HipSpec

Automating Inductive Proofs using Theory Exploration

Dan Rosén

Koen Claessen, Moa Johansson, Nicholas Smallbone

Chalmers University of Technology

May 31, 2013

Setting

- Automating proofs by induction of functional programs
- Terminating subset of Haskell

Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Z     xs     = xs
rotate (S n) []     = []
rotate (S n) (x:xs) = rotate n (xs ++ [x])
```

```
rotate 1 [1,2,3,4] = [2,3,4,1]
rotate 2 [1,2,3,4] = [3,4,1,2]
rotate 3 [1,2,3,4] = [4,1,2,3]
rotate 4 [1,2,3,4] = [1,2,3,4]
```

Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Z     xs     = xs
rotate (S n) []     = []
rotate (S n) (x:xs) = rotate n (xs ++ [x])
```

$$\forall \mathtt{xs}.\mathtt{rotate\ (length\ xs)\ xs} = \mathtt{xs}$$

```
rotate 1 [1,2,3,4] = [2,3,4,1]
rotate 2 [1,2,3,4] = [3,4,1,2]
rotate 3 [1,2,3,4] = [4,1,2,3]
rotate 4 [1,2,3,4] = [1,2,3,4]
```

Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Z     xs     = xs
rotate (S n) []     = []
rotate (S n) (x:xs) = rotate n (xs ++ [x])
```

$$\forall \texttt{xs}. \texttt{rotate (length xs) xs} = \texttt{xs}$$

hypothesis:

$$\texttt{rotate (length as) as} = \texttt{as}$$

conclusion:

$$\begin{aligned}
&\texttt{rotate (length (a:as)) (a:as)} &= \\
&\texttt{rotate (S (length as)) (a:as)} &= \\
&\texttt{rotate (length as) (as ++ [a])} &=
\end{aligned}$$

Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Z     xs     = xs
rotate (S n) []     = []
rotate (S n) (x:xs) = rotate n (xs ++ [x])
```

$$\forall \texttt{xs}.\texttt{rotate (length xs) xs} = \texttt{xs}$$

hypothesis:

$$\texttt{rotate (length as) as} = \texttt{as}$$

conclusion:

$$\begin{aligned}
&\texttt{rotate (length (a:as)) (a:as)} =\\
&\texttt{rotate (S (length as)) (a:as)} =\\
&\texttt{rotate (length as) (as ++ [a])} =
\end{aligned}$$

Stuck!

Rotate-length Generalisation

$$\forall \texttt{xs}, \texttt{ys}.\, \texttt{rotate (length xs) (xs ++ ys)} = \texttt{ys ++ xs}$$

Rotate-length Generalisation

$$\forall xs, ys.\, \texttt{rotate (length xs) (xs ++ ys)} = \texttt{ys ++ xs}$$

conclusion:

$$\begin{aligned}
&\texttt{rotate (length (a:as)) (a:as ++ bs)} &&=\\
&\texttt{rotate (S (length as)) (a:as ++ bs)} &&=\\
&\texttt{rotate (length as) (as ++ bs ++ [a])} &&=\{\mathrm{IH}\}\\
&\texttt{bs ++ [a] ++ as} &&=\\
&\texttt{bs ++ (a:as)} &&=
\end{aligned}$$

hypothesis:

$$\forall ys.\, \texttt{rotate (length as) (as ++ ys)} = \texttt{ys ++ as}$$

Rotate-length Generalisation

$$\forall \mathtt{xs}, \mathtt{ys}.\, \mathtt{rotate\ (length\ xs)\ (xs\ ++\ ys)} = \mathtt{ys\ ++\ xs}$$

conclusion:

$$\begin{array}{ll}
\mathtt{rotate\ (length\ (a:as))\ (a:as\ ++\ bs)} & = \\
\mathtt{rotate\ (S\ (length\ as))\ (a:as\ ++\ bs)} & = \\
\mathtt{rotate\ (length\ as)\ (as\ ++\ bs\ ++\ [a])} & = \{\mathsf{IH}\} \\
\mathtt{bs\ ++\ [a]\ ++\ as} & = \\
\mathtt{bs\ ++\ (a:as)} & =
\end{array}$$

hypothesis:

$$\forall \mathtt{ys}.\, \mathtt{rotate\ (length\ as)\ (as\ ++\ ys)} = \mathtt{ys\ ++\ as}$$

Bundy, Basin, Hutter, Ireland: automated induction challenge in *Rippling: meta-level guidance for mathematical reasoning*

QuickSpec: the Theory Exploration Phase

Generates well-typed terms up to some depth:

`rot (len xs) xs`	`len xs`	`xs++(ys++ys)`
`rot n (xs++xs)`	`rot n (rot m xs)`	`rot n xs++rot n xs`
`(xs++ys)++ys`	`rot Z (xs++ys)`	`rot m (rot n xs)`
`xs`	`len (rot m xs)`	`len (rot n xs)`
`xs++ys`	`len (ys++xs)`	`len (rot o xs)`
`rot Z xs`	`len (xs++ys)`	`[]++xs`
`(xs++ys)++[]`	`xs++[]`	`rot (len ys) (ys++xs)`

Partitioning into Equivalence Classes

```
xs
xs++[]
[]++xs
rot Z xs
rot (len xs) xs
```

```
xs++(ys++ys)
(xs++ys)++ys
```

```
rot n (xs++xs)
rot n xs++rot n xs
```

```
xs++ys
[]++(xs++ys)
rot Z (xs++ys)
(xs++ys)++[]
rot (len ys) (ys++xs)
```

```
len (xs++ys)
len (ys++xs)
```

```
len xs
len (rot n xs)
```

```
rot n (rot m xs)
rot m (rot n xs)
```

Hip: The Haskell Inductive Prover

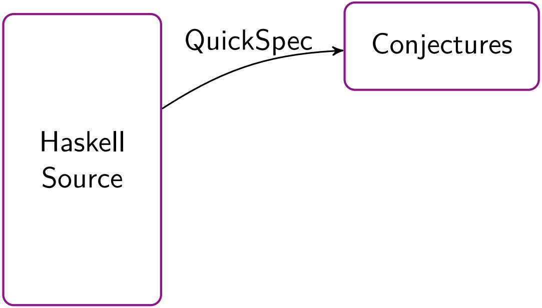
- Translate to typed first order logic
- Apply structural induction

Also supports higher-order functions and partial application

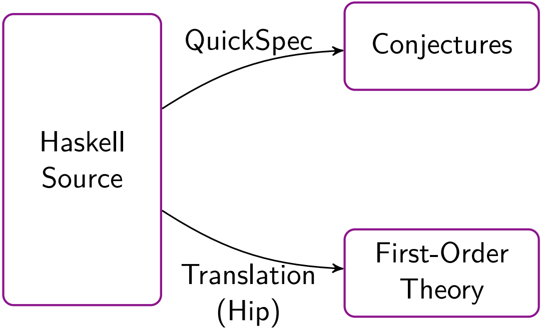
Overview of HipSpec

Haskell Source

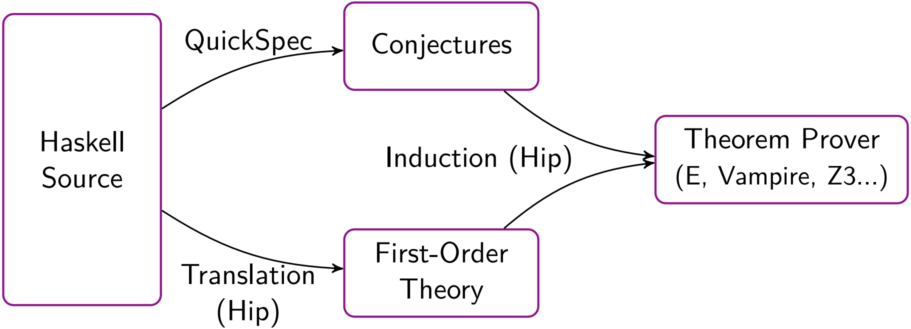
Overview of HipSpec



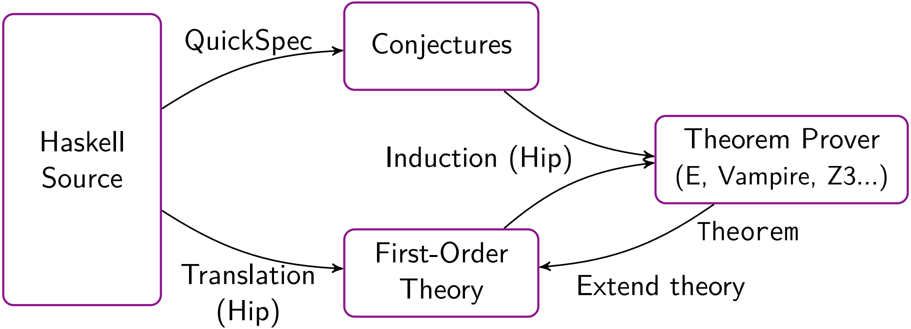
Overview of HipSpec



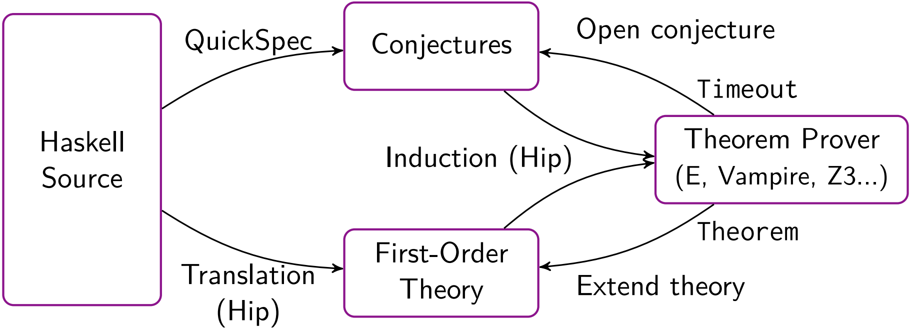
Overview of HipSpec



Overview of HipSpec

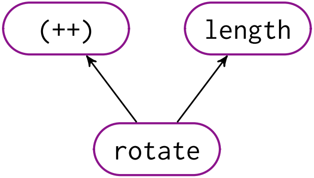


Overview of HipSpec



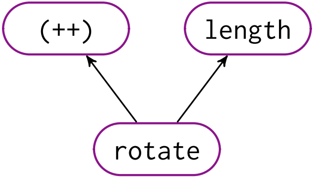
Prioritising Equations

1. Call graph



Prioritising Equations

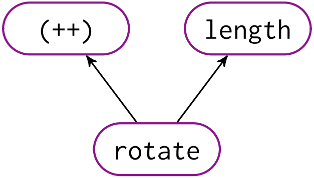
1. Call graph




```
xs++[] = xs
length (xs++ys) = length (ys++xs)
rotate (length xs) (xs ++ ys) = ys ++ xs
```

Prioritising Equations

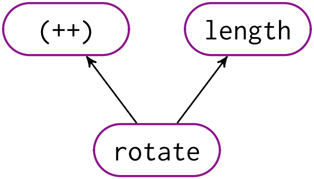
1. Call graph
2. Size of term




```
xs++[] = xs
(xs++ys)++zs = xs++(ys++zs)
```

Prioritising Equations

1. Call graph
2. Size of term
3. Number of variables




```
(xs++ys)++zs = xs++(ys++zs)
(xs++xs)++ys = xs++(xs++ys)
(xs++xs)++xs = xs++(xs++xs)
```

Evaluation Results

1st test suite from *Case-analysis for Rippling and Inductive Proof*:

#Props	HipSpec	Zeno	ACL2s	IsaPlanner	Dafny
85	80	82	74	47	45

Evaluation Results

1st test suite from *Case-analysis for Rippling and Inductive Proof*:

#Props	HipSpec	Zeno	ACL2s	IsaPlanner	Dafny
85	80	82	74	47	45

2nd test suite from *Productive use of Failure in Inductive Proof*:

#Props	HipSpec	CLAM	Zeno
50	44	41	21

Conjecturing Conditionals

$$\forall \texttt{xs}.\, \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

Conjecturing Conditionals

$$\forall\, \texttt{xs}.\, \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

Requires:

$$\forall\, \texttt{xs}.\, \texttt{sorted xs} = \texttt{True} \Rightarrow \texttt{sorted (insert x xs)} = \texttt{True}$$

$\top$ vs $\bot$

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$\top$ vs $\bot$

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$$\forall\, \mathtt{xs}.\, \mathtt{rotate\ (length\ xs)\ xs} = \mathtt{xs}$$

$$\begin{aligned}
\mathtt{rotate\ (length\ (a{:}as))\ (a{:}as)} &= \\
\mathtt{rotate\ (S\ (length\ as))\ (a{:}as)} &= \\
\mathtt{rotate\ (length\ as)\ (as\ {+}{+}\ [a])} &=
\end{aligned}$$

Stuck!

$\top$ vs $\bot$

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$$\forall\, \texttt{i}, \texttt{xs}.\, \texttt{rev (drop i xs)} = \texttt{take (length xs - i) (rev xs)}$$

$\top$ vs $\bot$

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$\forall$ `i, xs. rev (drop i xs)` $=$ `take (length xs - i) (rev xs)`

Required lemmas:

```
length (drop x xs)            = length xs - x
length (rev xs)               = length xs
take x xs ++ drop x xs        = xs
rev xs ++ rev ys              = rev (ys++xs)
take (length xs) (xs ++ ys) = xs
```

$\top$ vs $\bot$

- Top-down: Rippling/critics-based provers, ACL2, Zeno
- Bottom-up: IsaCoSy, IsaScheme, HipSpec

$$\forall\, \texttt{i}, \texttt{xs}. \texttt{rev (drop i xs)} = \texttt{take (length xs - i) (rev xs)}$$

Required lemmas:

```
length (drop x xs)            = length xs - x
length (rev xs)               = length xs
take x xs ++ drop x xs        = xs
rev xs ++ rev ys              = rev (ys++xs)
take (length xs) (xs ++ ys) = xs
```

HipSpec the Theory Exploration System

Precision/recall analysis against Isabelle standard library

HipSpec the Theory Exploration System

Precision/recall analysis against Isabelle standard library

What the other systems do in hours, HipSpec does *under a minute*!

HipSpec the Theory Exploration System

Precision/recall analysis against Isabelle standard library

What the other systems do in hours, HipSpec does *under a minute*!

- `data Integer = Positive Nat | Negative Nat`

Conclusions

- Evaluate your programs

Conclusions

- Evaluate your programs
- "Completeness" up to a certain depth

Conclusions

- Evaluate your programs
- "Completeness" up to a certain depth
- Progress in automated induction

github.com/danr/hipspec

Conditionals as Functions

$$\forall\, \texttt{xs}.\ \texttt{sorted xs} = \texttt{True} \Rightarrow \texttt{sorted (insert x xs)} = \texttt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall\, \texttt{x}, \texttt{xs}.\ \texttt{sorted (insert x (whenSorted xs))} = \texttt{True}$$

Conditionals as Functions

$$\forall\, \mathtt{xs}.\ \mathtt{sorted\ xs} = \mathtt{True} \Rightarrow \mathtt{sorted\ (insert\ x\ xs)} = \mathtt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall\, \mathtt{x}, \mathtt{xs}.\ \mathtt{sorted\ (insert\ x\ (whenSorted\ xs))} = \mathtt{True}$$

```
  sorted (insert x (whenSorted xs))
= sorted (insert x (if sorted xs then xs else []))
= if sorted xs then sorted (insert x xs)
               else sorted (insert x [])
```

What is HipSpec?

Haskell source

```
rev [] = []
rev (x:xs)
  = rev xs ++ [x]

prop_rev xs
  = rev (rev xs) =:= xs
```

Hip

Haskell Inductive Prover

- FOL translation
- Apply induction
- Success, or stuck!

QuickSpec

Eq-theory from testing:

```
rev (xs ++ ys)
  = rev ys ++ rev xs
xs ++ [] = []
xs ++ (ys ++ zs) =
  (xs ++ ys) ++ zs
```

HipSpec

Use these as lemmas!!