1 Lecture 2

A **quasi-projective variety** is a subset of projective space that

Example 1.1

$\mathbb{A}^n$ is birational to $\mathbb{P}^n$ via the map

$$(x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$$

with inverse

$$[x_0 : x_1 : \ldots : x_n] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

(not defined when $x_0 = 0$). check these are rational inverses

Lemma 1.1

Two irreducible quasi-projective varieties V, W are birational if and only if there are open subsets $A \subset V$ and $B \subset W$ such that A and B are isomorphic (as quasi-projectivevarieties).

Proof. Given $f : A \to B$ with inverse $g : B \to A$ isomorphisms, we can extend to rational maps $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ by defining them to be undefined outside of A and B respectively. These are clearly rational inverses. Open nonempty subsets of irreducible varieties are dense, so these are indeed birational maps.

$$A_1 := dom(\varphi) \subset V$$

$$B_1 := dom(\psi) \subset W$$

$$A = \varphi^{-1}|_{A_1}(B_1) \subset A_1$$

$$B = \psi^{-1}|_{B_1}(A_1) \subset B_1$$

□

If V be a quasi-projective variety, a **rational function** on V is a rational map from V to $\mathbb{A}^1$. The set of all rational functions on V is denoted $K(V)$, called the **function field** of V.
There is the same as a rational map $\phi : V \dashrightarrow \mathbb{P}^1$