I M P E R I A L

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Assessed Coursework 1

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Course: *MATH70029-Algebraic Geometry* – Lecturer: *Matt Booth*
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50 points: 10 + 10 + 10 + 10 + 10.
As always, we let k be an algebraically closed field. Some answers may depend on the characteristic of k.

Exercise 1.

Consider the following polynomials in $k[x, y, z]$:

$$f = y^2 - x^2, \qquad g = x^4 - yz, \qquad h = z^2 - x^3 y.$$

Find and describe the irreducible components of the varieties $V(f, g)$, $V(f, h)$, $V(f, g, h)$.

Solution. Throughout, note that $f = (y - x)(y + x)$.
$V(f, g)$**:** On $V(y - x)$: substituting $y = x$ into g gives $x^4 - xz = x(x^3 - z) = 0$, so $x = 0$ or $z = x^3$.

- $x = 0, y = 0$: the z-axis $V(x, y) = \{(0, 0, t) : t \in k\}$.
- $z = x^3, y = x$: the twisted cubic $C_1 = V(y - x,\ z - x^3) = \{(t, t, t^3) : t \in k\}$.

On $V(y + x)$: substituting $y = -x$ gives $x^4 + xz = x(x^3 + z) = 0$, so $x = 0$ (the z-axis again) or $z = -x^3$.

- $z = -x^3, y = -x$: the curve $C_2 = V(y + x,\ z + x^3) = \{(t, -t, -t^3) : t \in k\}$.

Each component is irreducible:

$$k[x, y, z]/(x, y) \cong k[z], \quad k[x, y, z]/(y - x, z - x^3) \cong k[x], \quad k[x, y, z]/(y + x, z + x^3) \cong k[x].$$

These are integral domains. Also, none is contained in the union of the others. For example,

$$(0, 0, 1) \in V(x, y), \qquad (0, 0, 1) \notin C_1 \cup C_2.$$

And when $\operatorname{char} k \neq 2$,

$$(1, 1, 1) \in C_1, \qquad (1, 1, 1) \notin V(x, y) \cup C_2.$$

Also, when $\operatorname{char} k \neq 2$,

$$(1, -1, -1) \in C_2, \qquad (1, -1, -1) \notin V(x, y) \cup C_1.$$

If $\operatorname{char} k = 2$: then $y + x = y - x$, so $C_1 = C_2$ and

$$V(f, g) = V(x, y) \cup V(y - x,\ z - x^3) \quad \text{(two components)}.$$

Here

$$(0, 0, 1) \in V(x, y) \setminus V(y - x, z - x^3), \qquad (1, 1, 1) \in V(y - x, z - x^3) \setminus V(x, y),$$

so neither component contains the other. If $\operatorname{char} k \neq 2$:

$$V(f, g) = V(x, y) \cup V(y - x,\ z - x^3) \cup V(y + x,\ z + x^3) \quad \text{(three components)}.$$

We are using explicitly:

[Definition, §2.1, p.3, Algebraic Geometry, 2026]: $V(S) = \{x \in \mathbb{A}^n_k : f(x) = 0\ \forall f \in S\}$,

and

[Lemma 2.1, §2.2, p.4, Algebraic Geometry, 2026]: $V(I) \cup V(J) = V(IJ)$.

Hence each piece above is an affine subvariety (a zero-locus), and their finite union is again Zariski closed.

$V(f, h)$**:** On $V(y-x)$: $h = z^2 - x^4 = (z - x^2)(z + x^2)$, giving components $V(y - x, z - x^2)$ and $V(y - x, z + x^2)$.
On $V(y + x)$:

$$h = z^2 + x^4.$$

Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Then

$$z^2 + x^4 = (z - ix^2)(z + ix^2),$$

giving components $V(y + x, z - ix^2)$ and $V(y + x, z + ix^2)$.
Each quotient $k[x, y, z]/(y \pm x, z - \alpha x^2) \cong k[x]$ is a domain, so each component is irreducible.
If $\operatorname{char} k = 2$: $i = 1$, $y + x = y - x$, and $z + x^2 = z - x^2$, so all four collapse to a single component:

$$V(f, h) = V(y - x,\ z - x^2) \quad \text{(one component)}.$$

If $\operatorname{char} k \neq 2$: four distinct irreducible components,

$$V(f, h) = V(y - x, z - x^2) \cup V(y - x, z + x^2) \cup V(y + x, z - ix^2) \cup V(y + x, z + ix^2).$$

$V(f, g, h)$**:** We intersect the components of $V(f, g)$ with the condition $h = 0$.
On $V(x, y)$: $h = z^2 = 0$, so $z = 0$. This gives the origin $(0, 0, 0)$.
On $C_1 = \{(t, t, t^3)\}$: $h = t^6 - t^4 = t^4(t^2 - 1) = 0$, so $t = 0$ or $t = \pm 1$. Points: $(0, 0, 0)$, $(1, 1, 1)$, $(-1, -1, -1)$.
On $C_2 = \{(t, -t, -t^3)\}$ (when $\operatorname{char} k \neq 2$):

$$h = t^6 + t^4 = t^4(t^2 + 1) = 0,$$

so $t = 0$ or $t = \pm i$. Hence the points are

$$(0,0,0),\ (i,-i,i),\ (-i,i,-i).$$

If $\operatorname{char} k \neq 2$: $V(f,g,h)$ consists of five points,

$$V(f,g,h) = \{(0,0,0),\ (1,1,1),\ (-1,-1,-1),\ (i,-i,i),\ (-i,i,-i)\}.$$

If $\operatorname{char} k = 2$: $t^2 - 1 = (t+1)^2$, so $t = 0$ or $t = 1$, giving

$$V(f,g,h) = \{(0,0,0),\ (1,1,1)\}.$$

In both cases, the irreducible components are the individual points. □

Exercise 2.

Let $f, g \in k[x,y]$ be two irreducible polynomials which are not multiples of each other.

(a) Suppose that at least one of f and g contains a nonzero term in y (i.e. is not an element of $k[x]$). Use Gauss's Lemma to show that f, g have no common factors in $k(x)[y]$.

Solution. Let

$$R := k[x], \qquad K := k(x), \qquad f, g \in R[y].$$

Assume $\deg_y f \geq 1$. If f were not primitive, then

$$f = c(x)\tilde{f}(x,y),$$

with non-unit $c(x) \in R$, contradiction. So f is primitive, hence Gauss implies

$$f \text{ irreducible in } K[y].$$

If $\deg_y g \geq 1$, the same gives g irreducible in $K[y]$. If f, g were associates in $K[y]$,

$$f = (a/b)g, \qquad a, b \in R \setminus \{0\},\ \gcd(a,b) = 1,$$

(after cancelling common factors in R), so

$$bf = ag.$$

Since $a \mid ag = bf$ and $\gcd(a,b) = 1$, Euclid's Lemma in the UFD $R[y]$ gives $a \mid f$. Since $\deg_y f \geq 1$ and f irreducible, $a \in k^\times$; similarly $b \in k^\times$. Hence f, g are scalar multiples in $k[x,y]$, contradiction. Therefore

$$\gcd_{K[y]}(f,g) = 1.$$

If $g \in k[x] \subset K$, then g is a unit of $K[y]$, so again

$$\gcd_{K[y]}(f,g) = 1.$$

□

(b) Show that there exist nonzero polynomials $h \in k[x]$ and $p, q \in k[x, y]$ such that $h = fp + gq$.

Solution. From (a),

$$\gcd_{k(x)[y]}(f, g) = 1.$$

Hence in the PID $k(x)[y]$,

$$1 = fP + gQ, \qquad P, Q \in k(x)[y].$$

Write

$$P = \frac{p_0}{d_1}, \qquad Q = \frac{q_0}{d_2},$$

with

$$p_0, q_0 \in k[x, y], \qquad d_1, d_2 \in k[x] \setminus \{0\}.$$

Multiply by $d_1 d_2$:

$$d_1 d_2 = f(d_2 p_0) + g(d_1 q_0).$$

Set

$$h := d_1 d_2, \qquad p := d_2 p_0, \qquad q := d_1 q_0.$$

Then

$$h = fp + gq, \qquad h \in k[x] \setminus \{0\}.$$

□

(c) Show that the set $\{x : (x, y) \in V(f, g)\}$ of first coordinates of points of $V(f, g)$ is finite.

Solution. By (b), there are

$$h \in k[x] \setminus \{0\}, \qquad p, q \in k[x, y], \qquad h = fp + gq.$$

If $(a, b) \in V(f, g)$, then

$$f(a, b) = g(a, b) = 0$$

and therefore

$$h(a) = f(a, b)p(a, b) + g(a, b)q(a, b) = 0.$$

So

$$\{a \in k : \exists b,\ (a, b) \in V(f, g)\} \subseteq V(h),$$

hence finite. □

(d) Show that the set $V(f, g)$ is finite.

Solution. Let

$$A := \{a \in k : \exists b,\ (a, b) \in V(f, g)\}.$$

By (c), A is finite.

For each $a \in A$, let

$$F_a := \{b \in k : (a, b) \in V(f, g)\}.$$

Then

$$F_a \subseteq \{b \in k : f(a, b) = 0\}.$$

w.l.o.g., assume $\deg_y f \geq 1$.[1] Write

$$f = \sum_{j=0}^{m} c_j(x) y^j, \qquad c_m \neq 0,\ m \geq 1.$$

If $f(a, y) \equiv 0$, then $c_j(a) = 0$ for all j, so

$$(x - a) \mid c_j(x)\ \forall j \Longrightarrow (x - a) \mid f,$$

contradiction (irreducibility of f and $\deg_y f \geq 1$).

Hence $f(a, y) \neq 0$, so each F_a is finite. Finally,

$$V(f, g) = \bigcup_{a \in A} \{a\} \times F_a$$

is finite. □

Exercise 3.

Let $n, m \geq 1$ and consider

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^2, \qquad t \mapsto (t^n, t^m).$$

Show that $\mathrm{im}(\varphi)$ is an affine subvariety of $\mathbb{A}^2$. Give the condition for which φ is bijective onto its image. In that case, give a birational inverse when $\mathrm{char}\, k = 0$.

[1] By part (a), at least one of f, g has positive y-degree. If instead $\deg_y f = 0$, then $\deg_y g \geq 1$, and the same argument below applies after swapping f and g. This is valid since $V(f, g) = V(g, f)$.

Solution. Let $d = \gcd(n, m)$, $a = n/d$, $b = m/d$, so $\gcd(a, b) = 1$.
Image is a subvariety: By [Definition, §2.1, p.3, Algebraic Geometry, 2026],

$$\text{it is enough to prove } \operatorname{im}(\varphi) = V(S) \text{ for some } S \subset k[x, y].$$

Since k is algebraically closed, the map $t \mapsto t^d$ is surjective on k, so

$$\operatorname{im}(\varphi) = \{(t^n, t^m) : t \in k\} = \{(s^a, s^b) : s \in k\}.$$

Claim. $\operatorname{im}(\varphi) = V(x^b - y^a)$.

$$(s^a, s^b) \in \operatorname{im}(\varphi) \Longrightarrow (s^a)^b - (s^b)^a = 0.$$

Hence

$$\operatorname{im}(\varphi) \subseteq V(x^b - y^a).$$

Conversely, let $(x, y) \in V(x^b - y^a)$. If $x = 0$, then $y^a = 0$, so $(x, y) = (0, 0) \in \operatorname{im}(\varphi)$. Assume $x \neq 0$ and choose $s_0 \in k$ with $s_0^a = x$. Then

$$s_0^{ab} = x^b = y^a,$$

so

$$\left(\frac{y}{s_0^b}\right)^a = 1.$$

Let $\zeta := y/s_0^b \in \mu_a$. Since $\gcd(a, b) = 1$, the map

$$\mu_a \to \mu_a, \qquad \eta \mapsto \eta^b$$

is bijective. Hence there exists $\eta \in \mu_a$ with $\eta^b = \zeta$. Set $s := \eta s_0$. Then

$$s^a = \eta^a s_0^a = x, \qquad s^b = \eta^b s_0^b = \zeta s_0^b = y.$$

Thus $(x, y) = (s^a, s^b) \in \operatorname{im}(\varphi)$.
Consider

$$k[x, y] \to k[t], \qquad x \mapsto t^a,\ y \mapsto t^b.$$

Let θ denote this map. Clearly

$$(x^b - y^a) \subseteq \ker \theta.$$

For $F \in k[x, y]$, reduce modulo $(x^b - y^a)$ to

$$F \equiv \sum_{j=0}^{a-1} f_j(x) y^j.$$

If $F \in \ker \theta$, then

$$0 = \sum_{j=0}^{a-1} f_j(t^a) t^{bj} = \sum_{j=0}^{a-1} \sum_{i \geq 0} c_{ij}\, t^{ai+bj}.$$

If

$$ai + bj = ai' + bj', \qquad 0 \leq j, j' < a,$$

then

$$a(i - i') = b(j' - j).$$

Since $\gcd(a,b)=1$, we get $a \mid (j'-j)$, hence $j=j'$ and then $i=i'$.[2] Therefore all $c_{ij}=0$, so every $f_j=0$, hence $F \in (x^b-y^a)$. Thus

$$\ker\theta = (x^b-y^a).$$

Hence

$$k[x,y]/(x^b-y^a) \cong k[t^a,t^b] \subset k[t]$$

is a domain. So (x^b-y^a) is prime, therefore x^b-y^a is irreducible.
Therefore

$$\mathrm{im}(\varphi) = V(x^b-y^a),$$

an affine subvariety by [Definition, §2.1, p.3, Algebraic Geometry, 2026].
Bijectivity:

$$\varphi(t)=\varphi(s) \iff t^n=s^n,\ t^m=s^m.$$

For $t,s\neq 0$, set $\omega=s/t$. Then

$$\omega^n=\omega^m=1 \iff \omega^d=1.$$

So injectivity is equivalent to

$$\mu_d(k)=\{1\}.$$

In characteristic zero,

$$\mu_d(k)=\{1\} \iff d=1.$$

Hence bijective iff

$$\gcd(n,m)=1.$$

In characteristic $p>0$,

$$\mu_d(k)=\{1\} \iff d=p^e,$$

so bijective iff $\gcd(n,m)$ is a power of p.
Birational inverse (char 0, $\gcd(n,m)=1$): Here we invoke the Chapter 7 framework:

[Definition, §7, p.22, Algebraic Geometry, 2026]: rational maps/functions and $k(V)$,

[Definition, §7, p.25, Algebraic Geometry, 2026]: birational equivalence via rational inverses.

Choose $\alpha,\beta\in\mathbb{Z}$ with

$$\alpha n+\beta m=1.$$

Define

$$\psi\colon V(x^b-y^a) \dashrightarrow \mathbb{A}^1, \qquad \psi(x,y)=x^\alpha y^\beta,$$

(here $a=n,\ b=m$ because $\gcd(n,m)=1$; negative exponents mean division). Then

$$\psi(\varphi(t)) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n+\beta m}=t,$$

In function fields,

$$\psi^*(t)=x^\alpha y^\beta, \qquad \varphi^*(x)=t^n,\ \varphi^*(y)=t^m,$$

so

$$(\varphi^*\circ\psi^*)(t) = (t^n)^\alpha (t^m)^\beta = t^{\alpha n+\beta m}=t.$$

[2] This uniqueness of $ai+bj$ for $0\le j<a$ is the only arithmetic input.

Also

$$(\psi^* \circ \varphi^*)(x) = x, \qquad (\psi^* \circ \varphi^*)(y) = y,$$

so

$$\psi^* \circ \varphi^* = \mathrm{id}_{k(V(x^b - y^a))}.$$

Hence ψ, φ are inverse rational maps ([Definition, §7, p.25, Algebraic Geometry, 2026]). □

Exercise 4.

(a) Let $S = V(x^2 + y^2 - 1)$ be the circle and $H = V(wz - 1)$ be the hyperbola. Show that either $S \cong H$ or $S \cong \mathbb{A}^1$.

Solution. **Case** $\mathrm{char}\, k \neq 2$**:** Since k is algebraically closed, choose $i \in k$ with $i^2 = -1$. Define

$$\Phi \colon S \to H, \qquad (x, y) \mapsto (x + iy,\, x - iy).$$

On S: $(x + iy)(x - iy) = x^2 + y^2 = 1$, so Φ maps into H. The inverse is

$$\Phi^{-1} \colon H \to S, \qquad (w, z) \mapsto \left(\frac{w + z}{2}, \frac{w - z}{2i} \right),$$

which is well-defined since $\mathrm{char}\, k \neq 2$. One verifies

$$\Phi^{-1} \circ \Phi = \mathrm{id}_S, \qquad \Phi \circ \Phi^{-1} = \mathrm{id}_H.$$

Both maps are morphisms, so $S \cong H$.

Case $\mathrm{char}\, k = 2$**:** In characteristic 2,

$$x^2 + y^2 - 1 = (x + y)^2 + 1 = (x + y + 1)^2,$$

since $(x + y + 1)^2 = x^2 + y^2 + 1$ and $-1 = 1$. Therefore

$$S = V((x + y + 1)^2) = V(x + y + 1),$$

which is a line in $\mathbb{A}^2$. The map

$$t \mapsto (t, t + 1)$$

is an isomorphism $\mathbb{A}^1 \xrightarrow{\sim} S$ with inverse $(x, y) \mapsto x$. So $S \cong \mathbb{A}^1$. □

(b) Let k have characteristic p. Show that

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^1, \qquad t \mapsto t^p$$

is a bijection, and show that φ is not a birational equivalence.

Solution. **Bijection:**

Surjectivity: For any $a \in k$, the polynomial $t^p - a$ has a root in k, since k is algebraically closed.

Injectivity: If $s^p = t^p$, then

$$(s-t)^p = s^p - t^p = 0,$$

because $\binom{p}{i} = 0$ in characteristic p for $0 < i < p$. Hence $s = t$.

Not a birational equivalence: By [Corollary 7.3, §7, p.26, Algebraic Geometry, 2026], irreducible affine varieties are birational iff their function fields are k-isomorphic. Here

$$\varphi^* \colon k(t) \to k(t), \qquad f(t) \mapsto f(t^p),$$

and

$$\operatorname{im}(\varphi^*) = k(t^p) \subsetneq k(t).$$

Put $s := t^p$. In $k(s)[X]$, consider

$$F(X) := X^p - s.$$

By Eisenstein in $k[s][X]$ with prime element s, F is irreducible in $k(s)[X]$. Hence the minimal polynomial of t over $k(t^p) = k(s)$ has degree p. Moreover

$$[k(t) : k(t^p)] = p > 1,$$

so φ^* is not surjective. Therefore φ^* is not an isomorphism, hence φ is not birational. □

Exercise 5.

Consider the cubic curve $C := V(y^2 - x^3 - x) \subseteq \mathbb{A}^2$.

(a) Prove that C is irreducible.

Solution. View $F = y^2 - x^3 - x$ as an element of $k[x][y]$. If F were reducible in $k[x, y]$, then since $\deg_y F = 2$, it would factor as

$$F = (y - p(x))(y - q(x))$$

for some $p, q \in k[x]$, giving

$$p + q = 0, \qquad pq = -(x^3 + x).$$

Thus $q = -p$ and $p^2 = x^3 + x$. But $\deg(p^2) = 2\deg p$ is even, while $\deg(x^3 + x) = 3$ is odd. Contradiction. Therefore F is irreducible. □

(b) Find the domain of definition of the rational map

$$\varphi\colon C \dashrightarrow \mathbb{A}^1, \qquad \varphi(x,y) = x/y.$$

Solution. On C, $y^2 = x^3 + x = x(x^2+1)$, hence

$$\frac{x}{y} = \frac{x}{y}\cdot\frac{y}{y} = \frac{xy}{y^2} = \frac{xy}{x(x^2+1)} = \frac{y}{x^2+1}.$$

So φ is regular on

$$C \setminus V(y) \quad \text{and} \quad C \setminus V(x^2+1),$$

hence on

$$C \setminus V(y, x^2+1).$$

Let $P \in C$ with

$$y(P) = 0, \qquad x(P)^2 + 1 = 0.$$

Let $\mathcal{O}_{C,P}$ be the local ring. Since $x(P)^2 = -1$, we have $x(P) \neq 0$, so x is a unit in $\mathcal{O}_{C,P}$.[3] If φ were regular at P, then

$$\frac{x}{y} \in \mathcal{O}_{C,P},$$

so there exists $r \in \mathcal{O}_{C,P}$ with

$$x = yr.$$

But x is a unit, hence y would be a unit (product of two elements equals a unit).[4] This is impossible because $y(P) = 0$, i.e. $y \in \mathfrak{m}_P$. Therefore φ is not regular at P.

Thus

$$\mathrm{Dom}(\varphi) = C \setminus V(y, x^2+1).$$

Explicitly:

$$\begin{cases} (i,0),\ (-i,0), & \operatorname{char} k \neq 2, \\ (1,0), & \operatorname{char} k = 2. \end{cases}$$

□

(c) Now consider $C' := V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2$. The same formula defines a rational map $\varphi : C' \dashrightarrow \mathbb{A}^1$. Find a dominant rational map $\psi : \mathbb{A}^1 \dashrightarrow C'$ with

$$\varphi\psi = \mathrm{id}_{\mathbb{A}^1}.$$

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[3] In a local ring, an element is a unit iff its value at P is nonzero.

[4] If ab is a unit, then both a and b are units.

Solution. Let
$$\psi(t) = (a(t), b(t)), \qquad \frac{a}{b} = t, \qquad a = tb.$$
Since $C' : y^2 = x^3 + x^2$,
$$\begin{aligned} b^2 &= a^3 + a^2 \\ &= (tb)^3 + (tb)^2 \\ &= t^3b^3 + t^2b^2. \end{aligned}$$
Hence
$$b^2(1 - t^2) = t^3b^3.$$
Choose
$$b = \frac{1 - t^2}{t^3}, \qquad a = tb = \frac{1 - t^2}{t^2}.$$
Define the rational map
$$\psi\colon \mathbb{A}^1 \dashrightarrow C', \qquad t \mapsto \left(\frac{1 - t^2}{t^2}, \frac{1 - t^2}{t^3}\right).$$

Verification: $\varphi(\psi(t)) = \dfrac{(1 - t^2)/t^2}{(1 - t^2)/t^3} = t.$ ✓ $\psi(t) \in C'$:
$$\begin{aligned} y^2 - x^3 - x^2 &= \frac{(1 - t^2)^2}{t^6} - \frac{(1 - t^2)^3}{t^6} - \frac{(1 - t^2)^2}{t^4} \\ &= \frac{(1 - t^2)^2\big(1 - (1 - t^2) - t^2\big)}{t^6} \\ &= 0. \end{aligned}$$

✓

Also, C' is irreducible: if $y^2 - x^3 - x^2$ were reducible in $k[x, y]$, then
$$y^2 - x^3 - x^2 = (y - p(x))(y - q(x))$$
for some $p, q \in k[x]$, so $q = -p$ and
$$p^2 = x^3 + x^2.$$
But $\deg(p^2)$ is even, whereas $\deg(x^3 + x^2) = 3$ is odd, contradiction.
Dominance: For $y \neq 0$, set
$$t = \frac{x}{y}.$$
Then
$$\psi(t) = (x, y).$$
So
$$C' \setminus V(y) \subseteq \operatorname{im}(\psi).$$
Since $C' \setminus V(y)$ is dense in irreducible C', ψ is dominant. □