HipSpec

Automating Inductive Proofs of Program Properties

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What is HipSpec?

Haskell source

```
rev [] = []
rev (x:xs)
  = rev xs ++ [x]

prop_rev xs
  = rev (rev xs) =:= xs
```

What is HipSpec?

Hip

Haskell Inductive Prover

- FOL translation
- Apply induction
- Success

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QuickSpec

Eq-theory from testing:

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  = rev ys ++ rev xs
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xs ++ (ys ++ zs) =
  (xs ++ ys) ++ zs
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  (xs ++ ys) ++ zs
```

HipSpec

Use these as lemmas!!

Example functional program and property

```
rev (x:xs) = rev xs ++ [x]
rev []     = []

qrev (x:xs) ys = qrev xs (x:ys)
qrev []     ys = ys
```

Goal: $\forall \texttt{xs}. \texttt{rev xs} = \texttt{qrev xs []}$

Structural induction

$$\frac{P(\texttt{[]}) \qquad \forall\, x, xs.\, P(xs) \implies P(x : xs)}{\forall\, xs.\, P(xs)}$$

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hypothesis: `rev xs` = `qrev xs []`

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Stuck!!!

How do we proceed?

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rev (x:xs) = rev xs ++ [x]
rev []     = []

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What about…

New Goal: $\forall$ xs, ys . `rev xs ++ ys` = `qrev xs ys`

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lhs: `rev (x:xs)` $=$ `rev xs ++ [x]` $=$ `qrev xs [] ++ [x]`

rhs: `qrev (x:xs) []` $=$ `qrev xs [x]`

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Then with `ys` $=$ `[]`, we get

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lhs: $\texttt{rev (x:xs)} = \texttt{rev xs ++ [x]} = \texttt{qrev xs [] ++ [x]}$

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What about...

New Goal: $\forall\, \texttt{xs}, \texttt{ys}\,.\, \texttt{rev xs ++ ys} = \texttt{qrev xs ys}$

Then with $\texttt{ys} = \texttt{[]}$, we get

$$\texttt{rev xs} = \texttt{rev xs ++ []} = \texttt{qrev xs []}$$

The crucial lemma

New Goal: $\forall\, \texttt{xs}, \texttt{ys}.\ \texttt{rev xs ++ ys} = \texttt{qrev xs ys}$

Induction on xs

The crucial lemma

$$\text{New Goal:} \quad \forall\, \mathtt{xs}, \mathtt{ys}.\, \mathtt{rev\ xs\ ++\ ys} = \mathtt{qrev\ xs\ ys}$$

Induction on xs

Base, to show: $\forall\, \mathtt{ys}.\, \mathtt{rev\ []\ ++\ ys} = \mathtt{qrev\ xs\ []}$

$$\begin{aligned}\text{lhs:} &\quad \mathtt{rev\ []\ ++\ ys} = \mathtt{ys} \\ \text{rhs:} &\quad \mathtt{qrev\ []\ ys} = \mathtt{ys}\end{aligned}$$

The crucial lemma

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Induction on $\mathtt{xs}$

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Hooray!

The crucial lemma

Step, to show: $\forall\, ys.$ `rev (x:xs) ++ ys` $=$ `qrev (x:xs) ys`

Hypothesis: $\forall\, ys.$ `rev xs ++ ys` $=$ `qrev xs ys`

The crucial lemma

Step, to show: $\forall\, \texttt{ys}.\, \texttt{rev (x:xs) ++ ys} = \texttt{qrev (x:xs) ys}$

Hypothesis: $\forall\, \texttt{ys}.\, \texttt{rev xs ++ ys} = \texttt{qrev xs ys}$

$$
\begin{array}{lll}
\text{lhs} & = \texttt{rev (x:xs) ++ ys} & = \{\text{definition of rev}\} \\
 & \texttt{(rev xs ++ [x]) ++ ys} & = \{\text{associativity of ++}\} \\
 & \texttt{rev xs ++ ([x] ++ ys)} & = \{\text{definition of ++}\} \\
 & \texttt{rev xs ++ (x:ys)} & = \{\text{induction hypothesis on (x:ys)}\} \\
 & \texttt{qrev xs (x:ys)} & = \{\text{definition of qrev}\} \\
 & \texttt{qrev (x:xs) ys} & = \text{rhs}
\end{array}
$$

The crucial lemma

Step, to show: $\forall\, ys.\, \texttt{rev (x:xs) ++ ys} = \texttt{qrev (x:xs) ys}$

Hypothesis: $\forall\, ys.\, \texttt{rev xs ++ ys} = \texttt{qrev xs ys}$

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& \texttt{qrev (x:xs) ys} & = \text{rhs}
\end{array}
$$

HOORAY!

Success

We managed to prove

$$\forall \mathtt{xs}.\, \mathtt{rev\ xs} = \mathtt{qrev\ xs\ []}$$

Using:

- $\forall \mathtt{xs}, \mathtt{ys}.\, \mathtt{rev\ xs\ ++\ ys} = \mathtt{qrev\ xs\ ys}$
- Induction

Success

We managed to prove

$$\forall \mathtt{xs}.\, \mathtt{rev\ xs} = \mathtt{qrev\ xs\ []}$$

Using:

- $\forall \mathtt{xs}, \mathtt{ys}.\, \mathtt{rev\ xs\ ++\ ys} = \mathtt{qrev\ xs\ ys}$
- Induction
- But we also needed associativity of ++ and $\forall \mathtt{xs}.\, \mathtt{xs\ ++\ []} = \mathtt{xs}$, which need induction to be proved

Setting

Prove properties of functional programs using rewriting and induction.

Problem: Some of these properties require lemmas, that

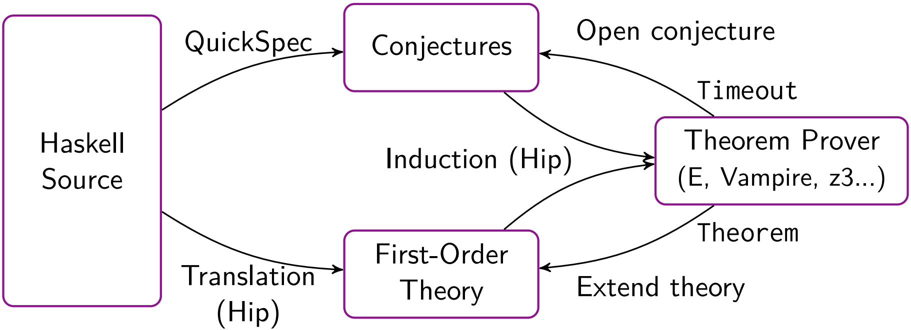
- Needs to be conjectured,
- Requires induction to be proved, and
- Might require lemmas themselves

Enter HipSpec

Solves this problems by:

- Generates an equational theory by counter-example testing,
- Try to prove this theory by applying induction
- Then, try to prove the user-stated properties
- Proof search with first-order theorem provers

Overview of HipSpec



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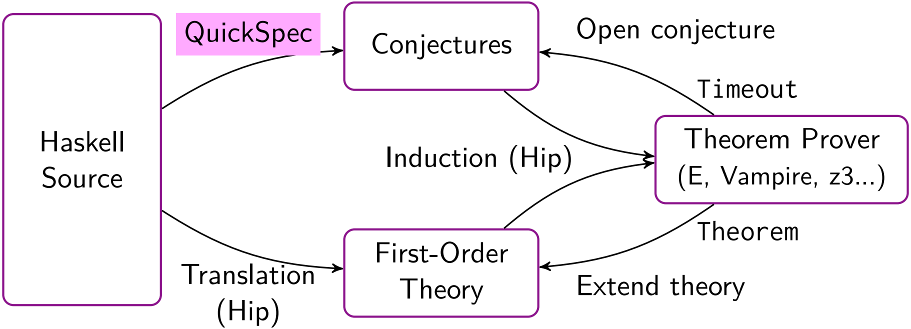

Equivalence classes partitioning

Generates a bunch of terms:

`[]`	`[]++[]`	`qrev [] []`	`qrev (rev xs) []`
`qrev [] (rev xs)`	`qrev (rev xs) ys`	`qrev [] xs`	`qrev xs []`
`[]++qrev xs ys`	`qrev [] (xs++ys)`	`(x:xs)++[]`	`qrev xs ys++[]`
`qrev (x:[]) xs`	`qrev [] (x:xs)`	`rev []`	`rev (qrev ys xs)`
`rev (rev xs)`	`[]++rev xs`	`rev xs`	`rev xs++ys`
`xs`	`[]++xs`	`xs++[]`	`(xs++ys)++[]`
`[]++(xs++ys)`	`xs++ys`	`(x:[])++xs`	`xs++(x:[])`

Equivalence classes partitioning

```
xs
xs++[]
[]++xs
qrev [] xs
rev (rev xs)
qrev (rev xs) []
```

```
[]
rev []
qrev [] []
[]++[]
```

```
qrev xs ys
rev (qrev ys xs)
rev xs++ys
[]++qrev xs ys
qrev [] (qrev xs ys)
qrev xs ys++[]
qrev (qrev ys xs) []
```

```
xs++ys
qrev (rev xs) ys
[]++(xs++ys)
qrev [] (xs++ys)
(xs++ys)++[]
```

```
x:xs
[]++(x:xs)
qrev [] (x:xs)
(x:xs)++[]
(x:[])++xs
qrev (x:[]) xs
```

```
rev xs
qrev xs []
[]++rev xs
qrev [] (rev xs)
```

Example of pruned equations from QuickSpec

```
Universe has 2893 terms, 1824 classes
== equations ==
 1: xs++[] == xs
 2: qrev xs [] == rev xs
 3: []++xs == xs
 4: qrev [] xs == xs
 5: (x:xs)++ys == x:(xs++ys)
 6: (xs++ys)++zs == xs++(ys++zs)
 7: qrev xs ys++zs == qrev xs (ys++zs)
 8: qrev (x:xs) ys == qrev xs (x:ys)
 9: qrev (xs++ys) zs == qrev ys (qrev xs zs)
10: qrev (qrev xs ys) zs == qrev ys (xs++zs)
```

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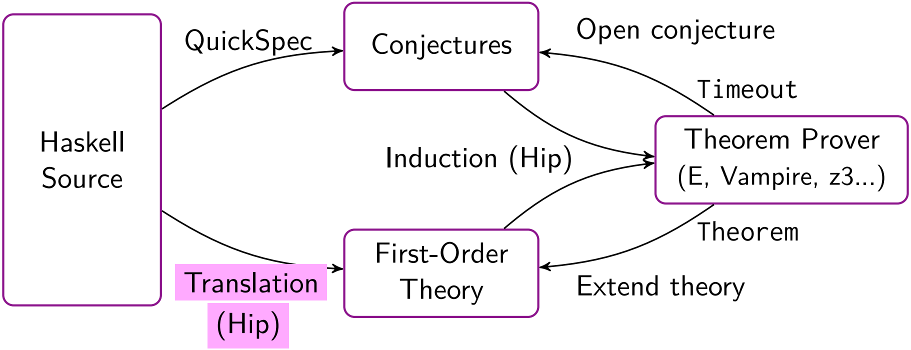


Hip : The Haskell Inductive Prover

- Translates the Haskell source definitions to first order logic

```
rev (x:xs) = rev xs ++ [x]
rev []     = []
```

Function definition axioms:

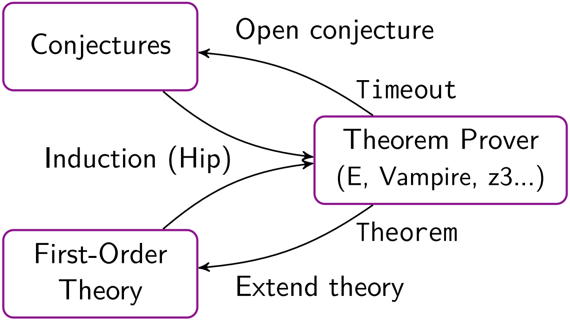
$$
\begin{array}{ll}
1 & \forall \mathsf{x}, \mathsf{xs}.\, \mathrm{rev}(\mathrm{cons}(\mathsf{x}, \mathsf{xs})) = \mathrm{append}(\mathrm{rev}(\mathsf{xs}), \mathrm{cons}(\mathsf{x}, \mathrm{nil})) \\
2 & \mathrm{rev}(\mathrm{nil}) = \mathrm{nil}
\end{array}
$$

Data type axioms:

$$
\begin{array}{ll}
3 & \forall \mathsf{x}, \mathsf{xs}, \mathsf{y}, \mathsf{ys}.\, \mathrm{cons}(\mathsf{x}, \mathsf{xs}) = \mathrm{cons}(\mathsf{y}, \mathsf{ys}) \implies \mathsf{x} = \mathsf{y} \wedge \mathsf{xs} = \mathsf{ys} \\
4 & \forall \mathsf{x}, \mathsf{xs}.\, \mathrm{nil} \neq \mathrm{cons}(\mathsf{x}, \mathsf{xs})
\end{array}
$$

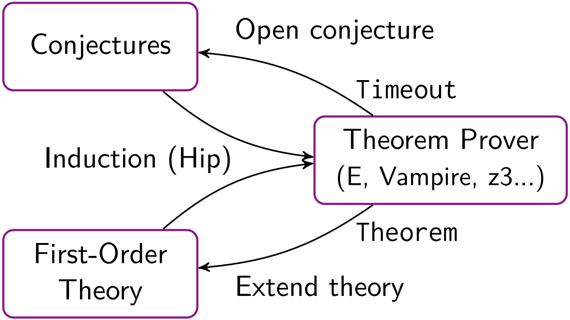
- Also supports higher-order functions and partial application
- Applies structural induction on properties

Picking a conjecture, and the main loop



1. Try to prove "smallest" unproved equation this round
2. Failure: save this for next round
3. Success: extend the theory
4. When a round did not lead to any successes, or everything proved, terminate.

Picking a conjecture, and the main loop



1. Try to prove "smallest" unproved equation this round
2. Failure: save this for next round
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We use light-weight reasoning by means of a congruence closure to prune away conjecture that can be proved without induction.

Demo!

Evaluation - First Test Suite

First suite from *Case-Analysis for Rippling and Inductive Proof* by Johansson, Dixon and Bundy (2010)

85 conjectures, 71 equational.

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Zeno	82
ACL2s	74
IsaPlanner	47
Dafny	45
HipSpec	67 (of 71)

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Unproved:

```
count n xs = count n (sort xs),   len (filter p xs) <= len xs
sorted (sort xs) = True,          len (delete n xs) <= len xs
```

But they require conditional lemmas!

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Two properties only proved by HipSpec!

```
rev (drop i xs) = take (len xs - i) (rev xs)
rev (take i xs) = drop (len xs - i) (rev xs)
```

Requires a bunch of quite far-fetched lemmas

Evaluation - Second Test Suite

Second test suite from *Productive Use of Failure in Inductive Proof* by Bundy and Ireland (1995)

Their tool CLAM supposedly proves all, but some properties contrived towards their tool, cf `rev (rev xs ++ []) = xs`

49 theorems, 38 equational.

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Unproved:

No	Conjecture
T14	`ordered (isort xs) = True`
T50	`count x (isort xs) = count x xs`

Zeno?

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Zeno? Proves 21/49

Success!

Success!?

There might be some limitations... ;)

Future work and current limitations

- Better heuristics (Equation order)
- Big theories and scalability
- Conditional properties
- Non-terminating programs and infinite values

Conclusion

Exploring the laws that hold through testing does not only help your understanding, but also helps to prove properties.

A form of completeness from QuickSpec: If there are laws up to a certain term size then QuickSpec is guaranteed to find them.

If the lemma is there, HipSpec will eventually try to prove it!

Extra slides

Obtaining HipSpec

- Clone the repository:
  `git clone http://github.com/danr/hipspec`
- Installation (requires GHC):
  ```
  cd hipspec
  git submodule update --init
  cabal install
  ```
- Install a theorem prover (say eprover)
- Try an example!
  ```
  cd testsuite/
  runghc Reverse.hs
  ```

Future work: Big theories

Taking all your functions from a big program:

- Testing takes a long time
- Lemmas become unrelated

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How do we know when functions are related?

$$\texttt{length (xs ++ ys)} = \texttt{length (ys ++ xs)}$$

Future work: Conditional properties

Lemmas with implications:

$$\texttt{sorted xs} = \texttt{True} \implies \texttt{sorted (insert x xs)} = \texttt{True}$$

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A trick: use a new data type, abstract for HipSpec:

```
data SortedList = SortedList { getSortedList :: [Nat] }

instance Arbitary SortedList where
  arbitrary = SortedList . scanl1 (+) `fmap` arbitrary
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```

```
instance Arbitary SortedList where
  arbitrary = SortedList . scanl1 (+) `fmap` arbitrary
```

Now, we can state the property in terms of a sorted list `sl`:

$$\texttt{sorted (insert x (getSortedList sl))} = \texttt{True}$$

Need a notation to HipSpec that `SortedList` has a sorted invariant.

Proof: `rev (drop i xs) = take (len xs-i) (rev xs)`

No	Conjecture
1	`len (drop x xs) = len xs-x`
2	`len xs = len (rev xs)`
3	`xs = take x xs++drop x xs`
4	`rev (ys++xs) = rev xs++rev ys`
5	`xs = take (len xs) (xs++ys)`

```
rev (drop i xs)                                                   = {5}
take (len (rev (drop i xs))) (rev (drop i xs)++rev (take i xs))   = {2}
take (len (drop i xs)) (rev (drop i xs)++rev (take i xs))         = {1}
take (len xs-i) (rev (drop i xs)++rev (take i xs))                = {4}
take (len xs-i) (rev (take i xs++drop i xs))                      = {3}
take (len xs-i) (rev xs)
```

Future work: Conditional properties II

What about

$$\mathsf{x} \ < \ \mathsf{y} = \mathsf{True} \wedge \mathsf{y} \ < \ \mathsf{z} = \mathsf{True} \implies \mathsf{x} \ < \ \mathsf{z} = \mathsf{True}$$

Future work: Conditional properties II

What about

$$\mathtt{x} < \mathtt{y} = \mathtt{True} \land \mathtt{y} < \mathtt{z} = \mathtt{True} \implies \mathtt{x} < \mathtt{z} = \mathtt{True}$$

Same trick?

```
data Pair = Pair { smaller :: Nat , larger :: Nat }
```

Can we state the property?

$$\mathtt{smaller\ p1} < \mathtt{larger\ p2} = \mathtt{True}$$

Future work: Conditional properties II

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Problem: how are p1 and p2 related?

Limitation: Expensive calculations

Imagine a program which does exponentiation, `**`, on unary nats
`data Nat = Zero | Succ Nat`

Too expensive to caluclate `x ** (y ** z)`.

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```
f x y = if y == 1000000000000
          then 0
          else x + y
```