

Catalan Number Recursive Formula

Franklin Chen

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1 Expression

The closed form expression for the n th Catalan number is

$$\frac{\binom{2n}{n}}{n+1}$$

2 Recursive Formula

The recursive formula for the $n+1$ th Catalan number is

$$\begin{aligned}\frac{\binom{2(n+1)}{n+1}}{n+2} &= \frac{(2n+2)!}{(n+1)!(n+1)!} \frac{1}{n+2} \\ &= \frac{(2n)!(2n+1)(2n+2)}{n!n!(n+1)(n+1)(n+2)} \\ &= \frac{(2n)!}{n!n!} \frac{(2n+1)(2n+2)}{(n+2)(n+1)(n+1)} \\ &= \frac{\binom{2n}{n}}{n+1} \frac{(2n+1)(2n+2)}{(n+2)(n+1)} \\ &= C_n \frac{(2n+1)(2n+2)}{(n+2)(n+1)} \\ &= C_n \frac{(2n+1)2(n+1)}{(n+2)(n+1)} \\ &= C_n \frac{2(2n+1)}{n+2} \\ &= C_n \frac{4n+2}{n+2}\end{aligned}$$

and $C_0 = 1$.

3 Recursive Formula

The recursive formula for the n th Catalan number is

$$\begin{aligned}\frac{\binom{2n}{n}}{n} &= \frac{(2n)!}{(n)!(n)!} \frac{1}{n+1} \\ &= \frac{(2n-2)!(2n-1)(2n)}{(n-1)!(n-1)!(n)(n)(n+1)} \\ &= \frac{(2n-2)!}{(n-1)!(n-1)!} \frac{(2n-1)(2n)}{(n)(n)(n+1)} \\ &= \frac{\binom{2n-2}{n-1}}{n} \frac{(2n-1)(2n)}{(n)(n+1)} \\ &= C_{n-1} \frac{(2n-1)2n}{(n)(n+1)} \\ &= C_{n-1} \frac{2(2n-1)}{n+1} \\ &= C_{n-1} \frac{4n-2}{n+1}\end{aligned}$$

and $C_0 = 1$.