American options and Pricing in Binomial Tree

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Overview

American Option Introduction

- An American option is one whose exercise can take place not only at expiry, but also at any time before expiry.
- In other words, American option differs from the European option only in that American option can be converted into its *intrinsic value* at any time before maturity. The optimal exercise strategy is studied in Page 22.
- Most traded stock options are American type while most index options are European type.

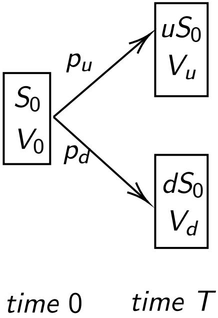
Pricing: Binomial Tree

- Risk-neutral Probability
- Cox - Ross - Rubinstein (CRR) Parameterization
- Issues and Enhancements

Binomial Tree and CRR Parameterization

- Binomial Tree
  It is a framework to model the **underlying's** behavior: from time 0 to expiry T, the underlying will have one of two possible values: either up to uS_0 with probability p_u or down to dS_0 with probability p_d, where S_0 is the underlying price at time 0.

  Thus we can associate the terminal payoff V_u and V_d with the underlying price at expiry.



Binomial Tree and CRR Parameterization

- Discount factor P_T (zero coupon bond price with tenor T) applies to cash flows occurring at T, and this quantity is not random, then we can calculate the expected present value of the option value V_0: $V_0 = P_T(p_u V_u + p_d V_d)$
- Risk-Neutral Probability: p_u, p_d

  For all models, the first step is to calibrate it: we must specify model's parameters to ensure that the prices it returns agree with known instrument values.
  For a equity forward contract, the fair strike for a forward contract is $\frac{S_0}{P_T}$, which makes the value of forward contract equals to zero at inception. We apply this zero forward contract :
  $0 = P_T[p_u(uS_0 - \frac{S_0}{P_T}) + p_d(dS_0 - \frac{S_0}{P_T})]$

- Risk-Neutral Probability: p_u, p_d

  Thus we have: $\frac{1}{P_T} = p_u u + p_d d$
  That is, our model must be constructed so that the expected return on the equity matches the risk-free return.The probabilities that satisfy this requirement are called risk-neutral probabilities.

  The risk-neutral probability is (in terms of the continuously compounded risk-free rate $e^{-rT} = P_T$):

$$p_u = \frac{e^{rT} - d}{u - d}, p_d = \frac{u - e^{rT}}{u - d} \quad (1)$$

Binomial Tree and CRR Parameterization

The probability p_u, p_d depends on u, d, which we still need to specify. There is a set of parameters (Cox - Ross - Rubinstein Parameterization) most commonly used to generate a binomial tree.

- $\Delta t = \frac{T}{N}$, where T is the time to maturity in years; N is the number of time steps.
- $u = e^{\sigma\sqrt{\Delta t}}$, $d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$

Binomial Tree and CRR Parameterization

- The main advantage of this choice is that valuation from the model converges to lognormal distribution of the asset price at time T as the number of time steps N increases. See Appendix (Page 29) for details.
- It is possible, for some choices of N, that this model may contain arbitrage. Since the single-step forward price returned by the model will of necessity lie between S_u and S_d at the next time step:

$$e^{-\sigma\sqrt{\Delta t}} < e^{(r-q)\Delta t} < e^{\sigma\sqrt{\Delta t}} \tag{2}$$

- If this inequality does not hold, then p_u or p_d would turn out negative, while the other will be greater than 1.
  Fortunately, the remedy for this situation is simple: By using a large enough N, $\sqrt{\Delta t} > \Delta t$.

Backward Pricing by Binomial Tree

- An American option can be exercised at any time before expiry. Thus, pricing requires modeling the optimal choice, at each time throughout the contract, of whether to exercise the option or not.
- This is the case where backward induction from the maturity nodes becomes particularly useful. At each node, we calculate the 1) value of non-exercise (discounted risk-neutral expectation from the nodes in next steps) and 2) value of exercise (intrinsic value). The greater value becomes the value of the option at that node, and pricing continues to previous nodes recursively.
  - Step 1: Calculate the value at maturity T;
  - Step 2: calculate 1 step before maturity: value of non-exercise: $V_t^{cont} = e^{-r\Delta t}(p_u V_{t+\Delta t} + p_d V_{t+\Delta t})$ and value of exercise: $V_t^{exe} = max(S_t - K, 0)$;
  - Step 3: Take the greater of two and as value of node: $V_t = max(V_t^{cont}, V_t^{exe})$;
  - Step 4: repeat this process to each node 1 step before until time t_0.

Pricing American Option by Binomial Tree: Sample

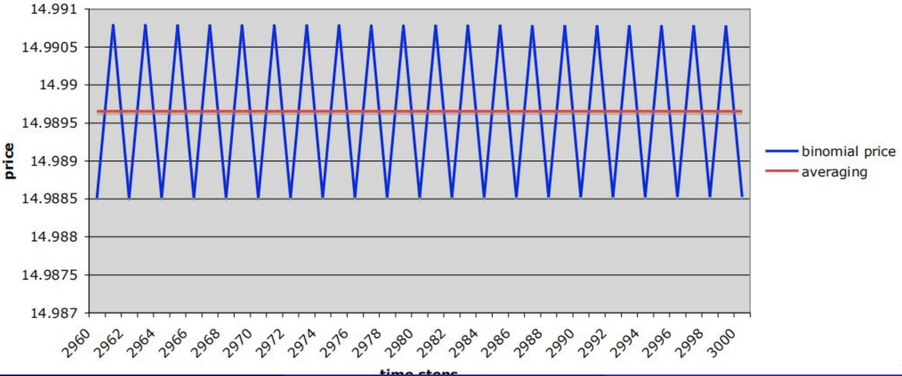
To illustrate this process, we use a put with $S_0 = \$30$, $K = \$40$:

				S	34.55729731
				V	5.442702695
		S	32.19811981	Continue	0
American Put		V	7.801880192	Exercise	5.442702695
S	30	Continue	7.702005088	S	30
V	10	Exercise	7.801880192	V	10
Continue	9.8503135	S	27.9519427	Continue	0
Exercise	10	V	12.0480573	Exercise	10
		Continue	11.94818219	S	26.04370336
		Exercise	12.0480573	V	13.95629664
				Continue	0
				Exercise	13.95629664

Figure: American Option Valuation Sample

Convergence of Binomial Tree

- It may be surprising to learn that the binomial tree, simple as it seems, is quite efficient. To use it, however, one must be aware of some of the model's oddities.
- The price shows oscillates around the value to which it converges. A simple solution is to average the output with N and N+1 steps. See Figure 2.



- OTM needs to increase N dramatically to guarantee the convergence (see Figure 3 and 4). Here we use European put option ($S_0 = 100, K = 120$) as we want to use BS as a benchmark.

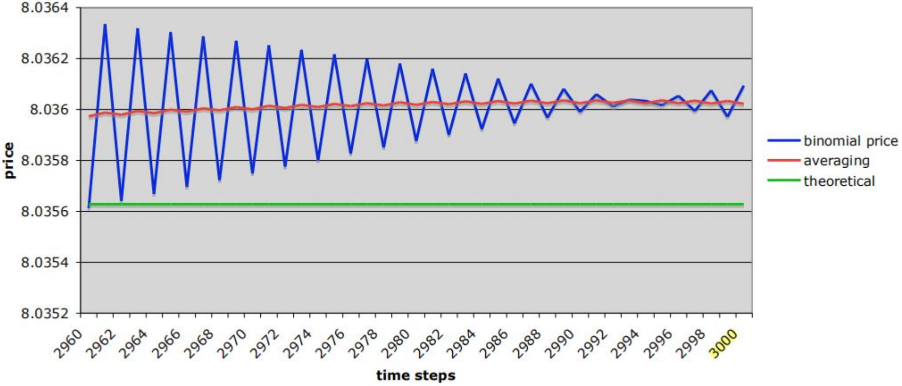


Figure: European OTM call does not converge to BS theoretical value

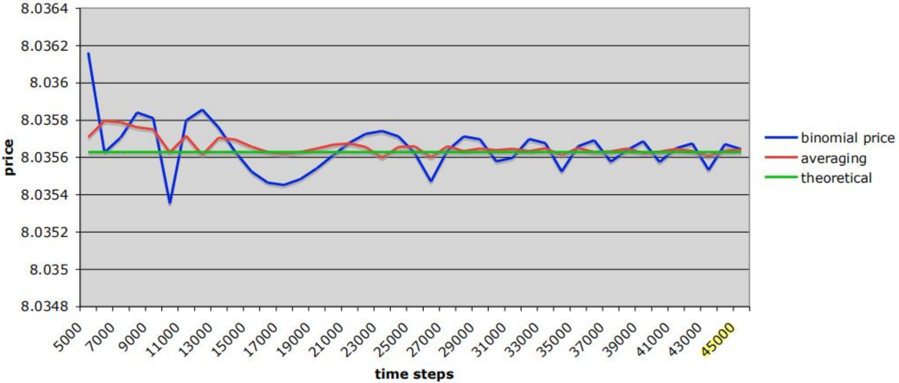
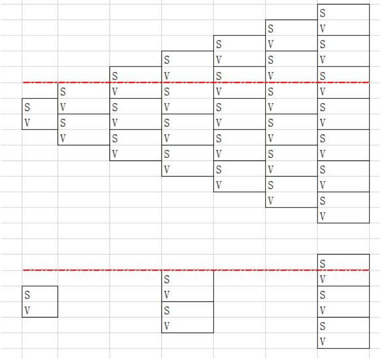


Figure: N increases from 3000 to 45000

Convergence of Binomial Tree

- The divergence arises from the fact that the set of points at the maturity - the discretization of S_T - assigns a set of discontinuous values to the option payoff. In particular, the nodes around the strike K have a great amount of influence on the price ultimately obtained.



- An extrapolation would solve this issue: using results from finite steps to estimate results from infinite steps.

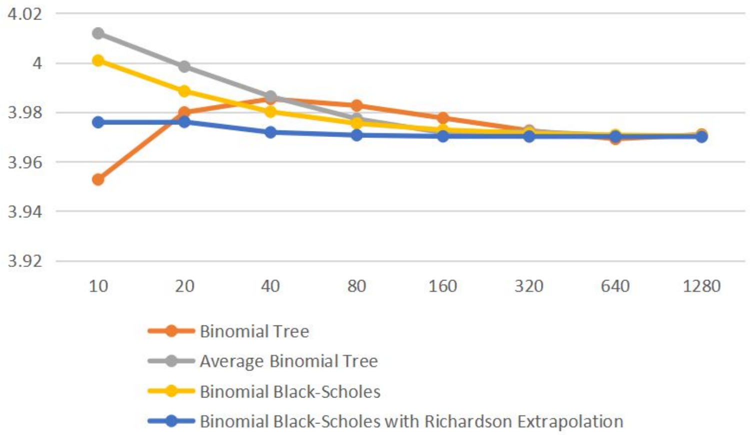
Binomial Tree Enhancements

- Average Binomial Tree
  Option value $V(N)$ is a function of number of steps N. Average binomial tree, is to output $\frac{V(N)+V(N+1)}{2}$ to remove oscillation in Figure 2.
- Binomial Black-Scholes with Richardson Extrapolation
  This is a variation of the binomial method, the Black–Scholes formula is applied to replace the continuation values at one time step before expiration and using extrapolation to solve the OTM convergence speed issue. $V(0) = 2V(2N) - V(N)$
- Unrealistic assumption of constant interest rate r and implied vol σ
  This is consistent with Black-Scholes assumption. Stochastic interest rate is often ignored in industry since rates vol is much smaller than equity vol. But constant implied vol assumption makes a big difference. Vol is usually an input from the calibrated vol surface. We will not discuss the vol surface here.

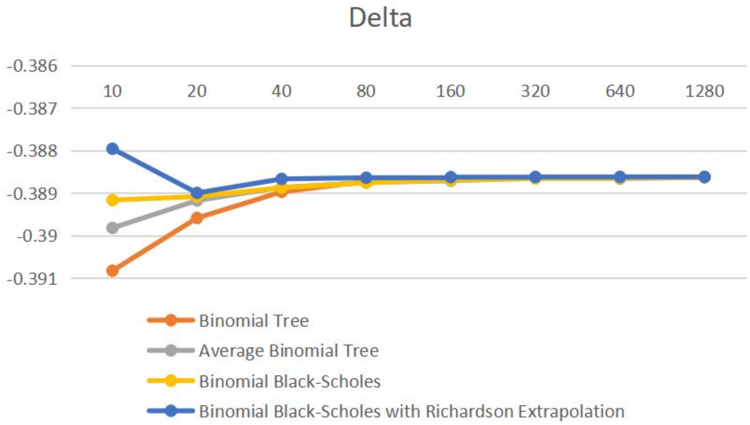
Brief Intro to Richardson Extrapolation

- Extrapolation is used as we want to estimate the option value with infinite steps by using finite N numbers (usually N = 3000);
- denote $x = \frac{1}{N}$ and option value V is function of x;
- Taylor expansion: $V(x) = V(0) + V'(0)x$ when x is close to zero;
- Similarly, we have $V(\frac{1}{2}x) = V(0) + V'(0)(\frac{1}{2}x)$;
- Thus we have $V(0) = 2V(\frac{1}{2N}) - V(\frac{1}{N})$, which means the final results can be extrapolated by involving results from time steps $2N$ and N

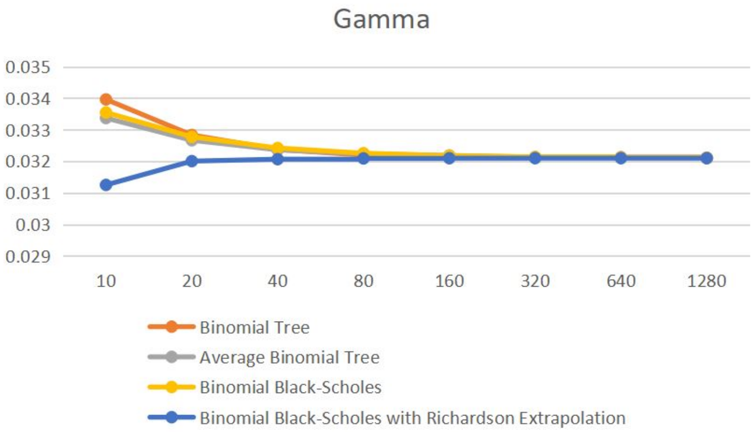
Binomial Tree Enhancements



Binomial Tree Enhancements



Binomial Tree Enhancements



Financial engineering and the choice of an appropriate method

- It seems that Richardson extrapolation can help at relatively low numbers of time steps, but at large numbers this technique adds relatively little value.
- This is not to say that Richardson extrapolation is a worthless technique in general; however, in cases where convergence to the theoretical value is slow, it can be quite helpful.

Optimal Exercise: American Call option

- ATM/OTM call: it is never optimal to early exercise since intrinsic value is zero compared to price of ATM/OTM is larger than zero.
- ITM call: we use a European call option as a lower bound of American call option and consider put call parity for European option:
  $C_0 - P_0 = S_0 e^{-qT} - Ke^{-rT}$
  $C_0 = S_0 e^{-qT} - K + K - Ke^{-rT} + P_0$
  Since American option is larger or equal to European option, the American call is always larger than ITM intrinsic value $S_0 - K$ given the no dividend $q = 0$ and the interest rate r is positive. Thus it is never optimal to early exercise ITM in this case.
- The cases above means American call is equal to European call.
- ITM with $q > 0$ and $r < 0$: consider a forward contract to be an approximation of a deep ITM option and the choice is between the continuation value and the intrinsic value. The continuation value is: $S_0 e^{-qT} - Ke^{-rT}$, whereas the intrinsic value of exercise is: $S_0 - K$

Optimal Exercise: American Put Option

- To begin with, consider a European put option

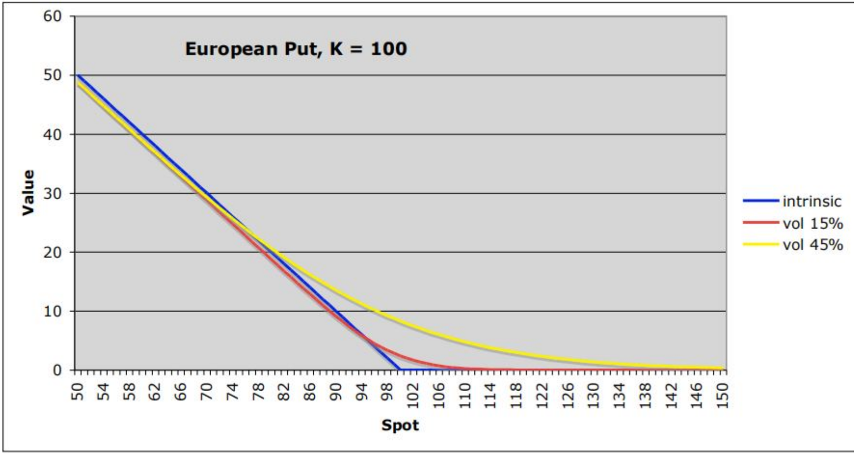


Figure:

Optimal Exercise: American Put Option

- ATM and OTM, if exercise, the option holder got zero. Thus it is not optimal to exercise.
- We see that the time value of a European put is negative below a threshold (ITM), and that this threshold occurs at a lower strike the greater the volatility; from Put-Call parity perspective: $P_0 = K - S_0 + Ke^{-rT} - K + C_0$
- Consider **one time step** before maturity for American option, at this time t, the American option becomes European option and we need to decide whether it should be exercised and there exist an threshold $S*$
- If $S_t \leq S*$, exercise put and get $K - S_t$; if $S_t > S*$, not exercise and get put option value V_t. See put option pricing sample in Page 11.
- $S*$ changes with time t, binomial tree provides an effective way to visualize this threshold - early exercise region;

Early Exercise Region

ATM put with $K = 100$ and $\sigma = 0.4$

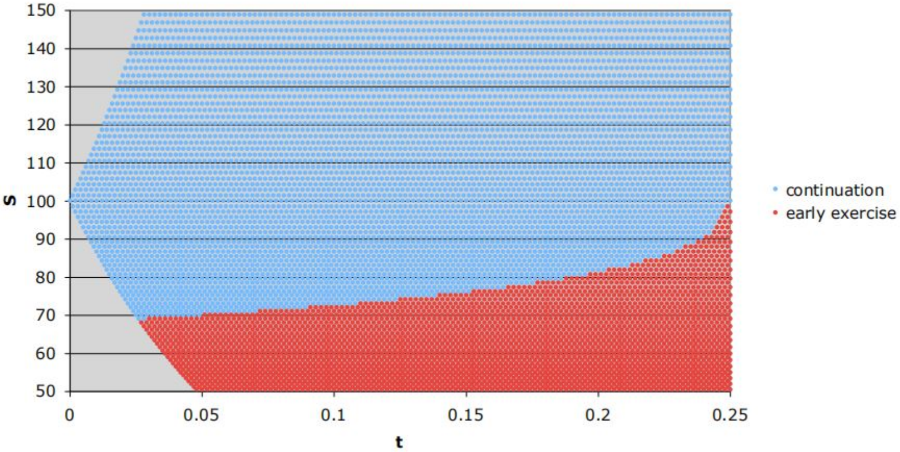


Figure: Early Exercise Region of American Put by Binomial Tree

Put-Call Parity for American Option

- Put-Call parity is model free;
- For European option, put-call parity is:
  $C_0 - P_0 = S_0 - Ke^{-rT}$
- For American option, put-call parity is different as. See details in Appendix C (Page 32):
  $S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$

References

1 J. Cox, S. Ross, and M. Rubinstein, "Option Pricing: A Simplified Approach," Journal of Financial Economics, 7(3), 1979 pp. 229–263.

2 M. Broadie and J. Detemple, "American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods," The Review of Financial Studies, 9(4), 1996 pp. 1211–1250.

Q&A

Appendix A: Deriving CRR tree parameters

The main thought here is to match equity mean and variance in u and d and given the equity price dynamics is lognormal $dS_t = rS_t dt + \sigma S_t dW_t$. Thus to show that CCR tree, essentially, is a risk-neutral lognormal process.

we know that equity price is assumed to be a lognormal process $dS_t = rS_t dt + \sigma S_t dW_t$. By applying *Ito's Lemma* to lnS_T, we have

$$lnS_T \sim N(lnS_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T)$$

$$\mathbb{E}[S_T] = S_0 e^{rT}, Var[S_T] = S_0^2 e^{2rT}(e^{\sigma^2 T} - 1)$$

Similarly, given a S_t the $S_{t+\Delta t}$ is:

$$lnS_{t+\Delta t} \sim N(lnS_t + (r - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t)$$

$$\mathbb{E}[S_{t+\Delta t}|S_t] = S_t e^{r\Delta t}$$

Appendix A: Deriving CRR tree parameters

$$
\begin{aligned}
Var[S_{t+\Delta t}|S_t] &= S_t^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) \\
&\approx S_t^2(1 + 2r\Delta t)(1 + \sigma^2 \Delta t - 1) \\
&= S_t^2 \sigma^2 \Delta t + S_t^2(2r\Delta)(\sigma^2 \Delta t) \\
&\approx S_t^2 \sigma^2 \Delta t
\end{aligned}
$$

To match $\mathbb{E}[S_{t+\Delta t}|S_t]$:

$$p_u u S_t + p_d d S_t = S_t e^{r\Delta t}$$

$$p_u = \frac{e^r \Delta t - d}{u - d}$$

To match the variance: $Var[S_{t+\Delta t}|S_t] = [S_{t+\Delta t}^2|S_t] - (\mathbb{E}[S_{t+\Delta t}|S_t])^2$

Appendix A: Deriving CRR tree parameters

$$
\begin{aligned}
\sigma^2 \Delta t &= p_u u^2 + p_d d^2 - (p_u u + p_d d)^2 \\
&= p_u(1 - p_u)(u - d)^2 \\
&= e^{r\Delta t}(u + \frac{1}{u}) - 1 - e^{2r\Delta t}
\end{aligned}
$$

$$
\Rightarrow u + \frac{1}{u} = e^{-r\Delta t}\sigma^2 \Delta t + e^{-r\Delta t} + e^{r\Delta t}
$$

$$
\Rightarrow u^2 - (\sigma^2 \Delta t + 2)u + 1 = 0
$$

Solve this we have:

$$
\begin{aligned}
u &= \frac{\sigma^2 \Delta t + 2 \pm \sqrt{(\sigma^2 \Delta t + 2)^2 - 4}}{2} \\
&= \frac{\sigma^2 \Delta t}{2} + 1 \pm \sigma\sqrt{\Delta t} \\
&\approx 1 \pm \sigma\sqrt{\Delta t} \approx e^{\sigma\sqrt{\Delta t}}
\end{aligned}
$$

Appendix B: American Put-Call Parity

American Put-Call Parity: $S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$

Proof:
For upper bound when $S_0 > K$:

$$\begin{aligned} P_0^{Amer} &\geq P_0^{Euro} \\ &= C_0^{Euro} - S_0 + Ke^{-rT} \\ &= C_0^{Amer} - S_0 + Ke^{-rT} \end{aligned} \tag{3}$$

$$C_0^{Amer} - P_0^{Amer} \leq S_0 - Ke^{-rT}$$

For lower bound when $S_0 < K$:
Given a portfolio $C_0 - P_0 - S_0 + K$, we will discuss 2 cases where put is either exercised or not (since OTM call is never optimal to early exercise).
If put is exercised:

$$C_0 - P_0 - S_0 + K = C_0 - (K - S_0) - S_0 + K = C_0 \geq 0$$

Appendix B: American Put-Call Parity

if put is not exercised early and hold to maturity:

$$(S_T - K)^+ - (K - S_T)^+ - S_0 + K = 0$$

Thus we proved both lower and upper bound.