1 segre embedding

Lemma 1.1 (Lemma 12.2)

Let $\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \Sigma_{m,n}$ be a bijection.

Proof. Let $N = (m+1)(n+1) - 1$ and write points of $\mathbb{P}^N$ as $z = [z_{ij}]_{0\le i\le m,\, 0\le j\le n}$. On $\Sigma_{m,n}$ we have the rank-1 relations

$$z_{ij}z_{k\ell} = z_{i\ell}z_{kj} \qquad (0 \le i,k \le m,\ 0 \le j,\ell \le n).$$

Fix $z \in \Sigma_{m,n}$ and choose (a,b) with $z_{ab} \neq 0$. Define

$$\pi_1(z) := [z_{0b} : \cdots : z_{mb}] \in \mathbb{P}^m, \qquad \pi_2(z) := [z_{a0} : \cdots : z_{an}] \in \mathbb{P}^n.$$

These are well-defined up to projective scaling, and independent of the choice of (a,b): if $z_{a'b'} \neq 0$, the above relations imply $[z_{0b} : \cdots : z_{mb}] = [z_{0b'} : \cdots : z_{mb'}]$ and $[z_{a0} : \cdots : z_{an}] = [z_{a'0} : \cdots : z_{a'n}]$.

Now compute, for any r, s,

$$(\sigma_{m,n}(\pi_1(z), \pi_2(z)))_{rs} = z_{rb}z_{as} = z_{rs}z_{ab},$$

so in projective coordinates $\sigma_{m,n}(\pi_1(z), \pi_2(z)) = [z_{rs}z_{ab}]_{r,s} = [z_{rs}]_{r,s} = z$. Hence $\sigma_{m,n} \circ (\pi_1 \times \pi_2) = \mathrm{id}_{\Sigma_{m,n}}$.

Conversely, let $([x],[y]) \in \mathbb{P}^m \times \mathbb{P}^n$ and put $z = \sigma_{m,n}([x],[y]) = [x_iy_j]_{i,j}$. Choose a, b with $x_a \neq 0$ and $y_b \neq 0$. Then

$$\pi_1(z) = [x_0y_b : \cdots : x_my_b] = [x_0 : \cdots : x_m] = [x],$$

$$\pi_2(z) = [x_ay_0 : \cdots : x_ay_n] = [y_0 : \cdots : y_n] = [y].$$

Therefore $(\pi_1 \times \pi_2) \circ \sigma_{m,n} = \mathrm{id}_{\mathbb{P}^m \times \mathbb{P}^n}$. So $\pi_1 \times \pi_2$ is the inverse of $\sigma_{m,n}$, and $\sigma_{m,n}$ is a bijection. □

Lemma 1.2

Let $V \subset \mathbb{P}^m \times \mathbb{P}^n$, then $\sigma_{m,n}(V)$ is closed in $\Sigma_{m,n}$ if and only if

$$V = \{[x_0 : \cdots : x_m], [y_0 : \cdots : y_n] | f_i() = 0, \text{for } 1 \le i \le s\}$$

where f_i are bihomogeneous polynomials in $x_0, \ldots, x_m$ and $y_0, \ldots, y_n$.

proof(not part of the course). □