d Veronese embedding** of $\mathbb{P}^n$.

By completeness, the image of $\nu_{n,d}$ is a projective algebraic subvariety $V_{n,d} \subseteq \mathbb{P}^N$. One can write down explicit polynomials defining this algebraic subvariety (they are determinants of 2×2 matrices). Importantly, $\nu_{n,d}$ is an isomorphism $\mathbb{P}^n \to V_{n,d}$ (proving this is elementary but the notation gets pretty complicated).

The benefit of doing all this is that, if $H \subseteq \mathbb{P}^n$ is a hypersurface defined by some homogeneous polynomial $f = \sum_I a_I \underline{X}^I$ of degree d, then because the monomials of degree d become individual homogeneous coordinates via the Veronese embedding, the equation for $\nu_{n,d}(H)$ is a linear equation $\sum_I a_I Z_I = 0$. Thus $\nu_{n,d}(H) = V_{n,d} \cap Z$ for some hyperplane $Z \subseteq \mathbb{P}^N$.

Therefore, instead of studying the intersection between $V \subseteq \mathbb{P}^n$ and a hypersurface $H \subseteq \mathbb{P}^n$, we can instead study the intersection between $\nu_{n,d}(V) \subseteq V_{n,d} \subseteq \mathbb{P}^N$ and a hyperplane $Z \subseteq \mathbb{P}^N$. Because $\nu_{n,d}$ is an isomorphism, we can use Theorem 16.3 to deduce the same result for intersections with hypersurfaces:

Theorem 16.6. Let $V \subseteq \mathbb{P}^n$ be irreducible and quasi-projective variety of dimension ≥ 1, and let H be a hypersurface of $\mathbb{P}^n$ not containing V. Then either $V \cap H$ is empty or every irreducible component of $V \cap H$ has dimension $\dim V - 1$. Moreover, if V is projective, $V \cap H$ is never empty.

17. Topological definition of dimension

Dimension of closed subsets.

Lemma 17.1. Let V be an irreducible quasi-projective variety and let W be a irreducible closed subset of V. Then $\dim W \leq \dim V$, with equality if and only if $W = V$.

Proof. For the first statement, replacing V with an affine open subset, we may embed V in $\mathbb{A}^n \subseteq \mathbb{P}^n$, and taking their projective closure, we can assume that V and W are projective. Then V admits a surjective, finite-to-one map $f : V \to \mathbb{P}^d$, where $d = \dim V$. Let $W' := f(W)$, which is closed in $\mathbb{P}^d$ by completeness of W, and such that $\dim W' = \dim W$ because $f|_W : W \to W'$ is finite to one and surjective. Because W' is a closed subvariety of $\mathbb{P}^d$, by considering successive projections it admits a finite-to-one map to $\mathbb{P}^{d-r}$ for $r \geq 0$, hence $\dim W = \dim W' = d - r \leq d = \dim V$. which finis the proof of the first part.

For the second part, let $V \subseteq \mathbb{P}^n$ be an irreducible quasi-projective variety and let W be a irreducible closed subset of V. Suppose W is a proper closed subset of V. Then $I(W)$ strictly contains $I(V)$, so there exists a polynomial f that vanishes on W but not on V. Then W is contained in $V \cap V(f)$ and $V(f) \subseteq \mathbb{P}^n$ is a hypersurface not containing V. From Theorem 16.6 each irreducible component of $V \cap V(f)$ have dimension $\dim V - 1$, hence from the first part of the proof $\dim W \leq \dim V - 1$. ☐

Topological definition of dimension.

Our previous definition of dimension was algebraic. We can also describe the dimension of a variety in terms of its topology.

Theorem 17.2. Let V be an irreducible quasi-projective variety.

The dimension of V is the maximum integer d such that there exists a chain of irreducible closed subsets

$$V = V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_0 \supsetneq \emptyset.$$

Some care is required in the statement of this theorem to get the numbering right! The point is that $\dim V_i = i$, so V_0 is still non-empty. In Theorem 17.2, it is essential to require all the V_i to be irreducible. Otherwise we could make the chain arbitrarily long by inserting reducible sets with more and more components, all of dimension i, in between V_i and V_{i+1}.

Proof. First we prove that such a sequence with $d = \dim V$ exists.

Choose V_d to be an irreducible component of V whose dimension is equal to $\dim V$. Choose H as in Theorem 16.3 applied to V_d. Let V_{d-1} be an irreducible component in $V_d \cap H$ such that

$$\dim V_{d-1} = \dim(V_d \cap H) = \dim V - 1.$$

We can repeat this procedure, getting $V_i \subsetneq V_{i+1}$ with $\dim V_i = i$ until we get to V_0 with $\dim V_0 = 0$.

In the other direction, to show that there is no such sequence with $d > \dim V$, this follows immediately from the fact that $\dim V_i < \dim V_{i+1}$ (Lemma 17.1). □

However, Theorem 17.2 is not really strong enough to be useful. For example, in $\mathbb{P}^n$, we can write down a chain of closed subsets

$$\mathbb{P}^n \supsetneq \mathbb{P}^{n-1} \supsetneq \mathbb{P}^{n-2} \supsetneq \cdots \supsetneq \mathbb{P}^1 \supsetneq \{\mathrm{pt}\} \supsetneq \emptyset.$$

This chain is *maximal* – we cannot insert another irreducible closed subset anywhere in the middle of it. But just exhibiting this chain is not enough to prove that $\dim \mathbb{P}^n = n$ – maybe there is a completely different chain which is longer.

It turns out that that can't happen: every maximal chain of irreducible closed subsets in an irreducible quasi-projective variety V has length equal to $\dim V$. This is another hard theorem, using ideas similar to Hilbert's Hauptidealsatz.

Krull dimension of rings. In particular, when V is an irreducible affine variety, we find that $\dim V$ is equal to the Krull dimension of the integral domain $k[V]$:

Definition. The Krull dimension of an integral domain R is the maximum integer d such that there exists a chain of prime ideals

$$\{0\} = I_d \subsetneq I_{d-1} \subsetneq \cdots \subsetneq I_0 \subsetneq R$$

The fact that the Krull dimension of a integral domain is equal to the transcendence degree of its field of fractions is an algebraic version of the geometric arguments that we used (with finite-to-one projections corresponding to algebraic extensions, and intersections with a hyperplane corresponding to adding a transcendent element).

18. Fibre dimension theorem

Motivation. Let $f : V \to W$ be a regular map of varieties, and $p \in W$ a point. The fibre of f over p is the subset $f^{-1}(p)$; it is a closed subvariety of V because $\{p\} \in W$ is closed and f is continuous. If f is the projection $V \times W \to W$, where V and W are irreducible, then the fibres $f^{-1}(p)$ are of the form $V \times \{p\}$, and are isomorphic to V. Recall that in this case we have seen that $\dim(V \times W) = \dim V + \dim W$, which we can rewrite in this case by saying that the dimension of the domain is the sum of the dimension of the target and the dimension of a fibre. One can then ask whether something like this holds for general surjective maps. (Surjectivity, or at least dominance, is necessary here, since one can otherwise compose with an embedding of Y in a variety of much higher dimension without changing X or the fibres.)

There are a number of technical issues that must be addressed in order to make a statement like this work for varieties. The first is that the fibre of a map $f : V \to W$ need not be irreducible, and its irreducible components need not even all have the same dimension: consider for instance the fibre over 0 of the map

$$\begin{aligned} &\mathbb{A}^3 \to \mathbb{A}^2 \\ &(x, y, z) \to (xy, xz) \end{aligned} \tag{30}$$

Over $(0, 0)$, the fibre is the union of the line $V(Y, Z)$ and of the plane $V(X)$, irreducible of dimension 1 and 2, but over any other points the fibre is irreducible of dimension 1. In this example, we see that the fibres can only have a dimension at least the expected dimension $\dim V - \dim W$, and that, generically on W, the fibres are of the expected dimensions.

Notice that in general $f^{-1}(w)$ is not irreducible: if we take the map $f : \mathbb{A}^1 \to \mathbb{A}^1$ which map x to x^n, for any $w \in \mathbb{A}^1 - \{0\}$, $f^{-1}(w)$ is made of n points (the n-th roots of w).

The fibre dimension theorem will generalize these observations.

The fibre dimension theorem.

Theorem 18.1. Let V, W be irreducible quasi-projective varieties and let $f : V \to W$ be a dominant regular map. Then:

(i) For every $w \in W$, each irreducible component of $\dim f^{-1}(w)$ has dimension at least $\dim V - \dim W$.
(ii) There exists a non-empty open subset $U \subseteq W$ such that each irreducible component of $f^{-1}(w)$ is of dimension $\dim V - \dim W$ for all $w \in U$.

Proof. First, we cover W with affine open subsets U_i, and then, we cover the open subsets $f^{-1}(U_i) \subseteq V$ by affine open subsets U'_{ij}. Because V, and then the $f^{-1}(U_i)$, are irreducible, each U_{ij} is dense in $f^{-1}(U_i)$, hence $f(U'_{ij})$ is dense in $f(f^{-1}(U_i))$, which is itself dense in U_i because f is dominant. Hence U'_{ij} and U_i are affine and irreducible, the induced map $f_{ij} : U'_{ij} \to U_i$ are dominant. If we assume that the theorem holds for V, W affine, we can apply if to f_{ij}, and the result follow for f.

We are now reduced to the case where V, W are affine, where we can work algebraically. Consider the injection of rings $k[W] \to k[V]$. Take a maximal family $f_1, ..., f_n$ of algebraically independent elements of $k[W]$: it gives a transcendence basis of $k(W)$, hence $n = \dim W$. We take a maximal family $g_1, ..., g_m$ of element of $k[V]$ algebraically independent over $k[W]$: $f_1, ..., f_n, g_1, ..., g_m$ gives a transcendence basis of $k(V)$, hence $n + m = \dim V$.

A point $u \in W$ corresponds to a maximal ideal $\mathfrak{m}_u \subsetneq k[W]$, and $k[f^{-1}(u)] = k[V]/\mathfrak{m}_u$. The $g_1, ..., g_m$ are algebraically independent elements of $k[V]$ over $k[W]$, so they give algebraically independent elements of $k[V]/\mathfrak{m}_u = k[f^{-1}(u)]$ over $k[W]/m_u = k$. Hence for every irreducible component V' of $f^{-1}(u)$, $k(V')$ has transcendence degree at least $m = \dim V - \dim W$, hence we have $\dim V' \geq \dim V - \dim W$. Notice that $g_1, ..., g_m$ is not always a transcendence basis, hence the dimension can be larger.

Now, we take $h_1, ..., h_r \in k[V]$, such that $f_1, ..., f_n, g_1, ..., g_m, h_1, ..., h_r$ generates $k[V]$: they are automatically algebraic in $f_1, ..., f_n, g_1, ..., g_m$. Then take polynomials $F_i(h_i; f_1, ..., f_n, g_1, ..., g_m) = 0$ which are non-constant in h_i. In particular, write:

$$F_i(h_i) = c_d h_i^d + c_{d-1} h_i^{d-1} + ... + c_0 \tag{31}$$

where $c_i \in k[f_1, ..., f_n, g_1, ..., g_m]$, and $c_d \neq 0$. Then take $U_i \in W$ to be the set of points $u \in W$ where c_d remain nonzero after substituting all the f_j by $f_j(u)$: it is an open subset (defined by the nonvanishing of c_d viewed as a polynomial in $f_1, ..., f_m$ with coefficients in $k[g_1, ..., g_m]$). Let U be the intersection of the U_i. For $u \in U$, we have that $k[f^{-1}(u)]$ is generated by $g_1, ..., g_m, h_1, ..., h_r$, with $F_i(h_i; f_1(u), ..., f_n(u), g_1, ..., g_m) = 0$ giving a non-constant polynomial equation for the h_i with coefficients in $k[g_1, ..., g_m]$. Then the h_i are algebraic over the $g_1, ..., g_m$ in $k[f^{-1}(u)]$, hence for any irreducible V' component of $f^{-1}(u)$, $g_1, ..., g_m$ gives a transcendence basis of $k(V')$, hence $\dim V' = \dim V - \dim W$. $\square$

We generally use this theorem in situations where we know the dimension of either V or W and want to work out the other. If we can work out $\dim \varphi^{-1}(w)$ for just a single $w \in W$, then we get an inequality. If we can work out $\dim \varphi^{-1}(w)$ for w in some open set then we can work out the desired dimension exactly.

An important special case: if there exists w such that $\dim \varphi^{-1}(w) = 0$, then $\dim V = \dim W$.

Example. Consider $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by

$$\varphi(x, y) = (x, xy).$$

Consider the vertical line $L_x = \{(x, y) : y \in k\}$. When $x \neq 0$, φ restricts to an isomorphism $L_x \to L_x$. But when $x = 0$, φ maps all of L_0 down to $(0, 0)$. Hence the image of φ is $\big(\mathbb{A}^2 \setminus \{(0, y)\}\big) \cup \{(0, 0)\}$.

We see that, above the open set $\{(x, y) : x \neq 0\}$, the fibres of φ are single points i.e. with dimension $2 - 2 = 0$. On the other hand, above the point $(0, 0)$, the fibre $\varphi^{-1}((0, 0))$ is a line, so has dimension $1 \geq 2 - 2$.

As an application, we give a criterion for irreducibility (recall that from Lemma 10.3, if $f : V \to W$ is dominant, V is irreducible implies that W is irreducible).

Proposition 18.2. Let $f : V \to W$ a surjective map, and suppose that V is projective and W is irreducible, and that the fibres of V are all irreducible of the same dimension d. Then V is irreducible.

Proof. Let $V_1, ..., V_n$ be the irreducible components of V. Because V is projective, hence complete, the $f(V_i)$ are closed, and, because f is surjective, W is he union of the $f(V_i)$. Because W is irreducible, at least one V_i must map surjectively onto W, and fix such a i. Denote $U_i = V_i - \bigcup_{j \neq i} V_j$, which is a nonempty open subset of V_i. Notice that U_i is dense in V_i, and $f_i : V_i \to W$ is continuous surjective, i.e. $f(U_i)$ is dense in W. For $y \in W$, $f^{-1}(y) = \bigcup_j f^{-1}(y) \cap V_j$, but $f^{-1}(y)$ is irreducible, hence $f^{-1}(y) \subseteq V_j$ for some j. If $y \in f(U_i)$ this cannot holds for $j \neq i$, hence $f^{-1}(y) \subseteq V_i$. As $f(U)_i$ is dense in W, it intersects the nonempty open subset of $ii)$ of the fibre dimension Thorem. For y in the intersection, the fibre $f_i^{-1}(y) = f^{-1}(y)$ has dimension $d = \dim V_i - \dim W$.

Then, by $i)$ of the fibre dimension theorem for each $w \in W$, each irreducible component Z of the fibre of the map $f_i : V_i \to W$ has dimension at least $\dim V_i - \dim W = d$. But this fibre is contained in $f^{-1}(w)$, which is irreducible of dimension d: we have then an inclusion of irreducible varieties $Z \subseteq f^{-1}(w)$ of decreasing dimension, hence $Z = f^{-1}(w)$. We thus have $f^{-1}(w) \subseteq V_i$ for all $w \in W$, which implies that V is contained in V_i (and hence equal to V_i), hence V is irreducible. $\square$

Note that this is false if V is not projective: let $V \subseteq \mathbb{A}^2$ be the union of the hyperbola $V(XY - 1)$ and of the origin $V(X, Y)$: V projects surjectively on $\mathbb{A}^1$, and each fibre is a single point, but V is not irreducible!

19. Parameter spaces

Universal family of hypersurfaces.

The fibre dimension theorem is particularly useful when applied to "families of algebraic varieties" and "parameter spaces." These are a powerful feature of algebraic geometry: often we can consider some collection of algebraic varieties, and construct another algebraic variety which has one point for each variety in the collection. We may also be able to fit all the varieties of the collection together into a single big algebraic variety. This is different from other forms of geometry, where a "family of objects" rarely forms an object of the same type.

Definition. Let B be a quasi-projective variety. A **family of projective algebraic subvarieties over** B is a Zariski closed subset $\mathcal{V} \subseteq B \times \mathbb{P}^n$.

For each $b \in B$, we write $\mathcal{V}_b = \{x \in \mathbb{P}^n : (b, x) \in V\}$ and call this a **fibre** of $\mathcal{V}$.

The set B is called the **base** or **parameter space** of the family.

This definition might seem rather abstract; to give some idea of what is going on, we will look at a simple example (we will see some more complex examples later). A hypersurface of degree d in $\mathbb{P}^n$ means the zero set of a non-zero homogeneous polynomial in $k[X_0, \dots, X_n]$ of degree d. These polynomials form a vector space of dimension $\binom{n+d}{d}$.

If one homogeneous polynomial is a scalar multiple of another, then they define the same hypersurface. Hence we get a hypersurface $\mathcal{H}_a$ associated with each point $a \in \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. (The homogeneous coordinates of a form the coefficients of the polynomial defining $\mathcal{H}_b$.)

These hypersurfaces form a family in $\mathbb{P}^N \times \mathbb{P}^n$: let $V_{n,d}$ denote the vector space of homogeneous polynomials in $k[X_0, \dots, X_n]$ of degree d. We can count the dimension of this vector space: $\dim V_{n,d} = \binom{n+d}{d}$. Let $P_{n,d}$ denote the projective space associated with $V_{n,d}$ i.e.

$$P_{n,d} = (V_{n,d} \setminus \{0\})/(\text{scalars}) \cong \mathbb{P}^N$$

where $N = \binom{n+d}{d} - 1$.

For a polynomial $f \in V_{n,d}$, let's write $[f]$ for the corresponding point in $P_{n,d}$ Using the basis for $V_{n,d}$ which consists of the monomials $X_0^{i_0} \cdots X_n^{i_n}$ (where $i_0 + \cdots + i_n = d$), we see that the homogeneous coordinates of $[f] \in P_{n,d}$ are given by the coefficients of f.

Each non-zero polynomial $f \in V_{n,d}$ defines a hypersurface $H_f \subseteq \mathbb{P}^n$. If f is a scalar multiple of g, then they define the same hypersurface: $H_f = H_g$. (This is not quite an if and only if, because things can go wrong with polynomials that do not generate a radical ideal. Try to come up with an example.)

Thus, instead of labelling hypersurfaces by polynomials $f \in V_{n,d}$ we can label them instead by points in $P_{n,d}$. This has two benefits:

(1) The association of hypersurfaces with points in $P_{n,d}$ is "almost" injective (it is injective for polynomials f which generate radical ideals – and these form a dense open subset of $P_{n,d}$).
(2) By using the projective base $P_{n,d}$ instead of the affine base $V_{n,d}$, we can take advantage of properties like completeness.

We can fit these hypersurfaces together into a family over the base $P_{n,d}$. In other words, there is a single closed set $\mathcal{H} \subseteq P_{n,d} \times \mathbb{P}^n$ such that the fibre

$$\mathcal{H}_{[f]} = \{x \in \mathbb{P}^n : ([f], x) \in \mathcal{H}\}$$

is the hypersurface defined by the polynomial f. To see that $\mathcal{H}$ is closed, we observe that it is defined by a polynomial equation which is bihomogeneous of degree $(1, d)$:

$$\mathcal{H} = \left\{ ([f], x) \in P_{n,d} \times \mathbb{P}^n : \sum_{\substack{0 \leq i_0, \dots, i_n \leq d \\ i_0 + \dots + i_n = d}} f_{i_0 \cdots i_n} X_0^{i_0} \cdots X_n^{i_n} = 0 \right\} \tag{*}$$

where $f_{i_0 \cdots i_n}$ denote the coefficients of the polynomial $f \in V_{n,d}$.

We call $\mathcal{H}$ the **universal family of hypersurfaces of degree d in $\mathbb{P}^n$**. We think of $P_{n,d} \cong \mathbb{P}^N$ as "the parameter space for hypersurfaces of degree d in $\mathbb{P}^n$."

Aside. The word "universal" here is related to the fact that every hypersurface of degree d appears as a fibre in this family, and most of them only appear once (if we work with schemes instead of varieties, then each hypersurface will really appear exactly once). However a rigorous definition of what it means for a family to be "universal" is more subtle than this, and too complicated to define in this course (it involves the notion of a "flat family of schemes").

Subsets of parameter spaces.

One of the benefits of parameter spaces and families of varieties is that they give us a way of talking about all varieties with some particular property at once. If we take a family $\mathcal{V} \subseteq B \times \mathbb{P}^n$ and consider the subset of fibres which satisfy an interesting geometric condition, then very often the corresponding set of points in the parameter space

$$\{b \in B : \mathcal{V}_b \text{ satisfies given condition}\}$$

is an open or closed subset of B.

As a simple example, if we fix a point $x \in \mathbb{P}^n$, then the set

$$\{b \in B : x \in \mathcal{V}_b\}$$

is a closed subset. This is the image of the closed set $(B \times \{x\}) \cap \mathcal{V} \subseteq B \times \mathbb{P}^n$ under the projection $p_1 \colon B \times \mathbb{P}^n \to B$, so it is closed because $\mathbb{P}^n$ is complete.

Another example: the set

$$\{[f] \in P_{n,d} : f \text{ is irreducible}\}$$

is an open set, and so is the set

$$\{[f] \in P_{n,d} : f \text{ generates a radical ideal}\}.$$

– this will be on problem sheet 4.

Dimension counting.

An important use of families of varieties, and the fact that the family is itself a variety, is that we can calculate the dimension of the parameter space, or of interesting subsets of it, using the fibre dimension theorem. By doing this, we can show that certain sets are empty/non-empty/finite/infinite/equal or not equal to the entire parameter space.

Example. Consider the intersection of $n+1$ hypersurfaces in $\mathbb{P}^n$. From our earlier discussions of dimension, we expect that usually such an intersection should be empty (because $n+1 > n$), but of course sometimes it will be non-empty. By counting dimensions of parameter spaces, we can be more specific about how often "sometimes non-empty" occurs: we will prove that the subset of the parameter space where this intersection is non-empty is a closed subset, and then we will compare its dimension with the dimension of the entire parameter space.

What is the appropriate parameter space? In order to get parameter spaces for hypersurfaces, we have to fix the degree of the defining polynomial (the dimension of $P_{n,d}$ increases as d grows, so if there was a single parameter space for all hypersurfaces in degree n it would have to be infinite-dimensional, which doesn't fit within our notion of varieties).

For simplicity, we will assume that all of the $n+1$ hypersurfaces we are intersecting have the same degree d. We are looking at sequences of $n+1$ hypersurfaces, so the parameter space we need is $(P_{n,d})^{n+1}$. (We could more generally pick a sequence of positive integers $d_0, \dots, d_n$ and look at intersections of the form $H_0 \cap \cdots \cap H_n$ where H_0 has degree d_0, H_1 has degree d_1, etc. Because we have fixed the degree of each H_i, we can still form a parameter space for such sequences: it would be the product $P_{n,d_0} \times P_{n,d_1} \times \cdots \times P_{n,d_n}$. The example below would still work, but would sometimes get a little more complicated.)

The subset of the parameter space which we want to study is

$$S = \Big\{(a_0, \dots, a_n) \in (P_{n,d})^{n+1} : \bigcap_{i=0}^{n} \mathcal{H}_{a_i} \neq \emptyset\Big\}.$$

The algebraic varieties we are interested in (intersections of $n+1$ hypersurfaces) should form a family over $(P_{n,d})^{n+1}$. More precisely, we want a family of algebraic varieties over $(P_{n,d})^{n+1}$ such that the fibre above $(a_0, \dots, a_n)$ is $\bigcap_{i=0}^{n} \mathcal{H}_{a_i}$. We can define this family by

$$\Sigma = \{(a_0, \dots, a_n, x) \in (P_{n,d})^{n+1} \times \mathbb{P}^n : x \in \mathcal{H}_{a_i} \text{ for all } i\}.$$

For each i, the condition $x \in \mathcal{H}_{a_i}$ is given by a polynomial (*), so Σ is a closed subset of $(P_{n,d})^{n+1} \times \mathbb{P}^n$. Thus it is a "family of algebraic varieties" in the sense we defined in the previous lecture.

Let π_1 denote the projection $\Sigma \subseteq (P_{n,d})^{n+1} \times \mathbb{P}^n \to (P_{n,d})^{n+1}$. By definition, $\pi_1^{-1}(a_0, \dots, a_n) \cap \Sigma \neq \emptyset$ if and only $(a_0, \dots, a_n) \in S$. In other words, $S = \pi_1(\Sigma)$. Therefore, because $\mathbb{P}^n$ is complete, S is a closed subset of $(P_{n,d})^{n+1}$.

(Why are we focussing on the set where $\bigcap_{i=0}^n \mathcal{H}_{a_i}$ is non-empty rather than the set where it is empty? Because closed sets are usually more interesting than open sets, e.g. it makes sense to ask what is the dimension of a closed subset, while the dimension of an open set is always the same as the dimension of the space it is contained in.)

What is $\dim S$?

We can work this out by two applications of the fibre dimension theorem: first we apply it to the projection $\Sigma \to \mathbb{P}^n$ to find $\dim \Sigma$, then we apply it to the projection $\Sigma \to S$ to find $\dim S$. The reason we can do this is that we know the dimension of $\mathbb{P}^n$ and we can work out the dimensions of the fibres of both projections from Σ.

To compute $\dim \Sigma$, we will apply the fibre dimension theorem to the projection $p\colon \Sigma \to \mathbb{P}^n$. This map is surjective: for any $x \in \mathbb{P}^n$, we can pick $a \in P_{n,d}$ such that $x \in \mathcal{H}_a$ and then $(a, \dots, a, x) \in p^{-1}(x) \subseteq \Sigma$. The fibres are

$$\begin{aligned} p^{-1}(x) &= \{(a_0, \dots, a_n) \in (P_{n,d})^{n+1} : x \in \mathcal{H}_{a_i} \text{ for all } i\} \\ &= \Big(\{a \in P_{n,d} : x \in \mathcal{H}_a\}\Big)^{n+1}. \end{aligned}$$

Thus

$$\dim p^{-1}(x) = (n+1) \dim\{a \in P_{n,d} : x \in \mathcal{H}_a\}.$$

In order to calculate $\dim\{a \in P_{n,d} : x \in \mathcal{H}_a\}$, make a linear change of coordinates so that $x = [0 : \dots : 0 : 1]$. This change of coordinates won't change the dimension of $\{a \in P_{n,d} : x \in \mathcal{H}_a\}$, so it suffices to work out the dimension for the special case of $[0 : \dots : 0 : 1]$.

Now $[0 : \dots : 0 : 1] \in \mathcal{H}_a$ if and only if the homogeneous polynomial f vanishes at $[0 : \dots : 0 : 1]$, where $a = [f]$. The value of f at $[0 : \dots : 0 : 1]$ is just the X_n^d coefficient of f. Thus

$$[0 : \dots : 0 : 1 \in \mathcal{H}_{[f]} \Leftrightarrow \text{the } X_n^d \text{ coefficient of } f \text{ is zero.}$$

In other words, $\{a \in P_{n,d} : x \in \mathcal{H}_a\}$ is a subspace of $P_{n,d}$ defined by one linear equation, so $\dim\{a \in P_{n,d} : x \in \mathcal{H}_a\} = \dim P_{n,d} - 1 = N - 1$ where $N = \binom{n+d}{d} - 1$. (Alternatively, we could have seen this without reducing to the case $x = [0 : \dots : 0 : 1]$ by observing that the condition $x \in \mathcal{H}_{[f]}$ is a single linear condition on the coefficients of f, as you see just by expanding out $f(x_0, \dots, x_n) = 0$.)

Therefore, for every $x \in \mathbb{P}^n$,

$$\dim p^{-1}(x) = (n+1)(N-1).$$

in particular $\mathbb{P}^n$ is irreducible, p is surjective, and the fibres of p are all irreducible of the same dimension: by Proposition 18.2, Σ is irreducible. We can apply the fibre dimension theorem (Theorem 18.1) to get

$$\dim \Sigma = \dim \mathbb{P}^n + \dim p^{-1}(x) = n + (n+1)(N-1) = N(n+1) - 1.$$

(By part (ii) of the fibre dimension theorem, this holds for all x in some non-empty open subset of $\mathbb{P}^n$. It doesn't matter which x we choose because we showed that all the fibres have the same dimension.)

Now to compute $\dim S$, we will apply the fibre dimension theorem to the projection $q\colon \Sigma \to S$. This map is surjective by construction. This time, the fibres do not all have the same dimension, but the minimum dimension of the fibres is zero – to see that there are fibres of dimension zero, observe that for suitable choices of $n+1$ homogeneous polynomials of degree d, the intersection of the corresponding hypersurfaces is finite and non-empty for example: $f_i = X_i^d$ for $0 \leq i \leq n-1$, $f_n = X_0^d$ (repeating f_0) have the unique common solution $[0 : \dots : 0 : 1]$.

Therefore the fibre dimension theorem implies that

$$\dim S = \dim \Sigma - 0 = N(n+1) - 1.$$

We recall how this follows from the fibre dimension theorem: by part (i), the fact that there is just a single fibre of dimension 0 implies that $0 \geq \dim \Sigma - \dim S$. By part (ii) of the fibre dimension theorem, there exists an open subset $U \subseteq S$ on which $\dim q^{-1}(s) = \dim \Sigma - \dim S$. But since $q^{-1}(s)$ can never be negative, this forces $0 \leq \dim q^{-1}(s) = \dim \Sigma - \dim S$. Combining these gives $\dim S = \dim \Sigma$ as we claimed.

In particular, we have $\dim S = \dim(P_{n,d})^{n+1} - 1$. This means that it is only slightly unusual for $n+1$ hypersurfaces to have non-empty intersection: this subset of the parameter space has dimension only 1 less than the entire parameter space.

As an application of this calculation, we see that S is a hypersurface in $(P_{n,d}^{n+1})$, and therefore it is defined by a single polynomial

$$F \in k[X_{iJ} : 0 \leq i \leq n, 0 \leq I \leq N].$$

In other words, there exists some polynomial F such that, when we evaluate it at the coefficients of $n+1$ homogeneous polynomials $f_0, \dots, f_n$ of degree d, we get zero if and only if the intersection $\bigcap_{i=0}^n \mathcal{H}_{[f_i]}$ is non-empty.

20. Tangent space and singular points (Mastery material)

In the mastery material, we will define the singular points of algebraic varieties. These are points where the variety is not smooth, for example the origin in the curves $y^2 = x^3$ or $y^2 = x(x+1)$.

To determine whether a point of a variety is singular, we use the tangent space of the variety at that point. This is a generalisation of the tangent line to a smooth curve (i.e. the line whose gradient is equal to the gradient of the curve). We say that the point is singular if the dimension of the tangent space at that point is different to the dimension of the variety (for an irreducible variety – there is a slight complication for reducible varieties).

So first we have to define tangent spaces. We start with the case of affine algebraic varieties, then generalise to quasi-projective varieties. We will finish with the proof that most points of a variety are not singular – more precisely, the singular points form a proper closed subset of the variety.

Tangent space in coordinates. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic subvariety. We consider V equipped with an embedding in $\mathbb{A}^n$ because we will use coordinates in our first definition of the tangent space, before turning to a coordinate-free definition.

Choose a point $x \in V$. For each polynomial $f \in k[X_1, ..., X_n]$, let df_x denote the linear map $k^n \to k$ given by:

$$df_x(a_1, ..., a_n) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial X_i}|_x \tag{32}$$

Informally: df_x sends a vector $a \in k^n$ to the "directional derivative" of f at x along that vector. Thus $df_x(a) = 0$ precisely for those directions in which f is stationary at x. Since the polynomials in $I(V)$ are zero on V , we should expect polynomials in $I(V)$ to be stationary along "tangent directions" to V. This motivates the following definition.

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic subvariety and let $x \in V$. The tangent space to V at x is:

$$T_xV = \bigcap_{f \in I(V)} \ker df_x \subseteq k^n \tag{33}$$

In our definition of df_x, we used partial derivatives. Because we are only differentiating polynomials, these can be defined purely algebraically and therefore make sense over any field, even in positive characteristic where there is no analysis. However derivatives of polynomials can behave surprisingly in positive characteristic: over a field of characteristic p we have:

$$\frac{\partial (X^p)}{\partial X} = pX^{p-1} = 0 \tag{34}$$

so it is possible for a non-constant polynomial to have derivative equal to zero.

Similarly the informal motivation for the definition relied on our intuition from analysis about what happens over $\mathbb{C}$. Even over $\mathbb{C}$, our analytic intuition only works correctly if the variety is non-singular at x.

Let $f_1, ..., f_n$ be a finite list of polynomials which generate $I(V)$. It is easy to prove that the tangent space T_xV can be calculated just by looking at this finite list of polynomials:

$$T_xV = \bigcap_{i=1}^{n} \ker d(f_i)_x \subseteq k^n \tag{35}$$

Thus it is straightforward to calculate tangent spaces in practice, from a list of generators for $I(V)$. There is just one thing to be careful of: it is not enough to take a list of functions which define V as an algebraic subvariety but do not generate the ideal (there is an issue with radical closure).

Example. As a very simple example, consider the line L in $\mathbb{A}^2$ defined by the polynomial X. At the point $(0,0)$, we have:

$$dX_{(0,0)}(a_1, a_2) = a_1 \frac{\partial X}{\partial} X|_{(0,0)} + a_2 \frac{\partial X}{\partial} Y|_{(0,0)} = a_1 \tag{36}$$

Since f generates $I(L)$, we get:

$$T_{(0,0)}V = \ker dX_{(0,0)} = \{(a_1, a_2) \in k^2 : a_1 = 0\} \tag{37}$$

This is what we should expect: the tangent space to a line is a line in the same direction.

However, if we were given the polynomial $g = X^2$, then $V(g)$ is again L. We could try to calculate:

$$d(X^2)_{(0,0)}(a_1, a_2) = a_1 \frac{\partial X^2}{\partial} X|_{(0,0)} + a_2 \frac{\partial X^2}{\partial} Y|_{(0,0)} = 0 \tag{38}$$

So $\ker dg_{(0,0)=k^2}$ which is too big. Thus using polynomials which do not generate the whole ideal of the variety may give the wrong answer for the tangent space.

Example. Consider the graph Γ of a polynomial function $g \in k[x]$. This is an affine algebraic subvariety in $\mathbb{A}^2$ defined by the polynomial $f(X,Y) = Y - g(X)$. Note that f is irreducible because it is monic of degree 1 in Y, so f generates $I(\Gamma)$. We can calculate:

$$df_{(x,g(x))}(a_1, a_2) = -g'(x)a_1 + 1 \cdot a_2 \tag{39}$$

so:

$$T_{(x,g(x))} = \ker df_{(x,g(x))} = k(1, g'(x)) \tag{40}$$

Thus the tangent space to Γ at the point $(x, g(x))$ is a line with gradient $g'(x)$, as we expect.

Example. Consider the cuspidal cubic curve defined by the polynomial $f(X,Y) = Y^2 - X^3$. We have

$$\frac{\partial f}{\partial X} = -3X^2 \quad \frac{\partial f}{\partial Y} = 2Y \tag{41}$$

Hence:

$$\frac{\partial f}{\partial X}|_{(0,0)} = 0 \quad \frac{\partial f}{\partial Y}|_{(0,0)} = 0 \tag{42}$$

and so $df_{(0,0)}V$ is the zero map. Therefore $T_{(0,0)}V = k^2$. We can't see this looking at a picture: this curve appears to have only the X-axis as a tangent line at the origin. This demonstrates that we cannot rely on geometric intuition to calculate the tangent space at singular points: it is necessary to use the algebraic definition.

Differentials in coordinates. We can also consider the differential of a regular map:

Lemma 20.1. Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be affine algebraic subvarieties. Let $\phi : V \to W$ be a regular map. For any $x \in V$, ϕ induces a linear map $d\phi_x : T_xV \to T_{f(x)}W$.

Proof. Choose polynomials $f_1, ..., f_n$ such that $\phi = (f_1, ..., f_n)$. Define a linear map $k^m \to k^n$ by the matrix

$$\left(\tfrac{\partial f_i}{\partial X_j}|_x\right)_{ij} \tag{43}$$

We define $d\phi_x$ to be the restriction of this map to T_xV. (Recall that T_xV is a subspace of k^m and T_yW is a subspace of k^n.) Using the chain rule for partial derivatives, one can check that:

i) $d\phi_x$ maps T_xV to T_yW.
ii) $d\phi_x$ is independent of the choice of polynomials representing ϕ.

□

In particular, if $\phi : V \to W$ is an isomorphism, $d\phi_x$ is an isomorphism of vector spaces: this gives that the tangent space does not depends on the embedding in $\mathbb{A}^n$. Similarly, if $U \subseteq V$ and $U' \subseteq V'$ are open and $V \subseteq \mathbb{A}^m$, $W \subseteq \mathbb{A}^n$ affine, and $\phi : U \to U'$ regular, one can define also for $x \in U$ $d\phi_x : T_xV \to T_{f(x)}W$, which is an isomorphism is ϕ is an isomorphism. Hence one can define T_xV for any V quasi-projective, by taking an affine neighborhood of x in V: by the above arguments, T_xV does not depends on this choice of affine neighborhood.

Definition of singular points. Intuition suggests that, at non-singular points, the dimension of the tangent space should be equal to the dimension of the algebraic variety. The above examples indicate that this breaks down at singular points. This motivates us to define a singular point to be a point $x \in V$ where $\dim T_xV \neq \dim V$. However this simple definition only works correctly for irreducible algebraic varieties.

To understand the definition in the reducible case, note that singularity and smoothness should be a "local" property. Hence take V a union of two irreducible components of different dimensions, say $V = V_1 \cup V_2$ where $\dim V_1 = 1$ and $\dim V_2$. For a point $x \in V_1 - V_1 \cap V_2$, whether x is a singular point of V should not care about V_2 – we ought to compare $\dim T_x V$ against $\dim V_1 = 1$.

In order to fix this and correctly define singular points of reducible algebraic varieties, we introduce a new definition:

Definition. Let V be a quasi-projective variety and let x be a point of V. The local dimension of V at x, written $\dim_x V$, is the maximum of the dimensions of those irreducible components of V which contain x.

Thus in our previous example $V = V_1 \cap V_2$, $\dim_x V = 2$ if $x \in V_2$ (including if $x \in V_1 \cap V_2$) while $\dim_x V = 1$ if $x \in V_1 - V_1 \cap V_2$. Now we can define singular points of a quasi-projective varieties by using local dimension.

Definition. Let V be an quasi-projective variety and let $x \in V$. Then x is a singular point of V if $\dim T_x V \neq \dim_x V$. Otherwise, x is said to be a smooth point of V.

(We may also express this as "V is singular at x", or conversely, "V is smooth at x").

We will prove later that $\dim T_x V \geq \dim_x V$ always, so we could equivalently state this definition as: x is a singular point of V if $\dim T_x V > \dim_x V$.

If $V = \bigcup_i V_i$ is a union of irreducible components, then for any point x which lies in only one irreducible component , we have $\dim_x V = \dim_x V_i$ by definition. A little algebra also shows that $T_x V = T_x V_i$ and so V is singular at x if and only if V_i is singular at x.

On the other hand, if x lies in an intersection of two or more irreducible components of V, then it turns out that x is always a singular point of V. This is intuitively sensible, but requires too much algebra to prove in this course (specifically, it requires Nakayama's lemma).

Singular locus of a variety. Let V be a quasi-projective variety. The set of singular points of V is called the singular locus of V and denoted $\operatorname{Sing} V$. The singular locus turns out to be a proper closed subset of V. We will prove this for irreducible quasi-projective varieties. Generalising to reducible varieties requires the fact that every point which lies in the intersection of two or more irreducible components is singular, and require more algebra (we will prove that in the next section).

Theorem 20.2. Let V be an irreducible quasi-projective variety. Then $\operatorname{Sing} V$ is a proper closed subset of V.

A key intermediate step in the proof of this theorem, which is also interesting in its own right, is the fact that $\dim T_x V \geq \dim_x V$ for every point $x \in V$.

The singular locus of a hypersurface. We begin by proving Theorem 20.2 for a hypersurface.

Let $V \subseteq \mathbb{A}^n$ be a hypersurface and let f be a polynomial which generates $I(V)$. Since f generates $I(V)$, the tangent space T_xV is just $\ker df_x$. In other words, T_xV is the kernel of a linear map $k^n \to k$, and so:

- $\dim T_xV = n-1$ if df_x is not the zero map;
- $\dim T_xV = n$ if df_x is the zero map.

Each irreducible component of a hypersurface is of dimension $n-1$, hence for any point $x \in V$, we have $\dim_x V = n-1$. Hence:

$$\operatorname{Sing} V = \{x \in V : df_x = 0\} \tag{44}$$

Going back to the definition of df_x, we can write this as:

$$\operatorname{Sing} V = \{x \in V : \frac{\partial f}{\partial X_i} = 0 \; for \; i = 0, ..., n\} \tag{45}$$

This may look like the definition of singular points which you have seen before for curves in $\mathbb{A}^2$. For each i, $\frac{\partial f}{\partial X_i}$ is a polynomial. Therefore, for any hypersurface $V \in \mathbb{A}^n$, $\operatorname{Sing} V$ is a closed subset of V.

Now we want to show that for a hypersurface V, $\operatorname{Sing} V \subsetneq V$.

This is a little harder in positive characteristic than in characteristic zero, so first we prove a lemma on derivatives in positive characteristic. Over a field of characteristic p, X^{ip} has derivative zero for any positive integer i. We prove that these span all the polynomials with zero derivative.

Lemma 20.3. Let k be a field of characteristic $p > 0$. Let $f \in k[X]$ be a polynomial. If $\frac{\partial f}{\partial X} = 0$, then for every term of f, the exponent of X is a multiple of p, that is,

$$f = \sum_{i=0}^{d} a_{ip} X^{ip} \tag{46}$$

Proof. Consider a term $a_j X^j$ in f. This term differentiates to $j a_j X^{j-1}$. No other term of f differentiates to a scalar multiple of X^{j-1}, so this term can never cancel with another term in $\frac{\partial f}{\partial X}$. Hence if $\frac{\partial f}{\partial X} = 0$, then $j a_j = 0$ (in k) for every j. If j is not a multiple of p, then j is invertible in k so this forces $a_j = 0$. Thus only terms where j is a multiple of p can appear in f. □

Proposition 20.4. If V is a non-empty hypersurface, then $\operatorname{Sing} V$ is strictly contained in V.

Proof. Assume for contradiction that $\operatorname{Sing} V = V$. Then $\frac{\partial f}{\partial X_1}, ..., \frac{\partial f}{\partial X_n}$ are all zero on V. Since f generates $I(V)$, this implies that f divides $\frac{\partial f}{\partial X_i}$ for each i. But $\frac{\partial f}{\partial X_i}$ has strictly smaller X_i-degree than f. This forces $\frac{\partial f}{\partial X_i} = 0$ for each i (as a polynomial in $k[X_1, ..., X_n]$).

Over a field of characteristic zero, this implies that f is constant. But then V would be empty, contradicting the hypothesis.

Over a field of characteristic $p > 0$, by Lemma 20.3, the fact that $\frac{\partial f}{\partial X_i} = 0$ implies that very term of f must have its X_i-exponent being a multiple of p. Since this holds for all i, each term of f is a p-th power (the constant in the term must be a p-th power because k is algebraically closed). But the binomial expansion implies that:

$$(a+b)^p = a^p + b^p \tag{47}$$

over a field of characteristic p. So if every term of f is a p-th power, then f itself is a p-th power. But then the ideal generated by f is not a radical ideal. Via the Nullstellensatz, this contradicts the assumption that f generates $I(V)$. □

The singular locus of an irreducible variety. We prove here Theorem 20.2. Consider V an irreducible quasi-projective variety.

Lemma 20.5. Let V a quasi-projective variety. For any integer d, the set:

$$\Sigma_d(V) = \{x \in V : \dim T_x V > d\} \tag{48}$$

is closed

Proof. $\Sigma_d(V)$ is closed if and only if its intersection with an affine open cover is closed: hence we can consider that $V \subseteq \mathbb{A}^n$ is an affine algebraic subvariety.

Choose polynomials $f_1, ..., f_m$ which generate $I(V)$. Recall that:

$$T_x V = \cap_{i=1}^m \ker d(f_i)_x \tag{49}$$

In other words, $T_x V$ is the kernel of the matrix

$$M_x = \left(\frac{\partial f_i}{\partial X_j}\right)_{ij} \tag{50}$$

which represents a linear map $k^n \to k^m$. By the rank–nullity theorem, $\dim T_x V$is equal to $n - rkM$. Hence:

$$\Sigma_d V = \{x \in V : rkM_x < n - d\} \tag{51}$$

By linear algebra, $rkM_x < n - d$ is equivalent to: every $(n-d) \times (n-d)$ submatrix of M_x has determinant zero. The determinant of a submatrix of M_x is a polynomial, hence this gives us polynomial equations defining $\Sigma_d(V)$. □

Lemma 20.6. Let V be an irreducible quasi-projective variety. Then the non-singular points of V are dense in V.

Proof. We consider a nonempty affine open subset U of V: by Proposition 7.6, U (and then V) is birational to a hypersurface $H \in \mathbb{A}^{d+1}$: in particular, an open U' subset of V is isomorphic to an open subset U' of H. By the previous subsection, the non-singular points of H form a nonempty open subset $H_{ns} \subseteq H$. Since H is irreducible, H_{ns} is dense in H and must intersect U': $H_{ns} \cap U'$ is then open and dense in U', hence also in V, which means that the set of non-singular points of V is dense in V. □

Lemma 20.7. Let V be an irreducible quasi-projective variety. For every $x \in V$, $\dim T_x V \geq \dim_x V$

Proof. Since V is irreducible, $\dim_x V = \dim V$ for every $x \in V$ so we can work with $\dim V$ instead of $\dim_x V$. (The lemma is true for reducible V as well, but we do not have the tools to prove it when $\dim_x V$ is not constant.)

Let $d = \dim V$ and consider the set $\Sigma_{d-1}(V)$ as in Lemma 20.5. By Lemma 20.5, $\Sigma_{d-1}(V)$ is closed. Every non-singular point of V is in $\Sigma_{d-1}(V)$, so Lemma 20.6 implies that $\Sigma_{d-1}(V)$ is dense in V. Since $\Sigma_{d-1}(V)$ is closed and dense in V, we conclude that $\Sigma_{d-1}(V) = V$. □

Theorem 20.8. Let V be an irreducible quasi-projective variety. Then $\operatorname{Sing} V$ is a proper closed subset of V.

Proof. By Lemma 20.7, $x \in V$ is a singular point if and only if $\dim T_x V > \dim_x V$. Again since V is irreducible, we can replace $\dim_x V$ by $d = \dim V$. Thus:

$$\operatorname{Sing} V = \Sigma_d(V) \tag{52}$$

So Lemma 20.5 tells us that $\operatorname{Sing} V$ is closed in V. Lemma 20.6 implies that $\operatorname{Sing} V$ is properly contained in V. □