HipSpec

Automating Inductive Proofs using Theory Exploration

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Rotate Example

```
rotate :: Nat -> [a] -> [a]
rotate Zero     xs     = xs
rotate (Succ n) []     = []
rotate (Succ n) (x:xs) = rotate n (xs ++ [x])
```

```
rotate 1 [1,2,3,4] = [2,3,4,1]
rotate 2 [1,2,3,4] = [3,4,1,2]
rotate 3 [1,2,3,4] = [4,1,2,3]
rotate 4 [1,2,3,4] = [1,2,3,4]
```

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hypothesis:

$$\texttt{rotate (length as) as} = \texttt{as}$$

conclusion:

$$\begin{aligned}
&\texttt{rotate (length (a:as)) (a:as)} &=\\
&\texttt{rotate (S (length as)) (a:as)} &=\\
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Stuck!

Rotate-length Helper Lemma

$$\forall \texttt{xs}, \texttt{ys}.\, \texttt{rotate (length xs) (xs ++ ys)} = \texttt{ys ++ xs}$$

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&\texttt{bs ++ [a] ++ as} && = \\
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Bundy, Basin, Hutter, Ireland: automated induction challenge in *Rippling: meta-level guidance for mathematical reasoning*

QuickSpec: the Theory Exploration Phase

Generates well-typed terms up to some depth:

```
rot (len xs) xs
rot n (xs++xs)
(xs++ys)++ys
xs
xs++ys
rot Z xs
(xs++ys)++[]
len xs
rot n (rot m xs)
rot Z (xs++ys)
len (rot m xs)
len (ys++xs)
len (xs++ys)
xs++[]
xs++(ys++ys)
rot n xs++rot n xs
rot m (rot n xs)
len (rot n xs)
len (rot o xs)
[]++xs
rot (len ys) (ys++xs)
```

Partitioning into Equivalence Classes

```
xs
xs++[]
[]++xs
rot Z xs
rot (len xs) xs
```

```
xs++(ys++ys)
(xs++ys)++ys
```

```
rot n (xs++xs)
rot n xs++rot n xs
```

```
xs++ys
[]++(xs++ys)
rot Z (xs++ys)
(xs++ys)++[]
rot (len ys) (ys++xs)
```

```
len (xs++ys)
len (ys++xs)
```

```
len xs
len (rot n xs)
```

```
rot n (rot m xs)
rot m (rot n xs)
```

Hip: The Haskell Inductive Prover

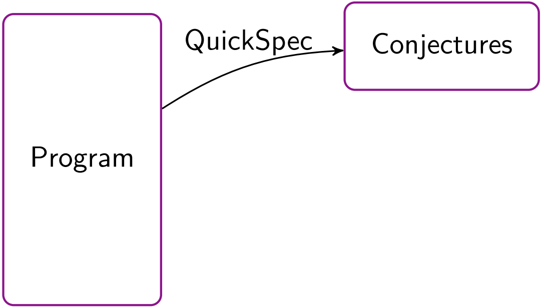
- Translate to typed first order logic
- Apply structural induction

Also supports higher-order functions and partial application

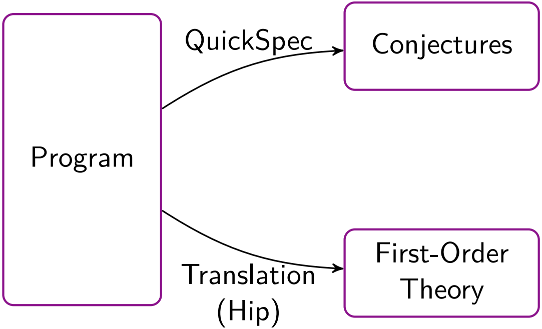
Overview of HipSpec

Program

Overview of HipSpec



Overview of HipSpec



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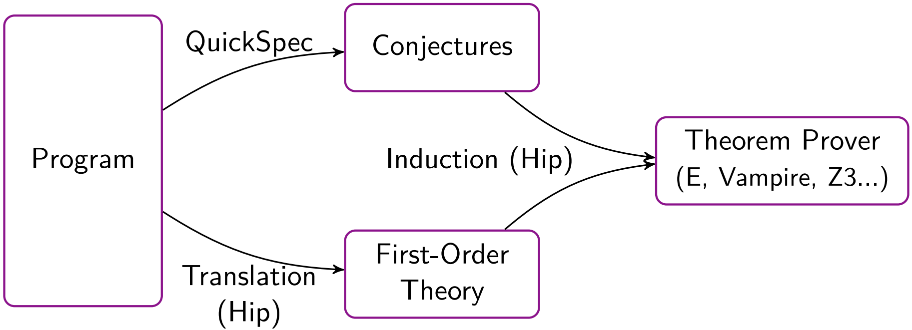

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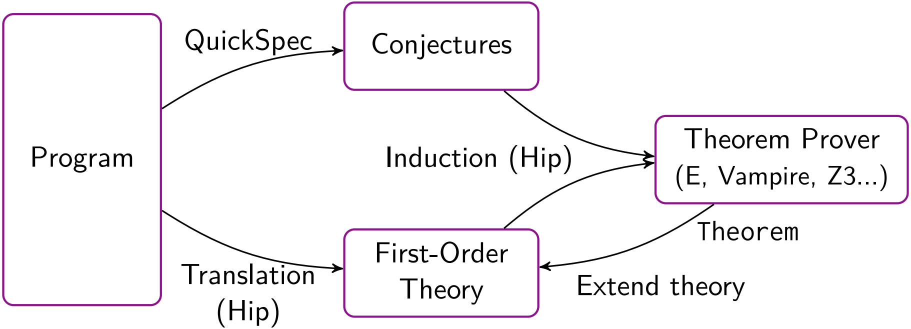

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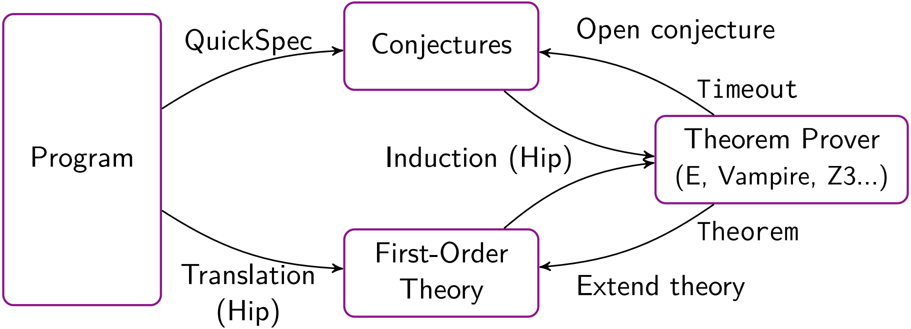
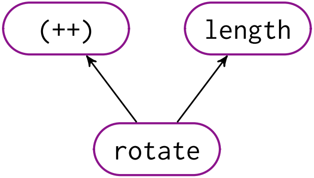


Prioritising Equations

1. Call graph



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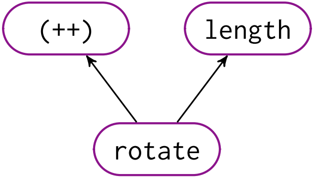


```
xs++[] = xs
length (xs++ys) = length (ys++xs)
rotate (length xs) (xs ++ ys) = ys ++ xs
```

Prioritising Equations

1. Call graph
2. Size of term

```
xs++[] = xs
(xs++ys)++zs = xs++(ys++zs)
```

Prioritising Equations

1. Call graph
2. Size of term
3. Number of variables

```
(xs++ys)++zs = xs++(ys++zs)
(xs++xs)++ys = xs++(xs++ys)
(xs++xs)++xs = xs++(xs++xs)
```

Evaluation Results I

1st test suite from *Case-analysis for Rippling and Inductive Proof*:

#Props	HipSpec	Zeno	ACL2s	IsaPlanner	Dafny
85	80	82	74	47	45

- Quite easy: around 60 provable without lemmas
- Some require conditional lemmas
  (can't generate with QuickSpec)

Evaluation Results II

2nd test suite from *Productive Use of Failure in Inductive Proof*:

#Props	HipSpec	CLAM	Zeno
50	44	41	21

- Harder test suite: need lemmas and generalisations

Conjecturing Conditionals

$$\forall \texttt{xs}.\, \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

Conjecturing Conditionals

$$\forall \texttt{xs}.\, \texttt{sorted (isort xs)} = \texttt{True}$$

```
isort :: [Nat] -> [Nat]

insert :: Nat -> [Nat] -> [Nat]

sorted :: [Nat] -> Bool
```

Requires:

$$\forall \texttt{xs}.\, \texttt{sorted xs} = \texttt{True} \Rightarrow \texttt{sorted (insert x xs)} = \texttt{True}$$

Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

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Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$$\forall\, \mathtt{i}, \mathtt{xs}.\, \mathtt{rev\ (drop\ i\ xs)} = \mathtt{take\ (length\ xs\ -\ i)\ (rev\ xs)}$$

Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno

$$\forall\, \mathtt{i}, \mathtt{xs}.\, \mathtt{rev\ (drop\ i\ xs)} = \mathtt{take\ (length\ xs\ -\ i)\ (rev\ xs)}$$

Required lemmas:

```
length (drop x xs)          = length xs - x
length (rev xs)             = length xs
take x xs ++ drop x xs      = xs
rev xs ++ rev ys            = rev (ys++xs)
take (length xs) (xs ++ ys) = xs
```

Two Approaches to Lemma Discovery

- Top-down: Rippling/critics-based provers, ACL2, Zeno
- Bottom-up: IsaCoSy, IsaScheme, HipSpec

$\forall$ `i, xs. rev (drop i xs)` $=$ `take (length xs - i) (rev xs)`

Required lemmas:

```
length (drop x xs)           = length xs - x
length (rev xs)              = length xs
take x xs ++ drop x xs       = xs
rev xs ++ rev ys             = rev (ys++xs)
take (length xs) (xs ++ ys)  = xs
```

Theory Exploration Results

- Produce background theory comparable to human.
- Comparison with Isabelle libraries.
- **HipSpec runs in minutes.** IsaCoSy/IsaScheme sometimes hours.

	HipSpec	IsaCoSy	IsaScheme	Isabelle
#Thms Naturals	10	16	16*	12
Precision	80%	63%	100%*	-
Recall	73%	83%	46%*	-
#Thms Lists	10	24	13	9
Precision	90%	38%	70%	-
Recall	100%	100%	100%	-

Table : Theory Exploration results. IsaScheme was evaluated on a natural number theory also including exponentiation.

- `data Integer = Positive Nat | Negative Nat`

Conclusions

- Evaluate your programs

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- Evaluate your programs
- "Completeness" up to a certain depth
- Progress in automated induction

github.com/danr/hipspec

Conditionals as Functions

$$\forall \mathtt{xs}.\, \mathtt{sorted\ xs} = \mathtt{True} \Rightarrow \mathtt{sorted\ (insert\ x\ xs)} = \mathtt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall \mathtt{x}, \mathtt{xs}.\, \mathtt{sorted\ (insert\ x\ (whenSorted\ xs))} = \mathtt{True}$$

Conditionals as Functions

$$\forall\, \mathtt{xs}.\, \mathtt{sorted\ xs} = \mathtt{True} \Rightarrow \mathtt{sorted\ (insert\ x\ xs)} = \mathtt{True}$$

```
whenSorted :: [Nat] -> [Nat]
whenSorted xs = if sorted xs then xs else []
```

$$\forall\, \mathtt{x}, \mathtt{xs}.\, \mathtt{sorted\ (insert\ x\ (whenSorted\ xs))} = \mathtt{True}$$

```
  sorted (insert x (whenSorted xs))
= sorted (insert x (if sorted xs then xs else []))
= if sorted xs then sorted (insert x xs)
               else sorted (insert x [])
```

What is HipSpec?

Haskell source

```
rev [] = []
rev (x:xs)
  = rev xs ++ [x]

prop_rev xs
  = rev (rev xs) =:= xs
```

Hip

Haskell Inductive Prover

- FOL translation
- Apply induction
- Success, or stuck!

QuickSpec

Eq-theory from testing:

```
rev (xs ++ ys)
  = rev ys ++ rev xs
xs ++ [] = []
xs ++ (ys ++ zs) =
  (xs ++ ys) ++ zs
```

HipSpec

Use these as lemmas!!