# HipSpec

## Automating Inductive Proofs using Theory Exploration

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# Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Zero     xs     = xs
rotate (Succ n) []     = []
rotate (Succ n) (x:xs) = rotate n (xs ++ [x])
```

```
rotate 1 [1,2,3,4] = [2,3,4,1]
rotate 2 [1,2,3,4] = [3,4,1,2]
rotate 3 [1,2,3,4] = [4,1,2,3]
rotate 4 [1,2,3,4] = [1,2,3,4]
```

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$$\forall \texttt{xs}.\, \texttt{rotate (length xs) xs} = \texttt{xs}$$

hypothesis:

$$\texttt{rotate (length as) as} = \texttt{as}$$

conclusion:

$$\begin{aligned}
&\texttt{rotate (length (a:as)) (a:as)} &=\\
&\texttt{rotate (S (length as)) (a:as)} &=\\
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$$\forall \, \texttt{xs}, \texttt{ys}. \, \texttt{rotate (length xs) (xs ++ ys)} = \texttt{ys ++ xs}$$

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&\texttt{rotate (length (a:as)) (a:as ++ bs)} &&= \\
&\texttt{rotate (S (length as)) (a:as ++ bs)} &&= \\
&\texttt{rotate (length as) (as ++ bs ++ [a])} &&=\{\text{IH}\} \\
&\texttt{bs ++ [a] ++ as} &&= \\
&\texttt{bs ++ (a:as)} &&
\end{aligned}$$

hypothesis:

$$\forall\, \texttt{ys}.\ \texttt{rotate (length as) (as ++ ys)} = \texttt{ys ++ as}$$

# Rotate-length Helper Lemma

$$\forall \mathtt{xs}, \mathtt{ys}.\, \mathtt{rotate\ (length\ xs)\ (xs\ ++\ ys)} = \mathtt{ys\ ++\ xs}$$

conclusion:

$$\begin{array}{ll}
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\mathtt{rotate\ (length\ as)\ (as\ ++\ bs\ ++\ [a])} & =\{\mathsf{IH}\} \\
\mathtt{bs\ ++\ [a]\ ++\ as} & = \\
\mathtt{bs\ ++\ (a:as)} &
\end{array}$$

hypothesis:

$$\forall \mathtt{ys}.\, \mathtt{rotate\ (length\ as)\ (as\ ++\ ys)} = \mathtt{ys\ ++\ as}$$

Bundy, Basin, Hutter, Ireland: automated induction challenge in *Rippling: meta-level guidance for mathematical reasoning*

# QuickSpec: the Theory Exploration Phase

Generates well-typed terms up to some depth:

```
rot (len xs) xs     len xs               xs++(ys++ys)
rot n (xs++xs)      rot n (rot m xs)     rot n xs++rot n xs
(xs++ys)++ys        rot Z (xs++ys)       rot m (rot n xs)
xs                  len (rot m xs)       len (rot n xs)
xs++ys              len (ys++xs)         len (rot o xs)
rot Z xs            len (xs++ys)         []++xs
(xs++ys)++[]        xs++[]               rot (len ys) (ys++xs)
```

# Partitioning into Equivalence Classes

```
xs
xs++[]
[]++xs
rot Z xs
rot (len xs) xs
```

```
xs++ys
[]++(xs++ys)
rot Z (xs++ys)
(xs++ys)++[]
rot (len ys) (ys++xs)
```

```
xs++(ys++ys)
(xs++ys)++ys
```

```
rot n (xs++xs)
rot n xs++rot n xs
```

```
len (xs++ys)
len (ys++xs)
```

```
len xs
len (rot n xs)
```

```
rot n (rot m xs)
rot m (rot n xs)
```

# Hip: The Haskell Inductive Prover

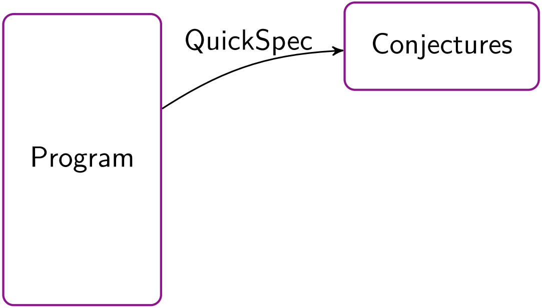
- Translate to typed first order logic
- Apply structural induction

Also supports higher-order functions and partial application

# Overview of HipSpec

Program

# Overview of HipSpec



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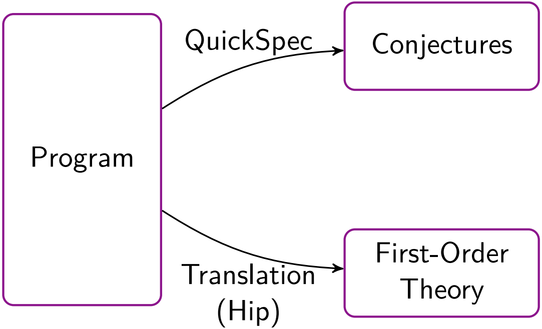

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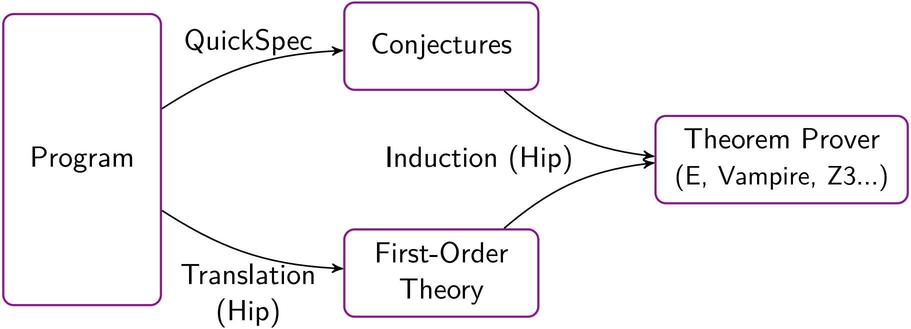

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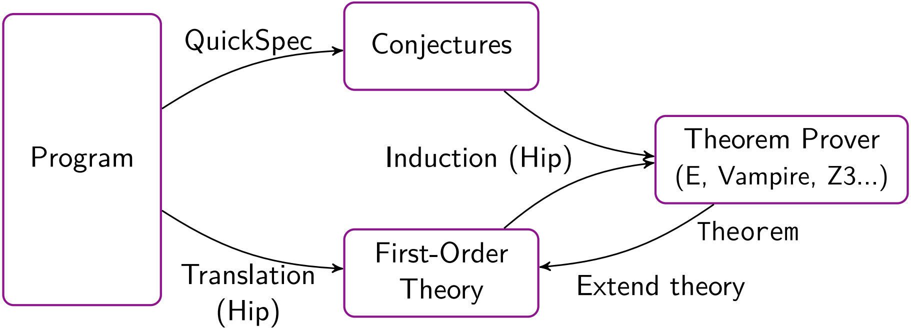

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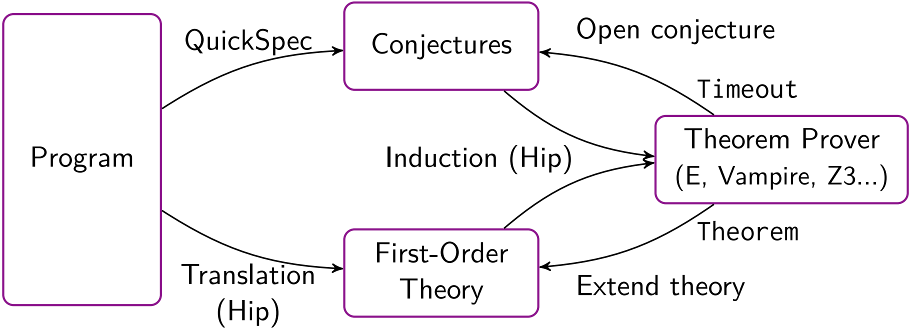
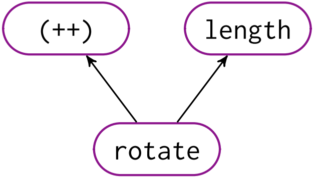


# Prioritising Equations

1. Call graph



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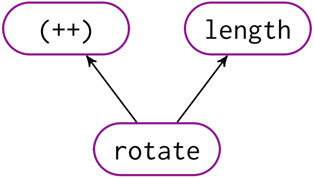


```
xs++[] = xs
length (xs++ys) = length (ys++xs)
rotate (length xs) (xs ++ ys) = ys ++ xs
```

# Prioritising Equations

1. Call graph
2. Size of term

```
xs++[] = xs
(xs++ys)++zs = xs++(ys++zs)
```

# Prioritising Equations

1. Call graph
2. Size of term
3. Number of variables

```
(xs++ys)++zs = xs++(ys++zs)
(xs++xs)++ys = xs++(xs++ys)
(xs++xs)++xs = xs++(xs++xs)
```

# Evaluation Results I

1st test suite from *Case-analysis for Rippling and Inductive Proof*:

| #Props | HipSpec | Zeno | ACL2s | IsaPlanner | Dafny |
|---|---|---|---|---|---|
| 85 | 80 | 82 | 74 | 47 | 45 |

- Quite easy: around 60 provable without lemmas
- Some require conditional lemmas
  (can't generate with QuickSpec)

# Evaluation Results II

2nd test suite from *Productive Use of Failure in Inductive Proof*:

| #Props | HipSpec | CLAM | Zeno |
|---|---|---|---|
| 50 | 44 | 41 | 21 |

- Harder test suite: need lemmas and generalisations

# Conjecturing Conditionals

$$\forall \texttt{xs}.\, \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

# Conjecturing Conditionals

$$\forall\, \texttt{xs}.\ \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

Requires:

$$\forall\, \texttt{xs}.\ \texttt{sorted xs} = \texttt{True} \Rightarrow \texttt{sorted (insert x xs)} = \texttt{True}$$

# Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

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\end{aligned}$$

Stuck!

# Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$$\forall\, \texttt{i}, \texttt{xs}. \texttt{rev (drop i xs)} = \texttt{take (length xs - i) (rev xs)}$$

# Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$\forall$ `i, xs. rev (drop i xs)` $=$ `take (length xs - i) (rev xs)`

Required lemmas:

```
length (drop x xs)          = length xs - x
length (rev xs)             = length xs
take x xs ++ drop x xs      = xs
rev xs ++ rev ys            = rev (ys++xs)
take (length xs) (xs ++ ys) = xs
```

# Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno
- Bottom-up: IsaCoSy, IsaScheme, HipSpec

$\forall$ `i, xs. rev (drop i xs)` $=$ `take (length xs - i) (rev xs)`

Required lemmas:

```
length (drop x xs)           = length xs - x
length (rev xs)              = length xs
take x xs ++ drop x xs       = xs
rev xs ++ rev ys             = rev (ys++xs)
take (length xs) (xs ++ ys) = xs
```

# Theory Exploration Results

- Produce background theory comparable to human.
- Comparison with Isabelle libraries.
- **HipSpec runs in minutes.** IsaCoSy/IsaScheme sometimes hours.

| | HipSpec | IsaCoSy | IsaScheme | Isabelle |
|---|---|---|---|---|
| **#Thms Naturals** | 10 | 16 | 16* | 12 |
| Precision | 80% | 63% | 100%* | - |
| Recall | 73% | 83% | 46%* | - |
| **#Thms Lists** | 10 | 24 | 13 | 9 |
| Precision | 90% | 38% | 70% | - |
| Recall | 100% | 100% | 100% | - |

Table : Theory Exploration results. IsaScheme was evaluated on a natural number theory also including exponentiation.

# Conclusions

- Evaluate your programs

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- Evaluate your programs
- "Completeness" up to a certain depth
- Progress in automated induction

github.com/danr/hipspec

# Conditionals as Functions

$$\forall \texttt{xs}.\, \texttt{sorted xs} = \texttt{True} \Rightarrow \texttt{sorted (insert x xs)} = \texttt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall \texttt{x}, \texttt{xs}.\, \texttt{sorted (insert x (whenSorted xs))} = \texttt{True}$$

# Conditionals as Functions

$$\forall\, \mathtt{xs}.\ \mathtt{sorted\ xs} = \mathtt{True} \Rightarrow \mathtt{sorted\ (insert\ x\ xs)} = \mathtt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall\, \mathtt{x}, \mathtt{xs}.\ \mathtt{sorted\ (insert\ x\ (whenSorted\ xs))} = \mathtt{True}$$

```
  sorted (insert x (whenSorted xs))
= sorted (insert x (if sorted xs then xs else []))
= if sorted xs then sorted (insert x xs)
               else sorted (insert x [])
```

# What is HipSpec?

**Hip**

*Haskell Inductive Prover*

- FOL translation
- Apply induction
- Success, or stuck!

**Haskell source**

```
rev [] = []
rev (x:xs)
  = rev xs ++ [x]

prop_rev xs
  = rev (rev xs) =:= xs
```

**QuickSpec**

Eq-theory from testing:

```
rev (xs ++ ys)
  = rev ys ++ rev xs
xs ++ [] = []
xs ++ (ys ++ zs) =
  (xs ++ ys) ++ zs
```

**HipSpec**

*Use these as lemmas!!*