Catalan Number Recursive Formula

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1 Expression

The closed form expression for the nth Catalan number is

$$\frac{\binom{2n}{n}}{n+1}$$

2 Recursive Formula

The recursive formula for the $n+1$th Catalan number is

$$\begin{aligned}
\frac{\binom{2(n+1)}{n+1}}{n+2} &= \frac{(2n+2)!}{(n+1)!(n+1)!}\frac{1}{n+2} \\
&= \frac{(2n)!(2n+1)(2n+2)}{n!n!(n+1)(n+1)(n+2)} \\
&= \frac{(2n)!}{n!n!}\frac{(2n+1)(2n+2)}{(n+2)(n+1)(n+1)} \\
&= \frac{\binom{2n}{n}}{n+1}\frac{(2n+1)(2n+2)}{(n+2)(n+1)} \\
&= C_n\frac{(2n+1)(2n+2)}{(n+2)(n+1)} \\
&= C_n\frac{(2n+1)2(n+1)}{(n+2)(n+1)} \\
&= C_n\frac{2(2n+1)}{n+2} \\
&= C_n\frac{4n+2}{n+2}
\end{aligned}$$

and $C_0 = 1$.

3 Recursive Formula

The recursive formula for the nth Catalan number is

$$\begin{aligned}
\frac{\binom{2n}{n}}{n} &= \frac{(2n)!}{(n)!(n)!}\frac{1}{n+1} \\
&= \frac{(2n-2)!(2n-1)(2n)}{(n-1)!(n-1)!(n)(n)(n+1)} \\
&= \frac{(2n-2)!}{(n-1)!(n-1)!}\frac{(2n-1)(2n)}{(n)(n)(n+1)} \\
&= \frac{\binom{2n-2}{n-1}}{n}\frac{(2n-1)(2n)}{(n)(n+1)} \\
&= C_{n-1}\frac{(2n-1)2n}{(n)(n+1)} \\
&= C_{n-1}\frac{2(2n-1)}{n+1} \\
&= C_{n-1}\frac{4n-2}{n+1}
\end{aligned}$$

and $C_0 = 1$.